AN INVESTIGATION INTO CHILDREN'S UNDERSTANDING OF THE ORDER OF OPERATIONS

CAROLINE HEADLAM

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AN INVESTIGATION INTO CHILDREN’S UNDERSTANDING OF
THE ORDER OF OPERATIONS

By

CAROLINE HEADLAM

A thesis submitted to Plymouth University
in partial fulfilment for the degree of

DOCTOR OF PHILOSOPHY

School of Computing and Mathematics
Faculty of Science and Technology

March 2013
ABSTRACT

Caroline Headlam

An investigation into children’s understanding of the order of operations

This thesis reports on the findings of an international study into the way in which children approach calculations which involve the order of operations. The study involved 203 pupils aged between 12 to 14 years from four different countries: England, The USA (New York State), Japan and The Netherlands.

Many pupils in England are taught to use mnemonics such as BODMAS or BIDMAS to remember the correct order of operations, and in the USA pupils are often taught to use PEMDAS. However in Japan and The Netherlands these methods are not used, and the approach to teaching mathematics differs considerably across the countries.

In this study pupils from classes in these four countries have been given calculations to perform and their work has been analysed for misconceptions. The analysis of their work has involved use of the Key Recorder software as a data collection tool, in which the pupils’ calculator keystrokes have been recorded and played back to give a unique insight into their thinking.
Analysis of the children’s work has resulted in the categorisation of the misconceptions that were observed, and suggests that the nature of the mathematics curriculum and the teaching methods employed may have a significant effect on the way in which children approach calculations of this sort.
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ACKNOWLEDGEMENTS

This journey began six years ago when I left my teaching post to become a Lecturer in Mathematics Education at Plymouth University. I had already achieved my MPhil with the support of Ted Graham and all the team in the Centre for Teaching Mathematics at Plymouth University, and now it was time to embark upon my PhD.

I would like thank the following people for their love, their help, their support and their professional advice which has made it all possible and worthwhile:

Without Ted Graham I don’t think I would have ever reached this point. Ted has been inspirational and such a supportive supervisor. Thank you, Ted, for all the time you have devoted to reading my work and for all your help and advice. Thanks too to Suki Honey and Taro Fujita for all your support and encouragement, and everyone at the Centre for Teaching Mathematics for their help and friendship; Jenny Sharp, Brian Watson, Stuart Rowlands and Julie Tombs who transcribed all the pupil interviews. Thank you to the teachers who welcomed me into their schools and allowed me to interview them, and to Mary Ann Harasymowycz for organising the American side of things, and for bringing all those worksheets across the Atlantic with her.

Sophie and Beth, my beautiful daughters, thank you for always being there with love and unconditional support, especially towards the end when things were feeling tough. Sophie your editing skills have been invaluable. My brother Adrian, for reading through this entire thesis and giving such detailed and helpful advice, and also for your love and support and for just being there whenever I have needed you.
There are so many friends who I would like to thank for supporting and encouraging me along the way – you know who you are – and especially Gina Tenquist for always taking such an interest in my work and for being an amazing friend. Thank you also Brian for all your encouragement.

Finally thank you to my mum and dad, Audrey and Bill Bowden, I know you are proud of me.

xxx
AUTHOR’S DECLARATION

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award.

A programme of advanced study was undertaken, which included regular presentations at the Centre for Teaching Mathematics at Plymouth University. The author also attended the bi-annual Graduate Student Week programme at Plymouth University throughout the duration of this research. Relevant seminars and conferences were attended.

Publications:

Word count of main body of thesis (excluding figures, tables and appendices): 72,924

Signed………………………………….

Date………………………………….
CHAPTER 1: INTRODUCTION

1.0 Biographical notes

I began my teaching career in 1983 having completed my BSc in Mathematics and Physics and a PGCE in Secondary Mathematics. I have always loved mathematics, and since the beginning of my teaching career I have been interested in finding out as much as I can about how children learn mathematics; I have also always been fascinated and concerned by the fact that mathematics seems to be generally regarded (in the UK at least) as a subject that is extremely difficult and, to many people, even frightening. The PhD study by Picker (2000) provided evidence that children in many countries find mathematics difficult and many have very negative images of what it is like to be a mathematician. My experiences as a mathematics teacher in secondary schools in the UK seems to have confirmed the fact that many children, and their parents, appear to regard mathematics as the most difficult subject in the curriculum, some having a real fear of the subject, and throughout my teaching career I have looked for ways of fostering both enjoyment and achievement in my pupils. I believe that teaching to avoid misconceptions, and developing the ability to recognise and address misconceptions is a fundamental part of this.

In 2004 I achieved my MPhil (Headlam, 2004) with a study relating to the ways in which mathematics teachers in the UK used graphics calculators in their teaching. This led me to investigate the work of Smith et al (2003) which pointed to the use of the graphics calculator as a potential research
tool. With the advice of my colleagues in the Centre for Teaching Mathematics at Plymouth University I decided to embark upon a study which utilised the graphics calculator as a way of looking into the way that children think, mathematically, and to look in depth into a very specific area of mathematics teaching and learning.

1.1 Background to the study

During my teaching career as a secondary mathematics teacher I worked in four very different secondary schools in and around Plymouth. Over the years I have experienced a great variety of different teaching and learning styles; sometimes because this was required of me by the school or department that I was working in, and sometimes because of my own personal beliefs about how effective learning can take place. I have used a variety of different teaching and learning resources, which have developed over the years as the curriculum and current thinking have changed. These have ranged from the individualised learning schemes, such as the SMP 11-16 scheme, that were very popular in the period immediately following the Cockroft report “Mathematics Counts” (1982) to text books produced to address particular National Curriculum Levels and particular GCSE and A-Level specifications, some of which have been aimed at whole-class teaching and others of which have been designed to encourage other learning styles such as collaborative work in pairs or in groups. For example, “National Curriculum Mathematics” (Vickers and Tipler, 1995)
was a series of textbooks, each aimed at a particular level of the National Curriculum at the time, and presented in a way that could be used flexibly. It was aimed at enabling ‘pupils, or groups of pupils, to work across different areas of mathematics at different levels’. These texts were popular and used in many schools in the 1990s, and were regarded as particularly useful resources for use with mixed-ability teaching groups. An example of another popular series of textbooks aimed at a particular examination board is “London GCSE Mathematics” (Pledger et al, 1996) which was produced to address each of the three tiers of GCSE mathematics that existed at the time, and was written in a way that encouraged whole-class teaching and learning, with pupils grouped according to ability.

As technology has developed, and in particular the use of interactive whiteboards, teaching resources have become increasingly interactive and mathematics lessons in the UK have been less dependent upon a set textbook and more likely to utilise a range of resources. Over recent years the internet has opened up an extensive range of teaching and learning resources, with many schools using web-based schemes such as “MyMaths”. Although this may seem to increase the choices that the class teacher has over the use of a range of resources, it could be argued that in some respects it can have the effect of narrowing down the flexibility, since some web-based schemes produce prescribed lessons with set examples, thus limiting the teacher’s choice of materials and methods of delivery.
This may also have the effect of limiting the teacher’s flexibility to explore different methods which may be suggested by the pupils.

With increasing experience and responsibility I have been able to make decisions about the most effective ways that I believe different mathematical concepts may be understood and consequently about the teaching methods that can be best employed to effect this understanding. Clearly, however, this has always been constrained by the requirements of the National Curriculum, in which required teaching methods have been set out very rigorously. Indeed, when the National Numeracy Strategy (DfEE, 1999a) was first introduced into primary schools in 1999 the “numeracy hour” which took place each day was prescribed down to the number of minutes spent on each part of the hour-long session, with a heavy emphasis on mental and written methods and a requirement for whole-class teaching.

Evaluations of the National Numeracy Strategy (NNS) within the Leverhulme Numeracy Research Programme revealed significant changes to the ways in which primary teachers planned and delivered numeracy lessons, for example Brown et al (2003) observed that teachers’ planning tended to focus more on objectives than activities, and that there was confusion about whether the NNS objectives should be regarded as teaching objectives or learning objectives. McNamara and Corbin (2001) reported variation in the way in which teachers engaged with the NNS but noted that the training given to teachers was considered important by many teachers, and this was echoed by Brown et al who concluded that “One extremely
positive result, we feel, of the NNS is that on-going professional
development now has the feel of common practice” (Brown et al, 2003, p. 18) Vorderman (2011, p. 40) voices concerns that this has constrained primary teachers as to the way in which they teach mathematics, forcing them into line with the documentation and guidance accompanying the National Strategy. Vorderman comments on the rigid way in which some primary teachers have interpreted the guidelines for teaching methods in arithmetic:

‘The Primary Strategy has specified ‘chunking’ as a form of long division and ‘grid multiplication’ as a form of long multiplication. These were meant to be staging methods, helping children’s understanding before going on to the more formal methods they will need in Key Stages 3 and 4, but many teachers have seen them as ends in themselves’

(Vorderman, 2011, p. 41)

The Numeracy Strategy fed through into secondary schools in the form of the Key Stage 3 Strategy and June 2001 saw the publication of the “Framework for Teaching Mathematics” (DfEE, 2001a) which set out the learning objectives of the National Curriculum in a year-by-year plan. A supplement of examples exemplified the learning outcomes for each year and encouraged a variety of approaches for delivering the curriculum, with an emphasis on appropriate use of ICT. These examples have been retained in exactly the same form in the latest version of the Mathematics Framework (DCSF, 2008b). Lesson timings were not prescribed so tightly as they had been in the primary “Numeracy Hour” and different teaching styles were encouraged, although a strong emphasis on whole-class teaching
prevailed with the three-part lesson as a requirement. This involves an oral or mental starter, a main teaching and learning activity and a plenary. Consequently the use of individualised learning schemes such as SMP (The Schools Mathematics Project) and KMP (The Kent Mathematics Project) decreased as whole-class teaching was accepted as the expected way to deliver the mathematics curriculum. This had a significant effect on the way in which pupils were grouped into classes; whereas many schools had taught mathematics to mixed attainment groups for mathematics in some or all of Key Stage 3 it became increasingly common to group pupils according to attainment from the beginning of year 7, in order to facilitate whole-class teaching approaches. This move from mixed attainment to whole class teaching in mathematics is noted by Gillard (2008) who asserts that in this period of time whilst most subjects were taught in mixed attainment groups, mathematics lessons were increasingly being taught in sets according to the pupils’ attainment.

From my experience of managing mathematics departments at this time in the development of the National Curriculum I believe that the strong emphasis on targets has encouraged a culture of “teaching to the test” so that pupils are able to jump through the hoops involved in achieving a certain level, without necessarily achieving a thorough conceptual understanding. Vorderman (2011) concludes that with regard to the Key Stage 2 National Curriculum Tests:
The test sets up a conflict of interests between children and their schools. The children’s interest is to develop their mathematical understanding continually, throughout their education; their school’s interest is to maximise the number of children obtaining level 4 in the test. The two are not the same.

(Vorderman, 2011, p.44)

Although the Key Stage 3 National Curriculum Test was abolished in 2009 it is rather too soon to judge whether this has had an effect on the way in which mathematics is taught in Key Stage 3. However, many schools are now choosing to begin GCSE mathematics in year 9 (age 13 – 14), which in some schools means that pupils will take at least one GCSE module whilst they are in year 9. These results will not be reported in league tables in the same way that the National Curriculum Test results were reported, but it still means that some pupils are being prepared for a public examination in year 9, which may well have an effect on the way in which mathematics is taught.

In my current role as a lecturer in Mathematics Education I have the privilege of visiting many secondary schools across the South West of England where I observe trainee teachers and, of course, children at work in their classrooms, doing mathematics. I have the responsibility of preparing my trainees to become effective teachers of mathematics and in order to do this one of the major areas for consideration is the appreciation of the nature of children’s misconceptions, and an awareness of effective teaching strategies which can be employed to prevent these misconceptions, or to address them if they are already there.
1.2 Order of Operations

One of the essential concepts in the understanding of arithmetic is the concept of “Order of Operations”. The current National Curriculum for Mathematics (DCSF, 2008a) sets out the Range and Content that should be covered in the mathematics curriculum, and splits this into three categories:

1. Number and algebra
2. Geometry and measures
3. Statistics

The content for **Number and Algebra** in Key Stage 3 is defined as follows:

*The study of mathematics should include:*

(a) *Rational numbers, their properties and their different representations*

(b) *Rules of arithmetic applied to calculations and manipulations with rational numbers*

(c) *Applications of ratio and proportion*

(d) *Accuracy and rounding*

(e) *Algebra as generalised arithmetic*

(f) *Linear equations, formulae, expressions and identities*

(g) *Analytical, graphical and numerical methods for solving equations*

(h) *Polynomial graphs, sequences and functions*

(DCSF, 2008a, p. 145)
Within the “explanatory notes” the rules of arithmetic are described as follows:

**Rules of arithmetic:** This includes knowledge of operations and inverse operations and how calculators use precedence. Pupils should understand that not all calculators use algebraic logic and may give different answers for calculations such as $1 + 2 \times 3$.

(DCSF, 2008a, p. 145)

These definitions are highly significant in determining the way in which number and algebra are taught in England. There is a clear sense of hierarchy and order within the list of topics, and crucially section (e) defines algebra as “generalised arithmetic”. The explanatory note provides an expectation that this work will include the use of calculators.

In the National Strategies Secondary Mathematics Exemplification (DCSF, 2008b) the learning objective relating to this states that pupils should be taught to: *Know and use the order of operations, including brackets*. (p. 86). This is exemplified as shown in figure 1.1:

![Diagram](image)

**Figure 1.1 Exemplification from National Curriculum Framework**

(DCSF, 2008b, p. 86)
Examples are given of learning outcomes that pupils in year 7 should be able to do. These include:

Calculate with mixed operations. For example:

Find mentally or use jottings to find the value of:

\[
\begin{align*}
16 ÷ 4 + 8 &= 12 \\
16 + 8 ÷ 4 &= 18 \\
14 × 7 + 8 × 11 &= 186 \\
32 + 13 × 5 &= 97 \\
(3² + 4²)² &= 625 \\
(5² - 7) / (2² - 1) &= 6
\end{align*}
\]

Use a calculator to calculate with mixed operations, e.g.

\[
(32 + 13) × (36 – 5) = 1395
\]

(DCSF, 2008b, p. 86)

By year 8, an example of a learning outcome is given as

Calculate with more complex mixed operations, including using the bracket keys on a calculator. For example:

Find the value of: \(2.1 – (3.5 + 2.1) + (5 + 3.5)\) \(= 5\)

Find, to two decimal places, the value of:

\[
\sqrt{(26² - 14²)} = 21.91\ \text{to 2 d.p}
\]

(DCSF, 2008b, p. 87)
Another example of a year 8 learning outcome for the same learning objective is that a pupil should understand that the position of brackets is important. A suggested task for this outcome is to make as many different answers as possible by putting brackets into the expression

\[ 3 \times 5 + 3 - 2 \times 7 + 1 \]

By year 9 the learning outcomes include the understanding of the effect of powers when evaluating an expression, for example to be able to evaluate

\[ \frac{7 \times 8^2}{7 \times 2} \]  

(DCSF, 2008b, p.87)

Another important example of a year 9 learning outcome is to recognise that

\[ (–a)^2 \neq –a^2 \]  

(DCSF, 2008b, p.87)

This objective is also linked to calculator methods, with the expectation that a pupil should be taught to *carry out more complex calculations using the facilities on a calculator* (DCSF 2008b, p. 108) and with the order of algebraic operations, where pupils are expected to *know that algebraic operations follow the same conventions and order as arithmetic operations*. (DCSF, 2008b, p. 114)

The Framework makes it clear that pupils are expected to be able to use a scientific calculator efficiently when evaluating more complex mixed operations.
More examples of learning outcomes for years 7, 8 and 9 are given in the appropriate pages from the Framework document, which are included in appendix A.

The importance of the understanding of the order of operations was emphasised in a report for the Channel 4 television documentary “Dispatches” entitled “Kids Don’t Count” in February 2010 (TimesOnline, 2010) which investigated the arithmetical abilities of primary school teachers. The first statistic that was reported was the finding that only 20% of the teachers surveyed were able to correctly work out the answer to the calculation $4 + 2 \times 5$. As well as emphasising the importance of children’s understanding of this concept the Dispatches report considers the necessity of children being taught important mathematical concepts by a teacher who has a clear understanding of the concepts. This relates to the recommendations of the Williams Review into the teaching of mathematics in primary schools (DCSF, 2008c). The questions used in this survey were produced by Richard Dunne who wrote a follow-up article on Symbolic Maths (Dunne, 2010) in which he discusses the importance of mathematics as a language and the link between arithmetic and the logic of algebraic understanding:

‘Mathematics is a logical, symbolic language. It is the clarity of its logic, the economy of the symbols and the fact that ‘it talks to you’ that makes it intrinsically exciting. It works like written English works (only twenty-six letters but thousands of words) or how music works (with crotchets and quavers etc for thousands of tunes)’

(Dunne, 2010 p. 1)
The link with number and language was noted in a recent comparison of mathematical attainment in high-performing countries by Askew et al (2010) for the Nuffield Foundation, who devote a section to “The Language of Number”. Within this they consider whether early differences in Asian languages could contribute to the different addition strategies that have been observed among Asian children based on partitioning digits rather than counting in ones, compared to those observed among their American peers. They speculate whether this may contribute to the higher attainment of the Chinese children.

Considering the way that algebra is defined within the National Curriculum Framework in England, in terms of being generalised arithmetic, it could be argued that the understanding of the concept of the order of operations is a first step to acquiring this feel for the logic of algebra. This is why the question $4 + 2 \times 5$ was regarded as such a fundamental part of a test of arithmetic and mathematical understanding. To feel the logic behind this calculation could be seen as a step towards understanding the meaning of an algebraic expression such as $a + bc$. This is certainly how the majority of texts set out these concepts in this country.

1.2.1 Using Mnemonics

The methods for teaching this topic can vary, but one common theme in some countries is to use a mnemonic to aid the memorisation of the order of
operations. In the UK this is commonly **BIDMAS** or **BODMAS**, as exemplified in figure 1.2:

<table>
<thead>
<tr>
<th>Brackets</th>
<th>Brackets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Indices</td>
<td>Of (Order)</td>
</tr>
<tr>
<td>Division</td>
<td>Division</td>
</tr>
<tr>
<td>Multiplication</td>
<td>Multiplication</td>
</tr>
<tr>
<td>Addition</td>
<td>Addition</td>
</tr>
<tr>
<td>Subtraction</td>
<td>Subtraction</td>
</tr>
</tbody>
</table>

**Figure 1.2** Exemplification of BODMAS and BIDMAS

The last four operations are sometimes remembered by **My Dear Aunt Sally** which interestingly gives the letters in a different order. This is not incorrect since the order of the D and the M does not matter, but it could be argued that it creates more opportunity for confusion in pupils who are learning this concept.

In the USA the mnemonic **PEMDAS** is commonly used, as exemplified in figure 1.3:

<table>
<thead>
<tr>
<th>Parentheses</th>
<th>Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication</td>
<td>Division</td>
</tr>
<tr>
<td>Addition</td>
<td>Subtraction</td>
</tr>
</tbody>
</table>

**Figure 1.3** Exemplification of PEMDAS used in the USA
and this is sometimes remembered by using

Please Excuse My Dear Aunt Sally

The use of mnemonic devices is quite commonly used in this country, in various contexts, for learning many kinds of facts, not just in mathematics. For example:

**Richard Of York Gave Battle In Vain**

For remembering the order of the colours of the rainbow:

**Red Orange Yellow Green Blue Indigo Violet**

In mathematics another commonly used mnemonic device is

**SOH CAH TOA**

Which is often used to remind pupils of the trigonometric ratios

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}
\]

Many mathematics teachers in the UK use the mnemonic “**CAST**” to remember which of the trigonometric ratios is positive in each quadrant of the unit circle, as shown in figure 1.4:

![Figure 1.4 “CAST” for remembering which trigonometric ratio is positive in each quadrant of the unit circle](image)

C: Cosine
A: All
S: Sine
T: Tangent

**Figure 1.4 “CAST” for remembering which trigonometric ratio is positive in each quadrant of the unit circle**
For expanding two brackets there is the “eyebrows and smile” approach as shown in figure 1.5:

![Figure 1.5 “Eyebrows and Smile” for remembering how to multiply two brackets together](image)

This is also sometimes taught using the mnemonic **FOIL** (First, Outside, Inside, Last) although clearly this is not at all necessary since the order in which the terms are multiplied together does not matter; indeed it seems confusing to suggest that there is a preferred order in which to multiply them.

If a mnemonic is to be used, it is clearly important that the pupil understands the context in which it is being applied; as pointed out by Tanner and Jones (2000)

> ‘**SOHCAHTOA** is no use without knowledge of the association between sides and letters. Thus it should be learned in tandem with the lettering convention for right-angled triangles’

(Tanner and Jones, 2000, p. 25)
However the order of arithmetic operations is not a fact, but a convention, where mathematical logic is applied, so it could be argued that the use of a mnemonic in this situation covers up the need for understanding the logic behind it; according to Tanner and Jones (2000)

‘this is the point at which we think the use of mnemonics becomes dubious. BODMAS strikes us as a mnemonic which reduces a principled operation to mindless rule following’

(Tanner and Jones, 2000, p. 25)

Although this approach is used by many teachers in this country to encourage pupils to remember the “rule” it is clearly the understanding of the underlying principles and conventions that is essential for the convention to be put into practice. This includes the understanding of index notation and the recognition of a fraction for division.

Thus it is far from merely being a case of learning a mnemonic; a sound understanding of mathematical notation and structure is required in order to carry out a calculation of the type given in the Framework for Teaching Mathematics. It is this deep understanding that lays the foundations for an understanding of algebraic structure.

It is interesting to note that the use of mnemonics is not referred to at all in the National Curriculum documentation, and yet many text books and other resources encourage it. Some examples from recent school textbooks and
other classroom resources are given in Appendix B. These text books and teaching resources are fairly typical of those used in secondary schools in the UK.

In my visits to different mathematics classrooms I have often seen BODMAS or BIDMAS displayed in various ways around the classrooms. Some examples are included in figures 1.6 to 1.9:

![BODMAS on display in a mathematics classroom in the UK](image)

**Figure 1.6** BODMAS on display in a mathematics classroom in the UK
Figure 1.7 BIDMAS over the whiteboard in a mathematics classroom in the UK

Figure 1.8 “BODMASTER” in a display in a mathematics classroom in the UK
In fact, BODMAS almost seems to have become accepted as a mathematical word in its own right. The instructions in this puzzle in a daily newspaper make the assumption that the readers will know what it means, as can be seen in figure 1.10:

![Figure 1.10 “Set Square” Puzzle in The Times (22nd July 2010)](image-url)
1.2.2 The use of calculators

The role of the calculator in the teaching and learning of this mathematical concept is an interesting one, and in the Framework for Teaching Mathematics (DCSF, 2008) there are learning objectives which link the use of the calculator with the understanding of order of operations. The key learning objective states that pupils should be taught to *carry out more complex calculations using the facilities of the calculator* (DCSF, 2008 p. 108) and an example of an expected outcome by year 9 is that pupils should be able to *use a calculator to evaluate more complex expressions such as those with nested brackets or where the memory function could be used*. (DCSF, 2008, p. 109)

An example of a year 9 learning outcome is

Find, to two decimal places, the value of

\[
\frac{(12 - 5)^2 (7 - 3)^2}{(8 - 3)^3}
\]

(DCSF, 2008a, p. 87)

What the learning objectives do not make entirely clear are the significant differences faced by students using different types (and makes) of calculator. Most scientific calculators will have brackets keys and operate with the correct conventions, and may be classified as “Multiplication and Division First” (MDF) whilst the majority of “basic” calculators will
operate on a Left to Right basis (LTR) which makes a calculation such as the one given above much more complex to carry out. Ironically, though, it is often the lower-achieving pupils who own a basic calculator and thus find the work even more complex.

1.3 Aims and Research Questions

My own experiences of teaching pupils about the order of operations have led me to see that many have difficulty with this concept; studies with adults (for example Dunne, 2010) confirm that these difficulties can remain through to adulthood. I therefore wanted to investigate why pupils find this difficult and whether pupils in other countries experience the same difficulties as those in the UK, and I was interested to consider the factors that might be significant in pupils’ understanding and achievement in this area.

The primary aim of this investigation was to examine the ways in which pupils perform calculations which require the correct use of the order of operations, to study the misconceptions and errors that may arise and to investigate the ways in which the children’s misconceptions and working styles may have been affected by the teaching methods employed.

An important tool that was utilised in order to carry out this investigation was a piece of software that was developed as a research tool by Texas Instruments in conjunction with the University of Plymouth. This software is called the Key Recorder and can be loaded onto the more recent models
of the TI graphics calculator. It has been used as a data collection tool in a small number of research projects (For example: Graham, Headlam, Honey, Sharp and Smith, (2003), Berry, Graham and Smith (2003), Smith (2003) Berry, Graham and Smith (2005), Berry, Graham and Smith (2006), Sheryn (2005), Sheryn (2006a), Sheryn (2006b) Graham, Headlam., Sharp and Watson (2007)).

The study has involved children in the UK, Japan, the USA and the Netherlands. All the children had been taught about the Order of Operations and could therefore be expected to be able to perform calculations based upon these principles. The children were all in the age range 12 – 14 (Years 8 and 9 in UK schools).

My overall research aim was to investigate whether it was possible to categorise the misconceptions that were observed, and to determine how these might have been related to the nature of the mathematics curriculum and the approaches to teaching mathematics in the different countries. This study involved the analysis of pupils’ workings and calculator key recordings, along with interviews with pupils and teachers in order to investigate how the nature of the curriculum influences teaching methods and approaches, and how this in turn influences the way in which pupils understand and carry out calculations involving order.
The research questions that were to be addressed can be summarised as follows:

- What misconceptions do children display when using the principles of order of operations in calculations?

- How can these misconceptions be classified?

- How might these misconceptions depend upon the teaching methods employed?
CHAPTER 2: A REVIEW OF THE LITERATURE

2.0 Introduction

The literature review in this chapter examines some of the relevant work carried out by other researchers on the themes of teaching and learning arithmetic and algebra, the theories about how children learn and the implications for teaching and for mathematics curriculum development.

The first section of this chapter discusses the theories relating to pupil misconceptions, how children learn mathematics, why they make mistakes and how good teaching can be planned to avoid misconceptions, to identify misconceptions that already exist and to address them.

The second section discusses numerical thinking and investigates some studies that have been carried out into the acquisition of ‘number sense’ and into teaching approaches and learning environments that can be effective in fostering it.

The third section investigates and discusses the relationship between arithmetic and algebra, contrasting different theories of when and how algebra should be introduced and discussing the statement in the current mathematics framework in England that algebra should be viewed as ‘generalised arithmetic’. Various studies into children’s acquisition of algebraic thinking are examined and discussed, and the implications for curriculum development are considered. The understanding of arithmetic
and algebraic structure is considered, along with the implications for the
nature of effective teaching and learning resources.

The fourth section investigates the ways in which technology and
calculators are currently used in the teaching and learning of number and
algebra, the advantages and disadvantages of different uses of technology
and the implications for teaching and learning and for the ways in which
pupils are tested and assessed.

The fifth section considers the theories behind the development of early
algebra, and discusses the findings of researchers who have studied the
development of algebraic thinking in children of primary age.

2.1 Misconceptions

Children can make mistakes in mathematics for a number of different
reasons. They can be careless errors, due to a lack of concentration, failure
to remember a mathematical fact or to pick out the correct information from
a problem. They may, however, be the result of an underlying
misconception.

One of the underlying principles of constructivism is that children construct
their own knowledge and understanding, and ‘as such we should not see
mathematics as something that is taught but rather something that is learnt’
(Hansen, 2005, p. 3). It follows that if children construct their own
meaning, they will at times make errors, and that a fundamental role of the
teacher is to understand how to make sense of the errors that children make and help them move to a correct understanding. In order to do this effectively it is important to consider learning mathematics with regard to various aspects which need to include: the teacher, the curriculum, the classroom environment and the individual child.

What is a misconception? According to Chambers (2010),

‘Misconceptions are not the same as mistakes. Anyone can make mistakes in mathematics; mistakes can occur even when the underlying work is thoroughly understood and are likely to be the result of carelessness or tiredness. Misconceptions are systematic errors. Misconceptions produce wrong answers but the arguments that lead to the answers can be explained, and the same error will be made time and time again. In other words, misconceptions are incorrect understandings of the mathematics’.

(Chambers, 2010, p.107)

Rowlands et al (2005) suggest that a misconception can be viewed as

‘A misunderstanding, an alternative understanding, an alternative concept, a different point of view?’

(Rowlands et al, 2005, p. 3)

Cockburn and Littler (2008) stress that misconceptions need not necessarily be viewed as ‘bad things’ and that ‘on the contrary, they often reveal much about children’s thinking and how they acquire – or not, as the case may be – an understanding of complex mathematical concepts’ (p. 3) and they make the point that children’s incorrect methods of tackling a problem can sometimes be used as a discussion point.
If a pupil brings a misconception to the learning of a new topic, this will affect the way in which they understand the new piece of mathematics that they are learning, and so the study of the nature of misconceptions is crucial to the way in which teachers plan their teaching. It is essential for a teacher to be able to expect, recognise and identify misconceptions, to understand how they come about and how they can be prevented. A teacher needs to consider how to avoid misconceptions being exacerbated by a certain teaching or learning approach, and it is also important to understand how misconceptions can be remedied. Shulman (1986, p.10) considers the study of misconceptions to be ‘the point at which research on learning and teaching coincide most closely’. He considers the ways in which students’ conceptions and preconceptions affect their learning, and points out that

‘if those preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners, because those learners are unlikely to appear before them as blank slates’

(Shulman, 1986, p. 9)

In considering how to teach to avoid misconceptions, Tanner and Jones (2000) point out that in order for learning to occur, then:

- ‘The pupils need to appreciate that something is not quite right – to have a sense of unease about their current understanding;
- The learning process needs to be important enough to the pupils for them to make the effort to change;
- Just telling them is not enough – pupils need help to construct new knowledge and to connect it to their existing concepts’

(Tanner and Jones, 2000, p. 92)
Since the 1980s a considerable amount of research has been carried out into pupils’ mistakes in mathematics. The earlier studies include the reports of the Assessment of Performance Unit (Foxman et al, 1980) and the Concepts in Secondary Mathematics and Science Project (Hart, 1981) and since the development of the National Curriculum more recent studies have looked at children’s work in National Curriculum tests, such as Ryan and Williams (2000). It is acknowledged that dealing with children’s misconceptions is an essential teaching skill for mathematics teachers and in 2002 the DfES produced a pack of training materials for teachers as part of the Key Stage 3 National Strategy, ‘Learning from mistakes, misunderstandings and misconceptions in mathematics at Key Stage 3’ (DfES, 2002) in which it is acknowledged that ‘mistakes are often the result of consistent, alternative interpretations of mathematical ideas’ and that ‘such misconceptions should not be dismissed as “wrong thinking” as they may be necessary stages of conceptual development.’ It refers to ‘local generalisations’ in which children use their existing knowledge to infer generalisations which work within the limited domain of their current knowledge, so for example a child who as only worked with natural numbers may conclude that the more digits a number has, the larger is its value, which does not necessarily extend to work with decimals.

The importance of being able to identify and deal with misconceptions is highlighted in Ofsted’s report in September 2008, ‘Mathematics: understanding the score’ in which it is acknowledged that in order to
improve pupils’ performance in mathematics ‘it is vitally important to shift from a narrow emphasis on disparate skills towards a focus on pupils’ mathematical understanding’ (Ofsted, 2008, p.4).

In setting out the characteristics of good mathematics teaching the first category that is considered is ‘meeting needs and addressing misconceptions’ and a number of key features are considered in this category. The listed features of good mathematics teaching include:

- ‘Teaching features a successful focus on each pupil’s learning.

- Teachers monitor all pupils’ understanding throughout the lesson, recognising quickly when pupils already understand the work or what their misconceptions might be.

- The teacher listens carefully and interprets pupils’ comments correctly, building on pupils’ contributions, questions and misconceptions to aid learning, flexibly adapting to meet needs and confidently departing from plans.

- Teachers’ marking identifies errors and underlying misconceptions and helps pupils to overcome difficulties’. (Ofsted 2008, p.4)

The importance of understanding, recognising and dealing effectively with pupils’ misconceptions was highlighted as one of the major elements in planning and teaching mathematics set out within the ‘Secondary
Mathematics Planning Toolkit’ produced by the DCSF(2008b) to provide guidance for teachers in delivering the most recent programmes of study in the National Curriculum in England. The components of this toolkit are shown in figure 2.1:

![Diagram of Secondary mathematics planning toolkit](image)

**Figure 2.1 Components of a secondary mathematics planning toolkit (DCSF, 2008b)**

Whilst it is acknowledged that identifying pupils’ misconceptions is a vital component of effective mathematics teaching, it is also the case that teachers do not always find it easy to draw the distinction between misconceptions and other errors, and it was noted by Ofsted (1994) that
many teachers have difficulty in distinguishing between ‘simple careless errors’ and ‘fundamental lack of understanding’ (p.4). In their study into effective teachers of numeracy, Askew et al (1997, p. 3) categorised teachers according to their beliefs, and this included beliefs about the possible difficulties that pupils may face when they learn. Some teachers were observed to put pupil errors down to a failure to grasp what has been taught, with the remedy being reinforcement of what is believed to be the correct method. Other teachers saw mistakes as an indication that the pupils are not ready for the ideas being taught, whilst the teachers who were considered to be the most effective saw the need to recognise and work on pupil misconceptions. Askew and Wiliam (1995) suggest that although it is impossible to avoid any misconceptions arising, the careful choice of examples can reduce the occurrence of misconceptions. They observe that the most common approach is to start with simple examples and move on to more complex examples, but suggest that this can be counter-productive due to the fact that pupils often solve simple examples intuitively, without knowing how they solved them, and are then unable to apply this method to more complex questions. As an example they consider teaching methods of solving equations, where they suggest that it might be better to start with equations that cannot be easily solved by “spotting” the solution or by trial and error.
It is acknowledged that trainee teachers should be required to consider the nature of children’s errors and misconceptions and this is a required component of Initial Teacher Training (DfEE 1997, DfEE 1998)

‘Trainees must be taught to recognise common pupil errors and misconceptions in mathematics, and to understand how these arise, how they can be prevented, and how to remedy them’

(DfEE, 1998, p. 57)

Swan (2001, p. 151) likens this simplistic approach to administering a medical ‘treatment’, as if some standard procedures exist which can be delivered to trainee teachers so that they can eradicate the problem. The main focus appears to be the preventability of mathematical errors. Educational researchers differ as to whether this is possible or even desirable. For example, Koshy et al (2000) report that when primary school children make mistakes they can express strong feelings of anger, frustration and disappointment, which suggests that it is beneficial to teach to avoid errors, and yet there is also research evidence to suggest that mathematical errors can provide an insight into the way that children think, and that the effective provision of assessment for learning can enable children to learn from their mistakes. For example Askew and William assert that ‘learning is more effective when common misconceptions are addressed, exposed and discussed in teaching’ (Askew and William, 1995, p.12) and they point to a feature which emerged from the Diagnostic Teaching Project conducted at Nottingham University’s Shell Centre for Mathematical Education (Bell, 1993) in which it was found that addressing mathematical misconceptions during teaching does improve achievement and long-term retention of
mathematical skills and concepts. Swan suggests that despite what they are
taught, children seem to make the same mathematical errors and construct
their own alternative meanings for mathematics, all over the world. It is
clearly important that the issue of dealing with misconceptions needs to be
addressed in Initial Teacher Training programmes. Houssart and Weller
(1999), noted that many student teachers found it difficult to engage with
the idea of the difference between a misconception and an error, but also
found that this type of categorisation could actually prove unhelpful at
times, when some students

’drew a sharp distinction between misconceptions, due to
misunderstanding and requiring action and “just errors” due to
carelessness and not requiring action’
(Houssart and Weller, 1999, p. 56)

Six key questions identified by Swan (2001, p. 147) are as follows:

- **Why do people make mistakes in mathematics?**
- **What does it mean to understand a concept in mathematics?**
- **What can we do about misconceptions? Can they be avoided?**
- **How do we help pupils to modify their misunderstandings?**
- **How can we structure lessons to facilitate this modification?**
- **What obstacles are there to this form of teaching?**

One important finding of the research into errors is that errors are usually
not just thoughtless or random attempts to solve an unfamiliar problem, but
are often the result of sensible and systematic strategies which a pupil has learned but then wrongly applied in a different situation.

There are various theoretical perspectives on how teachers should view the errors that their pupils make. Lannin et al (2007, p. 44) describe the way that many teachers in the USA apply a behaviourist view of learning when trying to eliminate pupil errors. This perspective involves the provision of positive reinforcement when pupils produce correct answers and either negative reinforcement or lack of positive reinforcement when errors are made. Errors are viewed as incorrect behaviours that need to be replaced with correct behaviours. The way in which mistakes are corrected involves repeated practice with correct procedures and the decomposition of procedures into small, manageable chunks. However as Dubinsky (1991, p. 121) suggests, whilst ‘doing examples’ can serve to reinforce the concepts that are already in the pupil’s mind, working with examples does not necessarily help with the construction of concepts. Dubinsky argues that

'It is a major aspect of our theory that understanding mathematical ideas comes from sources other than looking at many examples and ‘abstracting their common features’, which is what happens if there is only empirical abstraction. Something more is needed and we suggest that it is precisely the construction aspects of reflective abstraction'

(Dubinsky, 1991, p. 121)

Reflective abstraction is a concept that was introduced by Piaget (Piaget, 1969), who used it to describe the way in which mathematical structures are constructed in the process of cognitive development.
Another view of errors referred to by Lannin et al (2007, p. 44) is ‘repair theory’ which acknowledges that pupils may deal with unfamiliar situations by using and adapting a familiar strategy which appears to be a reasonable way of approaching an unfamiliar situation. So, pupils ‘repair’ a new situation by substituting what they view as a reasonable strategy.

2.1.1 Teaching approaches

Research into effective teaching of numeracy (Askew et al, 1997) highlighted the importance of discussion between pupil and teacher. This put forward the view that mathematics teaching and learning are one, emphasising the importance of focused discussion between teacher and pupil. This fits with the social constructivist view on learning and teaching.

The notion of ‘cognitive conflict’ is considered by Tanner and Jones (2000) who consider the process of restructuring thinking in order to accommodate new knowledge, and acknowledge that ‘such restructuring is hard work as it requires deliberate mental effort and a recognition by the pupils that their existing concepts are inadequate in some way’ (p. 92)

In earlier research into ‘cognitive conflict’ Bell (1993) suggested that children derive greater benefit when they encounter misconceptions through their own work than when teachers choose to draw attention to potential errors and misconceptions. The age of the child is an important factor here, and much of the research in this field (eg Swan, 2001) has been with children in Key Stages 3 and 4. However there is an expectation that all
children in a primary school should be encouraged to explain their mathematical ideas and methods (Askew et al, 1997). The emphasis should be on encouraging dialogue, allowing it to be sustained, and using the results to establish connections and address misconceptions.

In the Diagnostic Teaching Project based at Nottingham University Shell Centre, (Bell, 1993) a teaching methodology called Diagnostic Teaching was used with teaching packages that were designed to elicit and address children’s misconceptions. They set out to encourage a process of discussion and articulation of conflicting points of view so that a conflict is resolved and new learning is consolidated. This project reported the long-term retention of mathematical skills and improvements in achievement.

In considering the way in which teachers need to respond to children’s errors, Drews (2005) acknowledges that

‘A teacher’s response to dealing with a child’s mathematical error demands skill in diagnostic terms: different responses will be appropriate depending upon the nature (and frequency) of the error observed’

(Drews, 2005, p. 14)

She also points out that errors can sometimes be exacerbated by teachers making assumptions about children’s experiences. Incorrect resources can also result in children making errors. As an example of this Drews describes how a number line

‘can only be an effective tool for assisting “counting on” and “counting back” if children are shown, and understand, how to count on/back from the first number without including that number in the count’.

(Drews, 2005, p. 15)
In addressing the question of whether we should teach to avoid mathematical errors and misconceptions, she concludes that ‘teaching to avoid children developing misconceptions appears to be unhelpful and could result in misconceptions being hidden from the teacher (and from the children themselves)’ (Drews, 2005, p. 21). This means that in planning mathematics lessons, instead of trying to plan to avoid errors and misconceptions occurring, teachers should actively plan to confront children with carefully chosen examples that will allow for ‘accommodation’ which she describes as ‘presenting the children with “uncomfortable learning” as previously assimilated knowledge has to be revisited, reshaped and challenged’ (Drews 2005, p.17).

2.2 Numerical Thinking

Research studies increasingly point towards evidence that we have an inbuilt ‘number sense’ which supports approximate numerical operations (for example Cordes et al, 2005, Dehaene, 1997) Indeed it is argued by Attridge et al (2009) that it is the disconnect between this approximate number system and formal symbolic mathematics which can cause the difficulties exhibited in dyscalculia.

Verschaffel et al (2006) describe the learning of number as part of the process of developing a ‘mathematical disposition’ and put this in the context of what Lerman (2000) describes as the ‘social turn in mathematics education research’. In considering the key characteristics of learning
environments that enhance children’s early development of a mathematical disposition, Gravemeijer and Kindt (2001) point to the five design principles of the Realistic Mathematics Education (RME) approach in the Netherlands as providing a good summary of the key characteristics of these environments:

- **Learning mathematics is a constructive activity, which means that** children gather, discover and create their own mathematical knowledge and skills in the course of some social activity that has a purpose.

- **The use of meaningful or realistic context problems as anchoring points for the children to develop (or re-invent) their own mathematical knowledge and skills, and to prevent mathematics becoming separated from reality.**

- **Progressing towards higher levels of abstraction and formalisation,** using carefully chosen mathematical models and tools as scaffolds to bridge the gap between, on the one hand, children’s intuitive notions and informal strategies, and, on the other hand, the concepts and procedures of formal mathematics.

- **Learning through social interaction and cooperation,** which are considered as essential tools to mobilise children to reflect on their own constructions and thus to enhance their progressive development towards higher level concepts and strategies.
• Interconnecting the various learning strands within mathematics teaching (e.g., strategies for different arithmetic operations) to allow the construction of a well-organised knowledge base.

(Gravemeijer and Kindt, 2001, p. 200).

The development of the research strategy and the theory of mathematics pedagogy known as RME began in the Netherlands when the Freudenthal Institute was set up in 1971. It is now used throughout the Netherlands and has since been developed to some extent in the USA and the UK. In the USA the University of Wisconsin started to collaborate with the Freudenthal Institute in 1991 in order to develop an approach called Mathematics in Context which was adopted by a number of school districts and has been found to have produced a significant improvement in pupil achievement, which has been documented by Senk et al. (2003). In the UK the Mathematics in Context materials were used in a project set up by the Centre for Mathematics Education at Manchester Metropolitan University (MMU) in 2003 and over three years the project team found that the pupils’ problem-solving approaches changed, which influenced the way in which they understood mathematics (Dickinson and Eade, 2005). The summary of the trial of the RME approaches (Dickinson and Hough, 2011) suggests that misconceptions might be effectively addressed by using a contextual approach. The principles and approaches used in RME are described in further detail in chapter 9. In primary school, it is also suggested by
Vorderman et al (2011) that mathematics should be met in a variety of contexts as often as possible across the curriculum

2.3 Arithmetic and algebra

‘The language of mathematics, as any language, is the result of conventions agreed upon by a society. The conventions for symbol orderings and their meaning need to be generally accepted, consistently modelled and clearly taught’

(Lee and Messner, 2000, p. 173)

As children learn about the number system and start to work with arithmetic operations, they soon come across the need for convention and the use of symbols to express their thinking. From considering specific examples, ‘every learner who starts school has already displayed the power to generalise and abstract from particular cases, and this is the root of algebra’ (Mason, 2006, p.2) and when expressing generality, algebra provides the language and symbol system to do this. In this section the various views and philosophies that underpin the teaching and learning of arithmetic and algebra will be examined and discussed, with relation to the ways in which these philosophies have informed the development of mathematics curricula.

There is a considerable body of research into the nature of ‘the track from arithmetic to algebra’ (Lee and Wheeler, 1989) and into the problems that pupils experience in early algebra (eg Keiran, 1989, 2007, Slavit, 1998, Lins and Kaput, 2004) but it has been noted by Livneh and Linchevski that ‘what is lacking, however, is a sufficient theoretical definition as for what
will be considered as “difficulties with numerical structures” (Livneh and Linchevski, 2007, p. 217) and that there is a need for further research into
‘the underlying assumption that the understanding of the structural rules in
arithmetic is a key for understanding the corresponding parts in algebra’
(Livneh and Linchevski, 2007, p. 217) In Watson’s review of research into
children’s algebraic reasoning (Watson, 2009) she pointed to earlier
research studies such as Hart (1981) and the APU (Foxman et al, 1985)
which suggested that the understanding of operations is a greater problem
than the use of symbols, and in her conclusions recognised that ‘algebra is
not just generalised arithmetic; there are significant differences between
arithmetical and algebraic approaches’ (Watson, 2009, p. 11) In her
conclusions Watson also suggested that learning rules is not always
effective, and that pupils need to develop ‘new priorities’ (Watson, 2009, p.
19)

2.3.1 Structure and order in arithmetic and algebra

‘Traditionally the focus of elementary mathematics has been deeply
oriented to arithmetic and computation, with little attention given to
the relationships and structure underlying simple arithmetic tasks’
(Blanton and Kaput, 2002, p.105)

Many recent studies on the learning of arithmetic have focused on the
understanding of the structural properties of number systems and operations.
This shift in emphasis raises significant questions about the sequential view
that arithmetic comes first, followed by a transition to algebra. There are
different terminologies that reflect this shift in thinking, such as ‘pre-
algebra’, ‘emergent algebra’ (Ainley, 1999) ‘incipient algebraic thinking’ (Friedlander, Herschkowitz and Arcavi, 1989) and this leads to the consideration of the idea that algebra should be integrated with arithmetic from a young age (Freiman and Lee, 2004).

Various research studies (for example Herscovics and Linchevski, 1994) have pointed to a gap or separation between arithmetic and algebra, which many students struggle to negotiate successfully. However there is also much evidence to suggest that the relationship between arithmetic and algebra may be successfully and more usefully regarded as a continuum (for example Carraher, Schliemann and Brinzuela, 2001) and that algebraic thinking may be fostered and developed through appropriate and well-thought-through arithmetic activities.

In considering the ways in which algebra is developed in different countries, the relationship between algebra and arithmetic is frequently characterised by the definition of algebra as ‘generalised arithmetic’. This is the definition that is given in the current National Curriculum Framework for Mathematics (DCSF 2008a) within the Range and Content (section 3) of the Key Stage 3 Framework. Here it is stated that

‘The study of mathematics should include:
3.1 (e) algebra as generalised arithmetic’

(DCSF, 2008a, p. 145)

Thus, despite the more recent research evidence, the view of ‘arithmetic then algebra’ dominates school curricula in this and many other countries.
The reason for this, according to Lins and Kaput (2004) ‘can be found in the strong dominance of Piagetian constructivism. As algebra would require formal thinking, while arithmetic would not, and as formal thinking would correspond to a later developmental stage, algebra should come later than arithmetic’ (p. 50). This is seen in the work of Kuchemann (1981) for the Concepts in Secondary Mathematics and Science (CSMS) project in which he combined the view of algebra as generalised arithmetic with the Piagetian developmental view. Lins and Kaput (2004) argue that ‘the most visible result of Kuchemann’s work is a reported link between different uses of letters in “generalised arithmetic” and Piaget’s levels of intellectual development’

Kieran (2006) in her summary of research on the learning and teaching of algebra from the late-1970s describes the ‘early years’ of research into algebra as being a time when the community of algebra researchers

‘generally considered the algebra curriculum as a given, focusing attention primarily on students’ thinking and methods as they encountered the symbols and procedures that were the standard algebra fare in the beginning years of high school’

(Kieran, 2006, p. 13).

She describes the historical development of algebra as a backdrop to the way in which researchers thought about the development of students’ usage of symbolism, and points out that the perspective at that time was the view of algebra as generalised arithmetic. This viewpoint led to the belief that meaning for algebra was something that needed to be derived from numerical foundations. This, however, led to discontinuities for students
beginning algebra due to the fact that the signs and symbols in algebra were often interpreted differently from the ways in which they were interpreted in arithmetic.

Another discontinuity that has been recognised stems from the difficulty that students have been seen to have with the introduction of formal representations and methods needed to solve problems that have previously been solved intuitively. Kieran (2006) suggests that 'because arithmetic is largely procedural, students are used to thinking about the operations they use to solve a problem rather than the operations that they should use to represent the relations of the problem situation' (p. 15) She points to various studies which have found evidence of students’ preference for arithmetic reasoning and difficulties with the use of equations to solve word problems, and points to evidence that suggests that for many students learning early algebra, reasoning and symbolising appear to develop as independent capabilities. This is seen when students are able to write an equation to represent a problem, but then revert to informal methods to solve the problem rather than using the equation.

There have been a considerable number of research studies into the issue of students’ awareness of structure and order within arithmetic and algebraic expressions (e.g. Kieran, 1989, Wagner, Rachlin and Jensen ,1984, Linchevski and Livneh ,1999) and this has extended to work on the visual characteristics of the symbols used, and how this impacts on the understanding of algebraic rules and conventions. Hewitt (2003a) concluded
that the mathematical structure and visual impact of the notation had an effect on the way in which an equation was manipulated. He presented students with arithmetical equations written in words, and found that students generally ordered the operations as they read them: left to right. However the position of the equals sign was revealed to be significant in the way in which the students interpreted the written statements, and could encourage them to make a 'mental break' in statements which would lead to a correct interpretation of the order within the arithmetic. For example, when initially faced with the word statement ‘two plus one times three equals nine’, 20 out 29 pupils thought this was correct, but after being given statements that had the equals sign in the middle, for example ‘two plus seven equals three times three’ the pupils then had more success with the statement ‘three plus two times four equals eleven’ with 15 out 29 identifying this as correct. Hewitt conjectured that ‘a number of students had become practised at making such a mental break in statements and could see a way of creating a break in the final statement to make that statement correct’ (Hewitt, 2003a, p.67) In another study, Hewitt (2003b) considered students’ reading of formal algebraic notation and observed that many of the errors made by students could be accounted for by the strict left-to-right reading of formally written arithmetic statements. He also considered how students read word statements, acknowledging that ‘expressing non left-to-right order in written words can be problematic as well since words do not possess a set of notational conventions, such as brackets’ (Hewitt, 2003b, p. 34). Goldenberg et al (2010) pointed out that
when children read, they need to “decode” printed text and symbols, which involves unlearning a principle that applies to almost everything else that they see in their life. They give an example of how babies at first learn to recognise a bottle that is handed to them in the correct orientation, and at first will not recognise it if handed to them facing away, but then soon learn to recognise objects regardless of their orientation. Humans have evolved to see the same object despite different retinal images, as long as those images could be made “the same” under rotation, reflection, dilation or certain projective transformations. However when we start to read, the letters d, b, q and p are the same shape and differ only by rotation and reflection and we need to see them as different objects. The words was and saw need to be seen as different, and yet when it comes to $2 + 5$ this must be seen as the same as $5 + 2$, although on a number line they could be seen as different journeys, as demonstrated in figure 2.2:

![Figure 2.2](image)

**Figure 2.2** representations of $2 + 5 = 7$ and $5 + 2 = 7$ on a number line
Or as groups of objects, as demonstrated in figure 2.3:

| ●● | ●●●●●●●● the same as ●●●●●● | ●● |

**Figure 2.3 representations of 2 + 5 = 7 and 5 + 2 = 7 as groups of objects**

Hence, written symbols can present major challenges that spoken symbols do not. When Dickinson et al (2010) presented the question \( \frac{1}{2} + \frac{1}{4} \) to foundation level year 11 pupils just before taking their GCSE mathematics examination, 53% of them got it wrong, the most frequent incorrect response being \( \frac{2}{6} \). Pupils for whom the meaning is not already strongly established tend to see an addition sign and simply add everything in sight. If the question had been spoken, or illustrated by a picture, the attention would have been focused away from the whole number parts of the fractions, and the fractions would have been more likely to be seen as single objects. Mathematical reading and writing is very different to prose reading. Symbolic expressions require attention to vertical as well as horizontal position: for example \( 2^3 \) must be seen as different from 23.

Lee and Messner (2000) performed an analysis of concatenations and order of operations in written mathematics. They describe a concatenation as follows:
‘Concatenations are a characteristic of the written language of mathematics, the positioning of symbols next to each other to imply an operation without a symbol for the operation’.

(Lee and Messner, 2000, p. 173)

Some examples of this are given as:

The number 56 implies place value, the meaning of 5 as 50 and the operation of addition between 50 and 6

The mixed numeral $8 \frac{1}{3}$ which implies the operation of addition between 8 and $\frac{1}{3}$

$6y$ which implies the operation of multiplication between 6 and y

2.364E8 implies the operation of multiplication and exponentiation with the base 10.

(Lee and Messner, 2000, p.173)

This throws up a conflict between concatenations in arithmetic, such as place value and mixed numerals, and concatenations in algebra, where the implied operation is multiplication. Some examples of algebraic errors were given by Matz (1980) as:

1. Evaluating $4x$ when $x = 6$ as 46 or 46$x$
2. Evaluating $xy$ when $x = -3$ and $y = -5$ as -8
3. (a) Evaluating $2(-3)$ as -1
   (b) Evaluating $(-1)^3$ as -3
4. Parsing $3R^2$ as $3 + R^2$ or $(3R)^2$

(Matz, 1980, p.98)
If not addressed, these misconceptions will remain. Lee and Messner (2000) looked at the difficulties faced by adult learners who had studied applied mathematics over many years, and suggested that these difficulties ‘*may have been created by instruction*’ (p.174). They gave evidence of three particular examples of difficulties that these students displayed:

**Interpreting $8^{\frac{1}{3}}$ as a multiplication**

Discussion indicated that the adults were referring to the algorithm used to change mixed numbers to improper fractions, where the first step is to multiply the whole number part by the denominator of the fraction part.

This is partly explained by Matz (1980) in his theory of algebraic competence which proposes that ‘*errors are the results of reasonable, though unsuccessful, attempts to adapt previously acquired knowledge to a new situation*’ (p. 94)

**Interpreting $-8^{\frac{1}{3}}$ as $[(3) (-8) +1] / 3$ or $-\frac{23}{3}$**

This is another example of attempting to use a previously learned algorithm which treats the whole number, the numerator of the fraction and the denominator of the fraction as separate numbers, without seeing the number $-8^{\frac{1}{3}}$ as a number itself, with a place on the number line.
Interpretation of \(-3^2\) as 9 or -9?

There is confusion about the meaning of negative numbers with an exponent when there is no indication of grouping. It can be argued that the answer should be 9 since the negative, or additive inverse, sign should be considered as part of the number and should therefore give an answer of 9. However, if the negative sign is interpreted as being a multiplication by -1 then this should be carried out after the exponent and therefore give an answer of -9. This is further confused by the way in which different calculators carry out this operation. In order to avoid confusion, many text books will put negative numbers in brackets eg: \((-3)^2\) which clarifies the order and should yield an answer of 9.

A fourth level of difficulty is given by Lee and Messner as the “towering” of exponents. An example of this is

\[ 2^3^2 \]

Lee and Messner point out that the convention for dealing with this is not well established. This type of calculation is not dealt with specifically in the Mathematics exemplification of the National Curriculum Framework (DCSF, 2008b); the guidance for calculations without brackets is given as:

‘With strings of multiplications or divisions, or strings of additions and subtractions, with no brackets, the convention is to work from left to right’

(DCSF, 2008b, p.86)

So an extension of this statement would be to interpret the above statement as a string of multiplications and thus work from left to right.
A left to right ordering gives \((2^3)^2 = 8^2 = 64\)

A right to left ordering gives \(2^{(3^2)} = 2^9 = 512\)

Although the first answer of 64 is conventionally accepted to be the correct answer, an example of this inconsistency is given by the way in which different calculators deal with this calculation. At the time of Lee and Messner’s study the TI82 graphics calculator was in popular use, and this gave the answer as 64, whereas the TI92 graphics calculator gave the answer as 512. If brackets are not used, there is a need for conventions which are more generally and consistently accepted.

The principle of the Order of Operations is a fundamental cornerstone of the understanding of arithmetic and algebra. If algebra is regarded as generalised arithmetic, it could therefore be argued that a clear understanding of how to correctly perform arithmetic calculations is an essential prerequisite to the beginnings of the understanding of algebraic structure and to the ability to understand and apply the principles of algebraic convention correctly. This is certainly the approach that is promoted by the structure of the National Curriculum Framework, and which can be seen in the vast majority of classroom resources that are currently available in the UK. Despite the fact that in a recent report by Watson and Nunes (2009), commissioned by the Nuffield Foundation, it was recommended that
‘Teachers should avoid using published and web-based materials which exacerbate the difficulties by over-simplifying the transition from arithmetic to algebraic expression, mechanising algebraic transformation and focusing on “arithmetic with letters”’

(Watson and Nunes, 2009, p 34)

Nevertheless these types of classroom material are used extensively in mathematics classrooms in the United Kingdom, and in line with the structure of the Framework guidance, algebra is frequently developed in these terms as a transition from arithmetic to algebra. Examples of such classroom materials are given in appendix B.

However an alternative view is given by Hewitt (2010), who argues that despite the view of Vygotsky that … ‘algebra is harder than arithmetic’ (p.19), in fact ‘arithmetic is impossible without algebra’ (p. 19). Hewitt discusses the way in which finding structure in carrying out arithmetic calculations leads to an ‘awareness of awareness’ (p. 21) which is involved in working algebraically. He recognises that ‘finding structure in order to help carry out something efficiently and effectively is a human activity’ (p. 23) and describes how an awareness of algebraic structure can be promoted through activities which involve a ‘shift of attention’. This, he argues, is because

‘Arithmetic is concerned with getting answers. Algebra shifts attention from answers to what is required to be done to get an answer, namely operations’.

(Hewitt 2010, p. 21)

Within the review for the Nuffield Foundation in 2009, Anne Watson produced a section on Algebraic Reasoning. In this she states that
'In the United States, there is a strong commitment to arithmetic, particularly fluency with fractions, to be seen as an essential precursor to algebra’

(Watson, 2009, p. 9)

Whilst she accepts that number sense precedes formal algebra in age-related developmental terms, she describes this as a ‘one-way relationship’ and asserts that this is ‘far from obvious in mathematical terms’. Her argument is that in the United Kingdom, where secondary algebra is not taught separately from other mathematics,

‘Integration across mathematics makes a two-way relationship possible, seeing arithmetic as particular instances of algebraic structures which have the added feature that they can be calculated’.

(Watson, 2009, p. 9)

In considering the nature of algebra, Katz (2007, p.185) pointed out that very few secondary school textbooks give a definition of the subject, and looks back to its historical development for a definition:

‘Algebra is a General Method of Computation by certain Signs and Symbols which have been contrived for this Purpose, and found convenient. It is called a Universal Arithmetic, founded upon the same Principles’

(Maclaurin 1748, p. 1)

The National Numeracy Strategy (DfEE, 1999) in speaking of ‘Laying the Foundations for Algebra’ makes it clear that structural thinking about number and number operations might be considered to be the essence of algebra. It recognises that

‘Algebra is a compact language which follows precise conventions and rules. Formal algebra does not begin until Key Stage 3 but you need to lay the foundations in Key Stages 1 and 2 by providing early algebraic activities from which later work in algebra can develop’.

(DfEE, 1999, p. 9)
One important strategy in laying the foundations for algebra is to express relationships in words before moving on to interpreting this expression using symbols. Hewitt (2005) conducted a study in which he investigated the ways in which students interpret and write arithmetic equations in formal notation and their reading of equivalent word statements. He observed that

‘Words in English are written left-to-right so there is a natural temporal order created as someone reads. However, some mathematical expressions are difficult to express within such a linear order, hence non-linear symbolic conventions have been created which make use of vertical placement as well as horizontal placement’.

(Hewitt, 2005, p. 61)

The link between algebraic expressions and the associated verbalisation is an important one when looking at the ways in which pupils solve equations and draw upon their arithmetic understanding in doing so.

The perspective on the way in which pupils’ learning of algebraic symbolisation depends upon their language proficiency was explored by MacGregor and Price (1999) who investigated whether the three cognitive components of symbol, syntax and ambiguity are associated with success in learning algebraic notation. Their research was based upon findings that had identified a relationship between language proficiency and mathematical achievement (for example Secada, 1992) and they adopted the term ‘metalinguistic awareness’ which enables a pupil to view text as an object and to reflect upon structural and functional aspects of it. This
process is equated with analysing structure and manipulating expressions within algebra. In their conclusions they suggest that

‘Teachers need to know whether there is a route by which such students can overcome the barrier of the notation system and gain access to the realm of important algebraic ideas’

(MacGregor and Price, 1999, p.463)

The issues of verbalisation and the use of spoken and written language are clearly significant in the acquisition of an appreciation of convention and notation. Pirie (2005) addressed the teaching of linear equations and the approach of working with the notion of inverse operations. This approach relies upon a clear understanding of the order of operations so that the order can be reversed when the inverse approach is applied. In particular Pirie looked at the meaning of the equals sign. She investigated the solution of simple linear equations such as \(2x - 3 = 5\) looking at the interpretation as being verbalised in the form “twice something take away 3 gives 5”. This form of verbalisation would give rise to a solution being sought by carrying out the inverse operation of adding 3 followed by dividing by 2, provided that the principle of order of operations is understood. Pirie noted that

‘s since most school text books introduce algebra by linking it to arithmetic it should be no surprise that pupils should seek to transfer the associated linguistic understanding as well’.

(Pirie, 2005, p. 35)

This links to the work of Lins (1992) who proposed a framework within which it is possible to understand what algebraic thinking is, and in which is it possible to determine different facility levels between problems with the same underlying algebraic structure. Lins argued that ‘The arithmetical operations are a fundamental model for our understanding of the algebraic
operations’ (p. 56) and looked at the interpretation of an algebraic problem as an operation within different ‘Semantical Fields’, (p. 56) arguing that by operating within a ‘Numerical Semantical Field’ (p. 56) the arithmetic operations and the equals sign become ‘objects’ used as ‘tools’ (p. 57).

When considering the links between arithmetic and algebra it is necessary to consider how students use their previous arithmetic knowledge in dealing with new problems, and how the aspects of their understanding of arithmetic operations are brought to the surface in the resolution of new problems.

Sadovsky and Sessa (2005) refer to the ‘epistemological and didactic rupture’ (p. 91) involved in this process. This means that children need to use previous arithmetic knowledge in order to deal with new problems involving variables, and sometimes this involves a gap or rupture where there is a need to apply previous knowledge to a new situation. This can occur when the teacher does not match their teaching to the pupils’ knowledge. They investigated the possible strategies that students took when solving problems which were aimed at moving towards generalisations of arithmetic solutions to problems. In their conclusions they discuss the ‘elements of knowledge which emerge in the wake of the introduction of problems with one degree of freedom between their variables’ (p. 108). They look at the process of knowledge production in the class, where each number is the result of a specific calculation and where the pupils develop the idea of a variable through the process of producing solutions. In describing the process of knowledge production they identify
the emergence of ‘micro-knowledge’ which they describe as the ‘necessary fuel’ (p.108) for algebraic work. The understanding of arithmetic order may be considered to be such an element of ‘micro-knowledge’ (p. 108).

But is a solid understanding of numerical concepts sufficient for the effective acquisition of algebraic concepts? How do students think about algebra? Do they ‘encapsulate algebraic equations from process to object?’

This question was posed in a study by Nogueira de Lima and Tall (2007, p.3) who considered a theoretical framework which builds from natural human functioning in terms of ‘embodiment’, to shift to the use of symbolism to solve linear equations. In this analysis the conceptual framework was seen in terms of ‘conceptual embodiment’ which is essentially that moment when a pupil is able to fully understand a concept rather than working in a procedural manner, based on human perception, action and reflection and then broadened to include ‘proceptual symbols’ where mathematical symbols operate flexibly either as concepts to think about or as processes for calculation, and in which dynamic actions, such as counting, are symbolised in a way in which the symbols have a dual purpose as process and concept, which is defined as a ‘precept’ which can be described as a blend of process and concept.

The authors define the notion of ‘met-before’, where students are observed building upon experiences they have met before:

‘Experiences in counting affect our conceptions of number, experiences in arithmetic affect our conceptions of algebra, ways of operating that have been seen to work at one time are brought to mind in attempting to make sense of a new context’.

(Nogueira de Lima and Tall, 2007, p.6)
2.4 The Use of Technology in the Learning of Number and Algebra

As technology has become increasingly available in the classroom, many research studies have investigated the ways in which different technologies may be utilised to facilitate algebraic understanding. In the discussion document of the twelfth ICMI study on algebra education it is suggested that:

‘An algebra curriculum that serves its students well in the coming century may look very different from an ideal curriculum some years ago. The increased availability of computers and calculators will change what mathematics is useful as well as changing how mathematics is done. At the same time as challenging the content of what is taught, the technological revolution is providing rich prospects for teaching, and is offering students new paths to understanding’

(Stacey and Chick, 2000, p.216)

In terms of school mathematics, various studies have investigated the ways in which technological tools have been utilised in the learning of number and algebra. A number of studies have investigated the ways in which spreadsheets can be used as a means to learn and teach algebra, for example the ‘Purposeful Algebraic Activity Project’ (Bills and Wilson, 2005) developed a sequence of tasks based on spreadsheets, for the learning and teaching of algebra at ages 11 – 12, and other studies such as Ainley (1996) Filloy, Rojano and Rubio (2001) explored the use of the numerical-tabular and graphical facilities of a spreadsheet to develop algebraic understanding. However the limitations of the use of spreadsheets as a tool for learning algebra in school were recognised by Dettori et al (2001) who concluded
that ‘*spreadsheets can start the journey of learning algebra but do not have the tools to complete it*’ (p. 206).

The widespread introduction of interactive whiteboards in both primary and secondary schools has facilitated the use of software packages that have been specifically developed to encourage algebraic thinking. An example of this is ‘*Grid Algebra*’ which was developed by Dave Hewitt at Birmingham University. It is based upon a multiplication grid and enables teachers to use it flexibly in developing arithmetic and algebraic understanding through activities such as practising multiplication tables, developing an understanding of multiples and factors, expressing formal ways of representing arithmetic, finding equivalent expressions and developing an understanding of the order of operations. This software allows for the continuum from working with arithmetic to using letters to represent a number and creating algebraic expressions, and its role in helping young students to accept, rather than question, notation is described in a study by Hewitt (2009).

A study by Jones and Pratt in 2007 considered how children’s understanding of the equals sign could be taught in a manner which makes the richest possible meanings of mathematical equivalence which are relevant to the learner. They argued that this could be achieved through use of technology supported arithmetical systems. They utilised a Visual Fractions microworld in a series of arithmetic tasks which were used to identify and change children’s perception of the equals sign. Whilst accepting that technology
itself does not guarantee richer meanings of the equals sign, they concluded that ‘carefully designed and technologically supported mathematical environments offer the potential for re-visioning how arithmetical notation is taught in the classroom’ (Jones and Pratt 2007, p. 306). Ruthven (2003) in his report to the QCA discussed the role of dynamic geometry packages in the visualisation of algebraic relationships, using algebra tiles to model algebraic expressions. He acknowledged that the rationale of the National Curriculum Framework emphasises algebra as generalised arithmetic, but also that those aspects concern relationships between variables, functions and graphs. He concluded that ‘in respect of linkage between algebra and geometry, the published rationale for the Framework does not fully anticipate the character of progression to post-16 mathematics’ (p. 6)

The fact that children have great difficulties in the understanding of algebra as generalised arithmetic was addressed by Graham and Thomas (2000), who focused on the fact that one of the main conceptual obstacles to progress in algebra is the failure to understand the concept of a variable. They referred to Kuchemann’s (1981) research in which only 9% of 15-year-old students in his study had gained an appreciation of the use of letters in algebra beyond that of a specific unknown. They also refer to Skemp (1971, p 227) who expresses the view that ‘The idea of a variable is in fact a key concept in algebra – although many elementary texts do not explain it or even mention it’ and they argue that the concept of a variable is ‘the basis for the transition from arithmetic to algebra’. In their study, Graham and
Thomas (2000) produced a module of work based upon a graphic calculator in order to provide a learning environment which enabled children to experience the notion of variables and to enable them to build an understanding of them. This project, entitled ‘Tapping into Algebra’ used the lettered memory stores of the graphic calculator to model a variable. Each memory store is represented as a box in which changing values of the variable come and go, and each box is labelled with its letter. The project included children from the UK and New Zealand. The results of the study were positive in both countries and in their conclusions (p. 38) Graham and Thomas found an improvement in the children’s understanding of symbolic literacy. They noted that the improvement occurred regardless of the ability level of the students, but the greatest relative improvement was noted amongst the weakest students.

A similar study was carried out by Gage (2002), who described the graphics calculator as a mediating tool in the sense of Vygotsky’s theory of the mediation of tools in the development of higher mental processes. She argued that the student’s peers and the graphics calculator together formed a ‘zone of proximal development’ in which the students were able to achieve a higher level of understanding of a variable than they would have done if they had worked without the graphics calculator. She found that the graphics calculator formed a basis for reflective discussion, and proposed that it was the combination of the use of the calculator and the ensuing discussions that led to significant cognitive advance in the students.
The approach to the teaching of number in the current National Curriculum was strongly influenced by the ‘calculator-aware’ number (CAN) project (Schuard et al, 1991). This project advocated that pupils should be encouraged to develop their own informal methods of mental calculation through both practical and investigative activities and with unrestricted access to calculators.

However, in the development of the National Curriculum a conflict existed as to whether calculators should always be available and in the final statutory orders there is a requirement for non-calculator methods of computation to be taught, and for sections of the National Tests to be taken without a calculator. Parts of the GCSE mathematics examination also have to be taken without a calculator.

Hurts (2008) conducted a study involving computer-based learning in order to enable pupils to acquire long-division skills. He acknowledged the importance, in the teaching of arithmetic, of building on children’s informal problem-solving strategies based upon previous experiences, and he utilised a didactic method known as progressive schematisation. This method utilises contextualised problems and instructional aids to encourage pupils to employ their existing informal problem-solving strategies, with the expectation that these strategies would gradually evolve into more efficient versions, accompanied by a more formalised way of presenting the solution as it evolves. Hurts developed a computer-based learning environment based upon this method, offering the pupils a game-like environment for...
solving division problems. Hurts (2008) concluded that this had the potential to improve and supplement current didactic methods for teaching long division skills. Various other studies (for example, Lin and Chin, 2005), point to the fact that the use of calculators and computers in teaching number can provide pupils with an alternative non-counting-dependent procedure to develop number sense.

In the particular case of the teaching of order of operations, the use of calculators can provide a stimulating introduction to the topic. If the pupils have access to both a basic and a scientific calculator they can be asked to key in a calculation such as $2 + 3 \times 4$ and to compare answers. A basic calculator will give an answer of 20, a scientific or graphics calculator will give an answer of 14. This can be used to promote discussion about which is the correct answer, the fact that there must be only one correct answer, and the need for a convention to ensure that the correct answer is obtained. Schrock and Morrow (1993) advocate this approach and suggest that a good learning experience might involve pairing up pupils with different types of calculator so that they can ‘check, challenge, and justify their answer’.

Forrester and Searle (2000) describe a similar study in which calculators are used in this way, and Weibe (1989) gives a variety of examples some using calculators and others using programming languages such as BASIC or LOGO in order to provide experience with this type of problem. He makes the point that whilst pupils should be encouraged to utilise calculators and
computers in order to investigate mathematical problems and explore patterns and concepts:

‘most students, however, need some help in learning to use these tools, especially if they are using them with problems involving more than one operation, as many realistic applications do. They need to be taught how to enter multistep problems and evaluate the displayed result, that is, to do parallel mental calculations’. (Weibe, 1989, p. 36)

Clearly if pupils are to utilise calculators and computers as tools to investigate mathematical concepts, it is essential that they understand how the tool works and the need to question the output. ‘Most people assume that calculator and computer output are sacred’ (Ecker, 1989, p. 103) and so an understanding of the way in which different machines and different programming languages deal with arithmetic calculations is essential.

2.5 Early Algebra

Although the question of when to introduce the study of algebra has been debated since the 1960s, the research focus on the algebraic thinking of primary-aged children has been relatively recent, and due to the emergence of a body of research on this area a Research Forum on Early Algebra was held at the 2001 PME (Ainley, 2001). Many studies have investigated the difficulties in moving from an arithmetic to an algebraic form of reasoning (eg Herscovics and Linchesvski, 1991, 1994, Kieran, 1992) and have led to further consideration of the development of algebraic thinking in primary school. This has included the consideration of early relational thinking about numeric equalities and the symbolising of relationships between quantities.
The findings of Schliemann et al (2003) suggest that students as young as 9 and 10 are able to develop a sense of the equality sign and to represent unknown qualities with a letter and even solve letter-symbolic linear equations, although other studies disagree with this view (eg Fujii, 2003). Keiran (2006) points out that definitions of algebraic thinking in the early years rarely take both perspectives into consideration, and proposes that the activities involved in school algebra can be characterised according to three types: generational, transformational and global/meta-level. Although this characterisation was initially aimed at the secondary school curriculum, the global/meta-level activities of algebra provide context and a sense of purpose for letter-symbolic work in algebra and form a basis for the development of algebraic thinking in the primary school setting. The approach developed from the early work of Davydov (1962) has placed the emphasis not so much on early number as the basis for algebraic learning, but on setting up contextualised situations, and this approach is reflected in the development of Realistic Mathematics Education (RME) by Freudenthal in the Netherlands. Freudenthal suggested that children should be given the opportunity to experience a similar process to that by which a topic of mathematics was developed historically.

Goldenberg et al (2010) suggest that algebraic notation is used in two distinct ways: for describing what we know, and for deriving what we don’t know. In the first use we are using algebra as a language for describing the structure of a computation, or a numerical pattern, or a relationship between
varying quantities, and as they point out, ‘young children are phenomenal language learners!’

In describing the implications for teaching and learning ‘pre-algebra’ Nickson (2004, p.110) points to the fact that the teaching of algebra begins at primary school level when children learn the concepts of equivalence and the concept of an unknown. She recognises that the research evidence suggests that ‘there appears to be a need for more attention to be focused on the different uses of letters or other symbols to represent unknowns in the earlier years’ (p. 110). Nickson suggests that a greater awareness of the difference between the procedural and the structural aspects of the manipulation of numbers as literal terms could help to promote the understanding of algebraic expressions as ‘entities’ or ‘objects’ in their own right. She describes this as ‘the most crucial, but difficult aspect of children’s success in learning algebra’ (p. 123), and concludes that ‘the importance of the “order of operations” needs to be established more firmly in order for this to happen’. (p. 124)

In a recent study Hewitt (2012) argues that whilst many studies have identified the difficulties that students may experience with algebra, these difficulties are not inevitable and furthermore he asserts that children as young as 9 – 10 years old are able to engage with relatively complex, formal algebraic notation in a meaningful way. By using software which presents a numerical expression as a journey, rather than the result of that journey, he
argues that pupils can learn the order of operations without explicit instruction or the use of mnemonics.

The Algebraic Thinking Working Group which reported to the Seventh Congress of the European Society for Research in Mathematics Education (CERME7) in 2011 focused on the transition to algebraic symbolisation, and a number of studies investigated the ways in which algebraic thinking can be successfully developed with primary children (for example Dooley, Pytlak, Alexandrou-Leonidou and Philippou, Hewitt, and Barbosa, cited in Canadas, Dooley, Hodgen and Oldenburg, 2012) In this report it was acknowledged that early algebraic thinking is a ‘mature’ domain within mathematics education research and the group identified issues for further research. It was pointed out that ‘The early algebra debate in part reflects a current theme in the literature’ (Kaput, Carraher and Blanton, 2007) and ‘it also reflects the policy context in which some countries are introducing algebra earlier’. It was also noted that researchers ‘need to demonstrate the contribution they make to the field as a whole through stronger literature reviews’.

2.6 Summary

This review of the literature has investigated a number of points and raised a number of questions that are relevant to this thesis. It has established that there are varying theories and research evidence related to the ways in
which children learn arithmetic and algebra, and that there are different philosophies about the way that arithmetic and algebra are linked, which impact upon the way in which they may be learned. The impact of these theories on the development of the National Curriculum in England has been discussed, and the guidance given within the National Curriculum Framework has been referred to and considered, in the light of the research evidence. These different views and philosophies have impacted on the curricula and teaching methods in the four countries in the study, which all differ in various significant ways, along with the teaching and learning resources that are utilised. In chapter 9 the mathematics curricula in Japan, the Netherlands and the USA, the other three countries investigated in this study, will also be described and discussed, with relation to how the research evidence has influenced the content and teaching methods and resources used. They are all based upon a range of theories and research evidence. There is powerful evidence to suggest that the way in which pupils learn and understand of mathematics is heavily dependent on the structure of the curriculum, the way it is taught and the learning resources used by mathematics teachers.

This research study examines how the different curricula and teaching methods may affect and influence pupils’ understanding and methods of working when performing calculations involving an understanding of arithmetic and algebraic structure, and whether the misconceptions that may arise can be categorised and explained in terms of the teaching and learning environment.
In particular, the research questions that have emerged from the literature and which are being investigated in this study are as follows:

- What misconceptions do children display when using the principles of order of operations in calculations?
- How can these misconceptions be classified?
- How might these misconceptions depend upon the teaching methods employed?
CHAPTER 3: RESEARCH METHODOLOGY

3.0 Introduction

This chapter discusses the research methods that were employed in this study and the research instruments that were used. The use of a case study approach is discussed and examined for advantages and disadvantages.

3.1 The Research Questions

The research questions being addressed are:

- What misconceptions do children display when using the principles of order of operations in calculations?

- How can these misconceptions be classified?

- How might these misconceptions depend upon the teaching methods employed?

This study is essentially socio-cultural, and Lerman (2001, p. 87) describes that from this perspective an object of research on mathematics teaching and learning can be seen as ‘a particular moment in the zoom of a lens’. He uses the analogy of “zooming out” to examine the practices and meanings within the context that the children are learning, and “zooming in” to investigate how individual children learn. This study will use a case study method to
“zoom in” on individual pupils, and will “zoom out” to examine the contexts in which the pupils learn, with different contexts being provided by the curriculum and teaching methods in the different countries.

Whilst there are a number of theoretical research perspectives the two major educational research perspectives are the normative and interpretive, or quantitative and qualitative. The normative paradigm is based around the view that human behaviour is essentially rule-governed and that it should be investigated by the methods of natural science. Quantitative research involves the use of scientific techniques to produce quantified data and possibly leads to generalisable conclusions. The interpretive paradigm is characterised by a concern for the individual; qualitative research involves the investigation of individuals’ perceptions of the world and its aim is to use data to seek insight rather than to carry out statistical analysis.

In order to address the research questions, both quantitative and qualitative research methods were employed. Layder (1993) described this as ‘multi-strategy research’ and Bryman (2004) in examining the case for combining these research methods concluded that ‘when this is appropriate to the research question it may provide a better understanding of a phenomenon than if just one method were used’. (p. 464) This view is supported by others for example Cohen, Manion et al (2000), Hoyle, Harris and Judd (2002). A classification for multi-strategy research is given by Morgan (1998) who describes the ‘priority decision’ which is to decide which method is to be the principal method. Since the data collected in this study
contained some numerical results (the pupils’ worksheet scores) these were analysed statistically, but the most significant data in addressing the research questions was qualitative, emerging from the analysis of the pupils’ written work, the key-recorder data and the interviews with pupils and teachers. Thus the qualitative element provided the principal focus for this study. This was addressed by using a case study approach.

Yin (2009, p. 40) defines four criteria for the quality of research design: construct validity, internal validity, external validity and reliability. In order to achieve validity it is often the case that a triangulated study is carried out in which a combination of these perspectives is used. Triangulation may be defined as the use of two or more methods of data collection, with comparison of the results for these methods. This is one of the methods suggested by Lincoln and Guba (1985, p. 219) in addressing credibility in naturalistic inquiry. The multi-method approach may involve triangulation over time, space, investigators, theories or methods. It may utilise either normative or interpretive techniques or it may draw on methods from both of these approaches and use them in combination (Cohen et al, 2000, p.113). In this study triangulation was achieved by the use of different methods of data collection: pupil worksheets, Key Recorder data, pupil interviews and teacher interviews. The Key Recorder data was played back alongside the pupils’ work in order to establish methods of working that were sometimes impossible to infer from the data, and these were then discussed with the pupils who were interviewed, in order to gain
further insight into the conclusions that had been made from the analysis of their work. The interviews with the teachers provided an insight into how their teaching approaches had impacted upon the way in which the pupils carried out their work and how the findings from the analysis of the pupils’ work could be related back to their learning experiences.

In this chapter the methods which were utilised will be discussed, the research instruments described, and the selection of the use of a case study approach discussed and examined for advantages and disadvantages.

3.2 Rationale for the methods used

The first and most significant part of the data collection consisted of observing, collecting and analysing the work of the pupils involved in the study.

It was clearly important to find a way of observing the pupils’ work in a manner which was as unobtrusive as possible but which provided the opportunity to yield as much richness of information as possible. Collecting the pupils’ work and analysing their written methods of working elicited certain information, but would have been limited on its own in its effectiveness in gaining an understanding into the way in which the pupils were thinking as they completed the work. Video recording the pupils as they worked would have been one possible way of observing what they did, but this would have been very intrusive and may have introduced the
Hawthorne effect by influencing the way in which the pupils worked
knowing that they were being observed. It would not have been possible to
video all the children in the class in the detail that was needed for this study,
so the usefulness of the data would have been limited. The Key Recorder
software provided a way of “observing” the children by enabling the
researcher to play back the keystroke recordings and watching the
keystrokes they put into their calculators. In this study the children were
unaware that their keystrokes were being recorded until after they had
completed the worksheets, so this will not have influenced or affected their
methods of working.

Consideration of the teaching methods used was also an essential element of
this study. This could be partly addressed by examination of the relevant
curriculum documentation, which in England (DfSCF 2008a, DfSCF
2008b) points towards expected teaching approaches. It would have been
extremely useful to perform classroom observations of lessons being taught
on the order of operations, prior to giving the children the worksheets, but in
practice this would have been extremely difficult to arrange, even in the
schools in the United Kingdom. However the teacher interviews provided a
good insight into how their lessons were conducted and gave more general
information about how the teachers worked, which it was unlikely to have
been evidenced in a single or small number of lesson observations.
In this study a combination of normative and interpretive approaches has been used to explore children’s misconceptions. Data have been collected by means of worksheets, the Key Recorder software, interviews with pupils and interviews with teachers. Triangulation has been achieved in four main ways:

- The non-calculator worksheet was marked and analysed for evidence of misconceptions and errors, which were coded. This involved detailed consideration of the children’s written workings.

- The with-calculator worksheet was marked and analysed for evidence of misconceptions and errors by observation of the Key Recorder data. This was related question-by-question to the non-calculator worksheet.

- Some children were interviewed and their workings, as produced on the worksheet, were discussed in detail.

- Teachers were interviewed with respect to the teaching methods employed.

These methods triangulated because the written work of the pupils was observed alongside the key recordings of their work, and the conclusions of the analysis of this data could be tested against the responses of the pupils who were interviewed and backed up by accounts of their teaching and learning which was elicited from the teacher interviews.
3.3 Sampling methods

The population on which the research focused consists of children in the 12 – 14 age range in the United Kingdom, United States, Japan and the Netherlands. Each sample consisted of a class of pupils. The schools that were used in the study were all state-run schools and the classes were either mixed attainment or middle attainment.

In the three schools in the United Kingdom a set of TI-84+ graphics calculators was used as the tool for recording the children’s keystrokes, so it was important to ensure that the pupils in these classes were familiar with using graphics calculators. This was because the calculator was being used as a data collection instrument and it was important to ensure that the pupils’ work was not affected by working with a calculator with which they were unfamiliar. This was a restrictive condition as research has already revealed that graphics calculators are not utilised by many teachers in Key Stage 3 (Headlam, 2004).

It was essential to confirm with the class teachers that they had taught the class about the order of operations and that the questions on the worksheet were realistically attainable by the pupils in the class. The class teachers were given copies of the worksheets in advance to ensure that they were happy that the questions were appropriate and realistic for the pupils in their class to attempt.
For these reasons the samples were essentially purposive, although to an extent they were also convenience samples. From previous research (Headlam, 2004) I had developed an awareness of the schools in which graphics calculators were routinely used in Key Stage 3 and of some of the teachers who utilised them on a regular basis, and so the classes in the United Kingdom were chosen with this in mind.

3.4 Research methods

The research consisted of the following elements:

**Pilot Study**

This was carried out with one class of pupils in the United Kingdom and one class of pupils in Japan. The worksheets are included in Appendix F.

- One non-calculator worksheet was completed by pupils within a normal mathematics lesson.
- The worksheets were marked and analysed, and returned to the class teachers when copies had been made.
- The worksheet was revised in the light of the pilot study and two revised worksheets produced, one with and one without a calculator.
- Interviews took place with the class teachers.
Main study

This was carried out with three classes of pupils in the United Kingdom, one class of pupils in the Netherlands and two classes of pupils in New York State in the United States.

- Two worksheets were completed by pupils within one of their normal mathematics lessons, the first worksheet without a calculator and the second worksheet with a calculator
- For the schools in the United Kingdom, Key Recorder data was also collected while the pupils completed the second worksheet.
- An analysis of the pupils’ written methods on the first worksheet was carried out.
- An analysis of the pupils’ work on the second worksheet was carried out. For the schools in the United Kingdom this included analysis of the Key Recorder data
- Interviews took place with some of the pupils, based on the answers they produced for the worksheet questions. These interviews took place as soon as possible after the written work had been done.
- Interviews took place with the class teachers whenever possible.

3.4.1 Worksheets

The two worksheets were designed to be completed in class, one after the other. They were given to the class teachers to look at in advance as it was
important that the teachers were comfortable with the material that the pupils were being given, and could confirm that the children had been taught the appropriate work that should enable them to complete the questions on the worksheets. The instructions made it clear that this was a worksheet and not a test, although the children were asked to complete them in ‘test’ conditions i.e. without talking and without being able to ask questions. However the children were assured that their results would not contribute to any school assessments and that they would be able to ask their teacher questions about the work afterwards.

The worksheets are included in appendix G. The questions in worksheet 1 consisted of calculations that included single digit numbers, so that the pupils would not find the arithmetic difficult and should be able to complete the calculations easily if they applied the order of operations correctly. The questions in worksheet 2 all had the same structure as those in worksheet 1, but the numbers were all decimals with one decimal place, so that the pupils would need to use the calculator in order to carry them out. Examples of the questions in worksheet 1 and worksheet 2 are shown in figure 3.1:
3.4.2 Analysis of the worksheets

Initially the pupils’ answers were simply marked as right or wrong, and a score out of 14 allocated to each worksheet. Then the wrong answers were investigated. On the non-calculator worksheet this involved careful consideration of the answer and any ‘workings’ that the children had included. In each case I tried to determine the nature of the error or the misconception behind it, and attempted to categorise these.

For the second worksheet, each question corresponded in structure to a question in the first worksheet, but the ‘workings’ on the calculator had been captured by the Key Recorder software in addition to the written answers given by the pupils.
3.4.3 Analysis of the Key Recorder Data

The keystrokes were replayed on the graphics calculator and video recordings made of the screen, so that each section of the video could be related to a pupil’s work on a particular question. These recordings could then be replayed in order to interpret and explain the written workings on the pupil’s worksheet, with a view to aiding the categorisation of any misconceptions that are observed.

3.4.4 Interviews with pupils

Interviews were conducted with samples of pupils who had completed the worksheets in the schools in the United Kingdom. These samples were essentially purposive, as the pupils’ work had just been marked and questions had emerged which I hoped to discuss with particular pupils, but to an extent they were also a convenience sample as it depended upon the availability of the pupils at the time I was in their school.

According to Kvale (1996)

‘The use of the interview in research marks a move away from seeing human subjects as simply manipulable and data as somehow external to individuals, and towards regarding knowledge as generated between humans, often through conversations’.

(Kvale, 1996, p. 11)

Cohen, Manion and Morrison (2000) point out that

‘Interviews enable participants – be they interviewers or interviewees - to discuss their interpretations of the world in which they live, and to express how they regard situations from their own point of view. In these senses the interview is not simply concerned
with collecting data about life: it is part of life itself, its human embeddedness is inescapable.

(Cohen et al, 2000, p. 267)

Cohen Manion and Morrison (2000) use Woods (1986) to give three necessary attributes of ethnographers as interviewers. Although this research is not ethnography, the attributes described are nevertheless important features of this study

- **trust** There would have to be a relationship between the interviewer and the interviewee that transcendened the research, that promoted a bond of friendship, a feeling of togetherness and joint pursuit of a common mission rising above personal egos.

- **curiosity** There would have to be a desire to know, to learn people’s views and perceptions of the facts, to hear their stories, discover their feelings. This is the motive force, and it has to be a burning one, that drives researchers to tackle and overcome the many difficulties involved in setting up and conducting successful interviews.

- **naturalness** As with observation one endeavours to be unobtrusive in order to witness events as they are, untainted by one’s presence and actions, so in interviews the aim is to secure what is within the minds of interviewees, uncoloured and unaffected by the interviewer.

(Cohen et al, 2000, p. 286)

The issue of trust, as outlined above, is an important one and was addressed by the fact that all the interviewees were introduced to me by their class teacher. I then told them about myself, my background as a teacher in school, and explained in simple terms the purposes of my research. To an extent this enabled me to address the third issue of naturalness as outlined above by Woods. The issue of curiosity was inherent in my desire to try to achieve a deep understanding of how the children went about the
calculations, what they were thinking and why they sometimes revealed misconceptions. As a researcher I was in the fortunate position of being able to talk to the individual pupils in much greater depth and for a considerably longer period of time than a teacher would normally have the opportunity to do in a classroom situation, and so I was very curious to find out as much as I could from my conversations with the children in order to gain a deep perspective into their thinking. The interviews were intended to take approximately fifteen minutes to complete.

The interviews were semi-structured; there was no set of generic questions, since the questions asked in the interview depended upon what had been observed from the pupil’s work, and so the agenda for each interview was individual to each pupil, although the interviews all covered the issues arising from the same calculations. However the flexibility to build in follow-up questions left it open for further issues to be raised by the pupils which did not necessarily fit into any of the original research categories.

This type of interview is categorised by Cohen et al (2000, p. 271) as an ‘Interview Guide Approach’, in which the characteristics are that ‘topics and issues to be covered are specified in advance in outline form; interviewer decides sequence and working of questions in the course of the interview’. The strengths of this type of interview are described by Cohen et al (ibid): ‘The outline increases the comprehensiveness of the data and makes data collection somewhat systematic for each respondent. Logical gaps in data can be anticipated and closed. Interviews remain
conversational and situational’. The potential weaknesses of this type of interview include the fact that ‘important and salient topics may be inadvertently omitted. Interviewer flexibility in sequencing and wording questions can result in substantially different responses, thus reducing the comparability of responses’. In order to avoid this, I attempted to ensure that the questions were asked in the same manner and were similarly open-ended. For example “Can you tell me how you went about this calculation?” could be applied to the pupil’s response to any question on the worksheet, and so this approach helped to ensure comprehensiveness throughout the interviews. The follow-up questions would then become increasingly individualised. It was important that the language used in the interview was at the right level in order that the pupils clearly understood what they were being asked without feeling under either pressure or patronised. Alderson (2000, p. 244) describes the importance of this and explains the danger that ‘researchers’ over-complicated or poorly explained terms, topics and methods can also misleadingly make children (and some adults) appear to be ignorant or incapable’. Since the interview questions were focusing on the pupils’ mistakes it was therefore of great importance that they were approached in a sensitive manner and that where possible the pupil felt reassured that the interview had also provided them with a positive learning experience.

The pupil interviews were recorded and then transcribed.
3.4.5 Interviews with teachers

3.4.5.1 The teachers in the United Kingdom

Interviews with the class teachers in the United Kingdom were carried out as soon as possible after the pupil interviews. The purpose of the interviews was to discover how each teacher had taught their class about the order of operations and how their teaching methods fitted in with the specifications of the curriculum guidelines. They were also aimed at uncovering the teacher’s feelings and philosophies about how this topic fitted in with their teaching of algebra.

The interviews were semi-structured with a basic bank of key questions (see Appendix K) These were designed to discover how the teachers introduced the concept of order of operations, the types of activities and resources that they used, and to discover the extent to which they linked this activity to algebra, or whether they viewed it and taught it as an entirely arithmetical activity. The semi-structured nature of the interviews allowed for follow-up questions to be asked in order to pursue the teachers’ responses and also to discuss the analysis of the work of some of the pupils in their class. In a semi-structured interview ‘the questions are frequently somewhat more general in their frame of reference from that typically found in a structured interview; also the interviewer usually has some latitude to ask further questions in response to what are seen as significant replies’. (Bryman 2004, p 113). It is also described by Gillham (2000, p. 21) as the ‘key technique in real-world research’. The questions that the teachers were
asked were essentially open-ended, since according to Cohen et al (2000 p. 275) ‘they allow the interviewer to probe so that she may go into more depth if she chooses, or to clear up any misunderstandings’. This aspect of a qualitative research interview is characterised by Kvale (1996, p.31) as ‘Focused’ in which ‘the interview is focused on particular themes; it is neither strictly structured with standardized questions, nor entirely “non-directive”’. Merton et al (1990, p, 117) describe the central task of the focused interview to be ‘to learn how the prior experiences and dispositions of interviewees are related to their structuring of the stimulus situation’. They note the importance of an understanding of the personal context in being able to deal with an unanticipated response.

Towards the end of each interview there was a period spent on what Cohen et al (2000, p 271) describe as an ‘informal conversational interview’ characterised by the fact that questions emerge from the immediate context and are asked in the natural course of things; there is no predetermination of question topics or wording’. The purpose of this was to enable me to gain the best possible understanding of the professional views and philosophies that underpin the way in which they teach and which would determine the ways in which their pupils learn mathematics, and in having an informal professional discussion this enabled me to continue to provide ‘an appropriate atmosphere such that the participant can feel secure to talk freely’ (Cohen et al, 2000, p. 279)
The teacher interviews were not recorded, but field notes were made throughout the discussions. This sometimes included jotting down some mathematics that was being described by the teacher, and this added to the discussion that was taking place. In each case the teacher was informed in advance that I would be taking some notes, and was invited to look through what I was writing both during the interview and at the end of the interview. The issues involved in using the field note approach will be discussed in section 3.4.5.3.

3.4.5.2 Interviews with teachers from the other countries

There were a number of practical difficulties that needed to be addressed in attempting to interview teachers from each of the other countries involved in the study. In the case of Japan and the Netherlands these included language issues, and in the case of the classes from New York I had no information about the actual teachers of the classes involved in the study.

The class teacher of the pupils from the Netherlands was happy to take part in the study; at first I considered setting up an interview via Skype but she had concerns that her academic English might prevent her from answering questions fluently and quickly enough in this type of situation. It was therefore decided that I would send her a list of questions via email and that she would respond to these, giving her time to think about her responses and to translate them into English in her own time. This led to an exchange of emails and she provided me with examples of the resources that she had
used, in order to illustrate what she was telling me. This method of interviewing eliminates the ‘interpersonal, interactional, communicative and emotional aspects’ (Cohen et al, 2000, p. 279) since the effects of body language, facial expression and other means of non-verbal communication cannot come into play, but it did address the issue of ensuring that the teacher was comfortable and secure with the situation (Kvale, 1996, p.147) and it facilitated a written exchange of questions and responses.

It was not possible to interview the class teachers of the Japanese pupils or the pupils from the United States. Instead, I took the opportunity to meet with some teachers from both Japan and from New York who were visiting Plymouth University at different times. These interviews were essentially opportunistic, since I was constrained by the times when these particular colleagues visited the university. The interviews took the form of informal professional discussions, and I made field notes as we talked, in the same manner as with the UK class teachers.

The three Japanese teachers were all lecturers in mathematics education and had all been teachers in the secondary age range. I arranged to conduct a group interview with all three teachers together. The reasons for this were essentially practical. One of the group spoke English much more fluently than the other two, so he was able to translate the responses and suggestions of the other two teachers, which enabled me to involve all three in the discussion. Gillham (2000) points out that ‘what happens in a group interview is difficult to record’ (p. 88) but I took field notes and since we
were in a classroom we spent some time writing calculations on a whiteboard at one stage, so I was able to record this by photographing the whiteboard.

I used open ended questions in order to find out about teaching and learning strategies that might typically be used in Japan, and showed the teachers some of the pupils’ work, which enabled us to discuss the strategies that the pupils seemed to be using, and to link this to the teaching and learning strategies that would have been used.

The two teachers from New York State were both visiting the university on another occasion, so I took the opportunity to interview each of them. These interviews took the same format as those that I had conducted with the teachers in the UK, being semi-structured in nature and allowing for follow-up questions based upon the teachers’ responses to my initial questions.

3.4.5.3 The use of field notes in the teacher interviews

With the exception of the interview with the class teacher from the Netherlands, all the other teacher interviews were recorded by using field notes that I made throughout the interviews. The main reason for this was my desire to enable the teachers to feel that they were involved in a professional discussion rather than a formal interview; my aim was to encourage them to open up and reveal their philosophies about teaching and
learning mathematics in an atmosphere of discussion rather than simply feeling that they were answering my questions. Interviewees can often be reluctant to open up when being recorded with a voice recorder, whereas without a recorder there can be more a feel of a conversation between two people. It has been suggested that ‘interviewees frequently say much more once the tape recorder has been switched off, or give an entirely different viewpoint when having a chat over a cup of tea in the staffroom, than when they are confronted with a microphone’ (ResInEd, 2006)

In addition to this, my field notes often provided the means to promote further discussion, since they sometimes included jottings of mathematical ideas and calculations as the teachers described and discussed them. As I noted things down, it was sometimes helpful to discuss what I was writing and clarify the ideas with the teacher as I wrote.

Cohen et al (2000, p.311) consider a number of levels at which field notes may be written; within this study the notes were primarily at the level of ‘description’ and included some of the elements described in their list, namely:

- Quick, fragmentary jottings of key words/symbols
- Transcriptions and more detailed observations written out fully
- Descriptions that, when assembled and written out, form a comprehensive and comprehensible account of what has happened
- Pen portraits of participants
3.5 Ethical Issues

All aspects of this research have been carried out within the guidelines of the ethical principles of Plymouth University

(https://staff.plymouth.ac.uk/scitech/humanethics/intranet.htm)

and give careful consideration to the issues of

- Informed consent
- Openness and honesty
- Right to Withdraw
- Confidentiality

The study was approved by the Ethics Committee of the Faculty of Technology and the documentation is included in Appendix K.

In the case of the pupil worksheets, these were all done within normal mathematics lessons, with the class teacher present. The class teacher was informed by me about the nature and the purpose of the work, and given the worksheets in advance. The pupils were told that their work would be marked by me, and returned to their class teacher, but if they did not wish to put their name on the worksheet they could write something else, such as initials or a picture, so that they could identify the work as their own when it was returned. I informed all the pupils that I would be using their work to
help me in my research, but told them that if they did not wish to give it to me they could give it in to their class teacher to mark and that it would be kept in school and not used in my research. They were informed that I would not be naming any pupils, teachers or schools in my research.

At the end of the session the pupils were told about the Key Recorder software and it was explained how it would be used. They were told that if they did not wish me to use their data they could ask for it to be deleted at that stage, or else they could tell their class teacher later who would contact me and inform me. Each calculator was numbered and this corresponded to the numbers that the pupils had written on their worksheets, so that the Key Recorder data could be matched to the pupils’ work.

The pupils were told that after their work had been marked it would be returned it so that they could look at the marking and discuss their work with their class teacher. I explained that I would like to come back and talk to some of them about their work, but that they did not have to do this if they didn’t want to. An information sheet was sent to the parents of the children who I interviewed and parents were asked to contact the class teacher if they didn’t wish their child to be interviewed. The children were informed in advance that the interviews would be recorded, and were told that they could stop the interview at any time, and that they could ask for the tape to be deleted if they wished.
3.6 The Case Study Approach

In order to “zoom in” on individual pupils a case study approach is used within the study. Gillham (2000, p. 13) defines case study as a ‘main method’ when different sub-methods are used within a multi-method approach. Yin (2009, p.18) describes the scope of a case study and defines it in these terms:

A case study is an empirical enquiry that

- Investigates a contemporary phenomenon in depth and within its real-life context, especially when
- The boundaries between phenomenon and context are not clearly evident.

Cohen and Manion (2000, p.181) define a case study as a specific instance that is frequently designed to illustrate a more general principle. They refer to Nisbet and Watt (1984, p.72) in describing it as ‘the study of an instance in action and point out that a single instance is of a bounded system, for example a child, a clique, a class, a school, a community’. In this study the classes and the individual pupils are the ‘instances’ since the data have been analysed both at the class level and also at the level of some of the individual pupils. This has then been used to attempt to illustrate a general principle. I wanted to observe the pupils’ work and link what I observed to the context in which the class had been taught, and so the case study approach is appropriate to the nature of the investigation.
In this study the gathering of data involved different methods of ‘observatio’. Gillham (2000) describes the role of observation within case study research as having three main elements:

- **Watching what people do**;
- **Listening to what they say**;
- **Sometimes asking them clarifying questions**.

(Gillham, 2000, p.45)

In this case the pupils were “watched” by using the Key Recorder software, and listened to by interviewing them. The teachers were also listened to and asked clarifying questions in order to discover their teaching methods.

### 3.6.1 Advantages of using a Case Study Approach

Cohen et al (2000, p.184) describe a number of possible advantages of using a case study approach, some of which are particularly appropriate to this study. These include:

- **Case study data is ‘strong in reality’**

- **Case studies catch unique features that may otherwise be lost in larger scale data**
• Case studies allow generalizations either about an instance or from an instance to a class. Their peculiar strength lies in their attention to the subtlety and complexity of the case in its own right.

• Case studies recognise the complexity and ‘embeddedness’ of social truths. By carefully attending to social situations, case studies can represent something of the discrepancies or conflicts between the viewpoints held by the participants. The best case studies are capable of offering some support to alternative interpretations.

• Case studies are ‘a step to action’. They begin in a world of action and contribute to it. Their insights may be directly interpreted and put to use; for staff or individual self-development, for within – institutional feedback; for formative evaluation; and in educational policy making.

(Cohen et al, 2000, p.184)

These advantages are all possible in this study. The data is ‘strong in reality’ because it is gathered in context in real classroom situations. The use of the Key Recorder Software assists in catching the ‘unique features’, and studying the data from individual pupils and teachers enables generalisations to be put forward and for alternative viewpoints to be investigated. The ‘step to action’ is particularly important for me in this research. As a mathematics educator I wanted to investigate an area of
mathematics teaching and learning in order to discover the impact of different teaching methods in this particular topic, and to put forward suggestions regarding their effectiveness. I felt that any conclusions I might reach would then be relevant in my own role in Initial Teacher Training and also potentially in wider curriculum development.

3.6.2 Disadvantages of using a Case Study Approach

Although there are compelling reasons for using a case study approach, it must be acknowledged that there are also disadvantages which must be taken into account. Cohen et al (2000, p.184) give three main disadvantages:

- The results may not be generalizable except where other readers/researchers see their application
- They are not easily open to cross-checking, hence they may be selective, biased, personal and subjective
- They are prone to problems of observer bias, despite attempts made to address reflexivity.

(Cohen et al, 2000, p.184)

These weaknesses have been considered and despite attempts to guard against selectivity and personal bias, such weaknesses may have affected this study. To eliminate observer bias the interviews with pupils were recorded and transcribed, but due to the individual nature of the interviews it must be acknowledged that the pupils’ responses will have depended on the
way in which the questions were asked. The fact that the pupils were being asked to discuss their mistakes may have had differing effects on different pupils even if the questions were being asked in the same manner. Some pupils may have been defensive and unsure about discussing mistakes, whereas others may have seen it as an opportunity to learn from their mistakes and approach the interview as a learning experience. The teachers may also have felt defensive about being questioned about their teaching methods; I tried to avoid this effect by explaining clearly why I wanted to talk to them and describing the aims of the study, but it would be understandable for a teacher to want to put forward their “best” account of their teaching, focusing on the positives and possibly avoiding the negatives. The sample of teachers I interviewed in the United Kingdom were all very experienced teachers in positions of responsibility, so I anticipated that they would all be honest and reflective about their teaching, but it could be argued that the sample of teachers was itself biased as it was not truly representative of the population of mathematics teachers in the United Kingdom.

My own interpretation of the pupils’ work and Key Recorder data will have been subjective, since this was used in order to attempt to “see” what the pupils were thinking.
Despite these possible weaknesses in the study, the multi-strategy method provides the opportunity for triangulation, and any generalisations that can be made will be within the context of this being a small scale case study.

3.7 Summary

The design of this study involved a combination of normative and interpretive approaches and consequently the data analysis and conclusions reflected this perspective. The data were summarised by tabulating and analysing the quantitative data statistically, and by categorising the qualitative data to allow for comparisons and conclusions to be made. Although the disadvantages of the approach have been acknowledged and taken account of, various attempts have been made to ensure the reliability of the data and the many strengths of the methodology point to it being valid and reliable. Although the size of the sample means that the conclusions will not necessarily be generalisable, nevertheless the methods employed mean that valid conclusions can be made in order to link teaching strategies with pupils’ work and can therefore be used in order to evaluate the strategies used with the classes of pupils involved in the study.
CHAPTER 4: THE KEY RECORDER SOFTWARE

4.0 Introduction

In this chapter the Key Recorder software is described and discussed with respect to its use as a data collection tool in this study. The first section describes the development of the software and its use in a small number of other studies. In the first section the use of the software is explained and in the third section the advantages and disadvantages of using the software are discussed.

4.1 Background to the Key Recorder Software

The Key Recorder software was developed as a research tool by Texas Instruments in conjunction with researchers in the Centre for Teaching Mathematics (Berry et al, 2003) at the University of Plymouth who were investigating students’ working styles when using a graphics calculator. The stages of its development are described in detail by Smith (2003) who worked closely with the programming team at Texas Instruments in order to ensure that the software met the requirements of that particular study. It was originally designed to run on a Texas Instruments TI-83+ graphics calculator. When the program is running the calculator operates as normal and at the same speed, so that the student is not aware of any difference. At the end of the session the student’s work may then be played back,
appearing exactly as it did when the student entered the original commands.
the same as the way in which the student typed it onto the screen.
The motivation for its development was to produce a way of observing
students doing mathematics in an unobtrusive way, so that a student’s
calculator use can be “seen” without the student being aware of being
watched. Smith’s study (2003) was the first study to use such software on a
calculator, although some studies had made use of computer technology to
record students’ keystrokes on a computer (for example Thomas and Paine
(2000), Weigand and Weller (1999 and 2001)).

In Weigand and Weller’s study (2001) a program called ScreenCam was
used to record the students’ keystrokes whilst using a Computer Algebra
System (CAS) and they found this to be a successful research tool for the
analysis of students’ problem solving strategies. The KeyRecorder software
works on the same principles as ScreenCam.

Since its development the Key Recorder software has been used as a
research tool in a small number of studies (For example: Graham, Headlam,
Honey, Sharp and Smith, (2003), Berry, Graham and Smith (2003), Smith
(2003) Berry, Graham and Smith (2005), Berry, Graham and Smith (2006),
Sheryn (2005), Sheryn (2006a), Sheryn (2006b) Graham, Headlam, Sharp
and Watson, (2007))
4.2 Using the Key Recorder Software

The software is saved on the calculator as an Application file (APP). It is accessed by using a password, and can be set to record New Data as shown in figure 4.1.

![Key Recorder Main Menu](image)

**Figure 4.1 Key Recorder Main Menu**

When the calculator is switched on the software will run in the background, recording the keystrokes that are being made by the students.

The collected data can be viewed in two ways:

- **View Data** which provides a list of keystrokes.
- **Replay Data** which shows the screen that the student saw whilst using the calculator.

Figures 4.2 and 4.3 give examples of what would be seen when viewing the data in each of these ways.
The Key Recorder software was first used by Smith (2003). He started to analyse the data by playing back the students’ keystrokes and transcribing
them onto a data collection sheet. This proved to be a very time-consuming
method and so this was adapted so that the calculator screen was videoed.
As the video was played back, comments were made into a tape recorder.

4.3 Advantages and Disadvantages of using the Key Record Software

4.3.1 Advantages

This data collection tool is of significant importance to my work. It has not
been widely used before but in the studies already referred to it has provided
a unique insight into students’ ways of working and their thinking.

In this study the use of the Key Recorder software was significantly
different to its use in previous studies, which were all concerned with some
aspect of the use of the graphics calculator itself. I used the graphics
calculator simply as a research tool in order to gain an insight into the
arithmetical thinking of the children. The only reason I had for using a
graphics calculator is that it is the only calculator for which such Key
Recorder software is available. If it had been possible to obtain similar
software for a basic calculator or a scientific calculator then I would have
been able to use that in exactly the same way.

Figure 4.4 provides an example of how the Key Recorder data can be used
alongside a pupil’s work in order to gain an insight into how the pupil had
gone about a calculation and how the pupil had obtained the answer that
they had written on their worksheet.

105
The pupil’s written work with a wrong answer

<table>
<thead>
<tr>
<th>The pupil’s keystrokes</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First the pupil demonstrates that he understands how to square a number, although he did not use the “square” key.</strong> The fact that he performs the squaring first and then adds 2.4 indicates that he understands the correct order, and he correctly evaluates the numerator.</td>
<td></td>
</tr>
</tbody>
</table>
| 4.8*4.8 23.04  
  Ans+2.4 25.44 | |
| **Now in evaluating the denominator the pupil shows an understanding of the need to perform the calculation in the bracket first and then to square the value obtained, obtaining a correct value for the denominator. The pupil has written this on the worksheet under “workings”.** |
| 1.8+2.7 4.5  
  4.5*4.5 20.25 | |
| **The pupil now appears to go back and check the value of the numerator, but keys in 8.2 instead of 2.4. This is very likely to be because the number 8.2 appeared in the previous question, and so this is a careless error.** The pupil divides the new, incorrect value of the numerator by the value of the denominator and writes down the answer obtained. The pupil would have obtained the correct answer if he had used the first, correct, value for the numerator. |
| 4.8*4.8 23.04  
  Ans+8.2 31.24  
  Ans/20.25 1.542716049 | |

Figure 4.4 An example of how the Key Recorder data can be used to examine a pupil’s calculator work that led to a written answer.
The software provides an exact record of every keystroke that is made by the pupil. This can be used for many purposes. In Sheryn’s study (2006) the focus was on the appropriation of the graphics calculator by A-Level students and the Key Recorder software was used to identify how the students’ use of the graphics calculator changed over a period of time. In the study by Graham et al (2003) it was used specifically to examine how students used their graphics calculators in an A-Level Statistics examination and to provide a focus for the subsequent interviews with the students. Graham et al (2007) again used the Key Recorder software to record the way in which a teacher used the graphics calculator as a teaching resource, and to investigate whether the students’ use of their graphics calculators met the teacher’s expectations. In this study the keystrokes were video recorded and played back to the teacher during an interview, in order to examine the teacher’s aims and objectives behind the calculator use.

In this study the main advantage of using the Key Recorder software was that it enabled me to collect data from every single pupil in the class in an unobtrusive way, which would have been impossible to do by video recording or any other method. It enabled me to gain a unique insight into the children’s thinking, showing me the methods of calculation that they were employing and to gain some understanding of their thought processes. The data that this provided was a valuable supplement to the written “workings” that the children wrote on their worksheets as it tracked their attempts at each calculation on the way to writing down their answer.
Another major advantage is that the graphics calculators are portable and therefore it was relatively easy to take them into the different schools. When the pupils were familiar with using a graphics calculator they were able to use them comfortably and with confidence, even though the calculations I was asking them to do could be carried out on a basic calculator or scientific calculator. It also meant that I could take a calculator with me when interviewing the pupils and refer to their calculations during the interview.

4.3.2 Disadvantages

The main disadvantage of using the Key Recorder software in this particular study arose when pupils were not familiar with using a graphics calculator. I tried to minimise this problem by checking with the class teachers in advance that the class I was working with had used graphics calculators before, but it has to be recognised that even if they had been used for particular activities this would not have been the same as working with a calculator which they use on a daily basis. Also it was always possible that some pupils had joined the class more recently and missed these opportunities. Although the pupils were not required to use any functions that do not exist on a scientific calculator they may have been put off by the apparent complexity of the device. Some keys are slightly different on a graphics calculator as compared to some scientific calculators, for example the “power” key which appears as $^\wedge$. This is used on some makes of
scientific calculator but not many. Although it is the same as the symbol used in Excel, with which most year 8 and year 9 pupils will be familiar, they may not transfer this knowledge when using a calculator. The other key which is different on some, but not all, scientific calculators is the “negative” key which appears as \((-\)\) on the graphics calculator but is \(\pm\) on many scientific calculators. The graphics calculator also has an \(\text{ENTER}\) key instead of an \(=\) key. However since most pupils would be familiar with using a computer keyboard, this was not likely to cause confusion.

Even if the pupils have used graphics calculators before, they will probably not have used them in the place of a basic or scientific calculator, but instead will have met them for “special” lessons for example on drawing graphs or using the Coordinate functions. This was identified in my MPhil study (Headlam, 2004) where it was found that only 13% of the teachers questioned actually used class sets of graphics calculators in Key Stage 3, although 38% of the teachers stated that their use was expected in their department’s scheme of work. Furthermore it was evident from the pupils’ responses that those who had been exposed to graphics calculators in Key Stage 3 had used them mainly for drawing graphs, with a small number of pupils saying that they had used the “list” functions for calculating statistical measures. However, the vast majority of pupils questioned did have very positive attitudes to using a graphics calculator.
In order to minimise this potential problem I worked with classes of children who had used graphics calculators before and whose teachers had access to class sets of them which would have enabled them to be used regularly. I also made it clear to the pupils that if they needed help with a “calculator-related” issue then they could ask either me or their class teacher for any help that they needed. The main reason that pupils asked for help in fact turned out to be finding the “power” key.

One major disadvantage of the Key Recorder software is that when replaying the data it plays fairly quickly and it is not possible to pause and look at one screen shot. For this reason I decided to video record the replay data and then convert this to a media file using Movie Maker. This meant that I could pause at any time, and skip to different parts of the video. This proved to be extremely time consuming but was the best way of enabling me to view the data effectively and efficiently.

Using the View Data option is also a very time-consuming process. Only eight lines of data can be seen at a time. It is possible to scroll through the screens using the arrow keys, but it is not possible to skip through the data file.

There could potentially be problems with the amount of data reaching the maximum amount (1000 keystrokes is the default setting, although this can be changed) in which case the data will start to overwrite and the first part of the data will be lost. However, in my study this was not a problem since the worksheet is fairly short.
A significant issue with the use of the Key Recorder software in this study is that it could only be used with the pupils in the UK schools, due to the practical constraints involved. However it was felt that its use was very valuable even if it was restricted to the pupils in the UK.

4.4 Summary

Despite the fact that the process of data analysis was very time consuming, the Key Recorder software provided a unique opportunity to capture the calculator work of the pupils in the UK and thus to give an insight into their thinking and also to provide a focus for the ensuing discussions with some of the pupils.
CHAPTER 5: THE PILOT STUDY

5.0 Introduction

In this chapter the pilot study is described and analysed. The first section gives an overview of the study, describing the schools and classes of pupils who were involved, and the way in which the study was administered. In section 2 the results are summarised and analysed in terms of the raw scores obtained by the pupils on the pilot worksheet and also in terms of the types of errors and misconceptions that were observed in the pupils’ work. An initial attempt to categorise these errors and misconceptions is described. In the third section the results are discussed and the pupils’ misconceptions compared, in terms of both the types of misconception that were seen and also the frequencies with which they were observed. The fourth section describes the conclusions that were drawn from the pilot study and the implications in terms of developing the worksheets for the main study.

5.1 Overview of the Pilot Study

The pilot study was carried out in the UK and Japan. In each country one class of students was involved. In the UK this was a class of 20 middle-attaining students in year 8 (aged 12-13). The Japanese class consisted of 33 mixed-attaining students in grade 8 (aged 13-14). This is summarised in table 5.1:
<table>
<thead>
<tr>
<th>School</th>
<th>Number of pupils</th>
<th>Country</th>
<th>Age of pupils</th>
</tr>
</thead>
<tbody>
<tr>
<td>School X</td>
<td>20</td>
<td>UK</td>
<td>12 - 13</td>
</tr>
<tr>
<td>School Y</td>
<td>33</td>
<td>Japan</td>
<td>13 – 14</td>
</tr>
</tbody>
</table>

Table 5.1 Summary of the classes of pupils involved in the pilot study

Before carrying out the pilot study I confirmed with the class teachers that both classes had been taught the principles of the order of operations as part of their scheme of work, and had also been taught simple algebraic conventions, including substitution of letters for numbers in algebraic expressions. I gave the teachers a copy of the worksheet in advance, and they confirmed that they would reasonably expect the pupils to be able to attempt all the questions on the worksheet.

5.1.1 Administration of the worksheets

The worksheets were administered by the class teachers in a normal mathematics lesson. Each pupil in the class was given a copy of the pilot worksheet (see Appendix F for the UK and Japanese versions of the pilot worksheet). For the Japanese class the instructions were translated into Japanese but everything else was exactly the same. There were twelve questions altogether, eight questions being arithmetic and four involving substitution of numbers into algebraic expressions. The pupils were told that this was a worksheet and not a test and that if they couldn’t do a question they should just leave it and put a cross in the answer box. Although it was not a test, the worksheets were completed under test conditions, so that the pupils worked on their own without talking to each other and they were
given as much time as they needed. They were asked to answer all the questions that they were able to by working out the answers to the calculations without using a calculator. They were encouraged to write down their methods of working in the spaces provided. The pupils were told that their work would be marked and returned to their class teacher so that they would then have the opportunity to go through any of the questions with their teacher at that stage.

The pupils were also informed that they did not have to write their names on the worksheets but if they did their names would not be used in the study and neither would the name of their school or their teacher. They were each given a pupil number and encouraged to make a note of it if necessary for when their work was returned and they were assured that they would have the opportunity to discuss the questions with their teacher when the work was returned.

When the worksheets were given to me I performed an initial marking exercise, scoring each worksheet, and then copied the worksheets so that I could return the originals to the class teacher with a summary of scores.

Having performed this initial marking I then analysed the worksheets by investigating the pupils’ written working, in particular focusing on the incorrect answers, analysing these for the different types of errors and misconceptions that appeared.
5.2 Analysis of the results

5.2.1 Initial statistical analysis

Although I was more interested in the nature of the errors than the number of correct answers, it was helpful to start by looking at the worksheet scores, which were marked out of 12. A summary of the scores is given in table 5.2 and represented by boxplots in figure 5.1:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Minimum value</th>
<th>Maximum value</th>
</tr>
</thead>
<tbody>
<tr>
<td>UK School X n = 20</td>
<td>6.7</td>
<td>2.98</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>Japan School Y n = 33</td>
<td>11.2</td>
<td>0.88</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 5.2 Summary of Pilot Worksheet Scores

Figure 5.1 Boxplots to illustrate and compare the results of the pilot worksheets
A Mann-Whitney test was carried out in order to compare the results from the two schools. This test was chosen because the samples were small and not normally distributed, so a non-parametric test was needed. The scores were found to be significantly different, with a p-value of less than 0.001. However it must be acknowledged that with such small sample sizes this result must be treated with caution and can only be used to compare the two particular classes in the pilot study.

### 5.2.2 Analysis of the types of error

- The first class to complete the pilot task was the UK class. I analysed every incorrect answer and tried to decide on the nature of the misconception or error, and developed a list of emergent errors. I decided that these could be put into three categories associated with order of operations:
  - Problems with algebraic and notational conventions
  - Arithmetic errors

I was primarily concerned with the first two categories, as the third category contains errors that are due to minor errors rather than misconceptions or misunderstandings.
As the data analysis continued I built up a list of different types of error. These are shown below, together with examples of pupils’ work from UK school X:

**MISCONCEPTIONS IN ORDER OF OPERATIONS**

**M1  Works left to right**
These are cases where the calculation is performed in order from left to right ignoring the usual mathematical conventions, for example  \( 2 + 3 \times 4 + 5 = \)

25. Figure 5.2 shows an example of this type of misconception:

![Figure 5.2](image)

**M2 ignores bracket when with power**
In some cases the pupils seemed to “ignore” the brackets when there was a power outside the bracket. This resulted in calculating  \( (1 + 2)^2 \) as  \( 1 + 2^2 \) or  \( (2 \times 3)^2 \) as  \( 2 \times 3^2 \) . Examples of this type of misconception are shown in figure 5.3.
School X Pupil 4

School X Pupil 8

Figure 5.3  Examples of misconception M2 from School X Pupil 4 and Pupil 8

M3  Addition before power

If a calculation contained an addition and a power, some pupils performed the addition first, for example interpreting $2 + 4^2$ as $(2 + 4)^2$. This could also be regarded as a left-to-right interpretation, but when this was observed it usually appeared independently from misconception M1 in previous questions, therefore it was decided to categorise this separately. An example of this is shown in the numerator of the calculation in figure 5.4. This pupil had answered all questions correctly up to this question, and had shown no previous evidence of left-to-right thinking. Figure 5.4 also gives an example of this misconception from pupil 18, which cannot be seen as left-to-right thinking, since the value in brackets is calculated first and then added to the 4 before squaring.
M4 Addition before multiplication

In some cases the pupil demonstrated that they knew that an order was required, but put addition before multiplication. This was sometimes emphasised by the inclusion of brackets around the additions, as demonstrated in the example of work by pupil 16 in figure 5.5 in which the pupil has evaluated $2 + 3 \times 4 + 5$ as $(2 + 3) \times (4 + 5)$.
from pupil 3 in which this misconception does not appear until the calculations involve brackets and fraction notation, and work from pupil 5 who consistently multiplies by 2 instead of squaring:

**Figure 5.6 Examples of misconception C1 from School X Pupil 3 and Pupil 5**
C2 misunderstands algebraic convention for multiplication

Some pupils did not recognise $ab$ as a multiplication but interpreted $ab$ as a 2-digit number. This misconception was also demonstrated when a variable appeared outside a bracket. This can be seen in the work of pupil 1 and pupil 3 in figure 5.7. Pupil 1 was unable to calculate $2(3 +4)$ and gave no answer to this. Pupil 3 interpreted $ab$ as a 2-digit number but performed an addition when the variable appeared outside a bracket.

![Table of examples](image)

**Figure 5.7 Examples of misconception C2 from School X Pupil 3 and Pupil 5**
C3 **misunderstands fraction as a division**
Some pupils did not recognise fraction notation as a division, or became confused when this appeared as part of a more complex algebraic expression. Figure 5.8 shows the work of pupil 13, who adds the numerator and denominator (and also demonstrates misconceptions C1 and M3) and pupil 5, who also adds the numerator and the denominator but only when the fraction is algebraic, and pupil 4, who realises that a division is required, but is confused about which way to divide.

![Examples of misconception C3 from School X Pupil 13, Pupil 5 and Pupil 4](image)

Figure 5.8 Examples of misconception C3 from School X Pupil 13, Pupil 5 and Pupil 4
ARITHMETIC ERRORS

A1  error with multiplication tables or addition
Some pupils wrote down the correct calculation and the only mistake
appeared to be an incorrect multiplication or addition. For example, in
figure 5.9 pupil 14 writes $3 \times 3 = 6$. This appears to demonstrate that she
does understand that squaring involves multiplying the number by itself, and
she had successfully squared numbers previously on the worksheet, with no
errors, so this was taken to be a careless multiplication error. In Figure 5.9
it can also be seen that pupil 2 wrote $2 \times 3 = 5$, which was taken to be a
multiplication error.

![Image of pupil's work]

School X Pupil 13

![Image of pupil's work]

School X Pupil 2

Figure 5.9 Example of error A1 from School X Pupil 13 and pupil 2

A2  transcription error or careless error
Sometimes pupils made transcription errors. When this was observed it was
usually in one of the last four questions in which numbers were substituted
into algebraic expressions, as demonstrated in the work of pupil 9 and pupil
11 in figure 5.10. The only error that pupil 9 made was to substitute the
variable d incorrectly for the number 4 instead of 5 in the numerator of the expression. Similarly pupil 11 substituted the variable a incorrectly for 4 instead of 2, and although $4^2$ was correctly evaluated to be 16, there was then another careless error in failing to add 2 + 16 correctly. This was taken to be a “careless error” sometimes a pupil made an error that was classified as a “careless error”

![Image](image1.png)

**School X Pupil 9**

![Image](image2.png)

**School X Pupil 11**

**Figure 5.10 Examples of error A2 from School X Pupil 9 and Pupil 11**

Although at this stage I did not see the careless errors as being particularly significant to this study, I did notice as the main study progressed that a number of pupils seemed to “change sign” often changing a multiplication to an addition, as seen in the examples in figure 8, so at a later stage another category of misconception was introduced and this will be discussed in further detail in chapter 7.

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I was then able to complete a spreadsheet of results for each pupil with every incorrect answer coded. (Appendix H) In some cases I was unable to determine the nature of the error or misconception, in which case the code used was a question mark.

When the Japanese worksheets were returned I was able to carry out the same process with them, and although it was apparent that the Japanese pupils displayed far fewer misconceptions, their misconceptions appeared to be different, so I added two more categories. These are given below, with examples of the work of some of the Japanese pupils:

**M5 Incorrect interpretation of a power outside a bracket**

Some Japanese pupils were unable to successfully square the number in a bracket, as shown in the work of pupil 25 in figure 5.11. This pupil had successfully squared single numbers, but was unable to do this when the number was an expression inside a bracket, as seen in the denominator in figure 5.11. This could have been an unsuccessful attempt to square the bracket algebraically.

![Figure 5.11 Example of misconception M5 from School Y Pupil 25](image)
C4 Interprets divisions as fractions but then makes mistake in addition of fractions

Some Japanese pupils did not immediately see $\frac{8}{4}$ as a division but tried to calculate $8 + \frac{8}{4}$ as a problem involving the addition of two fractions. In many cases this was successful but sometimes this was not carried out correctly. The work of pupil 21 in figure 5.12 exemplifies this. Whilst this pupil has been unable to add fractions successfully, the workings demonstrate that he does recognise that $\frac{16}{4}$ is a division, and he correctly evaluates this to be 4.

![Figure 5.12 Example of misconception C4 from School Y Pupil 21](image)

A3 Tries to square brackets algebraically but makes an algebraic mistake

Many of the Japanese pupils evaluated $(1 + 2)^2$ by treating it algebraically and squaring the bracket. This was often completed successfully, but sometimes this included an algebraic mistake, as shown in the work of pupil 31 in figure 5.13 in which $(1 + 2)^2$ is evaluated as $(1 + 2)^2 = 2 + 4 + 4 = 10$ in the denominator of the fraction.
5.3 Discussion of results

The UK pupils displayed a range of errors and misconceptions. Many did not apply the correct order of operations even in the simple case of $2 + 3 \times 4 + 5$, where they used a left-to-right interpretation. Quite a few pupils misinterpreted or misunderstood the power notation and many displayed a poor understanding of algebraic notation in general, being unable to successfully substitute numbers into algebraic expressions. Appendix E gives a complete analysis of all these findings. The most common misconception was the “left to right” method of calculation, categorised as M1, with 10 of the pupils (50%) consistently applying this error. A summary of the frequency of errors is given in table 5.3:
<table>
<thead>
<tr>
<th>Category of error</th>
<th>Number of pupils who demonstrate this</th>
<th>Percentage of pupils</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>10</td>
<td>50.0</td>
</tr>
<tr>
<td>M2</td>
<td>1</td>
<td>5.0</td>
</tr>
<tr>
<td>M3</td>
<td>6</td>
<td>30.0</td>
</tr>
<tr>
<td>M4</td>
<td>2</td>
<td>10.0</td>
</tr>
<tr>
<td>M5</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>C1</td>
<td>6</td>
<td>30.0</td>
</tr>
<tr>
<td>C2</td>
<td>13</td>
<td>65.0</td>
</tr>
<tr>
<td>C3</td>
<td>2</td>
<td>10.0</td>
</tr>
<tr>
<td>C4</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>A1</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>A2</td>
<td>1</td>
<td>5.0</td>
</tr>
<tr>
<td>A3</td>
<td>0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 5.3 Frequencies of errors for pilot school X (UK)  \( n = 20 \)

The Japanese pupils achieved significantly higher scores than the UK pupils, and 14 of them answered all the questions correctly. This could be partly accounted for by the fact that they were a year older than the UK pupils, but the Japanese class was a mixed-ability class compared to the UK class which was grouped according to ability and categorised by the class teacher as “middle-attainment”. What was more interesting was the different types of errors that were seen in the work of the Japanese pupils. The main difference that emerged in the working methods employed by the Japanese pupils was that they appeared to view this as an algebraic exercise, employing algebraic techniques in order to calculate the arithmetic calculations. This was very much in contrast to the methods used by the
pupils in the UK. Often these techniques resulted in correct answers, but most incorrect answers appeared to be a result of making a mistake in multiplying out brackets algebraically.

The question that produced the greatest number of incorrect answers from the Japanese pupils was question 8:

\[
\frac{2 + 4^2}{(1 + 2)^2}
\]

and the next highest rate of errors was observed in question 7:

\[
4 + (2 \times 3)^2
\]

Some examples of the algebraic techniques employed by the Japanese pupils are shown in figures 5.14 and 5.15.

The work of three of the Japanese pupils is shown in figure 5.14 and 5.15 and these are typical of the methods that were employed by the majority of the Japanese pupils.
Figure 5.14 Examples of algebraic techniques giving correct answers School Y

Figure 5.15 Examples of algebraic techniques leading to incorrect answers School Y
The calculation \((1 + 2)^2\), in the denominator of the fraction, was frequently attempted by multiplying out brackets:

\[
(1 + 2)^2 = (1 + 2)(1 + 2) = 1 + 2 + 2 + 4 = 9
\]

This was often carried out correctly but sometimes resulted in arithmetic errors (see figure 5.15). This type of mistake was the main reason for wrong answers for the Japanese pupils.

The majority of incorrect answers for the Japanese pupils were the result of an incorrect attempt to multiply out the brackets in the denominator using an algebraic-style approach. They displayed very few actual misconceptions, with only one pupil demonstrating the “left to right” misconception. This can be seen in the complete summary of the analysis of their work in Appendix H.

The last four questions, involving substitution of numbers into algebraic expressions, were completed very successfully with many pupils answering them all correctly, and no Japanese pupil made more than one mistake in this set of questions. Only six mistakes were observed in total, and these were all categorised as arithmetic or careless errors. An example of this is shown in figure 5.16:
The results revealed that 27 of the 33 Japanese pupils got all four questions in the last sections correct, and figure 5.17 shows a typical set of responses, exemplifying the Japanese pupils’ methods of working which involve multiplying out the brackets, even after the numbers have been substituted into the expressions:

<table>
<thead>
<tr>
<th>問題</th>
<th>計算</th>
<th>答え</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a + bc + d$</td>
<td>$2 + 3 \times 4 + 5 = 19$ &lt;br&gt; $2 + 12 + 5 = 19$</td>
<td>19</td>
</tr>
<tr>
<td>$ab + cd$</td>
<td>$(2 \times 3) \div (4 \times 5) = 6 + 20 \div 26$</td>
<td>26</td>
</tr>
<tr>
<td>$a(b + c)$</td>
<td>$2(3 + 4) = 6 + 8 = 14$</td>
<td>14</td>
</tr>
<tr>
<td>$\frac{a(b + d)}{c + a^2}$</td>
<td>$\frac{2(3 + 5) + 10}{4 + 2} \div \frac{4 + 4}{8} = 2$</td>
<td>2</td>
</tr>
</tbody>
</table>

School Y Pupil 29

Figure 5.17 Example of Japanese pupils’ methods for the algebraic substitutions
A summary of the frequencies of the Japanese pupils’ errors is given in Table 5.4:

<table>
<thead>
<tr>
<th>Category of error</th>
<th>Number of pupils who demonstrate this</th>
<th>Percentage of pupils</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>1</td>
<td>3.0</td>
</tr>
<tr>
<td>M2</td>
<td>5</td>
<td>15.2</td>
</tr>
<tr>
<td>M3</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>M4</td>
<td>1</td>
<td>3.0</td>
</tr>
<tr>
<td>M5</td>
<td>3</td>
<td>9.1</td>
</tr>
<tr>
<td>C1</td>
<td>1</td>
<td>3.0</td>
</tr>
<tr>
<td>C2</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>C3</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>C4</td>
<td>1</td>
<td>3.0</td>
</tr>
<tr>
<td>A1</td>
<td>1</td>
<td>3.0</td>
</tr>
<tr>
<td>A2</td>
<td>7</td>
<td>21.2</td>
</tr>
<tr>
<td>A3</td>
<td>2</td>
<td>6.1</td>
</tr>
</tbody>
</table>

Table 5.4  Frequencies of errors for pilot school Y (Japan)  \( n = 33 \)

5.4 Conclusions from the pilot study and implications for the main study

It was clear that the class of pupils in Japan performed significantly better than the class in the UK, but that the misconceptions that did arise were different to those that emerged from the pupils in the UK class. The main difficulties that arose for the pupils in the Japanese class arose from their desire to tackle everything with an algebraic approach, despite the fact that the numbers were very simple to work with. The only question that the
Japanese pupils seemed to have any significant difficulties with was question 8, which involved a division with a bracket in the denominator.

It was also clear that the pilot questions had not really tackled the issue of working with negative numbers, in particular the ability to square negative numbers correctly. This can be a particular issue when working with scientific or graphics calculators, since it is often necessary to put the negative number in brackets before squaring, in order to obtain the correct positive answer. For example if \(-3^2\) is entered, many calculators will produce the answer \(-9\). There is also often confusion about the role of the “negative” key and the “subtract” key. These issues are exemplified in figure 5.18 in which the correct way of squaring a negative number on a graphic calculator is shown, along with the two most common incorrect ways. This also applies to scientific calculators.
Correct: The negative key is used and brackets are included

Incorrect: The negative key is used but brackets are not included

Incorrect: The subtract key is used instead of the negative key

The resulting error message when the subtract key is used incorrectly

Figure 5.18 The correct method for evaluating \((-3)^2\) on a calculator, and some common incorrect methods
If the pupils do not have a clear idea of these issues in their non-calculator work, they will find the use of the calculator to be confusing and this may well lead to misconceptions being reinforced. I therefore felt it necessary to include at least one question in the main study that would address this issue. I also felt that since the pilot study had not included any calculations involving subtractions it could be improved by including at least one question involving a subtraction when planning the main study. These issues are described in chapter 6.

5.5 Summary

The pilot study was important in terms of trialling the non-calculator worksheet and developing it in order to produce the non-calculator worksheet that was to be used in the main study, and to develop a corresponding worksheet that would be used with calculators. The results of the pilot study were significant in highlighting the differences between the work of the classes of pupils in the UK and Japan.
CHAPTER 6: IMPLEMENTING THE MAIN STUDY

6.0 Introduction

In this chapter the implementation of the main study is described and the detail of the research methodology is discussed with regard to the practical issues that needed to be considered. The worksheet scores are analysed by considering the summary data and performing appropriate hypothesis tests in order to investigate any differences between the pupils’ performances in the two worksheets and also for any differences between the results from the different classes. Whilst it is not possible to make any generalisations from these results it is nevertheless a good starting point to investigate the available quantitative data before moving on to an analysis of the qualitative data which will be described in chapters 7 and 8.

6.1 The revised worksheets

Having analysed the results of the pilot study the non-calculator worksheet was revised and a with-calculator worksheet produced. These were called worksheet 1 (non-calculator) and worksheet 2 (calculator allowed). These are given in appendix G. The questions in worksheet 2 were identical in structure to those in worksheet 1, but contained decimal numbers instead of integers. This was to encourage the pupils to use their calculators. The pupils were provided with TI84 calculators which had the Key Recorder
software running in order to do worksheet 2, so that a record of their calculations could be obtained.

Question 6 (originally $2 \times 3^2$) from the pilot worksheet was adapted as shown below.

Question 6  

\[
2 \times (-3)^2 \quad \text{(worksheet 1)} \\
2.6 \times (-3.7)^2 \quad \text{(worksheet 2)}
\]

This was done to include a question involving a negative number, since this is emphasised in the mathematics framework exemplification (DCSF, 2008, p. 87).

The first new question to be introduced was question 7, shown below.

Question 7  

\[
2 \times (6 - 4) \quad \text{(worksheet 1)} \\
2.47 \times (4.26 - 1.79) \quad \text{(worksheet 2)}
\]

This was included in order to ensure that there was one question which involved a subtraction.

The second new question to be introduced was question 9, shown below.

Question 9  

\[
\frac{8+6}{3+2^2} \quad \text{(worksheet 1)} \\
\frac{8.2+6.3}{3.7+2.8^2} \quad \text{(worksheet 2)}
\]

This was included so that there was a question involving a division but no brackets.
Apart from these changes the structure of the pilot study worksheets was retained, with the final four questions still involving the need for substitution of numbers into algebraic expressions.

The worksheets were translated into Dutch by a colleague in the Netherlands so that they could be given to pupils in school D.

6.2 Conducting the main study

The main study took place over a period of two years and involved 148 pupils altogether, from five schools in three different countries (the UK, the USA and the Netherlands). Each school involved in the study was a state-run school and the classes chosen were all middle-attaining classes of pupils in the age range 12 – 14.

Table 6.1 summarises the information on the classes of pupils involved:

<table>
<thead>
<tr>
<th>School</th>
<th>Number of pupils</th>
<th>Country</th>
<th>Age of pupils</th>
</tr>
</thead>
<tbody>
<tr>
<td>School A</td>
<td>17</td>
<td>UK</td>
<td>12 - 13</td>
</tr>
<tr>
<td>School B</td>
<td>27</td>
<td>UK</td>
<td>12 - 13</td>
</tr>
<tr>
<td>School C</td>
<td>29</td>
<td>UK</td>
<td>12 - 13</td>
</tr>
<tr>
<td>School D</td>
<td>22</td>
<td>Netherlands</td>
<td>13 - 14</td>
</tr>
<tr>
<td>School E class 1</td>
<td>28</td>
<td>USA</td>
<td>12 - 13</td>
</tr>
<tr>
<td>School E class 2</td>
<td>27</td>
<td>USA</td>
<td>13 – 14</td>
</tr>
</tbody>
</table>

Table 6.1 Summary of the classes of pupils involved in the study
6.2.1 Conducting the study in the UK schools

In the case of schools A, B and C in the UK I was able to visit each of the schools and conduct the research with the class teacher present. I made two visits to each school. The first visit was to give the pupils the worksheets to complete. Each session took one lesson, and in each of the three schools the length of a lesson was one hour. This proved to be long enough for all pupils to complete both worksheets with time to spare. The second visit took place between a week and two weeks later, and the purpose of this visit was to conduct interviews with some of the pupils who had completed the worksheets, and with the class teacher. The pupils to be interviewed were selected after the analysis of the work had been completed; I wanted the opportunity to discuss a range of the misconceptions that had been identified within the class and I selected a sample of pupils whose work had demonstrated different misconceptions across the range. The marked work was also returned to the class teacher at this stage, so that it could be returned to the pupils in their mathematics lesson and discussed with them as appropriate.

6.2.1.1 The first visit

Before the session I had to set up the calculators so that the Key Recorder software was running. Each calculator was numbered. I had already briefed the class teachers about my research and about the Key Recorder software (Appendix K). The teachers had been given copies of the worksheets and
they were fully informed with regard to how I intended to conduct the
session.

I started each session by introducing myself to the pupils, briefly explaining
my role at the University of Plymouth and why I was interested in finding
out about how pupils go about certain aspects of their work in maths. I
explained that I would not be using their names, or the names of their
teacher or their school, and that they did not have to write their name on the
worksheets if they did not want to. They did need to write their pupil
number, which was the same as the number on the calculator that they
would be given. I asked them to make a note of their number in their maths
book, so that they could identify their work when it was given back. I made
it clear that this was not a test and would not be used by their teacher in any
form of assessment within school, but that they might find it helpful to go
through the worksheets with their teacher after I had marked them and
returned them.

With the help of the class teacher I distributed the worksheets and the
calculators. I instructed the pupils that they should first complete worksheet
1, the non-calculator worksheet, and encouraged them to write down their
workings in the spaces provided. There was no time limit and they could
take as much time as they needed. They were told that if they could not do
a question they should put a cross in the answer box. The pupils were told
that when they had finished worksheet 1 they should start on
worksheet 2 and use the calculator that I had provided them with. Again there was no time limit. In each school I had previously checked with the teacher that the pupils had used graphics calculators before, but I explained to the pupils that they would be using the calculator in the same way as their own scientific calculators, and that if they needed any help operating the calculator, such as finding a particular button, they could ask either me or their teacher. In practice the main reason for asking for help was to find the negative key \( (-) \) and the power key \( ^\wedge \), as some pupils chose to use this key rather than the squared key \( ^2 \). The vast majority of pupils in all three schools appeared to be confident with using the calculators and needed very little help.

When they had finished both worksheets the pupils gave them in to me or to their teacher, and they were able to spend the remainder of the lesson on a mathematical activity that I had arranged with the class teacher.

Towards the end of the lesson, after all the pupils had completed both worksheets and handed them in, with the calculators, I spoke to the whole class. I explained that I would mark their work and return it to their teacher to give back to them, and that if they had found any of the work difficult they would have the opportunity to go over it with their teacher. I then described the Key Recorder software and explained that I would be able to replay all their key strokes and see their calculator screens exactly as they had used them. I made it clear that my reason for doing this was to gain an understanding of their ways of working and their methods of carrying out
the calculations on their calculators. I then explained that if anyone did not want me to view their calculator data they could tell either myself or their class teacher and I would erase their data without viewing it. I also explained that if anyone did not want me to use their worksheet data in my study they could tell me or their class teacher. In practice none of the pupils requested that their data be erased or not used. Finally I told the pupils that I would be coming back again to interview a few of them about their work. I did not tell them how I would be selecting them for interview but I assured them that they would not have to be interviewed if they did not want to.

6.2.1.2 Between the first and second visit

It was important that I marked and returned the worksheets promptly. I wanted to conduct the pupil interviews as soon as possible after they had completed the work, and I wanted to return the marked work promptly to the class teacher so that they could return it to the pupils and go through the work as appropriate. I also needed to video record the Key Recorder data quickly, so that I could review the recordings before interviewing the pupils, and also because the calculators would be needed by other members of staff.

I initially marked and scored the worksheets out of 14, and photocopied them so that I could return the originals to the class teacher on my second visit. I also used the Key Recorder software to play back the pupils’ keystrokes on each calculator, and video-recorded each one as I played them
back. Each calculator was numbered so that I would be able to identify them with the pupil numbers on the worksheets. I then analysed the pupils’ work for misconceptions leading to incorrect answers. Analysing worksheet 1 involved inspection of the workings that the pupils had written on the worksheets; analysing worksheet 2 involved playing back the video recordings of the Key Recorder data for each pupil alongside their answers in worksheet 2. In some cases the pupils had also written some workings on worksheet 2, which could be used in the analysis alongside the keystroke recordings, but this was not done as frequently as in worksheet 1, as most pupils just wrote down the answer that they had obtained on the calculator. On the basis of an initial analysis of the pupils’ work I selected a number of pupils who I hoped to interview.

6.2.1.3 The second visit

Having analysed the pupil worksheets I then made a second visit to the school, between one and two weeks after my first visit, and carried out the interviews with some of the pupils who I had selected. The pupils were told that they did not have to be interviewed and that they could withdraw from the interview at any time. They were told that the interviews would be recorded but that I would not be addressing them by name, and they were also told that if at any stage afterwards they decided that they did not want me to use their interview data they could tell their teacher who would be able to inform me.
The interviews were conducted during one of the pupils’ mathematics lessons, with the agreement of the class teacher. In each school I was able use either an empty classroom or an office and the pupils were able to leave their mathematics lesson in order to be interviewed by me.

The interviews were semi-structured and open. I started each interview by thanking the pupil for agreeing to be interviewed and checking that they were still happy to be interviewed. On one occasion, a pupil did reply that he did not wish to be interviewed, so I thanked him anyway and allowed him to return to his maths lesson. The questions that I asked the pupils depended upon their own workings and answers. I always started by pointing out their correct answers, and praising them for answering these questions correctly, before looking at the questions that they had got wrong. It was clearly important to conduct this sensitively, particularly if the pupil had got a number of questions wrong. The questions I asked depended upon the individual pupil’s workings, but were of the form “Can you explain to me what you did here?”, “Can you see what went wrong with this question?” “What do you think you needed to do in this question?” Some pupils had written “BODMAS” or “BIDMAS” on their worksheet, so I asked them why they had written it and what it meant. In addition to trying to find out what the pupil had been thinking I also took care that whilst discussing the misconceptions I tried to address them with the pupil, as far as possible in the time allocation, in order to leave the pupil with a positive feeling that they understood where they had gone wrong. This lengthened
some of the interviews, but I saw it as an important part of the process, and indeed I was thanked by some pupils for helping them to understand the work more clearly. Each pupil interview lasted about 10 minutes. The interviews were later transcribed.

On my second visit to each school I also interviewed the class teacher. I did not record these interviews but made field notes as we talked. I wanted to create the atmosphere of a professional chat rather than a formal interview. The notes sometimes involved diagrams and examples, and I was able to share these with the teacher and discuss them as we talked. I ensured that each of the teachers was aware that I would be using our conversation to inform my research and from time to time I would check with them that they were in agreement with the notes that I was writing down. I informed the teachers that I would not be naming them, or their school, or any of their pupils in my research, and that they could at any stage request that I do not use their interview data in my research.

The interviews were open and semi-structured. I had some set questions which I asked each teacher (appendix K) but the tone of the interview was conversational and did not adhere to a strict interview format. Often the response of the teacher would lead to some follow-up questions, and at times the interviews became more of a conversation between professional colleagues. I considered this to be very helpful, as it enabled me to gain greater insight into how their views and opinions about the curriculum and the resources they used impacted on their teaching methods, and I did not
feel that I would have achieved this depth of insight in a more formal, recorded interview.

The teacher interviews took between 30 – 40 minutes and at the end I thanked them for their time and promised that I would make my research findings available to them on its completion.

6.2.2 Conducting the study in the other countries

In the case of schools D and E (Netherlands and New York State, US) I was unable to visit the schools myself so I relied upon colleagues in each country to administer the worksheets with classes in their schools. I had chosen to approach these teachers because I had met them in professional settings and had asked them if they would be willing to help me with this aspect of my research. I ascertained that they would be able to conduct this study with classes of similar ages and attainment groups to those that I had worked with in the UK. I showed them both worksheets and confirmed that they felt confident that the pupils in the classes that they intended to work with should reasonably be expected to tackle all the questions. I briefed the teachers fully about the conditions that the pupils would need to work in, and obtained the appropriate permission and ethical consent. Since the Key Recorder software was not being used this was not an issue when obtaining ethical consent.
Both worksheets were completed by the pupils, in the same conditions as in the UK schools. When working on worksheet 2 the pupils were able to use their own calculators. In each school I was informed by the teachers that all the pupils would have access to a scientific calculator when completing the worksheet, and if they did not have a calculator of their own, they would be provided with a scientific calculator by the school.

The staff in schools D and E were fully briefed with regard to the ethical considerations outlined previously, although the issue of the Key Recorder data was clearly not relevant in these cases.

The worksheets were completed within a mathematics lesson and sent to me. I marked and returned them, keeping photocopies for further analysis. I was then able to analyse both worksheets for misconceptions in the incorrect answers. Although I did not have any Key Recorder data to inform my analysis of worksheet 2, I was able to use my experience of analysing the Key Recorder data in schools A, B and C to identify many of the methods that led to particular wrong answers, and therefore to make an informed assumption about what the pupil was likely to have keyed into the calculator.

In the case of school D in the Netherlands the class teacher expressed her willingness to take part in the study and initially I had considered using Skype or some form of video conferencing to conduct the interview. However this teacher was concerned that the shortcomings of her academic
English may have made this difficult and suggested that she would feel more comfortable if I sent her questions by email, allowing her time to think about her answers and ensure that her English was correct. This approach facilitated an exchange of emails and I was able to pose further questions in the same way that I had in my interviews with teachers in the UK, and so I was confident that I had been able to obtain sufficient depth of information by interviewing her in this manner. She also sent me examples of teaching resources that she had used when teaching her class about the order of operations.

I was unable to do this in school E in New York State. The colleague who I had approached and discussed my work with taught mathematics in a High School, and in order to carry out the study with children of the required age group she needed to approach colleagues in a middle school in the same school district. This meant that I had no direct contact with the teachers of the two classes that were used for my study, and therefore was unable to interview them. However I was able to talk to two colleagues from New York State who visited the University of Plymouth in connection with their own research. One teaches in a middle school and the other is involved in teacher education in middle schools, so both were able to talk to me about the teaching methods, approaches and resources that are used across the state. There is a considerable degree of common practice across the state so I felt confident that these conversations would enable me to elicit the sort of
information that would be relevant to the teaching and learning of the pupils who took part in the study.

6.3 Analysis of the pupil worksheets

My main objective was to analyse the worksheets for misconceptions, but as an initial part of the data analysis I investigated the total scores, out of 14, for the pupils in each of the classes, before trying to analyse and categorise the pupils’ misconceptions that I observed.

6.3.1 Analysis of the worksheet scores

An initial summary was made of the worksheet scores (out of 14) for each class, given in table 6.2

<table>
<thead>
<tr>
<th></th>
<th>Worksheet 1 (non-calculator)</th>
<th>Worksheet 2 (with calculator)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>min</td>
<td>max</td>
</tr>
<tr>
<td><strong>School A</strong></td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td><strong>School B</strong></td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td><strong>School C</strong></td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td><strong>School D</strong></td>
<td>8</td>
<td>14</td>
</tr>
<tr>
<td><strong>School E class 1</strong></td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td><strong>School E class 2</strong></td>
<td>7</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 6.2 Summary Data for worksheet scores
These data are displayed using boxplots in Figure 6.1

![Boxplot summary and comparison of worksheet scores](image)

**Figure 6.1** Boxplots to illustrate the worksheet scores for each class

### 6.3.2 Comparison of scores in worksheet 1 and worksheet 2

Since each worksheet contains calculations which are structurally identical, it is first interesting to investigate pupils who achieve significantly different scores in each worksheet. This comparison was carried out for each school with a paired $t$-test. This was used because the sample sizes were small and from inspection of the data the distributions were seen to be reasonably normal and it was assumed that they came from normal populations. In each case the null and alternative hypotheses take the form:
\( H_0: \) There is no difference between the mean worksheet scores

\( H_1: \) There is a difference between the mean worksheet scores

The test was carried out at the 5% significance level. Table 6.3 summarises the results of the paired t-tests. The significant values are labelled with an asterix:

\[
\begin{array}{|c|c|c|}
\hline
\text{School} & n & t-value & p-value \\
\hline
\text{School A} & 17 & -0.60 & 0.558 \\
\text{School B} & 27 & 0.00 & 1.000 \\
\text{School C} & 29 & -2.77 & 0.01^* \\
\text{School D} & 22 & -0.72 & 0.482 \\
\text{School E class 1} & 28 & 1.26 & 0.215 \\
\text{School E class 2} & 27 & 2.87 & 0.008^* \\
\hline
\end{array}
\]

Table 6.3 Results of paired t-tests for differences between scores in worksheet 1 and worksheet 2 for each school

In the UK, only one of the three schools, school C, exhibited a statistically significant difference between the worksheet scores and in this case the pupils performed better in worksheet 2 than in worksheet 1. In school B the mean scores were identical for the two worksheets, and in school A the mean score was slightly higher for worksheet 2 although this was not statistically significant. This would suggest that the overall performance of the UK pupils appeared to be largely unaffected by the use of the calculator,
despite the fact that the pupils were working with calculators that were not their everyday calculators, and that if the use of calculators did have an effect, this was a positive effect.

In the US, the pupils in class 1 (grade 7) scored slightly better on worksheet 2 than worksheet 1, although this was not significant. In class 2 (grade 8) the mean score for worksheet 1 was significantly higher than that for worksheet 2, indicating that the pupils performed better on the non-calculator questions than when using a calculator. In school D (the Netherlands) there was no statistically significant difference between the scores in each worksheet.

6.3.3 Comparison of the scores from the different classes

The boxplots in figure 6.1 suggested that it would be worth looking into possible differences between the scores from class to class, therefore a one-way Analysis of Variance was carried out. The assumptions were that the variances of the samples were approximately equal and the population distributions were assumed to be normal. As this was an opportunity sample, it was not random but is used to enrich and inform the case study approach. In order to compare like with like I carried this out on the scores from each worksheet separately, ie an analysis was made for the scores in worksheet 1 in each class, and then again for the scores in worksheet 2. The 95% confidence intervals for the mean scores for each worksheet are shown in figures 6.2 and 6.3 and demonstrate that in both worksheet 1 and
worksheet 2 there is a significant difference (no overlap of confidence
interval) between the scores of the class in school D compared to those
classes in schools A, B and C. The results of the ANOVA are given in table

6.4

<table>
<thead>
<tr>
<th>Individual 95% CIs For Mean Based on Pooled StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>school A sheet 1</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>school B sheet 1</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>school C sheet 1</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>school D sheet 1</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>School E class 1 sheet 1</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>School E class 2 sheet 1</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>7.5 9.0 10.5 12.0</td>
</tr>
</tbody>
</table>

**Figure 6.2 Individual 95% confidence intervals for the mean for Worksheet 1**

<table>
<thead>
<tr>
<th>Individual 95% CIs For Mean Based on Pooled StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level</td>
</tr>
<tr>
<td>school A sheet 2</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>school B sheet 2</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>school C sheet 2</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>school D sheet 2</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>School E class 1 sheet 2</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>School E class 2 sheet 2</td>
</tr>
<tr>
<td>(--------*-----)</td>
</tr>
<tr>
<td>9.0 10.5 12.0 13.5</td>
</tr>
</tbody>
</table>

**Figure 6.3 Individual 95% confidence intervals for the mean for Worksheet 2**
Table 6.4 Analysis of Variance comparing the mean scores of the classes for worksheet 1 and worksheet 2

Worksheet 1

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
<td>5</td>
<td>272.74</td>
<td>54.55</td>
<td>11.01</td>
<td>0.000</td>
</tr>
<tr>
<td>Error</td>
<td>142</td>
<td>703.82</td>
<td>4.96</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>147</td>
<td>976.56</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

S = 2.226  R-Sq = 27.93%  R-Sq(adj) = 25.39%

<table>
<thead>
<tr>
<th>Level</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>school A sheet 1</td>
<td>17</td>
<td>9.412</td>
<td>2.647</td>
</tr>
<tr>
<td>school B sheet 1</td>
<td>27</td>
<td>8.852</td>
<td>2.365</td>
</tr>
<tr>
<td>school C sheet 1</td>
<td>27</td>
<td>8.407</td>
<td>2.206</td>
</tr>
<tr>
<td>school D sheet 1</td>
<td>22</td>
<td>12.227</td>
<td>1.901</td>
</tr>
<tr>
<td>School E class 1 sheet 1</td>
<td>28</td>
<td>10.750</td>
<td>2.474</td>
</tr>
<tr>
<td>School E class 2 sheet 1</td>
<td>27</td>
<td>11.222</td>
<td>1.739</td>
</tr>
</tbody>
</table>

Worksheet 2

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
<td>5</td>
<td>181.31</td>
<td>36.26</td>
<td>6.67</td>
<td>0.000</td>
</tr>
<tr>
<td>Error</td>
<td>142</td>
<td>771.98</td>
<td>5.44</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>147</td>
<td>953.30</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

S = 2.332  R-Sq = 19.02%  R-Sq(adj) = 16.17%

<table>
<thead>
<tr>
<th>Level</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>school A sheet 2</td>
<td>17</td>
<td>9.882</td>
<td>1.799</td>
</tr>
<tr>
<td>school B sheet 2</td>
<td>27</td>
<td>8.852</td>
<td>2.568</td>
</tr>
<tr>
<td>school C sheet 2</td>
<td>27</td>
<td>10.037</td>
<td>2.738</td>
</tr>
<tr>
<td>school D sheet 2</td>
<td>22</td>
<td>12.591</td>
<td>1.368</td>
</tr>
<tr>
<td>School E class 1 sheet 2</td>
<td>28</td>
<td>9.893</td>
<td>2.629</td>
</tr>
<tr>
<td>School E class 2 sheet 2</td>
<td>27</td>
<td>9.926</td>
<td>2.218</td>
</tr>
</tbody>
</table>

This analysis also indicates that there were no statistically significant differences in performance between any of the schools in the UK, and no statistically significant differences between the two classes in the US. This demonstrates a degree of consistency in the results for the classes in each of these countries.
6.4 Summary

In this chapter the main study has been described and the worksheet scores analysed statistically, demonstrating that the pupils in the class in the Netherlands achieved scores that were higher and more consistent than the classes of pupils in either the US or the UK. It must be remembered that this is a case study and therefore these results cannot be generalised but they do suggest that there may be differences in the work of the pupils in the Dutch class compared to that of the pupils in the classes in the UK and the US, and that a qualitative analysis of the pupils’ work might reveal differences in their ways of working. This will be described and addressed in chapter 7 and chapter 8. The interviews that were carried out with the teachers in the different countries would also reveal differences in their teaching approaches, and these are discussed and compared in chapter 9. A description and comparison of the mathematics curricula in the different countries is also described in chapter 9.
CHAPTER 7: MISCONCEPTIONS OBSERVED

7.0 Introduction

In this chapter I will examine in detail the work of a selection of pupils in order to exemplify the themes that emerged from the analysis of their data, and to comment in detail on some of the pupils’ work and keystrokes that were observed and the misconceptions that were seen and categorised.

7.1 Analysis of Misconceptions

Having marked and allocated scores to the worksheets, I subsequently looked at each question that had been answered wrongly, and attempted to determine what the pupil had done wrong, whether this could be described as a misconception, and how this misconception could be categorised. I had already built up a bank of different types of misconceptions that had been observed in the pilot study, and I added to this as I investigated the pupils’ work from the main study. In order to do this it was necessary to examine the pupils’ written workings in worksheet 1. In worksheet 2 some pupils gave written workings some of the time, but the main method for analysing the pupils’ work was the Key Recorder data for the UK schools. For the schools in the Netherlands and New York I was able to analyse worksheet 2 to some extent by comparing the values of the wrong answers with the values of wrong answers obtained by the pupils in the UK schools and thus making an inference about the pupil’s method of obtaining their answer. This approach appeared to work in many cases but in some cases it was simply not possible to work out what the pupil might have done to get a particular wrong answer, and so it was impossible to categorise.
The categories included those already observed in the pilot study with the pupils in the UK and Japan.

7.1.1 Categorising the misconceptions

The incorrect methods that were observed fell into four categories:

- **A misconception about the order of operations**
  For example, simply working from left to right

- **A misinterpretation or misunderstanding of notation**
  For example, interpreting $3^2$ as $3 \times 2$

- **An arithmetic error**
  For example, writing a correct multiplication but evaluating it incorrectly

- **A calculator error (Worksheet 2 only)**
  For example, not being able to work correctly with negative numbers or fractions on the calculator, or incorrect use of brackets on the calculator.

A full list of all observed errors was gradually compiled as the analysis progressed. This will now be given with examples of pupils’ work or Key Recorder data to exemplify some of the ways in which each misconception was demonstrated in each worksheet. Some of these are consistent with those already observed in the pilot study, and some were only observed in the main study.
MISCONCEPTIONS ABOUT THE ORDER OF OPERATIONS

M1 Works from left to right

These are cases where the calculation is performed in order from left to right ignoring the usual mathematical conventions, for example

\[ 2 + 3 \times 4 + 5 = 25 \]. Figure 7.1 shows examples of this type of misconception which is demonstrated by one pupil and is evident from both his non-calculator work in worksheet 1, and also from his written workings and keystrokes in worksheet 2.

\[
\begin{array}{|c|c|}
\hline
1 & 2 + 3 \times 4 + 5 \\
\hline
2 & 4 + 10 \div 2 \\
\hline
3 & 8 + \frac{8}{4} \\
\hline
\end{array}
\]

School A Pupil 2 worksheet 1 questions 1, 2, 3

\[
\begin{array}{|c|c|}
\hline
2 & 4.45 + 10.8 + 2 \\
\hline
3 & 8.9 + \frac{8.6}{4} \\
\hline
\end{array}
\]

School A Pupil 2 worksheet 2 questions 2, 3

\[
\begin{array}{|c|c|}
\hline
4.45+10.8  & 15.25 \\
15.25/2 & 7.625 \\
\hline
\end{array}
\]

School A Pupil 2 Keystrokes for question 2

\[
\begin{array}{|c|c|}
\hline
8.9+8.6  & 17.5 \\
17.5/4 & 4.375 \\
\hline
\end{array}
\]

School A Pupil 2 Keystrokes for question 3

Figure 7.1 Examples of misconception M1 by Pupil 2 from School A
M2 Ignores bracket when with power

In some cases the pupils seemed to “ignore” the brackets when there was a power outside the bracket. This resulted in calculating

\[(1 + 2)^2 \text{ as } 1 + 2^2 \text{ or } (2 \times 3)^2 \text{ as } 2 \times 3^2\].

Examples of this type of misconception, exemplified by written workings and keystrokes, are shown in figure 7.2.

Figure 7.2 Examples of misconception M2 by Pupil 3 and Pupil 4 from School A, Pupil 21 from School C and Pupil 7 from School D
M3 Performs addition before power

If a calculation contained an addition and a power, some pupils performed the addition first. In some cases this could also be regarded as a left-to-right interpretation, for example interpreting $2 + 3^2$ as 25, but the examples in figure 7.3 reveal that this is not always the case as the same misconception can sometimes be seen in question 8 where the multiplication in the bracket is carried out first, but then the addition is performed before squaring, which does not involve left-to-right thinking.

Figure 7.3 Examples of misconception M3 by Pupil 8 and Pupil 7 from School C
M4 Performs addition before multiplication

In some cases the pupils demonstrated that they knew that an order was required, but put addition before multiplication. This might suggest that they had an awareness of a rule, but no familiarity with it. An example of this is shown in figure 7.4 in which the pupil consistently demonstrates this misconception in worksheet 1 and worksheet 2.

![Figure 7.4](image1)

School E class 1 Pupil 2 worksheet 1 question 1

School E class 1 Pupil 2 worksheet 2 question 1

Figure 7.4 Examples of misconception M4 by Pupil 2 from School E Class 1

M5 Incorrect interpretation of power outside bracket

Although the previous misconceptions had been observed with pupils in all the countries in the study, this misconception had been observed and classified only with some of the Japanese pupils in the pilot study. It was difficult to determine whether the issue simply lay with an incorrect interpretation of squaring, but in the example given in figure 7.5 this pupil
had given a correct interpretation of squaring in three previous questions, so
the error observed when trying to square the bracket in evaluating the
denominator in question 10 appeared to arise because it was with a bracket,
and so it was classified as M5. This pupil also interpreted the fraction as an
addition (C3, described later).

Example 7.5 Example of misconception M5
by Pupil 7 from School E class 1 worksheet 1 question 10

M6 Performs multiplication before power

When a calculation involved a multiplication and a power some pupils
performed the multiplication first. This is exemplified in figure 7.6 with the
work of two pupils from school C.
M7 Performs division before power

The workings shown in figure 7.7 show that although the pupil has correctly dealt with the power in the numerator of the fraction, the power on the denominator has not been evaluated until after the numerator has been divided by the bracket in the denominator. This could be seen as a left-to-right interpretation over two lines.
M8 Changes operation

In some cases the pupil had given the correct operation in their written workings but had calculated using a different operation, as shown in the first two examples in figure 7.8. In the first example from pupil 24 in school C, the denominator has been correctly written as $3 + 4$ but then the pupil has multiplied to give a value of 12. In the second example pupil 24 from school C has correctly written $4 \times 5$ but has evaluated this as 9. In these cases it could be argued that this is not a misconception but a careless error as a result of the brain seeing two numbers and being “hard wired” to perform an operation, most frequently an addition as in the example from pupil 24 in figure 7.8.

In other cases, however, the pupils’ written workings show that the operation has been exchanged before the calculation has been written down. This is exemplified in the third example in figure 7.8 where the pupil has exchanged the addition sign for a multiplication symbol. This happened most frequently with the pupils from school E in the Netherlands, who used a dot for the multiplication symbol.
MISCONCEPTIONS WITH ALGEBRAIC AND NOTATIONAL CONVENTIONS

C1 Misunderstands power notation

Some pupils did not correctly interpret index notation, the most frequent interpretation being that they would multiply by 2 instead of squaring.

Examples of this, exemplified by both written working and by calculator keystrokes, are given in figure 7.9. This differs from the misconception categorised as M5, in which the pupils seemed to be aware of what squaring meant, but seemed unable to carry it out correctly when a bracket was involved.
School B Pupil 10 worksheet 1 questions 5 and 6

School B Pupil 25 worksheet 1 question 6

School B Pupil 25 worksheet 2 question 5 and question 6

School B Pupil 25 worksheet 2 keystrokes for question 5 and question 6

Figure 7.9 Examples of misconception C1 by Pupil 10 and Pupil 25 from School B
C2 Misunderstands algebraic convention for multiplication

Some pupils did not recognise \( ab \) as a multiplication but interpreted \( ab \) as a 2-digit number. This misconception was also demonstrated when a variable appeared outside a bracket. This can be seen in the work of pupil 10 from school B, shown in figure 7.10, who consistently displays this misconception even when there is a bracket involved, for example in question 13 where the bracket is evaluated first, then the result treated as the second digit of a 2-digit number. Some pupils only misunderstood the algebraic notation for multiplication when a bracket was involved. For example pupil 10 from school D successfully interpreted \( ab \) as a multiplication in questions 11 and 12, but did not recognise that when the number is outside a bracket it is also a multiplication, and interpreted this as an addition, as shown in question 13 in which he wrote \( 2(3 + 4) \) and then interpreted this as \( 2 + 7 \). The same pupil also wrote \( 2(3 + 5) \) as the numerator in question 14 of worksheet 1, then evaluated this as \( 2 + 8 \), as shown in figure 7.10.

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
</table>
| 11 \( a + bc + d \) | \( 2 + 34 + 5 \)  
\( 2 + 34 = 36 + 5 = 41 \) | 41 \( \times \) c2 |
| 12 \( \frac{23}{ab} + \frac{45}{cd} \) | \( 24 + 45 = 69 \) | 69 \( \times \) c1 |
| 13 \( a(b + c) \) | \( 2(8 + 4) = 28 \)  
\( 2(3 + 4) = 27 \) | 27 \( \times \) c2 |

School B Pupil 10 worksheet 1 questions 11, 12 and 13
C3 Does not recognise a fraction as a division

Some pupils did not recognise fraction notation as a division, or became confused when this appeared as part of a more complex algebraic expression. This is exemplified in figure 7.11 by the work of pupil 15 from school C and pupil 1 from school D. Although both pupils had correctly interpreted a single fraction as a division, both appeared confused when the fraction contained calculations on the numerator and denominator, and interpreted the fraction as an addition in these cases.
C4 Tries to add fractions and makes a mistake

Some pupils tried to calculate question 3 by adding two fractions, and then carried this out incorrectly. This links with the misconception previously identified as C3, where fraction notation is not always associated with a division. This is exemplified by the work of pupil 19 in school E class 2, shown in figure 7.12. This pupil tries to perform an addition with a common denominator of 4, but does not write the whole number 8 as a fraction with a denominator of 4.

![School E class 2 Pupil 19 worksheet 1 question 3](image)

**Figure 7.12 Example of misconception C4 by Pupil 19 from School E class 2**

C5 Cannot square negative numbers

Some pupils were not able to correctly square a negative number, even though the negative numbers in the questions had been put in brackets. Sometimes the pupils did not even attempt question 6, which contained \((-3)^2\). This can be seen in figure 7.13 by pupil 15 from school B, who put a cross in the answer box for question 6 on both worksheets, indicating that she could not attempt the question. Many pupils gave a negative answer, and this was often consistent across the two worksheets, whether working
with or without a calculator, as exemplified by the written work and keystrokes of pupil 5 from school A, and the written work of pupil 23 from school E class 2. This is an example of how it was possible to infer the pupils’ keystrokes from the incorrect answer, even though there was no Key Recorder data available.

School B Pupil 15 worksheet 1 question 6

School B Pupil 15 worksheet 2 question 6

School A Pupil 5 worksheet 1 question 6

School A Pupil 5 worksheet 2 question 6

School A Pupil 5 Keystrokes for worksheet 2 question 6
Figure 7.13 Examples of misconception C5 by Pupil 5 from School A and Pupil 23 from School E Class 2

C6 Divides fractions the wrong way round

In some cases, pupils recognised that a fraction represented a division, but carried out the division the wrong way round, dividing the denominator by the numerator. In many cases where this occurred, it occurred inconsistently, and occurred more frequently in worksheet 2, working with a calculator. This is exemplified by the work of pupil 22 from school B, shown in figure 7.14. His keystrokes reveal that he has first calculated the denominator of the fraction, and has then attempted to divide this by the expression in the numerator, although this has been done inefficiently by failing to put the numerator in brackets, or by evaluating it separately.
C7 Interprets $ab$ as $a + b$

Some pupils misinterpreted the algebraic notation for multiplication and performed an addition. This was often observed as consistent, even when the multiplication was with a bracket. This is exemplified by the work of pupil 14 in school B shown in figure 7.15. In worksheet 1, question 11, this pupil has tried to evaluate $a + bc + d$ by adding all four values together, and has done the same thing in question 12 for the expression $ab + cd$, obtaining the same answer in each case. In question 13 he has interpreted $a(b + c)$ to be $a + b + c$. His keystrokes for the equivalent questions in worksheet 2, shown in figure 7.15, reveal that he has done exactly the same thing when working with the calculator.
School B Pupil 14 worksheet 1 questions 11, 12, 14 and 14

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>$a + bc + d$</td>
<td>$\times c_7$</td>
</tr>
<tr>
<td>12</td>
<td>$ab + cd$</td>
<td>$\times c_7$</td>
</tr>
<tr>
<td>13</td>
<td>$a(b + c)$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$\frac{a(b + d)}{c + a^2}$</td>
<td>$\times c_7$</td>
</tr>
</tbody>
</table>

School B Pupil 14 worksheet 2 questions 11, 12, 13 and 14

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings (if needed)</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>$a + bc + d$</td>
<td>$16 \times c_7$</td>
</tr>
<tr>
<td>12</td>
<td>$ab + cd$</td>
<td>$16 \times c_7$</td>
</tr>
<tr>
<td>13</td>
<td>$a(b + c)$</td>
<td>$10 \div c_7$</td>
</tr>
<tr>
<td>14</td>
<td>$\frac{a(b + d)}{c + a^2}$</td>
<td>$\times c_7$</td>
</tr>
</tbody>
</table>

School B Pupil 14 Keystrokes for worksheet 2 questions 11, 12, 13 and 14

$\begin{array}{c}
2.4+3.2+4.6+5.8 \\
16 \\
\end{array}$  
$\begin{array}{c}
2.4+3.2+4.6+5.8 \\
16 \\
\end{array}$  

Question 11  
Question 12

$\begin{array}{c}
2.4+3.2+4.6 \\
10.2 \\
\end{array}$  

Question 13

$\begin{array}{c}
2.4+3.2+5.8 \\
4.6+2.4^2 \\
11.4+10.36 \\
3.1003861 \\
\end{array}$  

Question 14

**Figure 7.15 Example of misconception C7 by Pupil 14 from School B**
C8 Reverses the numbers in an operation

In some cases, the pupil appeared to reverse the numbers in an operation, for example interpreting $6 - 4$ as $4 - 6$ as demonstrated by pupil 28 from school B shown in figure 7.16.

![School B Pupil 28 worksheet 1 question 7](image)

Figure 7.16 Example of misconception C8 by Pupil 28 from School B

C9 Separates out fractions

In the calculations that involved an operation in the numerator and denominator, some pupils did not appreciate that these operations needed to be done first, as if in brackets, and tried to deal with the calculation as two separate fractions. Figure 7.17 gives examples of the work and keystrokes of pupil 15 in school B, demonstrating this misconception being applied both with and without a calculator. In worksheet 1 this pupil wrote down $\frac{8}{3}$ as a single fraction, but appears to have given up at this stage, possibly because he felt unable to evaluate this.
ARITHMETIC ERRORS

**A1  Error with multiplication tables or other single operation**

In some cases the pupils appeared to make an error with a multiplication or addition, as demonstrated in figure 7.18 where the first pupil appears to
have evaluated \(2 + 4\) to be 7, and the second pupil has written \(3 + 4\) in brackets but has evaluated this to be 6.

**Figure 7.18 Examples of error A1 by Pupil 15 from School C and Pupil 1 from School E Class 2**

**A2 Transcription error or careless error**

Sometimes pupils made transcription errors. Sometimes these were between stages in their working, and sometimes through misreading the question. These tended to occur in the last four questions where the pupils needed to substitute values into the algebraic expressions, and sometimes they substituted an incorrect value. This type of error is exemplified in figure 7.19 in the work of pupil 5 from school B, who substitutes the values that have been given for \(c\) and \(d\) into the bracket instead of the correct values for \(b\) and \(c\). Similar errors could be described as careless errors, as demonstrated by pupil 15 from school B in her work on question 11 in worksheet 2 shown in figure 7.19. Her keystrokes reveal that she has made two errors in this calculation; she has made a transcription error in copying
the number 14.72 incorrectly from her calculator display as 14.76, and then she has left out the decimal point in the number 5.8 and has entered this on her calculator as the number 58. Both errors have been categorised as A2.

Figure 7.19 Examples of error A2 by Pupil 5 and Pupil 15 from School B

A3 Tries to multiply out brackets algebraically and makes a mistake

In the pilot study this misconception had only been observed in the work of the Japanese pupils in school Y, and it was not observed very frequently in the main study but a small number of pupils attempted to multiply out brackets algebraically and then made a mistake. This is shown in figure 7.20 in the work of pupil 14 from school E (class 2) in question 4. This pupil’s written work shows that he is trying to multiply the expression in the bracket by 3, but although he correctly multiplies the first number in the
bracket by 3 he fails to multiply the second number in the bracket by 3. In
other words, thinking algebraically, he has interpreted \(a(b+c)\) as \(ab + c\).

Figure 7.20 Examples of misconception A3 by Pupil 14 from School E

**A4 Confusion about operations with negative numbers**

In some cases the pupil’s written working suggested a deeper confusion
about working with negative numbers which appeared to be more than being
unable to square a negative number. An example of this is shown in figure
7.21 where the pupil has drawn a number line in an attempt to make sense
of the calculation \(-9 \times 2\) in question 6, and incorrectly gives the answer as 9.

Figure 7.21 Example of misconception A4 by Pupil 6 from School A
A5 Confusion when fraction notation involved

In some cases pupils appeared to become confused when fraction notation was involved in the more complex calculations which included an operation in the numerator and the denominator. This is exemplified by the work of pupil 23 in school B, shown in figure 7.22. This pupil had demonstrated in question 4 of worksheet 1 that he recognised that $\frac{8}{4}$ means $8 \div 4$, that is he recognises that a fraction represents a division, but when he gets to question 9 he appears to be very unsure about dealing with a fraction which involves calculations within the numerator and the denominator. He is unable to deal with the numerator and denominator separately, and instead of dividing he multiplies values from the numerator by values from the denominator. It is interesting to notice that when he performs the equivalent calculation in worksheet 2, with a calculator, he does this efficiently and correctly, as demonstrated by his written workings and keystrokes in question 9 of worksheet 2 shown in figure 7.22 even though he uses effectively the same method of evaluating both the numerator and denominator and then dividing.
CALCULATOR ERRORS

Analysis of the pupils’ work on worksheet 2, and the accompanying keystrokes, revealed that a number of incorrect answers were the result of incorrect or inefficient use of the calculator. In some cases this was due to incorrect syntax, in other cases it appeared to be due to the fact that the pupil could not determine an efficient sequence of keystrokes. This was
particularly evident in questions 9 and 10 which required the evaluation of the numerator and denominator of a fraction before performing the division. The calculator errors were categorised as follows:

**BC  Brackets not dealt with correctly on calculator**

Some pupils did not deal correctly with brackets on their calculator, despite the fact that the graphics calculator, like a scientific calculator, has brackets keys. This difficulty is exemplified in figure 7.23 by the work and keystrokes of pupil 9 in school B. This pupil had correctly evaluated the calculation in question 4 of worksheet 1; when he attempted the corresponding question in worksheet 2 his keystrokes reveal that he knew he should evaluate the calculation in brackets first, but because he didn’t press “enter” after this step, or make use of the calculator brackets keys, his answer was incorrect.

![School B Pupil 9 worksheet 1 Question 4](image1)

![School B Pupil 9 worksheet 2 question 4](image2)

**Figure 7.23 Examples of error BC by Pupil 9 from School B**
NC  Problems with negative numbers on a calculator

Some pupils were able to correctly square a negative number when working without a calculator, but when working with a calculator they did not put the negative number in brackets, despite the fact that the question included brackets, and so their answer to question 6 was negative. An example of this is shown in fig 7.24 in the work and keystrokes of pupil 23 from school B. Other pupils, such as pupil 4 from school C, shown in figure 7.24, used the “subtract” key instead of the “negative” key, which resulted in a syntax error, and so he was unable to evaluate an answer, leaving the answer box blank.

Figure 7.24  Examples of error  NC by Pupil 23 from School B and Pupil 4 from School C
IC Problems with indices on a calculator

In some cases the pupil became confused when using the calculator to square numbers, despite being able to successfully square numbers without a calculator. In the example shown in figure 7.25, from pupil 4 in school B, the keystrokes reveal that the pupil has used the reciprocal key instead of the squaring key. This could be because the reciprocal key is immediately above the squaring key on the keyboard of the graphics calculator.

![School B Pupil 4 worksheet 2 question 8](image)

Figure 7.25 Example of error IC by Pupil 4 from School B

FC Cannot deal with fractions on a calculator

In some cases the pupils were unable to use the calculator efficiently to perform the calculations which involved operations on the numerator and denominator of a fraction. The most frequent mistake was to key in the expression as it appeared, without putting the denominator in brackets. This would suggest that the pupil knew that the numerator had to be divided by the denominator, but did not appreciate the fact that the calculator would
perform this using the correct order of operations as keyed in. This is exemplified by the work and keystrokes of pupil 9 from school A, shown in fig 7.26

![Example of error FC by Pupil 9 from School A](image)

**Figure 7.26 Example of error FC by Pupil 9 from School A**

**KC Key error**

In some cases a careless error resulted from apparently pressing the wrong key by mistake. This was regarded as a “careless error” as opposed to a confusion about the meaning of two keys such as the $x^2$ key and the $x^{-1}$ key. This could usually be inferred by observing the pupil’s keystrokes alongside their written working, as demonstrated in figure 7.27

This pupil has written down the correct calculation on the worksheet, with brackets indicating an understanding that the multiplication in the middle needs to be done first. This is reflected in the order of his keystrokes. He has probably pressed the addition key accidentally instead of the multiplication key.
7.1.2 Summary of Categories of Misconceptions

To summarise, the misconceptions that had been observed were categorised as follows:

**MISCONCEPTIONS ABOUT THE ORDER OF OPERATIONS**

M1 Works from left to right
M2 Ignores bracket when with power
M3 Performs addition before power
M4 Performs addition before multiplication
M5 Incorrect interpretation of power outside bracket
M6 Performs multiplication before power
M7 Performs division before power
M8 Changes operation
MISCONCEPTIONS WITH ALGEBRAIC AND NOTATIONAL CONVENTIONS

C1 Misunderstands power notation
C2 Misunderstands algebraic convention for multiplication
C3 Does not recognise a fraction as a division
C4 Tries to add fractions but makes a mistake
C5 Cannot square negative numbers
C6 Divides fractions the wrong way round
C7 Interprets $ab$ as $a+b$
C8 Reverses the numbers in an operation
C9 Separates out fractions

ARITHMETIC ERRORS

A1 Error with multiplication tables or other single operation
A2 Transcription error or careless error
A3 Tries to multiply out brackets algebraically and makes a mistake
A4 Confusion about operations with negative numbers
A5 Confusion when fraction notation involved

CALCULATOR ERRORS

BC Brackets not dealt with correctly on calculator
NC Problems with negative numbers on calculator
IC Problems with indices on calculator
FC Cannot deal with fractions on a calculator
KC Key error
These categories were recorded for each pupil’s work and the full set of data analysis is given for each class in Appendix I.

7.2 Summary

Having categorised the misconceptions and errors that had been observed, I wanted to discover how the types of misconception might vary according to the classes and schools, so I recorded the number of pupils in the class who had demonstrated each misconception on at least one occasion. Since the classes contained different numbers of pupils I also recorded this as a percentage of the number of pupils in the class. These data summaries are included in chapter 9.
CHAPTER 8: THE PUPIL INTERVIEWS

8.0 Introduction

Interviews were carried out with pupils from two of the UK schools. In the first section of this chapter the pupils are described and in the second section the interviews are described in terms of the main themes that emerged from these discussions. The pupils’ responses to the questions they were asked about the work are compared and contrasted and triangulated with the themes that have emerged from the analysis of their written work and keystroke data. The third section of the chapter contains more detailed case studies of two of the pupils who were interviewed, and in the fourth section the emergent themes are summarised.

8.1 The pupils

The pupil interviews were all carried out with pupils from school B and school C in the UK. The interviews took place within two weeks of the pupils completing the worksheets. They were recorded and subsequently transcribed; the implementation of the interviews has been described in detail in chapter 6 in section 6.2.1.3. On the basis of my initial marking and analysis of the pupils’ work I had listed a number of pupils from each school who I thought would be interesting to talk to, due to their apparent strategies and wrong answers, so there was a purposive element to my sampling strategy, but from my list of possible interviewees the final sample of pupils was then essentially a convenience sample since it depended on
which pupils were available and consented to be interviewed. I also attempted to achieve a balance of male and female pupils. The time constraints limited the number of pupils I was able to interview; some interviews lasted longer than others, depending upon the way in which the conversation went, and how long it took for the pupils to answer the questions. I was anxious not to rush the pupils, and wanted them to feel that they had enough time to think about their answers as well as giving them the opportunity to ask me any questions about their work if they wanted to. Each interview lasted for between 10 minutes and 20 minutes.

The result was that five pupils were interviewed from each of school B and school C, and of the ten pupils five were male and five female. The pupils were:

**School B:**
Pupil 4 (female), Pupil 7 (female), Pupil 8 (female), Pupil 11 (male), Pupil 18 (male)

**School C:**
Pupil 16 (male), Pupil 23 (female) Pupil 25 (female) Pupil 27 (male) Pupil 30 (male)

When referring to these pupils in the selections which have been included from the interview transcripts, these pupils will be referred to using the letter of the school followed by the pupil number, so for example Pupil B4 refers to pupil 4 from school B. The researcher is referred to in the transcripts as CH.
8.2 Emergent Themes

From the analysis of the interview transcripts a number of common themes emerged:

- Rule-based ways of working
- Not recognising a fraction as a division
- Difficulties when working with negative numbers
- Difficulties when working with a calculator
- Algebraic notation helps to establish order

These themes will now be described and considered with respect to the pupils’ responses in the interviews.

8.2.1 Rule-based ways of working

The interviews contained a great deal of evidence of rule-based ways of working; sometimes this resulted in pupils obtaining the correct answers through the correct application of the rules they had learned, but in other cases there was evidence that a rule had been mis-remembered, partially remembered or mis-applied.

This rule-based approach is evident from the responses of a number of the pupils. For example, Pupil B11 was questioned about his answers to the first three questions on worksheet 1, shown in figure 8.1:
Figure 8.1 School B Pupil 11 worksheet 1 questions 1, 2 and 3

Discussing question 1, this pupil appeared to know that there was a rule but that he was uncertain about it:

_Pupil B11:_ I did the 2 plus the 3 which equals 5 then multiplied by 4 which equalled 20 then just added 5 which makes 25

_CH:_ And that is how you got 25? Can you see why that isn’t right?

_Pupil B11:_ Do you have to multiply the 3 and the 4 first then add the 2 and the 5?

_CH:_ Yes, you are right. So what should you have got?

_Pupil B11:_ 19

_CH:_ Good – that’s right. Do you remember doing some work on this?

_Pupil B11:_ Yes, now I remember you have to multiply first

Once this pupil had been reminded about the order of operations, he was able to explain the correct methods for the next question:
CH: If you look at question 2 now, you might be able to work out why you got it wrong and what the correct answer should have been.

Pupil B11: Well I was supposed to divide 10 by 2 which is 5 and then add 4 which is 9

CH: Yes that is right, so you know you should have done the division first?

Pupil B11: Yes I remember now

Pupil 25 in school C also worked from left to right in question 1 and obtained the answer 25. When asked about this she said “I should have timesed first” and then:

CH: How did you remember that?

Pupil C25: BIDMAS

Similarly Pupil 27 in School C also worked from left to right in question 1 and knew that there was a rule to remember, but seemed less clear:

CH: Can you think why that answer is not right?

Pupil C27: Is it the BODMAS thing?

He then continued to give the answer correctly as 19, explaining “I should have done the 3 times 4 then added the 2 and the 3”.

The need to “remember” rules was emphasised by a number of other pupils, for example Pupil 4 in School B, when asked about an incorrect answer,
replied “I think I forgot to use BIDMAS”. When questioned about this she demonstrated an understanding of what this meant:

CH : So if you had used BIDMAS what would you have done?

Pupil B4: Done the multiplication first

A number of pupils placed an emphasis on the need to remember, and seemed to feel that getting the questions correct was largely dependent on remembering things. Pupil B18 said “I had to remember about BIDMAS and I started remembering things”. This pupil clearly regarded “BIDMAS” as the mathematical rule that needed to be applied; when talking to her about the second question I asked “can you have a look and see what you think you should have done here?” She replied “BIDMAS”.

This pupil also remembered a rule for multiplying negative numbers together, and when asked about question 6 she said “I should have got 9 because negative times negative makes a positive” and she continued by explaining “I never really remember until I get it wrong”. Similarly, Pupil 27 in school C obtained an answer of -9 for (-3)^2 but when asked “what happens when you square a negative number?” he immediately replied “it turns into a positive number”. The nature of this response strongly suggests that he sees this as a rule, which he has now remembered, but which he failed to apply in the calculation on the worksheet.
The use of “BIDMAS” or “BODMAS” as a “rule” resulted in varying degrees of success with the calculations. Those pupils who mentioned it were asked about what it meant; most pupils knew what the letters stand for, although many were unsure about the I and the O, and even if they remembered the words “Index” or “Order” they were not always clear about what these words meant. The pupils’ replies often suggested an incomplete understanding of what BODMAS or BIDMAS actually referred to. For example, with Pupil 30 in School C:

CH: Do you remember what to do?
Pupil C30: Partly
CH: Which part do you remember?
Pupil C30: Is it brackets, indices, division, multiplication, addition and can’t remember

Pupil 8 from School B was quick to mention BIDMAS but struggled to make any sense of the letter “I” even when told that it stood for “Index”:

CH: Can you tell me what you were thinking about when you were doing these?
Pupil B8: I was thinking of BIDMAS
CH: What do you remember about that?
Pupil B8: Brackets, individual, divided, multiply, addition and subtraction
CH: I am interested in the “I”
Pupil B8:  *I think that was the wrong word.*

CH:  *Can you think of another word that it might be?*

Pupil B8:  *I don’t know what it is*

CH:  *If I tell you that it stands for Index, does that help?*

Pupil B8:  *Ah, that was the one.*

CH:  *Do you know what “Index” means?*

Pupil B8:  *(pause) does it mean look it through and put it in the correct order?*

CH:  *Can you look at the worksheet and see if you can find any number that you might describe as an index?*

Pupil B8:  *(after some time) is it 1? Right at the beginning of the number line?*

For this pupil the BIDMAS “rule” had enabled her to obtain the correct answer to many of the calculations but she clearly had no understanding of what the word Index is referring to. Another pupil who referred to BIDMAS, and was asked about the meaning of the letters, said “*I thought the I was for integer*”.

In some cases, despite remembering “BIDMAS”, the pupils still went on to carry out the calculations in an incorrect order. Pupil 18 in School B obtained incorrect answers to the first three questions on worksheet 1, performing the additions first in question 1 and working from left to right in questions 2 and 3. His work is shown in figure 8.2.
When asked about these questions in the interview, this pupil immediately mentioned BIDMAS and, talking about question 1, explained “I think what I should have done is used BIDMAS and timesed 3 and 4 together first and then added on the 2 and the 5, which would probably have given me a different answer to what I got. Yes, I was doing it the wrong way round”.

When asked about question 9, this pupil answered “I had to remember about BIDMAS and I started remembering things”.

Only two of the pupils who were interviewed did not mention BIDMAS or BODMAS at some point. One of these two, Pupil 23 in school C, when asked whether she had ever heard of BIDMAS or BODMAS said that she had not. When asked how she worked out the order in which to do the calculations, she replied “I think I did the hardest bit first”. This intuitive approach to using the hierarchy of the operations meant that she correctly calculated those questions which involved indices, but her “hardest first” approach did not enable her to establish an order for those calculations that only involved additions, multiplications and divisions. Her work is shown in figure 8.3 and it is interesting to see that when a division sign was used, in
question 2, she used a left to right approach, but when the division was
given in the form of a fraction, as in questions 3 and 9, she used the correct
order. She also correctly answered all four questions involving algebraic
expressions, as seen in figure 8.3. The other pupil who did not mention
BIDMAS or BODMAS was Pupil 11 in school B, and he also said that he
had never heard of it. When asked how he knew which order to use he said
“*I just know it*”.

![Table](image)

**Figure 8.3 School C Pupil 23 worksheet 1 questions 1, 2, 3, 8 and 9**

In summary, the majority of the pupils who were interviewed demonstrated
a view that mathematics is about remembering rules; some pupils were able
to apply the “rules” correctly, but there were many instances of pupils either
forgetting a “rule”, remembering it incorrectly, or mis-applying it even if
they appeared to have remembered it correctly. In the case of the mnemonic
BIDMAS or BODMAS the meaning of the letter I or O was not always understood.

8.2.2 Not recognising a fraction as a division

Another aspect of the work in which many of the pupils demonstrated an incorrect or incomplete understanding was the concept of a fraction as a division. Even if they did recognise that a fraction meant that a division was required, this was still not always carried out correctly; for example some pupils divided the denominator by the numerator. One pupil who talked about this was Pupil 8 in school C. In her work on question 3 of worksheet 1 she interpreted \( \frac{8}{4} \) as 0.5. When asked about this she explained “Yes I know what went wrong. I divided 4 by 8 but I forgot to divide the top number by the bottom number so it should have been 2”.

Some pupils, on seeing a fraction, tried to apply the “rules” that they had learnt for dealing with fractions. For example, Pupil 25 in School C was asked about her working in question 3, which is shown in figure 8.4. She said “I don’t think I saw that as a division. I saw it as a fraction. I tried to add the whole number and the fraction”. Other pupils seemed to demonstrate a similar mismatch between the concept of fractions and division; Pupil 25 in school C was describing his working for question 10 in worksheet 1 and explained “I had done that on the top and I had done the indices thing first which was 16 then added that to the top. Then I added that to the bottom and then I did the fraction”.

201
Pupil 16 in School C also dealt with question 3 as a fraction, leaving the answer as a mixed number without making the connection that \( \frac{8}{4} \) is equal to 2. When asked about this he appeared to be unaware that \( \frac{8}{4} \) might be regarded as a calculation, but seemed to regard it as a single object:

*CH:* Your answer to question 3 is correct but could you give your answer as a whole number? (Pause, pupil does not answer)

*What would you need to calculate first?*

*Pupil C16:* 8 over 4?

*CH:* Yes that’s right. What answer would that give you?

*Pupil C16:* 2?

Despite being so unsure about the connection between a fraction and a division in this context, this pupil correctly calculated the answer to question 3 on worksheet 2, as shown in figure 8.4, and his keystrokes revealed that he was aware that he needed to calculate the division first. When asked about this question he said that “it does not look like a fraction” and he was clear that “a fraction has to have whole numbers”
Figure 8.4 School C Pupil 25 and Pupil 16 worksheet 1 question 3 worksheet 2 question 3 and keystrokes

8.2.3 Difficulties when working with negative numbers

Another common theme that emerged from the pupil interviews was the fact that many of the pupils experienced difficulties when working with negative numbers. It seemed clear that the majority of the pupils relied upon remembering rules for working with negative numbers but frequently mis-remembered or mis-applied their rules. Question 6 had been included to investigate the pupils’ ability to square negative numbers, and out of the ten
pupils who were interviewed, only one gave the correct answer to this on worksheet 1, although most of them were able to correctly remember the “rule” when prompted. When working with a calculator on worksheet 2, six of the pupils obtained a correct answer for question 6. In the interviews, the pupils were asked “what happens when you square a negative number?” and the responses included “it turns into a positive number” (Pupil C27) “It makes a plus” (Pupil C16) “positive” (Pupil B4) and “you square negative numbers as a positive number” (Pupil B18). These responses all suggest that a rule has been learnt but not always remembered or correctly applied.

The response of Pupil C30 was interesting; his initial reply was “when you do a negative number by a negative number it does the opposite” suggesting that he knew he needed to apply a rule, but when this was pursued and he was asked what sort of number he would get, he said “a minus number?” indicating that he was very unsure of what his rule actually meant.

Pupil B7 appeared to have very little knowledge or understanding of negative numbers and although her written workings included $2 \times -3 = -6$, her responses during the interview revealed that she was in fact very confused about negative numbers, and also about the fact that the negative number was inside brackets in question 6. Her written work for question 6 on worksheet 1 is shown in figure 8.5:
In her interview responses this pupil appeared to ignore the negative sign completely:

**CH:** *Can you see what you should have done for question 6?*

**Pupil B7:** *I should have done 3 times 3 which is 9 but I did 2 times 3 instead*

**CH:** *You started off by doing 2 times -3. What should you have done?*

**Pupil B7:** *minus 3, but I didn’t get it, it was just one number in brackets*

**CH:** *Sometimes we put minus numbers in brackets like that. So what it is saying is minus 3 squared. What should you have got?*

**Pupil B7:** *I should have timesed the 3 first which is 9 then timesed it by 2 which is 18.*

**CH:** *Did the brackets put you off?*

**Pupil B7:** *Yes. What is the minus? Does it have a name?*

**CH:** *It is telling you where it is on the number line. (Draws horizontal number line) If the number is here to the left of...*
zero then we call it negative and put a negative, or minus
sign, in front of the number.

Pupil B7: But doesn’t that affect the multiplication?

It was clear at this stage that this pupil would first need to establish a clear understanding of the nature of negative numbers before moving on to operations with negative numbers, despite the fact that her written work appeared to suggest that she was able to multiply a negative number by a positive number.

The fact that six of the pupils obtained the correct answer to question 6 on worksheet 2 would suggest that they did have a basic understanding of negative numbers and that they were able to correctly identify the negative key on the calculator as opposed to using the subtract key. They had also used the brackets keys correctly in order to enter the calculation as it appeared on the worksheet, and obtained the correct positive answer. Their lack of success in the non-calculator worksheet appeared to be down to the fact that they had not remembered or applied their “rule” for squaring negative numbers.

8.2.4 Issues arising from working with a calculator

Some of the pupils who were interviewed demonstrated a significant lack of confidence with using a calculator. For example, Pupil 18 from School B, in discussing an incorrect answer in worksheet 2, said “I think I just put my calculations in wrong” and followed this up by explaining “I didn’t know
whether to just type it into the calculator or to do it by writing it out but
using the calculator as well” and then explained “I just don’t think I can
work with calculators”. Similarly Pupil 4 in School B had problems using
the calculator to square negative numbers. Her keystrokes for question 6 on
worksheet 2, shown in figure 8.6, revealed that she had used the subtract
key instead of the negative key, resulting in a syntax error message.
Although her keystrokes reveal that she had numerous attempts at this
question, she was unable to determine what she should key in, and
eventually left the question blank, as shown in figure 8.6.

![Figure 8.6 School B Pupil 4 work and keystrokes for question 6
worksheet 2](image_url)

8.2.5 Algebraic notation helps to establish order

Another theme that emerged from the pupil interviews was that although the
pupils did not all understand algebraic notation, those who were confident
with algebraic notation seemed to perform better on the algebraic questions
than the arithmetic questions, and seemed to find algebra “easier” than arithmetic, with the algebraic notation helping to establish order. This is exemplified by Pupil 23 from School C, whose work is shown in figure 8.7.

She used the incorrect order in questions 1 and 2 on worksheet 1, performing the additions first on question 1 and working from left to right in question 2, although in question 3 where fraction notation was used she used the correct order. Her answers to questions 11-14, also shown in figure 8.7, were all correct. This would suggest that the use of algebraic notation enabled her to deduce the correct order, even though she had not been able to do this with the purely arithmetic examples. When interviewed she said that she preferred the algebra because “I prefer harder questions”; although she perceived the algebra as harder, this did in fact enable her to perform the calculations correctly, although her work contained a substantial amount of incorrect notation.

![Worksheet Image]

Figure 8.7 School C Pupil 23 worksheet 1 questions 1, 2, 3, 11, 12, 13 and 14
8.3 Case Studies

The themes explored in this chapter will now be exemplified in two case studies of pupils in school B: Pupil 11 (male) and pupil 8 Female

8.3.1 Pupil 11 School B

This pupil provides an interesting example of how an understanding of algebraic structure seems to be much more effective than trying to learn arithmetic rules.

8.3.1.1 Analysis of worksheets and keystrokes

He scored 10 out of 14 in worksheet 1 and 12 out of 14 in worksheet 2. Analysis of his work, summarised in table 8.1, revealed that both incorrect answers in worksheet 2 were the result of careless mistakes. In the table a 1 denotes a correct answer, and the incorrect answers are coded according to the categories described in section 7.1.2.

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Worksheet 1</strong></td>
<td>M1</td>
<td>M1</td>
<td>M1</td>
<td>1</td>
<td>1</td>
<td>C5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td><strong>Worksheet 2</strong></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>A2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>M8</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Table 8.1 School B Pupil 11 Question analysis

In worksheet 1 he got the first three questions incorrect, clearly demonstrating a Left to Right misconception as can be seen in table 8.1.

This work also shows misuse of the equals sign.
He then completed the remainder of the questions correctly, getting all four of the algebraic questions correct as shown in figure 8.9:

It is interesting to notice that this pupil uses a very confusing notation. He omits the multiplication signs and does not use any alternative notation such as a dot to indicate multiplication in questions 11 and 12, but his working
indicates that he clearly understands that he needs to multiply the numbers together.

The analysis of this pupil’s keystrokes in worksheet 2 also indicates how his algebraic understanding provides the sense of structure. Figure 8.10 shows the pupil’s work for question 10 on the worksheet. This working only shows one intermediate step, which is the value of the denominator, but the keystrokes shown in figure 8.11 reveal the work that he carried out on his calculator:

![Figure 8.10](image)

**Figure 8.10 School B Pupil 11 worksheet 2 question 10**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.8*4.8</td>
<td>23.04</td>
</tr>
<tr>
<td>Ans+2.4</td>
<td>25.44</td>
</tr>
</tbody>
</table>

First the pupil squares the 4.8, but does not use the square command on the calculator. The 2.4 is then added to find the value of the numerator.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8+2.7</td>
<td>4.5</td>
</tr>
<tr>
<td>4.5*4.5</td>
<td>20.25</td>
</tr>
</tbody>
</table>

The value of the bracket in the denominator is then calculated and squared. This value (20.25) is then written on the worksheet (see Figure 7.3).

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.8*4.8</td>
<td>23.04</td>
</tr>
<tr>
<td>Ans+8.2</td>
<td>31.24</td>
</tr>
<tr>
<td>Ans/20.25</td>
<td>1.542716049</td>
</tr>
</tbody>
</table>

The denominator is then recalculated, but with 8.2 (from question 9) used instead of 2.4. Finally the division is carried out, but using the wrong value for the numerator.

![Figure 8.11](image)

**Figure 8.11 School B Pupil 11 worksheet 2 question 10 keystrokes**
These keystrokes demonstrate a clear sense of the structure of the expression; the pupil clearly understands the fraction as a division and evaluates the numerator and the denominator separately before dividing. Within the numerator he performs the squaring operation first and then adds 2.4. However, when he has correctly evaluated the denominator he again tries to evaluate the numerator, this time making a transcription error and adding 8.2 instead of 2.4. He then divides this incorrect value for the numerator by his correct value for the denominator, obtaining his final answer.

His understanding of the correct structure is demonstrated again in the next group of questions which can be seen in figure 8.12:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>$a + bc + d$</td>
<td>22.92</td>
</tr>
<tr>
<td>12</td>
<td>$ab + cd$</td>
<td>34.36</td>
</tr>
<tr>
<td>13</td>
<td>$a(b + c)$</td>
<td>18.72</td>
</tr>
<tr>
<td>14</td>
<td>$\frac{a(b + d)}{c + a^2}$</td>
<td>31.96</td>
</tr>
</tbody>
</table>

Figure 8.12 School B Pupil 11 worksheet 2 questions 11, 12, 13, 14
Figure 8.13  School B Pupil 11 worksheet 2 question 11 keystrokes

Figure 8.13 shows how, for question 11, he performs the multiplication first and then uses this answer in the addition to get the final correct answer.

Figure 8.14  School B Pupil 11 worksheet 2 question 12 keystrokes

Figure 8.14 shows that, for question 12, he calculates the two multiplications first then adds the answers together. This demonstrates his sound understanding of the need to perform the multiplications before the addition.

Figure 8.15  School B Pupil 11 worksheet 2 question 13 keystrokes
In figure 8.15 we see that the pupil clearly understands the role of the brackets in question 13 by evaluating the expression in the brackets first then utilising the “Ans” key to multiply this by the number outside the bracket.

<table>
<thead>
<tr>
<th>3.2+5.8</th>
<th>9.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ans*2.4</td>
<td>21.60</td>
</tr>
</tbody>
</table>

First the student calculates the value of the bracket on the numerator and then multiplies this by the value of $a$. The value for the numerator is then noted on the worksheet (see Figure 8.12).

<table>
<thead>
<tr>
<th>2.4*2.4</th>
<th>5.76</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ans+4.6</td>
<td>10.36</td>
</tr>
</tbody>
</table>

The squared term in the denominator is then calculated and the value of $c$ is added to obtain the correct value of the denominator. Again this number is recorded on the worksheet.

| 21.6+10.36 | 31.96 |

Finally he adds the numerator and denominator instead of dividing.

Figure 8.16 School B Pupil 11 worksheet 2 question 14 Keystrokes

The keystrokes for question 14 shown in figure 8.16 are an interesting illustration of this pupil’s clear understanding of structure, but also reveal a final mistake in which he adds the numerator and denominator instead of dividing. He has already demonstrated a clear understanding of a fraction as a division in question 10, so it is likely that this change of sign is a careless error rather than a misconception.
8.3.1.2 Interview with Pupil 11 from School B

Pupil B11 was later interviewed and asked about his responses to questions 1, 2 and 3 on worksheet 1, which he had got wrong because he had been working from left to right. He explained how he got the answer, and then asked “do you have to multiply the 3 and the 4 first and then add the 2 and the 5?”

**CH:** Yes that’s right. So what should the answer be?

**Pupil B11:** 19

**CH:** Yes that’s right. Can you remember doing some work on this?

**Pupil B11:** Yes

**CH:** How could you remember that? Is there a way that you could know what to do first?

**Pupil B11:** No, not really.

**CH:** What about question 2. Do you think you know why you got that one wrong?

**Pupil B11:** Well I was supposed to divide 10 by 2 which is 5 and then add 4 which is 9

**CH:** Yes that’s right, you should have done the division first. So can you look at question 3 and tell me what you should have done?

**Pupil B11:** I should have divided 8 by 4 then added it onto 8 which makes 10
CH: Good – so you have got the correct answer now. You got most of the other questions correct. What were you thinking when you did question 4?

Pupil B11: Because it was in brackets I remembered that you always work out the brackets first.

CH: How do you remember that?

Pupil B11: I just know it.

This pupil was very quickly able to work out what the correct answers should have been once he had been reminded about the convention for calculating multiplications and divisions first, although he was not very clear about how he could remember to do this.

He was then asked about the questions that involved algebra:

CH: The rest of the questions were all correct. Were you happy about doing the questions with algebra?

Pupil B11: Yes

CH: Were you using the same rules as before?

Pupil B11: Yes. When you see ab together like that it means multiplied

CH: So when you saw this question you knew you had to do the multiplying first?

Pupil B11: Yes.

CH: Do think the algebra questions were easier or harder?

Pupil B11: Easier, because you know what to do.
CH: Do you think that if I gave you this worksheet again you would get all the questions right?

Pupil B11: Yes

CH: Is this something you need to revise?

Pupil B11: No I don’t think so. It’s just common sense. Just think a bit harder.

8.3.1.3 Summary comments for Pupil 11 from School B

This pupil’s work, keystrokes and his responses in the interview, show that although he demonstrates a left-to-right misconception in the numerical calculations, he uses the correct order when the calculations are given as algebraic expressions. He is also able to use the calculator functions efficiently to apply the correct order.

His response to the question about whether the algebra questions were easier or harder was very revealing. He said that the algebra questions were “easier, because you know what to do”. This suggests that the algebraic notation has made it easier, indeed entirely logical, for him to see the order in which to perform the calculations. He no longer needed to rely upon memory, because the order was clear to him. He has also shown that although he originally forgot the correct order he was easily able to obtain the correct answers once he had been reminded about the convention, and simply needed a way of remembering what he described as “common
He did not rely upon a mnemonic, but he had not reached a point at which the “common sense” pervaded.

8.3.2 Pupil 8 School B

8.3.2.1 Analysis of worksheets and keystrokes

Pupil 8 scored 7 out of 14 on worksheet 1 and 10 out of 14 on worksheet 2.

The analysis of her answers is given in table 8.2:

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worksheet 1</td>
<td>1</td>
<td>1</td>
<td>C6</td>
<td>1</td>
<td>1</td>
<td>C5</td>
<td>1</td>
<td>1</td>
<td>M3</td>
<td>M3</td>
<td>C7</td>
<td>C7</td>
<td>1</td>
<td>X</td>
<td>7</td>
</tr>
<tr>
<td>Worksheet 2</td>
<td>C7</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>C7</td>
<td>C7</td>
<td>1</td>
<td>C7</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Table 8.2 School B Pupil 8 Question analysis

This pupil seemed to demonstrate in her work in questions 1 and 2 on worksheet 1 (figure 8.17) that she knew the convention of multiplication and division before addition and subtraction. However in question 3 she performed the division incorrectly by dividing the denominator by the numerator. She discussed this in the interview and said “yes I know what went wrong. I divided 4 by 8 but I forgot to divide the top number by the bottom number so it should have been 2” This would suggest that she had an appreciation of a fraction as a division but an incomplete consolidation of this concept.
Her workings for questions 9 and 10, shown in figure 8.18, demonstrate that she is not able to apply the convention of evaluating indices first. In both questions she consistently adds before squaring, in the denominator in question 9 and the numerator in question 10. When interviewed she revealed that although she could remember that BIDMAS was applicable here, she was unaware of the meaning of the “I” in BIDMAS and was unaware of the convention for evaluating powers first.

When faced with the algebraic calculations this pupil interpreted $ab$ as an addition, as demonstrated in questions 11 and 12 (figure 8.19) although when brackets were involved in question 13 she evaluated this correctly.
Unlike Pupil 11, the algebra did not make it easier for her to complete the calculations in the correct order, because her understanding of algebraic notation was incomplete.

When working with a calculator on worksheet 2 this pupil’s keystrokes (figure 8.21) reveal an interesting error in question 1. The pupil keys in the calculation correctly as it appears and obtains the correct answer of 23.9 on her calculator. However in the worksheet she writes the working $2.7 + 7.9 + 5.9$ as can be seen in figure 8.20 giving an answer of 16.5. She has clearly evaluated this mentally as this does not appear on her keystroke recording. She has internally changed the multiplication sign to an addition sign and this conviction is so strong that she ignores the answer given on her calculator.
In question 2 this pupil seems to know that she must do the division first, evaluating this and then adding 4.45. Her calculator work can be seen in Figure 8.22:

She takes the same approach for question 3, evaluating the division first and then adding, which can be seen in figure 8.23:
In question 4 she does not use the brackets on the calculator but evaluates the addition within the brackets before multiplying by 3.1 as shown in figure 8.24. This suggests that she is aware of the need to evaluate the calculation in the brackets first, but unable or unaware of how to use them on the calculator.

Also in question 5 she correctly uses the “square” key to square 3.2 before adding 2.6, demonstrating that she knows she must square first before adding, as shown in figure 8.25:

It is interesting to see that this pupil goes on to get both questions 9 and 10 correct in worksheet 2, as shown in figure 8.26, even though she consistently displayed a misconception in worksheet 1 by adding before squaring. However her keystrokes for question 9, shown in figure 8.27, indicate that she is not entirely sure, as she evaluated the denominator in
two different ways before finishing the calculation. The fact that she has been able to experiment on the calculator has enabled her to convince herself that her final answer is correct.

![Figure 8.26 School B Pupil 8 worksheet 2 questions 9, 10](image)

| 9 | 8.2 + 6.3  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3.7 + 2.8²</td>
</tr>
</tbody>
</table>
| 10 | 2.4 + 4.8²  
|   | (1.8 + 2.7)² |

First the pupil calculates the numerator and writes this value on her worksheet (see Figure 7.19).

She then calculates the denominator in two different ways, presumably as a check. One typing in the calculation as given and secondly reordering it. She writes the value of the numerator on her worksheet (see Figure 7.19).

Finally she carries out the division to obtain the correct answer.

![Figure 8.27 School B Pupil 8 worksheet 2 question 9 keystrokes](image)

By the time she gets to question 10 she seems happy that the calculator is producing the correct values for the numerator and denominator and she produces the answer in an efficient manner, as shown in Figure 8.28.
Figure 8.28 Pupil 8 School B worksheet 2 question 10 keystrokes

Figure 8.29 shows her answers to questions 11, 12, 13 and 14. In the first two of these questions she demonstrates the same misconception that she did previously and interprets $ab$ as $a + b$. When faced with brackets in question 13 she gets this correct, although she gets question 14 wrong. Her keystrokes, shown in Figure 8.30, reveal that this is only because she thinks that the numerator is the same as the expression in question 13, so she uses this value in the division of numerator by denominator.

![Figure 8.29 Pupil 8 School B worksheet 2 questions 11, 12, 13, 14](image)
8.3.2.2 Interview with Pupil 8 from School B

This pupil got the first two questions correct on worksheet 1, and, looking at the worksheet, I asked her about this.

CH: Can you tell me what you were thinking about when you were doing those questions?

Pupil B8: I was thinking of BIDMAS

CH: OK. What do you remember?

Pupil B8: Brackets, individual, divided, multiply, addition and subtraction

CH: That's nearly right – but what about the I?

Pupil B8: I don't know what it is

CH: Would it help if I tell you that it stands for Index?

Pupil B8: That was the one

CH: Can you tell me what Index means?

Pupil B8: Look it through and put it in the correct order

CH: Could you look at the worksheet and find any number that you might describe as an index number?
Pupil B8: 1, right at the beginning of the number line.

CH: OK – could you tell me what we mean by “power” in maths? Do you know what that means?

Pupil B8: No

It was clear from the interview that this pupil did not know what the words “index” or “power” meant, although she did know what squaring meant and she correctly evaluated all the “squared” terms both with and without a calculator, even when the value was negative.

I then asked her to look at question 11.

CH: Can you figure out why that answer is wrong?

Pupil B8: I forgot that when you put them together you times the 3 and the 4 together.

CH: Can you tell me what the answer should have been?

Pupil B8: (after some time thinking) 19

CH: Well done – that is correct. And what about question 12? Do you think you could do that now?

Pupil B8: (thinks) 26

CH: Good. That’s right. You can do them now. Do you think that if I gave you this worksheet again you would get them right?
8.3.2.3 Summary comments for Pupil 8 from School B

This pupil’s work and keystrokes demonstrated that she had a partial understanding of the application of the order of operations. She was able to use brackets correctly but was unable to perform calculations in the correct order when powers were involved. She also had an incomplete understanding of algebraic notation which led to incorrect calculations when algebra was involved. However her answers to the interview questions revealed that she was able to do the calculations once she had been reminded of the rules and the algebraic conventions, but she was not confident in her ability to remember them. She saw this as something that needed to be remembered rather than understanding it as logical.

8.4 Summary

To summarise, there is evidence that has been discussed in this chapter of five main themes that emerged from the interviews with the 10 pupils from the UK schools:

8.4.1 Rule-based ways of working

The pupils referred to mnemonics, BODMAS or BIDMAS, and placed an emphasis on the need to “remember” things in mathematics. They were
often able to answer the questions correctly during the interviews once they had been reminded of the “rules”.

8.4.2 Not recognising a fraction as a division

There was considerable evidence that the pupils had an incorrect or incomplete understanding of the concept of a fraction as a division; this theme had been observed in the work of many of the pupils in the study and was reinforced by some of the pupils who were interviewed. Even if they knew that a division was required there was sometimes confusion about which number to divide by which.

8.4.3 Difficulties when working with negative numbers

There was considerable evidence of the pupils struggling to work with negative numbers, with attempts to remember rules but an incomplete or mis-remembered application of the rules, such as “two positives makes a negative”. Squaring a negative number was not always seen as a case of multiplying two negative numbers together, but when pupils were guided to consider this in the interview they were frequently then able to calculate the correct answer to \((-3)^2\).

8.4.4 Difficulties when working with a calculator

Although the pupils had been encouraged to use calculators in their mathematics lessons there was evidence that some were not confident or
competent with using the calculator beyond the basic operations; calculations involving negative numbers, brackets and powers often revealed problems with the syntax and with the order. The pupils’ comments suggested that this did not seem limited specifically to the graphics calculators used in the research, but suggested a more general lack of confidence with using a calculator.

8.4.5 Algebraic notation helps to establish order

Although many pupils in the UK schools had demonstrated a lack of understanding of algebraic notation, the interview data revealed that those who were confident with algebraic notation found the algebraic questions easier than the arithmetic questions, and that substituting numbers into an algebraic expression helped them to “see” logically the order in which to evaluate the calculation. This strongly suggests that an understanding of algebraic notation is a far more powerful way to underpin the understanding and appreciation of order than the remembering of rules.
CHAPTER 9: INTERNATIONAL COMPARISONS

9.0 Introduction

The first section of this chapter describes the mathematics curriculum in each country included in the study, looking at the development of the curriculum and the philosophies underpinning it. Consideration is given to the effect that the curriculum documentation and guidance has on the resources used in schools and on the ways in which classroom teaching is influenced by this. The curricula are then compared and contrasted. In the second section the work of the pupils in each class is summarised, with an analysis of the misconceptions that were observed. This is related to the written work of the pupils and their keystrokes, where available, in order to describe any patterns observed in their ways of working. These findings are then compared and discussed. The discussion also refers to the findings of the interviews that were conducted with pupils in the UK schools and which were described fully in chapter 8. It must be remembered that this is a case study and therefore the comparisons are between the classes that were sampled and cannot be generalised. In the third section of this chapter, the interviews with the teachers from each country are described and discussed in order to understand the teaching methods and resources that they use, and to determine the extent to which these are influenced by the curriculum guidance and the philosophy behind it, in the countries in which they teach. The findings from the teacher interviews are then compared.
9.1 The mathematics curricula and teaching approaches in the other countries involved in the study

9.1.1 Japan

9.1.1.1 The school system in Japan

The following information has been taken from the “Mathematics Program in Japan” (Japan Society of Mathematical Education, 2000)

Primary education in Japan starts at the age of six and continues until the age of eleven. Lower secondary education is from the age of twelve through to fifteen and both primary and lower secondary education are compulsory.

All public compulsory education is co-educational.

There are various types of school for each educational level:

| Pre-primary: | Kindergarten (Youchien) | Ages 3 – 5 years |
| Primary:     | Elementary school (Sho-gakko) | Ages 6 – 11 years |
| Secondary:   | Lower Secondary School (Chu-gakko) | Ages 12 – 15 years |
|              | Secondary education school (Chuto-kyouiku-gakko) | Ages 12 – 17+ years |
|              | Upper secondary school (Koto-gakko) | Ages 15 – 17 years |
|              | College of Technology (Koto-senmon-gakko) | Ages 15 – 19 years |
|              | Specialised Training College (Senshu-gakko) | Ages 15+ years |
| Tertiary:    | Junior College (Tanki-daigaku) | Ages 18 – 19 or 20 years |
|              | University (Daigaku) | Ages 18 – 21+ years |
| Special Education: | Special Education School (Tokushu-kyouiku-gakko) | Ages 3 – 17+ years |
Pupils are grouped in mixed-ability classes for all lessons, including mathematics.

National standards for education in kindergartens, elementary schools, lower secondary schools, upper secondary schools and special education schools are prescribed in the Courses of Study and within these are set out the objectives and content for each subject. These are set by the Ministry of Education, Science, Sports and Culture, in which the Central Council is advised by the Curriculum Council. Along with the Courses of Study, the Ministry also produces instructional materials in which teaching methods are explained. Although commercial publishers can produce textbooks freely, the textbooks must be authorised by the Ministry.

9.1.1.2 Main Features of Japanese Mathematics Curricula

The national standards for Mathematics for elementary school (Sansuu in Japanese: often translated as Arithmetic in English), Mathematics for lower secondary schools (Suugaku) and Mathematics for upper secondary schools (Suugaku) are prescribed in the Courses of Study. (Japan Society of Mathematical Education, 2000)

“Zest for living” is emphasised in all subjects, and “spontaneous problem solving” is emphasised in mathematics. In all levels of mathematics, “mathematical activities” are introduced, and “enjoyment of mathematics” is described in elementary and lower secondary levels, and “fostering
creativity” is described in upper secondary levels. (Japan Society of Mathematical Education, 2000)

The time allocation and number of credits for each subject are prescribed in the Regulations for Enforcement of School Education Law and the Courses of Study. (Japan Society of Mathematical Education, 2000)

For elementary schools and lower secondary schools, mathematics is a required subject in each grade. The standard number of school weeks per year, the standard total number of classes per year, the standard number of class periods per year and week for mathematics and the standard time of a class period are all prescribed. In lower secondary schools mathematics can be taken as one of several optional subjects at each grade, in addition to mathematics as a compulsory subject.

For each level of schooling there is an overall objective, with three specific objectives for each grade. The content is then defined for each grade, grouped into four categories. For example, for elementary school the overall objectives are defined as:

‘Through mathematical activities concerning numbers, quantities and geometrical figures, children should get basic knowledge and skills, should get abilities to think logically and to think with good perspectives, should notice the pleasure of doing activities and the value of mathematical methods, and should get attitudes to make use of mathematics in daily life situations’.

(Japan Society of Mathematical Education, 2000, p.7)
The objectives for grade 1 (6 – 7 years) are then given:

(1) ‘Through activities using concrete materials and so on, children should get a good sense of numbers. Children should understand the meaning of numbers and how to represent them, should understand the meaning of addition and subtraction, and should make use of them’.

(2) ‘Through activities using concrete materials and so on, children should enrich their basic experiences to understand the meaning of quantities and measurements and should have good sense of quantitie’.

(3) ‘Through activities using concrete materials and so on, children should enrich their basic experiences to understand the meaning of geometrical figures and should have a good sense of geometry’

(Japan Society of Mathematical Education, 2000, p.7)

The content for each grade is then given within the following categories:

A. Numbers and Calculations

B. Quantities and Measurements

C. Geometrical Figures

At grade 3, a fourth category is introduced,

D. Mathematical Relations

In each case there are between two and five objectives in each category. An interesting example of an objective within section A for grade 1 (6 – 7 years) is.

A (2) (a) ‘To know the situations where addition and subtraction is used, and to express them in algebraic expressions, and to read these expressions’

(Japan Society of Mathematical Education, 2000, p.8)
So the use of algebra is introduced as early as grade 1, within the content for
“Numbers and Calculations”. This continues throughout elementary school,
as more operations are introduced, so that for example in grade 2 (7 – 8
years) the pupils are expected to

A (3) (a) To know the situations where multiplication is used, to
represent them by algebraic expressions, and to read them.
(Japan Society of Mathematical Education, 2000, p.9)

And by grade 4 (9 – 10 years) the expectation is that

D (2) ‘Children should express quantitative relations clearly in a
formula, and should be able to read them,
(a) To understand expressions that mix together brackets and the
four fundamental rules of arithmetic, and calculating them
correctly’

(b) ‘To understand formulas and to use them’
(Japan Society of Mathematical Education, 2000, p.14)

Within the Japanese curriculum there is an intense coverage of both algebra
and geometry from elementary school onwards, and throughout the entire
curriculum, in secondary as well as elementary school, there is an emphasis
on the use of concrete materials, ‘hands-on learning’, and the opportunity
to apply mathematics to real-life situations. There is also frequent reference
to the promotion of enjoyment of mathematical activities, and the
cultivation of positive and insightful views of mathematics as a human
activity.
9.1.1.3 The role of text books

Text books are an integral part of the Japanese mathematics curriculum. The School Education Law in Japan states that the use of textbooks is compulsory (Ministry of Education, Culture, Sports, Science and Technology [MEXT], 2010) and the textbooks must be centrally approved. There are six major mathematics textbooks available in Japan, and they are all very similar, written as a series with one book per grade. They are written based on lesson study findings; textbook company executives attend lesson study open houses and consult many published research findings in order to establish what to incorporate in their books. The most commonly used series of textbooks is “Mathematics” (pub. Tokyo Shoseki) which covers all grades in elementary and secondary school. They are thin, lightweight books with colourful cartoon illustrations, but the illustrations have a purpose and are there to give helpful hints and mathematical ideas. The philosophy is to make them child friendly, and so instead of a statement like ‘Solve the problem below’ the Japanese book will use ‘Let’s solve the problem below’ reflecting the philosophy of collaborative pupil-centred problem solving. This is described by Takahashi (2006) who explains that a unit in a Japanese textbook is a ‘series of problems and activities’ (p. 6) and he points to the cohesiveness of these carefully selected activities. The textbooks are inexpensive and they are given to the pupils to keep, so that they may write notes in them. The textbooks are fairly small and, in line
with the national curriculum, embody the Japanese philosophy of teaching a few important topics per year in depth.

A particular characteristic of the textbooks for elementary grades is that they are designed for teachers who do not have any specialist subject knowledge, and they include teachers’ guides with suggestions for how to teach each lesson, based upon research lessons. They even contain examples of questions that the pupils may ask, with suggestions for how to deal with these questions and examples of typical pupil responses and errors.

9.1.1.4 The approach to teaching and learning algebra

The course of study introduces the concept of algebra in elementary school from grade 1, and textbooks promote algebraic thinking with notes in the accompanying teacher’s manual to explain how to use the activities in the textbook in order to introduce algebraic ideas, and the problems are set out to encourage children to look for patterns. For example, in the grade 2 textbook (Gakko Tosho, 2006, p.34) in the section on multiplication tables, the tables are described in terms of variables, represented by empty boxes; the pupils are then given a multiplication square and prompted to look for “secrets” in the square in the form of patterns and relationships. Some examples of this are shown in figure 9.1:
By grade 5 the pupils are using formulas involving words as variables and this is linked to the work in other topics, such as geometry, as illustrated in the example in figure 9.2:
By the beginning of secondary school pupils are expected to work with formal algebra, using letters as variables, working with algebraic expressions and solving equations, and making use of algebra to describe situations throughout the mathematics curriculum.

### 9.1.1.5 The role of calculators and ICT

Calculators are not used until grade 4 of elementary school (9 – 10 years) and the guidelines in the course of study are given as follows:

> ‘When handling calculations involving large numbers or complex calculations in the problem-solving process, soroban (abacus) and held-held calculators may be used from Grade 4 on. In this, appropriate opportunities should be established to estimate the results of calculations and check the results. In lower grades, teaching materials such as abacuses and physical objects may be used with care to deepen understanding of the meaning of numbers and calculations’

> ‘Consideration should be given to the effective use of computers, enriching children’s sense of numbers, quantities and geometrical figures and improving their ability to use tables and graphs for expression’

(Japan Society of Mathematical Education, 2000, p.20)
Whilst in the guidelines for lower secondary school it is stated that

‘In instruction for each grade, soroban (abacus) calculators, computers and information communication networks may be used when necessary with care given to improving learning results. In particular, consideration should be given in this regard when carrying out instruction of the contents involving numerical calculations or instruction involving observation, manipulation and experimentation’

(Japan Society of Mathematical Education, 2000, p.27)

By upper secondary school the expectation is that

‘The teacher should make active use of computers and information communication networks etc for understanding mathematical-scientific phenomena and for recognizing rules through computational trials and for the collecting, searching, measuring and controlling of information in the process of observations and experiments, for simulation and for the calculation and management of results’.

(Japan Society of Mathematical Education, 2000, p.43)

Thus within the course of study there is an emphasis on careful and judicious use of calculators and ICT, with computers being regarded mainly as a teaching aid, and calculators only used when necessary rather than being seen as a tool for the pupils to use readily. In conversations with Japanese teachers (details of these conversations are given in section 9.3.2) it would appear that in many Japanese secondary mathematics lessons calculators are not used at all.
9.1.1.6 Teaching methods and staff development

The description of Japanese mathematics teaching was given by Miyakawa (2006) as ‘structured problem-solving’ and in the TIMMS Video Study carried out in 1997 (Stigler and Hiebert, 1997) typical mathematics lessons in lower secondary school in Japan were seen to have the following characteristics:

- Teacher poses a complex thought-provoking problem
- Students struggle with the problem
- Various students present ideas or solutions to the class
- Class discusses the various solution methods
- Teacher summarises the class’ conclusions
- Students practise similar problems

(Stigler and Hiebert, 1997)

these characteristics are contrasted with those observed in many mathematics lessons with the same age group in the U.S., which had a much greater emphasis on skill acquisition:

- Teacher instructs students in a concept or skill
- Teacher solves example problems
- Students practise on their own while the teacher assists individual students

(Stigler and Hiebert, 1997)
These findings were summarised by Jones (1997) in the table given in table 9.1:

<table>
<thead>
<tr>
<th><strong>Typical Year 9 Japanese mathematics lesson</strong></th>
<th><strong>Typical Year 9 U.S. mathematics lesson</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>is with a mixed-ability class</td>
<td>is with a class set by ability</td>
</tr>
<tr>
<td>begins with a complex problem</td>
<td>relies on a textbook</td>
</tr>
<tr>
<td>focuses on developing mathematical thinking</td>
<td>focuses on developing a mathematical skill</td>
</tr>
<tr>
<td>devotes most time to mathematical reasoning and understanding</td>
<td>devotes most available time to practising routine procedures</td>
</tr>
<tr>
<td>makes explicit links between concepts</td>
<td>features isolated tasks</td>
</tr>
</tbody>
</table>

Table 9.1 Comparison of mathematics lessons in Japan and the U.S. (Jones, 1997)

Jones (1997) also identified the fact that Japanese mathematics teachers have a substantially lighter teaching load than mathematics teachers in the US, with more time to prepare lessons, better working conditions and more time to develop professionally. He also suggests that mathematics classrooms in the UK are likely to be much closer in style to US classrooms than Japanese ones. Stigler and Hiebert (1999) went on to compare mathematics teaching in three countries, Japan, America and Germany, and identified the Japanese practice of ‘lesson study’ as being a very effective form of professional development, and point to the fact that in Japan it is the teachers themselves who have primary responsibility for the improvement of classroom practice. *Kounaikenshuu* is the continuous process of school-based professional development, in groups, that all Japanese teachers have to participate in as part of their job. These groups, according to Stigler and Hiebert (1999), perform a dual role: ‘not only do they provide a context in

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which teachers are mentored and trained; they also provide a laboratory for the development and testing of new teaching techniques.' (p. 110) Thus, the majority of educational research in Japan is done by teachers and not by university researchers, which is the case in many other countries including the United States and the United Kingdom.

9.1.2 The Netherlands

9.1.2.1 The school system in the Netherlands

The Dutch education system consists of primary, secondary and higher education, and education is compulsory from the age of five to sixteen, although many children start primary school at the age of four.

Primary school includes: Kindergarten 1 Age 4-5

Kindergarten 2 Age 5-6

Primary Grades 3 – 8 Age 6-12

Secondary education, which covers secondary grades 1 – 6 (age 12 – 18), is split into three different types:

VMBO is pre-vocational education, preparing pupils for vocational education (MBO).

HAVO is general secondary education, which prepares pupils for higher vocational education (HBO).

VWO is general secondary education which prepares pupils for university.
All three types of secondary school begin with a foundation phase of two years (grades 1 and 2)

9.1.2.2 Recent curriculum development and Realistic Mathematics Education

In the Netherlands, current mathematics education policy has been inspired by the work of Hans Freudenthal and his colleagues who developed a philosophy known as Realistic Mathematics Education (RME). This characterises mathematics as being an activity rather than a set of rules, and therefore seen as a creative, sociable and organising activity, in which unknown relationships and structures are to be discovered. In 1971 The Dutch Ministry of Education established an institute for the development of mathematics education, and this later became known as the Freudenthal Institute. At the same time, a reform of the primary curriculum began and ten years later there was a major reform of the secondary curriculum. The impact on the mathematics curriculum was the introduction of two strands of mathematics, named “Mathematics A” and Mathematics B”. Mathematics A was introduced as a subject that prepared students for the social sciences, and reduced abstract mathematics to a minimum, focusing more on statistics and discrete mathematics, throughout secondary schooling. Mathematics B had an emphasis on abstract functions, vectors etc. and was targeted at future engineers. All schools had to offer both strands. In 1993 the next curriculum reform established a new common core curriculum at junior secondary level (grades 7/8) and for the final years of
the lower ability track (grades 9/10 of the VMBO, the vocational track which serves the lower 60% of Dutch students). According to Vos (2009) this was developed ‘not as a watering- down of curricula at a higher level, but it intended to give all future citizens basic, mathematical abilities’ (p. 612). This was developed from the ‘Wiskunde 12 – 16’ project (Mathematics 12 – 16) by a team of teachers, trainers and RME researchers from the Freudenthal Institute. The emphasis was placed on the fact that the usefulness of mathematics should be experienced during the learning process and should not just be justified by its role in future adult life.

Moreover, inherent to RME is the concept that it can never be considered as a fixed or finished theory of mathematics education. RME is seen as ‘work in progress’ (Van den Heuvel-Panhuizen, 1998). The present form of RME has developed from Freudenthal’s view of mathematics (Freudenthal, 1973) as something which should be connected to reality, stay close to children’s experience and be relevant to society. Mathematics lessons should give children the opportunity to be guided towards re-inventing mathematics by doing it. The new curriculum emphasised data modelling and interpretation, through graphs, tables and diagrams, the use of ICT, and topics considered relevant to daily life. Mathematical concepts were introduced through a range of examples of mathematical models, and open problems were introduced. National assessment was developed in line with this approach, with questions based on a daily-life situation, usually with authentic photographs. The examples in Appendix D were developed for 14-year old students by the W12-16 curriculum designers. The problems in the context-
based Dutch mathematics curriculum share a number of similarities with the mathematics PISA-tests (Decker et al., 2006): a title which indicates the theme, the clarifying text, the photographs and the provided mathematical model.

The theoretical model for this approach was described by Treffers (1987) in the form of two types of ‘mathematization’. He distinguished ‘horizontal mathematization’ in which the students come up with the mathematical tools to organise and solve a problem set in a real-life situation, and ‘vertical mathematization’ as the process of reorganisation within the mathematics itself, for example by finding connections between concepts and strategies and applying these to the problem. In other words, horizontal mathematization involves going from the world of life to the world of symbols, whereas vertical mathematization involves working within the world of symbols.

The reason why the word “Realistic” is used in the context of mathematics education in the Netherlands is not simply due to the way in which mathematics is connected to the real world, but because the mathematics is related to something that the students can imagine. The Dutch translation of ‘to imagine’ is ‘zich realiseren’ and the emphasis is on making it real in your mind. Thus the context of problems given to students does not necessarily come from the real world, but needs to be something they can imagine. ‘The fantasy world of fairy tales and even the formal world of
mathematics can provide suitable contexts for a problem, as long as they are real in the student’s mind’ (Van den Heuvel-Panhuizen, 2000, p. 4).

RME reflects a certain view of what mathematics is, how students learn mathematics and how mathematics should be taught. This is characterised by Van den Heuvel-Panhuizen (2000) in terms of six principles:

- **Activity principle**  This can be described as “learning by doing”. Instead of students being receivers of ready-made mathematics, they are treated as active participants in the educational process, in which they develop mathematical tools and insights by themselves.

- **Reality principle**

  In RME this is not just recognisable at the end of a learning process, but reality is seen as a source for learning mathematics. Even in the early years of RME it was emphasised that if children learn mathematics in an isolated fashion they will quickly forget it and be unable to apply it. Thus mathematics should be learned whilst working on context problems, in which the students can develop their mathematical tools and understanding.

- **Level principle**

  As students learn mathematics they pass through various levels of understanding, and the condition for arriving at the next level is the ability to reflect upon the activities conducted. This reflection can
be elicited by interaction. The strength of this is that it guides growth in mathematical understanding and gives the curriculum a longitudinal coherence. This long-term perspective is a characteristic of RME. A powerful example of this longitudinal model is the number line. It begins in first grade as (a) a beaded necklace on which students can practise counting activities, and in higher grades it then successively becomes (b) an empty number line for supporting additions and subtractions, (c) a double number line for supporting work on ratios, and (d) a fraction/percentage bar for supporting work with fractions and percentages. These ideas are illustrated in figure 9.3.

Figure 9.3 Different ways a number line can appear (Van den Heuvel-Panhuizen, 2000, p. 7)
• **Intertwinement principle**

Another characteristic of RME is that mathematics in school should not be split into distinctive learning strands. Solving rich context problems often means that you have to apply a broad range of mathematical tools and understandings. The strength of this principle is that it lends coherence to the curriculum. In the number strand, for example, topics like number sense, mental arithmetic, estimation and algorithms are closely related.

• **Interaction principle**

Within RME, the learning of mathematics is regarded as a social activity. Pupils should be given opportunities to share their strategies and inventions with each other. By listening to others and discussing their findings the students can find ideas for improving their own strategies, and since this interaction may invoke reflection, this enables pupils to reach a higher level of understanding. Whilst whole-class teaching plays an important role in the RME approach, this does not mean that the whole class proceeds collectively. On the contrary, the pupils are seen as individuals, each following their own learning path. This is done by providing the pupils with problems which can be solved at different levels of understanding.
• **Guidance Principle**

One of Freudenthal’s key principles for mathematics education is that pupils should be given a ‘guided’ opportunity to ‘re-invent’ mathematics. This means that both the curriculum and the teachers play a crucial role in how the pupils learn. They steer the learning process, not in a fixed way by demonstrating what the pupils need to learn, but by providing the pupils with a learning environment in which they have room to construct their own mathematical insights and tools.

9.1.2.3 The role of textbooks

In many countries the use of textbooks is gradually being discouraged, and it is noted by Van den Heuvel-Panhuizen (2000) that ‘many reform movements are aimed at getting rid of textbooks’ (p. 10). However in the Netherlands the improvement of mathematics education is seen largely to depend on new textbooks as the most important tool in guiding the teacher’s teaching, in terms of both content and teaching methods. However Dutch teachers are encouraged to be ‘fairly free with their teaching’ (Van den Heuvel-Panhuizen, 2000, p. 10) and schools are free to decide which textbook they use. An important aid in the development of Dutch textbooks is a series of publications called the ‘Proeve’ or ‘National Program’ (Treffers, De Moor and Feijis, 1989) which was introduced in 1989 and has been developed since then, providing support for textbook authors, teacher
educators and school advisors. An important element of this program is that it has been developed with significant contributions from researchers in mathematics education.

9.1.2.4 The approach to teaching and learning algebra

In primary school, the ultimate goal of RME arithmetic is that children are able to make sense of numbers and numerical operations. This means that they should be able to decide for themselves what calculation is appropriate for solving a particular problem, and they should know when mental calculation is adequate, when to do a written calculation and when to use a calculator. The primary curriculum, published in 1993 by the Ministry of Education, Culture and Science (OCenW) contained 23 Core Goals for mathematics, split into six domains. In 2004 this list was revised and shortened to just 10 Core Goals grouped into three domains. This reflected recent developments in mathematics education in the Netherlands, and the main changes were:

- More attention paid to mental arithmetic and estimation
- Formal operations with fractions were replaced with fractions in context situations
- The insightful use of a calculator was included as a Core Goal

The full list of core goals is as follows:
**General mathematical insights and abilities**

Learn to use mathematical language

Learn to solve practical and formal mathematical problems and express their reasonings in a clear way

Learn to support and judge solution strategies

**Numbers and operations**

Learn to understand in a general way the structure and relationships of whole numbers, decimal numbers, fractions, percentages and ratios and are able to calculate with them in practical situations

Learn to carry out mentally and quickly the basic operations with whole numbers at least up to 100; know the additions and subtractions up to 20 and the multiplication tables by heart

Learn to estimate and calculate by approximation

Learn to add, subtract, multiply and divide in a clear way

Learn to write additions, subtraction, multiplications and divisions in standardized ways

Learn to use the calculator with insight

**Measurement and geometry**

Learn to solve simple geometric problems

Learn to measure and to calculate with measurement units and measures such as appear in time, money, perimeter, area, volume, weight, speed and temperature

Throughout the curriculum the aim is that pupils will acquire mathematical concepts and skills by representing and analysing real and realistic situations. In his description of the RME curriculum, Romberg (2001, p.5) describes how in primary school this involves the ‘progressive formalization of the mathematics’ in which the pupils ‘first approach problems and acquire algebraic skills in an informal way’. He explains that they ‘use
words and pictures and/or diagrams of their own invention to describe mathematical situations, organize their own knowledge and work, solve problems and explain their strategies.’ He then describes how pupils move on and ‘gradually begin to use symbols to describe situations’, ’ (Romberg, 2001, p 5). and emphasises that they can ‘devise their own symbols or learn certain non-conventional notation (e.g arrow language)’ (Romberg, 2001, p 5). This builds the foundation for moving on to the use of formal algebraic notation in secondary school.

9.1.2.5 The role of calculators and ICT

The use of calculators in learning mathematics in the Netherlands is strongly encouraged from primary school onwards, with one of the ten Core Goals for primary mathematics being ‘Learn to use the calculator with insight’.

Scientific calculators are used in mathematics lessons in secondary school, and pupils are encouraged to determine when a calculation could be done using mental or written methods, or whether the use of a calculator is the most appropriate method. The varied use of ICT is also encouraged within the secondary mathematics curriculum, and researchers in the Freudenthal Institute have developed a suite of applets called Wisweb to support mathematics teaching and learning in the secondary school, and these are widely used. This includes a Digital Learning Environment (DLE) in which pupils can set up their own account and save their work to continue either at home or at school. This is used in most secondary schools.
9.1.3 The USA (New York State)

In the USA, each state has its own curriculum and methods of assessment, and so since the school in this study was in New York State, this section will focus on the mathematics curriculum in this state.

9.1.3.1 The school system in New York State

The education system in New York State consists of elementary school, middle school and high school. Children start elementary school at the age of 5, then move to middle school then high school, the grades and age groups being as shown in table 9.2:

<table>
<thead>
<tr>
<th>Grade</th>
<th>Age</th>
</tr>
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<tbody>
<tr>
<td>K</td>
<td>5 - 6</td>
</tr>
<tr>
<td>1</td>
<td>6 - 7</td>
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<td>2</td>
<td>7 - 8</td>
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<td>3</td>
<td>8 – 9</td>
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<td>4</td>
<td>9 - 10</td>
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<table>
<thead>
<tr>
<th>Grade</th>
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<tr>
<td>5</td>
<td>10 - 11</td>
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<table>
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<tr>
<th>Grade</th>
<th>Age</th>
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<td>6</td>
<td>11- 12</td>
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<td>7</td>
<td>12 - 13</td>
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<td>8</td>
<td>13 - 14</td>
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<table>
<thead>
<tr>
<th>Grade</th>
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<tbody>
<tr>
<td>9</td>
<td>14 - 15</td>
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<tr>
<td>10</td>
<td>15 - 16</td>
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<tr>
<td>11</td>
<td>16 – 17</td>
</tr>
<tr>
<td>12</td>
<td>17 - 18</td>
</tr>
</tbody>
</table>

Table 9.2 Grades and Ages in schools in New York State
9.1.3.2 The Mathematics Curriculum

In January 2005 the New York State Department of Education Mathematics Department released ‘performance indicators pre-K-12’, which are the standards to be implemented starting in grade 3 (in elementary school) until grade 8 (end of middle school). These are set out by the State Education Department in the ‘Mathematics Core Curriculum MST Standard 3’ in 2005. In the introduction to this document it is stated that

‘Every teacher of mathematics, whether at the elementary, middle or high school level, has an individual goal to provide students with the knowledge and understanding of the mathematics necessary to function in a world very dependent upon the application of mathematics. Instructionally, this goal translates into three components’:

Conceptual understanding

Procedural fluency

Problem solving

(New York State Education Department, 2005, p. 1)

It is stressed that these components are integrally related and need to be taught simultaneously, and that they should be a component of every mathematics lesson.

The standards are arranged in 10 strands, consisting of five Process Strands and five Content Strands:
Process Strands:

- Problem Solving Strand
- Reasoning and Proof Strand
- Communication Strand
- Connections Strand
- Representation Strand

Content Strands:

- Number Sense and Operations Strand
- Algebra Strand
- Geometry Strand
- Measurement Strand
- Statistics and Probability Strand
Each strand contains a list of specific standards which the pupils are expected to achieve in this grade. In grade 7, for example, there are 120 standards altogether in the 10 strands. The way that these strands overlap is shown in figure 9.4:

![Figure 9.4 The Process and Content Strands in the New York State Core Curriculum](image)

**Figure 9.4 The Process and Content Strands in the New York State Core Curriculum** (New York State Education Department, 2005)

Once a year, in May, all pupils in grades 3 to 8 must take a ‘*state mathematics assessment*’. The questions are multiple-choice and each question in this assessment is linked to one of the standards. An example of a recent assessment for grade 7 is given in appendix E along with the corresponding mark scheme which contains the standards that are being assessed in each question. As an example that is particularly relevant to this
research, one of the standards in the Number and Operations Strand for grade 7 is:

7.N.11  Simplify expressions using order of operations  

*Note: Expressions may include absolute value and/or integral exponents greater than 0*  

(New York State Learning Standards for Mathematics, 2005, p.75)

The question that assessed this standard in the 2009 assessment was:

Simplify the expression below:

\[ |7 - 3^2| + 4 \]

A   2    B  3    C  5    D  6

(New York State Education Department, 2009, p. 20)

A recent study (Moyer et al, 2011) into the impact of standards-based curricula in middle school mathematics in the United States investigated nearly 600 algebra-related lessons over a period of three years, approximately half of which were in schools which had adopted a Standards-based mathematics curriculum. The authors suggested that a standards-based curriculum had a significant effect on the mathematics teaching that took place, with increased emphasis on conceptual understanding when compared to what they described as a ‘traditional’ curriculum. They also found that teachers following a standards-based curriculum were more likely to follow the lessons exactly as laid out in the set textbook, and that they ‘were more likely to implement the curriculum as intended by the textbook authors’ (p. 97). However, Schmidt et al (2006, p. 259)
3) criticised the U.S. curriculum content for being highly repetitive, with the standards providing ‘long laundry lists of seemingly unrelated, separate topics’.

9.1.3.2 The role of textbooks

In the US textbooks are regarded as an important resource to be used alongside a range of other teaching and learning resources, in elementary schools, middle schools and high schools. There are numerous textbooks available and usually a school district will agree on a particular series of textbooks which will be used by all the schools in that district. In the school district in which I conducted my study, all schools use the ‘Maths Connect’ programme (Macmillan, 2009) in grade 5 (last year of elementary school) and the ‘Middle School Math’ series (Holt, Rinehart, Winston, 2002) to deliver the district curriculum. An example of exercises from the ‘Middle School Math’ series is shown in Appendix E. The mathematics curriculum for each grade is set out in detail, lesson by lesson, with each lesson linked to a particular standard and with reference to the appropriate pages of the set textbook. Textbooks tend to be very long, and in the report of the National Mathematics Advisory Panel ‘Foundations for Success’ (U.S. Department of Education, 2008) it was stated that

‘U.S. mathematics textbooks are extremely long—often 700–1,000 pages. Excessive length makes books more expensive and can contribute to a lack of coherence’.

9.1.3.4 The approach to teaching and learning algebra

The National Council of Teachers of Mathematics (NCTM) set out its position on the teaching of algebra in 2008 in a paper entitled ‘Algebra: What, When and for Whom?’ In answer to the question “What is algebra, when should it be taught, and to whom?” it starts by stating:

‘Algebra is a way of thinking and a set of concepts and skills that enable students to generalize, model, and analyze mathematical situations. Algebra provides a systematic way to investigate relationships, helping to describe, organize, and understand the world. Although learning to use algebra makes students powerful problem solvers, these important concepts and skills take time to develop. Its development begins early and should be a focus of mathematics instruction from pre-K through grade 12. Knowing algebra opens doors and expands opportunities, instilling a broad range of mathematical ideas that are useful in many professions and careers. All students should have access to algebra and support for learning it’.

(NCTM, 2008, p.1)

The paper acknowledges that the development of algebraic concepts and skills happens over time, and that children should encounter algebraic ideas from elementary school. However, in terms of more formal algebra the recommended approach is ‘algebra when ready’ which recommends that:

‘Only when students exhibit demonstrable success with prerequisite skills—not at a prescribed grade level—should they focus explicitly and extensively on algebra, whether in a course titled Algebra 1 or within an integrated mathematics curriculum. Exposing students to such coursework before they are ready often leads to frustration, failure, and negative attitudes toward mathematics and learning’

(NCTM 2008, p. 1)
9.1.3.5 The role of calculators and ICT

The use of calculators and computers has been strongly recommended by the NCTM since the 1980s, and in its Curriculum and Evaluation Standards for School Mathematics (1989) it recommended that ‘calculators and computers be freely available to elementary school students for solving mathematical problems, exploring patterns and concepts, and investigating realistic applications’ (NCTM, 1989, p.4).

As technology has advanced, its use in the teaching and learning of mathematics in the USA has been strongly encouraged, and in 2008 the NCTM recommended that

‘Teachers should also help students develop skills in the strategic use of a range of technological tools, including graphing calculators, spreadsheets, statistical software, and computer algebra systems’.

(NCTM, 2008, p. 1)

9.1.4 Comparisons of curricula and key findings

The main feature that the Japanese and Dutch curricula have in common is that of introducing problems in context and encouraging a problem-solving approach within classroom activities, developing mathematical reasoning and understanding and solving rich-context problems that require a broad range of mathematical skills and understandings. In contrast with this, the curricula in the UK and the US focus more on developing specific
mathematical skills, and are more likely to feature isolated tasks. In each of the UK and the US the curriculum is defined by a set of learning objectives (UK) or standards (US) which effectively break down the curriculum into individual units of work, whereas in each of the Netherlands and Japan the curriculum is defined in terms of much more general objectives which lend coherence to the curriculum and encourage the practice of making explicit links between topics.

In each of the UK and the US the approach to algebra is characterised by the view of algebra as generalised arithmetic, and algebra is not explicitly introduced to the curriculum until the age of 12. An emphasis is placed on the acquisition and development of arithmetic skills before the introduction of algebraic symbolism. In Japan, formal algebra is explicitly introduced in the objectives for grade 1 (age 6 – 7 years) and continues to feature significantly from then onwards, meaning that all pupils start using formal algebraic symbolism as early as the age of 6. In the Netherlands, although algebra is not formally introduced until secondary school (age 12) the primary curriculum goals place an emphasis on the use of mathematical language and the formal expression of mathematical reasoning, and when complex calculations are introduced in the secondary school they are taught alongside algebraic notation rather than beforehand. Thus the philosophy behind the approach to the development of algebraic concepts and skills varies significantly when Japan and the Netherlands are compared to the UK and the US; in both Japan and the Netherlands the emphasis is more on
concepts than skills, whereas in the UK and the US the emphasis is on the
development of algebraic skills through generalised arithmetic.
9.2 Comparison of misconceptions observed by pupils in each country

9.2.1 Japan

A summary of the misconceptions demonstrated by the Japanese pupils in the pilot study is given in table 9.3 and illustrated by a bar chart given in figure 9.5:

\( n = 33 \)

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>2</td>
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<td>0</td>
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<tr>
<td>Percentage of pupils who demonstrate this error</td>
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<td>0</td>
<td>3.0</td>
<td>6.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>1</td>
<td>7</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>3.0</td>
<td>21.2</td>
<td>6.1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9.3 Analysis of Misconceptions School Y Japan (pilot study)
This analysis reveals that only a small number of the Japanese pupils demonstrated misconceptions with the order of operations, with only 6 pupils (18%) showing any evidence of misconceptions with the order of operations. Only one pupil demonstrated the ‘Left to Right’ misconception. This pupil also demonstrated the ‘Addition Before Multiplication’ misconception. However this was the only pupil in the class to demonstrate either of these. These are shown in the pupil’s work on the first two calculations, shown in figure 9.6:
The only other misconception relating to the order of operations revealed by the Japanese pupils was categorised as M2, in which the power appeared to have been calculated before the bracket. This was observed with 6 of the pupils (18.1%) and some examples of this are shown in Figure 9.7:

![Examples of misconception M2 by Pupils 15, 22 and 23 in School Y](image)

This could, however, be explained by the manner in which many of the Japanese pupils seemed to approach arithmetic, which was in an algebraic manner. It appeared that many of the Japanese pupils treated the arithmetic as a specific case of doing algebra. They manipulated the arithmetic expressions as if they were algebraic expressions. Some typical examples of
this are shown in figure 9.8. Pupil 29’s working in the first calculation shows that he seems to have a clear understanding of the order of operations, but has made a mistake in treating the bracket as an addition rather than a multiplication. In the second calculation he has multiplied out the brackets in the denominator correctly. Pupil 18 has multiplied out the brackets correctly but has made a mistake in adding the terms together, and Pupil 31 has incorrectly squared the brackets in the denominator of the fraction.
In general the Japanese pupils carried out these algebraic manipulations correctly, but the main source of incorrect answers was due to mistakes in multiplying out brackets. It therefore seemed likely that the pupils who exhibited the misconception M2, some of whose work was shown in figure 8.3, were not simply “ignoring” the brackets in the denominator but were perhaps trying to square the bracket and doing this incorrectly as follows:

\[(1 + 2)^2 = (1 + 2)(1 + 2) = 1^2 + 2^2\]

**Figure 9.9 Example of an “algebraic mistake” by Japanese pupils**

This type of mistake appears to account for the vast majority of the incorrect answers that were given by the Japanese pupils.

### 9.2.2 The UK

In the UK there was one class in the pilot study, in school X, and three in the main study, in schools A, B and C. Summaries of the misconceptions observed in these classes are given in tables 9.4 to 9.10 and are shown graphically in figures 9.10 to 9.16:
School X  n = 20

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
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<td>1</td>
<td>6</td>
<td>2</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>50.0</td>
<td>5.0</td>
<td>30.0</td>
<td>10.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>6</td>
<td>13</td>
<td>2</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Percentage of pupils who demonstrate this error</td>
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<th>A3</th>
<th>A4</th>
<th>A5</th>
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</thead>
<tbody>
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<td>Number of pupils who demonstrate this error</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
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<td>5.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9.4 Analysis of Misconceptions UK School X (pilot study)

![Figure 9.10 Barchart showing pupil misconceptions in UK School X (pilot study)](image-url)

270
School A  n=17

Worksheet 1 (non-calculator)

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>10</td>
<td>3</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>58.8</td>
<td>17.6</td>
<td>35.3</td>
<td>5.9</td>
<td>0</td>
<td>11.8</td>
<td>5.9</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
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<table>
<thead>
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<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>3</td>
<td>4</td>
<td>1</td>
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<td>14</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
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<td>23.5</td>
<td>5.9</td>
<td>0</td>
<td>82.4</td>
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<th>A5</th>
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<tbody>
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<td>Percentage of pupils who demonstrate this error</td>
<td>0</td>
<td>17.6</td>
<td>0</td>
<td>5.9</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9.5 Analysis of Misconceptions UK School A worksheet 1

UK School A worksheet 1

Figure 9.11 Barchart showing pupil misconceptions in UK School A worksheet 1 (non-calculator)
### Table 9.6 Analysis of Misconceptions UK School A worksheet 2 (with calculator)

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
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<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>23.5</td>
<td>29.4</td>
<td>5.9</td>
<td>0</td>
<td>0</td>
<td>5.9</td>
<td>0</td>
<td>5.9</td>
<td>5.9</td>
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</thead>
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<td>0</td>
<td>0</td>
<td>4</td>
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<td>23.5</td>
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<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
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<td>Percentage of pupils who demonstrate this error</td>
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<th>KC</th>
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<td>0</td>
<td>52.9</td>
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</tr>
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</table>

**Figure 9.12** Barchart showing pupil misconceptions in UK School A worksheet 2 (with calculator)
School B  \( n = 29 \)  Worksheet 1 (non-calculator)

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
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<td>Percentage of pupils who demonstrate this error</td>
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<th>C2</th>
<th>C3</th>
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<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
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<td>0</td>
<td>4</td>
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<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
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<td>6.9</td>
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<td>58.6</td>
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<th>A4</th>
<th>A5</th>
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<td>0</td>
<td>3</td>
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<td>10.3</td>
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<td>10.3</td>
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</table>

Table 9.7  Analysis of Misconceptions UK School B worksheet 1 (non-calculator)

![UK School B worksheet 1](image)

Figure 9.13  Barchart showing pupil misconceptions in UK School B worksheet 1 (non-calculator)
School B  n = 29  Worksheet 2 (with-calculator)

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
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<td>3</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>0</td>
<td>24.1</td>
<td>10.3</td>
<td>0</td>
<td>0</td>
<td>6.9</td>
<td>0</td>
<td>17.2</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>9</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>3.4</td>
<td>6.9</td>
<td>3.4</td>
<td>0</td>
<td>31.0</td>
<td>3.4</td>
<td>13.8</td>
<td>0</td>
<td>3.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>0</td>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>0</td>
<td>37.9</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>BC</th>
<th>NC</th>
<th>IC</th>
<th>FC</th>
<th>KC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>3.4</td>
<td>3.4</td>
<td>3.4</td>
<td>24.1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9.8  Analysis of Misconceptions UK School B worksheet 2 (with calculator)

Figure 9.14  Barchart showing pupil misconceptions in UK school B worksheet 2 (with calculator)
School C  n = 30  Worksheet 1 (non-calculator)

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>23</td>
<td>0</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>76.7</td>
<td>0</td>
<td>30.0</td>
<td>3.3</td>
<td>0</td>
<td>16.7</td>
<td>0</td>
<td>20.0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>6.7</td>
<td>16.7</td>
<td>6.7</td>
<td>0</td>
<td>20.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>10.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9.9  Analysis of Misconceptions UK School C worksheet 1 (non-calculator)

Figure 9.15  Barchart showing pupil misconceptions in UK School C worksheet 1 (non-calculator)
School C  \( n = 30 \)  Worksheet 2 (with-calculator)

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>7</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>23.3</td>
<td>10.0</td>
<td>3.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>16.7</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>0</td>
<td>16.7</td>
<td>0</td>
<td>0</td>
<td>3.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>0</td>
<td>20.0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>BC</th>
<th>NC</th>
<th>IC</th>
<th>FC</th>
<th>KC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>3.3</td>
<td>6.7</td>
<td>0</td>
<td>36.7</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Table 9.10 Analysis of Misconceptions UK School C worksheet 2 (with calculator)

![UK School C Worksheet 2 Barchart](image-url)

Figure 9.16 Barchart showing pupil misconceptions in UK school C worksheet 2 (with calculator)
This analysis reveals that in all four classes studied in the UK, many of the pupils demonstrated a range of misconceptions with regard to the order of operations, with 14 pupils (70%) in School X, 14 pupils (82%) in School A, 18 pupils (62%) in School B and 29 pupils (97%) in School C demonstrating at least one misconception with regard to the order of operations, the most frequent being working from left to right (categorised as M1). It appeared from the pupils’ work that many were aware that there was a convention regarding the order, since a number of pupils from each school had written “BODMAS” or BIDMAS” somewhere on the worksheet. Some examples of this are shown in figure 9.17. In some cases this seemed to help, and the pupils who had written this did not work from left to right, but it did not ensure that the pupils would be able to deal with the more complex calculations or those involving powers. It appeared that some pupils were unsure about the meaning of the letter “I” in BIDMAS or “O” in BODMAS; this is clearly evident in the work of Pupil 17 from School B and also Pupil 4 from School C, who both wrote down their interpretations of the meanings of the letters in BIDMAS, but omitted an interpretation of the letter I. This failure to completely understand what the mnemonic represented had been evident in some of the interviews with the pupils in the UK schools, described in section 8.2).
**School A Pupil 6 worksheet 1**

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
</table>
| $2 + 3 \times 4 + 5$ | $2 + 12 + 5$  
                       | $14 + 5 = 19$           | 19     |
| $4 + 10 \div 2$   | $10 - 2 \div 5 + 4 = 9$ | 9      |
| $8 + \frac{8}{4}$ | $8 + 2 = 10$              | 10     |

**School B Pupil 15 writing on the top of worksheet 1**

School B Pupil 17 writing on the top of worksheet 1 and worksheet 2
<table>
<thead>
<tr>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 + 3 \times 4 + 5$</td>
<td>$\frac{2+12+5}{10}$</td>
<td>$19$</td>
</tr>
<tr>
<td>$4 + 10 + 2$</td>
<td>$\frac{8+4}{10+2+4}$</td>
<td>$9$</td>
</tr>
<tr>
<td>$8 + \frac{8}{4}$</td>
<td>$9 + 2$</td>
<td>$10$</td>
</tr>
<tr>
<td>$3 \times (2 + 4)$</td>
<td>$6 \times 3$</td>
<td>$18$</td>
</tr>
<tr>
<td>$2 + 3^2$</td>
<td>$9 + 2$</td>
<td>$11$</td>
</tr>
<tr>
<td>$2 \times (-3)^2$</td>
<td>$2 \times 2$</td>
<td>$7$</td>
</tr>
<tr>
<td>$2 \times (6 - 4)$</td>
<td>$2 \times 2$</td>
<td>$4$</td>
</tr>
<tr>
<td>$4 + (2 \times 3)^2$</td>
<td>$(6+4)^2$</td>
<td>$20$</td>
</tr>
<tr>
<td>$\frac{8+6}{3+2^2}$</td>
<td>$\frac{14}{12}$</td>
<td>$7$</td>
</tr>
<tr>
<td>$\frac{2+4^2}{(1+2)^2}$</td>
<td>$\frac{26}{5}$</td>
<td>$2$</td>
</tr>
<tr>
<td>$a + bc + d$</td>
<td>$\frac{2+5\times 4}{10}$</td>
<td>$19$</td>
</tr>
<tr>
<td>$ab + cd$</td>
<td>$(2 \times 3 + (4 \times 5))$</td>
<td>$26$</td>
</tr>
<tr>
<td>$a(b + c)$</td>
<td>$2 \times (3+4)$</td>
<td>$14$</td>
</tr>
<tr>
<td>$\frac{a(b + d)}{c + a^2}$</td>
<td>$\frac{5}{16}$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

Figure 9.17 Examples of pupils in the UK trying to use BIDMAS or BODMAS
Even when the word BIDMAS or BODMAS had not been written explicitly, there was evidence to suggest that many of the pupils in the UK were aware of a need for a convention, but were often unsure about what this should be. This is exemplified in figure 9.18 by the work of Pupil 5 from School A. Her work in question 1 suggests that she cannot make up her mind whether to work from left to write or to calculate the multiplication first. Having done this in both ways she finally decides to use the left to right calculation, and continues to do this in the subsequent questions. It is interesting to see that she even continues to do this when evaluating the first of the algebraic expressions, but subsequently carries out the correct calculations with the last three algebraic questions, suggesting perhaps that it is the algebraic expressions themselves that have prompted her to use the correct convention.
### School A Pupil 5 worksheet 1 questions 1 – 3 and 11 – 14

<table>
<thead>
<tr>
<th>Question</th>
<th>Expression/Equation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2 + 3 \times 4 + 5)</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>(4 + 10 + 2)</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{8 + 8}{4})</td>
<td>4</td>
</tr>
<tr>
<td>11</td>
<td>(a + bc + d)</td>
<td>25</td>
</tr>
<tr>
<td>12</td>
<td>(ab + cd)</td>
<td>26</td>
</tr>
<tr>
<td>13</td>
<td>(a(b + c))</td>
<td>14</td>
</tr>
<tr>
<td>14</td>
<td>(\frac{a(b + d)}{c + a^2})</td>
<td>2</td>
</tr>
</tbody>
</table>

### School A Pupil 9 worksheet 1 question 1, question 2 and question 11

<table>
<thead>
<tr>
<th>Question</th>
<th>Expression/Equation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2 + 3 \times 4 + 5)</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>(4 + 10 + 2)</td>
<td>16</td>
</tr>
<tr>
<td>11</td>
<td>(a + bc + d)</td>
<td>19</td>
</tr>
</tbody>
</table>

**Figure 9.18** Examples of evidence of knowledge of the need for a convention, and how algebraic thinking can prompt the correct convention
It was interesting to note that many of the pupils who exhibited the ‘Left-to-Right’ misconception, categorised as M1, demonstrated this consistently even when working with a calculator. This often resulted in inefficient use of the calculator in order for the pupil to carry out the calculation in a left-to-right manner. An example of this is given in figure 9.19, in which Pupil 2 from School A works very deliberately from left to right in questions 1, 2 and 3 in both worksheets, as revealed by her written workings and keystrokes. She also makes a mistake in question 1 of worksheet 2 involving a change of sign from a multiplication to an addition, but her keystrokes reveal that she is attempting to work from left to right.

School A Pupil 2 worksheet 1 questions 1, 2 and 3

School A Pupil 2 worksheet 2 questions 1, 2 and 3
A range of errors with notational conventions was observed across the pupils in all four classes in the UK, a particularly common error amongst the pupils in the UK was being unable to correctly square a negative number, which was categorised as C5. The percentage of pupils who demonstrated this error in worksheet 1 ranged from 20% in School C to 82% in School A. This error was also highly evident in worksheet 2; even when working with a calculator the percentage of pupils who squared a negative number incorrectly was as high as 30% in School B, although only one pupil (3%) in School C did this incorrectly with a calculator. Another relatively frequent misconception amongst the pupils in the UK, regarding convention, was an incorrect use of power notation, categorised as C1. A
number of pupils in each of the UK schools, as many as 30% in pilot school X, incorrectly interpreted a power of 2 to mean multiply by 2. This was less frequent in worksheet 2, possibly because the graphics calculator has an key which many of the pupils used correctly, but some pupils demonstrated a strong conviction in both worksheets that the power of 2 should be interpreted as a multiplication by 2. This is exemplified in figure 9.20 by the work and keystrokes of Pupil 13 from School B.

School B Pupil 13 worksheet 1 question 5

![Image of pupil's work for worksheet 1 question 5]

School B Pupil 13 worksheet 2 question 5

![Image of pupil's work for worksheet 2 question 5]

School B Pupil 13 worksheet 2 keystrokes for question 5

![Image of pupil's keystrokes for worksheet 2 question 5]

Figure 9.20 An example of a UK pupil’s misunderstanding of index notation
### 9.2.3 The Netherlands

There was one class of pupils in the Netherlands, in School D. A summary of the misconceptions observed from the pupils in this class is given in table 9.10 and illustrated graphically in figure 9.21. This is given for worksheet 1 only, as without the key record data the analysis for worksheet 2 is less reliable.

**School D  n = 22**

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>9.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>18.2</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>18.2</td>
<td>9.1</td>
<td>9.1</td>
<td>0</td>
<td>9.1</td>
<td>0</td>
<td>4.5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>0</td>
<td>13.6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 9.11 Analysis of Misconceptions Netherlands School D**

**Worksheet 1 (non-calculator)**
The first noticeable difference in the Dutch pupils’ work was their use of notation for multiplication and division. Many, but not all, of these pupils used a dot for multiplication and a colon for division, although this was not always consistent. Some pupils used a mixture of dots, multiplication signs, colons and division signs, but when their class teacher was interviewed (described in section 9.3.3) she suggested that this was probably due to the notation that was used in the worksheets, and that the pupils would generally be consistent about using dots for multiplication and colons for division. The class teacher did not feel that the use of multiplication signs and division signs in the worksheet would have caused problems for the pupils, and their work generally seemed to bear this out, but it may well have accounted for the fact that 7 of pupils (32%) made a “change of sign” error, classified as M8, at some point in one of the two worksheets. The Dutch pupils consistently used a comma instead of a decimal point, but
again the teacher did not feel that they would have been confused by the use of decimal points in the worksheets. This is exemplified in figure 9.22 by the work of Pupil 1 who uses a dot for multiplication in question 1, a colon for division in questions 2 and 3, and a multiplication sign in question 4. This pupil only answered one question incorrectly, and does not appear to have been confused with the mixture of notation.

![Table](image)

<table>
<thead>
<tr>
<th>Vraag</th>
<th>Uitwerking (indien nodig)</th>
<th>Antwoord</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.7 + 3.4 x 4.5 + 5.9</td>
<td>23.9</td>
</tr>
<tr>
<td>2</td>
<td>4.45 + 10.8 + 2</td>
<td>9.85</td>
</tr>
<tr>
<td>3</td>
<td>8.9 + \frac{8.6}{4}</td>
<td>11.05</td>
</tr>
<tr>
<td>4</td>
<td>3.1 \times (2.3 + 4.6)</td>
<td>21.39</td>
</tr>
</tbody>
</table>

School D Pupil 1 worksheet 2 questions 1, 2, 3 and 4

**Figure 9.22** An example showing Dutch pupils’ use of symbols for multiplication and division

Apart from those pupils who had exhibited a ‘Change of Sign’ error, only 2 pupils in this class (9%) demonstrated any misconceptions with the order of operations, and these 2 pupils worked from left to right (categorised as M1). There was no evidence of any other misconception with the order of operations in either of the worksheets, and the pupils’ scores in worksheet 2
suggested that they were confident and competent when working with a
calculator.

Also 4 pupils (18%) changed the operation sign from that in the question.
This is a higher percentage of the class than observed in any of the other
schools in the study. This had been categorised as M8, and is demonstrated
in the work of pupils 7, 11, 17 and 21 in figure 9.23. The pupils’ workings
in some cases, such as pupil 7, reveal that they have written down the
correct calculation but have changed the sign when evaluating it. This could
have just been a careless error in some cases, and not a misconception, and
as suggested in the discussion regarding notation, it is possible that the
difference in notation for multiplication and division on the worksheet,
compared to what they were used to, could have contributed to these errors

School D Pupil 7 worksheet 1 question 1

School D Pupil 11 worksheet 1 question 12

School D Pupil 17 worksheet 1 question 6
A small number of pupils demonstrated difficulties with algebraic notational conventions, with 4 pupils (18%) misinterpreting the notation for squaring, and 2 pupils (9%) being unable to correctly interpret the algebraic notation in the last four questions.

9.2.4 The USA (New York State)

There were two classes of pupils from school E in New York State. Class 1 was grade 7 and Class 2 was grade 8. Summaries of the misconceptions observed from the pupils in these classes are given in tables 9.12 and 9.13, and are illustrated graphically in figures 9.24 and 9.25:

School D Pupil 21 worksheet 1 questions 13 and 14

**Figure 9.23 Examples of pupils from School D ‘changing the sign’ of the operation**
### Table 9.12 Analysis of Misconceptions USA School E Class 1 (Grade 7)

#### USA School E class 1 worksheet 1

#### Figure 9.24 Barchart showing pupil misconceptions in USA school E class 1 worksheet 1 (non-calculator)
School E Class 2 (grade 8)  n = 27 Worksheet 1

<table>
<thead>
<tr>
<th>Category of error</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
<th>M7</th>
<th>M8</th>
<th>M9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>22.2</td>
<td>11.1</td>
<td>3.7</td>
<td>11.1</td>
<td>0</td>
<td>3.7</td>
<td>0</td>
<td>33.3</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
<th>C9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Percentage of pupils who demonstrate this error</td>
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<td>0</td>
<td>40.7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Category of error</th>
<th>A1</th>
<th>A2</th>
<th>A3</th>
<th>A4</th>
<th>A5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of pupils who demonstrate this error</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Percentage of pupils who demonstrate this error</td>
<td>11.1</td>
<td>3.7</td>
<td>3.7</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9.13 Analysis of Misconceptions USA School E Class 2 (Grade 8)

Figure 9.25 Barchart showing pupil misconceptions in USA school E Class 2 worksheet 1 (non-calculator)
The range and frequency of misconceptions observed in the work of the pupils in both classes in School E in the USA was similar to those observed in the UK, with all the order of operations misconceptions being observed. The left-to-right misconception was demonstrated by 4 pupils (14%) in Class 1 and 6 pupils (22%) in Class 2.

Like the pupils in the UK, many pupils in School E wrote a mnemonic at the top of their worksheet, and these pupils did not tend to work from left to right. However, unlike the pupils in the UK, the American pupils appeared to be confident about the fact that the letter E in the mnemonic stood for “Exponent”. This is evidenced, for example, in the jotting of Pupil 14 shown in figure 9.26, and the fact that this pupil correctly calculated the indices in the correct order in all the questions that required this.
Figure 9.26  Examples showing PEMDAS written on their work by pupils in the USA

Some pupils did not write PEMDAS but had a method of annotating their calculations by circling or using symbols to show the order of the calculation, and some combined this approach with writing PEMDAS. This
method often proved successful even with the more complex calculations, as demonstrated in figure 9.27:

School E Class 1 Pupil 1 worksheet 1 questions 8, 9 and 10

School E Class 1 Pupil 8 worksheet 1 questions 1, 2 and 3
School E Class 1 Pupil 10 worksheet 1 questions 1 - 5

School E Class 1 Pupil 16 worksheet 2 question 1

School E Class 1 Pupil 18 worksheet 2 question 1

Figure 9.27 Examples of pupils in the US trying to use PEMDAS
Some, but not all, of the pupils in the USA used the dot notation for multiplication. Most used a division sign but a small number of pupils were not able to recognise a fraction as a division. This can be seen in the work of Pupil 8 shown in figure 9.24 who uses the dot notation and division sign successfully in questions 1 and 2, but who seems unable to proceed with question 3 which includes fraction notation, and does not write an answer in the answer box.

9.2.5 Comparison of the pupils’ work and the misconceptions observed
Of the 22 distinct misconceptions and errors that had been observed and categorised, excluding the calculator errors, the work of the pupils in the four schools in the UK contained 21 of these. The work of the pupils in the classes in the US contained 16 of the 22 categories of error, whilst in the work of the pupils in the Dutch class only 8 of these were observed and with the Japanese class of pupils 9 distinct errors were observed. It can therefore be seen that the pupils in the classes in the UK demonstrated the greatest range and variety of errors and that the pupils in the classes in Japan and the Netherlands demonstrated the least. The US pupils demonstrated a considerably greater range of errors than the pupils in Japan and the Netherlands, but less than the pupils in the UK. The frequencies of occurrence of misconceptions were highest in the classes in the UK; for example the percentages of pupils exhibiting misconception M1 (left-to-right thinking) varied from 24% to 77% in the UK classes, whereas the corresponding percentages for the other countries were 3% in Japan, 9% in the Netherlands and 18% in the US. The corresponding percentages of
occurrence of each of the categorised errors are summarised in table 9.14.

The percentages are given from worksheet 1, which is the worksheet that
was taken and analysed for all pupils in the study, and in the cases of the
UK and the US the percentages are the mean of the classes.

<table>
<thead>
<tr>
<th>Category of error</th>
<th>Percentage of pupils demonstrating this error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Japan</td>
</tr>
<tr>
<td>M1</td>
<td>3.0</td>
</tr>
<tr>
<td>M2</td>
<td>18.1</td>
</tr>
<tr>
<td>M3</td>
<td>0</td>
</tr>
<tr>
<td>M4</td>
<td>3.0</td>
</tr>
<tr>
<td>M5</td>
<td>6.1</td>
</tr>
<tr>
<td>M6</td>
<td>0</td>
</tr>
<tr>
<td>M7</td>
<td>0</td>
</tr>
<tr>
<td>M8</td>
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</tr>
<tr>
<td>C1</td>
<td>3.0</td>
</tr>
<tr>
<td>C2</td>
<td>0</td>
</tr>
<tr>
<td>C3</td>
<td>0</td>
</tr>
<tr>
<td>C4</td>
<td>0</td>
</tr>
<tr>
<td>C5</td>
<td>0</td>
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<tr>
<td>C6</td>
<td>0</td>
</tr>
<tr>
<td>C7</td>
<td>0</td>
</tr>
<tr>
<td>C8</td>
<td>0</td>
</tr>
<tr>
<td>C9</td>
<td>0</td>
</tr>
<tr>
<td>A1</td>
<td>3.0</td>
</tr>
<tr>
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<td>21.2</td>
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<td>A3</td>
<td>6.1</td>
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<td>A4</td>
<td>0</td>
</tr>
<tr>
<td>A5</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9.14 Comparison of percentages of pupils demonstrating each error
Another key feature of the pupils’ work across the samples of classes was their use of mathematical notation. The Dutch pupils consistently used a dot • to represent a multiplication and a ratio symbol : to represent a division. Their workings were characterised by an algebraic approach, such as multiplying out brackets when evaluating an arithmetic expression. The Japanese pupils used multiplication signs and used fractions for division, and, like the Dutch pupils, they used algebraic ways of working.

The pupils in the US also sometimes used a dot for a division but not consistently; they were also inconsistent about their use of notation for division; some used a division sign and others used fraction notation. Their working focused on arithmetic rules rather than taking an algebraic approach. The pupils in the UK used multiplication signs and division signs, and some but not all were able to recognise a fraction as a division. Their ways of working, like the US pupils, were characterised by the use of rules for arithmetic, and their use of notation tended to be inconsistent. Even when they achieved the correct answer to a calculation, many pupils in the classes in the UK tended to write statements that were not mathematically correct, such as $2 + 4 = 6 \times 3 = 18$ (question 4 on worksheet 1). This was observed more frequently with the UK pupils than with those in any of the other countries.
9.3 The teacher interviews

The purpose of the teacher interviews was to discuss the ways in which the order of operations was introduced and consolidated, to find out about the types of teaching and learning methods that the teachers used, to discuss the types of resources that they had used and to look into the ways in which the links between arithmetic operations and algebraic operations may or may not be made at this stage in their schemes of work.

9.3.1 Interview with three Japanese teachers

Although I was unable to interview the class teacher of the Japanese pupils who participated in the pilot study, I did have the opportunity to have a discussion with three teachers from Japan who were visiting the University of Plymouth. I was able to show them the worksheets of the Japanese pupils and discuss these with the three teachers, along with a discussion relating to the Japanese mathematics curriculum and how this topic might be taught in Japan.

The first important point that was made by the Japanese teachers related to the age at which the pupils would have first been taught this concept, which is by grade 4 (age 9 – 10 years). By this age pupils will be performing calculations involving all four operations including the use of brackets. By grade 7 (12 – 13 years) they would be using indices in their calculations.

One highly significant factor that the teachers identified is the strong emphasis that is placed in Japan upon learning algebra alongside arithmetic.
This is emphasised throughout the mathematics curriculum (JSME, 2000) from grade 4 onwards. As a result, the concept of order of operations is not contained within the content for section A “Numbers and Calculations” but within section D “Mathematical Relations” where the objective is given as follows:

‘Children should express quantitative relations clearly in the formula and should be able to read them.

(a) To understand expressions that mix together brackets and the four fundamental rules of mathematics, and calculate them correctly.

(b) To understand formulas, and to use them’. (JSME, 2000, p. 14)

Thus the curriculum emphasises the inherent link between arithmetic and algebra from an early age and ensures that algebra is met at the same time as arithmetic in this context.

Another important point that was made by the Japanese teachers was the way in which the Japanese curriculum encourages teachers to introduce arithmetic and algebra in a practical and meaningful context. To illustrate how they might use introduce work on the order of operations they gave an example involving shopping for sweets:

‘If a boy buys 4 sweets which cost 90 yen each, how much change will he have from 500 yen?’

This would then lead to the calculation

\[
\frac{500 - 90 \times 4}{300}
\]
and the context of the situation would ensure that the pupils understood that the multiplication would have to be carried out before the subtraction.

A similar example they said they might use would be

‘If a girl buys 4 cakes which cost 120 yen each and 3 cakes which cost 150 yen each, how much money does she spend altogether?’

Leading to the calculation

\[ 120 \times 4 + 150 \times 3 \]

and again, they argued that the context would make it clear that the multiplications need to be done before the addition.

Appendix C gives some examples of a Japanese textbook used in grade 4. The example in question 1 on page 14 illustrates a similar example in a cake shop, in which the use of brackets is explained in order to demonstrate the structure of the calculation

\[ 500 - (180 + 90) \]

in the context of calculating change.

The feelings of the Japanese teachers were that if a calculation was met in context then the order of operations would be logical and clearly understood by the pupils. They all agreed that mnemonics are never used.
9.3.2 Interviews with the class teachers in the UK

9.3.2.1 Class Teacher from School A

This teacher was an experienced, specialist mathematics teacher who had taught mathematics in a number of different secondary schools. She explained that in School A there was a departmental scheme of work which ensured that all the teachers in the mathematics department were teaching topics in the same sequence, and that they would all be teaching the same topics at about the same time. A series of set textbooks was used and all the pupils had been issued with a copy of the textbook that was relevant to their level of study. Although the scheme of work was referenced to this textbook, the teacher explained that everyone in the department had the flexibility to use any resources that they wanted to, and that they were not required to use the textbook if they preferred to use their own resources. She explained that in practice she made some use of the textbook and supplemented this with a variety of ideas and resources of her own, and that this was typical of the practice of all the mathematics teachers in the school. She did not have an Interactive Whiteboard in her classroom.

When asked about how she teaches the order of operations, this teacher described how she had introduced the topic through an investigation. When the class were in the previous year she had introduced them to the “Four Fours” investigation, in which the pupils were challenged to produce all the numbers between 1 and 10 (initially) by using four fours and any mathematical symbols. In the current year she re-visited this investigation.
and extended it to include numbers between 1 and 50, and finally all the numbers up to 100. This investigation was worked on by the pupils partly in class and partly for homework. By discussing the pupils’ calculations she was able to encourage the pupils to talk about the use of brackets and the ways in which they could write their calculations using correct mathematical notation. She talked about ‘writing correct mathematical sentences’ and described how she would use hand gestures to indicate the presence of brackets, and vocabulary such as ‘and then’ to emphasise the order of the operations. She explained ‘I want them to write the calculations correctly, with correct use of the equals sign’ and she described how she consistently focussed on this throughout her teaching, in order to ensure that the pupils learned how to write mathematical statements correctly and unambiguously.

This teacher said that she did not initially talk about BODMAS or BIDMAS but that some pupils remembered meeting these mnemonics before, some in primary school, and that when they asked about it she explained what they were intended to mean. She did not actively encourage the pupils to use them but did not try to discourage them if they wanted to use the mnemonics to help them to remember the correct order. She did encourage them to put brackets in to emphasise the order, even if they were not explicitly needed, so for example she would encourage the pupils to write $2 + (3 \times 4) + 5 = 19$ and $10 - (2 \times 3) = 4$ with brackets included. She referred to these as ‘hidden brackets’ and always put the brackets in when
working with the class on the board or in individuals’ books. Similarly in a
calculation such as \( 4 \times 3^2 \) she would write this as \( 4 \times (3^2) \).

In order to practise and consolidate these ideas this teacher explained how
she used games and puzzles, including calculator puzzles which involved
using scientific calculators with brackets. One example of the puzzles she
used was to ask the pupils to write down the digits

\[
1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9
\]

and then challenge the pupils to insert any mathematical symbols and
brackets in order to achieve an answer as close as possible to 100.
Pupils could work together in groups. She would then discuss their methods
and write the keystrokes on the board before writing the calculation in
correct mathematical notation, focussing on the order and the use of
brackets.

She used the calculator puzzles as a way of promoting discussion and to
help focus on the parts of the calculations that ‘belong together’.

When asked about the use of textbooks and published resources this teacher
explained that ‘\textit{I don’t always teach in the same way as in the books}’ and
that she preferred to use her own resources and ideas in order to promote
discussion, so that the ideas and concepts ‘\textit{come from the pupil}’.

To sum up, this teacher had led these pupils to learn about the conventions
for the order of operations by investigation, discovery and discussion.
9.3.2.2 Class Teacher from School B

This teacher was also an experienced specialist mathematics teacher who has taught mathematics in a number of secondary schools and had been a Head of Mathematics. She explained that there was a departmental scheme of work that all the mathematics teaching staff were expected to follow, but that this did not specify any particular resources to be used and that she made use of a variety of resources of her own choice. Many teaching and learning resources were available in the school and a mathematics technician was available to assist staff in the preparation of any teaching and learning materials that they wished to produce. She had her own classroom with an Interactive Whiteboard which she used regularly, and frequently used resources from the internet.

When asked about teaching the order of operations, this teacher explained that she introduced this topic by means of a calculator investigation as a lesson starter. She gave the pupils a few calculations and asked them to use their calculators to work out the answers.

A typical calculation that she would have used would be $2 + 3 \times 4$

Some pupils had basic calculators which work on a left-to-right basis (LTR) and would give the answer 20. Some pupils had scientific calculators which would calculate multiplications and divisions first (MDF) and give the answer 14.
The teacher then discussed the pupils’ different answers and asked them to think about what the calculators were calculating. She asked some key questions: ‘Why were some answers different?’ ‘Can we have two different answers to a calculation?’ ‘Which answer is correct?’ ‘How do we decide?’

The teacher used the discussion to establish the need for a ‘rule’ or ‘convention’ and explained that we always perform calculations in a particular order. She asked them to write some calculations of their own and to predict the answers that the calculator would give them.

She explained that she had treated this as an arithmetic exercise, but encouraged the pupils to write their calculations using a fraction as a division, in a step towards algebraic notation. She focussed on writing mathematical “sentences” correctly, with correct use of the equals sign. She saw algebraic notation as the next step, and said that she moved on to the idea of introducing algebraic variables and substituting numbers for letters once the pupils had demonstrated an understanding of arithmetic conventions.

This teacher did not tell her pupils to use BODMAS or BIDMAS to remember the order, but she acknowledged that some pupils had heard of this already and asked her about it. She then explained what it referred to and wrote the letters for BIDMAS vertically on the board, so that she could write in the appropriate operation:
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Brackets
Indices
Division
Multiplication
Addition
Subtraction

This teacher did not use textbooks or published resources but produced her own activities in order to lead up to the type of discussion that she wanted the pupils to have.

To sum up, this teacher had used an investigatory approach to challenge her pupils to discover the need for a convention, and she had utilised calculators as a way of promoting this discovery. She had made extensive use of discussion in order to lead the pupils to appreciate the need for a convention. She saw the arithmetic understanding as a prerequisite for the introduction of algebra.

9.3.2.3 Class Teacher from School C

This teacher was an experienced specialist mathematics teacher who had the role of Assistant Head of Mathematics within the school, with particular responsibility for the development of the Key Stage 3 mathematics curriculum within the school. The mathematics department had a scheme of work, developed by this teacher, and which all the mathematics staff were
required to follow. Each mathematics class had a set of textbooks available in the classroom, but the textbooks were not issued to the pupils and there was no requirement for these to be used if the teachers wished to make use of other resources. This teacher explained that she sometimes set exercises from the textbooks but that she often made use of other resources such as worksheets and games, and often used resources straight from the internet, particularly for starter activities.

When asked about how she had taught the class about the order of operations, she said that she had started with an activity that involved her showing the class some calculations on the board. The calculations did not contain brackets. Some example of the types of calculations that she used would be:

\[
3 + 5 \times 6 - 4 \\
10 - 28 \div 7 \\
5 + 3^2 \times 2
\]

She then separated the pupils into two groups, one group working without calculators and the other group working with calculators, including both basic and scientific calculators in order to try and work out the answers to the calculations, working in pairs. She explained that ‘some pupils remembered doing some work on this in Primary school – some of them had been taught to use MY DEAR AUNT SALLY, but many of them did not remember this, and none of them had come across working with indices in this type of calculation before’.

She encouraged the pupils to discuss the
work in their pairs, and to try and agree upon what they thought was the correct answer.

When the pupils had had enough time to agree upon answers to the calculations, the teacher then initiated a whole-class discussion about the answers that they had obtained, and talked about how they had gone about working them out, some with and some without a calculator. She picked out some pupils who said they had typed the calculations directly into their calculator in left-to-right order, and asked them to think about the order in which the calculator had performed the operations. This differed according to whether they were working with a basic or scientific calculator. She then compared these answers to those that had been obtained by the pupils working without calculators, and continued to discuss the order in which the operations had been performed. She said that ‘I try to get the children to realise that we have to have a rule to stick to, to make sure that everyone gets the same answer. That there can only be one right answer. We talk about the way that scientific calculators are programmed to follow this rule, so that you will always get the correct answer if you type your calculation into a scientific calculator. I try to get them to come up with the rules themselves.’

Once the teacher was happy that the pupils had established the correct rules she then introduced some calculations containing brackets, and discussed with the whole class what they thought the brackets meant. She encouraged them to investigate with the brackets keys on the scientific calculators. She
then gave the pupils worksheets with some calculations to work on in pairs, to consolidate their understanding of the rules for the correct answer. She set them a few of the calculations from the worksheet given in appendix B but did not set all of them as there are very many calculations on the worksheet. After some time working on these she stopped and asked some of the pupils to give their answers, again taking time to talk about the correct order.

At this stage she introduced BIDMAS as a way of remembering the correct order of operations. She said that ‘I used to use BODMAS but I use BIDMAS now as it makes more sense and the children find it easier to remember what the I stands for.’

The lesson was concluded with an activity in the form of a game: Four-in-a-line BIDMAS (Appendix B). This involved pupils playing in pairs, throwing three dice and using the three numbers, and any operations, including brackets, to obtain an answer that was on the grid, and cover it with a counter. The aim of the game was to get four counters in a line, and the first person to do this was the winner.

The teacher said that she then gave the pupils some further questions for homework, but these took the form of puzzles, in which the pupils had to fill in the gaps in some calculations. These would have been similar to those shown on the worksheet BIDMAS Boxes in Appendix B but only a few of those on the worksheet would be set for homework.
When this teacher was asked about when she introduced the idea of operations in algebra, she explained that this was done later in the year, when the pupils had been given the time and opportunities to consolidate their understanding of number operations. This was established in the scheme of work that was used in the school, and in line with the material in the National Curriculum Framework. She felt that ‘they need to be confident with the number work before they move on to algebra.’

9.3.3 Interview with the class teacher in the Netherlands

This teacher said that she had introduced the concept of order of operations in the first semester of the first grade (equivalent to year 8 in the UK). She described how she linked this with algebra and gave an example of a resource that she had used called “Algebra Arrows” (see Appendix D). This required the pupils to make “arrow chains” using input/output machines, and gave the option of the input being either a number or an algebraic variable. Thus the pupils were led to see the equivalence of the arithmetic structure and the algebraic structure.

When asked whether she used mnemonics in her teaching the teacher responded:

‘No! At primary school some children still learn a wrong mnemonic (Meneer Van Dalen Wacht Op Antwoord), but that’s wrong nowadays and therefore old-school’. 

‘I always write down on the whiteboard’:
From left to right:

1. Erase curved brackets
2. Raise to a power
3. Multiply and divide
4. Add and subtract

She went on to explain that as well as linking the arithmetic to algebra within this topic she also used ‘story questions’ which would enable the pupils to produce answers to calculations which were presented in a context. This is consistent with the Dutch approach to teaching mathematics, ‘Realistic Mathematics Education’ (RME), which was described and discussed in section 9.1.2.2.

9.3.4 Interviews with teachers from New York State

It was not possible to interview the class teachers of the two classes that participated in the study but interviews were held with two mathematics educators from New York in order to discuss the state curriculum, the teaching methods that are encouraged and the types of resources that are used.

Teacher 1 is a mathematics teacher currently working in a public middle school. Teacher 2 is a lecturer in mathematics education at New York University. Both were asked about how and when the order of arithmetic operations was addressed in the state curriculum, about the way in which
they would teach it in grade 7, the resources that they would use and about
the teaching methods that were generally encouraged within the state.

9.3.4.1 Teacher 1 New York State

Teacher 1 had a detailed lesson plan, some of which is given in Appendix F.
He described how he would start his lesson by giving the pupils a
calculation to do, for example

\[ 3 + 4 \times 5 \]

After giving the pupils some time to calculate this he would then discuss
their answers, anticipating that some would obtain the answer 35 and others
would obtain the answer 23 and promoting a discussion about why there
seemed to be two different answers. He would then establish the need for a
convention, explain the order in which calculations are conventionally
calculated, and introduce the rules for the order of operations. At this point
he would also introduce two mnemonics for remembering these rules:

<table>
<thead>
<tr>
<th>Parentheses ( )</th>
<th>Please</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponents</td>
<td>Excuse</td>
</tr>
<tr>
<td>Multiplication</td>
<td>My</td>
</tr>
<tr>
<td>Division</td>
<td>Dear</td>
</tr>
<tr>
<td>(from left</td>
<td>Aunt</td>
</tr>
<tr>
<td>to right)</td>
<td>Sally</td>
</tr>
</tbody>
</table>

313
or

**PEMDAS**

He described how he would present this as a checklist, with tick boxes to enable the pupils to track through the calculation:

- P
- E
- M
- D

From left to right

- A
- S

From left to right

He would then provide the pupils with a variety of activities aimed at using this in arithmetic calculations. This might include setting them to work in pairs with one pupil coming up with a “wrong” calculation and the other pupil trying to “spot the mistake”. He encouraged the use of scientific calculators at this stage, asking the pupils to do the calculations first without a calculator and to then use the calculator as a “checking tool”.

He explained that the use of multiplication signs and division signs is discouraged beyond elementary school, using a dot for multiplication and a fraction for division, as this promotes a feel for algebraic structure.

### 9.3.4.2 Teacher 2 New York State

This teacher described how she would start the lesson using an activity she described as “name the number” in which the pupils were asked to produce strings of calculations using a set number in order to produce a target
answer. Figure 9.28 shows an example of a pupil’s work on this activity in which they have been asked to find ways of using twelve threes. This is a very similar activity to the “four fours” described by the teacher in UK School A.

Figure 9.28 Example of a pupil’s work in New York State on the “Name the Number” activity

It is interesting to note from the work shown in Figure 9.25 that this pupil has mainly used fractions as divisions, producing correct answers, but on the ninth calculation uses division signs, getting the answers incorrect since the order of operations has not been applied correctly. This would suggest that the use of fractions as divisions gives the pupil an intuitive logical feel for the correct order of operations.
This teacher explained that she would then discuss the need for a “correct” order in which to carry out calculations and introduce it with the idea that we do the most complex calculations first, with powers being the most complex, then multiplication and division, and then addition and subtraction, which she saw as the most simple. Parentheses could then be used if this order needed to be changed. She would refer to the mnemonic PEMDAS if pupils had met it before, which was often the case, but she did not put this forward unless asked.

When asked about the use of calculators this teacher explained that ‘I encourage the pupils to use them to check their answers, and to appreciate that there can only be one correct answer to a calculation’. She would set the pupils various puzzle-based activities, one example, “bowl-a-fact” being given in Appendix E, allowing them to use calculators if they wished. She did however explain that the state assessment papers include some non-calculator papers and that the use of calculators was not encouraged in activities that were seen as being non-calculator work, and that ‘calculator skills may not be so good’.

9.3.5 Comparisons and key findings from the teacher interviews

The teachers in the UK and the US have described very similar teaching approaches; all had made use of investigations and puzzles to introduce the concept of order of operations and all had encouraged the use of a calculator as a tool for investigating the ‘rules’. They had all described a rule-based
approach to the concept, emphasising that some sort of rule needs to be accepted and remembered. Sometimes this involved reference to BIDMAS, BODMAS or PEMDAS, and even when the teacher had not mentioned this themselves they discovered that many pupils had come across it already, and continued to want to use it. This emphasises the fact that a pupil’s ways of working and attitude to mathematics will have been influenced by a number of teachers, including primary teachers and will not simply depend upon the attitudes and teaching methods of any one teacher. The UK and US teachers all utilised exercises in textbooks which encouraged the ability to perform numerical calculations out of context, such as the examples given in Appendices B and E. In contrast, the Japanese and Dutch teachers described a more contextual approach, emphasising structural understanding rather than learning rules, and that mnemonics were never used. The Japanese teachers did not use calculators at all, whilst the Dutch teacher emphasised the use of a particular piece of computer software to enhance the understanding of structure rather than taking a calculator-based approach. The tasks they set their pupils were based in context.
9.4 Summary

In this chapter the four countries involved in the study have been discussed and compared in terms of their curricula, the analysis and observations of pupils’ work collected in the study, and the interviews with the teachers from each country involved in the study. In making comparisons it has emerged that there are many similarities between Japan and the Netherlands, and in contrast there are also similarities between the UK and the US. The key findings are summarised in table 9.15:

<table>
<thead>
<tr>
<th>Country</th>
<th>Curriculum</th>
<th>Teaching methods</th>
<th>Pupils’ work observed</th>
</tr>
</thead>
</table>
| UK      | Learning objectives defined discretely in separate curriculum areas.  
Algebra defined as generalised arithmetic and introduced after arithmetic in secondary school (age 12).  
Guidance given in terms of lesson structure with emphasis on “three part lesson” but schemes of work developed by individual schools. | Classes usually but not always grouped by ability.  
Textbooks sometimes used but usually not as a main resource.  
Schools and individual teachers decide on teaching resources.  
Lessons are usually based on a discrete topic.  
Puzzles, games and investigations often used as a teaching strategy.  
Use of calculators encouraged. | Evidence of a wide range and frequency of misconceptions.  
Inconsistent use of mathematical notation.  
Pupils use a variety of methods; some evidence that working with a calculator can help investigate different approaches.  
Evidence of rule-based thinking. |
| US      | Curriculum defined in terms of discrete standards within separate curriculum areas, with testing at the end of each grade.  
Algebra defined as generalised arithmetic | Classes are grouped by ability.  
Textbooks often used alongside teaching resources produced by individual teachers. | Evidence of a moderate range and frequency of misconceptions.  
Inconsistent use of mathematical notation. |
and introduced after arithmetic in middle school (age 12).

School districts decide on schemes of work.

Pupils use similar approaches; some evidence that they work better without a calculator. Evidence of rule-based thinking.

<table>
<thead>
<tr>
<th>Japan</th>
<th>Mixed-ability classes</th>
<th>Evidence of a low range and frequency of misconceptions.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All teachers use the same set textbooks and work collaboratively in planning and evaluating lessons.</td>
<td>Consistent use of correct mathematical notation.</td>
</tr>
<tr>
<td></td>
<td>Calculators used rarely or not at all</td>
<td>Pupils use consistent approaches.</td>
</tr>
<tr>
<td></td>
<td>Strong emphasis on the use of correct mathematical notation.</td>
<td>Evidence of algebraic thinking to inform arithmetic work.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Netherlands</th>
<th>Pupils taught in ability groups.</th>
<th>Evidence of a low range and frequency of misconceptions.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Textbooks used and seen as an important tool in terms of content and teaching methods, but no one set text.</td>
<td>Fairly consistent use of correct mathematical notation.</td>
</tr>
<tr>
<td></td>
<td>Algebra introduced in secondary school and taught alongside arithmetic with an emphasis on structure and context.</td>
<td>Pupils generally use consistent approaches.</td>
</tr>
<tr>
<td></td>
<td>Calculators used “with insight”</td>
<td>Some evidence of algebraic thinking to inform arithmetic work.</td>
</tr>
</tbody>
</table>

Table 9.15 Summary of comparisons between the countries involved in the study
CHAPTER 10: CONCLUSIONS AND IMPLICATIONS FOR TEACHING AND LEARNING

10.0 Introduction

This chapter summarises the findings of this research project and considers how they could have implications for the teaching and learning of arithmetic and early algebra. In the first section the research questions are reviewed and discussed, with consideration given to the extent to which it has been possible to address and answer them. In the second section this research is considered in the wider context and linked to the findings of earlier studies. In section 3 the findings are summarised in terms of the international comparisons that have been made and the implications that the findings could have for teaching and learning, with regard to curricula, teaching strategies and teaching and learning resources. The fourth section looks at the limitations of this research project, and the final section considers suggestions for further research.

10.1 Addressing the research questions and implications from the research

The research questions that were initially posed have resulted in some interesting and thought-provoking ideas and conclusions, which will be investigated in this section.
10.1.1 What misconceptions do children display when using the principles of order of operations in calculations?

In chapter 7 the misconceptions that were observed in the study have been described and discussed, with reference to the pupils’ written work and their calculator keystrokes which were observed by utilising the Key Recorder software. In all, 27 distinct types of error were observed, some of which could be regarded as misconceptions. From the pupil interviews, described in chapter 8, some themes emerged with regard to the misconceptions that had been observed with the sample of pupils in classes in the UK. These pointed to a prevalence of left-to-right thinking, difficulties with recognising a fraction as a division, difficulties with negative numbers and problems associated with working with calculators, and these difficulties appeared to be associated with rule-based methods of learning.

10.1.2 How can these misconceptions be classified?

As the study progressed and the pupils’ work was analysed it emerged that the misconceptions and errors that the pupils had made seemed to fall into four distinct categories:

- Misconceptions about the order of operations
- Misconceptions with algebraic and notational conventions
- Arithmetic errors
- Calculator errors
The first two categories contained what could be described as misconceptions, whilst the last two categories consisted of those, which were either careless mistakes, or, in the case of some of the calculator questions, an inability to identify the appropriate calculator function.

Although it was clearly the first two categories which were of main interest within this study, all the incorrect answers were categorised where possible, for completeness. It was therefore decided to code each incorrect answer within one of these four categories, and a list of codes emerged. These have been described in section 7.1.1 and summarised in section 7.1.2.

10.1.3 How might these misconceptions depend upon the teaching methods employed?

The summary in section 9.5 of chapter 9 points to the fact that there are key similarities between the teaching methods employed in Japan and the Netherlands, with corresponding similarities in the work of the samples of pupils from these countries. One highly significant common theme is the teaching of mathematics, including arithmetic, in context. This underpins the mathematics curriculum in both Japan and the Netherlands and is a key aspect of all mathematics lessons. Also common to both countries is the fact that the correct use of mathematical language and notation is encouraged and developed from an early age, and that when algebra is introduced it is done so alongside arithmetic and not regarded as a “jump” or transition from the study of arithmetic. Another key similarity is that the use of
textbooks is regarded as highly important in guiding both the content and teaching methods, and that the textbooks in each country are underpinned by a consistent philosophy. In Japan the textbooks are set and used in every school; in the Netherlands there is a wider range of textbooks for schools to choose from but all are based upon the principles of RME, giving them a consistent approach. Hence there is a significant degree of consistency across schools and from teacher to teacher. Another significant similarity is the view of mathematics as a social activity, with a focus on problem solving and the development of mathematical thinking.

The work of the pupils in the classes in the Netherlands and Japan displayed some striking similarities; in both classes the pupils displayed a small range and frequency of misconceptions, and the errors that were observed were more often categorised as mistakes than misconceptions. Their work was very consistent in terms of the symbols and notation used, and generally showed correct use of mathematical notation. In addition to this, there was evidence from pupils in both classes of algebraic thinking, even when the calculations were purely arithmetic.

The comparisons in chapter 9 also point to key similarities between the teaching methods employed in the UK and the US. The curricula in each of these countries are underpinned by a set of detailed and discrete objectives or standards, grouped under topic areas and leading to assessments which are objective-led. This can encourage teaching approaches which tend to focus on addressing one objective at a time and which can tend to feature
isolated tasks. This is consistent with the findings of Jones (1997), which were discussed in section 9.1.1 and summarised in table 9.1. The approach to the introduction of algebra is also very similar in the UK and the US, being underpinned by the view of algebra as generalised arithmetic, and with curricula that focus first on the development of arithmetic skills and which then make ‘a transition from arithmetic to algebra’ (National Research Council, 2001, p 419) at about the age of 12.

Another similarity between the teaching methods observed by talking to the sample of teachers in the UK and the US is the use of “rules”. In this study the rules took the form of mnemonics such as BODMAS, BIDMAS and PEMDAS, but there are many other cases, common to both the UK and the US, of concepts in mathematics being “remembered” such as SOH CAH TOA for the trigonometric ratios within a right-angled triangle, the use of “CAST” to remember which of the trigonometric ratios is positive in each quadrant of the unit circle, and the “eyebrows and smile” approach to multiplying brackets, that was described in section 1.2.1.

The work of the pupils in the classes in the UK and the US also displayed significant similarities; the range and frequency of misconceptions that were observed with these pupils were much higher than in the work of the pupils in the Dutch and Japanese classes, and the pupils in the classes in the UK and the US were much less consistent in their use of mathematical notation and more prone to writing incorrect mathematical statements.
The findings of this study would therefore suggest that there could be key aspects of mathematics teaching methods in both Japan and the Netherlands which may have led to the pupils in the classes in this study developing a deeper understanding of the concepts underlying arithmetic and algebraic thinking without relying upon memorising rules, and therefore leading to the pupils displaying fewer misconceptions. The findings also suggest that there are aspects of mathematics teaching in both the UK and the US that may have led to the pupils in the classes observed having a less developed “feel” for the algebraic logic which can underpin arithmetic thinking and working, and to place a considerable reliance upon the memorisation of rules and techniques, which may subsequently lead to the occurrence of more misconceptions.

10.1.4 How can these misconceptions be addressed?

The findings of this study suggest that an appreciation of algebraic structure is more effective than the rote learning of rules, and therefore if misconceptions are observed it seems that an effective teaching strategy is to focus on structure, pattern and the development of an appreciation of algebraic notation. Recent research findings, discussed in section 2.2, suggest that this is most effective when met in a problem-solving context. Vorderman et al (2011) suggest that in primary school, mathematics should be met as often as possible when covering the rest the curriculum, in a variety of different contexts, and that basic number work should be reinforced constantly by applying it within other subjects and in many
different activities. The summary of the trial of RME approaches carried out by the Centre for Mathematics Education at Manchester Metropolitan University (Dickinson and Hough, 2011) points to ‘significant gains in terms of correct answers’ (p.17) within the pupils’ work and acknowledges that a notable feature of the work of the pupils involved with the project was that they were able to employ strategies that made sense to them, rather than focusing on manipulating numbers, which was more commonly observed in the control classes. This suggests that misconceptions might be effectively addressed by adopting a contextual approach, and points to the most effective ways of teaching to prevent misconceptions in the first place, which will be considered in the next section.

10.1.5 What teaching strategies seem to be most effective in preventing misconceptions?

Evidence from this research study points to some significant factors in teaching and learning that would appear to reduce the range and frequency of misconceptions related to the order of arithmetic operations:

- **The use of context.**
  
  It has been seen that a key aspect of mathematics teaching in both Japan and the Netherlands is the philosophy underpinning the curriculum that mathematics should be taught and learned in context. This does not necessarily mean that all topics in mathematics need to have a “real-life” or functional application, but that they should be
introduced and used in a realistic or problem-solving context. In terms of arithmetic calculations this could arise from a real-life situation, or from an algebraic context. For example, if a pupil is faced with the calculation $200 - 3 \times 45$ he or she will need to rely upon “rules” in order to perform the calculation. When relying upon rules, this can often result in the rules being mis-remembered or mis-applied, and may sometimes mean that the topic was not fully understood in the first place. If, however, the pupil is given the problem “how much change would you get from two pounds if you buy 3 bars of chocolate at 45p each?” the context of the question may lead the pupil to consider the correct order in which to carry out the calculations involved. The importance of the structure of the arithmetic in this calculation is that it provides an essential link in developing the notion of an algebraic expression, so that this calculation is seen as an ‘object’, i.e. $200 - 3 \times 45$ and not merely the result of two calculations or ‘processes’ $3 \times 45 = 135$ followed by $200 - 135$.

Alternatively, if a pupil is faced with a problem that involves use of an algebraic expression, such as substituting values into an expression like $a - bc$, an appreciation of the logic of algebra can lead to an inherent understanding of the term $bc$ as an object and therefore an appreciation of the need for it be evaluated first before
being subtracted from $a$. This appreciation of algebraic thought and notation can provide the context for an arithmetic calculation.

- **The timing and nature of the introduction to algebra**
  
  Another key aspect that has been observed is that the early emphasis on algebraic thought and on the correct and consistent use of algebraic notation is a consistent theme in both Japan and the Netherlands. Informal algebraic notation is introduced very early in Japan as a natural component of the study of arithmetic and formal algebraic notation is introduced earlier in Japan than in any of the other countries in the study. In the Netherlands formal algebraic notation is introduced later than this, but once it has been introduced there is a very consistent approach to the correct use of notation, and an emphasis on algebraic structure even when approaching arithmetic problems. This approach would appear to make the concept of structure more evident in arithmetic and to enable pupils to see the inherent structure without relying upon the memorisation of “rules”.

- **The nature and consistency of the delivery of the mathematics curriculum as a whole, including the use of text books**
  
  The mathematics curricula in both Japan and the Netherlands are underpinned by a consistent philosophy based upon research evidence, with text books produced by mathematics education
researchers in line with the philosophy behind the curriculum. In both countries, ongoing staff development is regarded as a priority and is funded accordingly. The themes of consistency and the provision of high-quality classroom resources seem to be very effective in ensuring that mathematics teachers are supported to ensure that their teaching strategies are consistent, underpinned by research evidence and effective in teaching to avoid and deal with misconceptions. The importance of staff development in using resources effectively is pointed out by Dickinson and Hough (2011, p. 20) in the evaluation of the RME pilot study, in which it was noted that ‘it is essential that teachers understand the philosophy and are trained in the use of the materials’ and that ‘you can’t just pick up the books and use them; it will not be effective’.

10.2 How this research supports the work of other researchers

This thesis has drawn upon the work of many earlier studies but the intention is that it will support and advance the work of other researchers.

Research into the nature of ‘the track from arithmetic to algebra’ (Lee and Wheeler, 1989) and into the problems that pupils experience in early algebra (eg Keiran, 1989, 2007, Slavit, 1998, Lins and Kaput, 2004) has been discussed in section 2.3 and it was acknowledged by Livneh and Linchevski that ‘what is lacking, however, is a sufficient theoretical definition as for what will be considered as “difficulties with numerical
structures’ (Livneh and Linchevski, 2007, p. 217) and that there is a need for further research into ‘the underlying assumption that the understanding of the structural rules in arithmetic is a key for understanding the corresponding parts in algebra’ (Livneh and Linchevski, 2007, p. 217) Watson’s review of research into children’s algebraic reasoning (Watson, 2009) has been discussed in section 2.3. Importantly she felt that ‘algebra is not just generalised arithmetic; there are significant differences between arithmetical and algebraic approaches’ (Watson, 2009, p. 11) In her conclusions Watson also suggested that learning rules is not always effective, and that pupils need to develop ‘new priorities’ (Watson, 2009, p. 19), and my research findings support this view.

I embarked upon this study wanting to investigate the misconceptions that arose when pupils worked with calculations requiring arithmetic structure and in doing so I came to conclusions which encompass much more than what might be seen as a single topic within the mathematics curriculum. My conclusions lend support to the recent findings of Dickinson and Hough (2012) in the evaluation of their project to introduce Realistic Mathematics Education into some schools in the UK in placing emphasis on the need for “purpose” in the learning of mathematics and also studies such as Fujii (2006) which advocate a problem-solving approach in order to foster algebraic thinking.
10.3 Implications for teaching and learning

The findings of this study lead to some implications and recommendations for teaching and learning, and are in agreement with Watson (2009), who made recommendations that ‘require a change from a fragmented, test-driven system that encourages an emphasis on fluent procedure followed by application’.

- **Context**
  Calculations involving arithmetic order are most effectively understood by teaching them using a context

- **Resources**
  Text books can be a highly effective teaching and learning resource when they are informed by research, and can be particularly effective in enabling teachers to anticipate and recognise the problems that the pupils are likely to encounter. The use of a well written text book, backed up with appropriate guidance and staff development, would appear to be preferable to using a mixture of many different worksheets and web-based resources that may not have been informed by research and which may be of varying quality.

- **Rules**
  Pupils should not be encouraged to learn rules and rely upon memorisation; emphasis should be given to the appreciation of
structure and it needs to be recognised that this takes time and multiple experiences in different contexts.

- **Curriculum**

  An early emphasis on algebraic structure can lead to a more effective understanding of the structure of calculations; algebra should not be regarded as generalised arithmetic and should be encountered and developed alongside arithmetic.

**10.4 Limitations of this research**

This research study took the form of a case study and therefore the comparisons made can only be related to the classes of pupils who participated in the study and are not generalisable. Nevertheless it was possible to make some interesting and insightful observations from the results of the research.

This study was carried out with a relatively small sample of pupils (203 pupils including both the pilot study and the main study). The samples were essentially convenience samples: ‘*The researcher simply chooses the sample from those to whom she has easy access*’ (Cohen, et al, 2000, p. 102) although there was a purposive element since the researcher wanted to ensure that the pupils in the UK were familiar with the graphics calculators that were used as a research tool, and also wanted to ensure that the classes of pupils sampled were either of mixed attainment or middle attainment, in order to provide a suitable comparison.
The only pupils whose calculator keystrokes were recorded using the Key Recorder software were the pupils in the three UK schools in the main study. This revealed a great deal about their thinking, and proved very useful in informing the interviews with some of the pupils. For practical reasons it was not possible to use the Key Recorder software with the pupils in the other countries. Due also to practical reasons it was not possible to interview any of the pupils in countries other than in the UK.

The teacher interviews were all very informative and insightful, but although I was able to interview each of the class teachers of the pupils who participated in the study in the UK, there were limitations to the teacher interviews for the other countries, due to both language issues and physical practicalities. Consequently the teachers interviewed from the US and Japan were not the class teachers of those pupils in the study.

Despite the limitations I have made comparisons and drawn some conclusions and comparisons relating to this specific area of mathematics; these comparisons are in broadly in keeping with the findings of a number of other international comparisons including the Programme for International Student Assessment (PISA, 2009)) and the Trends in Mathematics and Science Study (TIMSS, 2011), in which both the UK and the US are currently performing below the mean score and both the
Netherlands and Japan perform significantly above the mean score. For example, in the PISA report for 2009 (PISA, 2009) the mean score for the mathematics tests was reported as 496; Japan was ranked 8th with a score of 529, the Netherlands ranked 10th with a score of 526. These scores are both statistically significantly above the mean score. The US ranked 33rd with a score of 492 and the UK ranked 34th scoring 487 although neither of these scores was statistically significantly below the mean score. This study is consistent with my findings that both Japan and the Netherlands are achieving similar high results, with both the UK and the US scoring similar lower results.

It must be remembered, however, that merely by making comparisons we cannot conclude that it is possible to directly take on board the practices observed in other countries and expect them to fit in to a different cultural system. It is acknowledged by Vorderman et al (2011) that whilst ‘the attainment of students in top performing countries presents a target that we should aspire to reach, however, lessons from international comparisons must be applied in the context of this country’ (p. 13) and it is noted that ‘due to large cultural differences in society, home and education, we cannot merely import a system from one of these nations and expect it to work’ (p. 7).

Furthermore, the whole practice of making international comparisons based upon raw test results, such as those carried out within the PISA and TIMSS studies, must be viewed with caution. This is noted by Askew et al (2010)
who describe the ‘horse-race approach’ of looking at the relative rankings of different countries as being not particularly meaningful, partly because the absolute differences in scores are not that great and partly because the countries involved in these studies vary from year to year. Askew also acknowledges, in line with Vorderman et al (2011) that it is difficult to ‘cherry pick’ particular parts of successful systems to transfer and adopt elsewhere, and suggests that ‘piecemeal adoptions are not likely to succeed if features selected are not looked at in terms of their relationships with other aspects of the national culture and school system’ (Askew et al, 2010, p. 13).

Furthermore, the qualities that are assessed in tests such as those used in the TIMMS and PISA studies does not tell the whole picture. In the Pearson Report (Keilstra, 2012) it is recognised that factors such as income, school choice and the recruitment of good teachers all play an important part in the quality of educational outcomes. For example the UK ranks third (after China and the Netherlands) in an index of “School Responsibility and Autonomy” (p. 29) which measures the extent to which schools are able to provide a variety of choice to the curriculum, and it is acknowledged that in this respect the US is still developing. This report also acknowledges the importance of “softer skills”(p.36) in which some of the most successful school systems, particularly in Asia, consider how best to equip students for a changing employment market and an uncertain future.
10.5 Suggestions for further research

The process of researching for this thesis has raised some major questions for me, relating to the way that I have taught mathematics over the years, and to the guidance I now give to the trainee mathematics teachers for whom I have responsibility. The first questions, most specific to this piece of research, are:

- Should we be teaching the “order of operations” as we do in the UK, as a purely arithmetic topic which relies upon memorisation and application of “rules” of convention? When do we ever have to perform a calculation such as $2 + 3 \times 4 + 5$ in “real life”?

- Should we consider how to deal more effectively with the problems and difficulties that pupils appear to experience with fractions?

- Should we consider how to deal more effectively with the problems and difficulties that pupils appear to experience with negative numbers?
The more fundamental questions that have arisen from this are:

- Should we be placing a much greater emphasis on learning mathematics in context?

- Should the National Curriculum guidelines define algebra as generalised arithmetic?

- Should we be placing a much greater emphasis on the development of algebraic thinking in primary school and on the introduction of formal, consistent algebraic notation at an earlier age?

- Should mathematics teachers and non-specialist teachers be provided with much more guidance and support in terms of teaching materials and approaches that they should be using, in which the issues of dealing with pupil misconceptions are addressed thoroughly and effectively?

These are important questions which need to be considered with a larger body of research evidence than currently exists and which requires significant international comparisons to be made, not only in terms of specific mathematical topics but also in terms of factors such as attitudes and enjoyment of mathematics which can play a significant part in overall mathematical achievement.
REFERENCES


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Davydov, V.V. (1962) An experiment in introducing elements of algebra in elementary school *Soviet Education* 1: 27 – 37

DCSF (2008a) *The Standards Site* [online] 

DCSF (2008b) *National Strategies Secondary Mathematics Exemplification* [online] 

DCSF (2008c) *Independent Review of Mathematics Teaching in Early Years Settings and Primary Schools: Final Report- Sir Peter Williams*


DfEE (1999a) *The National Numeracy Strategy* DfEE.


New York State Education Department (2005) Learning Standards for Mathematics

New York State Education Department (2009) Grade 7 Mathematics Test Book 1: McGraw Hill, California


Appendix A

Examples of learning outcomes from the Key Stage 3 Framework

CALCULATIONS

A pupil should be taught to:

- Carry out more complex calculations using the facilities on a calculator.
- Interpret the display on a calculator in different contexts (fractions, decimals, money, metric measures, time).
- Use a calculator to work out.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Convert units of time.
- Use the formula to add and subtract from a given number.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Interpret the display on a calculator in different contexts (fractions, decimals, money, metric measures, time).
- Use a calculator to work out.

As outcomes, Year 3 pupils should, for example:

- Use vocabulary from previous years and extend to sign change key on calculator.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Convert units of time.
- Use the formula to add and subtract from a given number.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Interpret the display on a calculator in different contexts (fractions, decimals, money, metric measures, time).
- Use a calculator to work out.

As outcomes, Year 4 pupils should, for example:

- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Convert units of time.
- Use the formula to add and subtract from a given number.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Interpret the display on a calculator in different contexts (fractions, decimals, money, metric measures, time).
- Use a calculator to work out.

Calculation methods

As outcomes, Year 3 pupils should:

- Use vocabulary from previous years and extend to sign change key on calculator.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Convert units of time.
- Use the formula to add and subtract from a given number.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Interpret the display on a calculator in different contexts (fractions, decimals, money, metric measures, time).
- Use a calculator to work out.

As outcomes, Year 4 pupils should:

- Use vocabulary from previous years and extend to sign change key on calculator.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Convert units of time.
- Use the formula to add and subtract from a given number.
- Choose the correct key sequence to carry out calculations involving more than one step, e.g., to calculate 20 ÷ 5 × 3.
- Interpret the display on a calculator in different contexts (fractions, decimals, money, metric measures, time).
- Use a calculator to work out.
CALCULATIONS

As outcomes, Year 7 pupils should, for example:

* use, read and write, spelling correctly, order of operations, brackets.

Know the conventions that apply when evaluating expressions:

- Order of operations are evaluated first.
- In the absence of brackets, multiplication and division have precedence over addition and subtraction.
- A negative sign can be a bracket in expressions such as:
  
  \[
  \frac{5-2}{5-2} = \frac{3}{3} = 1
  \]

Number operations and the relationships between them

As outcomes, Year 9 pupils should, for example:

Use vocabulary from previous years.

Recognise that, for example:

\[
\begin{align*}
100 & \div 4 \times 5 = 6 \\
5 & \div 13 \\
\end{align*}
\]

Can calculate more complex mixed operations, including using the bracket laws on a calculator.

For example:

- Find the value of:
  \[a. \quad \frac{5}{2} \div \frac{3}{5} \times 3 \div 6 = 5\]
  \[b. \quad \frac{5}{2} \div \frac{3}{5} \times 3 \div 6 \div 5 = 1\]

Can calculate with mixed operations, e.g.:

\[
(20 + 10) \div (30 - 10) = 10
\]

Can evaluate expressions using nested brackets, such as:

\[
(12 - (25 - (2 - 7)))
\]

Understand the position of the brackets is important for example:

- Make 2 different answers as possible by rearranging into the expression:
  \[\frac{5}{2} \div 10 \times 5\]

Link to calculation methods (pages 168-69), order of operations (pages 168-169), and substitution in expressions and formulae (pages 168-170).
Appendix B

Examples of classroom resources used in the UK


(v) CIMT(2007)[online]

http://www.cimt.plymouth.ac.uk/projects/mepres/book8/bk8i4_4i3.htm

(vi) Ten Ticks (2012) [online] http://www.10ticks.co.uk/

(vi) TES Resouces  BIDMAS Blaster game [online]

http://www.ies.co.uk/teaching-resource/Bidmas-Blaster-number-calculating-game-6029281/

(vi) E-Lab Order of Operations Game [online]

2 Order of operations and brackets

This lesson will help you to know the order for doing addition, subtraction, multiplication and division and how to do calculations with brackets.

There is an order for doing addition, subtraction, multiplication and division. Multiply and divide before you add and subtract.

Example 1 Work out $5 + 6 \times 7$.

$5 + 6 \times 7 = 5 + 42 = 47$, because you work out $6 \times 7$ first.

If there are brackets, always work out expressions inside the brackets first.

Example 2 Work out $2 \times (3 + 4)$.

$2 \times (3 + 4) = 2 \times 7 = 14$, because you work out the brackets first.

Exercise 2

1. For each calculation, predict the result your calculator will give. Check with your calculator.
   - a $8 + 2 \times 5$
   - b $25 - 15 \div 3$
   - c $5 + 2 \times 3$
   - d $18 - 7 \div 2$
   - e $6 + 12 \div 4$
   - f $16 - 3 \times 5$
   - g $4 \times 10 \div 5$
   - h $9 + 3 \times 6$

2. Without using a calculator, work out:
   - a $5 + 10 \times 5$
   - b $20 - 16 \div 4$
   - c $19 - 10 - 5$
   - d $5 \times 8 + 1$
   - e $24 + 2 + 6$
   - f $15 - 5 \times 2$
   - g $21 + 9 + 3$
   - h $5 \times 9 + 3$

3. Without using a calculator, find the missing number in each of these calculations.
   - a $20 - \square \times 2 = 10$
   - b $27 \div 3 + \square = 13$
   - c $\square - 45 \div 5 = 3$
   - d $3 \times \square - 2 = 28$
   - e $20 - \square + 2 = 15$
   - f $8 + 12 \div \square = 10$
   - g $\square + 4 + 2 = 2$
   - h $\square + 3 \times 3 = 16$

4. Without using a calculator, work out:
   - a $5 \times (10 - 5)$
   - b $5 \times 10 - 5$
   - c $(9 + 6) \div 3$
   - d $9 + 6 + 3$
   - e $20 \div (3 - 1)$
   - f $20 \div 5 - 1$
   - g $14 - (1 + 3)$
   - h $14 - 1 + 3$

4 N2.1 Properties of numbers
3 Solving problems

This lesson will help you to solve problems involving integers.

You can find the value of \((3 + 4) \times 2\) with these calculator key presses:

\(1 \quad 3 \quad + \quad 4 \quad = \quad 1 \quad \times \quad 2\)

Exercise 3A

You need a copy of N3.1 Resource sheet 3.1.

Use the digits 1, 2, 3 and 4 with any of the four operations +, −, × and ÷ to make the numbers 1 to 30.

- Use each of the four digits once each time.
  - You can use them in any order but must not repeat a digit.
- You can use any operation. You don’t need to use all four operations
  and you can repeat an operation.
- Use brackets where they are needed.

Extension problem

Try to make the numbers from 30 to 40.

Exercise 3B

1. For each calculation, predict and write down the result your calculator will give. Check with your calculator.
   - a 16 − (4 − 3)
   - b 12 ÷ (6 − 2)
   - c (5 + 3) × 3
   - d (20 − 4) ÷ 2
   - e (8 + 4) ÷ (2 + 1)
   - f (8 + 4) ÷ 2 + 1
   - g (5 + 1) × (6 − 3)
   - h (5 + 1) × 6 − 3
Order of operations and brackets

**Explain points:**
1. Write down evaluated expressions.
2. Ask for the calculation and adjust brackets accordingly.

**Starter**
Show slide 2.1. Discuss the objectives for this and the next lesson. Say that this lesson is about the order for doing calculations, including those with brackets.
Explain the order of operations and work some examples on the board, e.g.

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 5 + 2$ Multiply first.</td>
<td>$8 - 12 - 4$ Divide first.</td>
</tr>
<tr>
<td>$= 20 + 2$ Then add.</td>
<td>$= 8 - 3$ Then subtract.</td>
</tr>
<tr>
<td>$= 22$ Answer: 22</td>
<td>$= 5$ Answer: 5</td>
</tr>
</tbody>
</table>

Write some more examples on the board for pupils to do, such as:

$5 + 4 \times 10$  $16 + 4 + 5$  $12 - 9 + 3$  $4 \times 3 - 6$

Show slide 2.2. Ask pupils to find pairs with the same answer (AG, BD, CJ, EI, FH).

**Main activity**
Show slide 2.3, expressions with missing operations. Ask pairs to replace each circle with an operation sign to make each statement correct. Demonstrate that $7 - 6 + 2$ would give 4 (not one of the answers).

Discuss solutions:

$7 + 6 - 2 = 11, 7 \times 6 - 2 = 40, 7 + 6 \times 2 = 19, 7 + 6 + 2 = 10$

Say that sometimes we want to add and subtract before we multiply or divide.
We use brackets to show this. Write on the board:

<table>
<thead>
<tr>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5 \times (4 + 3)$ Brackets first.</td>
<td>$(4 + 6) + (5 - 3)$ Brackets first.</td>
</tr>
<tr>
<td>$= 5 \times 7$ Then multiply.</td>
<td>$= 10 + 2$ Then divide.</td>
</tr>
<tr>
<td>$= 35$ Answer: 35</td>
<td>$= 5$ Answer: 5</td>
</tr>
</tbody>
</table>

Say that brackets show that you must first work out what is inside them.
Demonstrate that brackets make a difference by working out:

$3 \times (4 + 5) \text{ and } (3 \times 4) + 5$

Write some examples on the board for pupils to do, such as:

$(12 - 4) \times 3$  $15 + (7 - 4)$  $6 + (8 - 4) \times (1 + 3)$

Use the Calculator tool to show pupils how to use the bracket keys.
Show slide 2.4. Ask pupils to discuss the problem in pairs.


What was the total cost of the books?

Invite a pair to explain their method to the class.

Did anyone do it in a different way?

Draw out that one method is to work out the cost of Tom’s books (5 × 8) and the cost of Sara’s books (5 × 2), and to add the two amounts:

\[ (5 \times 8) + (5 \times 2) = 40 + 10 = 50 \]

A second method is first to add the number of Tom’s books to the number of Sara’s books, and then to find the total cost of all the books at £5 per book:

\[ 5 \times (8 + 2) = 5 \times 10 = 50 \]

Get the class to repeat both calculations using their calculators.

Review

Show the four calculations on slide 2.5. Tell pupils that the answers are correct but the brackets are missing. Ask them to put in the brackets.

\[ (2 + 7) \times 3 = 27 \quad 2 + (7 \times 3) = 23 \]

\[ (8 - 4) - 2 = 0 \quad 8 - (4 - 2) = 4 \]

Sum up by stressing the points on slide 2.6.

Homework

Ask pupils to do N3.1 Task 2 in the home book (p. 2).
6.2 Order of operations

What's the BIG idea?

→ The order of operations is
  - Brackets → you need to calculate whatever is in brackets first
  - Indices → powers like $^2$
  - Division
  - Multiplication
  - Addition
  - Subtraction

We call this BIDMAS to help us remember. Level 3 & Level 4

→ $2(5 + 6) = 2 \times 5 + 2 \times 6$
This is called the distributive law. Level 4

Watch out!

When you enter $x^2$ in a calculator, some answers are $25$, but others are $35$. What answer do you give? Is it correct? If you need to put brackets in the calculation.

Practice, practice, practice!

1. Match up the pairs of calculations that give the same answer:

   - $2 \times 10$
   - $2 \times 10$
   - $10 - 3$
   - $10 - 3$
   - $2 + 10$
   - $2 + 10$
   - $10 \times 3$
   - $10 \times 3$
   - $3 \times 10$
   - $3 \times 10$

2. Calculate
   - $a \ 4 \times 5 + 3$
   - $b \ 10 + 2 \times 7$
   - $c \ 20 - 3 \times 5$
   - $d \ 10 \times 6 - 7$
   - $e \ 0.5 \times 8 - 1.5$
   - $f \ 4.5 + 3 \times 1.5$

3. Calculate
   - $a \ 6 \times 3 - 7 \times 2$
   - $b \ 8 + 6 \times 3 - 2$
   - $c \ 8 \times 10 + 3 + 2$
   - $d \ 10 + 5 \times 2 \times 3$
   - $e \ 18 + 36 + 7 \times 4$
   - $f \ 2 \times 9 - 3 \times 6$

4. Use these numbers and the operations $+,-,\times$ and $+$ to make number sentences that give the target answers:
   - $a \ 4, 5, 3$ target = 7
   - $b \ 18, 2, 6$ target = 30
   - $c \ 18, 2, 6$ target = 6
   - $d \ 3, 7, 9$ target = 54

5. Calculate
   - $a \ (6 + 2) \times 5$
   - $b \ (5 \times 3) + 6$
   - $c \ 4 \times (6 - 3)$
   - $d \ (10 - 7) \times 2$
   - $e \ (6 + 3) + 10$
   - $f \ (10 + 3) + 6$

   g. Which one of these did not need brackets? Explain your answer.
### Level 3

**Calculate**
- **a** \((17 \times 2) - (6 \times 4)\)
- **b** \((2 \times 26) - (8 \times 3)\)
- **c** \(7 + (48 + 3) + 5\)
- **d** \(\frac{7 + 4}{2 + 10}\)
- **e** \((120 - 5) \times (3 + 5)\)
- **f** \((120 - 90) \times (3 + 5)\)

**7 Use these numbers, +, -, \times, \div and brackets to make number sentences that give the target answers.**
- **a** 5, 5, 2 **target = 25**
- **b** 10, 8, 6 **target = 60**
- **c** 20, 3, 4 **target = 8**
- **d** 16, 2, 5 **target = 63**
- **e** 5, 5, 2, 4 **target = 26**
- **f** 4, 4, 3, 2 **target = 26**

### Level 4

**Calculate**
- **a** \(2^2 \times 5 \times 6\)
- **b** \((10 \times 6) + 3\)
- **c** \(5^2 + 3\)
- **d** \((6 - 2)^2 + 3\)
- **e** \((6 + 3)^2 + 5\)
- **f** \(4^2 \times (6 - 1) + 5\)

**8 Put brackets into these calculations to make them correct.**
- **a** \(4 + 2^2 \times 1 = 8\)
- **b** \(2 + 3^2 = 25\)
- **c** \(4 \times 3^2 + 6 = 60\)
- **d** \(6^2 + 2^2 \times 4 = 160\)
- **e** \(10^2 - 7 - 2 = 95\)

**9 Calculate**
- **a** \(5 \times (80 - 2)\)
- **b** \(4 \times (30 + 3)\)
- **c** \(4 \times (60 + 5)\)
- **d** \(8 \times (20 + 7)\)

**10 Show how brackets and the distributive law can be used to work out these products.**
- **a** \(3 \times 42\)
- **b** \(5 \times 64\)
- **c** \(18 \times 5\)
- **d** \(7 \times 16\)
- **e** \(3 \times 998\)

### Now try this!

A **Target numbers**
Work in pairs. One person picks five numbers under 10, and then the other chooses a target number between 50 and 100. The first person to write a mathematical sentence that gives the target answer using some or all of the five numbers wins 1 point.
Take it in turns to pick the target number. The first person to 3 points wins.

B **Number sentences**
Make five number sentences that give an answer of 17 - the more complicated the better!

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<table>
<thead>
<tr>
<th>Addition</th>
<th>Subtraction</th>
<th>BIDMAS</th>
</tr>
</thead>
</table>

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9.8 More on order of operations

- Understand and use the order of operations
- Develop calculator skills and use a calculator effectively
- Check a result by working the problem backwards

What's the BIG idea?

→ Calculations must be done in the right order:
  - Brackets → Indices (powers) → Division → Multiplication → Addition → Subtraction

→ If there are several multiplications and divisions, or additions and subtractions, do them one at a time from left to right.

→ In order: BS (bracket), ED (exponents), MD (multiplication), AS (addition and subtraction).

→ Order of operations: BIDMAS

→ 5² is said '5 squared' and means 5 × 5.
→ You can use the [x²] keys on your calculator to work out squares.

→ You can check an answer by working it backwards.
→ 25 × 6 = 150
→ 150 ÷ 6 = 25
→ ÷ is the inverse (opposite) of ×.

→ Finding the square root is the inverse of squaring.
→ Use the [√] key on your calculator.

Learn this

BIDMAS
- Brackets
- Indices (powers)
- Division
- Multiplication
- Addition
- Subtraction

Practice, practice, practice!

1. Do these calculations on your calculator. Work from left to right.
   a) 7 + 5.1 + 4.8
   b) 57 - 8.6 - 3.9
   c) 2.4 × 3.5 × 8.7
   d) 728 ÷ 8 + 7

2. 180 students are going on a school trip. Each coach holds 49 people. How many coaches are needed?
3. Carly has £43. A ticket for the cinema costs £7.50. How many tickets can Carly afford to buy?

4. Work these out without using a calculator.
   a) 6 + 4 × 5
   b) 3 × 8 + 100
   c) 13 + 40 ÷ 10
   d) 8 - 2 × 11
   e) 50 - 6 × 7
   f) 10 - 16 ÷ 4

→ Do multiplications and divisions before additions and subtractions.

5. Use the bracket keys on your calculator to work these out.
   a) (4.1 + 5.2) × 6.3
   b) 10.1 × (9.3 + 3.2)
   c) 270 ÷ (15 + 7.5)
   d) (6.2 + 3.8) ÷ 4
6 Work these out using the memory keys on your calculator.
   a. $2.7 \times (5.2 + 3.0)$  
   b. $4.9 \times (9.5 - 6.8)$  
   c. $15.18 + (14.7 + 6.4)$  
   d. $34.52 + (14.2 - 7)$

   Work out the calculation inside the brackets first and put the answer in the calculator memory.

7 Work these out without using a calculator.
   a. $2 \times 6 + 3 \times 6$  
   b. $72 ÷ 8 - 2 \times 4$  
   c. $(3 + 4) \times (6 - 1)$  
   d. $36 + (10 + 5 - 6)$

8 Use a calculator to work these out.
   a. $202.53 + (5.4 + 3.7 + 3.8)$  
   b. $9.6^2 + 3 \times 7.4$  
   c. $(3.7 + 4.9)^2 - 27.5$  
   d. $39.7 \div 50 - 22.1$  

Work out $50 - 22.1$ first and use the calculator memory.

9 One minute is 60 seconds. One hour is 60 minutes.
   How many seconds are there in 12 hours?
   Use a calculator. Check your answer by working the calculation backwards.

10 A biscuit weighs 12 g. There are 15 biscuits in a packet.
   There are 24 packets of biscuits in a box.
   What is the total weight of the biscuits?
   Use a calculator. Check your answer by working the calculation backwards.

11 Use the $\sqrt{ }$ button on your calculator to find the square roots of these numbers.
   a. $\sqrt{289}$  
   b. $\sqrt{625}$  
   c. $\sqrt{529}$  
   d. $\sqrt{60.84}$  
   e. $\sqrt{15.6816}$  
   f. $\sqrt{0.5625}$

Now try this!

A Calculator words
Do each calculation on your calculator. Turn the calculator upside down to find the answers to the riddles.
   a. $134.5 \times 6$  
     Knees are in the middle of these.
   b. $877.67 \times 15.2$  
     Ding dong!
   c. $216 \times 1460 + 648$  
     Get down on the dance floor!

Investigate other number words on your calculator.

B The value of brackets
Put brackets in this calculation to make the highest possible answer.
   $24 ÷ 12 - 8 \times 5 + 3$

What is the lowest possible answer?
6.2 Order of operations

Starter (1) Oral and mental objective
Display the following grid. Split the class into two teams: one team is noughts, the other crosses.

<table>
<thead>
<tr>
<th>0.3</th>
<th>0.56</th>
<th>0.71</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12</td>
<td>0.92</td>
<td>0.04</td>
</tr>
<tr>
<td>0.67</td>
<td>0.28</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Teams take it in turn to choose a square on the grid and give the complement of the number (the number that needs to be added to give 1) to win that square and then put an O or X in it.

Starter (2) Introducing the lesson topic
Ask pupils to make target numbers using digit and operations cards – pupils are likely to use informal methods. Possible questions include:

3, 5, 2; target = 13 (answer: 3 × 5 − 2)
9, 3, 2; target = 5 (answer: 9 ÷ 3 − 2)
4, 5, 3; target = 27 (answer: (4 ÷ 5) × 3)
6, 3, 2, 1, 10; target = 67 (answer: (6 ÷ 3) × (10 − 2) − 1)

Differentiation: 9, 8, 5, 7, 3, 75; target = 412 (closest answers)

Main lesson
- Discuss why we need to know which operation to do first.
- Discuss the order of operations, e.g. 4 ÷ 5 × 3. Use examples to show how the order of operations can affect the answer to a calculation.

Which of the operations would you do first?
Introduce pupils to the order of operations (BIDMAS): Brackets, Indices, Division, Multiplication, Addition, Subtraction.
Display some calculations and go through them, e.g. 6 ÷ 3 + 2; 10 ÷ 2 × 5 − 4
Demonstrate the use of brackets.
In the calculation \(5 - 2 \times 25\), how can we make sure that the \(5 - 2\) is calculated first? Q1-7

- Go through examples of the use of indices in the order of operations using examples such as:
  \((2^2 + 5) 	imes 10 = (9 + 5) \times 10 = 14 \times 10\)
  \((6 + 2)^2 - 11 = (8)^2 - 11 = 64 - 11\) Q8-9

- Explain the distributive law to pupils by considering simple numeric examples such as:
  \(3 \times (2 + 1) = 3 \times 2 + 3 \times 1\)

Discuss with pupils how the distributive law could be used to make more complex multiplications easier to work out e.g. \(3 \times 53 = 3 \times (50 + 3) = 3 \times 50 + 3 \times 3\).

Q10-11

Activity A

Pupils practise using brackets and order of operations to reach target numbers. This activity could also be carried out in small groups, to see who can get the target number the quickest.

Activity B

Pupils practise using brackets and order of operations to reach target numbers. Differentiation: Pupils see how many ways they can find to make the target number. They score points for any way that only one of them has used.

Plenary

Ask pupils to write mathematically correct sentences to get as near to the target numbers as possible, using order of operations and brackets.

5, 7, 2, 3, 25; target = 142 6, 7, 4, 2, 9, 100; target = 551

Homework

Homework Book section 6.2.

Challenging homework: Using 4, 4, 4, 4 and the operations \(+, -, \times, \div\) and brackets, try to make all the numbers from 1 to 20. Can it be done? Which numbers can’t you make?

Answers

1 2 \times 10 \text{ and } 10 \times 2; 2 \times 10 \text{ and } 10 \times 2; \text{ the divisions and subtractions give different answers.}
2 a) 23 b) 24 c) 5 d) 55 e) 25 f) 83 g) 9
3 a) 4 b) 24 c) 55 d) 40 e) 117 f) 83 g) 9
4 a) 4 \times 3 = 5 + 7 b) 18 + 2 \times 6 = 30 c) 18 - 2 \times 6 = 6 d) 5 + 9 \times 3 = 34
5 a) 40 b) 21 c) 12 d) 6 e) 9 f) 3
6 a) 10 b) 28 c) 28 a) 7 d) 80 e) 80 f) 240
g) Part b), because BRIMAS would ensure that the multiplication is first in any case.
7 a) 5 \times (5 - 2) = 20 b) \((b + 6) + 10 = 140\) c) \((20 - (3 \times 4) = 8\) d) \((8 - (5 - 2) = 13\) e) \((5 + 2) \times 5 + 30 f) \((4 \times 4 - 3) + 2 = 26\)
8 a) 120 b) 63 c) 28 d) 19 e) 3 f) 85
9 a) \((4 + 2)^2 \times 1 = 8\text{ or } 4 \times (2^2) = 8\text{ or } no\text{ change}\) BIOMAS applies.
b) \((2 + 2)^2 \times 20 a) 4 \times (2^2) = 16\) c) \((b + 2) - 10 = 40\)
d) \((b^2 - 2) \times 4 = 160\) e) \((2^2 - 2) = 95\)
10 a) 95 b) 132 c) 260 d) 218 e) 2994
11 a) 128 b) 345 c) 90 d) 112 e) 2994

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9.8 More on order of operations

Starter (1) Oral and mental objective
Ask questions involving square numbers, varying the vocabulary as much as possible.
What is 3 multiplied by 3? Two 2s? 5 times 5? 6 squared? The square of 4?
Repeat for square roots, but do not use the term 'square root' yet.
What number multiplied by itself is 100?
A number squared is 64. What is the number?
The area of a square is 36 cm². What is its side length?
Differentiation: Use squares beyond 12 x 12.

Starter (2) Introducing the lesson topic
What other numbers can you make using the digits 1, 2, 3? (11, 12, 13, 21, 22, 23, 31, 32, 33, 111, 112, ...)
How can you make 6 using 1, 2, 3 and + or -? (3 + 3 is the easiest way.)
Ask pupils to investigate what other numbers they can make, using only 1, 2, 3, + and -. Differentiation: Repeat using only 4, 5, 6, +, -.

Main lesson
- Revise work on using a calculator (decimal point and the four operation keys) and giving an answer dependent on the context of the problem.
- Pose the following problem: Izo is making bracelets using some wire and plastic beads. He has 194.6 m of wire remaining. If each necklace uses 0.15 m of wire, how many bracelets can he make? Ask for a volunteer to draw on the board the keys (in order) that he/she used on the calculator. Why is the answer not 12.37? Q1-3
- Write on the board: 6 + 5 x 4
- What is the answer to this problem? How did you work it out? Q 17
- Ask pupils to work in pairs to establish which operation has been done first in each of these clues.
  - Clue 1: 18 + 4 ÷ 2 = 20
  - Clue 2: 20 - 2 x 8 = 2
  - Clue 3: 20 - 9 ÷ 3 = 17
  - Clue 4: 2 ÷ 4 x 2 = 10
- Stress the importance of carrying out calculations in the right order.

Work it out
Emphasise that if there are several multiplications and divisions, or additions and subtractions, they should be done one at a time from left to right. Q4

Consider the use of brackets. Explain that brackets are just a way of showing which bit to do first and are the most important in terms of order of operations. Introduce the bracket keys on a calculator.

What is $5 \times 3 \times 10$ on the calculator?

What is $(5 \times 3) \times 10$ on the calculator?

Suggest pupils use the acronym BIDMAS to remember the correct order of operations. Explain that a scientific calculator works out the order of operations by itself.

Using the bracket keys on your calculator, what is $3 \times (14 - 7)$?

Demonstrate using the memory keys as an alternative method. Q5–10

Remind pupils that $5^2$ is shorthand for "5 squared" which is $5 \times 5$. Explain that the "^" is known as an index power and that finding the square root is the opposite of squaring. Demonstrate how to work out $5^2 = 25$ and $\sqrt{25} = 5$ using a calculator.

Explain that powers are more powerful than division, multiplication, addition and subtraction, but not as powerful as brackets. Draw attention to the position of the "^" representing 'indices' in the acronym BIDMAS.

Go through a more complex calculation one step at a time. Q11

Activities A and B

Pupils practise carrying out calculations in the correct order.

(Activity A: a LEGS; b BELL; c BOOGIE. Activity B: highest answer (24 - (12 - 8)) \times (5 + 3) = 48; lowest answer (24 + 12) - (6 \times (5 + 3)) = -62.)

Plenary

Display "6 \times 4 + 3 = 42" on the board. Brackets are needed to make this correct. Where should they go? Repeat for other calculations.

Homework

Homework Booklet section 8.8.

Challenging homework: Using only 2, 4, 7, +, \times, – and brackets, ask pupils to find ten calculations with answers 11 to 20. They can use the numbers 2, 4 and 7 more than once.

Answers

1. a) 22.2 b) 44.5 c) 72.00 d) 33

2. 3 5

4. a) 26 b) 132 c) 17 d) 15 e) 8 f) 6

5. a) 58.59 b) 126.25 c) 30 d) 5 e) 30 f) 8.8

6. a) 22.41 b) 13.23 c) 5.8 d) 7.6 e) 30 f) 1

7. a) 30 b) 1 c) 35 d) 4 e) 15.7 f) 115.1

8. a) 46.46 b) 46.46 c) 46.46 d) 12.6 e) 43.30 f) 43.30

9. a) 43.30 b) 43.30 c) 43.30 d) 43.30 e) 43.30 f) 43.30

10. 43.30 g) 43.30 h) 43.30 i) 43.30 j) 43.30

11. a) 17 b) 25 c) 23 d) 7.8 e) 3.96 f) 0.75

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Unit 4 Section 3 : Order of operations

This section covers the order in which operations are carried out.
The two main methods taught for working out the order of operations are BODMAS and BIDMAS.

<table>
<thead>
<tr>
<th>B</th>
<th>Brackets</th>
<th>D</th>
<th>Division</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>Order or Of (as in ¼ of 6)</td>
<td>M</td>
<td>Multiplication</td>
</tr>
<tr>
<td>M</td>
<td>Multiplication</td>
<td>A</td>
<td>Addition</td>
</tr>
<tr>
<td>A</td>
<td>Addition</td>
<td>S</td>
<td>Subtraction</td>
</tr>
</tbody>
</table>

If more than one operation in a calculation has the same precedence, the operations are carried out from left-to-right.

Example Question

Calculate:
(a) \((3 + 2) \times 6 - 8\)
(b) \(4 \times 6 + 18 - 2\)
(c) \((17 - 2) + 5 + 6\)

(a) \[
(3 + 2) \times 6 - 8 \quad \text{(brackets first)}
\]
\[
= 5 \times 6 - 8 \quad \text{(multiplication second)}
\]
\[
= 30 - 8 \quad \text{(subtraction last)}
\]
\[
= 22
\]

(b) \[
4 \times 6 + 18 + 2 \quad \text{(multiplication and division must be done before addition)}
\]
\[
= 24 + 9
\]
\[
= 33
\]

(c) \[
(17 - 2) + 5 + 6 \quad \text{(brackets first)}
\]
\[
= 15 + 5 + 6 \quad \text{(division second)}
\]
\[
= 3 + 6 \quad \text{(addition last)}
\]
\[
= 9
\]

Practice Questions

Work out the answer to each of these questions then click on the button marked Show me to see whether you are correct.


373
Put brackets in each statement below to make it correct:
(a) 3 \times 6 + 1 = 21 \text{ Check}
(b) 5 + 6 \times 2 = 22 \text{ Check}
(c) 46 + 6 + 3 = 5 \text{ Check}
(d) 49 - 3 - 2 = 44 \text{ Check}
(e) 7 \times 3 + 2 = 35 \text{ Check}
(f) 13 - 4 \times 2 = 18 \text{ Check}

**Question 4**
Fill in the missing numbers in each of the questions below:
(a) 3 \times \_ + 2 = 17 \text{ Check}
(b) \_ \times 5 \times 2 = 22 \text{ Check}
(c) (4 + \_) \times 2 = 20 \text{ Check}
(d) 6 - \_ \times 2 = 0 \text{ Check}
(e) (7 - \_) \times 4 = 20 \text{ Check}
(f) \_ + 4 = 8 \text{ Check}

**Question 5**
Decide whether each of the statements below is true or false.
(a) (3 \times 6) \times 2 = 3 \times (6 \times 2) \text{ Check}
(b) (4 + 2) + 7 = 4 + (2 + 7) \text{ Check}
(c) (8 - 2) - 1 = 8 - (2 - 1) \text{ Check}
(d) (8 + 4) \times 2 = 8 \times (4 + 2) \text{ Check}

**Question 6**
Put brackets in each statement below to make it correct:
(a) 13 - 4 \times 1 = 10 \text{ Check}
(b) 30 - 9 + 2 = 13 \text{ Check}
(c) 60 \times 6 + 3 = 30 \text{ Check}

**Question 7**
Calculate:
(a) 8.2 \div 0.2 - 0.1 \text{ Check}
(b) 3.6 \times 0.2 - 0.1 \text{ Check}
(c) 8.2 \times (6 - 5.4) \text{ Check}
(d) 2.2 \times 0.7 \times 0.2 \text{ Check}

17/02/2007
Order of Operation (BIDMAS).

A. Find the answer to these questions.

1. \( 7 - 6 \times 2 \)
2. \( 5 \times 3 + 4 \)
3. \( 9 \times 3 + 5 \)
4. \( 7 - 10 + 2 \)
5. \( 7 - 12 - 4 \)
6. \( 21 + 7 - 2 \)
7. \( 12 - 42 + 6 \)
8. \( 14 + 30 + 5 \)
9. \( 19 - 15 + 3 \)
10. \( 12 + 8 - 6 \)
11. \( (3 + 5) \times 2 \)
12. \( 12 + (7 - 3) \)
13. \( 15 \times (9 - 7) \)
14. \( (16 - 13) + 3 \)
15. \( (11 + 9) + 4 \)
16. \( 7 + 24 - 6 \)
17. \( 22 - 6 - 3 \)
18. \( 4 \times 5 - 12 \)
19. \( 40 + (12 - 4) \)
20. \( (24 - 9) + 3 \)
21. \( 4 \times 3^2 \)
22. \( 17 - 4^2 \)
23. \( 10 - 2^2 \)
24. \( 7 + 9 \)
25. \( (3 - 2)^3 \)
26. \( (14 - 2)^2 \)
27. \( (6 - 2)^3 \)
28. \( 6 - 2^2 \)
29. \( (2 \times 4)^2 \)
30. \( 10 - 7^2 \)
31. \( 3 - 7 \)
32. \( 7 - 20 \)
33. \( 2 \times 2^3 \)
34. \( 20 + 2^3 \)
35. \( 36 - 3^3 \)
36. \( (16 - 8)^2 \)
37. \( 6^2 - 4 \)
38. \( (4 + 6)^2 \)
39. \( 4^3 + 8 \)
40. \( 4 \times 3^2 \)
41. \( 6 + 12 - 4 - 2 \)
42. \( (3 + 9) + (2 + 1) \)
43. \( 6 + 4 + 2 + 3^3 \)
44. \( 6 + 2^3 - 1 \)
45. \( 30 + (4 + 2) + 3 + 46 \)
46. \( 5 \times (2 - 3) - 4 \)
47. \( 36 + (6 + 2)^2 \)
48. \( (8 + 4) \times 3 - 2^3 \)
49. \( 2 + (4 + 3)^2 \)
50. \( (1 + 4)^3 \times 2 - 4 \)
51. \( 7 - 3 + 6 + 8 \)
52. \( (4 + 2)^3 + 4 \)
53. \( 15 + 4 - 2^2 \)
54. \( 4 + 6 - 3 - 3 \)
55. \( 10^2 - 4 \times 7 \)
56. \( 28 + (4 + 2)^2 \)
57. \( (8 - 2)^3 \)
58. \( 20 \times (6 - 2)^2 + 1 \)
59. \( 5 \times (2 + 35) - 4 \)
60. \( (6 + 4)^3 \times 9 \)
61. \( 7 + 5 \times (2 + 5)^2 \)
62. \( 2 + 3 \times 7 \)
63. \( 68 + 2 \times 7 \)
64. \( 7^2 + 7 - 3 \)
65. \( 3 \times 5 + 15 + 10 \)
66. \( 4^2 - 13 \times 4 \)
67. \( 3 - 2^2 + 1 \)
68. \( 3 \times (3^2 - 4) \)
69. \( (7 + 1)^2 \times (10 - 6)^2 \)
70. \( 9^2 + 4^2 \times (4 + 2)^2 - 1 \)
71. \( 6 \times 6 + 3^2 - 3 + 7 \)
72. \( (9 - 3)^2 = (18 + 6) \)
73. \( (3 + 1) \times 2^3 - 5^2 \)
74. \( (3 \times [12 - 9])^3 + 3 \)

B. Write out these questions and correct them by putting in one or two pairs of brackets. Some may be correct already and therefore do not need any brackets at all.

1. \( 3 \times 1 \times 5 = 20 \)
2. \( 5 \times 18 + 6 = 2 \)
3. \( 7 + 4 \times 3 = 19 \)
4. \( 6 + 20 \times 5 = 10 \)
5. \( 7 \times 3 - 10 = 11 \)
6. \( 12 - 2 \times 4 = 4 \)
7. \( 11 - 3 \times 6 = 16 \)
8. \( 14 - 3^2 = 6 \)
9. \( 27 + 3 \times 6 = 3 \)
10. \( 3 \times 10 - 4 = 18 \)
11. \( 12 + 8 + 5 = 4 \)
12. \( 17 + 8 + 3 = 15 \)
13. \( 1 + 4^3 = 17 \)
14. \( 21 - 3 \times 3 - 2 = 10 \)
15. \( 23 + 3 \times 4 + 1 = 21 \)
16. \( 24 + 3 \times 4 + 1 = 12 \)
17. \( 27 + 18 + 3 = 1 \)
18. \( 3 + 2^2 \times 3 - 1 = 74 \)
19. \( 32 + 3 \times 1 = 50 \)
20. \( 30 + 3 \times 2 \times 3 = 3 \)
21. \( 33 - 7 + 2^2 = 4 = 24 \)
22. \( 27 + 4 + 1 = 21 \)
23. \( 28 + 2 \times 4 + 1 = 25 \)
24. \( 27 - 18 + 3 = 1 \)
25. \( 29 + 5 \times 2 \times 4 = 6 \)
26. \( 30 + 3 \times 1 = 9 \)
27. \( 31 + 2 \times 3 = 1 \)
28. \( 32 + 3 \times 3 - 1 = 11 \)
29. \( 36 + 4^3 + 1 = 56 \)
30. \( 33 + 2^2 \times 3 - 1 = 24 \)
31. \( 34 + 2 \times 1 = 13 \)
32. \( 36 + 4 \times 3 = 9 \)
33. \( 37 + 4 \times 3 = 9 \)
34. \( 38 + 2 \times 6 = 27 \)
35. \( 39 + 7 \times 2 \times 3 = 9 \)
36. \( 40 + 2 \times 4 = 2 \)
37. \( 41 + 7 \times 2 = 28 \)
38. \( 42 + 6 = 2 \)
39. \( 43 + 1 \times 4 = 4 \)
40. \( 44 + 2 + 4 = 11 \)
41. \( 45 + 7 - 18 = 6 \)
42. \( 46 + 2 \times 4 + 2 = 23 \)
43. \( 47 + 10 + 5 + 1 = 7 \)
44. \( 48 + 7 + 12 + 2^2 = 23 \)
45. \( 49 + 4 \times 3 + 50 = 50 \)
46. \( 51 + 4 + 2^3 + 1 = 5 \)
47. \( 52 + 4 \times 7 - 4 \times 2 = 40 \)
48. \( 53 + 40 + 3 \times 2 = 2 \)

Level 5 Pack 3, Page 3.
Four in a Line-BIDMAS

Rules.

Each player throws the 3 dice in turn.

Using the 3 numbers make any number on the grid below.

Remember the BIDMAS procedure. Brackets, Indices, Multiplication, Division, Addition and Subtraction.

You can cover up that number on the grid below with your colour counter after your opponent has checked it.

If it is already covered - hard luck!

The first person to get 4 counters in a line wins.
Word Search 1. (BIDMAS).

Solve the sum and write the answer in words in the space provided.

Now search for the words in the answer grid below, the answer may be in any direction!!

The first one has been done for you.

1). $2 + 3 \times 5 = \text{SEVENTEEN}$
2). $(2 + 3) \times 5 = \hspace{2cm}$
3). $9 + 20 \div 4 = \hspace{2cm}$
4). $(2 + 3) \times (12 - 5) = \hspace{2cm}$
5). $(40 + 20) + 5 = \hspace{2cm}$
6). $5 \times 8 \times 3 = \hspace{2cm}$
7). $38 + 12 \div 4 = \hspace{2cm}$
8). $(72 + 16) \div 8 = \hspace{2cm}$
9). $4 + 7 \times 0 - 6 = \hspace{2cm}$
10). $9^2 - 5 = \hspace{2cm}$
11). $9 \times 11 - 16 - 8 = \hspace{2cm}$
12). $48 - 2 \times 7 = \hspace{2cm}$
13). $90 + 10 - 3^3 = \hspace{2cm}$
14). $(57 - 12) + 5 = \hspace{2cm}$
15). $3 + 8^2 - 10 = \hspace{2cm}$
16). $4 + 3^3 = \hspace{2cm}$
17). $(4 + 3)^2 = \hspace{2cm}$
18). $56 + 8 - 9 + 3 = \hspace{2cm}$
19). $9^3 + 2^4 = \hspace{2cm}$
20). $2 \times 7^2 - 50 = \hspace{2cm}$
21). $6 \times (13 - 4) = \hspace{2cm}$
22). $9 \times (16 - 14)^2 = \hspace{2cm}$
23). $3^3 + (10 - 4)^2 = \hspace{2cm}$
24). $7 + (2 + 4)^3 = \hspace{2cm}$
25). $(13 - 5)^2 + 4 = \hspace{2cm}$
26). $7 + 3 \times 2 + 10 = \hspace{2cm}$
27). $78 - 5 \times 9 + 5 = \hspace{2cm}$
28). $3^3 - 4 \times 5 = \hspace{2cm}$
29). $5 \times 9 + 2^4 - 56 + 7 = \hspace{2cm}$
30). $(51 - (3 + 4)^2)^3 = \hspace{2cm}$

NE I G H T Y N I N E A R U O F D M L N
T U W K T A A E N A U N D L E I R K S E
H I E E H F F F K I Y T I X E F T J I V
I T R D G O H T O W N I H N N T Y G R E
R S U C I R D N W E W E Y H Y Y H R T S
T I O V E T F E T E R M T S P T O F X R
E X F B Y Y H Y H N B I Y O H R D C T
E T Y N T T T T T T T N U T W W S R P O G V
N Y T U R H H $\text{S N I E W Y Q M E A S F E}$
S T F L O R G A E T I E D T V E V A Y N
T H I O F E I I V P H G T I H H Z E E O
U R F Q U E E S E E P I F X W R X V N Y
Y E B W K R Y D S Y O Y R U I E E F U T
H E N I N Y T N E W T S Z T E S C E Y R
M E R R O H R E O N L X X P Y L V G F O
N E V E L E I H E U A D I T R F B H G F
D P N L A M H W P N L F F S D K O J H R
E S E V E N T Y S I X I A A F J N U V Y
S L S Y S W X E V I F Y T R I H T K R T
P E T H I R T Y T W O D K Y G X N I N E
Word Search 2. (BIDMAS).

Solve the sum and write the answer in words in the space provided. Now search for the words in the answer grid below, the answer may be in any direction!!

The first one has been done for you.

1. \(6 \times 9 = \) \_FI\_TY\_\_

2. \((43 + 13) - 8 = \)

3. \(6 \times (20 - 8) = \)

4. \(33 + 24 + 3 = \)

5. \(20 + 3 \times 4 = \)

6. \((100 - 4) + 8 = \)

7. \(7 \times 9 - 16 + 2 = \)

8. \(5 + 31 = \)

9. \((5 + 3)^2 = \)

10. \(9 \times (15 - 9) = \)

11. \(44 + 2^2 = \)

12. \(91 - 27 + 9 = \)

13. \(2 \times 6 + 4 = \)

14. \(20 + (3 + 4)^2 = \)

15. \(48 - 6 + 9 = \)

16. \(6 \times 7 - 3 \times 8 = \)

17. \((7 + 16) \times 2 = \)

18. \(8^4 + 14 \times 2 = \)

19. \(3^2 + 2^2 + 6 = \)

20. \((3 \times 4)^3 - 2^4 = \)

21. \((5 \times 6)^2 = \)

22. \((30 - 6) + (3 + 1) = \)

23. \(5 + 8 \times 3 = \)

24. \(2 \times (4 + 3)^2 = \)

25. \(3^4 - (5^2 + 2) = \)

26. \((3 \times 9 + 2)^2 + 12 = \)

27. \(2^3 - (5^2 + 2) = \)

28. \((4^2 + 8^2 + 3 = \)

29. \((2 + 3)^2 \times 4 = \times 3 = \)

30. \((2^2 + 5)^3 + 2^2 = \)

---

NE VE LE DG T WEN TY SE VEN S
U S T N B I E T Q I Z V R P Q P S L M A
N F I S E O L R G Z S T I E V E I G H T
I G H X G E L H U K R E F F X V E P Q Y
N H I D T W T E X O F I V I Y Z X O Z W
E S G T H Y A R Z L F Z S E D T Y G X T
S E S H E F N R U T X Y U R N Y R F Y H
E I H I J N T I Y O T X T X S E T O T G
N G G R X R O F N R F E H R Z E R F W I
R H P T K T O Y O E O R I Z I R I Z E
T T O Y L U Y F T Z X W R N Y H F E O Y
W Y I T R C T F Z R C H T O V T T P E T
E N U H W V Y Y O B O Z Y Y Y Y Y N R E
N I Y R E B U U G U T F F F T T O I V N
T N T E R S I X H Y R X I G M N N L B I
Y E R E T N P V J U Y V V H N E E E A O N
N I N E T Y T W O I E S E R O W T V R E
I V A D H M U B K Z S E O Y T T W S E R
N X S P Y B H H N E E T H G I E E B Z S
E E R H T Y T X I S S T H I R T Y T W O

---

BIDMAS-Boxes.

By each problem is a set of numbers. You may only use these numbers. No number is to be used twice.

Fill in the boxes to make the number at the end.

Remember BIDMAS!

Example. \((1, 2, 3, 4, 5) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 11.\)

Answer \[1 \quad + \quad \boxed{5} \quad \boxed{\times} \quad \boxed{2} = 11.\]

1). \((1, 2, 3, 4, 5) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 7.\)
2). \((1, 2, 3, 4, 5) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 6.\)
3). \((1, 2, 3, 4, 5) \quad \boxed{x} \quad \boxed{-} \quad \boxed{-} = 7.\)
4). \((1, 2, 3, 4, 5) \quad \boxed{+} \quad \boxed{-} \quad \boxed{-} = 10.\)
5). \((1, 2, 3, 4, 5) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 3.\)
6). \((1, 2, 3, 4, 5) \quad \boxed{+} \quad \boxed{-} \quad \boxed{+} = 17.\)
7). \((1, 2, 3, 4, 5) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 5.\)
8). \((1, 2, 3, 4, 5) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 9.\)
9). \((2, 3, 4, 5, 6) \quad \boxed{x} \quad \boxed{-} \quad \boxed{-} = 9.\)
10). \((2, 3, 4, 5, 6) \quad \boxed{x} \quad \boxed{-} \quad \boxed{-} = 17.\)
11). \((2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 4.\)
12). \((2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 15.\)
13). \((2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 8.\)
14). \((2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 0.\)
15). \((2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 21.\)
16). \((2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 29.\)
17). \((2, 3, 6, 7, 12) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 19.\)
18). \((2, 3, 6, 7, 12) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 6.\)
19). \((2, 3, 6, 7, 12) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 3.\)
20). \((2, 3, 6, 7, 12) \quad \boxed{+} \quad \boxed{+} \quad \boxed{-} = 3.\)

Now there are 4 boxes to fill in!

21). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 9.\)
22). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 4.\)
23). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 18.\)
24). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 22.\)
25). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 14.\)
26). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 13.\)
27). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 6.\)
28). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 17.\)
29). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 10.\)
30). \((1, 2, 3, 4, 5, 6) \quad \boxed{+} \quad \boxed{x} \quad \boxed{-} = 15.\)

Now there are brackets in the problems!

31). \((1, 2, 3, 4, 5) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{+} \quad \boxed{-}) = 25.\)
32). \((1, 2, 3, 4, 5) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{+} \quad \boxed{-}) = 12.\)
33). \((1, 2, 3, 4, 5) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{+} \quad \boxed{-}) = 1.\)
34). \((1, 2, 3, 4, 5) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{+} \quad \boxed{-}) = 2.\)
35). \((1, 2, 3, 4, 5) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{+} \quad \boxed{-}) = 9.\)
36). \((1, 2, 3, 4, 5) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{+} \quad \boxed{-}) = 35.\)
37). \((2, 3, 4, 6, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 12.\)
38). \((2, 3, 4, 6, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 12.\)
39). \((2, 3, 4, 6, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 100.\)
40). \((2, 3, 4, 6, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 4.\)
41). \((2, 3, 4, 6, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 5.\)
42). \((2, 3, 4, 6, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 90.\)

Now there are 4 boxes and brackets in each problem!

43). \((2, 3, 4, 6, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 70.\)
44). \((2, 3, 4, 6, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 4.\)
45). \((2, 5, 6, 9, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 4.\)
46). \((2, 5, 6, 9, 10) \quad (\boxed{+} \quad \boxed{+}) \quad (\boxed{-} \quad \boxed{-}) = 30.\)
47). \((2, 5, 6, 9, 10) \quad (\boxed{+} \quad \boxed{-}) \quad (\boxed{-} \quad \boxed{-}) = 35.\)
48). \((2, 5, 6, 9, 10) \quad (\boxed{+} \quad \boxed{-}) \quad (\boxed{-} \quad \boxed{-}) = 5.\)
Order of Operations

Problem 1

$(2 + 1) \times 11 = 33$
Appendix C

Examples of classroom resources used in Japan


式と計算のじゅんじょ

1 180円のジュースと90円のドーナツを買って、500円出しました。

おつりをもとめる式をかきましょう。

\[
500 - 180 - 90 = 230 \\
180 + 90 = 270 \\
500 - 270 = 230
\]

上のつばささんの考えを、一つの式に表すことを考えましょう。

次のようなことはの式をもとにして、\( ( ) \)を使って表すことを考えましょう。

出したおかね - 代金 = おつり

\[
500 - (\underline{\hspace{2cm}} - \underline{\hspace{2cm}}) = 230
\]
次のおつりや代金をもとめる計算を、( )を使って、
1つの式にかきましょう。

(1) 1こ90円のドーナツを4こ買って、500円出したときのおつり

式

(2) 1こ120円のプリンを4こ、1こ150円のチーズケーキを3こ買ったときの代金

式

ただし算やひき算と、かけ算やわり算とがまじった式
では、かけ算やわり算をさきに計算するきまりになって
います。

このきまりを使って、上の(1), (2)の式は( )をとり、
それぞれ
500−90×4, 120×4+150×3
とかくのがふつうです。
次の計算をしてみましょう。
④ 12 ÷ 2 × 3
⑤ 12 ÷ (2 × 3)
⑥ 12 + 2 × 3

どんなじゅんじょで計算すればよいでしょうか。

いろいろな計算のまじっている式では、計算のじゅんじょは、次のようになります。

・ふつう、左からじゅんにします。
・( )があるときは、( )の中をさきにします。
・+、−と、×、÷とでは、×、÷をさきにします。

④ 16 ÷ 4 + 2 ⑤ 16 ÷ 4 × 2 ⑥ 16 + 4 ÷ 2
④ 16 − (4 + 2)
④ 16 ÷ (4 × 2)
② (16+4) ÷ 2

⑤ 4 × 7 − 6 ÷ 2 ④ 4 × (7 − 6) ÷ 2
③ (4 × 7 − 6) ÷ 2 ④ 4 × (7 − 6 ÷ 2)

けんたさんは、60 + 40 ÷ 5 と 8 × 4 − 10 ÷ 2 の計算を右のようにしました。
計算のまちがいをみつけて、正しい答えをもとめましょう。

① 60 + 40 ÷ 5 = 20
② 8 × 4 − 10 ÷ 2 = 11
My Math Notes

When studying mathematics, keep a good record of your learning in your notes so that you can always look back.

In your notebook, write:

Date
Problem
My Idea
Friends' Ideas
Summary
Reflection
etc.

Write down your friends' ideas that are different from yours.

As reflection, record
What you've come to understand
Any strategies you or your friends came up with
What you thought as you listened to your friend's ideas
What you want to think about next, etc.

<table>
<thead>
<tr>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use many different ways to find the average height of the 8 players.</td>
</tr>
<tr>
<td>152cm, 148cm, 152cm, 155cm, 158cm, 155cm, 147cm, 151cm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>My Idea</th>
</tr>
</thead>
<tbody>
<tr>
<td>All 8 people are taller than 100cm so if we think about the average of the last two digits:</td>
</tr>
<tr>
<td>( 5 + 8 + 5 + 2 + 5 + 5 + 0 + 5 + 7 + 5 + 2 = 86 )</td>
</tr>
<tr>
<td>( 86 \div 8 = 52 )</td>
</tr>
<tr>
<td>Therefore the average height of the 8 players is 52cm higher than 100cm.</td>
</tr>
<tr>
<td>150 + 52 = 202</td>
</tr>
<tr>
<td>Ans. 152cm</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Yuto's Idea</th>
</tr>
</thead>
<tbody>
<tr>
<td>The height of all 8 people are close to 100cm so think about the difference that are obtained by subtracting 100cm from each person's height.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Height (cm)</th>
<th>152</th>
<th>148</th>
<th>152</th>
<th>155</th>
<th>158</th>
<th>155</th>
<th>147</th>
<th>151</th>
</tr>
</thead>
<tbody>
<tr>
<td>Difference that are obtained by subtracting 100cm from each person's height (cm)</td>
<td>2</td>
<td>-2</td>
<td>-2</td>
<td>5</td>
<td>0</td>
<td>-7</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ \frac{2 + (-2) + (-2) + 5 + 0 + (-7) + (-1) + 2}{8} = 0.16 \]

\[ 150 + 0.16 = 150.16 \]

Therefore the average height of the 8 people is 2cm higher than 100cm. |
| 150 + 2 = 152 |
| Ans. 152cm |

<table>
<thead>
<tr>
<th>Summary</th>
</tr>
</thead>
<tbody>
<tr>
<td>When trying to find the average, we may be able to simplify the calculation by changing the point of reference. If we use negative numbers, we can set the point of reference anywhere.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Since I used 100cm as the point of reference, I didn't use negative numbers. But, Yuto used 150cm as the point of reference and we get the same result. Yuto's method was good because none of the positive and negative numbers cancelled each other out, so I think the calculation became easier. Next time, I'd like to try and use negative numbers in my calculation.</td>
</tr>
</tbody>
</table>

My Math Notes 49
Letters in Algebraic Expressions

How many matchsticks do we need?

We are making squares by lining up matchsticks as shown below. When we make 20 squares, how many matchsticks will we need?

1  2  3  20

Should I actually use the matchsticks and try to line them up?

It is not easy to count them one by one, is it?

Sakura  Yuto

Think about it by keeping the number of squares small.

When we make 5 squares, how many matchsticks will we need? Think about it by drawing a diagram.
Fill in the with the appropriate math sentence in Sakura’s idea as shown below.

If I think about it by dividing it like below, I can represent the number of matchsticks with the following math sentence.

How did Yuto think to come up with the math sentence $1 + 3 \times 5$? Think about it using the diagram below.

The number of matchsticks can be found with the following number sentence.

$1 + 3 \times 5$

What are some other methods that can be used? Try using math sentences and diagrams to explain your ideas.

In this chapter, let’s think about math sentence using letters.
Algebraic expressions using letters

Use of letters

On the previous page, Yuto thought about the number of matchsticks for 1 square, 2 squares, and 3 squares by drawing the following diagram. What will happen to the math sentence for finding the number of matchsticks?

<table>
<thead>
<tr>
<th>Number of squares</th>
<th>Math sentences for finding the number of matchsticks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Just like how we thought in Q above, the number of matchsticks can be represented with the math sentence

\[ 1 + 3 \times \text{(number of squares)} \]

The number of squares will be changed 1, 2, 3, etc. in variety of ways but if we show the math sentence using the letter \( x \), we can express it as the following.

\[ (1 + 3 \times x) \text{ matchsticks} \]

How many matchsticks will we need if we make 20 squares by lining up the matchsticks?

The number of matchsticks will change depending on the number of squares that will be made. The "\( x \)" used in the algebraic expression \( 1 + 3 \times x \) represents all cases. This expression can show how to find the number of matchsticks and at the same time can represent the actual results.

Represent Saki’s solution method on the previous page in an algebraic expression using the letter \( x \).

Try to use an algebraic expression with an "\( x \)" to show your own method.
Let's use letters to express different quantities.

**Ex. 1** When buying any number of 90 yen notebooks, the total cost will be:

\[ 90 \times \text{(number of notebooks)} \]

So, if \( x \) number of notebooks are purchased, the total cost can be expressed as:

\[ (90 \times x) \text{yen} \]

<table>
<thead>
<tr>
<th>Number of notebooks</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>90 \times 1</td>
</tr>
<tr>
<td>2</td>
<td>90 \times 2</td>
</tr>
<tr>
<td>3</td>
<td>90 \times 3</td>
</tr>
<tr>
<td>4</td>
<td>90 \times 4</td>
</tr>
<tr>
<td>( x )</td>
<td>( 90 \times x )</td>
</tr>
</tbody>
</table>

☐ (Check) I bought \( x \) m of string that costs 60 yen per meter. Show this cost as an algebraic expression.

**Ex. 3** Represent the following (1) to (4) in algebraic expressions.

1. 3 students were absent in a class of \( x \) students. How many students were present?
2. How many cm is the perimeter of an equilateral triangle when the length of one side is \( a \) cm?
3. When you divide \( x \) m of ribbon by 4 people equally, how much m of ribbon does 1 person get?
4. The temperature at 9:00 was \( t \) °C. At 10:00 the temperature had risen 3°C since 9:00. What is the temperature in °C at 10:00?

In Ex. 1, instead of the natural numbers, 1, 2, 3 — the letter \( x \) is used. Letters are also used to represent decimal numbers and negative numbers such as 0.5 and \(-4\).

**Ex. 4** Of the letters used in Prob. 3, which were used to represent numbers including decimal numbers? Also, which were used to represent numbers including negative numbers?
How to write algebraic expressions

Let's learn how to represent different quantities using letters and the rules for writing algebraic expressions using letters.

How to represent products

In multiplication with a letter such as $3 \times x$, the multiplication sign $\times$ is left out and it is written as $3x$. Therefore, the expression $1 + 3 \times x$ on page 52 can be written as $1 + 3x$.

In the product of a number and a letter, the number should be written in the front.

EX. 1 In order to buy 2 items that cost $a$ yen each, a 1000 yen bill is paid.

The amount of change can be represented as:

$$1000 - 2a$$

Omit the $\times$ and the expression becomes

$$1000 - 2a$$

Check

$x$ apples were bought at 120 yen per apple. Represent the price using an algebraic expression with a letter.

EX. 1 Represent the following quantities using algebraic expressions with letters.

1. The length of tape remaining when two pieces of $a$ cm tape were cut off from a piece of 80 cm tape.
2. The area of a rectangle that has a $5$ cm length and $y$ cm width.
3. The total cost when you buy six $x$ yen pencils and a 200 yen sketchbook.

Be sure to take out the $\times$ sign.

54 Chapter 2 Letters in Algebraic Expressions
[Example 2] x sheets of 50 yen stamps and y sheets of 80 yen stamps were bought.
The total cost of the purchase was:
\[(50x + 80y)\text{ yen}\]
If we omit the × sign and rewrite the algebraic expression, it becomes:
\[(50x + 80y)\text{ yen}\]

2. Represent the following quantities using algebraic expressions with letters.
(1) The weight of three a kg items and two b kg items.
(2) The change we get when we purchase 300 yen apples with the money collected from 5 people that gave 50 yen each.

The way to represent a product using algebraic expressions can be summarized as the following.

**How to represent a product**

1. In multiplication with letters, omit the × sign.
2. In a product with numbers and letters, write the number in front of the letter.

**Example 3**

1. \[a \times b = ab\]
2. \[a \times b \times c = abc\]
3. \[x \times 3 \times y = 3xy\]
4. \[(a - 5) \times 2 = 2(a - 5)\]
5. \[\frac{2}{5} \times x = \frac{2}{5}x\]
6. \[x \times \frac{2}{3} = \frac{2}{3}x\]

**Note:** Although \(b \times a\) is \(ba\), in a case like this the product is often written in alphabetical order, \(ab\).

**Check**

Rewrite the following algebraic expressions by following the rules for representing products.

1. \[x \times y\]
2. \[c \times a \times 5\]
3. \[a \times x \times 2\]
4. \[(a - b) \times 5\]
5. \[\frac{a}{3} \times a\]
6. \[x \times \frac{7}{4}\]

**Exercise 3**

Write the following quantities as algebraic expressions.

1. Twice the product of \(x\) and \(y\)
2. Six times the sum of \(x\) and \(y\)
Appendix D

Examples of classroom resources used in the Netherlands

(i) An example of RME based materials taken from Dickinson and Hough (2012, p. 2)

(ii) Wisweg Algebra Arrows [online]

http://www.fi.uu.nl/wisweb/en/
Shown here are some of the displays of goods that can be seen at a local market. In each case, write down how many items you think there are in the display. Also write down whether you think each answer is exact or an estimate.

An example of RME-based materials relating to volume
Appendix E

Examples of classroom resources used in the US (New York State)

Math Solutions (2009) [online]

Holt Mathematics Course (2010) [online]
http://my.hrw.com/math06_07/nsmedia/homework_help/msm1_2010/msm1_2010_ch01_03.html

Thinkmath (2008) [online]

Examples from New York State Testing Program(2010) Grade 7 p. 5 – 8
Order of Operations
A Lesson for Grades 6–8
Featured in Math Solutions Online Newsletter, Issue 33

This lesson was developed and refined by Math Solutions Education Specialists.

Lesson Objective
Students will understand the need for an agreed upon order in which to perform operations and apply the standard order of operations to a variety of number expressions.

Reference
- “Writing a PEMDAS Story,” by Vadim Golombek (Mathematics Teaching in the Middle School 5, no. 6 [2000]: 574–79)

Materials
- Paper and pencil for each student
- Newsprint or chart paper, 1 sheet per pair of students
- Markers, 2 different colors per pair of students

Reviewed Vocabulary
punctuation, number expression, number sentence, order of operations, social conventions

Overview of Lesson
Students are introduced to two sentences that involve the same words. The meanings of the sentences are different because of a different use of punctuation. A number expression is introduced to the students that results in a variety of solutions based on the order in which operations are performed. A comparison between the word sentences and the number sentence provides a context for students to understand the need for an agreed upon order to perform operations. An agreed upon “meaning” or value for number sentences is as important as an agreed upon meaning for word sentences. Working in pairs, students use the standard order of operations to create a variety of number expressions that have specific values.

Lesson Outline

Focus or Warm-Up
1. Introduce the idea of using conventions to clarify meaning by displaying the following sentences:
   - “Paul,” said the teacher, “is very intelligent.”
   - Paul said the teacher is very intelligent.

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2. Ask students to talk to a partner about the meaning of each sentence and be ready to share their ideas with the whole class.

Important ideas to come from this discussion include how the words are the same but the punctuation changes the meaning. Explain that punctuation is something that has developed over time to help us communicate in writing.

Introduction

1. Next, ask the students to complete the following number sentence:

\[ 4 \times 4 \times 4 = \_ \_ \_ \_ \]

Many students are unaware of the convention of the order of operations and the answers students offer might vary, including 28, 16, and 0. As students share how they arrived at their answers, record their thinking for the class. Some examples:

<table>
<thead>
<tr>
<th>28</th>
<th>16</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 \times 4 = 16</td>
<td>4 \times 4 = 16</td>
<td>4 \times 4 = 16</td>
</tr>
<tr>
<td>8 \times 4 = 32</td>
<td>4 \times 4 = 16</td>
<td>4 \times 4 = 0</td>
</tr>
<tr>
<td>32 - 4 = 28</td>
<td>8 \times 0 = 0</td>
<td></td>
</tr>
</tbody>
</table>

2. Continue by discussing how confusing this could be when we are trying to communicate our thinking in writing. Explain that, just like in using agreed upon punctuation to clarify meaning with written words, we use an agreed upon order of operations to clarify meaning with written math sentences.

Introduce the order of operations for the four basic operations:

- First: Simplify all operations inside parentheses.
- Then: Simplify all exponents, working from left to right.
- Next: Perform all multiplications and divisions, working from left to right.
- Finally: Perform all additions and subtractions, working from left to right.

3. Ask the students to again complete the number sentence but this time use the agreed upon order, or conventional order, in each sentence to compare the solution with that of another student sitting nearby. Finally, ask for students to share their solutions with the whole class. This time most of the solutions should be the same.

Confirm the process and solution using the order of operations.

4. Refer back to the record of their thinking and model how to record their original thinking using number sentences. Use parentheses to communicate the order of operations used by each method:

<table>
<thead>
<tr>
<th>28</th>
<th>16</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4 + 4) \times 4 = 28</td>
<td>(4 \times 4) - 4 = 16</td>
<td>(4 + 4) \times (4 - 4) = 0</td>
</tr>
<tr>
<td>4 \times 4 = 16</td>
<td>4 \times 4 = 16</td>
<td>4 \times 4 = 0</td>
</tr>
<tr>
<td>8 \times 4 = 32</td>
<td>8 \times 4 = 16</td>
<td>8 \times 4 = 0</td>
</tr>
<tr>
<td>32 - 4 = 28</td>
<td>32 - 4 = 28</td>
<td>32 - 4 = 28</td>
</tr>
</tbody>
</table>

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Exploration

1. Next, introduce the Four 4s challenge. Note with students that in this lesson, three different values have already been created using only four 4s—28 and 16 and 0. Ask students to work with a partner to complete the challenge.

Four 4s Challenge

How many of the numbers from 1 to 10 can you create using four 4s?

1 =
2 =
3 =
4 =
5 =
6 =
7 =
8 =
9 =
10 =

Record your findings on newsprint using markers and large letters so others can read it from across the classroom.

This challenge engages students in exploring how changes in operations affect the value of a numerical expression. In searching for expressions equal to given solutions, they develop strategies for manipulating the value of an expression and record expressions using conventional methods.

2. As partners complete their work, direct them to post their newsprint representations so they can be viewed by the whole class.

Summary

1. Ask the following questions as a way for students to demonstrate new learning and to reflect on the lesson objective.

Q. What do you notice is the same about all of the newsprint posters?

Q. What is different?

Q. Which number sentences, if any, have the same value even though the operations were performed in a different order?

Q. With which solution, if any, do you disagree? How might you correct it?

Q. Why is it important that we have an agreed upon order of operations?

Q. Who can state, using your own words, the standard order of operations?

Q. The saying “Please excuse my dear Aunt Sally” has been used by many students to recall the standard order of operations. How do you think it helps?
Lesson Notes for the Teacher

Even though students can recite the correct order of operations, they need repeated opportunities to apply the order of operations in contexts that make sense.

A follow-up assignment might include writing a PEMDAS (parentheses, exponents, multiplication, division, addition, subtraction) story (see reference article by Vladim Golomb)

Here's an example from the article:

Problem: Write a PEMDAS story about \((4 + 2) \times 2 = 4\).

Possible Solution: Four friends were playing ball in the park. They were having a great day because it was the weekend. Later, two more of their friends from that neighborhood joined them. Now there were six friends playing in the park. Another group of six kids saw the group of six playing and asked if they could join to make two teams. Everyone agreed and now there were twice as many people playing; this made the game more competitive. Everyone was out to win. The group stayed in the park long after the game was over, just talking about their favorite topics.

As it was getting late, everyone was getting tired and hungry. When they were ready to go home, the large group of twelve friends divided into four groups. Each group had the same number of people. This way four groups of three kids walked each other home.

Find more classroom lessons online at mathsolutions.com.
Visit the "Books and Resources" section and click on "Free Classroom Lessons."

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Chapter 1
Whole Numbers and Patterns

Online Support for selected exercises from Lesson 1-3 Homework

See Example 1
Simplify each expression.
1. \(36 - 10 \times 6\)
2. \(7 + 24 \div 6 \times 2\)
3. \(62 - 4 \times (15 - 5)\)

See Example 2
4. \(11 + 2^3 \times 5\)
5. \(5 \times (28 + 7) - 8^2\)
6. \(5 \times 3^2 \times 6 - (10 - 9)\)

See Example 3
7. Coach-Mcfox fed the team after the game by buying 24 Chicken Deals for $4 each and 7 Burger Deals for $5 each. Simplify \(24 \times 4 + 7 \times 6\) to find the cost of the food.

See Example 4
8. \(9 + 27 \div 3\)
9. \(2 \times 7 - 32 \div 8\)
10. \(45 - (3 + 6) \times 3\)
11. \((6 + 21) \times 4\)
12. \(9 + 3 \times 6 \div 2\)
13. \(5 \times 3 \times 2 \times 12 \div 4\)

See Example 5
14. \(x^2 + 36 \div 12 - 4\)
15. \(160 \div 2^2 \times 7 \div 3\)
16. \(6 \times 2^2 + 28 - 5\)
17. \(2^2 - 12 \times 3 \div (15 - 7)\)
18. \(23 \div 13 \div 4 \div 9 - 3^2 19. (3^2 + 6 \div 2) \times 135 \div 6 - 4\)

See Example 6
20. The nature park has a pride of 5 adult lions and 3 cubs. The adults eat 9 lb of meat each day and the cubs eat 4 lb. Simplify \(5 \times 9 + 3 \times 4\) to find the amount of meat consumed each day by the lions.
21. Angle read 4 books that were each 130 pages long and 2 books that were each 325 pages long. Simplify \(4 \times 130 + 2 \times 325\) to find the total number of pages Angle read.

http://my.hrw.com/math06_07/nsmedia/homework_help/msm1_2010/msm1_2010_ch... 05/01/2012
Simplify each expression.

25. \(60 + 110 + 21 + 4^3 = 23\)

27. \(15 - 31 + 2\)

31. \(115 - 60^3 = 34 + 2\)

Add parentheses so that each equation is correct.

29. \(2^3 = 2 \times 15 + 15 - 8 = 11\)

41. \(4 \times 3 = 2 + 4 = 6\)

Archaeologists study cultures of the past by uncovering items from ancient sites. An archaeologist has chosen a site in Colorado for her team’s next dig. She divides the location into two equal plots and labels each plot so that uncovered items can be identified by the plot in which they were found.

43. The archaeologist must order a cover for the plot where the team is digging. Simplify the expression \(3 \times 12^2 + 63\) to find the area of the plot in square meters.

45. After the next two weeks, the archaeology team digs down an additional 27 meters. Simplify the expression \(3 \times 11^2 + 60 \times 12 + 27\) to find the total volume of dirt removed from the plot after 3 weeks.
Order of operations

From Thinkmath

Contents
- 1 Reducing ambiguity by agreement
- 2 Conventions for reading and writing mathematical expressions
- 3 Common Misconceptions

Reducing ambiguity by agreement

In general, nobody wants to be misunderstood. In mathematics, it is so important that readers understand expressions exactly the way the writer intended that mathematics establishes conventions, agreed-upon rules, for interpreting mathematical expressions.

- Does $10 - 5 - 3$ mean that we start with 10, subtract 5, and then subtract 3 more leaving 2? Or does it mean that we are subtracting 5 - 3 from 10?
- Does $2 + 3 \times 10$ equal 50 because $2 + 3$ is 5 and then we multiply by 10, or does the writer intend that we add 2 to the result of $3 \times 10$?

To avoid these and other possible ambiguities, mathematics has established conventions (agreements) for the way we interpret mathematical expressions. One of these conventions states that when all of the operations are the same, we proceed left to right, so $10 - 5 - 3 = 2$, so a writer who wanted the other interpretation would have to write the expression differently: $10 - (5 - 2)$. When the operations are not the same, as in $2 + 3 \times 10$, some may be given preference over others. In particular, multiplication is performed before addition regardless of which appears first when reading left to right. For example, in $2 + 3 \times 10$, the multiplication must be performed first, even though it appears to the right of the addition, and the expression means $2 + 30$.

See full rules for order of operations below.

Conventions for reading and writing mathematical expressions

The basic principle: "more powerful" operations have priority over "less powerful" ones.

Using a number as an exponent (e.g., $5^3 = 390625$) has, in general, the "most powerful" effect; using the same number as a multiplier (e.g., $5 \times 8 = 40$) has a weaker effect; addition has, in general, the "weakest" effect (e.g., $5 + 8 = 13$). Although these terms (powerful, weak) are not used in mathematics, the sense is preserved in the language of "raising 5 to the 8th power." Exponentiation is "powerful" and so it comes first! Addition/subtraction are "weak," so they come last. Multiplication/division come in between.


20/01/2012

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When it is important to specify a different order, as it sometimes is, we use parentheses to package the numbers and a weaker operation as if they represented a single number.

For example, while $2 + 3 \times 8$ means the same as $2 + 24$ (because the multiplication takes priority and is done first), $(2 + 3) \times 8$ means $5 \times 8$, because the $(2 + 3)$ is a package deal, a quantity that must be figured out before using it. In fact $(2 + 3) \times 8$ is often pronounced "two plus three, the quantity, times eight" (or "the quantity two plus three all times eight").

Summary of the rules:
- Parentheses first. Referring to these as "packages" often helps children remember their purpose and role.
- Exponents next.
- Multiplication and division next. (Neither takes priority, and when there is a consecutive string of them, they are performed left to right.)
- Addition and subtraction last. (Again, neither takes priority and a consecutive string of them are performed left to right.)

Common Misconceptions

Many students learn the order of operations using PEMDAS (Parentheses, Exponents, Multiplication, Division...) as a memory aid. This very often leads to the misconception that multiplication comes before division and that addition comes before subtraction. Understanding the principle is probably the best memory aid.


* This page was last modified on 8 March 2008, at 02:01.
1. Which measure is equivalent to 1.5 kilograms?

   1 kilogram = 1,000 grams

   A 15 grams
   B 150 grams
   C 1,500 grams
   D 15,000 grams

2. At noon on Monday in Minneapolis, the temperature, in degrees Fahrenheit (°F), was –4°F. At noon on Tuesday, the temperature was 6 degrees higher. What was the temperature at noon on Tuesday?

   A 2°F
   B –2°F
   C 10°F
   D –10°F

3. The prices of plasma televisions at an electronics store are shown below.

   $1,544 $1,242 $2,285 $1,116 $1,899 $1,649 $1,423 $1,242

   What is the range of the prices of these plasma televisions?

   A $1,043
   B $1,169
   C $1,242
   D $1,484

Go On

Page 5
4. Which expression is a binomial?
   A $p^2$
   B $3w$
   C $3w + 1$
   D $3p^2 + 2p + 2$

5. A box contains 6 red pens and 4 blue pens. Cory randomly picks a pen from the box and keeps it. Then Todd randomly picks a pen from the box. What is the probability both boys will pick red pens?
   A $\frac{1}{3}$
   B $\frac{9}{25}$
   C $\frac{1}{30}$
   D $\frac{1}{36}$

6. Simplify the expression below.
   $-4x - (-x)$
   A $5x$
   B $3x$
   C $-3x$
   D $-5x$
7. What is the value of the expression?

\[ 2 + 3^2 + |-4| \]

A 7
B 12
C 15
D 29

8. A rectangular pyramid is shown below.

Which shape could be the base of the pyramid?

A square
B pentagon
C triangle
D trapezoid
Gary and Thomas are playing a game with number cards. At the end of the game, Thomas still has 5 cards. If the value of each card is -50 points, how many points does Thomas have?

A  -250
B  -10
C  10
D  250

Find the value of $a$ in the equation below.

$3a + 2 = a - 6$

A  4
B  2
C  -2
D  -4
Appendix F

The Pilot Worksheets

(i) English version

(ii) Japanese version
Name...........................................................(optional)

Year Group..................... Age............... Male/Female
(please circle one)

This is a worksheet, not a test. You will be asked to complete it on your own and give in your completed sheet, but afterwards you will be able to ask your teacher about any of the questions.

Please work out the answers to these calculations, without using a calculator. It would be helpful if you could write down your workings-out in the box in the middle, writing your answer in the answer box on the right. If there are any questions that you cannot do, then just put a cross in the answer box.
<table>
<thead>
<tr>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 + 3 \times 4 + 5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4 + 10 \div 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{8 + 8}{4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2 + 3^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2 \times 3^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2 \times (3 + 4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4 + (2 \times 3)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{2 + 4^2}{(1 + 2)^2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In the last four questions, $a = 2$, $b = 3$, $c = 4$, and $d = 5$

Work out the value of the expression in each question:

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a + bc + d$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ab + cd$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a(b + c)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a(b + d)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c + a^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
名前:  
学年/年齢:  

これはテストではありません。問題は自分の力で解いてください。問題を解いている間は先生に質問をしないようにしてください。
以下の問題を電卓を使わず計算してください。答えだけでなく、途中の式も省略せずに書いてください。もし分からないときは何も書かずにそのままにしておいてください。

<table>
<thead>
<tr>
<th>問題</th>
<th>計算</th>
<th>答</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 + 3 × 4 + 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 + 10 ÷ 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 + 8/4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 + 3²</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 × 3²</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 × (3 + 4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 + (2 × 3)²</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 + 4²/(1 + 2)²</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

417
$a = 2$, $b = 3$, $c = 4$, $d = 5$ を次の四つの式に代入して計算してください。答えだけでなく、途中の式も省略せずに書いてください。もし分からないときは何も書かずにそのままにしておいてください。

<table>
<thead>
<tr>
<th>問題</th>
<th>計算</th>
<th>答え</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a + bc + d$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ab + cd$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a(b + c)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{a(b + d)}{c + a^2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ありがとうございました

数学教育センター　プリマス大学
Appendix G

The Main Study Worksheets

(i) English version
(ii) Dutch version
WORKSHEET 1  NO CALCULATOR

Name...........................................................................(optional)

Year Group.................. Age.................. Male/Female
(please circle one)

Pupil number......................

This is a worksheet, not a test. You will be asked to complete it on your own and give in your completed sheet, but afterwards you will be able to ask your teacher about any of the questions.

Please work out the answers to these calculations, without using a calculator. It would be helpful if you could write down your workings-out in the box in the middle, writing your answer in the answer box on the right. If there are any questions that you cannot do, then just put a cross in the answer box.

Please turn over
<table>
<thead>
<tr>
<th></th>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2 + 3 \times 4 + 5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$4 + 10 \div 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\frac{8 + 8}{4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$3 \times (2 + 4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$2 + 3^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$2 \times (-3)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$2 \times (6 - 4)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$4 + (2 \times 3)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$\frac{8 + 6}{3 + 2^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$\frac{2 + 4^2}{(1 + 2)^2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In the last four questions, \( a = 2 \)  \( b = 3 \)  \( c = 4 \)  and \( d = 5 \)

Work out the value of the expression in each question

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 ( a + bc + d )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 ( ab + cd )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13 ( a( b + c) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14 ( \frac{a(b + d)}{c + a^2} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
WORKSHEET 2 WITH CALCULATOR

Name..................................................(optional)

Year Group.................... Age.................. Male/Female
(please circle one)

Pupil Number..............................

This is a worksheet, not a test. You will be asked to
complete it on your own and give in your completed sheet,
but afterwards you will be able to ask your teacher about
any of the questions.

Please work out the answers to these calculations. You
may use the calculator provided, and when you are happy
with your answer you should write it clearly in the answer
box. There is space for any workings that you wish to
write down.

When you write your answers, please write down all the
numbers that are shown on the calculator display.

If there are any questions you cannot do, just put a cross
in the answer box.

Please turn over
<table>
<thead>
<tr>
<th></th>
<th>Question</th>
<th>Workings (If needed)</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.7 + 3.4 \times 4.5 + 5.9$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$4.45 + 10.8 \div 2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\frac{8.9 + 8.6}{4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$3.1 \times (2.3 + 4.6)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$2.6 + 3.2^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$2.6 \times (-3.7)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$2.47 \times (4.26 - 1.79)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$4.9 + (2.4 \times 3.6)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$\frac{8.2 + 6.3}{3.7 + 2.8^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$\frac{2.4 + 4.8^2}{(1.8 + 2.7)^2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In the last four questions,

\[ a = 2.4 \quad b = 3.2 \quad c = 4.6 \quad \text{and} \quad d = 5.8 \]

Work out the value of the expression in each question

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings (if needed)</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>( a + bc + d )</td>
<td></td>
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<tr>
<td>12</td>
<td>( ab + cd )</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>( a(b + c) )</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>( a(b + d) )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c + a^2 )</td>
<td></td>
</tr>
</tbody>
</table>
WERKBLAD 1  ZONDER REKENMACHINE

Naam.................................................................

Klas...................... Leeftijd...................... Jongen/Meisje

(omarkeer)

Leerlingnummer.................................

Dit is een werkblad, geen toets. Je wordt gevraagd het blad zelf te maken en deze in te leveren. Na afloop kun je je docent vragen stellen over de vragen.

Geef je antwoorden zonder de rekenmachine te gebruiken. Het zou helpen als je je uitwerking in het vak in het midden schrijft en je antwoord in het vak aan de rechterkant. Als er vragen zijn die je niet kunt maken, geef dan een kruis als antwoord.

Zie ommezijde
In de laatste vier vragen geldt dat, $a = 2 \ b = 3 \ c = 4$ en $d = 5$

Werk de waarde van elke expressie in elke vraag uit

<table>
<thead>
<tr>
<th>Vraag</th>
<th>Uitwerking</th>
<th>Antwoord</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>$a + bc + d$</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$ab + cd$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>$a(b + c)$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$\frac{a(b + d)}{c + a^2}$</td>
<td></td>
</tr>
</tbody>
</table>
WERKBLAD 2  MET REKENMACHINE

Naam..........................................................

Klas.............. Leeftijd............... Jongen/Meisje
(omcirkel)

Leerlingnummer.............................

Dit is een werkblad, geen toets. Je wordt gevraagd het blad zelf te maken en deze in te leveren. Na afloop kun je je docent vragen stellen over de vragen.

Werk de antwoorden van deze berekeningen uit. Je mag rekenmachine gebruiken, en als je tevreden bent met het antwoord, schrijf dit dan op in het antwoordvak. Er is ook ruimte voor uitwerkingen en berekeningen.

Wanneer je je antwoorden opschrijft, schrijf dan alle getallen op die zichtbaar zijn op het scherm van de rekenmachine.

Als er vragen zijn die je niet kunt maken, geef dan een kruis als antwoord.
In de laatste vier vragen geldt dat,
\[ a = 2.4 \quad b = 3.2 \quad c = 4.6 \quad \text{en} \quad d = 5.8 \]

Werk de waarde van elke expressie in elke vraag uit

<table>
<thead>
<tr>
<th>Vraag</th>
<th>Uitwerking (indien nodig)</th>
<th>Antwoord</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>( a + bc + d )</td>
<td></td>
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<tr>
<td>12</td>
<td>( ab + cd )</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>( a(b + c) )</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>( \frac{ab + d}{c + a^2} )</td>
<td></td>
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Appendix H

Results of the Pilot Study

(i) UK School X
(ii) Japan School Y
<table>
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</table>

Pilot Study Japan School Y
Appendix I

Results of the Main Study

(i) UK School A
(ii) UK School B
(iii) UK School C
(iv) The Netherlands School D
(v) US School E Class 1
(vi) US School E Class 2
## MAIN STUDY

### UK School A, Worksheet 1

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### MAIN STUDY

**UK School B**  **Worksheet 1**

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## MAIN STUDY

### UK School C  Worksheet 2

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n=27  Mean  9.93  Standard Deviation  2.22
Appendix J

Case Study Data

(i) Completed Pupil Worksheets
(ii) Interview Transcripts
WORKSHEET 1 NO CALCULATOR

Name: ...........................................(optional)

Year Group: 8LI Age: 13 (Male/Female)

Pupil number: 11

This is a worksheet, not a test. You will be asked to complete it on your own and give in your completed sheet, but afterwards you will be able to ask your teacher about any of the questions.

Please work out the answers to these calculations, without using a calculator. It would be helpful if you could write down your workings-out in the box in the middle, writing your answer in the answer box on the right. If there are any questions that you cannot do, then just put a cross in the answer box.

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<th>Answer</th>
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<tr>
<td>1</td>
<td>$2 + 3 \times 4 + 5$</td>
<td>$2+3=5 \times 4 = 20 + 5 = 25$</td>
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<tr>
<td>2</td>
<td>$4 + 10 \div 2$</td>
<td>$4+10=14 \div 2 = 7$</td>
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<td>3</td>
<td>$\frac{8 + 8}{4}$</td>
<td>$8+8=16 \div 4 = 4$</td>
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<tr>
<td>4</td>
<td>$3 \times (2 + 4)$</td>
<td>$2+4=6$ $3 \times 6 = 18$</td>
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<td>$2 + 3^2$</td>
<td>$3 \times 3 = 9$ $2+9=11$</td>
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<td>6</td>
<td>$2 \times (-3)^2$</td>
<td>$-3 \times -3 = 9$ $2 \times -9 = -9$</td>
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<td>7</td>
<td>$2 \times (6 - 4)$</td>
<td>$6 - 4 = 2$</td>
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<tr>
<td>8</td>
<td>$4 + (2 \times 3)^2$</td>
<td>$2 \times 3 = 6 \times 6 = 36$ $36 + 4 = 40$</td>
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<td>9</td>
<td>$\frac{8 + 6}{3 + 2^2}$</td>
<td>$8+6=14$ $2 \times 4 = 8$ $14-7=7$</td>
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<td>10</td>
<td>$\frac{2 + 4^2}{(1 + 2)^2}$</td>
<td>$2+16=18$ $4 \times 4 = 16$ $18=9=2$</td>
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In the last four questions, \( a = 2 \), \( b = 3 \), \( c = 4 \), and \( d = 5 \)

Work out the value of the expression in each question

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<th>Answer</th>
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| 11| \( a + bc + d \) | \( \begin{align*} 2 + 3 \cdot 4 + 5 \\
12 & 2 + 12 + 5 = 19 \end{align*} \) | 19     |
| 12| \( ab + cd \)  | \( \begin{align*} 2 \cdot 3 + 4 \cdot 5 \\
6 + 20 \end{align*} \) | 26     |
| 13| \( a(b + c) \) | \( \begin{align*} 3 + 4 \cdot 7 \\
8 \times 7 = 56 \end{align*} \) | 14     |
| 14| \( \frac{a(b + d)}{c + a^2} \) | \( \begin{align*} 2\times 8 = 16 \\
n + 4 + 8 \end{align*} \) | 2      |
WORKSHEET 2  WITH CALCULATOR

Name... .......................................................... (optional)

Year Group. 8L .......................... Age 13 .......................... (Male/Female)

Pupil Number 11 ..........................

This is a worksheet, not a test. You will be asked to complete it on your own and give in your completed sheet, but afterwards you will be able to ask your teacher about any of the questions.

Please work out the answers to these calculations. You may use the calculator provided, and when you are happy with your answer you should write it clearly in the answer box. There is space for any workings that you wish to write down.

When you write your answers, please write down all the numbers that are shown on the calculator display.

If there are any questions you cannot do, just put a cross in the answer box.

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<td>$4.45 + 10.8 \div 2$</td>
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<td>$3.1 \times (2.3 + 4.6)$</td>
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<td>$2.6 + 3.2^2$</td>
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</table>
In the last four questions,

\[ a = 2.4 \quad b = 3.2 \quad c = 4.6 \quad \text{and} \quad d = 5.8 \]

Work out the value of the expression in each question.

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<td>( 22.92 )</td>
</tr>
<tr>
<td>12 ( ab + cd )</td>
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<td>( 34.36 )</td>
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<tr>
<td>13 ( a(b + c) )</td>
<td></td>
<td>( 18.72 )</td>
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<tr>
<td>14 ( \frac{a(b + d)}{c + a^2} )</td>
<td>( 21.6 ) ( 10.36 )</td>
<td>( 31.96 )</td>
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</table>
WORKSHEET 1  NO CALCULATOR

Name. .................................. (optional)

Year Group. 8 .................. Age 13 .......... (Female)  

Pupil number. 8 ......................

This is a worksheet, not a test. You will be asked to complete it on your own and give in your completed sheet, but afterwards you will be able to ask your teacher about any of the questions.

Please work out the answers to these calculations, without using a calculator. It would be helpful if you could write down your workings-out in the box in the middle, writing your answer in the answer box on the right. If there are any questions that you cannot do, then just put a cross in the answer box.

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<td>$6 \times 6 = 36 + 4$</td>
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</tr>
<tr>
<td>9</td>
<td>$\frac{8 + 6}{3 + 2^2}$</td>
<td>$\frac{14}{(25 \times)}$</td>
<td>? \times m3</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{2 + 4^2}{(1 + 2)^2}$</td>
<td>$\frac{36}{9}$</td>
<td>4 \times m3</td>
</tr>
</tbody>
</table>
In the last four questions, $a = 2$, $b = 3$, $c = 4$, and $d = 5$

Work out the value of the expression in each question

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a + bc + d$</td>
<td>$2 + 7 + 5$</td>
<td>14</td>
</tr>
<tr>
<td>$ab + cd$</td>
<td>5 + 9</td>
<td>14</td>
</tr>
<tr>
<td>$a(b + c)$</td>
<td>$(7)$</td>
<td>14</td>
</tr>
<tr>
<td>$\frac{a(b + d)}{c + a^2}$</td>
<td>?</td>
<td>7</td>
</tr>
</tbody>
</table>
WORKSHEET 2 WITH CALCULATOR

Name. ......................................(optional)

Year Group.................. 8 ............... Age........ 13 ............... ♂ (please circle one)

Pupil Number.................. 8

This is a worksheet, not a test. You will be asked to complete it on your own and give in your completed sheet, but afterwards you will be able to ask your teacher about any of the questions.

Please work out the answers to these calculations. You may use the calculator provided, and when you are happy with your answer you should write it clearly in the answer box. There is space for any workings that you wish to write down.

When you write your answers, please write down all the numbers that are shown on the calculator display.

If there are any questions you cannot do, just put a cross in the answer box.

Please turn over
<table>
<thead>
<tr>
<th>Question</th>
<th>Workings (if needed)</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $2.7 + 3.4 \times 4.5 + 5.9$</td>
<td>$2.7 + 7.9 + 5.9$</td>
<td>$16.5$</td>
</tr>
<tr>
<td>2. $4.45 + 10.8 \div 2$</td>
<td>$4.45 + 5.4$</td>
<td>$9.85$</td>
</tr>
<tr>
<td>3. $\frac{8.9 + 8.6}{4}$</td>
<td>$8.9 + 2.15$</td>
<td>$11.05$</td>
</tr>
<tr>
<td>4. $3.1 \times (2.3 + 4.6)$</td>
<td>$3.1 \times 6.9$</td>
<td>$21.39$</td>
</tr>
<tr>
<td>5. $2.6 + 3.2^2$</td>
<td>$2.6 + 10.24$</td>
<td>$12.84$</td>
</tr>
<tr>
<td>6. $2.6 \times (-3.7)^2$</td>
<td>$2.6 \times 13.69$</td>
<td>$35.694$</td>
</tr>
<tr>
<td>7. $2.47 \times (4.26 - 1.79)$</td>
<td>$2.47 \times 2.47$</td>
<td>$6.109$</td>
</tr>
<tr>
<td>8. $4.9 + (2.4 \times 3.6)^2$</td>
<td>$4.9 + 74.6496$</td>
<td>$79.5496$</td>
</tr>
<tr>
<td>9. $\frac{8.2 + 6.3}{3.7 + 2.8^2}$</td>
<td>$\frac{14.5}{11.54}$</td>
<td>$1.256491$</td>
</tr>
<tr>
<td>10. $\frac{2.4 + 4.8^2}{(1.8 + 2.7)^2}$</td>
<td>$\frac{25.44}{25.25}$</td>
<td>$1.2856296$</td>
</tr>
</tbody>
</table>
In the last four questions,

\[ a = 2.4 \quad b = 3.2 \quad c = 4.6 \quad \text{and} \quad d = 5.8 \]

Work out the value of the expression in each question

<table>
<thead>
<tr>
<th>Question</th>
<th>Workings (if needed)</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 ( a + bc + d )</td>
<td>( a + 7.8 + 5.8 )</td>
<td>16 * C7</td>
</tr>
<tr>
<td>12 ( ab + cd )</td>
<td>5.6 + 10.4</td>
<td>16 * C7</td>
</tr>
<tr>
<td>13 ( a(b + c) )</td>
<td>( 2 \times (7.8) )</td>
<td>18.72 ✓</td>
</tr>
<tr>
<td>14 ( \frac{a(b + d)}{c + a^2} )</td>
<td>( \frac{18.72}{10.56} ) ( \approx 1.80 )</td>
<td>0.7807 X C7</td>
</tr>
</tbody>
</table>
Transcription Tape for Carrie Headlam

School B

CARRIE: This is School B. This is pupil number 11.
Now what I am interested in is these first three questions, you have got them all wrong. You have shown all your workings nice and clearly so I have been trying to work out why you got them wrong. So can you just have a look at your workings here on question 1 because you have written it out nice and clearly what you did. I wonder if you can tell just by looking at it what you think you may have done wrong?
PUPIL: I did the 2 plus the 3 which equalled 5 then multiplied that by 4 which equalled 20 then just added the 5 which makes 25.
CARRIE: And that is how you got 25? Can you see why that isn't right? Because it looks as if it should be, doesn't it?
PUPIL Do you have to multiply the 3 and the 4 first then add the 2 and the 5?
CARRIE Yes, you are quite right. So what should you have got?
PUPIL 19
CARRIE Yes, the correct answer is 19. Can you remember now doing some work on that?
PUPIL Yes.
CARRIE How could you remember that? Is there a way you have been taught that you could remember what to do first?
PUPIL No. Not really.
CARRIE Have you heard of BIDMAS or BODMAS?
PUPIL No.
CARRIE Well it is either one of these ways of remembering which way round to do things. So you know that you should have done the multiplication first?
PUPIL Yes.
CARRIE It is interesting because if you had tapped that sum into a scientific calculator or the calculator I gave you just as it is you would have got the correct answer which is 19. If you had tapped it into a normal calculator you would have got the wrong answer of 25. You probably made the same sort of mistake in question 2 so if you have a look at number 2 again you might be able to work out why you got it wrong and what the correct answer should have been.
PUPIL Well I was supposed to divide 10 by 2 which is 5 and then add 4 which is 9.
CARRIE: Yes, that is right. So you should have done the division first, even though it looked like it should be done the other way round, that is quite right. So again can you look at question 3 and tell me what you should have done there?

PUPIL: I should have divided 8 by 4 which is 2 then added it onto 8 which makes 10.

CARRIE: So you have got the correct answer now. Are you happy now that even though you got them wrong at the time you know what you did wrong and have got the correct answers now?

PUPIL: Yes,

CARRIE: Good. Now you got most of the rest of the questions correct. What were you thinking about when you did question number 4?

PUPIL: Because it was in brackets I remembered that you always work out the brackets first.

CARRIE: So you know you do anything in brackets first? How do you remember that or do you just know it?

PUPIL: I just know it.

CARRIE: Good. Now I have a question mark here, I was trying to work out what you were doing here because this was wrong and I wasn’t sure about your working. I think there are 2 mistakes here. What should minus 3 times minus 3 be?

PUPIL: 9 because two negatives multiplied make a positive.

CARRIE: Yes, that is right. So that really when you did that first bit, three multiplied by itself you should have got 9 shouldn’t you? I think you have done things a bit funny here. But even if that had been right you said 2 times minus 9 was 9. So even you had got that you should have got minus 18 shouldn’t you? I think there were a couple of things gone funny there. What should the answer have been?

PUPIL: It should have been 18.

CARRIE: So you know what you should have done there. In fact the rest of it was completely correct. Were you happy doing the questions with algebra?

PUPIL: Yes.

CARRIE: Were you using the same rules as before?

PUPIL: Yes. When you see ab together like that it means multiplied.

CARRIE: So when you see ab it means multiplied and even though you didn’t get them correct there you did actually know that you had to do the multiplying first? So you did get
the rules right when you were doing the algebra? Did you find the algebra easier than the numbers?

PUPIL      Yes and I think at the start I rushed a bit.

CARRIE    So you just weren't thinking? Looking back at it you should really have got them all right then?

PUPIL        Yes

CARRIE    Yes, well that is important if you got the questions wrong you know what you are doing wrong. Again with the calculator paper you did very well and you got all of these questions correct but you did get this one wrong. Now there is not very much working to see there but one of things I did do if you remember was to record your calculator data so I could see what you did. I have a calculator here and I just wonder if you could do it again and I can look at what you got. I know you are not as familiar with these calculators but do you want to have a go at this?..... So you have worked out what is in the bracket... Then you have squared it.

PUPIL        Then you add it up.

CARRIE    Do you want to jot that down - oh that is what you have done. Good I can see now why you have written that down.

PUPIL        Then I divide........

CARRIE    You can use the arrow and overwrite it...... You can just type that again and that will overwrite it. So that is a different answer to what you had got before. Let us have a look and see what you got wrong. You did the bottom correctly. You were quite right in that you worked out what was in the bracket and then squared it. You were thinking about what to do on the top there. You did the 2.4 plus 4.8.

PUPIL        I think I squared the 4.8 instead of the answer.

CARRIE    That is what you should do, the squaring first. Do you just want to have another go at doing the top because you have done the bottom correctly? The bottom of the fraction you have got correct so really you put it in again the other way around. Start with the 4.8. I think that is what you got last time isn't it? What we need to do is square the 4.8 first. So we were talking about the order you do things.

PUPIL        Then 20 =

CARRIE    That is what you should have on the top of the fraction then divide it by 20. That would have been the right answer then. There were lots of people who got that wrong. You are happy that you always do brackets first then? You know that. When you have an addition and when you have a power you do the power first, that is the rule, you do not necessarily work from left to right. OK, I think that is basically all I wanted to ask you about. Basically you did it very well and it was interesting to see where you did make mistakes. The first few I think was probably because you weren't concentrating. If you had to do something like that again do you feel confident you would be able to get them all right?
PUPIL: Probably most of them.
CARRIE: Is this something you need to revise or do you know it now?
PUPIL: Yes, just common sense. Just think a bit harder.
CARRIE: Is there anything else you need to ask me?
PUPIL: No, not really.
CARRIE: Well thank you very much.
Transcription Tape for Carrie Headlam

School B Pupil 8

CARRIE OK. This is pupil Number 8 in School B. Right, let's look at the first two. You got those right. Can you tell me what you were thinking about when you were doing those.

PUPIL I was thinking of BIDMAS.

CARRIE Right. What do you remember?

PUPIL Brackets, individual, divided, multiply, addition and subtraction.

CARRIE That was nearly right. It was I that I was asking about. I was writing that down because someone else said that. Do you know what it is? Can you think now?

PUPIL I don't know what it is.

CARRIE Well if I tell you now that might help while we are going through these. It actually means index.

PUPIL That was the one.

CARRIE That might help. Do you know what index means? What is that telling you to do?

PUPIL Look it through and put it in the correct order.

CARRIE Can you look through the worksheet and see if you can find any number you might describe as an index?

PUPIL 1, right at the beginning of the number line.

CARRIE So you are thinking of it in terms of the number line?

PUPIL Yes.

CARRIE Actually in this meaning here, index means power. Do you know what power means?

PUPIL No.

CARRIE I will tell you because it might help in our discussions. That little number there for example. When you see 3 with a little 2...

PUPIL Oh, to the power of 2?
CARRIE: Yes, that is right and another word for the 2 is index. Another way of saying that is 2 squared. So when we are talking about index we are talking about the squared and the cubed numbers. That is what we are really getting at. So that is something you were not too sure about?

PUPIL: No.

CARRIE: Actually, interestingly you got that question right so you knew how to do it in the right order even though you weren't thinking it through using BIDMAS. OK, let us start from the beginning. You got that right so just explain to me what you did.

PUPIL: I know that multiply comes before addition in BIDMAS so I timesed 3 by 4 then I added the 2 and 5 then added the 12.

CARRIE: So you got the correct answer of 19. You did the same thing here. You did the 10 divided by 2 first and then added the 4.

PUPIL: Yes, because divided is before multiply.

CARRIE: Good, so you are remembering BIDMAS. You didn't write it down but you are remembering that. Now this one went wrong. Let's just look at question 3 and see if we can figure out what went wrong. Now, I am quite interested in why this went wrong. Have a little look at what you did and see if you can tell me why it went wrong.

PUPIL: Yes, I know what went wrong. I divided 4 by 8 but I forgot to divide the top number by the bottom number so it should have been 2.

CARRIE: Yes, so you should have got that right really. You did it in the right order. You used BIDMAS so you did that bit first. You just did the division wrong. I wonder why you did that. Can you think why you did that?

PUPIL: Because I thought it was the bottom number divided by the top number. Only when I was doing the test. Afterwards I realised.

CARRIE: So that was thinking about fractions and what you need to do. But you reckon you know that really?

PUPIL: Yes.

CARRIE: That is fine so that should have been right. Now these are all right. Now here you are doing some funny things. I can see what you did here that was wrong. On question 6 you are using BIDMAS correctly so you have done what you need to do first. You did minus 3 times minus 3.

PUPIL: I got minus 9. But I know that is wrong and it should have been 9.

CARRIE: So you know that is wrong and that should have been 9. Why is that?

PUPIL: Because when numbers are below zero and they are multiplied together they make a number above zero.
CARRIE: Yes. Good. You remember that absolutely right. So how you do you remember that? Do you just try and remember?

PUPIL: I try.

CARRIE: So, all of these things are about trying to remember?

PUPIL: Yes.

CARRIE: So you just remembered wrong. So actually I am a bit intrigued here. You then did something else funny because although you got that wrong you knew you had to multiply by 2 so what should you have got then if you did minus 9 by 2?

PUPIL: I should have got minus 18 instead of minus 4.5.

CARRIE: It would still have been wrong but that is what you should have got. So how did you get minus 4.5?

PUPIL: Well after I timesed minus 3 by minus 3 and got minus 9. I timesed 2. I remembered with minus numbers if you minus them they go up.

CARRIE: So you thought you ought to have a smaller number so you divided by 2 instead?

PUPIL: Yes.

CARRIE: So, it was really about being confused with negative numbers?

PUPIL: Yes.

CARRIE: So can you see if you did them again now could you do them correctly?

PUPIL: Yes.

CARRIE: OK. Well I think I have figured out what you were doing on question 9 but I wonder if you can figure that out? I have been putting in codes here to help me figure out what you were doing.

PUPIL: I know what I did. I did 3 add 2 and then I divided by 2 then I timesed it by 5.

CARRIE: OK. What should you have done?

PUPIL: 2 times 2 add 3.

CARRIE: Yes. So it is really this index thing isn’t it. There are no brackets so the next thing is indexing. You are quite right, what you should have done is 2 squared then added 3. What should you have got?

PUPIL: I should have got 7.
CARRIE  Good. You would then have got 14 divided by 7.

PUPIL    It is 2.

CARRIE  That would have been nice and easy to do. You have put a big question mark because that one wasn’t easy to do. OK. Having done that you can probably look at question 10 and tell me what you should have done there.

PUPIL    Yes. I did 2 and 4 makes 6 and then timesed it by 6.

CARRIE  Yes, you kind of put a bracket in there didn’t you? You were pretending there was a bracket around the 2 and 4 and added them together. You saw some brackets that weren’t there. It was the same mistake. In fact you did the bottom correctly because there were brackets. You did the 1 and 2 then squared it. You got the correct answer of 9 then. So it was the same mistake which is why I put the same code down. That is what I thought. OK, good. Although there were a few mistakes you know now what you did wrong. Finally let us just have a look at these two. I think I know what you were doing wrong here. This was algebra so you were having to put numbers in. You were right, a is 2, b is 3 and c is 4 and you have put 7.

PUPIL    I forgot when you put them together you times the 3 and the 4 together.

CARRIE  Yes, you added them. If I had wanted you to add them I would have put an addition sign there. What should you have got?

PUPIL    I should have got 12.

CARRIE  Yes, then you would have added the a and then the d. So, again it is just remembering that. I think you did the same thing wrong here didn’t you? You added a and b.

PUPIL    Yes, I should have timesed a and b together.

CARRIE  Yes. Now this one you just put a big question mark. I don’t think we will bother with question 14 because that is really quite difficult. We will leave that for now. Actually although there were a few mistakes it is just remembering rules. Now on this one you did much better. You got 10 out of 14 on the calculator paper…….

END OF SIDE ONE.
Appendix K

**Ethical Documentation**

(i) Application for ethical approval
(ii) Letters to parents
(iii) Letter to teachers
(iv) Outline interview questions
Faculty of Technology Ethics Committee (FTEC)

APPLICATION FOR ETHICAL APPROVAL OF RESEARCH OR OTHER PROJECTS INVOLVING HUMAN PARTICIPANTS

In completing this form reference should be made to Faculty Procedures and the University of Plymouth document Ethical Principles for Research Involving Human Participants

Title of Research:

1. Nature of Approval Sought (please tick relevant box)
   RESEARCH [X] OTHER PROJECT: [ ]
   Please tick which category:
   - Funded Research Project
   - MPhil/PhD Project [X]
   - Other Project (please specify below): [ ]

2. Name of Principal Investigator or Project Leader or Director of Studies*:
   (Name of Director of Studies is required where Principal Investigator is a postgraduate student)

   Mrs Caroline Headlam
   Contact Details**: Room 115 School of Mathematics and Statistics ext 2782
   carrie.headlam@plymouth.ac.uk

   *Principal Investigators, Directors of Studies and Project Leaders are responsible for ensuring that all staff employed on research or projects (including research assistants, technicians and clerical staff) act in accordance with the University’s ethical principles in the design and conduct of the research or project described in this proposal and any conditions attached to its approval.

   **Please indicate department of each named individual, including collaborators external to the Faculty.

3. Funding Body and Duration of Project/Programme with Dates:
   Funded by University of Plymouth. Start date 1st October 2006

4. Aims and Objectives of Research Project/Programme:
   To investigate children's understanding of the Order of Operations

5. Brief Description of Research Methods and Procedures:
   Please specify subject populations and recruitment method. Please indicate also any ethically sensitive aspects of the methods. Continue on an attached sheet if required.
   I have completed my pilot study and now intend to proceed with my main study.
   I have written two worksheets (attached) each consisting of twelve questions to be
attempted by class groups of secondary school children from years 8 to 11. (Ages 13 – 16) Worksheet 1 is to be done without a calculator. For Worksheet 2 a calculator may be used. This will be provided by me and will have software installed which will enable me to record the calculator keystrokes. Since I wish to make international comparisons I intend to give the worksheets to classes of children in the UK and also to classes of children in Japan. My worksheets will be translated by Dr Taro Fujita, my third supervisor, and he will make contact with schools in Japan. I will also seek the cooperation of colleagues in other countries including the USA. We will each contact schools that we already work with and seek the cooperation of teachers in the school to administer the worksheets with one of their classes. Where possible I will carry out short interviews with the class teachers and with some of the pupils in the class.

6. Ethical Protocol:
   Please indicate how you will ensure this research conforms with each clause of the University of Plymouth’s Principles for Research Involving Human Participants. Please include a statement which addresses each of the ethical principles set out below.

(a) Informed Consent:
The pupils will be asked to complete the worksheets as a part of their mathematics lesson, and the teachers will incorporate it into their lesson. The pupils will be told that their work will be marked by me, and they will not be required to put their name on it, although they may wish to if they would like feedback, in which case their work will be returned to their teacher. If the children do not wish their work to be given to me they may ask their teacher to retain their worksheet in school. The pupils will be asked if they would talk to me about their work and I will ask them if I may record the conversation. They will be given the opportunity to decline or to withdraw at any stage.

(b) Openness and Honesty:
The class teachers will be informed about the purpose of the study and will offer the opportunity to receive feedback with the marked worksheets. The pupils will be informed that their work will be used by me, marked by me, and returned to their teacher. They will be told that they can ask the class teacher afterwards about any of the questions they were given. The pupils will be informed about the key-record software and they will have the opportunity to decline to use the calculator or to have the data erased if they wish. The parents will be sent an information sheet regarding the interviews and asked to write to the class teacher if they do not wish their child to be interviewed. The teacher and pupils will be informed that their names and the name of the school will not be named in my study. The teacher will be told that I will provide them with a copy of my study if they would like me to.

Note that deception is permissible only where it can be shown that all three conditions specified in Section 2 of the University of Plymouth's Ethical Principles have been met in full. Proposers are required to provide a detailed justification and to supply the names of two independent assessors whom the Sub-Committee can approach for advice.

(c) Right to Withdraw: The worksheet will form part of the pupils' mathematics lesson but they will be told that they have the right to retain their work in school if they wish. The pupils will have the right to decline to be interviewed. They will be asked if they mind the conversation being taped, and they will have the right to decline or withdraw at any stage.

(d) Protection From Harm: The teachers and the children will not be exposed to any harm. The questions on the worksheet will all be based upon work that the children have already done in school and will be similar to the work that they do in school, but they will also be reassured that they may simply leave any questions that they cannot do. During the interviews I will not necessarily be asking each child exactly the same questions, as the questions will arise from their work, but I will ensure that I talk to
each child for about the same length of time and ask about the same number of questions.

(e) Debriefing: The teachers will be sent copies of the marked worksheets and will be sent a copy of the completed project. The pupils will be given the opportunity to ask about any of the questions on the worksheets after they have completed them.

(f) Confidentiality: The schools, the teachers and the children will not be named in the project. The children will not be required to put their names on their worksheets and they will be reassured that their work will only be seen by me, their teacher, and possibly by other researchers such as my supervisor. I will provide each child with a number which they will be asked to write on each worksheet and which will correspond to a number on the calculator, so that all the work can be investigated together.

(g) Professional Bodies Whose Ethical Policies Apply to this Research or Project:

<table>
<thead>
<tr>
<th>Principal Investigator:</th>
<th>Name</th>
<th>Email (s)</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Caroline Headlam</td>
<td><a href="mailto:carrie.headlam@plymouth.ac.uk">carrie.headlam@plymouth.ac.uk</a></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Completed Forms should be forwarded BY EMAIL to the Secretary. These will be forwarded to members of the Committee. Responses will normally take two weeks to process so please ensure applications are submitted in sufficient time for approval to be considered before the start of the proposed research or project. You will be notified by the Ethics Committee once your application has been considered and approved.
Ethical principles for research involving human participants

1. Informed consent

The researcher should, where possible, inform potential participants in advance of any features of the research that might reasonably be expected to influence their willingness to take part in the study.

Where the research topic is sensitive, the ethical protocol should include verbatim instructions for the informed consent procedure and consent should be obtained in writing.

Where children are concerned, informed consent may be obtained from parents or teachers acting in loco parentis, or from the children themselves if they are of sufficient understanding. However, where the topic of research is sensitive, written informed consent should be obtained from individual parents.

2. Openness and honesty

So far as possible, researchers should be open and honest about the research, its purpose and application.

Some types of research appear to require deception in order to achieve their scientific purpose. Deception will be approved in experimental procedures only if the following conditions are met:

a. Deception is completely unavoidable if the purpose of the research is to be achieved.

b. The research objective has strong scientific merit.

c. Any potential harm arising from the proposed deception can be effectively neutralised or reversed by the proposed debriefing procedures (see section 5).

Failing to inform participants of the specific purpose of the study at the outset is not normally considered to be deception, provided that adequate informed consent and debriefing procedures are proposed.

Covert observation should be resorted to only where it is impossible to use other methods to obtain essential data. Ideally, where informed consent has not been obtained prior to the research it should be obtained post hoc.

3. Right to withdraw

Where possible, participants should be informed at the outset of the study that they have the right to withdraw at any time without penalty.

In the case of children, those acting in loco parentis or the children themselves if of sufficient understanding, shall be informed of the right to withdraw from participation in the study.

4. Protection from Harm
Researchers must endeavour to protect participants from physical and psychological harm at all times during the investigation.

Note that where stressful or hazardous procedures are concerned, obtaining informed consent (1) whilst essential, does not absolve the researcher from responsibility for protecting the participant. In such cases, the ethical protocol must specify the means by which the participant will be protected, e.g. by the availability of qualified medical assistance.

Where physical or mental harm nevertheless does result from research procedure, investigators are obliged to take action to remedy the problems created:

5. Debriefing

Researchers should, where possible, provide an account of the purpose of the study as well as its procedures. If this is not possible at the outset, then ideally it should be provided on completion of the study.

6. Confidentiality

Except with the consent of the participant, researchers are required to ensure confidentiality of the participant’s identity and data throughout the conduct and reporting of the research.

Ethical protocols may need to specify procedures for how this will be achieved. For example, transcriptions of the interviews may be encoded by the secretary so that no written record of the participant’s name and data exist side by side. Where records are held on computer, the Data Protection Act also applies.

7. Ethical principles of professional bodies

This set of principles is generic and not exhaustive of considerations which apply in all disciplines. Where relevant professional bodies have published their own guidelines and principles, these must be followed and the current principles interpreted and extended as necessary in this context.
Dear Parent/Carer,

I am a lecturer in Mathematics Education at the University of Plymouth and I am carrying out a research project with the cooperation of a number of schools internationally in order to investigate children's ways of working in certain aspects of arithmetic and algebra.

I will be spending a short time in your child's school working with the mathematics staff in their lessons. During this time I will be asking the children to complete two short worksheets and I will also be asking some of the children to talk to me afterwards about how they worked out the answers. I will ask the children if I may record our conversations. The children will not be put under any pressure to agree to this, and they may opt out at any time. The questions will all be based upon work which has already been covered in mathematics lessons this year.

When I write up my study I will not refer to any of the schools, teachers or pupils by name.

If you do not wish your child to participate in this study would you please put this in writing to your child's mathematics teacher using the tear-off slip below.

Yours sincerely,

---

I do not wish my child to participate in the mathematics research study outlined in the letter from Mrs Headlam of the University of Plymouth.

Child's name: ..............................................................

Signature of parent/carer: ...................................................

This slip may be returned to your child's mathematics teacher.
Department of Computing and Mathematics  
University of Plymouth  
Devon. PL4 8AA.  
United Kingdom.

Dear Parent/Guardian,

I am a lecturer in Mathematics Education at the University of Plymouth, UK, and I am conducting a PhD research study into the way in which children go about certain types of arithmetic calculations, both with and without calculators. This is an international study and I am hoping to include children from a number of different countries in order to compare their approaches.

I would like to ask some students in the Williamsville School District to complete two worksheets within one of their mathematics lessons. This will be supervised by a mathematics teacher and the work will be sent to me to score and to make comments. I will return the work so that the students may see how they did, and they will have the opportunity to ask their mathematics teacher any questions about the work. All student responses will be kept confidential. Student names will only be recorded on the worksheets for purposes of providing the mathematics teacher and student with feedback. When I record and analyze the results all students will be assigned a coded number which cannot be ‘traced back’ to the student or serve as an identifier.

The questions will all be on work that the students have done already, and they will be reassured that this is not a test. The students will be given the right to withdraw if they do not wish to participate.

The data collected will be used to further my knowledge and to explore possible implications for the improvement of mathematics education. When my thesis is published I will not be referring to any student or teacher or school by name and I will make my thesis available to all schools who participated.

In order for your child to take part in my research project, the permission slip below must be signed and returned to Mr. Chris McGinley, Instructional Specialist for Mathematics in the Mathematics Department of Williamsville North High School. Returned consent forms will be kept in a secure, confidential location and will not be available to anyone but myself and school district administrators. If you have any questions relating to this study you may contact by email at carrie.headlam@plymouth.ac.uk.

Sincerely,
Carrie Headlam

I give my permission for __________________________ to take part in this research project.

Student Name

Student Signature __________________________ Date _____________

Parent/Guardian Signature __________________________ Date _____________
Carrie Headlam  
School of Mathematics and Statistics  
University of Plymouth  
Drake Circus  
Plymouth  
Devon. PL4 8AA.  
UK  
Tel +44(0)1752 232782

Dear

Order of Operations: Research study

I am conducting a research project in which I am investigating childrens’ understanding of the order of operations, with the aim of carrying out international comparisons and reflecting upon the teaching methods used in different countries.

I would be very grateful for your help with my study.

I have written two short worksheets, each consisting of twelve questions. They could be given to any class who have been introduced to the concept of order of operations and who have been taught how to substitute letters for numbers in algebraic expressions. I would anticipate that this could involve age groups from year 8 through to year 11. Each worksheet is likely to take a maximum of 20 minutes to complete.

I would like the pupils to have a go at as many of the questions as they can, and they should be encouraged to jot down their workings in the spaces provided. They should do this on their own in “test” conditions, and give in their own attempts, but if you would like to go through the questions with them afterwards then that would be fine.

The first worksheet is to be completed without a calculator. For the second worksheet a calculator may be used and I would like the pupils to use a calculator provided by me, which will be running a piece of software which will capture the pupils’ keystrokes. This will enable me to replay and observe their keystrokes.

The pupils are not required to write their names on the papers, but they may wish to do so if they would like feedback. They have the right to withdraw from the exercise if they wish. They should be told that the key-record software is on their calculators and they may request that the data be deleted afterwards if they wish.
The questions that I ask the pupils will depend upon their written responses on the worksheets. These are examples of the type of questions I will ask:

Were there questions that you found easy?

Were there questions that you found difficult?

Why did you find this question difficult? Can you explain which part of it you were not sure about?

Can you talk to me about the working-out that you wrote here?

What do the brackets mean?

What does this number mean? (referring to an index number)

Which worksheet did you find easier? Why?

Did you feel more or less confident when you were allowed to use a calculator? Why?
1. Please tell me about how you introduce this topic:
   - What teaching strategies do you use?
   - What resources do you use?

2. Do you refer to a mnemonic such as BODMAS?

3. Please outline the ways in which you reinforce and consolidate this topic

4. How and when do you introduce the use of a calculator in performing complex calculations:
   - Basic or scientific
   - Use of brackets? Memories? Power key?

5. In your experience, what difficulties do you think that pupils have with this topic?

6. Any other comments?
Some initial findings from a study of children’s understanding of the Order of Operations

Carrie Headlam and Ted Graham
University of Plymouth

This paper presents some of the initial findings of a study into the strategies used by children to solve arithmetic and algebraic problems requiring the appropriate use of the order of arithmetic operations. The research has utilised graphics calculators which have been programmed with Key Recorder Software as a data collection tool. This has enabled the researchers to analyse the children’s approaches to some of the questions posed by observing their calculator keystrokes. Interviews with both teachers and pupils will be used to link the pupils’ strategies with the teaching methods used, and an initial analysis of observed misconceptions has been carried out. Initially this study has involved children in the UK and in Japan, where teaching methods differ substantially.

Introduction

The principle of the Order of Operations is a cornerstone of the understanding of arithmetic. It is necessary in order to correctly perform arithmetic calculations and it is also an essential prerequisite to the beginnings of the understanding of algebraic structure and the ability to understand and apply the principles of algebraic convention correctly.

More fundamentally, it could be argued that an acknowledgment of the need for a convention in arithmetic is an important step in the development of an appreciation of mathematical convention in other areas of mathematics and indeed to the sense of learning the language of mathematics, where standard rules are necessary in order to assist in clear communication.

In considering the ways in which algebra is developed in different countries, the relationship between algebra and arithmetic is usually characterised by the definition of algebra as generalised arithmetic. Thus the view of ‘arithmetic then algebra’ dominates school curricula in most countries. The reason for this, according to Lins and Kaput (2004), can be found in the strong dominance of Piagetian constructivism. As algebra would require formal thinking, while arithmetic would not, and as formal thinking would correspond to a later developmental stage, algebra should come later than arithmetic (page 50). This is seen in the work of Kuchemann in Hart (ed) (1981) for the Concepts in Secondary Mathematics and Science (CSMS) project who combined the view of algebra as generalised arithmetic with the Piagetian developmental view. Lins and Kaput (2004) argue that the most visible result of Kuchemann’s work is a reported link between different uses of letters in ‘generalised arithmetic’ and Piaget’s levels of intellectual development (page 50).

Hewitt (2003) considered students’ reading of formal algebraic notation and he observed that many errors made by students could be accounted for by the strict left-to-right reading of formally written arithmetic statements. He also considered how students read word statements, acknowledging that expressing non left-to-right order in written words can be problematic as well since words do not possess a set of notational conventions, such as brackets (page 34).

In the National Strategies Secondary Mathematics Exemplification (DCSF, 2008) the learning objective relating to this states that pupils should be taught to: Use the order of
operations, including brackets. (Page 86) One example of a learning outcome is that a year 7 student should be able to perform the calculation
\[ \frac{1}{2} - \frac{1}{3} \]
either mentally or using jottings. This objective is also linked to calculator methods, with the expectation that a pupil should be taught to carry out more complex calculations using the facilities on a calculator (page 108) and with the order of algebraic operations, where pupils are expected to understand that algebraic operations follow the same conventions and order as arithmetic operations. (Page 114) The exemplification makes it clear that pupils are expected to be able to use a scientific calculator efficiently when evaluating more complex mixed operations.

The methods for teaching this topic can vary, but one common theme in some countries is to use a mnemonic to aid the memorisation of the order of operations. In the UK this is commonly BIDMAS or BODMAS:

- Brackets
- Index
- Division
- Multiplication
- Addition
- Subtraction

In the USA the mnemonic PEMDAS is commonly used:

- Parentheses
- Exponents
- Multiplication
- Division
- Addition
- Subtraction

Clearly this may have its uses in remembering the "rule" once the concept has been understood, but it is the clear understanding of the underlying principle and conventions that enable it to be put into practice. This includes the understanding of index notation and the recognition of a fraction for division.

Thus it is far from merely being a case of learning a mnemonic; a sound understanding of mathematical notation and structure is required in order to carry out a calculation of the type given in the National Strategies Mathematics Exemplification. It is this deep understanding that lays the foundations for an understanding of algebraic structure.

It is interesting to note that the use of mnemonics is not referred to at all in the National Strategies documentation, and yet many text books and other resources used in the UK and in the USA encourage it.

In contrast, from conversations with Japanese teachers it would seem that mnemonics are never used in Japan. Indeed in the Japan National Mathematics Program (2000) the order of operations is not specifically referred to at all. The teaching methods are very didactic with a large emphasis on whole-class teaching and repetition of questions, focusing on algebraic structure.

The study: Context and Methods

The primary aim of this study is to examine the ways that pupils perform calculations which require the correct use of the order of operations and to study the misconceptions that may arise. One tool that will be utilised in order to carry out the research will be a piece of software that was developed as a research tool by Texas Instruments in conjunction with the University of Plymouth. This software is called the Key Recorder and can be loaded onto the more recent models of the TI graphics calculator. It has been used as a data collection tool in a small number of research projects (For example: Graham, Headlam, Honey, Sharp and Smith, (2003); Berry, Graham and Smith (2005); Smith (2005) Berry, Graham and Smith (2005); Berry, Graham and Smith (2006); Sheryn (2005a); Sheryn (2006b); Graham, Headlam, Sharp and Watson (2007)).

This study involves classes of children who have been taught about the Order of Operations and who would therefore be expected to be able to perform calculations based
upon these principles. These children were in the age range 12 - 14 (Years 8 and 9 in UK schools). The children complete worksheets of questions involving a variety of calculations, some with and some without a calculator. For the calculator-based questions the children are provided with a TI-84 graphics calculator which has the Key Record software running.

When the children's work has been initially analysed, some of the children are then interviewed in order to follow up and pursue questions that arise.

**Initial Findings**

A pilot study was carried out in the UK and in Japan. In each country one class of students was involved. In the UK this was a class of 20 middle ability students in Year 8 (age 13). The Japanese class consisted of 33 mixed ability students aged 14.

Both classes had been taught the principles of the order of operations as part of their scheme of work, and had also been taught simple algebraic conventions, including substitution of letters for numbers in algebraic expressions.

In the pilot study the graphics calculators were not used; the children were given one worksheet to complete without using a calculator. As a result of this study the worksheets were adapted and a second worksheet produced. The second worksheet contained questions which were identical in structure to those in the first worksheet but involving decimal numbers which would encourage the use of a calculator. The children were given a graphics calculator with the Key Record Software running which they were asked to use when completing the second worksheet. The main study has now been carried out in a further two classes in UK schools, both middle ability Year 8 classes. The children completed both worksheets and afterwards their worksheets were analysed alongside the Key Record data. Some children were then interviewed and the teachers were also interviewed.

From the pilot study it was interesting to investigate the questions that the Japanese children got wrong, and to analyse their ways of working. There was a general tendency to treat all the questions as algebraic, even though they were mainly numerical. The calculation that was answered incorrectly by most Japanese pupils was question 10:

\[
2 - 4 \div (1 - 3)^2
\]

the calculation \((1 + 2)^2\) was in many cases calculated by expanding the brackets first:

Figure 1 Examples of three Japanese pupils’ work on question 10

\[
\begin{align*}
2 \times 4^2 & \quad \text{\underline{(1 + 2)^2}} \\
2 \times 4 & \quad \text{\underline{(1 + 2)^2}} \\
2 \times 4^2 & \quad \text{\underline{(1 + 2)^2}}
\end{align*}
\]

and it was observed that the incorrect answers were more likely to be due to careless errors rather than revealing misconceptions. When calculated in this way, the need to remember a rule such as BIDMAS becomes unnecessary, although there is still a need to know that indices are evaluated first in the numerator.
This question was approached differently by the children in the UK. One girl’s attempts at questions 9 and 10 of the non-calculator worksheet are shown in figure 2:

Figure 2: One UK pupil’s work on questions 9 and 10 on non-calculator worksheet

\[
\begin{array}{c}
\frac{9}{2} + \frac{3}{2} \\
\frac{2}{3} + \frac{1}{6} \\
\frac{1}{2} + \frac{2}{3}
\end{array}
\]

In question 10, although she correctly evaluated the denominator, involving brackets, she did not evaluate the index first in the numerator. When interviewed, she was asked what she was thinking about when doing the worksheet, she immediately answered “I was thinking of BIDMAS” but when asked what this stood for she hesitated and then answered “Brackets, index, divided multiple, addition and subtraction.” When prompted about what the “I” stood for she did not know, and even when asked about the word “Index” she was not sure what this meant, although when she was shown the number she immediately said “oh to the power of 2!” which revealed that she understood what a power was, but had not related this to the word “index” and therefore could make no sense of the 1 in BIDMAS. The same misconception is also revealed in her answer to question 9. Her attempts at the corresponding questions on the calculator paper are shown in figure 3:

Figure 3: The same pupil’s work for questions 9 and 10 of the with-calculator worksheet

\[
\begin{array}{c}
3.4 + 0.3 \\
1.7 + 2.7 \\
2.4 + 4.8 \\
1.0 + 2.7
\end{array}
\]

Analysing her keystrokes revealed that she used her calculator efficiently with a good group of the need to evaluate each of the numerator and denominator first before dividing. Her incomplete understanding of the BIDMAS rule was overcome by using the calculator efficiently. For question 9 the pupil correctly evaluated the numerator on her calculator, and then evaluated the denominator:

\[
\begin{array}{c}
2.774281 \\
11.56 \\
2.84 \\
7.36 \\
\text{Press } = \text{.7} \\
11.56
\end{array}
\]

Once happy with the denominator she proceeded to finish the calculation:

\[
\begin{array}{c}
14.591125 \\
1.125499913
\end{array}
\]

It would seem that she wanted to check that the answer from the first line was the same as evaluating the power first, then adding, thus indicating that she had an idea of the correct order, even though she had got the equivalent non-calculator question wrong. When she was using her calculator she was able to investigate the effect of calculating the power first and successfully confirm that this was the correct way to carry out the calculation. In
question 10 the pupil now seemed satisfied that the calculator would produce the correct result for both numerator and denominator and worked efficiently to produce the correct answer.

\[
\begin{array}{c}
4.444\ldots \\
1.452.72 \times 25.44 \\
1256.296296
\end{array}
\]

Conclusions

In this first part of the study it would seem that there are substantial differences between Japanese and British children's ability to carry out arithmetic calculations and the approaches used. The Japanese pupils relied upon algebraic approaches which they generally employed correctly, but this sometimes caused them to make the calculation unnecessarily complicated and they made algebraic mistakes. The children in the UK relied heavily upon remembering HOMAS which worked well if they remembered it correctly but broke down if they did not fully understand what all the letters stood for. However the use of a calculator enabled the pupils to experiment and discover the conventions, which reflects the teaching approaches used.

References


Sheryn, S.L. 2006a. What do Students Do with Personal Technology and How Do We Know? How one student uses her graphical calculator International Journal of Technology in Mathematics Education 13 no.3 151 – 158.
