Dynamics of an Isolated Intrathermocline Eddy

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The z-plane approximation is used to discuss the dynamics of a thin vortical liquid lens of homogeneous density floating on a free surface, on the bottom, or at the boundary in a liquid with a two-layer stratification. The orbital speeds of the liquid particles are assumed to be considerably greater than the speeds produced by the z-effect of drift of the lens, and the Kibel number is assumed to be much less than unity. An equation for the dynamics of a lens in steady-state motion is derived and its exact analytical solution is found. The constraints on the unperturbed characteristics of the lens (i.e., those calculated in the f-plane approximation) are identified. Calculations are made for specific examples of liquid circulation in the lens and are compared with the theory of nonlinear Rossby waves.

In 1976 an anticyclonic vortical lens of anomalous water, presumably of Mediterranean origin [10], was found in the Atlantic Ocean near the Bahamas and was carefully studied. It was in the main thermocline at a depth of about 1000 m, with a maximum thickness at its center \( h = 450 \), a radius \( R \approx 80-100 \) km, and a maximum orbital velocity \( V = 30 \) cm/s at \( r = 50 \) km. When detected, this lens had moved more than 6000 km. A vortical lens with a quasihomogeneous nucleus of radius \( R \approx 15 \) km and a thickness \( h = 200 \) m was found in the Sargasso Sea at a depth of 750 m during the POLYMODE experiment in 1978 [11]. It was estimated [11] to have moved more than 5000 km from its place of origin off the west coast of Africa. About 20 intrathermocline mesoscale lenses up to 220 m thick and with radii of up to 35 km have been found at depths of 550-800 m [8]. There is also other evidence supporting the hypothesis [6], proposed about a quarter-century ago, that large numbers of mesoscale quasihomogeneous lenses exist in the ocean [3].

Theoretically, there is a certain analogy between the dynamics of vortical lenses and Rossby solitons, which have been studied intensively in recent years (see [1] and its citations). But there are also highly significant differences, which prevent the rather well-developed theory of nonlinear Rossby waves from being applied to lenses. The most important difference is that the thickness of the lenses changes from a finite value at the center to zero at the margins. This produces additional significantly nonlinear effects in the dynamics of the lenses, which cannot be allowed for by small nonlinear (with respect to wavelength) corrections to the linear theory, as is generally done in the theory of weakly nonlinear waves. The strong nonlinearity appears especially in the fact that the lens transports all of the water present in it.

There have thus far been few theoretical studies of lens-like eddies. Nof [12] uses integral equations to calculate the zonal displacement velocity of a homogeneous-density lens in an ideal liquid stratified in two layers. It is assumed that a stationary system of the problem exists, but this solution is not found. Killworth [9] introduces a system of zeroth- and first-order (in the inverse of the z-effect) approximations to determine the characteristics of a lens in steady-state motion, on the assumption that the Kibel number is close to unity, and finds its solution for one special case, namely linear increase of orbital velocity with radius. Shapiro [5] discusses non-stationary dynamics of a thin near-surface anticyclonic vortical lens at small Kibel numbers. With the exception of certain special cases, the equation derived, which describes the evolution of the lens with reference to the effects of wind, turbulence, and the z-effect, can be solved only numerically.

We note that according to field observations [10, 11] the Kibel number (\( K_i \)) is between 0.06 and 0.2 even in relatively small vortical lenses and may be treated as a small parameter. Nonetheless, ageostrophic effects are considerable and may not be entirely neglected. Below the asymptotic method is used to derive an equation for the dynamics of a lenslike eddy in steady-state motion, its exact analytic solution is found, and the conditions under which a solution actually exists are indicated.

Consider a thin liquid lens of uniform density floating at the surface of a heavier homogeneous liquid or at the boundary of a two-layer stratified ocean, or at the bottom under a layer of lighter liquid. Since such a lens is assumed to be thin compared with the layers outside it, the speeds of the currents produced in them by the motion of the lens are small (they are proportional to the ratio of the thickness of the lens to the thickness of the layer outside it) and their effect on the
dynamics of the lens can be neglected.* Under these conditions the pressure can be calculated from hydrostatic equations, and the equation of motion within the lens can be written for the shallow-water ς-plane approximation:

\[
\frac{\partial u}{\partial t} + (u \nabla) u = -g' \nabla h + \mathbf{f}[u \times k],
\]

(1)

where \(u(x, y, t)\) is the horizontal velocity vector; \(\nabla\) is the two-dimensional Hamiltonian operator; \(g' = g_1 + \beta y\) is the Coriolis parameter; \(k\) is the unit vector directed vertically upward; \(h(x, y, t)\) is the thickness of the lens; \(\rho\) is the reduced gravitational acceleration, equal to \(g_0 \Delta \rho \Delta \rho_0 / \rho_0 (\Delta \rho + \rho_0)\) for a lens moving at a boundary between liquids with density \(\rho_0 - \Delta \rho\) and \(\rho_0 + \Delta \rho\). When the lens moves along a surface or along the bottom, \(g' = g_0 \Delta \rho\), where \(\Delta \rho\) is the density difference.

We write the continuity equation in the vertically integrated form with allowance for the full (non-linear) kinematic boundary conditions [1]:

\[
\frac{\partial h}{\partial t} + \nabla (hu) = 0.
\]

(2)

We now change over from the dimensional quantities \(h, x, y, u\) and \(t\) to dimensionless quantities by making use of the following scale factors: \(h_0\) is the maximum thickness of the lens; \(L_0\) is its characteristic radius; \(U_0 = g_0 h_0 / L_0\) is the scale of the orbital velocity; \(\eta_0\) is the characteristic time, equal to \(L_0 / C_L\), where \(C_L\) is the scale of the speed of the lens' displacement relative to the surrounding liquid. We introduce the dimensionless parameters \(\varepsilon = U_0 / L_0\), \(\gamma = \beta L_0 / U_0\), and the Kibel number \(\mu = U_0 / L_0\), \(R_0 = (g_0 h_0) / L_0^2\), \(\xi_0 = \text{internal deformation radius}\). Equations (1) and (2) then take the form

\[
\varepsilon \frac{\partial u}{\partial t} + \mu (u \nabla) u = -\nabla h + (1 + \mu \beta y) [u \times k],
\]

(3)

\[
\varepsilon \frac{\partial h}{\partial t} + \mu \nabla (hu) = 0,
\]

(4)

where for simplicity the dimensionless variables are represented by the same signs as the dimensional variables. We shall determine \(\gamma, \mu, \varepsilon\) for the lens described in [10]. We use as \(L_0\) the radius at which the orbital velocity is at its maximum, i.e., \(L_\star = 50\) km; since \(g' = 0.27\) cm/s² [12], we obtain \(U_0 = 24\) cm/s, \(\gamma = 0.2\), and \(\mu = 5 \cdot 10^{-2}\); \(\varepsilon\) is also much less than unity (it cannot be determined any more precisely from the data in [10]).

It is therefore natural to seek the velocity field \(u(x, y, t)\) as an expansion in three independent small parameters \(\varepsilon, \gamma, \mu\):

\[
u = \sum_{k, \lambda, \mu} \varepsilon^k \gamma^\lambda \mu^\mu u^{k, \lambda, \mu}(x, y, t).
\]

(5)

We shall solve system (3)-(4) by expressing \(u\) in terms of \(h\) and its derivatives from equation (3) and substituting the resulting expression into continuity equation (4). We shall obtain the closed approximate equation in terms of \(h(x, y, t)\). A similar asymptotic method was used previously in [4, 7]. After substituting Eq. (5) into Eq. (3) and setting the terms in the same powers of \(\varepsilon, \gamma, \mu\) equal, we have

\[
u = k \times \nabla h + \mu J(\nabla h, h) - \gamma u v - \gamma \mu y [k \times \nabla h] + \xi \frac{\partial h}{\partial t} + \mathbf{f}[u \times k],
\]

(6)

where \(J\) is the Jacobian determinant in \(x\) and \(y\). When necessary, the higher-order components of \(u\) can be written in explicit form.

Now, substituting Eq. (6) into Eq. (4), we obtain

\[
u \frac{\partial h}{\partial t} + \mu \nabla [J(\nabla h, h)] - \mu \nabla \cdot (\mu \nabla h) = \xi \frac{\partial h}{\partial t} + \mathbf{f}[u \times k].
\]

(7)

Note that in some cases Eq. (6) is more conveniently substituted into a different scalar equation that is a consequence of the shallow-water equations (3) and (4), into the equation for conservation of potential vorticity:

\[
u \frac{\partial h}{\partial t} + \mu (u \nabla) h = 0,
\]

(8)

where the potential vorticity \(\xi\) is given by the equation

\[
\xi = \frac{\mu}{\gamma} \cdot \nabla \cdot u + 1 + \gamma \mu y.
\]

The resulting equation for \(h\) agrees with Eq. (7) to within the inverse second order.

Discarding the small terms on the right side of Eq. (7) and converting to the dimensional variables, we obtain

\[
u \frac{\partial h}{\partial t} + \frac{g_0}{L_0^2} \nabla \left( \nabla h \frac{\partial h}{\partial t} + \beta \nabla h \frac{\partial h}{\partial x} + (g_0^2) \nabla [J(\nabla h, h)] = 0.\right.
\]

(9)

This equation is valid for perturbations of the thickness \(h\) of arbitrary (not necessarily small) amplitude. For small perturbations \(\delta h < L_\star\), Eq. (9) becomes the ordinary equation for the evolution of a quasigeostrophic potential eddy,* which is widely used to study Rossby waves [1].

Let us consider the stationary movement of a lens in the zonal direction at a constant velocity \(C\), with \(h = H(\xi, y), \xi = x - Ct\). As a boundary condition we use the condition of impermeability of the horizontal boundary \(\Gamma\) of the lens to liquid:

\[
u h|_{\Gamma} = 0, \quad h|_{\Gamma} = 0.
\]

(10)

*We recall that the thickness of layer \(h\) is proportional to the deviation of the pressure from the equilibrium value.
dynamics of the lens can be neglected.** Under these conditions the pressure can be calculated from hydrostatic equations, and the equation of motion within the lens can be written for the shallow-water \( \nu \)-plane approximation:

\[
\frac{\partial u}{\partial t} + (u \nu) u = -g \nabla h + f(u,k),
\]

where \( u(x,y,t) \) is the horizontal velocity vector; \( \nu \) is the two-dimensional Hamiltonian operator; \( f = f_1 + \nu \) is the Coriolis parameter; \( k \) is the unit vector directed vertically upward; \( h(x,y,t) \) is the thickness of the lens; \( g' \) is the reduced gravitational acceleration, equal to \( g \Delta \rho \Delta \rho \rho / [\rho (\Delta \rho + \Delta \rho)] \) for a lens moving at a boundary between liquids with density \( \rho - \Delta \rho \) and \( \rho + \Delta \rho \). When the lens moves along a surface or along the bottom, \( g' = -\Delta \rho \), where \( \Delta \rho \) is the density difference.

We write the continuity equation in the vertically integrated form with allowance for the full (non-linear) kinematic boundary conditions [1]:

\[
\frac{\partial h}{\partial t} + \nabla (hu) = 0.
\]

We now change over from the dimensional quantities \( h, x, y, u \) and \( t \) to dimensionless quantities by making use of the following scale factors: \( h_* \) is the maximum thickness of the lens; \( L_* \) is its characteristic radius; \( U_* = g' h_* L_* \) is the scale of the orbital velocity; \( \epsilon_* \) is the characteristic time, equal to \( L_*/c_* \), where \( c_* \) is the scale of the speed of the lens' displacement relative to the surrounding liquid. We introduce the dimensionless parameters

\[
e = \frac{1}{L_*}, \gamma \equiv \frac{h_*}{L_*}, \mu \equiv \frac{\nu}{U_*}, \text{ and the Kibble number } \mu = \frac{U_*}{h_*} = \frac{h_*}{L_*}, \text{ where } R_* \equiv \left( \frac{g' h_*}{\nu}\right)^{1/2}. \]

Equations (1) and (2) then take the form

\[
\varepsilon \frac{\partial u}{\partial t} + \mu (u \nu) u = - \nabla h + (1 + \gamma \mu u) [u \times k], \tag{3}
\]

\[
\varepsilon \frac{\partial h}{\partial t} + \mu (u \nu) h = 0, \tag{4}
\]

where for simplicity the dimensionless variables are represented by the same signs as the dimensional variables. We shall determine \( \gamma, \mu, \) and \( \epsilon \) for the lens described in [10]. We use as \( h_* \) the radius at which the orbital velocity is at its maximum, i.e., \( h_* = 50 \text{ km} \); since \( g' = 0.27 \text{ cm/s}^2 \) [12], we obtain \( U_* = 24 \text{ cm/s}, \gamma = 0.2, \) and \( \mu = 5 \times 10^{-2} \); \( \epsilon \) is also much less than unity (it cannot be determined any more precisely from the data in [10]).

It is therefore natural to seek the velocity field \( u(x,y,t) \) as an expansion in three independent small parameters \( \epsilon, \mu, \) and \( \gamma \):

\[
u = \sum_{k,n=0} e^n \gamma^m u^n(x,y,t) = \sum_{k,n=0} e^n \gamma^m u^n \text{ even}(x,y,t). \tag{5}
\]

We shall solve system (3)-(4) by expressing \( u \) in terms of \( h \) and its derivatives from equation of motion (3) and substituting the resulting expression into continuity equation (4). We shall obtain the closed approximate equation in terms of \( h(x,y,t) \).

A similar asymptotic method was used previously in [4, 7]. After substituting Eq. (3) into Eq. (3) and setting the terms in the same powers of \( \epsilon, \mu, \) and \( \gamma \) equal, we have

\[
u = k \times \nabla h + \mu (\nabla h \times h) - \gamma \mu u \nabla h \times h + \frac{\partial}{\partial t} (\epsilon^2, \mu^2, \epsilon^2, \mu^2, \gamma^2).
\]

where \( \partial \) is the Jacobian determinant in \( x \) and \( y \).

When necessary, the higher-order components of \( u \) can be written in explicit form.

Now, substituting Eq. (6) into Eq. (4), we obtain

\[
\frac{\partial h}{\partial t} + \mu (u \nu) h = 0
\]

Note that in some cases Eq. (6) is more conveniently substituted into a different scalar equation that is a consequence of the shallow-water equations (3) and (4), into the equation for conservation of potential vorticity:

\[
\frac{\partial \zeta}{\partial t} + \mu (u \nu) \zeta = 0
\]

where the potential vorticity \( \zeta \) is given by the equation

\[
\zeta = \frac{h k \cdot \text{rot} u + \gamma \mu u}{h}
\]

The resulting equation for \( h \) agrees with Eq. (7) to within the inverse second order.

Discarding the small terms on the right side of Eq. (7) and converting to the dimensional variables, we obtain

\[
\frac{\partial h}{\partial t} - \frac{g' h \nabla h}{f_0^2} + \frac{g' h \nabla h}{f_0} \frac{\partial h}{\partial x} + \left( \frac{g' h}{f_0} \right)^2 \nabla [h (\nabla h \times h)] = 0.
\]

This equation is valid for perturbations of the thickness \( h \) of arbitrary (not necessarily small) amplitude. For small perturbations \( \partial h \ll h \), Eq. (9) becomes the ordinary equation for the evolution of a quasigeostrophic potential eddy,* which is widely used to study Rossby waves [1].

Let us consider the stationary movement of a lens in the zonal direction at a constant velocity \( C \), with \( h = h(\xi, \eta), \xi = x - Ct \). As a boundary condition we use the condition of impermeability of the horizontal boundary \( \Gamma \) of the lens to liquid:

\[
u h |_{r = 0} = 0, h |_{r = 0} = 0.
\]

*We recall that the thickness of layer \( h \) is proportional to the deviation of the pressure from the equilibrium value.
We shall seek solutions of Eq. (7) that are nearly axially symmetrical; for this purpose we represent \( h \) as an axially symmetrical part and a small additional term:

\[
h = h^0(r) + \gamma h^1(r, \theta),
\]

(11)

where the function \( h^1 \), which in general is dependent on the parameters \( \varepsilon, \gamma, \mu \), does not exceed \( O(1) \) as \( \varepsilon, \gamma, \mu \to 0 \); \( r, \theta \) are the polar radius and angle in a system of coordinates moving along with the lens. Thus far the small parameters \( \varepsilon, \gamma, \mu \) and \( \theta \) have been assumed to be arbitrary. We now relate them by means of the equation \( \varepsilon = \gamma \mu \) and substitute Eq. (11) into Eq. (7). By virtue of the identity \( \nabla \cdot [\nabla (\nabla h^0, h^1)] = 0 \) we obtain

\[
\gamma \mu \left[ -\frac{\partial h^0}{\partial r} + \nabla \cdot (\nabla h^0, h^0) + \nabla \cdot (\nabla h^1, h^0) + \nabla \cdot (\nabla h^1, h^1) - h^0 \frac{\partial h^0}{\partial r} \right] = O(\gamma \mu^2, \gamma \mu^3, \mu^4).
\]

(12)

It is very important that for the nearly axially symmetrical lenses under consideration, the terms on the right side of Eq. (12) that are proportional to \( \gamma \mu^3, \gamma \mu^4, \mu^5 \) vanish. The terms in Eq. (3) that do not contain \( \varepsilon \) and \( \gamma \) give

\[
\mu (u^0 \cdot \nabla) u^0 = -\nabla h^0 + [u^0 \times k],
\]

(13)

where \( u^0 \) is given by the first equality in (5).

Since \( h^0(r) \) is not dependent on the angle \( \theta \), according to Eq. (13), the vector \( u^0 \) (and its components \( u^0(r) \)) will have only an azimuthal component not dependent on \( \theta \). It is readily seen that in this case \( \nabla \cdot (\nabla h^0, h^0) = 0 \), and that these are the terms that appear in Eq. (12) with the coefficient \( \gamma \mu^3 \). This enables us to divide Eq. (12) by \( \gamma \mu^3 \) and to discard the small terms \( O(\gamma^2, \mu) \) for any relative size of \( \gamma \) and \( \mu \). We obtain the final equation for calculating \( h^1(r, \theta) \) for a specified unperturbed \( h^0(r) \):

\[
-C \frac{\partial h^0}{\partial r} + \nabla \cdot (h^0 \nabla h^0, h^1) + h^0 \nabla \cdot (h^1 \nabla h^0, h^1) + h^0 \nabla \cdot (h^1 \nabla h^1, h^1) - h^0 \frac{\partial h^0}{\partial r} = 0.
\]

(14)

The velocity of the liquid particles is expressed in terms of \( h^0 \) and \( h^1 \) by Eq. (6), and the velocity \( C \) produced by the \( \beta \)-effect of drift of the lens is determined from the condition of smoothness of the solution of \( h^1 \).

Consider boundary conditions (10). By virtue of Eq. (6), we see that they are satisfied to within \( O(1, \gamma, \mu, \mu^2) \).

Despite its cumbersome form, Eq. (14) has an analytical solution of \( h^1(r, \theta) \) for a rather extensive class of unperturbed distributions of \( h^0(r) \) (or, equivalently, of the orbital velocities \( V^0(r) = d h^0/dr \)). We shall seek a solution of Eq. (14) of the form \( h^1 = H(r) \sin \theta \). We obtain an ordinary differential equation for \( H(r) \):

\[
\frac{d}{dr} \left( \frac{h^0 V^0}{V^0} \frac{d}{dr} \left( \frac{H}{V^0} \right) + r V^0 (h^0 + C) \right) = 0.
\]

(15)

We integrate Eq. (15) from \( r = 0 \) to \( r = 1 \). From the condition that there be no singularities at the boundary of the lens, we obtain an expression for the velocity \( C \) of the zonal \( \beta \)-drift of the lens,

\[
C = \frac{\frac{1}{2} \int \left( h^0 V^0 r^2 dr \right)}{\frac{1}{2} \int r V^0 dr} = \frac{1}{2} \int \frac{h^0 V^0 r^2 dr}{h^0 V^0 dr}.
\]

(16)

Eq. (16) agrees with the equation for \( C \) obtained in [9, 12], since for \( K \ll 1 \) the kinetic energy is small compared to the available potential energy.

By elementary integration of Eq. (15),

\[
H(r) = V^0 \int_0^r \varphi(r) dr + C V^0,
\]

(17)

where

\[
\varphi(r) = -\frac{1}{h^0 V^0} \int_0^r V^0 (h^0 + C) dr.
\]

(18)

For definiteness, we choose the constant of integration \( c_1 \) such that \( H(0) = 0 \).

Analysis of Eq. (15) shows that its solution \( H(r) \) is not physically meaningful for all \( h^0(r) \). As often happens, the properties of a higher-level approximation impose constraints on the arbitrary choice of a lower-level approximation, namely, on the behavior of the orbital velocity \( V^0(r) \) close to the edge of the lens \( r = 1 \). Let \( V^0 = A(1-r) \) as \( r \to 1 \), \( p > 0 \). The function \( H(r) \) does not tend to infinity as \( r \to 1 \) provided that \( p < 1 \). The symmetrical component \( V^0 = \gamma [k \times \nabla h^0] \) is finite only for \( p = 0 \) and \( p = 1 \). At intermediate values \( 0 < p < 1 \), it tends to infinity at the edge of the lens, but the kinetic energy remains finite. It is possible that this solution too should be discarded, since dissipative processes in the real ocean will of course constrain the growth of the orbital velocity at the periphery of the lens. This result most likely should be treated as the appearance of a rapidly rotating thin ring at the edge of the lens. For \( p > 1 \) the solution of Eq. (14) has no physical meaning.

Fig. 1. Distribution of unperturbed lens thickness \( h^0(r) \), magnitude of orbital velocity \( |V^0(r)| \), and radial component of unperturbed lens thickness \( h^0 \) for \( V^0 = 6 \varepsilon (1-r) \).
In conclusion we note that in contrast to the dynamics of almost axially symmetrical Rossby solitons, which were studied theoretically in [2], the lenses always move westward under the influence of the $\beta$-effect. A qualitative explanation is as follows. In an immobile lens with a strictly axially symmetrical velocity distribution, the total Coriolis force over the volume of the lens is not zero and, because of the $\beta$-effect, is directed toward the equator. It may be compensated by the Coriolis force resulting from movement of the lens as a whole only when it moves westward. Slow Rossby solitons (in which the absolute values of the phase velocities are in the linear range) move only eastward [2]. They could move westward only at velocities greater than the maximum velocity of linear waves and with rather high free-surface elevations. Under these conditions a certain analogy between Rossby waves and vortical lenses appears.

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REFERENCES


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