Superintegrable relativistic systems in spacetime-dependent background fields

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Abstract. We consider a relativistic charged particle in background electromagnetic fields depending on both space and time. We identify which symmetries of the fields automatically generate integrals (conserved quantities) of the charge motion, accounting fully for relativistic and gauge invariance. Using this we present new examples of superintegrable relativistic systems. This includes examples where the integrals of motion are quadratic or nonpolynomial in the canonical momenta.

Keywords: superintegrability, integrability, relativistic dynamics, electromagnetic fields

1. Introduction

The majority of known superintegrable systems correspond to dynamics on the low-dimensional Euclidean spaces E_2 or E_3 and are non-relativistic, see [1] for a recent review. Here we present new examples of superintegrable systems in the relativistic dynamics of charged particles in background electromagnetic fields with nontrivial space-time dependendence.

Recall that a classical system with $2n$-dimensional phase space is integrable if it admits n conserved quantities Q_j which are functionally independent and in involution, so $\{Q_i, Q_j\} = 0 \,\forall\, i, j = 1, \ldots, n$, see e.g. [2, 3]. For autonomous systems the Hamiltonian itself, H, may be taken as one of the Q_j. If there are a further k conserved quantities, $1 \le k \le n-1$, then the system is superintegrable [4]. If $k = 1$ the system is minimally superintegrable, if $k = n-1$ it is maximally superintegrable.

These definitions may seem to present an obstacle in the case of relativistic systems, as reparameterisation invariance of the relativistic particle action implies that the Hamiltonian is identically zero [5]. The solution to this problem is though well known; one 'gauge fixes' the reparameterisation invariance and singles out a preferred time co-ordinate [6]. The disadvantage (from a physicist's perspective) is that in doing so one loses manifest Lorentz invariance, but the benefit is that by choosing a preferred time one obtains a well-defined Hamiltonian system.

Although the purpose of this paper is to flag the existence of novel relativistic superintegrable examples, expanding the literature, the methods behind these examples could be turned into a systematic study exhausting all possibilities, following e.g. [7, 8].

This paper is organised as follows. We begin in Sect. 2 by briefly reviewing the necessary elements of relativistic particle dynamics in the Lagrangian and Hamiltonian formalisms. We show here that if a background field is symmetric under a Poincaré transformation then charge motion in that field automatically admits a related conserved quantity. Based on this result we present a number of superintegrable relativistic systems in Sections 3 and 4, in order of decreasing number of Poincaré symmetries. We conclude in Sect 5, where we also comment on the extension of our results to quantum mechanics.

2. Relativistic dynamics

2.1. Integrals of motion from Poincaré symmetries

We consider a relativistic particle of unit mass and charge with spacetime coordinates $x^\mu(\tau)$ moving in a background electromagnetic field $A_\mu(x)$. The relativistic particle action is

$$S = \int \mathrm{d}\tau \; L = -\int \mathrm{d}\tau \; \left(\sqrt{\dot{x}^\mu \dot{x}_\mu} + \dot{x}^\mu A_\mu(x)\right) \, , \tag{1}$$

where τ is proper time, $\dot{x}^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\tau$ and Einstein summation convention is used for repeated indices throughout, unless explicitly stated otherwise. Varying the action functional one finds that it becomes stationary for particle worldlines obeying the Lorentz equation of motion,

$$\dot{p}_\mu = \dot{x}^\nu \partial_\mu A_\nu \, , \tag{2}$$

in which p_μ is the canonical momentum defined by

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} + A_\mu(x) \, . \tag{3}$$

(The minus sign ensures the correct nonrelativistic limit in our conventions.) A free particle, $A_\mu = 0$, has Poincareé symmetry. The infinitesimal form of a Poincaré transformation is described by $\xi_\mu(x) = a_\mu + \omega_{\mu\nu}x^\nu$, where a_μ and $\omega_{\mu\nu}$ are constant, and $\omega_{\mu\nu} = -\omega_{\nu\mu}$; these parametrise, respectively, the four translations and six Lorentz transformations comprising the Poincaré group (and corresponding to the 10 Killing vectors of flat Minkowski space [9]). In the free theory, the Poincaré symmetry is generated by 10 conserved 'Noether charges' $\xi^\mu p_\mu$. In the presence of the background A_μ, however, these acquire a proper time dependence, which may be found directly from the equation of motion (2),

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi^\nu p_\nu = \dot{x}^\mu \mathcal{L}_\xi A_\mu \, , \tag{4}$$

where $\mathcal{L}_\xi$ is the Lie derivative of the background field under the Poincaré transform,

$$\mathcal{L}_\xi A_\mu \equiv \xi^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \xi^\nu \, . \tag{5}$$

A_μ is called 'symmetric' if it is invariant under the action of the Lie derivative up to a $U(1)$ gauge transformation [10], i.e. if

$$\mathcal{L}_\xi A_\mu = \partial_\mu \Lambda \ , \tag{6}$$

where the scalar field Λ depends implicity on ξ. This is equivalent to the physical electromagnetic fields, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, being strictly invariant,

$$\mathcal{L}_\xi F_{\mu\nu} \equiv \xi^\sigma \partial_\sigma F_{\mu\nu} + F_{\sigma\nu}\partial_\mu \xi^\sigma + F_{\mu\sigma}\partial_\nu \xi^\sigma = 0 \ . \tag{7}$$

If A_μ is symmetric then (4) becomes an exact differential with respect to τ and can be integrated. Thus a Poincaré symmetric background automatically implies an integral of motion (conserved quantity) Q,

$$Q = \xi^\mu p_\mu - \Lambda \ . \tag{8}$$

It follows that if we can identify a background with sufficiently many Poincaré symmetries, charge dynamics in that background will be (super) integrable. To make this concrete, we turn to the Hamiltonian picture.

2.2. Hamiltonian formulation of relativistic dynamics

The action (1) is the proper time integral of a Lagrangian which is homogeneous of first degree in velocities, $L[\lambda \dot{x}] = \lambda L[\dot{x}]$. Euler's homogeneous function theorem then implies that the Hamiltonian vanishes [5] (see [11] for historical context and additional references), as is easily verified:

$$H = -p_\mu \dot{x}^\mu - L = 0 \ . \tag{9}$$

(Again, minus signs follow from conventions.) This has long been understood to be due to the reparametrisation invariance of (1) under $\tau \to f(\tau)$, $\dot{f} > 0$. The most convenient solution to this problem is to give up manifest Lorentz covariance and 'gauge fix' the reparametrisation invariance by choosing τ to be a physical time coordinate. (This is unrelated to the gauge choice for the background potential A_μ.) It turns out that there are basically three choices of time (as pointed out by Dirac [12]), leading to different possible Hamiltonians, all of which should give equivalent descriptions of the dynamics. The choice is dictated by the symmetries of the system under consideration.

The details of the gauge fixing procedure have been discussed at length in [6], so we only give a brief outline of the method. To gauge fix, one chooses a time variable, $\tau = G(x)$ and identifies the Hamiltonian as the variable conjugate to $G(x)$. Thus, $\tau = G(c) =$ constant defines hypersurfaces of equal time τ, which provide a foliation of Minkowski space. The Hamiltonian then generates a flow connecting these hypersurfaces, which hence describes time evolution. Typical hypersurfaces of equal time will be space-like, which implies that their normal, $N^\mu \equiv \partial^\mu G$, will be time-like. A somewhat degenerate case arises for light-like hypersurfaces such as null planes, the normal of which lies in the plane due to the indefinite Minkowski scalar product and the ensuing non-vanishing null vectors. We will consider both space-like and light-like hyperplanes below. The Poisson bracket between the gauge fixing G and the mass-shell constraint, $p^\mu p_\mu - 1 = 0$, may be interpreted as a Faddeev-Popov (FP) expression [13], which has to be non-zero for consistency, $\mathrm{FP} \neq 0$. Zeros of FP would correspond to either fixed points of the time evolution ('frozen time') or particle orbits parallel to the hypersurface ('instantaneous motion'). These two cases thus correspond to gauge fixing ambiguities ('Gribov problems' [14]) which render the Hamiltonian flow, hence time evolution, ill-defined, cf. [6, 15]. In terms of the hypersurface normal N^μ, the FP expression is found to be

$$\mathrm{FP} = \{G, p^\mu p_\mu\} = -2N^\mu p_\mu \ . \tag{10}$$

The Hamiltonian H_τ is then obtained by solving the mass-shell constraint, $p^\mu p_\mu = 1$, for the momentum component that generates (time) translations off the equal-τ hyperplane. One can show explicitly that this indeed requires $\mathrm{FP} \neq 0$. General expressions can be found in [6]. Here we will focus on two gauge fixings, hence time choices, that are particularly useful for later developments.

(i) Instant form

We gauge fix $\tau = t$, i.e. time is t and the surface normal is time-like, $N^\mu = (1, \mathbf{0})$. Phase space is then six dimensional, spanned by the spatial co-ordinates, $\mathbf{x} = (x^j) = (x, y, z)$, and their conjugate momenta, $\mathbf{p} = (p_j) = (p_1, p_2, p_3)$. The Poisson bracket is

$$\{X, Y\} = \frac{\partial X}{\partial x^j}\frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial x^j} \ . \tag{11}$$

The FP expression is $\mathrm{FP} = -2p_0$, which is nonvanishing and proportional to the Hamiltonian, the generator of evolution in t,

$$H = p_0 = \sqrt{1 + (p_j - A_j)^2} + A_0 \ . \tag{12}$$

This can be simplified by adopting Weyl gauge, $A_0 = 0$. The time evolution of any quantity Q is determined by

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} - \{Q, H\} \ , \tag{13}$$

where we have allowed for explicit time dependence since the Hamiltonian will typically depend explicitly on time, through the background field.

The advantage of the instant form is that the time and canonical phase space variables are those familiar from non-relativistic mechanics. The disadvantage is the complicated square root in the Hamiltonian. In fact the majority of the superintegrable systems we will present are better discussed using the 'front form' [6, 16, 17], to which we now turn.

(ii) Front form

We gauge fix $\tau = x^+ \equiv t+z$, i.e. time is x^+, phase space is six dimensional, spanned by the 'longitudinal' coordinate $x^- \equiv t - z$, 'transverse' coordinates $\boldsymbol{x}^\perp \equiv (x, y)$, and their conjugate momenta p_- and $\boldsymbol{p}_\perp \equiv (p_1, p_2)$. The surface normal in this case is light-like, $N^\mu = (1, \mathbf{0}_\perp, -1)$, $N^\mu N_\mu = 0$. The Poisson bracket is (summation convention is used throughout for the index $\perp$)

$$\{A, B\} = \frac{\partial A}{\partial x^-}\frac{\partial B}{\partial p_-} - \frac{\partial A}{\partial p_-}\frac{\partial B}{\partial x^-} + \frac{\partial A}{\partial x^\perp}\frac{\partial B}{\partial p_\perp} - \frac{\partial A}{\partial p_\perp}\frac{\partial B}{\partial x^\perp} \; . \tag{14}$$

The FP expression is FP $= -4p_-$, which *can* be zero for massless particles moving in the negative z-direction. This is not an issue here because we always take nonzero particle mass. As the normal N^μ lies within the hypersurface $x^+ = \mathit{const}$, the Hamiltonian is *not* given by $N^\mu p_\mu$ but rather in terms of a second light-like vector, $n^\mu \equiv (1, \mathbf{0}_\perp, 1)$ such that $N^\mu n_\mu = 2$. Thus, the Hamiltonian generating evolution in x^+ is $n^\mu p_\mu = p_+$, or

$$H = p_+ = \frac{(p_\perp - A_\perp(x))^2 + 1}{4(p_- - A_-)} + A_+(x) \; . \tag{15}$$

Choosing light-front gauge, $A_- = 0$, simplifies the denominator. Time evolution is determined by

$$\frac{\mathrm{d}Q}{\mathrm{d}x^+} = \frac{\partial Q}{\partial x^+} - \{Q, H\} \; . \tag{16}$$

We list below a convenient basis of the 10 Poincaré generators P_μ and $M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$ in the canonical variables of the front form, in which we will mostly work.

$$\begin{aligned}
&\text{4 translations } (P_\mu): && p_+ \equiv H \, , \; p_- \, , \; p_\perp \, , && (17)\\
&\text{1 rotation } (M^{12}): && L_z \equiv x p_2 - y p_1 \, , && (18)\\
&\text{1 boost } (M^{-+}): && K_z \equiv x^+ p_- - x^- H \, , && (19)\\
&\text{2 null rotations } (M^{i+}): && T_i \equiv 2x^i p_- + x^+ p_i \, , && (20)\\
&\text{2 null rotations } (M^{i-}): && U_i \equiv 2x^i H + x^- p_i \, . && (21)
\end{aligned}$$

3. Superintegrable systems with more than three Poincaré symmetries

3.1. Plane waves

We work in the front form. The gauge potential for a plane wave can be taken to have only two nonzero components,

$$A_j(x) = f_j'(x^+) \; , \quad j \in \{1, 2\} \; , \tag{22}$$

in which the f_j are arbitrary functions and the prime, an x^+-derivative, is for notational convenience. The Hamiltonian $H = p_+$, (15), is then explicitly time (x^+) dependent.

The translational invariance of (22) leads to the conservation of all three of the canonical momenta:

$$Q_1 = p_1 \ , \quad Q_2 = p_2 \ , \quad Q_3 = p_- \ , \tag{23}$$

which are in involution, i.e. the system is integrable [18].

What has seemingly not been noticed before is that a plane wave is invariant under the action of the two null rotations (20), i.e. $\mathcal{L}_\xi F_{\mu\nu} = 0$, for ξ such that $\xi^\mu p_\mu = T_1$ or T_2. This implies that there are two further integrals of motion. It is easily verified that the potential (22) is, under T_j, symmetric up to a gauge transformation with $\mathcal{L}_\xi A_\mu = \partial_\mu f_j(x^+)$. Hence the two additional integrals Q_4 and Q_5 are, combining (20) and (8)

$$Q_4 = 2xp_- + x^+ p_1 - f_1(x^+) \ , \qquad Q_5 = 2yp_- + x^+ p_2 - f_2(x^+) \ . \tag{24}$$

This may be verified directly by taking Poisson brackets with H. Thus a particle in a background plane wave is a *maximally superintegrable* relativistic system. Q_4 and Q_5 are in involution with each other and with Q_1, but not with Q_2 and Q_3.

The solution of the equations of motion proceeds as follows. All three momenta are conserved. From the conservation of Q_4 and Q_5 in (24) we are able to read off the transverse orbits immediately:

$$x(x^+) = \frac{Q_4 + f_1(x^+) - Q_1 x^+}{2Q_3} \ , \qquad y(x^+) = \frac{Q_5 + f_2(x^+) - Q_2 x^+}{2Q_3} \ . \tag{25}$$

It remains only to identify x^-, Hamilton's equation for which is

$$\frac{\mathrm{d}x^-}{\mathrm{d}x^+} = -\{x^-, H\} = \frac{1 + \big(Q_1 - f_1'(x^+)\big)^2 + \big(Q_2 - f_2'(x^+)\big)^2}{4Q_3^2} \ . \tag{26}$$

This can be integrated directly. The orbits given by this elegant method agree exactly with those found by standard methods, see [19] and references therein.

3.2. TM-mode model

For our next example we consider fields which are symmetric under translations in x^- and under two null rotations, as for plane waves, but we abandon transverse translation invariance. A potential symmetric (without gauge term) under p_- and T_j is

$$A_+ = -\frac{\boldsymbol{x}^\perp \boldsymbol{x}^\perp}{2x^{+2}} f(x^+) \ , \quad A_- = -\frac{1}{2} f(x^+) \ , \quad \boldsymbol{A}_\perp = \frac{\boldsymbol{x}^\perp}{x^+} f(x^+) \ , \tag{27}$$

where f is arbitrary. The electric and magnetic fields are, for $\mathcal{E}(x) := f'(x^+)/x^+$ [20],

$$\mathbf{E} = \mathcal{E}(x^+)\big(x, y, x^+\big) \ , \qquad \mathbf{B} = \mathcal{E}(x^+)\big(y, -x, 0\big) \ . \tag{28}$$

The fields describe a radially (azimuthally) polarised electric (magnetic) field transverse to the propagation direction, along with a longitudinal electric field. This is a toy model of a transverse magnetic (TM) laser beam near the beam axis [21].

As the potential (27) is symmetric without gauge term, the three quantities

$$Q_1 = 2xp_- + x^+p_1 \ , \qquad Q_2 = 2yp_- + x^+p_2 \ , \qquad Q_3 = p_- \ , \tag{29}$$

are conserved under the action of the lightfront Hamiltonian as in (16), and are in involution. Hence charge motion is integrable. One can verify directly that angular momentum

$$Q_4 = L_z = xp_2 - yp_1 \ , \tag{30}$$

is also conserved, and so that charge motion becomes *minimally superintegrable* due to Poincaré symmetries. A direct computation shows that this exhausts the possible Poincaré symmetries ξ of the field giving $\mathcal{L}_\xi F_{\mu\nu} = 0$. However, there is a fifth integral, which is hence not related to Poincaré invariance. This is found, following [1, 22, 23], by making the ansatz that Q is of a certain order in the momenta, and then imposing (16). Here, we begin by assuming that Q is *linear* in $p_\perp$ but otherwise depends arbitrarily on the coordinates and on p_-, i.e. we assume

$$Q = c_1p_1 + c_2p_2 + c_3 \ , \qquad c_j \equiv c_j(x^+, x^-, x^\perp, p_-) \ . \tag{31}$$

Demanding that the time derivative (16) is zero, one obtains an expression cubic in the $p_\perp$ which should vanish. Equating coefficients of powers of $p_\perp$ gives a finite number of overdetermined partial differential equations which specify the coefficient functions c_j; one recovers the Poincaré generators $Q_1, \ldots, Q_4$ above, as well as the non-Poincaré generator

$$Q_5 = \frac{x}{x^+} + Q_1 \int^{x^+} \mathrm{d}s \, \frac{1}{s^2(2p_- + f(s))} \ , \tag{32}$$

which in general is not polynomial in p_-. The apparent asymmetry between x and y is due to our choice of basis of independent Q's: instead of $Q_4 = L_z$ we could equivalently take

$$\widetilde{Q}_4 = \frac{y}{x^+} + Q_2 \int^{x^+} \mathrm{d}s \, \frac{1}{s^2(2p_- + f(s))} \ , \tag{33}$$

since

$$Q_2Q_5 - Q_1\widetilde{Q}_4 = Q_4 \ . \tag{34}$$

The explicit solution of the equations of motion proceeds as follows, and is almost entirely algebraic, as for the plane wave case above. It is convenient to use (33) rather than (30) as part of the set of independent quantities. From the conservation of (32) and (33) we read off the transverse coordinates,

$$\begin{aligned} x(x^+) &= Q_5x^+ - Q_1x^+ \int^{x^+} \mathrm{d}s \, \frac{1}{s^2(2Q_3 + f(s))} \ , \\ y(x^+) &= \tilde{Q}_4x^+ - Q_2x^+ \int^{x^+} \mathrm{d}s \, \frac{1}{s^2(2Q_3 + f(s))} \ . \end{aligned} \tag{35}$$

Now that these have been determined, we can read off the transverse momenta from (29),

$$p_1(x^+) = \frac{Q_1 - 2x(x^+)p_-}{x^+} \ , \qquad p_2(x^+) = \frac{Q_2 - 2y(x^+)p_-}{x^+} \ . \tag{36}$$

Since p_- is conserved, it remains only to solve for x^-. Hamilton's equations for x^- give

$$\frac{\mathrm{d}x^-}{\mathrm{d}x^+} = -\{x^-, H\} = \frac{1 + \big(p_\perp(x^+) - A_\perp(x^+)\big)^2}{(2p_- + f(x^+))^2} \ . \tag{37}$$

Everything on the right hand side has been determined explicitly as a function of x^+, so the equation can be integrated directly – this is the only integral needed, all other coordinates and momenta have been determined algebraically.

3.3. The undulator

In this example we consider the spatially oscillating magnetic field

$$\mathbf{B} = B_0(\cos\omega z, \sin\omega z, 0) \ , \tag{38}$$

modelling a helical undulator. Similarities and differences between relativistic dynamics in undulator and plane wave fields have recently been discussed in [24]. Non-relativistic charge dynamics in the field (38) was shown to be superintegrable in [8]. One may choose a gauge in which the only non-vanishing components of A_μ are transverse,

$$A_1 = b_0 \cos(\omega z) \ , \qquad A_2 = b_0 \sin(\omega z) \ , \qquad b_0 \equiv \frac{B_0}{\omega} \ . \tag{39}$$

In this system, energy p_0 is conserved, making it convenient to work in the instant form with Hamiltonian

$$H = p_0 = \sqrt{1 + p_3^2 + (p_1 - b_0\cos(\omega z))^2 + (p_2 - b_0 \sin(\omega z))^2} \ , \tag{40}$$

which is time-independent and conserved. As in the non-relativistic limit of this system, the transverse momenta remain conserved,

$$\{p_1, H\} = \{p_2, H\} = 0 \ . \tag{41}$$

The set $\{H, p_1, p_2\}$ are three independent integrals in involution, and the system is integrable,

$$Q_1 = p_1 \ , \qquad Q_2 = p_2 \ , \qquad Q_3 = H \ . \tag{42}$$

A fourth Poincaré integral, not in involution with p_1 and p_2, is given by the helical generator

$$Q_4 = p_3 + \omega L_z = p_3 + \omega(x p_2 - y p_1) \ , \tag{43}$$

leading to minimal superintegrability. The four integrals $Q_1 \ldots Q_4$ are the relativistic generalisations of those found in the non-relativistic limit [8]. The same is true for a fifth integral, which can be identified as follows. The equations of motion for y, z and p_3 are, writing a dot for a time (t) derivative,

$$\begin{aligned} \dot{y} &= -\frac{\partial H}{\partial p_2} = -\frac{p_2 - b_0 \sin \omega z}{H} \, , \\ \dot{z} &= -\frac{\partial H}{\partial p_3} = -\frac{p_3}{H} \, , \\ \dot{p}_3 &= \frac{\partial H}{\partial z} = \frac{\omega b_0}{H}\big(p_1 \sin \omega z - p_2 \cos \omega z\big) \, . \end{aligned} \tag{44}$$

The only difference between these expressions and their nonrelativistic limit resides in the factors of (conserved) H in the denominators. Combining the first two equations in (44) gives

$$\frac{\mathrm{d}y}{p_2 - b_0 \sin \omega z} = \frac{\mathrm{d}z}{p_3} \, . \tag{45}$$

Multiplying the third equation of (44) by p_3 and integrating up gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_3^2 - 2b_0(p_1 \cos \omega z + p_2 \sin \omega z)) = 0 \, . \tag{46}$$

Inserting this into (45) gives

$$\frac{\mathrm{d}y}{p_2 - b_0 \sin \omega z} = \frac{\mathrm{d}z}{\sqrt{2b_0(p_1 \cos \omega z + p_2 \sin \omega z) + u}} \, , \tag{47}$$

in which $u = Q_3^2 - Q_1^2 - Q_2^2 - 1 - b_0^2 =$ constant. This implies, as in the non-relativistic limit [8], the existence of a fifth conserved quantity which is non-polynomial in the canonical momenta. Noting that the extra factors of H in (44) have dropped out, the only difference between (47) and its nonrelativistic limit is in the definition of u. Consistency is easily verified – using that

$$Q_3^2 - 1 = H^2 - 1 = 2H_{\text{non-rel}} + \text{relativistic corrections,} \tag{48}$$

one recovers, in the non-relativistic limit, the definition of u in [8].

4. Superintegrable systems with fewer than three Poincaré symmetries

In the examples above superintegrability was realised in terms of Poincaré symmetries; here we present two examples in which the number of Poincaré symmetries is not enough even to give integrability, but in which there exist additional symmetries in phase space analogous to the Laplace-Runge-Lenz vector of the Kepler problem, see [1, 25] and references therein. (For a historical account see [26, 27].)

4.1. Helical boosts

We return to the front form and consider the electromagnetic fields

$$\mathbf{E} = F_0\big(y, x - \omega x^{+2}, 0\big) \, , \qquad \mathbf{B} = F_0\big(x - \omega x^{+2}, -y, 0\big) \, , \tag{49}$$

where F_0 is a constant. The fields are given by a potential with nonzero components

$$A_1 = F_0 x^+ y \ , \qquad A_2 = F_0 x^+ (x - \frac{\omega}{3} x^{+2}) \ . \tag{50}$$

Taking the Lie derivative of the electromagnetic fields with respect to a general Poincaré transformation ξ, one finds that there are only two Poincaré symmetries, $\mathcal{L}_\xi F_{\mu\nu} = 0$, for ξ such that $\xi^\mu p_\mu = p_-$ or $\xi^\mu p_\mu = p_+ + 2\omega T_1$, the latter of which might be called a generator of 'helical boosts'. The two corresponding conserved quantities are

$$Q_1 = p_- \ , \qquad \widetilde{Q}_2 = H + 2\omega(2p_- x + x^+ p_1) - F_0 y(x + \omega x^{+2}) \ , \tag{51}$$

where the final term results from a gauge term in the transformation of (50). The system is, though, superintegrable, which we can show by searching directly for conserved quantities Q, making the same ansatz as in (31). Imposing time independence of Q, one finds five conserved quantities with lengthy expressions. To compactify them define

$$\Omega = \sqrt{\frac{F_0}{(2p_-)}} \ , \quad \Delta_x = x - y, \quad \Sigma_x = x + y, \quad \Delta_p = \frac{p_1 - p_2}{2p_-}, \quad \Sigma_p = \frac{p_1 + p_2}{2p_-} \ . \tag{52}$$

The five integrals are then

$$\begin{aligned}
Q_1 &= p_- \ , \\
Q_2 &= \Omega \left(\Sigma_x - \omega x^{+2} - \frac{2\omega}{\Omega^2} \right) \sinh \Omega x^+ + \left(\Sigma_p + x^+ \left(\Omega^2 \left(\tfrac{\omega}{3} x^{+2} - \Sigma_x \right) + 2\omega \right) \right) \cosh \Omega x^+ \ , \\
Q_3 &= \Omega \left(\Sigma_x - \omega x^{+2} - \frac{2\omega}{\Omega^2} \right) \cosh \Omega x^+ + \left(\Sigma_p + x^+ \left(\Omega^2 \left(\tfrac{\omega}{3} x^{+2} - \Sigma_x \right) + 2\omega \right) \right) \sinh \Omega x^+ \ , \\
Q_4 &= \Omega \left(\Delta_x - \omega x^{+2} + \frac{2\omega}{\Omega^2} \right) \cos \Omega x^+ + \left(\Delta_p - x^+ (\Omega^2 \left(\tfrac{\omega}{3} x^{+2} - \Delta_x \right) - 2\omega) \right) \sin \Omega x^+ \ , \\
Q_5 &= \Omega \left(\Delta_x - \omega x^{+2} + \frac{2\omega}{\Omega^2} \right) \sin \Omega x^+ - \left(\Delta_p - x^+ (\Omega^2 \left(\tfrac{\omega}{3} x^{+2} - \Delta_x \right) - 2\omega) \right) \cos \Omega x^+ \ ,
\end{aligned} \tag{53}$$

depending nonlinearly on p_-. The helical boost generator in (51), which is quadratic in $p_\perp$, is a combination of these five,

$$\widetilde{Q}_2 = \frac{Q_1}{2}(Q_2^2 - Q_3^2 + Q_4^2 + Q_5^2) + \frac{1}{4Q_1} \ . \tag{54}$$

Note that $Q_1 \dots Q_5$ are linear in the transverse momenta, which suggests that the Lorentz boost symmetry $\widetilde{Q}_2$ in (51), quadratic in the transverse momenta, is perhaps less 'fundamental'.

4.2. Vortex beams

We continue to work in the front form. The potential of an electromagnetic vortex has the nonzero components [28]

$$A_1 = B_0(x \sin\phi - y \cos\phi) \ , \quad A_2 = B_0(-x \cos\phi - y \sin\phi) \ , \tag{55}$$

where $\phi := \omega x^+$ (ω is a frequency) and B_0 is an amplitude. The front form Hamiltonian is (15) with $A_\pm = 0$. As in the previous subsection, there are only two Poincaré symmetries, under which the potential is invariant, giving the two conserved quantities

$$p_- \quad \text{and} \quad \widetilde{Q} := H + \frac{\omega}{2}(xp_2 - yp_1) , \tag{56}$$

the second of which (also a generator of helical boosts) is quadratic in the transverse momenta by virtue of the form of the light-front Hamiltonian. Both the classical and quantum equations of motion can be solved exactly in the field (55), see [28]; we will demonstrate here that the classical system is superintegrable.

We first review some useful properties of the classical orbits. Due to the conservation of p_- and the form of the field, the equations of motion for $\{x, y, p_1, p_2\}$ decouple from those for x^-. If the 'transverse' subsystem of equations for $\{x, y, p_1, p_2\}$ is soluble (for example if it is integrable), then Hamilton's equations for x^-,

$$\frac{\mathrm{d}x^-}{\mathrm{d}x^+} = -\{x^-, H\} = \frac{1}{p_-}H , \tag{57}$$

can be integrated since the Hamiltonian will be determined explicitly in terms of the transverse orbit and the conserved p_-, as in (37). Calculating the time derivatives of the transverse coordinates and momenta yields the equations of motion, where a prime denotes a ϕ-derivative and where $\epsilon := B_0/(2\omega p_-) > 0$ will act as an effective coupling,

$$x'' = \epsilon(x\cos\phi + y\sin\phi) , \qquad y'' = \epsilon(x\sin\phi - y\cos\phi) . \tag{58}$$

There are four independent solutions to these equations [28]. Label these by $\{x_j, y_j\}$ for $j = 1 \ldots 4$. Introducing the shorthand notation

$$\begin{aligned} &\mathrm{Sh} = \sinh(\sqrt{-\epsilon_-}\phi) \, , \mathrm{Ch} = \cosh(\sqrt{-\epsilon_-}\phi) \, , \mathrm{C} = \cos(\sqrt{\epsilon_+}\phi) \, , \mathrm{S} = \sin(\sqrt{\epsilon_+}\phi) \, , \\ &\epsilon_\pm = \frac{1}{4} \pm \epsilon \, , \end{aligned} \tag{59}$$

we have:

$$x_1 = \cos(\tfrac{\phi}{2})\,(4\epsilon_+\mathrm{Ch} - \mathrm{C}) + 2\frac{\sin\left(\frac{\phi}{2}\right)(\epsilon_+\mathrm{Sh} - \sqrt{-\epsilon_-\epsilon_+}\mathrm{S})}{\sqrt{-\epsilon_-}} \tag{60}$$

$$y_1 = \sin(\tfrac{\phi}{2})\,(4\epsilon_+\mathrm{Ch} - \mathrm{C}) - 2\frac{\cos\left(\frac{\phi}{2}\right)(\epsilon_+\mathrm{Sh} - \sqrt{-\epsilon_-\epsilon_+}\mathrm{S})}{\sqrt{-\epsilon_-}} \tag{61}$$

$$x_2 = \sin(\tfrac{\phi}{2})(\mathrm{Ch} - \mathrm{C}) + \frac{\cos\left(\frac{\phi}{2}\right)(4\sqrt{-\epsilon_-\epsilon_+}\mathrm{Sh} + \mathrm{S})}{2\sqrt{\epsilon_+}} \tag{62}$$

$$y_2 = \cos(\tfrac{\phi}{2})(-\mathrm{Ch} + \mathrm{C}) + \frac{\sin\left(\frac{\phi}{2}\right)(4\sqrt{-\epsilon_-\epsilon_+}\mathrm{Sh} + \mathrm{S})}{2\sqrt{\epsilon_+}} \tag{63}$$

$$x_3 = \sin(\tfrac{\phi}{2})\,(-\mathrm{Ch} + 4\epsilon_-\mathrm{C}) - 2\frac{\cos\left(\frac{\phi}{2}\right)(\sqrt{-\epsilon_-\epsilon_+}\mathrm{Sh} + \epsilon_-\mathrm{S})}{\sqrt{\epsilon_+}} \tag{64}$$

$$y_3 = \cos(\tfrac{\phi}{2})\,(\mathrm{Ch} - 4\epsilon_-\mathrm{C}) - 2\frac{\sin\left(\frac{\phi}{2}\right)(\sqrt{-\epsilon_-\epsilon_+}\mathrm{Sh} + \epsilon_-\mathrm{S})}{\sqrt{\epsilon_+}} \tag{65}$$

$$x_4 = \cos(\tfrac{\phi}{2})(\mathrm{Ch} - \mathrm{C}) + \frac{\sin\left(\frac{\phi}{2}\right)(\mathrm{Sh} - 4\sqrt{-\epsilon_-\epsilon_+}\mathrm{S})}{2\sqrt{-\epsilon_-}} \tag{66}$$

$$y_4 = \sin(\tfrac{\phi}{2})(\mathrm{Ch} - \mathrm{C}) - \frac{\cos\left(\frac{\phi}{2}\right)(\mathrm{Sh} - 4\sqrt{-\epsilon_-\epsilon_+}\mathrm{S})}{2\sqrt{-\epsilon_-}} \tag{67}$$

Having identified the transverse orbit, one uses (57) to solve for the remaining coordinate x^-. Let us turn to the superintegrability of the system. We again search for conserved quantities Q using the ansatz (31). We find in this case that Q, linear in the transverse momenta, is conserved if it has the form

$$Q = f_x p_1 + f_y p_2 + 2p_-\omega x(f_x' + \epsilon f_y\cos\phi - \epsilon f_x \sin\phi) + 2p_-\omega y(f_y' + \epsilon f_x\cos\phi + \epsilon f_y\sin\phi)\ , \tag{68}$$

in which the functions $f_x \equiv f_x(\phi)$ and $f_y \equiv f_y(\phi)$ obey

$$f_x'' = \epsilon(f_x\cos\phi + f_y\sin\phi)\ , \qquad f_y'' = \epsilon(f_x\sin\phi - f_y\cos\phi)\ . \tag{69}$$

which are precisely the same equations as (58); thus there are four possible independent choices of $\{f_x, f_y\}$ given by the four pairs of classical solutions $\{x_j, y_j\}$, which give four corresponding conserved quantities Q_j. Define $Q_5 := p_-$. It is directly verified that

$$\{Q_1, Q_4\} = \{Q_1, Q_5\} = \{Q_4, Q_5\} = 0\ , \tag{70}$$

giving integrability, while the addition of Q_2 and Q_3 (which are in involution with Q_5, but not with Q_1 and Q_4) gives maximal superintegrability.

We briefly comment on the solution of the transverse subsystem itself. To pursue this, write $z = x + iy$, so that the decoupled transverse equations of motion reduce to [28, 19]

$$z'' = \epsilon e^{i\phi}\bar{z}\ . \tag{71}$$

Following [28] we trade z for a new variable χ defined by

$$z = e^{i\phi/2}\chi\ , \tag{72}$$

which makes the system autonomous, as in terms of χ the equations of motion become

$$\chi'' = -i\chi' + \frac{1}{4}\chi + \epsilon\bar{\chi}\ . \tag{73}$$

Taking real and imaginary parts of this equation for $\chi =: \alpha + i\beta$, the equations are seen to describe two coupled oscillators, again with $\epsilon_\pm = \frac{1}{4} \pm \epsilon$,

$$\alpha'' = \beta' + \epsilon_+\alpha\ , \qquad \beta'' = -\alpha' + \epsilon_-\beta\ . \tag{74}$$

These equations can be derived from a 'nonrelativistic' action S_E, where $\{\alpha, \beta\}$ are Cartesian coordinates on a plane and $\phi = \omega x^+$ acts as time,

$$S_E = \frac{1}{2}\int \mathrm{d}\phi\ \alpha'^2 + \beta'^2 + \alpha\beta' - \beta\alpha' + \epsilon_+\alpha^2 + \epsilon_-\beta^2\ . \tag{75}$$

The canonical momenta are

$$p_\alpha = \alpha' - \frac{1}{2}\beta \; , \qquad p_\beta = \beta' + \frac{1}{2}\alpha \; , \tag{76}$$

and the Hamiltonian is,

$$H_E = \frac{1}{2}\bigg[(p_\alpha + \tfrac{1}{2}\beta)^2 + (p_\beta - \tfrac{1}{2}\alpha)^2 - \epsilon_+\alpha^2 - \epsilon_-\beta^2\bigg] \; . \tag{77}$$

The Hamiltonian H_E is conserved‡. We can now search for conserved quantities of this new Hamiltonian system. Let these be X, rather than Q. One finds here that there are no conserved quantities linear in the momenta $\{p_\alpha, p_\beta\}$, but that there are conserved quantities quadratic in the momenta,

$$\begin{aligned} X_1 &= (p_\alpha + \tfrac{1}{2}\beta)^2 - \epsilon_+\alpha^2 - \frac{\epsilon_+}{\epsilon}\alpha p_\beta + \frac{\epsilon_-}{\epsilon}\beta p_\alpha \; , \\ X_2 &= (p_\beta - \tfrac{1}{2}\alpha)^2 - \epsilon_-\beta^2 + \frac{\epsilon_+}{\epsilon}\alpha p_\beta - \frac{\epsilon_-}{\epsilon}\beta p_\alpha \; . \end{aligned} \tag{78}$$

These are in involution with the Hamiltonian (by construction) and with each other,

$$\{H_E, X_1\} = \{H_E, X_2\} = \{X_1, X_2\} = 0 \; . \tag{79}$$

Note that the X_j contain terms nonperturbative in the coupling ϵ, and that their sum is

$$X_1 + X_2 = 2H_E \; . \tag{80}$$

Thus we have only two independent conserved quantities in involution, but this makes the transverse system of coupled oscillators *integrable*. Finally, it is interesting to note that

$$H_E = \widetilde{Q} + \text{constant} \; , \tag{81}$$

so that the change of variables from $\{x, y\}$ to $\{\alpha, \beta\}$ corresponds to choosing the conserved quantity $\widetilde{Q}$ of the full theory as a new Hamiltonian for the transverse subsystem.

5. Discussion and conclusions

5.1. Choice of Background

The construction of further superintegrable relativistic systems begins with the choice of background field. In many physical applications it may be desirable for the field, as well as having as many symmetries as possible, to obey Maxwell's equations in vacuum, i.e. to obey the wave equation $\Box A_\mu = 0$ (in Lorenz gauge, $\partial.A = 0$). An example of such a field is a plane wave, discussed above. To generalise this, consider first the scalar

‡ Our H_E is equivalent to the Hamiltonian in [28], but not equal to it. The difference is due to using integration by parts to simplify the action S_E, which does not affect the equation of motion but which does lead to a different definition of the canonical momenta.

wave equation, $\Box\Phi = 0$, and make the additional assumption that the solution has the particular product form $\Phi(x) = \exp(ik_- x^-)\Psi(x^+, x^\perp)$, with a plane wave phase factor separated off. This implies that the field Ψ obeys

$$(4ik_-\partial_+ - \triangle_\perp)\Psi = 0 \; , \tag{82}$$

which is the 2D Schrödinger equation with light-front time x^+ as the time coordinate. The 2D Schrödinger equation has been extensively studied by Miller and collaborators as reviewed in [29], which shows how the symmetry group of (82), the 2D Schrödinger group $Sch(2)$ [30], can be used to classify and determine its solutions. (See [31, 32, 33] for recent applications of $Sch(2)$ in the context of holography.) There are precisely 17 co-ordinate systems for which (82) separates, the simplest yielding plane wave solutions for Ψ. Important for our discussion is the possible impact of the $Sch(2)$ symmetry on the integrability of the charge equation of motion (2): $Sch(2)$ is made up of 5 transformations acting on the transverse (xy) plane (two translations, a rotation around the z axis and two null rotations corresponding to Galilei boosts), a time translation in x^+, a dilation, a conformal transformation and the identity. The first six of these nine generators form a Galilei subgroup of the Poincaré group as has long been known in the context of light-front field quantisation [34, 35, 36]. Dilations and conformal transformations are symmetries of massless particles and hence are not shared by the massive charge obeying the Lorentz equation (2).

Nevertheless, focussing on Poincaré generators only, it seems straightforward to find a background obeying the wave equation *and* having multiple Poincaré symmetries. However, one will typically lose a number of these symmetries upon generalising from scalar to vector solutions, $\Phi \to A_\mu$, which is unsurprising as any choice of vector field singles out a preferred direction. The obvious question is thus whether a sufficient number of Poincaré symmetries survives. We therefore plan to analyse Miller's list of 17 coordinate systems [29] in order to identify backgrounds obeying the wave equation with sufficiently many symmetries to give (super)integrability.

5.2. Conclusions

It has long been known in the physics community that relativistic charge motion in a background electromagnetic plane wave is exactly solvable. To the best of our knowledge it has not though been pointed out that this is in fact a maximally superintegrable system.

We have presented several further examples of maximally superintegrable systems in background fields. These examples are relativistic, and the backgrounds depend nontrivially on both space and time.

It has been conjectured that all maximally superintegrable systems are also exactly solvable quantum mechanically [37]. The appropriate quantum extension of our results is not though to quantum mechanics, but quantum field theory, as the quantum theory of a single relativistic particle suffers from e.g. the Klein paradox. Nevertheless, the first step in such a programme is to solve the Klein-Gordon or Dirac equations in the given background, in order to obtain 'first quantised' wavefunctions which provide the input needed for scattering calculations. In this sense the superintegrable systems we have presented here are indeed also solvable quantum mechanically: the plane wave case is well known [38, 39], the vortex beam case has been solved in [28], and we solve the TM case in [40].

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References

[1] W. Miller, Jr., S. Post, and P. Winternitz. Classical and quantum superintegrability with applications. *J. Phys. A*, 46:423001, 2013.

[2] V. Kozlov. Integrability and non-integrability in Hamiltonian mechanics. *Russian Mathematical Surveys*, 38:1–76, 1983.

[3] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov. *Integrable Systems. I*, pages 173–280. Springer, Berlin, Heidelberg, 1990.

[4] S. Wojciechowski. Superintegrability of the Calogero-Moser system. *Phys. Lett. A*, 95(6):279 – 281, 1983.

[5] P. A. M. Dirac. Generalized Hamiltonian dynamics. *Can. J. Math.*, 2:129–148, 1950.

[6] T. Heinzl. Light cone quantization: Foundations and applications. *Lect. Notes Phys.*, 572:55–142, 2001.

[7] J. W. van Holten. Covariant Hamiltonian dynamics. *Phys. Rev.*, D75:025027, 2007.

[8] A. Marchesiello, L. Šnobl, and P. Winternitz. Three-dimensional superintegrable systems in a static electromagnetic field. *J. Phys. A*, 48(39):395206, 2015.

[9] H. Stephani. *Relativity.* Cambridge University Press, 2004.

[10] R. Jackiw. *Diverse topics in theoretical and mathematical physics.* Singapore: World Scientific (1995) 514 p, 1995.

[11] D. Salisbury and K. Sundermeyer. Léon rosenfeld's general theory of constrained hamiltonian dynamics. *The European Physical Journal H*, pages 1–39, 2017.

[12] P. A. M. Dirac. Forms of Relativistic Dynamics. *Rev. Mod. Phys.*, 21:392–399, 1949.

[13] L. D. Faddeev and V. N. Popov, Feynman Diagrams for the Yang-Mills Field. *Phys. Lett.*, 25B:29, 1967.

[14] V. N. Gribov. Quantization of Nonabelian Gauge Theories. *Nucl. Phys.*, B139:1, 1978.

[15] M. M. Amaral and M. Bojowald. A path-integral approach to the problem of time. 2016. arXiv:1601.07477[gr-qc].

[16] R. A. Neville and F. Rohrlich. Quantum electrodynamics on null planes and applications to lasers. *Phys. Rev.*, D3:1692–1707, 1971.

[17] B. L. G. Bakker et al. Light-Front Quantum Chromodynamics: A framework for the analysis of hadron physics. *Nucl. Phys. Proc. Suppl.*, 251-252:165–174, 2014.

[18] L.D. Landau and E.M. Lifshitz. *The Classical Theory of Fields.* Butterworth-Heinemann, Oxford, 1987. (Course of Theoretical Physics, Vol. 2).

[19] T. Heinzl and A. Ilderton. A Lorentz and gauge invariant measure of laser intensity. *Opt. Commun.*, 282:1879–1883, 2009.

[20] V.G. Bagrov and D.M. Gitman. *Exact solutions of relativistic wave equations.* Dordrecht, Netherlands: Kluwer, 1990.

[21] K. T. McDonald. Axicon Gaussian laser beams. arXiv:physics/0003056.

[22] J. Fris, V. Mandrosov, Ya.A. Smorodinsky, M. Uhlir, and P. Winternitz. On higher symmetries in quantum mechanics. *Physics Letters*, 16(3):354 – 356, 1965.

[23] I. Marquette and P. Winternitz. Superintegrable systems with third-order integrals of motion. *J. Phys. A*, 41(30):304031, 2008.

[24] T. Heinzl, A. Ilderton, and B. King. Classical and quantum particle dynamics in univariate background fields. *Phys. Rev.*, D94(6):065039, 2016.

[25] N. W. Evans. Superintegrability in classical mechanics. *Phys. Rev. A*, 41:5666–5676, May 1990.

[26] H. Goldstein. Prehistory of the Runge-Lenz vector. *American Journal of Physics*, 43:737–738, 1975.

[27] H. Goldstein. More on the prehistory of the Laplace or Runge-Lenz vector. *American Journal of Physics*, 44:1123–1124, 1976.

[28] I. Bialynicki-Birula. Particle beams guided by electromagnetic vortices: New solutions of the Lorentz, Schrödinger, Klein-Gordon and Dirac equations. *Phys. Rev. Lett.*, 93:020402, 2004.

[29] W. Miller, Jr. *Symmetries and Separation of Variables.* Addison-Wesley, Reading, Mass. 1977.

[30] U. Niederer. The maximal kinematical invariance group of the free Schrödinger equation. *Helv. Phys. Acta*, 45:802–810, 1972.

[31] D. T. Son. Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrödinger symmetry. *Phys. Rev.*, D78:046003, 2008.

[32] M. Guica, K. Skenderis, M. Taylor, and B. C. van Rees. Holography for Schrödinger backgrounds. *JHEP*, 02:056, 2011.

[33] M. Taylor. Lifshitz holography. *Class. Quant. Grav.*, 33(3):033001, 2016.

[34] L. Susskind. Model of self-induced strong interactions. *Phys. Rev.*, 165:1535–1546, 1968.

[35] K. Bardakci and M. B. Halpern. Theories at infinite momentum. *Phys. Rev.*, 176:1686–1699, 1968.

[36] J. B. Kogut and D. E. Soper. Quantum Electrodynamics in the Infinite Momentum Frame. *Phys. Rev.*, D1:2901–2913, 1970.

[37] P. Tempesta, A. V. Turbiner, and P. Winternitz. Exact solvability of superintegrable systems. *Journal of Mathematical Physics*, 42(9):4248–4257, 2001.

[38] D.M. Volkov. Über eine Klasse von Lösungen der Diracschen Gleichung. *Z. Phys.*, 94:250–260, 1935.

[39] V. B. Berestetsky, E. M. Lifshitz, and L. P. Pitaevsky. *Quantum Electrodynamics.* Pergamon, Oxford, UK, 1982. (Course Of Theoretical Physics, Vol. 4).

[40] T. Heinzl and A. Ilderton. Exact quantum and classical dynamics in electromagnetic background fields. To appear.