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SHAPIRO, GI

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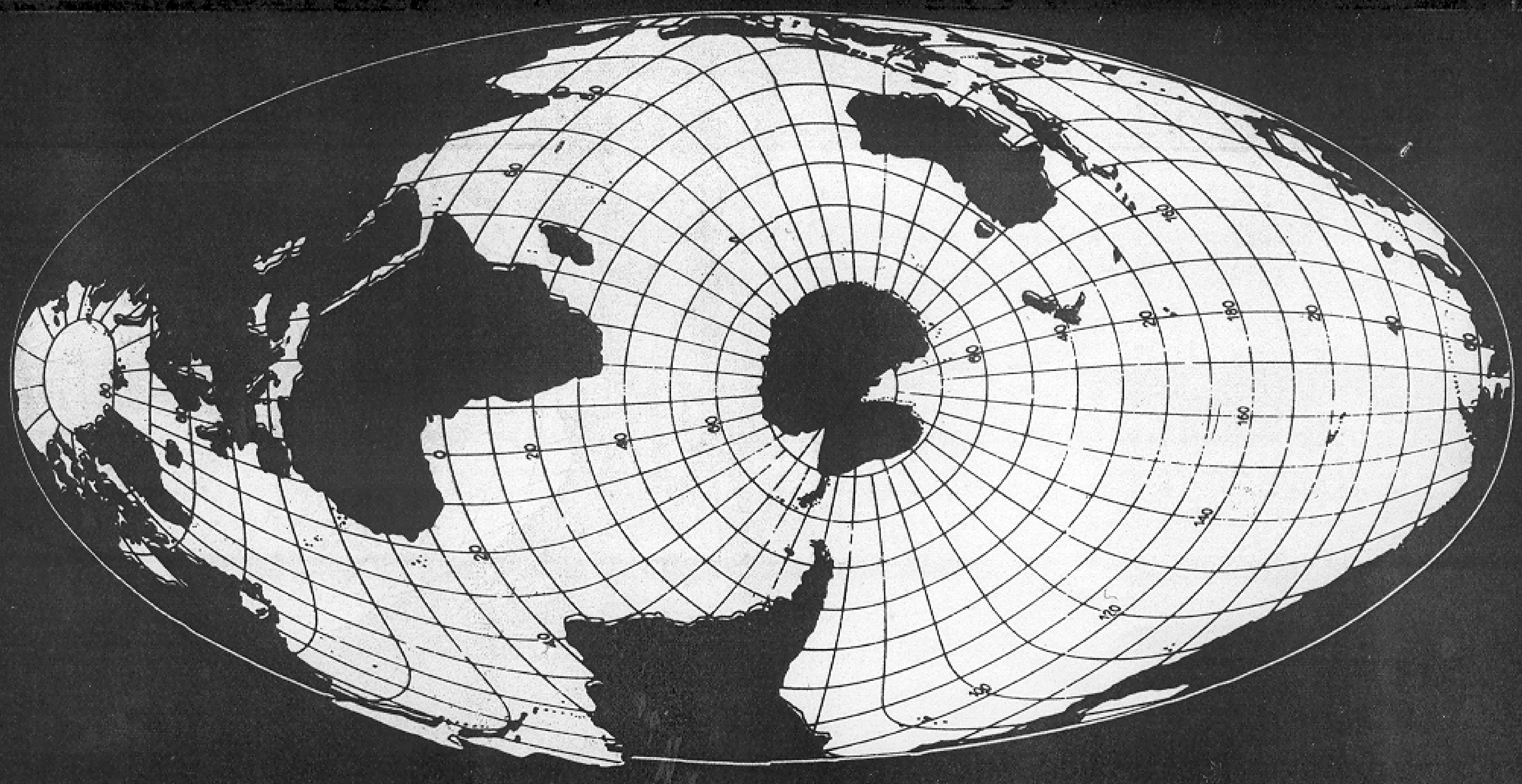

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MARINE PHYSICS

Theory of Axisymmetric Rossby Solitons

G. I. SHAPIRO

Shirshov Institute of Oceanology, USSR Academy of Sciences, Moscow

Estimates are given of the rate of evolution for axisymmetric Rossby solitons under the influence of additional effects not ordinarily included in the soliton theory. It is shown that for a given disturbance amplitude there are infinitely many soliton solutions with differing horizontal size, but having approximately the same lifetime. These results have made it possible to explain why the correlation between the amplitude of a soliton and its radius established in laboratory experiments does not always satisfy the nonlinear dispersion relation derived in a number of theoretical papers.

Rossby solitons are nonlinear solitary waves that propagate with a constant speed and without changing their shape. Interest in the study of Rossby solitons has come about from attempts to explain the long time for the existence of the Great Red Spot on Jupiter and strong localized eddies in the ocean [1-6, 9]. Several types of solitons are known for Rossby waves [1-6, 9, etc.]. In this paper we consider axisymmetric anticyclones for which solutions were found in [4, 6, 9] for greater clarity considering only barotropic motions over a horizontal bottom in the β-plane approximation. Baroclinic motions in a stratified fluid can be considered in an analogous manner. The theory in [4, 6, 9] makes it possible to calculate the elevation in the free surface as a function of the coordinates, as well as the phase velocity for the motion of the soliton. According to [4, 6, 9], the diameter and speed of a Rossby soliton are related by a nonlinear dispersion relation and are uniquely determined by its amplitude (and the parameters of the medium), as are the well-studied Korteweg de Vries solitons.

Rossby soliton research has also been carried out under laboratory conditions [1, 5]. In the experiments, in particular, the height h was determined for the free surface of the fluid over a uniform bottom. It turned out that the variation in h as a function of x and y often did not agree with the predictions of the theory in [6, 9] or with its generalization given in [7]. In particular, the correlation between the amplitude of a soliton and its radius did not always satisfy the theoretical nonlinear dispersion relation.

The present article is devoted to an explanation of this circumstance. It is shown that the dispersion relation for Rossby solitons contains an additional "hidden" parameter. The reason for this is the fact that for a given amplitude there is not one but a whole family of solitons with a different horizontal size and propagation phase velocity. The actual value for the "hidden" parameter is determined by the history of formation for the soliton.

We consider a thin fluid layer with a free surface in the shallow water equations approximation on the β-plane whose motions are close to geostrophic [9]:

$$\varepsilon_T \frac{\partial \mathbf{u}}{\partial t} + \varepsilon (\mathbf{u}\nabla)\mathbf{u} + (1+\hat{\beta} y)[\mathbf{k}\times\mathbf{u}] = -\nabla H, \tag{1}$$

$$\sigma\varepsilon_T \frac{\partial H}{\partial t} + \varepsilon\nabla[(1+\sigma H)\mathbf{u}] = 0. \tag{2}$$

Equations (1) and (2) are written in dimensionless variables, for the conversion to which we used the following dimensional scales: length L, geostrophic velocity V, time T, undisturbed layer thickness h_0, and Coriolis parameter f_0; $\mathbf{u}$ is the horizontal velocity component, $H=(h-h_0)/(\sigma h_0)$, $h(x, y, t)$ is the total thickness of the layer, σ is the relative amplitude of the disturbances, $\mathbf{k}$ is the unit vector for the Oz axis, and ∇ is the horizontal gradient operator. The parameters are $\varepsilon_T=1/f_0T$, $\varepsilon=V/f_0L$, and $\hat{\beta}=\beta L/f_0$; $V=\sigma f_0 L_R^2/L$, $L_R=(gh_0)^{1/2}/f_0$ is the deformation radius, $\sigma = \varepsilon/s$ and $s = L_R^2/L^2$.

We will assume that the parameters ε_T, ε and $\hat{\beta}$ are small, and in this case the amplitude σ should not necessarily be small. Following [8], we substitute for the velocity $\mathbf{u}$ the asymptotic series $\mathbf{u}=\mathbf{u}_0+\varepsilon\mathbf{u}_1+\varepsilon_T\mathbf{u}_2+\ldots$ in Eq. (1), solve it relative to $\mathbf{u}_i$, and substitute the result into Eq. (2). We have

$$\sigma\varepsilon_T\left(\frac{\partial H}{\partial t} - s\nabla\left((1+\sigma H)\nabla\frac{\partial H}{\partial t}\right)\right) + \varepsilon^2 J\left((1+\sigma H)\Delta H + \right.$$
$$\left. + \sigma\frac{(\nabla H)^2}{2},\ H\right) - \varepsilon\hat{\beta}\frac{\partial H}{\partial x}(1+\sigma H)(1-2\hat{\beta}y) =$$
$$\varepsilon O(\varepsilon^2, \varepsilon\varepsilon_T, \varepsilon_T^2) + \varepsilon\hat{\beta} O(\varepsilon_T, \varepsilon, \hat{\beta}^2). \tag{3}$$

A more detailed derivation and analysis of this equation is given in [8]. An analogous equation has been obtained independently in [10]. We change in Eq. (3) to a moving coordinate system using the equations $\xi = x - ct$ and $t' = t$, where c is a constant to be determined later, and we carry out certain algebraic transformations on the right side using the well-known properties of Jacobians. We have

$$\varepsilon_T\left(\frac{\partial H}{\partial t} - s\nabla\left(\left(1+\frac{\varepsilon}{s}H\right)\nabla\frac{\partial H}{\partial t}\right)\right) = -s\varepsilon J\left(\Delta H + \frac{3}{2}\frac{\varepsilon}{s^2}H^2 - \right.$$
$$\left. - 2\frac{\hat\beta}{s}Hy + \frac{1+c'}{s}H, H - \frac{\hat\beta s}{\varepsilon}y\right) - \hat\beta s^2(1+c')\Delta\frac{\partial H}{\partial\xi} -$$
$$- \varepsilon\hat\beta s c'\nabla\left(H\nabla\frac{\partial H}{\partial\xi}\right) - \varepsilon^2 J\left(H\Delta H + \frac{(\nabla H)^2}{2}, H\right) -$$
$$- 2\varepsilon\beta^2 yH\frac{\partial H}{\partial\xi} + sO(\varepsilon^2, \varepsilon\varepsilon_T, \varepsilon_T^2) + \varepsilon\hat\beta O(\varepsilon_T, \varepsilon, \hat\beta^2), \tag{4}$$

where $c' = \varepsilon_T c/s\hat\beta$. Up to now the parameters ε, s, $\hat\beta$, and c' have been arbitrary. Following [9], we restrict ourselves to a specific class of motions that contains Rossby solitons, and we tie together these parameters by the relations

$$\hat\beta = Bs, \varepsilon = Es^2, 1 + c' = Cs, \quad E, C \sim 1, B \leqslant 1, s \ll 1. \tag{5}$$

From this we have, in particular, constraints on the horizontal scale L: $L_R \ll L \lessdot L_I = (f_0 L_R^2/\beta)^{1/3}$.

The scale L_I was evidently first derived in [9]. It plays an important role in the classification of motions in the ocean that are close to geostrophic. For $L > L_I$ the traditional quasi-geostrophic approximations on the β-plane turns out be be incorrect. Taking into account the conditions (5), Eq. (4) becomes

$$\varepsilon_T\left(\frac{\partial H}{\partial t} - s\nabla\left((1+EsH)\nabla\frac{\partial H}{\partial t}\right)\right) =$$
$$= -Es^3 J\left(\Delta H + \frac{3}{2}EH^2 - 2ByH + CH, H - \frac{B}{E}y\right) +$$
$$+ O(s^4) + sO(E^2s^4, Es^2\varepsilon_T, \varepsilon_T^2) + s^2 O(\varepsilon_T, Es^2, B^2s^2). \tag{6}$$

From a comparison of the terms in Eq. (6) it is easily seen that $\varepsilon_T \leqslant O(s^3)$. We drop in Eq. (6) small terms on the order of $O(s^4)$ and higher, and we consider the steady-state solutions for this equation. We have

$$J\left(\Delta H + \frac{3}{2}EH^2 - 2ByH + CH, H - \frac{B}{E}y\right) = 0. \tag{7}$$

To within the accuracy of the notation, Eq. (7) agrees with the steady-state form of Eq. (18.20) from [9]. Equation (7) can be integrated in the form

$$\Delta H + \frac{3}{2}EH^2 - 2BHy + CH = f\left(H - \frac{B}{E}y\right), \tag{7'}$$

where f is an arbitrary and sufficiently smooth function of its argument. For soliton solutions $H \to 0$ as $x^2 + y^2 \to \infty$, from which it follows that $f = 0$. Further, the authors in [9] assume that $B \ll 1$, drop the term $2BHy$ on the left side of Eq. (7'), and obtain the fundamental equation of soliton theory

$$\Delta H + \frac{3}{2}EH^2 + CH = 0. \tag{8}$$

Equation (8) has an axisymmetrically rapidly decreasing solution (a soliton) that satisfies the conditions $H(0) = 1$ and $H(\infty) = 0$:

$$H(r) = G(kr), C = -k^2, r = (\xi^2 + y^2)^{1/2}, \tag{9}$$

$$k = \sqrt{\frac{3E}{2d_0}}, \ d_0 \approx 2{,}39, \tag{10}$$

where $G(kr)$ is a bell-shaped function whose curve is given in Fig. 18.6 in [9]. The parameter C is defined as the eigensolution of the problem. It has been shown in [6] that the function $G(kr)$ can be approximated by the simple equation

$$G(kr) \approx G_0(k_0 r) = \mathrm{ch}^{-4/3}(k_0 r), \ k_0 = \sqrt{\frac{3E}{5}}. \tag{9'}$$

Equations (9) and (10) determine the nonlinear dispersion relation, i.e., the relation between the soliton diameter and phase velocity, and its amplitude. Actually, if we take for the scale L the width of the soliton at half-height, i.e., at $h - h_0 = \frac{1}{2}\sigma h_0$, then from Eqs. (9') and (10) we have $L \approx 3\frac{L_R}{\sqrt{\sigma}}$ and $C_p = \beta L_R^2 c' \approx -\beta L_R^2(1 + 0{,}6\sigma)$, where C_p is the value for the phase velocity in dimensional variables.

We emphasize that the soliton (9) and (10) from [9] and also the analogous solitons from [4, 6] are solutions of approximate equations. The asymptotic derivation given above for the fundamental equation (8) makes it possible to trace the character of the approximations made and to evaluate the order of magnitude for the omitted terms. If the shape (9) and (10) is substituted as the initial condition in a more exact equation, such as in Eq. (4), then it does not remain constant but begins, generally speaking, to slowly evolve under the influence of terms not included in Eq. (8).

It is analogous to the theory of long waves described by the Korteweg de Vries (KDV) equation. A characteristic of the KDV solitons is that they evolve (within the framework of the more complete equations of hydrodynamics) much more slowly than other initial disturbances of the same type. Actually, suppose the nonlinear and dispersion terms in the KDV equation have the same order of magnitude for μ (μ is a small parameter). Then the subsequent terms not included in this equation are on the order of μ^2. Therefore, the characteristic time for the evolution of the solitons is $t \sim \Delta/\mu^2$. For other

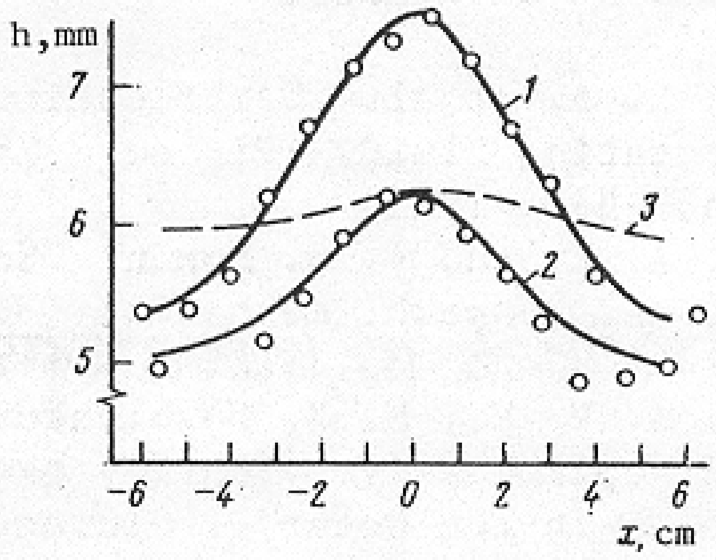

Fig. 1. Variation in the free surface height h as a function of the horizontal coordinate x according to data in the experiment in [5] and Eq. (14). The undisturbed layer thickness is h_0 = 0.5 cm and the deformation radius is L_R = 2.1 cm: *1)* q = 2.4; *2)* q = 4.3. Shown for comparison is the profile *3* calculated from Eq. (18.19c) from [9], $q = \sqrt{1.5}$.

initial disturbances with the same characteristic dimensions it is much less, $t \sim 1/\mu$.

We shall show that the Rossby solitons (9) and (10) do not have this special property. To do this, we estimate the time derivative in Eq. (6) using the solution (9) and (10) at the initial time. Substituting Eqs. (9) and (10) on the right side of Eq. (6), we have

$$\varepsilon_T\left(\frac{\partial H}{\partial t}-s\nabla\left((1+EsH)\nabla\frac{\partial H}{\partial t}\right)=-2EBs^3\frac{\partial H}{\partial \xi}\left(H+\frac{B}{E}y\right)+O(s^4).\tag{11}$$

Equating the orders of magnitude for quantities on both sides of the equation we have

$$\frac{1}{f_0T}=\varepsilon_T\approx 2EBs^3=\frac{2\beta H}{f_0^2},\; T\approx\frac{f_0}{2\beta V}.\tag{12}$$

Equation (12) determines the characteristic time for the evolution of the Rossby soliton constructed in [9]. The question as to what happens over the course of this time, decay of the soliton or adjustment of its internal structure and transition to a more stable state, is still open. We note that as a result of the conditions (5) the time T cannot exceed $f_0/(2E\beta^2L^2_R)$.

We now construct a one-parameter family of approximate solutions for Eq. (7) that have the same properties as the solution (9) and (10). In order to do this we rewrite Eq. (7) in the form

$$J\left(\Delta H+q^2EH^2+CH, H-\frac{B}{E}y\right)+$$
$$2B\left(q^2-\frac{1}{2}\right)H\frac{\partial H}{\partial \xi}+\frac{2B^2}{E}y\frac{\partial H}{\partial \xi}=0.\tag{7''}$$

We consider the equation

$$\Delta H+q^2EH^2+CH=0,\tag{13}$$

whose form* is suggested by the expression in the integral near the symbol for the Jacobian on the left side of Eq. (7"). When $q^2={}^3/_2$ Eq. (13) changes to Eq. (8). The same as for Eq. (8), it has an axially-symmetric solution (9, except that the equation for the "wave number" k now has the form

$$k=q\sqrt{\frac{E}{d_0}}.\tag{10'}$$

For different q the solutions of Eq. (13) differ only in spatial scale. In dimensional variables the resulting family is determined by the equations

$$h(r)-h_0=\sigma h_0G\left(\frac{r}{L_R}q\sqrt{\frac{\sigma}{d_0}}\right);\; c_p=-\beta L_R^2\left(1+\frac{\sigma q^2}{d_0}\right)\tag{14}$$

We recall that the constraint $\sigma q^2/d_0\ll 1$ follows from the conditions (5)

By direct substitution we can see that the solutions of Eq. (13) when q^2 = 1/2 satisfy Eq. (7") with a discrepancy on the order of B^2/E, and for arbitrary $O(B^2/E)$, including q^2 = 3/2, with a discrepancy $O(B^2/E)+O(B)$. Consequently, the value q^2 = 3/2 has no particular advantanges in this case. Finally, if Eq. (7') with f = 0 and $B \neq 0$ were to have localized solutions, they would satisfy Eq. (6) with less discrepancy than Eq. (14), and the value c^2 = 3/2 would turn out to be distinct. However, as has been shown by V. I. Petviashvili and G. M. Reznik, such localized solutions do not exist.

Proceeding in a similar manner to the preceding, we estimate the characteristic evolution time for the solution (14). We have

$$\frac{1}{f_0T}=\varepsilon_T=2\left(\frac{\beta L_R}{f_0}\right)^2\left\{O\left[\left(q^2-\frac{1}{2}\right)\frac{\sigma f_0}{\beta L}\right]+O(1)\right\}.\tag{15}$$

For the scale L we now have from Eqs. (9') and (10') that $L\approx 4{,}6L_R/q\sqrt{\sigma}$. Substituting this equation into Eq. (15) we find that for the given properties of the medium (i.e., the values for f_0, β and L_R) and for a given amplitude σ the smallest value for ε_T (i.e., the maximum in T) is reached when q^2 = 1/2. Solutions with other values for q on the order of unity, including q^2 = 3/2, have approximately the same evolution time.

Thus, there has been constructed a one-parameter family of localized solutions, which for the

*This equation can also be obtained from Eq. (7') if for B << 1 we neglect the terms containing B, and set $f(H)=\left(\frac{3}{2}-q^2\right)EH^2$.

parameter value $q^2 = 3/2$ contains an axially-symmetric Rossby soliton [9]. The solutions of this family corresponding to different values for q on the order of unity (but $q^2 \neq 1/2$) satisfy Eq. (7), and thus also the initial system (1) and (2) with the same degree of accuracy, and have evolution times that are on the same order of magnitude. From this point of view they can on the same basis be called (or not called) solitons.

In order to illustrate these results, numerical modeling* has been carried out for soliton evolution according to Eq. (3) with $\beta L_R/f_0 = 2 \cdot 10^{-3}$ and $\sigma = 0.3$ for the two values $q^2 = 1$ and $q^2 = 3/2$. In both cases there was distortion in the initially axially-symmetric shape, the gradients being steeper in the southern part of the anticyclones. Cyclonic satellite-eddies were formed in the wake, and at $t = 32\,(\beta L_R)^{-1}$ with $q^2 = 3/2$ the amplitude of the satellite (in terms of H) was 3% of the amplitude of the main eddy, and for $q^2 = 1$ it was only 1%.

Rossby soliton profiles $h(r)$ were measured in the laboratory experiments in [5]. Two of them (for $h_0 = 0.5$ cm and $L_R = 2.1$ cm with amplitudes $\sigma_1 = 0.25$ and $\sigma_2 = 0.51$) are shown in Fig. 1. These profiles are quite well approximated by Eqs. (9') and (10') with $q_1 = 4.3$ and $q_2 = 2.4$. In other words, the radii for the solitons observed in experiment are correspondingly 3.5 and 2 times less than what follows from the theory in [6, 9]. Some of the constraints (5) were not satisfied for these conditions. Nevertheless, Eqs. (14) lead to a qualitative understanding of the reason for the discrepancies between the theoretical and experimental [5] relations for $h(r)$: The theoretical profiles were calculated under the assumption $q^2 = 3/2$, whereas solitons with a different value for q were observed in experiment.

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*The calculations were done jointly with V. N. Kon'shin.

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