# The numerical solution of the dynamic fluid-structure interaction problem. 

Harris, Paul John

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University of Plymouth

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# THE NUMERICAL SOLUTION OF THE DYNAMIC FLUID-STRUCTURE INTERACTION PROBLEM 

By
Paul John Harris

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# THE NUMERICAL SOLUTION OF THE DYNAMIC FLUID-STRUCTURE INTERACTION PROBLEM <br> By <br> Paul John Harris. <br> Abstract 

In this thesis we consider the problem of the dynamic fluid-structure interaction between a finite elastic structure and the acoustic field in an unbounded fluid-filled exterior domain.

We formulate the exterior acoustic problem as an integral equation over the structure surface. However, the classical boundary integral equation formulations of this problem do not have unique solutions at certain characteristic frequencies (which depend on the surface) and it is necessary to employ modified boundary integral equation formulations which are valid for all frequencies. The modified integral equation formulation used here involves certain arbitrary parameters and we shall study the effect of these parameters on the stability and accuracy of the numerical methods used to solve the integral equation.

We then couple the boundary element analysis of the exterior acoustic problem with a finite element analysis of the elastic structure to investigate the interaction between the structure and the acoustic field. Recently there has been some controversy over whether or not the coupled problem suffers from the non-uniqueness problems associated with the classical integral equation formulations of the exterior acoustic problem. We resolve this question by demonstrating that the solution to the coupled problem is not unique at the characteristic frequencies and that we need to employ an integral equation formulation valid for all frequencies.

We discuss the accuracy of our numerical results for both the acoustic problem and the coupled problem, for a number of axisymmetric and fully three-dimensional problems. Finally, we apply our method to the problem of a piezoelectric sonar transducer transmitting an acoustic signal in water, and observe reasonable agreement between our theoretical predictions and some experimental results.

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Some of the material contained in this thesis has been published or accepted for publication $[6,7,8]$.

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## 1 INTRODUCTION

The problem of the interaction between a finite elastic structure and the acoustic field in an unbounded exterior domain occurs in many areas of mathematical physics. In particular, we are concerned with the field of underwater acoustics where it is desirable to determine the acoustic field either radiated by a submerged vibrating elastic structure or scattered by a submerged elastic structure. Here, the impedance mis-match between the structure and the acoustic medium is less than that between the structure and air, and hence it is not feasible to assume that the structure is perfectly rigid. Examples of this problem include that of finding the radiated or scattered sound field around ships and submarines. A practical problem that we shall consider is to determine the acoustic field radiated by a sonar transducer and the frequency for which the maximum response is obtained.

In general, there are very few structures for which an analytical solution to this problem can be found and numerical methods appear to be the only feasible way of obtaining solutions for arbitrary shaped structures. Our aim is to design an efficient and accurate numerical method to determine the acoustic field about an arbitrary three-dimensional structure which we can use to solve the transducer problem.

The independent development of general numerical methods for both the structural dynamics, governed by the linear law of elasticity, and the acoustic field in the fluid, governed by the Helmholtz equation, has made it possible for us to solve this coupled fluid-structure interaction problem. This is achieved by matching a bound-
ary element analysis of the exterior acoustic field with a finite element analysis of the elastic structure at the structure surface.

Because the exterior region is unbounded, obvious diffculties arise in the use of domain techniques, such as finite elements or finite differences, in numerically solving the partial differential equation which governs the exterior acoustic field. For this reason many workers have chosen to re-formulate the problem as an integral equation over the structure surface, and solve it using the boundary element method. This has the two major advantages that the domain of interest is reduced from the infinite three-dimensional region exterior to the structure to the two-dimensional finite surface of the structure, and that the Sommerfeld radiation condition at infinity is automatically satisfied.

Integral equation formulations for the exterior acoustic field fall into two categories. In Section 2.2.1 we consider the indirect methods which assume that the solution can be represented in the form of a layer potential. This yields an integral equation which it is necessary to solve for the unknown density function. The acoustic pressure in the exterior domain may then be computed from the density function. In Section 2.2.2 we consider direst methods which use Green's Second Theorem to derive a formula directly relating the pressure and its normal derivative on the structure surface. Once both the pressure and its normal derivative are known on the surface, Green's formula can be used to compute the acoustic pressure in the exterior domain.

Unfortunately, at the standing wave frequencies of the corresponding interior
problem, which we will refer to as characteristic frequencies, the classical integral equation formulations either fail to have a unique solution or have no solution. This problem is due to the integral equation formulation, and is not a feature of the physical problem, which will always have a unique solution. In Section 2.4 we give a survey of the more recent and important techniques for overcoming the nonuniqueness or non-existence problems. However, some of the formulations, such as those by Schenck [49] and Jones [36], are only likely to yield the required solution for low frequencies, whilst other methods, such as those of Panich [47] and Burton and Miller [13] which are valid for all frequencies, require much more computational effort.

In Chapter 3 we will describe a numerical scheme for solving the exterior Neumann problem for Helmholtz equation using the direct method of formulation via Green's theorem. Here the Burton and Miller formulation [13? is employed to ensure that the integral equation has a unique solution for all frequencies.

Section 3.1 briefly considers some alternative numerical methods that could be used for solving integral equations. The following sections investigate different methods for representing the surface and the accuracy of various numerical quadrature rules for evaluating the surface integrals.

Section 3.4 pays particular attention to the value of the arbitrary coupling parameter present in the Burton and Miller formulation. We shall study the effect that different values of this coupling parameter have on the conditioning of the integral operators involved, since it is easier to obtain accurate numerical approxi-
mations to well conditioned integral operators.

The finite element method is a well known and tested method for analysing the motion of an elastic structure. A method of coupling a finite element analysis of the structure to a boundary element analysis of the exterior acoustic field is described in Chapter 4. This is achieved by enforcing the conditions that the normal velocity is continuous at the surface, and that the acoustic pressure equals the stresses in the structure at the surface.

Recently it has been suggested that it is possible to couple one of the classical integral equation formulations of the exterior Helmholtz equation with a finite element analysis of the structure with no significant loss of accuracy at the characteristic frequencies of the exterior problem [19, 33, 48; However, it has also been suggested that it is necessary to employ an integral equation formulation which is valid for all frequencies in order to obtain an accurate solution to the coupled fluid-structure interaction problem [43, 62]. In Section 4.4 we shall attempt to resolve this apparent controversy by treating the relationship between the surface pressure and velocities due to the presence of the elastic structure as a Robin type boundary condition for the exterior field equations. Further, it is clear from our numerical results given in Section 4.5 that a numerical scheme based on one of the classical integral equation formulations of the exterior Helmholtz problem does not yield accurate results for frequencies close to one of the characteristic frequencies. It is also clear from our results that the numerical scheme employed suffers from ill-conditioning problems whenever the frequency is close to a natural frequency of
the structure.
Finally, in Chapter 5, the numerical techniques that have been developed in earlier chapters are employed to analyse the acoustic field radiated by a simple piezoelectric sonar transducer. In particular we shall predict the frequency at which the maximum response is obtained from the transducer and compare the results to some experimental data.

## 2 INTEGRAL EQUATION FORMULATIONS

## OF THE EXTERIOR HELMHOLTZ PROB-

## LEM

### 2.1 Introduction

In this chapter we will consider various integral equation formulations for the Helmholtz equation in the infinite region exterior to a bounded three-dimensional structure. In later chapters the integral equation we feel is the most suitable will be used to provide an impedance type relationship between the acoustic pressure and the normal particle velocity on the surface of the structure.

Let $D_{\text {- denote }}$ the region in $\mathcal{R}^{3}$ occupied by the structure, with closed surface $S$. We assume $S$ consists of a number of sub-surfaces each of which is of class $C^{2}$. The unbounded region exterior to $S$ is denoted by $D_{+}$and is filled with a homogeneous acoustic medium of density $\rho_{f}$ and with speed of sound $c$. The unit normal to $S$, directed into $D_{+}$is denoted $\underline{n}$.

It is well-known that small amplitude acoustic waves propagate through $D_{+}$ according to the linear wave equation

$$
\begin{equation*}
\nabla^{2} \Phi(p, t)=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \Phi(p, t) \tag{2.1}
\end{equation*}
$$

where $\Phi(p, t)$ is the excess pressure at the point $p \in D_{\text {- and }}$ at time $t$. If we assume single frequency harmonic time dependence of the form $e^{-i \omega t}$ where $\omega$ is the angular frequency ( $=2 \pi$ times the frequency in hertz) then (2.1) becomes the Helmholtz
(or reduced wave) equation

$$
\begin{equation*}
\Gamma^{2} \phi(p)+k^{2} \phi(p)=0 \quad p \in D_{\mp} \tag{2.2}
\end{equation*}
$$

where $\Phi(p, t)=\phi(p) e^{-i \omega t}$ and $k=\omega / c$ is called the acoustic wavenumber.
On the surface $S$ we assume one of the following boundary conditions

$$
\begin{array}{ccc}
\text { (a) (Dirichlet condition) } & \phi(p)=f(p) & p \in S \\
\text { (b) (Neumann condition) } & \frac{\partial \phi}{\partial n}(p)=f(p) & p \in S \\
\text { (c) (Robin condition) } & \frac{\partial \phi}{\partial n}(p)+h(p) \phi(p)=f(p) & p \in S \tag{2.5}
\end{array}
$$

where $f$ and $h$ are known functions. Two important practical boundary conditions are given by $\phi(p) \equiv 0$ for a perfectly acoustically soft scatterer, and $\frac{\partial \phi}{\partial n}(p) \equiv 0$ corresponding to an acoustically rigid structure. Another special case is when the normal particle velocity $v_{n}$ is known on $S$, as $\frac{\partial \phi}{\partial n}(p)=i w \rho_{f} v_{n}(p), p \in S$, which follows from

$$
\begin{equation*}
\nabla \Phi(p, t)=-\rho_{f} \frac{\partial \underline{v}(p, t)}{\partial t} \tag{2.6}
\end{equation*}
$$

where $\underline{v}(p, t)$ is the particle velocity.
In addition to the above boundary conditions, we need a suitable condition at infinity, corresponding to the physical requirement that all scattered and radiated waves are out-going at infinity. This was first expressed by Sommerfeld as the radiation condition [20]

$$
\begin{equation*}
\lim _{r=\{r \mid \rightarrow \infty} r\left\{\frac{\partial \phi(r)}{\partial r}-i k \phi(r)\right\}=0 \tag{2.7}
\end{equation*}
$$

uniformly in $\underline{\underline{r}}$ where $\underline{r}$ is the vector of length $r=|\underline{r}|$ from a fixed origin in $D_{-}$to a general field point, and $\underline{\underline{~}}$ is the unit vector in this direction. A function satisfying (2.2) and (2.7) is known as a radiating wave function.

## THEOREM 2.1

The solution to the exterior Helmholtz equation (2.2) subject to the radiation condition (2.7) and one of the boundary conditions (2.3), (2.4) or (2.5) is unique, provided $r e(k)>0, i m(k) \geq 0$ and in the case of (2.5), $h$ satisfies $\operatorname{im}(h) \geq 0$ or $i m(h) \geq \frac{i m(k)}{r e(k)} r e(h)$.

Proof. See [14, 20].
In. the practical situations considered in this thesis $k$ is the ratio of two real valued physical quantities, the angular frequency and the speed of sound. It follows that $k$ will always be a positive real number and so the exterior Helmholtz problems under consideration will always have unique solutions.

### 2.2 Basic Integral Equation Formulations

Before we formulate the exterior Helmholtz problem as an integral equation, let us define the single and double layer Helmholtz potentials and investigate their properties. Let

$$
\begin{equation*}
L_{k} \sigma(p)=\int_{S} \sigma(q) G_{k}(p, q) d S_{q} \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{k} \sigma(p)=\int_{S} \sigma(q) \frac{\partial}{\partial n_{q}} G_{k}(p, q) d S_{q} \tag{2.9}
\end{equation*}
$$

where $L_{k}$ and $M_{k}$ are the single and double layer Helmholtz potential operators respectively. The function $G_{k}(p, q)$ is the free space Green's function for the Helmholtz equation. That is, it satisfies in both variables

$$
\begin{equation*}
\nabla^{2} G_{k}(p, q)+k^{2} G_{k}(p, q)=-\delta(\vdots-q \mid) \tag{2.10}
\end{equation*}
$$

(where $\delta$ is the Dirac delta function) and the radiation condition (2.7). The function $G_{k}(p, q)$ represents the effect at $q$ of a unit point source at $p$ radiating into freespace, and vice-versa. In three dimensions $G_{k}$ is given by

$$
\begin{equation*}
G_{k}(p, q)=\frac{e^{i k|p-q|}}{4 \pi \mid p-q!} \tag{2.11}
\end{equation*}
$$

It is well known that both $L_{k} \sigma$ and $M_{k} \sigma$ define radiating wave functions, see Burton [14].

It is clear from the definitions (2.8) and (2.9) of the potentials that whether or not the integrals exist depends on the behaviour of the layer density $\sigma$, the singularities in the kernel functions and the nature of the surface $S$. If $S=\bigcup_{i=1}^{n} S_{i}$ and each $S_{i} \in C^{2}$, then provided $p$ is not on an edge or vertex of $S$, it can be shown that $L_{k} \sigma(p)$ and $M_{k} \sigma(p)$ exist for all other $p \in \mathcal{R}^{3}$ provided $\sigma$ is continuous over $S$. The existence and continuity of the normal derivative of $M_{k}$ requires that $\sigma \in C^{2}$ $[14,20]$.

The Green's function $G_{k}$ is closely related to the corresponding Green's function $G_{0}(p, q)$ for Laplace's equation $(k=0)$ and possesses the same singularity at $p=q$, since $[16,20,35]$

$$
\begin{equation*}
G_{k}(p, q)=\frac{1}{4 \pi r}+o(1)=G_{0}(p, q)+o(1) \quad \text { as } r \rightarrow 0 \tag{2.12}
\end{equation*}
$$

where $r=|p-q|$. It is this singularity that is responsible for the jump properties of the layer potentials at the surface $S$, obtained from classical potential theory, which are outlined below [14, 20]

Before considering the jump discontinuity properties of the layer potentials we introduce the following notation. Denote $p \in D_{+}$by $p_{+}$and the limit as $p_{+}$tends to $p \in S$ by $\lim _{p_{+} \rightarrow p}$. Similarly denote $p \in D_{-}$by $p_{-}$and the limit as $p_{-}$tends to $p \in S$ by $\lim _{p-p}$. The jump discontinuity properties of $L_{k} . M I_{k}$ and their normal derivatives can now be stated.

## LEMIMA 2.1

The layer potential $L_{k} \sigma(p)$ is continuous throughout $\mathcal{R}^{3}$. that is

$$
\begin{equation*}
\lim _{p \rightarrow p} L_{k} \sigma\left(p_{+}\right)=L_{k} \sigma(p)=\lim _{p_{-} \rightarrow p} L_{k} \sigma\left(p_{-}\right) \quad p \in S \tag{2.13}
\end{equation*}
$$

whilst its normal derivative has a jump discontinuity at $S$ of the form

$$
\begin{array}{ll}
\lim _{p \rightarrow p} \frac{\partial}{\partial n_{p}} L_{k} \sigma\left(p_{+}\right)=-\frac{\sigma(p)}{2}+\frac{\partial}{\partial n_{p}} L_{k} \sigma(p) & p \leqq S  \tag{2.14}\\
\lim _{p \rightarrow p} \frac{\partial}{\partial n_{p}} L_{k} \sigma\left(p_{-}\right)=\frac{\sigma(p)}{2}+\frac{\partial}{\partial n_{p}} L_{k} \sigma(p) & p \doteq S .
\end{array}
$$

The double layer potential $M_{k} \sigma(p)$ has a discontinuity at $S$ of the form

$$
\begin{array}{ll}
\lim _{p_{+} \rightarrow p} M I_{k} \sigma\left(p_{+}\right)=\frac{\sigma(p)}{2}+M_{k} \sigma(p) & p \in S  \tag{2.15}\\
\lim _{p \rightarrow p} M I_{k} \sigma\left(p_{-}\right)=-\frac{\sigma(p)}{2}+M I_{k} \sigma(p) & p \in S
\end{array}
$$

and its normal derivative is continuous throughout $\mathcal{R}^{3}$, that is

$$
\begin{equation*}
\lim _{p+-p} \frac{\partial}{\partial n_{p}} M_{k} \sigma\left(p_{-}\right)=\frac{\partial}{\partial n_{p}} M_{k} \sigma(p)=\lim _{p-} \frac{\partial}{\partial n_{p}} M_{k} \sigma\left(p_{-}\right) \quad p \in S \tag{2.16}
\end{equation*}
$$

We denote the normal derivatives of $L_{k}$ and $M_{k}$ by $M_{k}^{T}$ and $V_{k}$ respectively, where

$$
\begin{equation*}
M_{k}^{T} \sigma(p)=\frac{\partial}{\partial n_{p}} L_{k} \sigma(p)=\int_{S} \sigma(q) \frac{\partial}{\partial n_{p}} G_{k}(p, q) d S_{q} \tag{2.17}
\end{equation*}
$$

and

$$
\begin{equation*}
N_{k} \sigma(p)=\frac{\partial}{\partial n_{p}} M_{k} \sigma(p)=\frac{\partial}{\partial n_{p}} \int_{S} \sigma(q) \frac{\partial}{\partial n_{q}} G_{k}(p, q) d S_{q} . \tag{2.18}
\end{equation*}
$$

Strictly speaking, the derivative with respect to $\underline{n}_{p}$ in (2.18) cannot be taken inside the integral sign because

$$
\begin{equation*}
\frac{\partial^{2}}{\partial n_{p} \partial n_{q}} G_{k}(p, q)=\frac{1}{4 \pi|p-q|^{3}}+o(1) \quad \text { as } q \rightarrow p \tag{2.19}
\end{equation*}
$$

and the resulting kernel function is non-integrable. However, we are allowed to change the order of integration and differentiation (which is always done in practice), provided the resulting hyper-singular integral is interpreted in the sense of a Hadamard finite part [30].

### 2.2.1 Indirect Integral Equation Formulations of the Exterior Helmholtz Problem.

Since both $L_{k} \sigma$ and $M_{k} \sigma$ are radiating wave functions, it seems reasonable to attempt to express the solution as a layer potential. An integral equation for the unknown density function is obtained by enforcing the appropriate boundary condition on $S$. If we seek a solution to the Dirichlet problem in the form of a single layer potential, that is

$$
\begin{equation*}
\phi\left(p_{\dot{+}}\right)=L_{k} \sigma\left(p_{+}\right) \quad p_{+} \in D_{-}, \tag{2.20}
\end{equation*}
$$

then taking the limit $p_{+} \rightarrow p \in S$, using the continuity of $L_{k}$ and applying the boundary condition (2.3) leads to

$$
\begin{equation*}
L_{k} \sigma(p)=f(p) \quad p \in S \tag{2.21}
\end{equation*}
$$

This is a first kind Fredholm integral equation for the surface density $\sigma$. For the Robin problem differentiate (2.20) with respect to $\underline{n}_{p}$, take the limit as $p_{+} \rightarrow p \in S$, and apply the boundary condition (2.5) to obtain

$$
\begin{equation*}
\left(-\frac{1}{2} I+M_{k}^{T}+h L_{k}\right) \sigma(p)=f(p) \quad p \in S \tag{2.22}
\end{equation*}
$$

which is a second kind Fredholm integral equation for the surface density $\sigma$. The Neumann problem is a special case of the Robin problem with $h(p) \equiv 0$. In all of the above cases, once $\sigma$ has been determined we can use (2.20) to obtain $\phi\left(p_{+}\right)$.

Alternatively we can seek a solution of the form

$$
\begin{equation*}
\phi\left(p_{+}\right)=M_{k} \sigma\left(p_{+}\right) \quad p_{+} \in D_{-} . \tag{2.23}
\end{equation*}
$$

Taking the limit $p_{+} \rightarrow p \in S$ and applying the Dirichlet condition (2.3) yields the second kind Fredholm integral equation

$$
\begin{equation*}
\left(\frac{1}{2} I+M_{k}\right) \sigma(p)=f(p) \quad p \in S \tag{2.24}
\end{equation*}
$$

Similarly, differentiating (2.23), taking the limit as $p_{-} \rightarrow p \in S$ and applying the Robin boundary condition (2.5) yields

$$
\begin{equation*}
\left[N_{k}+h\left(\frac{1}{2} I+M I_{k}\right) \sigma(p)=f(p) \quad p \in S .\right. \tag{2.25}
\end{equation*}
$$

Again, we obtain the Neumann problem by setting $h(p) \equiv 0$ in (2.25). Once $\sigma$ is known (2.23) can be used to obtain $\phi\left(p_{+}\right)$. These layer potential methods for obtaining an integral equation representation of the solution of Helmholtz equation are referred to as indirect formulations since the integral equation is not solved directly for a field quantity such as $\phi$ or $\frac{\partial \phi}{\partial n}$, but for a layer density $\sigma$ on $S$ which has no physical significance.

### 2.2.2 Direct Integral Equation Formulations of the Exterior Helmholtz Problem.

An alternative way of obtaining an integral equation formulation is to use the direct method based on Green's Second Theorem

$$
\begin{equation*}
\int_{\delta V}\left\{\phi_{1} \frac{\partial \phi_{2}}{\partial n}-\phi_{2} \frac{\partial \phi_{1}}{\partial n}\right\} d S=\int_{V}\left\{\phi_{1} \nabla^{2} \phi_{2}-\phi_{2} \nabla^{2} \phi_{1}\right\} d V \tag{2.26}
\end{equation*}
$$

where $V$ is a domain surrounded by a surface $\delta V^{\circ}$ which is not necessarily simply connected. Since $D_{-}$is unbounded, this result cannot be applied directly with $V=D_{+}$. Let $\Sigma_{R}$ be a sphere of radius $R$ completely surrounding the surface $S$, and let $V$ be the region between $S$ and $\Sigma_{R}$. Putting $\phi_{1}=\phi$ and $\phi_{2}=G_{k}$ leads to

$$
\begin{align*}
\int_{S}\left\{\phi(q) \frac{\partial G_{k}(p, q)}{\partial n_{q}}\right. & \left.-G_{k}(p, q) \frac{\partial \phi(q)}{\partial n_{q}}\right\} d S_{q}+\int_{\Sigma_{R}}\left\{\phi(q) \frac{\partial G_{k}(p, q)}{\partial n_{q}}-G_{k}(p, q) \frac{\partial \phi(q)}{\partial n_{q}}\right\} d S_{q} \\
& =\int_{V}\left\{\phi(q) \nabla^{2} G_{k}(p, q)-G_{k}(p, q) \nabla^{2} \dot{\phi}(q)\right\} d V \tag{2.27}
\end{align*}
$$

Writing the integral over $\Sigma_{R}$ as

$$
\begin{equation*}
\int_{\Sigma_{R}} \phi(q)\left\{\frac{\partial G_{k}(p, q)}{\partial n_{q}}-i k G_{k}(p, q)\right\} d S_{q}-\int_{\Sigma_{R}} G_{k}(p, q)\left\{\frac{\partial \phi(q)}{\partial n_{q}}-i k \phi(q)\right\} d S_{q} \tag{2.28}
\end{equation*}
$$

and on taking the limit $R \rightarrow \infty$ the region $V$ becomes $D_{+}$and the integral over $\Sigma_{R}$ becomes zero since both $G_{k}$ and $\phi$ satisfy the radiation condition. Hence

$$
\begin{equation*}
\int_{S}\left\{\phi(q) \frac{\partial G_{k}(p, q)}{\partial n_{q}}-G_{k}(p, q) \frac{\partial \phi(q)}{\partial n_{q}}\right\} d S_{q}=\int_{D_{+}}\left\{\phi(q) \Gamma^{2} G_{k}(p, q)-G_{k}(p, q) \nabla^{2} \phi(q)\right\} d \upharpoonright_{q} . \tag{2.29}
\end{equation*}
$$

Now write the integral over $D_{+}$as

$$
\begin{equation*}
\int_{D_{+}} \phi(q)\left\{\nabla^{2} G_{k}(p, q)+k^{2} G_{k}(p, q)\right\} d V_{q} . \tag{2.30}
\end{equation*}
$$

If $p \in D_{-}$then $G_{k}$ satisfies Helmholtz equation and so the integral over $D_{+}$is zero. If $p \in D_{+}$then $G_{k}$ is singular when $p=q$ and the use of Green's theorem is not valid. However, if we surround the point $p$ with a sphere of radius $\varepsilon$ with surface $S_{e}$, Green's theorem can be applied to the remaining volume. This gives an extra surface integral over $S_{\varepsilon}$, which on taking the limit $\varepsilon \rightarrow 0$ gives $[14,20]$

$$
\begin{equation*}
\int_{D_{+}} \phi(q)\left\{\nabla^{2} G_{k}(p, q)+k^{2} G_{k}(p, q)\right\} d V=\phi(p) \quad p \in D_{+} . \tag{2.31}
\end{equation*}
$$

If $p \in S$ then we proceed as for $p \in D_{+}$except that we surround $p$ by a small hemisphere, and obtain $[14,20]$

$$
\begin{equation*}
\int_{D_{+}} \phi(q)\left\{\nabla^{2} G_{k}(p, q)+k^{2} G_{k}(p, q)\right\} d V=\frac{\phi(p)}{2} \quad p \in S \tag{2.32}
\end{equation*}
$$

provided $p$ is not on an edge or vertex of $S$.
Combining (2.29) with (2.30), (2.31) and (2.32) gives the Helmholtz formula

$$
\int_{S}\left\{\phi(q) \frac{\partial G_{k}(p, q)}{\partial n_{q}}-G_{k}(p, q) \frac{\partial \phi(q)}{\partial n_{q}}\right\} d S_{q}= \begin{cases}0 & p \in D_{-}  \tag{2.33}\\ \frac{\phi(p)}{2} & p \in S \\ \phi(p) & p \in D_{+}\end{cases}
$$

The equation for $p \in S$ can be written in operator notation as

$$
\begin{equation*}
\left(-\frac{1}{2} I+M_{k}\right) \phi=L_{k} \frac{\partial \phi}{\partial n} \quad p \in S \tag{2.34}
\end{equation*}
$$

This equation is known as the Surface Helmholtz Equation (SHE) which is a first kind Fredholm integral equation for $\frac{\partial \phi}{\partial n}$ if we have a Dirichlet boundary condition, or a second kind Fredholm integral equation for $\phi$ if we have a Neumann or Robin boundary condition. This method of obtaining an integral equation is known as a
direct method since the physical quantities $\phi$ and $\frac{\partial \phi}{\partial n}$ are related directly through (2.34).

For the problem of a structure scattering an incident acoustic wave, the total wave $\phi$ can be written as $\phi=\phi_{s}+\phi_{i n c}$ where $\phi_{s}$ is the scattered wave and $\phi_{i n c}$ is the incident wave. Since $\phi_{s}$ is a radiating wave function, it must satisfy (2.33). The incident wave does not satisfy (2.33) since it does not satisfy the radiation condition, but a similar condition [20]

$$
\begin{equation*}
\lim _{r=i r \mid \rightarrow \infty} r\left\{\frac{\partial \phi(r)}{\partial r}+i k \phi(r)\right\}=0 \tag{2.35}
\end{equation*}
$$

uniformly in all direction $\underline{r} /|\underline{r}|$. Performing a similar analysis to that above yields

$$
\int_{S}\left\{\phi_{i n c}(q) \frac{\partial G_{k}(p, q)}{\partial n_{q}}-G_{k}(p, q) \frac{\partial \phi_{i n c}(q)}{\partial n_{q}}\right\} d S_{q}= \begin{cases}0 & p \in D_{-}  \tag{2.36}\\ -\frac{\phi_{i n c}(p)}{2} & p \in S \\ -\phi_{i n c}(p) & p \in D_{-}\end{cases}
$$

To obtain an integral equation for the total wave function, we add (2.33) with $\phi$ replaced by $\phi_{s}$ to (2.36) to obtain

$$
\int_{S}\left\{\phi(q) \frac{\partial G_{k}(p, q)}{\partial n_{q}}-G_{k}(p, q) \frac{\partial \phi(q)}{\partial n_{q}}\right\} d S_{q}= \begin{cases}-\phi_{i n c}(p) & p \in D_{-}  \tag{2.37}\\ \frac{\phi(p)}{2}-\phi_{i n c}(p) & p \in S \\ \phi(p)-\phi_{i n c}(p) & p \in D_{+}\end{cases}
$$

We now briefly consider the integral equation formulations of the interior Helmholtz problem, which are required in the next section for determining whether or not unique solutions to the integral equation formulations of the exterior problem exist. Let $\phi_{i}(p)$ be a solution of the interior Helmholtz equation with a boundary
condition of the form (2.3), (2.4) or (2.5) on $S$. It can be shown, by applying Green's Second Theorem in the interior domain $D_{-}$, that the appropriate integral relation between $\phi_{i}$ and $\frac{\partial \phi_{i}}{\partial n}$ is equation (2.36) with $\phi_{i n c}$ replaced by $\phi_{i}$.

### 2.3 Basic Integral Equation Theory

In this section we review some of the classical theory for operator equations of the form

$$
\begin{equation*}
(-\lambda I+\mathcal{K}) \phi=f \tag{2.38}
\end{equation*}
$$

where $\lambda$ is a complex constant and $\mathcal{K}$ is a compact linear operator. Clearly the integral equations introduced in the previous section may be written in the form (2.38). Later in this section it will be established whether or not the operators $L_{k}$, $M_{k}, M_{k}^{T}$ and $N_{k}$ are compact. The theory may then be used to deduce that the solutions to the integral equations either do not exist or are non-unique for certain values of the wavenumber $k$. All the results that follow are classical results whose proofs can be found in $9,20,46]$.

## DEFINITION 2.1

A linear operator $K: X \rightarrow Y$ from a normed space $X$ to a normed space $Y$ is called compact if it maps any bounded set in $X$ into a relatively compact set in $Y$, where a set is said to be relatively compact if every sequence in the set contains a convergent subsequence.

## THEOREM 2.2

All compact operators are bounded. In addition, any linear combination of
compact linear operators is compact and the product of $t$ wo bounded operators is compact if one of the operators is compact.

## DEFINITION 2.2

Let $X$ be a normed space and $Y$ be a Banach space. Let $\mathcal{K}_{n}: X \rightarrow Y$ be a sequence of bounded linear operators. $\left\{\mathcal{K}_{n}\right\}$ is said to converge pointwise to $\mathcal{K}$ if

$$
\begin{equation*}
\left\|\left(\mathcal{K}-\mathcal{K}_{n}\right) x\right\| \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty \tag{2.39}
\end{equation*}
$$

for all $x \in X$, and said to converge uniformly if

$$
\begin{equation*}
\left\|\mathcal{K}-\mathcal{K}_{n}\right\| \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty \tag{2.40}
\end{equation*}
$$

If $\left\{\mathcal{K}_{n}\right\}$ converges uniformly to $\mathcal{K}$ then clearly $\left\{\mathcal{K}_{n}\right\}$ converges pointwise to $\mathcal{K}$. However the converse is not true.

## THEOREM 2.3

Let $X$ and $Y$ be as in Definition 2.2. If the sequence $\left\{\mathcal{K}_{n}\right\}$ of compact integral operators converges uniformly to $\mathcal{K}$ then $\mathcal{K}$ is compact.

## THEOREM 2.4

Let $\mathcal{K}: X \rightarrow Y$ be a bounded linear operator with finite-dimensional range $\mathcal{K}(X)$. Then $\mathcal{K}$ is compact.

We now introduce the following related operators which are required in our analysis.

## DEFINITION 2.3

Let $\mathcal{K}: X \rightarrow Y$ be an integral operator given by

$$
\begin{equation*}
(\mathcal{K} \sigma)(p)=\int_{S} k(p, q) \sigma(q) d S_{q} \quad \sigma E X \tag{2.41}
\end{equation*}
$$

Then the transposed operator, $\mathcal{K}^{T}$, is given by

$$
\begin{equation*}
\left(\mathcal{K}^{T} \sigma\right)(p)=\int_{S} k(q, p) \sigma(q) d S_{q} \quad \sigma \in X \tag{2.42}
\end{equation*}
$$

and the adjoint operator, $\mathcal{K}^{*}$, is given by

$$
\begin{equation*}
\left(\mathcal{K}^{*} \sigma\right)(p)=\int_{S} \overline{k(q, p)} \sigma(q) d S_{q} \quad \sigma \in X^{-} \tag{2.43}
\end{equation*}
$$

where $\overline{k(p, q)}$ denotes the complex conjugate of $k(p, q)$. An integral operator $\mathcal{K}$ is said to be symmetric if $k(p, q)=k(q, p)$ for all $p$ and $q$, and is said to be self-adjoint if $k(p, q)=\overline{k(q, p)}$ for all $p$ and $q$.

It is obvious from (2.8) and (2.18) that $L_{k}$ and $V_{k}$ are symmetric operators, and from (2.9) and (2.17) that $M_{k}^{T}$ is the transpose of $M_{k}$.

## DEFINITION 2.4

The null space of a linear operator $\mathcal{K}: X \rightarrow X$, is given by

$$
\begin{equation*}
N u l l(\mathcal{K})=\{\phi \in X \quad \text { such that } \quad \mathcal{K} \phi=0\} . \tag{2.44}
\end{equation*}
$$

## THEOREM 2.5

The null space of the operator $-\lambda I+\mathcal{K}$, where $\mathcal{K}$ is a compact operator, is of finite dimension.

We are now able to state an important result concerning the existence and uniqueness of the solution of (2.38).

THEOREM 2.6 (Fredholm Alternative)
Let $\mathcal{K}: X \rightarrow X$ and $\mathcal{K}^{*}: Y \rightarrow Y$ be compact adjoint linear operators. Then, either $A$ :

$$
N u l l(-\lambda I+\mathcal{K})=\{0\} \quad \text { and } \quad N u l l\left(-\bar{\lambda} I+\mathcal{K}^{*}\right)=\{0\}
$$

and

$$
(-\lambda I+\mathcal{K})(X)=(X) \quad \text { and } \quad\left(-\bar{\lambda} I+\mathcal{K}^{*}\right)(Y)=(Y)
$$

or B :

$$
\operatorname{dim}(\operatorname{Null}(-\lambda I+\mathcal{K}))=\operatorname{dim}\left(\operatorname{Null}\left(-\bar{\lambda} I+\mathcal{K}^{*}\right)\right)
$$

and

$$
(-\lambda I+\mathcal{K})(X)=\left\{f \in X \quad \text { such that } \quad(f, \psi)=0 \psi \in \operatorname{Null}\left(-\bar{\lambda} I+\mathcal{K}^{*}\right)\right\}
$$

and

$$
\left(-\bar{\lambda} I+\mathcal{K}^{*}\right)\left(Y^{*}\right)=\left\{f \in Y^{\prime} \quad \text { such that } \quad(f, \phi)=0 \quad \phi \in \operatorname{Null}(-\lambda I+\mathcal{K})\right\}
$$

The values of $\lambda$ for which $\operatorname{Null}(-\lambda I+\mathcal{K})$ is not $\{0\}$ are called the eigenvalues of $\mathcal{K}$ and the $\phi \in \operatorname{Null}(-\lambda I+\mathcal{K})$ are called the eigenfunctions. A consequence of this theorem is that if $\lambda$ is not an eigenvalue of $\mathcal{K}$ and provided $\mathcal{K}$ is compact then (2.38) will always have a unique solution for $f \in X$. If $\lambda$ is an eigenvalue of $\mathcal{K}$, then (2.38) will have a solution provided

$$
\begin{equation*}
(f, \psi)=0 \quad \text { for all } \quad \psi \in \operatorname{Null}\left(\bar{\lambda} I-\mathcal{K}^{*}\right) \tag{2.45}
\end{equation*}
$$

That is, provided the right hand side of (2.38) is orthogonal to all the eigenfunctions of the adjoint operator corresponding to the eigenvalue $\bar{\lambda}$. However in this case the solution obtained will not be unique.

## THEOREM 2.7

If $\lambda$ is an eigenvalue of $\mathcal{K}$, then it is also an eigenvalue of $\mathcal{K}^{T}$, and $\bar{\lambda}$ is an eigenvalue of $\mathcal{K}^{*}$.

Proof.
The fact that $\bar{\lambda}$ is an eigenvalue of $\mathcal{K}$ • follows from the Fredholm Alternative Theorem (Theorem 2.6). Hence

$$
\begin{equation*}
\left(-\bar{\lambda} I+\mathcal{K}^{*}\right) \phi=0, \quad \phi \in \operatorname{null}\left(-\bar{\lambda}+\mathcal{K}^{*}\right), \quad \phi \neq 0 . \tag{2.46}
\end{equation*}
$$

Taking the complex conjugate of this gives the result for $\mathcal{K}^{T}$.
Before discussing the compactness properties of the layer potential operators we must define the appropriate function spaces in which we will be working.

## DEFINITION 2.5

Let $C(S)$ denote the space of continuous functions, with the norm $\|\phi\|_{\infty}=$ sup : $\phi(q) \mid, q \in S$, and let $C^{n}(S)$ denote the sub-space of $C(S)$ consisting of functions with continuous derivatives of up to order $n$ with norm

Let $L_{2}(S)$ denote the space of functions which satisfy

$$
\begin{equation*}
\|\phi\|_{2}^{2}=\int_{S}|\phi(q)|^{2} d S_{q}<\infty \tag{2.47}
\end{equation*}
$$

and $H^{r}(S)$ denote the Sobolev space of functions whose gencralised derivatives of order $n$ exist and are in $L_{2}(S)$ and with norm $\mid 32$;

$$
\|\phi\|^{2}=\sum_{n_{1}+n_{2}+n_{3} \leq n} \int_{S}\left|\frac{\partial_{1}^{n_{1}+n_{2}+n_{3}}\langle(q)}{\partial x^{2} 1 y_{3} y_{3} \partial_{2} \partial_{3} n_{3}}\right|^{2} d S_{q} \quad n_{1}, n_{2}, n_{3} \geq 0 .
$$

## THEORE. 2.8

An integral operator with a kernel function which is either continuous for all
$p, q \in S$, or which is continuous for all $p, q \in S, p \neq q$ and

$$
\begin{equation*}
|k(p, q)| \leq M|p-q|^{\alpha-2} \quad p \neq q \tag{2.48}
\end{equation*}
$$

where $M$ is a positive constant and $\alpha \in(0,2]$, is compact on $C(S)$ and $L_{2}(S)$.
Any kernel function which satisfies condition (2.48) above is said to be weakly singular. It is obvious that the free space Green's function $G_{k}(p, q)$ defined by (2.11) satisfies (2.48) with $M=1$ and $\alpha=1$ and hence $L_{k}$ is compact. Writing

$$
\begin{equation*}
\frac{\partial G_{k}(p, q)}{\partial n_{q}}=\frac{(1-i k|p-q|) G_{k}(p, q)(p-q) \cdot n_{q}}{4 \pi|p-q|^{2}} \tag{2.49}
\end{equation*}
$$

and noting that $(p-q) . n_{q} \leq L|p-q|^{2}\left(\right.$ see $[20 j)$ shows that $M_{k}$ is also compact. Similarly it is possible to show that $M_{k}^{T}$ is compact. However, in view of (2.19) $N_{k}$ can be shown to be a non-compact operator on $C(S)$ or $L_{2}(S),[20]$. However, it can be shown that

$$
\begin{equation*}
\lim _{q \rightarrow p} \frac{\partial^{2}}{\partial n_{p} \partial n_{q}}\left\{G_{k}(p, q)-G_{0}(p, q)\right\}=\frac{1}{4 \pi \mid p-q}+o(1) \tag{2.50}
\end{equation*}
$$

and hence the operator $N_{k}-N_{0}$ is compact if it is treated as a single operator $[14,20]$.

Concentrating on the Sobolev space $H^{r}(S)$ we have the following result, the proof of which may be found in Colton and Kress [20;.

## THEOREM 2.9

The operators $L_{k}: H^{r}(S) \rightarrow H^{r+1}(S), M_{k}: H^{r}(S) \rightarrow H^{r}(S)$ and $M_{k}^{T}:$ $H^{r}(S) \rightarrow H^{r}(S)$ are compact. The operator $N_{k}: H^{r}(S) \rightarrow H^{r-1}(S)$ is bounded.

Having established that the operators $L_{k}, M_{k}$ and $M_{k}^{T}$ are compact operators, it is necessary to consider whether or not the null-spaces of the operators in (2.21), (2.22) and (2.34) contain non-zero elements. If a null-space only contains the zero element then we shall refer to it as a trivial null-space. Consider the interior Helmholtz problem with a homogeneous Dirichlet boundary condition. It is well known that this problem has a non-trivial solution $\phi_{i}$ for a countably infinite set of values of $k$, denoted $I_{D} ;[14]$. According to the remarks at the end of Section 2.2.2, the funcions $\phi_{i}$ must satisfy the interior Helmholtz formula (2.36) for $p \in D_{-}$ since this equation was derived from Green's theorem. Hence, for these values of $k$ we must have $\frac{\partial \phi_{i}}{\partial n} \not \equiv 0, p \in S$. Writing (2.36) for $p \in S$ in operator notation and applying the Dirichlet condition gives

$$
\begin{equation*}
L_{k} \frac{\partial \phi_{i}}{\partial n}=0 \tag{2.51}
\end{equation*}
$$

implying that the null-space of $L_{k}$ is non-trivial for $k \in I_{D}$.
If $k \nexists I_{D}$, let $u=L_{k} \frac{\partial \phi_{i}}{\partial n}$. Hence by (2.36) $u(p)=0$ for $p \in D_{-}$since $\phi_{i}(p)=0$ for $p \in S$ and $p \in D_{\text {.. }}$. Then, by the continuity of $L_{k}$, it is clear that $u(p)=0$ for $p \in S$. Since $u(p)$ is a solution of the exterior Helmholtz equation, which is alway unique, (see Theorem 2.1), we have that $u(p)=0, p \in D_{-}$. Let $\frac{\partial u}{\partial n_{+}}$and $\frac{\partial u}{\partial n_{-}}$denote the normal derivative of $u$ for $p_{+} \in D_{+}$and $p_{-} \in D_{-}$respectively, then $\frac{\partial u}{\partial n_{-}}=\frac{\partial u}{\partial n_{-}}=0$. Taking the appropriate limits as $p_{ \pm} \rightarrow S$, and applying the jump conditions gives

$$
\begin{equation*}
\frac{\partial u}{\partial n_{-}}+\frac{\partial u}{\partial n_{+}}=\frac{\partial \phi_{i}}{\partial n} \quad p \in S \tag{2.52}
\end{equation*}
$$

which implies $\frac{\partial \phi_{i}}{\partial n}=0$. Therefore, the null-space of $L_{k}$ is non-trivial if and only if $k \in I_{D}$.

Differentiating (2.36) with respect to $\underline{n}_{p}$, and replacing $\phi_{i n c}$ with $\phi_{i}$, gives

$$
\frac{\partial}{\partial n_{p}} \int_{S} \phi_{i}(q) \frac{\partial G_{k}(p, q)}{\partial n_{q}} d S_{q}-\int_{S} \frac{\partial \phi_{i}(q)}{\partial n_{q}} \frac{\partial G_{k}(p, q)}{\partial n_{p}} d S_{q}= \begin{cases}-\frac{\partial \phi_{i}(p)}{\partial n_{p}} & p \in D_{-}  \tag{2.53}\\ -\frac{1}{2} \frac{\partial \phi_{i}(p)}{\partial n_{p}} & p \in S \\ 0 & p \in D_{+}\end{cases}
$$

Clearly $\phi_{i}$ and $\frac{\partial \phi_{i}}{\partial n}$ satisfy (2.53) if they satisfy (2.36). Applying the homogeneous Dirichlet boundary condition to (2.53), and writing the equation for $p \in S$ in operator notation yields

$$
\begin{equation*}
\left(-\frac{1}{2} I+M M_{k}^{T}\right) \frac{\partial \phi_{i}}{\partial n}=0 \tag{2.54}
\end{equation*}
$$

If $k \in I_{D}$ we know that a non-trivial $\frac{\partial \phi_{i}}{\partial n}$ exists, and so for these values of $k$, the null-space of $-\frac{1}{2} I+M_{k}^{T}$ is non-trivial. Using a similar technique to above, it is possible to show that for all other values of $k$, the null-space of $-\frac{1}{2} I+M_{k}^{T}$ is trivial.

We recall from (2.22) for the Neumann problem that the operator concerned is $-\frac{1}{2}+M I_{k}^{T}$. If $k \notin I_{D}(2.22)$ will always have a unique solution (by the Fredholm Alternative Theorem). However, if $k \in I_{D}$ then a non-unique solution will exist if and only if the right hand side of (2.22), $\frac{\partial \phi}{\partial n}=f$, is orthogonal to all the eigenfunctions of the adjoint operator. As $f$ can in general be any function, this condition will not hold and so (2.22) will not possess a solution for $k \in I_{D}$. We call any $k \in I_{D}$ a characteristic value of the integral operators $L_{k}$ and $-\frac{1}{2} I+M I_{k}^{T}$ and emphasise that $I_{D}$ denotes the set of values of $k$ for which the operators $L_{k}$ and $-\frac{1}{2}+M_{k}^{T}$ are
singular (ie the values of $k$ for which they have a zero eigenvalue).

Let us now consider the direct formulation, from Green's Theorem, of the Neumann problem. The operator on the left hand side of (2.34) is the transpose of the operator in (2.54) and so, by Theorem 2.8, they have the same eigenvalues and characteristic values. To see if a solution of (2.34) exists, apply Green's second theorem to $L_{k} f$, the right-hand side of (2.34), and $\phi_{i}$, an eigenfunction of the interior Dirichlet problem, to obtain

$$
\begin{equation*}
\int_{S} \frac{\partial \phi_{i}}{\partial n} L_{k} f d S-\int_{S} \phi_{i}\left(\frac{\partial}{\partial n} L_{k} f\right) d S=\int_{D_{-}}\left\{\phi_{i} \Gamma^{2}\left(L_{k} f\right)-L_{k} f \nabla^{2} \phi_{i}\right\} d V . \tag{2.55}
\end{equation*}
$$

Since both $L_{k} f$ and $\phi_{i}$ satisfy Helmholtz equation in $D_{-}$, and $\phi_{i}(p)=0$ for $p \in S$ then

$$
\begin{equation*}
\int_{S}\left(L_{k} f\right) \frac{\partial \phi_{i}}{\partial n} d S=0 . \tag{2.56}
\end{equation*}
$$

But, if $\phi_{i}$ is an interior eigenfunction $\left(-\frac{1}{2} I+M_{k}^{T}\right) \frac{\partial \phi_{1}}{\partial n}=0$ and hence $\left(-\frac{1}{2} I+\right.$ $M_{k}^{*} \frac{\overline{\phi_{i}}}{\partial n}=0$. That is, $\frac{\overline{\partial \phi_{1}}}{\partial n}$ is an eigenfunction of the adjoint operator. Therefore, in view of (2.56), the orthogonality condition will be satisfied for all $f$, and the formulation (2.34) for the Neumann problem will always have a solution, but it will be non-unique if $k \in I_{D}$.

We have shown that the integral equation formulations of the exterior Neumann problem will have a unique solution provided $k \nsubseteq I_{D}$. The problems associated with the characteristic wavenumbers are only due to the integral equation formulation since, by Theorem 2.1, the exterior Helmholtz problem will always have a unique solution if $k$ is real and positive. Clearly, for a given value of $k=\alpha$ say, the
conditioning of the integral equation (2.22) or (2.34) is inversely related to the distance of $\alpha$ from $I_{D}$. If $\operatorname{dist}\left(\alpha, I_{D}\right)$, the distance between $\alpha$ and the nearest element of $I_{D}$, is small then the formulation (2.22) or (2.34) is ill-conditioned and hence the results obtained from an approximate solution of (2.22) or (2.34) are likely to bear little or no resemblance to the exact solution $\phi(p)$. It is well-known that asymptotically ( as $k \rightarrow \infty$ ) the number of terms in $I_{D}$ less than a given value of $k$ is proportional to $k^{3}$ [14]. Clearly, as $k$ increases, $\operatorname{dist}\left(k, I_{D}\right)$ is likely to be small for all values of $k$. Hence we need to develop formulations which have unique solutions for all values of the wavenumber $k$.

### 2.4 Integral Equation Formulations Valid for all Wavenumbers.

As shown in the previous section, the classical integral equation formulations of the exterior Helmholtz problem at best do not possess unique solutions for certain values of the wavenumber $k$. A survey of the formulations valid for all wavenumbers, which were proposed before 1973, is given in Burton [14]. Here we present a brief review of the more recent suggestions for the exterior Neumann problem, as well as those which are often used in practice.

### 2.4.1 The Methods of Ursell and Jones.

Some of the most elegant methods for obtaining a uniformly valid integral equation formulation in $k$ use a modified Green's function. Ursell [56, 57] observed that we
can replace $G_{k}(p, q)$ by any fundamental solution to Helmholtz equation in the exterior domain $D_{+}$which satisfies the radiation condition. The derivation of the integral equations using either a direct or indirect formulation now follows as before, but by choosing an appropriate fundamental solution it is possible to eliminate all the interference from the interior eigenfunctions. Write the modified Green's function $G(p, q)$ as

$$
\begin{equation*}
G(p, q)=G_{k}(p, q)+\Gamma(p, q) \tag{2.57}
\end{equation*}
$$

where $\Gamma(p, q)$ is an analytic wave function in $D_{-}$and let $S_{R}$ denote a sphere of radius $R$ which is totally enclosed within $S$. If $\Gamma$ satisfies

$$
\begin{equation*}
\frac{\partial \Gamma(p, q)}{\partial n}+C \Gamma(p, q)=0 \quad p E S_{R} \tag{2.58}
\end{equation*}
$$

where $C$ is a complex constant with $\operatorname{im}(C)>0$, then the solution to the integral equations introduced in Section 2.2 with $G$ replacing $G_{k}$ are unique. ©rsell $[56,57$, shows that $\Gamma(p, q)$ can be chosen to be an infinite series of spherical wave functions, which converges quickly for small $k$, but more slowly as $k$ increases.

A modification to this method is to replace the infinite series with a finite one. Jones [36] suggests the choice

$$
\begin{equation*}
\Gamma(p, q)=\sum_{m=0}^{M} \sum_{n=0}^{m} b_{m n} \imath_{m n}^{\prime}(p) \Psi_{m n}(q) \tag{2.59}
\end{equation*}
$$

where

$$
\begin{align*}
& \psi_{m n}^{\prime}(p)=h_{m}\left(k r_{p}\right) P_{m}^{n}\left(\cos \theta_{p}\right) \cos \phi_{p}  \tag{2.60}\\
& \Psi_{m n}(p)=h_{m}\left(k r_{p}\right) P_{m}^{n}\left(\cos \theta_{p}\right) \sin \phi_{p} \tag{2.61}
\end{align*}
$$

and the point $p$ has spherical polar coordinates $\left(r_{p}, \theta_{p}, \phi_{p}\right)$. The functions $h_{m}\left(k r_{p}\right)$ are the spherical Hankel functions of the first kind and $P_{m}^{n}\left(\cos \theta_{p}\right)$ are the associated Legendre functions. If we label the elements of $I_{D}$ such that $k_{1}<k_{2}<\ldots<k_{n}<$ ... then it is possible to show that if $k<k_{M+2}$ the integral equations (2.22) and (2.34) with Green's function (2.57) and $\Gamma(p, q)$ given by (2.59) have a unique solution provided the $b_{m n}$ 's are real and non-zero. In Kleinmann and Kress [37] it is suggested that the $b_{m n}$ 's are chosen so to minimise the condition number of the resulting integral operator.

This method has the deficiency that, for an arbitrary surface, we do not know how large to take $M$ since, in general, we do not know the elements of $I_{D}$. Also, as mentioned previously, the number of elements of $I_{D}$ less than a given value of $k$ increases proportionately to $k^{3}$, so for a moderately large value of $k$ it is necessary to take a large number of terms. The excessive cost of evaluating $\Gamma(p, q)$ and hence $G(p, q)$ for a large number of $p$ and $q$ has meant that there has been little or no use of the modified Green's function methods for practical problems.

### 2.4.2 Layer Potential Formulations Valid for all Wavenumbers.

Methods for obtaining a unique solution to the Neumann problem using all indirect formulation typically consist of taking a combined layer potential of the form (Panich i47)

$$
\begin{equation*}
\phi(p)=\left(L_{k}+\mu M I_{k}\right) \sigma(p) \quad p \in D_{+} \tag{2.62}
\end{equation*}
$$

where $\mu$ is a complex constant, chosen to be $+i$ if $\operatorname{Re}(k) \geq 0[14,16]$. Taking the appropriate limit as $p_{+} \rightarrow S$, using the jump properties given in Lemma 2.1 and applying the boundary condition (2.4) gives

$$
\begin{equation*}
\left(-\frac{1}{2} I+M_{k}^{T}+\mu N_{k}\right) \sigma=\frac{\partial \phi}{\partial n}=f \quad p \in S \tag{2.63}
\end{equation*}
$$

This expression now contains the hyper-singular operator $N_{k}$, which is not compact.
A similar formulation, proposed by Panich [47], is to take a hybrid potential of the form

$$
\begin{equation*}
\phi(p)=\left(L_{k} \sigma \div \mu_{k} M_{k} L_{0}\right) \sigma(p) \quad p \cong D_{+} \tag{2.64}
\end{equation*}
$$

where the suffix 0 denotes putting $k=0$ in the definition of the potentials. The resulting integral equation corresponding to (2.64) is

$$
\begin{equation*}
\left(-\frac{1}{2} I+M_{k}^{T}+\mu V_{k} L_{0}\right) \sigma=\frac{\partial \phi}{\partial n}=f \quad p \in S . \tag{2.65}
\end{equation*}
$$

Writing $N_{k} L_{0}=\left(N_{k}-N_{0}\right) L_{0}+N_{0} L_{0}$ and using the identity [47]

$$
\begin{equation*}
N_{0} L_{0}=\left(M_{0}^{T}+\frac{1}{2} I\right)\left(M_{0}^{T}-\frac{1}{2} I\right) \tag{2.66}
\end{equation*}
$$

gives

$$
\begin{equation*}
\left(-\frac{1}{2} I+M I_{k}^{T}+\mu\left[\left(N_{k}-N_{0}\right)+\left(M M_{0}^{T}\right)^{2}-\frac{1}{4} I\right]\right) \sigma=\frac{\partial \phi}{\partial n}=f \tag{2.67}
\end{equation*}
$$

in which all the operators are weakly singular and hence compact. It is now possible to use the Fredholm theory to deduce that (2.67) has a unique solution for all real values of $k \geq 0$ provided $\operatorname{im}(\mu)>0$, see Colton and Kress 20].

### 2.4.3 The Method of Schenck.

Various methods for obtaining a unique solution in the case of a direct formulation, based on taking advantage of equation (2.33) for $p \in D_{-}$, have been proposed. Equation (2.33) for $p \in D_{\text {- }}$ provides an integral relationship of the form

$$
\begin{equation*}
\left(M_{k} \phi\right)(p)=\left(L_{k} \frac{\partial \phi}{\partial n}\right)(p) \quad p \in D_{-} . \tag{2.68}
\end{equation*}
$$

In Schenck [49] it is shown that for any $k$, there is only one solution to (2.34) which also satisfies (2.68) for all $p \in D_{\text {_ }}$. Copley [21] suggested a scheme for enforcing (2.68) at a finite number of points $\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ in $D_{-}$. But for a bad choice of $p \in D_{-}$this does not eliminate the interference of the interior eigenfunctions $\phi_{i}$. Clearly $\phi_{i}$ must satisfy the corresponding interior equation (see equation (2.36) with $\phi_{i}$ replacing $\phi_{i n c}$ )

$$
\begin{equation*}
\left(I+M I_{k}\right) \phi_{i}=L_{k} \frac{\partial \phi_{i}}{\partial n} \quad p \in D_{-} . \tag{2.69}
\end{equation*}
$$

If $p$ lies on a nodal surface of $\phi_{i}$, that is, a point where $\phi_{i}(p)=0$, then (2.69) is the
 will also satisfy (2.68) and will not have eliminated the non-uniqueness, [21, 49].

A method for overcoming this problem, suggested in Cunefare et al [22], is to take a linear combination of (2.68) and its normal derivative. However, as the points $\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ do not lie on the surface $S$ it is not clear how to define the normal derivative of (2.68). Further, it can be shown that any numerical scheme based on (2.68) has undesirable characteristics as the number of points increases, see [49].

Shenck (49) devised a scheme which supplements a suitable discretisation of (2.34) (see Chapter 3) with some extra equations derived from discretising (2.68) for only a few points in $D_{\text {. }}$. The resulting non-square system of equations is then solved by a least-squares procedure. This method is known as the Combined Helmholtz Integral Equation Formulation, or by the acronym CHIEF. The main disadvantage of this scheme is that there is no way of telling how many interior points are needed, or where to place them, since if they are placed on a nodal surface of an eigenfunction, the method will still fail for the reasons given above. However, since this method is relatively simple to implement, it is often successfully used practice for modest values of $k[7,50]$.

### 2.4.4 The Exterior Over-Determination Formulation.

Recently a method similar to the CHIEF but using the exterior relation

$$
\begin{equation*}
\phi(p)=\left(M_{k} \phi\right)(p)-\left(L_{k} \frac{\partial \phi}{\partial n}\right)(p) \quad p \in D_{+} \tag{2.70}
\end{equation*}
$$

has been proposed in Piaszczyk and Klosner :48]. Since $\dot{\varphi}(p)$ for $p \in D_{+}$is not known, we assume that there is some function $Z(p)$ which gives a simple impedance relationship of the form

$$
\begin{equation*}
\dot{\phi}(p)=Z(p) \frac{\partial \phi(p)}{\partial n} \quad p \in S . \tag{2.71}
\end{equation*}
$$

Substituting (2.71) into (2.70) enables $\phi(p)$ to be computed for a few points $p \in$ $D_{+}$. Using these values and a suitable discretisation of (2.34) we form an overdetermined system for $\phi(p)$ on $S$, which is then solved using a least-squares pro-
cedure. Once $\phi(p)$ on $S$ has been found, we can re-compute the $\phi(p)$ in $D_{+}$and repeat the process until a convergence criterion is satisfied. A proof of the uniqueness of this solution is given in Piaszczyk and Klosner [48]. This method obviously requires the solution of an over-determined system at each iteration, which is expensive. The rate of convergence depends on the choice of the function $Z(p)$, and there are doubts as to whether the process will converge [22].

### 2.4.5 The Method of Burton and Miller.

The method of Burton and Miller is the direct formulation counterpart of the indirect formulation of Panich [47]. On formally differentiating the surface Helmholtz equation (2.34) in the direction of the normal to $S$ at $p$, we obtain

$$
\begin{equation*}
N_{k} \phi=\left(\frac{1}{2} I+M_{k}^{T}\right) \frac{\partial \phi}{\partial n} \quad p \in S \tag{2.72}
\end{equation*}
$$

Burton and Miller [13] suggest taking a linear combination of (2.i2) and (2.34) in the form

$$
\begin{equation*}
\left(-\frac{1}{2} I+M_{k} \div \alpha V_{k}\right) \phi=L_{k}+\alpha\left(\frac{1}{2} I-M M_{k}^{T}\right) \cdot \frac{\partial \phi}{\partial n} \quad p \in S \tag{2.73}
\end{equation*}
$$

where $\alpha$ is an arbitrary coupling parameter. It has been shown $[13,41]$ that if $\alpha$ is taken as a constant, with $i m(\alpha)>0$ then the homogeneous form of (2.73)

$$
\begin{equation*}
\left(-\frac{1}{2} I+M I_{k}+\alpha V_{k}\right) \phi=0 \tag{2.74}
\end{equation*}
$$

has only the trivial solution $\phi \equiv 0$ for real values of $k$, hence ensuring that the solution of (2.73) is unique. Below we shall extend this result to the case where $\alpha$
is taken to be a function of $p$ (see also [41]). This result will be used later in the study of the conditioning of this formulation.

## THEOREM 2.9

Provided $\operatorname{im}(\alpha(p))>0$ for all $p \in S$, then the only solution to (2.74) is the trivial solution $\phi(p) \equiv 0$.

Proof.
Let $v_{i}\left(p_{-}\right)=M_{k} \phi\left(p_{-}\right)$for $p_{-} \in D_{-}$. Taking the limit as $p_{-} \rightarrow S$ and substituting $v_{i}$ into (2.74) gives

$$
\begin{equation*}
-v_{i}(p)+\alpha(p) \frac{\partial v_{i}(p)}{\partial n}=0 \quad p \in S . \tag{2.75}
\end{equation*}
$$

Applying Green's Second Theorem to $v_{i}$ and $\overline{v_{i}}$ we obtain

$$
\begin{equation*}
\int_{D_{-}}\left\{v_{i}\left(\Gamma^{2} \overline{v_{i}}\right)-\left(\Gamma^{2} v_{i}\right) \overline{v_{i}}\right\} d V=\int_{S}\left\{v_{i} \frac{\partial \overline{v_{i}}}{\partial n}-\overline{v_{i}} \frac{\partial v_{i}}{\partial n}\right\} d S . \tag{2.76}
\end{equation*}
$$

Since both $v_{i}$ and $\bar{v}_{i}$ satisfy Helmholtz equation, the integral over $D_{-}$is zero. Hence, from (2.75),

$$
\begin{equation*}
\int_{S} \alpha(p) \frac{\partial v_{i}}{\partial n} \frac{\partial \bar{v}_{i}}{\partial n}-\overline{\alpha(p)} \frac{\partial v_{i}}{\partial n} \frac{\partial \overline{v_{i}}}{\partial n} d S=2 i \int_{S} i m(\alpha(p))\left|\frac{\partial v_{i}}{\partial n}\right|^{2} d S=0 \tag{2.77}
\end{equation*}
$$

If we choose $\alpha(p)$ to be any function satisfying $\operatorname{im}(\alpha(p))>0$ for all $p \in S,(2.77)$ implies that $\frac{\partial v_{i}}{\partial n}=0$, and hence from (2.75), $v_{i}=0$. Since $\frac{\partial M_{k}}{\partial n}$ is continuous throughout $\mathcal{R}^{3}$, then

$$
\begin{equation*}
\frac{\partial v_{e}}{\partial n}=0 \quad p \in S \tag{2.78}
\end{equation*}
$$

where $v_{e}(p) \doteq M_{k} \phi(p)$ for $p \in D_{+}$. Now since $v_{e}$ is a radiating wave function, and since (2.78) holds, then $v_{e}(p)=0$ for $p \in D_{+}$as the solution to the exterior
problem is unique. Hence it follows that $v_{\mathrm{e}}(p)=0$ for $p \in S$, see [14]. Now, for $p \in S$ we have

$$
\begin{equation*}
v_{e}=M_{k} \phi-\frac{\phi}{2} \tag{2.79}
\end{equation*}
$$

and

$$
\begin{equation*}
v_{i}=M_{k} \phi+\frac{\phi}{2} \tag{2.80}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\phi(p)=\left(v_{i}(p)-v_{e}(p)\right)=0 \quad p \text { ミS. } \tag{2.81}
\end{equation*}
$$

Hence the only solution to (2.74) is $\phi(p)=0$ provided $\operatorname{im}(\alpha(p))>0$.
We note that $i m(\alpha(p))>0$ is a sufficient condition for a unique solution, but it is not necessary. It seems, therefore, that taking $\operatorname{im}(\alpha(p))=0$ over a set of measure zero can be acceptable. The formulation (2.73) is robust since, with an appropriate choice of $\alpha(p)$, (see Section 3.4) it will in general be well conditioned and will always yield a unique solution.

However, the formulation (2.73) introduces the non-compact operator $V_{k}$. To allow us to work with compact operators, and hence use the Fredholm theory to establish the existence of a solution, we regularise $N_{k}$ as follows [15].

Consider the operator equation

$$
\begin{equation*}
\mathcal{K} \phi=f \tag{2.82}
\end{equation*}
$$

where $\mathcal{K}$ is not a compact operator. We may regularise $\mathcal{K}$ by multiplying it by another operator $\mathcal{K}_{1}$ to obtain an operator equation of the form

$$
\begin{equation*}
\mathcal{K}_{1} \mathcal{K} \phi=\left(I-\mathcal{K}_{2}\right) \phi=\mathcal{K}_{1} f \tag{2.83}
\end{equation*}
$$

where $\mathcal{K}_{2}$ is compact. We say that it is an equivalent regularisation if and only if the equation (2.83) and(2.82) have the same solutions. In particular, there must be no change in the number of solutions.

Similar to Panich [47], we can write $N_{k}=\left(N_{k}-N_{0}\right)+N_{0}$ and use the operator identity

$$
\begin{equation*}
L_{k} N_{k}=\left(M I_{k}-\frac{1}{2} I\right)\left(M I_{k}+\frac{1}{2} I\right) \tag{2.84}
\end{equation*}
$$

with $k=0$ to obtain from (2.72)

$$
\begin{equation*}
\left(L_{0}\left(N_{k}-N_{0}\right)+M_{0}^{2}-\frac{1}{4} I\right) \phi=L_{0}\left(\frac{1}{2} I+M_{k}^{T}\right) \frac{\partial \phi}{\partial n} . \tag{2.85}
\end{equation*}
$$

We cannot multiply (2.72) by $L_{k}$ and apply (2.84) directly since it would not be an equivalent regularisation for all $k$, ie $L_{k}$ may be singular for certain values of $k$. Equation (2.85) may now be coupled with (2.34) to yield the regularised Burton and Miller formulation

$$
\left\{-\frac{1}{2} I+M_{k}+\alpha(p)\left[L_{0}\left(N_{k}-N_{0}\right) \div M_{0}^{2}-\frac{1}{4} I \vdots\right\} \phi(p)=L_{k}+\alpha(p) L_{0}\left(\frac{1}{2} I+M_{k}^{T}\right)\right] \frac{\partial \phi(p)}{\partial n}
$$

where all the integral operators are weakly singular and compact for $S \subseteq C^{2}$. We refer to (2.86) as the regularised Burton and Miller formulation. The choice of $\alpha(p)$ is arbitrary, except that $\operatorname{im}(\alpha(p))>0$ for $k>0$ (see Theorem 2.10 ). We will discuss the choice of $\alpha(p)$ in Section 3.4.

The main drawback of this regularisation method, from a practical point of view, is having to compute the products of operators, usually by multiplying their matrix approximations together which is expensive. Clearly it would be much more
efficient to compute a direct approximation to $N_{k}$. The original reason for the use of this regularisation technique was to establish the existence of a unique solution using the classical Fredholm theory for compact operators (see Panich [47]).

However, the regularised method is often used in practice since it is not clear how to interpret the $N_{k}$ operator directly, and consequently there is relatively little reported experience with (2.73). Recently some methods for the approximation of $N_{k}$ have been proposed to allow the direct use of (2.73).

One method for interpreting $N_{k}$ is to assume that the surface $S$ consists of a number of flat sub-surfaces $S_{i}$. If $p \in S_{i}$, it is possible to write

$$
\begin{equation*}
\int_{S_{i}} \frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}} \phi(q) d S_{q}=\phi(p) \int_{S_{i}} \frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}} d S_{q}+\int_{S_{i}}\{\phi(q)-\phi(p)\} \frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}} d S_{q} . \tag{2.8i}
\end{equation*}
$$

If we assume $\phi(p)$ is a constant for $p \in S_{i}$, the second integral on the right hand side of (2.87) is zero. Terai [55] shows that the first integral can be written as

$$
\begin{equation*}
\int_{S_{i}} \frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}} d S_{q}=\int_{\theta=0}^{\theta=2 \pi} \frac{e^{i k \rho(\theta)}}{4 \pi \rho(\theta)} d \theta+\frac{i k}{2} \tag{2.88}
\end{equation*}
$$

where $(r, \theta)$ are local polar coordinates on the surface $S_{i}$ with their origin at $p$ and $\rho(\theta)$ is the distance from $p$ to the edge of $S_{i}$ in the direction $\theta$. This method has been implemented by Terai [55], but the results are much more satisfactory when $S$ consists of a number of flat regions, such as when $S$ is a cube.

This method has the disadvantage that the surface must consist of a number of flat elements, and cannot be applied to curved surfaces without some further analysis. Also, the assumption that $\phi(p)$ is a constant for $p \in S_{i}$ is not generally
valid. To overcome this, Terai [55] suggests expanding $\phi$ in the second integral on the right hand side of (2.87) in a Taylor's series about the point $p$.

An alternative method for interpreting $N_{k}$, which is applicable to any surface, uses a pointwise variational principle introduced in [53], to show that [44]

$$
\begin{equation*}
\int_{S} \frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}} d S_{q}=k^{2} \int_{S} G_{k}(p, q) \underline{n}_{p} \cdot \underline{n}_{q} d S_{q} \tag{2.89}
\end{equation*}
$$

We can now write

$$
\begin{equation*}
\int_{S} \phi(q) \frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}} d S_{q}=\int_{S}\{\phi(q)-\dot{\phi}(p)\} \frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}} d S_{q}+k^{2} \phi(p) \int_{S} G_{k}(p, q) \underline{n}_{p} \cdot \underline{n}_{q} d S_{q} . \tag{2.90}
\end{equation*}
$$

The second integral on the right hand side is weakly singular and may be approximated by an appropriate quadrature rule. The first integral contains a $\frac{1}{p-q^{2}}$ singularity, and must be interpreted in the sense of a Cauchy principal value. An appropriate choice of basis functions and quadrature rule for evaluating (2.90) is considered in Section 3.3.

In this section we have discussed some different integral equation formulations of the exterior Helmholtz equation which are valid for all wavenumbers. Of the different formulations considered here, the one due to Panich 47 , provides an efficient method of obtaining a unique solution using an indirect formulation. Likewise, the Burton and Miller formulation [13 is an efficient method for obtaining unique solution when employing a direct formulation. Here we shall only consider the Burton and Miller formulation since it is simpler to couple a direct integral equation formulation to a finite element analysis of an elastic structure for solving the
fluid-structure interaction problem considered in Chapter 4.
A numerical scheme for solving the Neumann problem using the direct formulation is discussed in the next chapter where we shall give a comparison between the regularised and direct Burton and Miller formulations. A similar comparison has been carried out in Amini and Harris [7].

## 3 NUMERICAL SOLUTION OF THE NEU-

## MANN PROBLEM FOR THE EXTERIOR

## HELMHOLTZ EQUATION

### 3.1 Numerical Methods for Solving Integral Equations

This section will consider some of the most commonly used methods for solving second kind integral equations of the form

$$
\begin{equation*}
-\lambda \phi(p)+\int_{S} k(p, q) \phi(q) d S_{q}=f(p) \quad p \in S \tag{3.1}
\end{equation*}
$$

or in equivalent operator form

$$
\begin{equation*}
(-\lambda I+\mathcal{K}) \phi=f \tag{3.2}
\end{equation*}
$$

The collocation scheme that we have chosen to implement will then be discussed in some detail.

### 3.1.1 Nyström Method.

For the Nyström or quadrature method the integral operator in (3.1) is replaced by a numerical quadrature rule. Let

$$
\begin{equation*}
\int_{S} f(q) d S \approx \sum_{j=1}^{n} w_{j} f\left(q_{j}\right) \tag{3.3}
\end{equation*}
$$

denote a numerical quadrature rule, where $w_{j}$ denote the weights and $q_{j}$ the nodes.
(3.1) is approximated by

$$
\begin{equation*}
-\lambda \phi_{n}(p)+\sum_{j=1}^{n} w_{j} k\left(p, q_{j}\right) \phi_{n}\left(q_{j}\right)=f(p) \tag{3.4}
\end{equation*}
$$

where $\phi_{\mathrm{n}}$ is an approximation to $\phi$. Choosing $p=q_{1}, q_{2}, \ldots, q_{n}$ leads to the linear system

$$
\begin{equation*}
-\lambda \phi_{i}+\sum_{j=1}^{n} w_{j} k\left(q_{i}, q_{j}\right) \phi_{j}=f\left(q_{i}\right) \quad i=1,2, \ldots, n \tag{3.5}
\end{equation*}
$$

where $\phi_{i}=\phi_{n}\left(q_{i}\right) \approx \phi\left(q_{i}\right)$.
Once the linear system (3.5) has been solved for $\phi_{1}, \phi_{2}, \ldots, \phi_{n}$ (3.4) may be written as (the Nyström extension)

$$
\begin{equation*}
\phi_{n}(p)=\frac{1}{\lambda}\left[\sum_{j=1}^{n} w_{j} k\left(p, q_{j}\right) \phi_{j}-f(p)\right] \tag{3.6}
\end{equation*}
$$

to find an approximation to $\phi(p)$ for all $p \in S$. Clearly the function $\phi_{n}(p)$ defined by (3.6) and the solution $\left\{\phi_{1}, \ldots, \phi_{n}\right\}$ of (3.5) is unique and both solutions agree at the nodes $q_{1}, \ldots, q_{n}$.

However, this method is not best suited for use on the exterior Helmholtz problem since the kernel functions of the integral operators involved are not continuous. For example, if this method is used to discretise (2.34) then the elements of the matrix approximating ( $-\frac{1}{2} I+M I_{k}$ ) are given by

$$
\begin{equation*}
-\frac{1}{2} \delta_{i j}+\frac{\partial G_{k}\left(q_{i}, q_{j}\right)}{\partial n_{q}} w_{j} \tag{3.7}
\end{equation*}
$$

where $\delta_{i j}$ is the Krondecker delta function. If $i=j$ then $\frac{\partial G_{k}}{\partial n_{q}}$ cannot be evaluated since it is unbounded and the simple application of this method breaks down. It is possible to overcome this problem by using a product integration technique [9], although this has not been done here.

### 3.1.2 Degenerate Kernel Methods.

If the kernel function can be expressed in the form

$$
\begin{equation*}
k(p, q)=\sum_{i=1}^{n} a_{i}(p) b_{i}(q) \tag{3.8}
\end{equation*}
$$

then $k(p, q)$ is said to be a degenerate or separable kernel. Substituting (3.8) into (3.1) gives

$$
\begin{equation*}
-\lambda \phi(p)+\left[c_{1} a_{1}(p) \div c_{2} a_{2}(p) \div \cdots+c_{n} a_{n}(p)\right]=f(p) \tag{3.9}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{i}=\int_{S} \phi(q) b_{i}(q) d S_{q} . \tag{3.10}
\end{equation*}
$$

Taking the inner-product of (3.9) with each of the functions $b_{i}$ gives

$$
\begin{equation*}
-\lambda c_{i}+\sum_{j=1}^{n} c_{j} \int_{S} a_{j}(q) b_{i}(q) d S_{q}=\int_{S} f(q) b_{i}(q) d S_{q} \quad i=1,2, \ldots, n \tag{3.11}
\end{equation*}
$$

This is a system of linear equations for the constants $c_{1}, c_{2}, \ldots, c_{n}$ and once they have been found (3.9) can be used to find $\phi(p)$ exactly.

In general, the kernel function $k(p, q)$ of a given integral operator $\mathcal{K}: X \rightarrow X$ is not separable. If we let $\left\{\psi_{1}^{\prime}, \psi_{2}, \ldots, \psi_{n}, \ldots\right\}$ denote an orthonormal set in $X$, $k(p, q)$ may be expressed as a series of the form

$$
\begin{equation*}
k(p, q)=\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i j} \psi_{i}^{\prime}(p) \psi^{\prime} j(q) \tag{3.12}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{i j}=\int_{S} \int_{S} k(p, q) \overline{\psi_{i}(p)} \overline{\dot{\psi}_{j}(q)} d S_{q} d S_{p} \tag{3.13}
\end{equation*}
$$

To obtain a degenerate kernel approximation the series in (3.12) is truncated after $n$ terms in both $i$ and $j$.

This method can be used for the exterior Helmholtz problem where $S$ is a sphere and the orthogonal functions $\psi_{i}$ are the spherical harmonics given by

$$
\begin{equation*}
S_{n}^{m}(p)=\left[\frac{1}{2 \pi}\left(n+\frac{1}{2}\right) \frac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} P_{n}^{m}\left(\cos \theta_{p}\right) e^{i m \phi_{p}} \quad m=-n, \ldots, 0, \ldots, n, \tag{3.14}
\end{equation*}
$$

where the point $p$ has spherical polar coordinates $\left(r_{p}, \theta_{p}, \phi_{p}\right)$ and the $P_{n}^{m}$ 's are the associated Legendre functions. It can be shown that (see Morse and Feshback [45:)

$$
\begin{equation*}
G_{k}(p, q)=i k \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_{n}\left(k r_{p}\right) S_{n}^{m}(p) j_{n}\left(k r_{q}\right) \overline{S_{n}^{m}(q)} \quad r_{p}>r_{q} \tag{3.15}
\end{equation*}
$$

where $j_{n}$ is the spherical Bessel function and $h_{n}$ is the spherical Hankel function of the first kind. The expansion for the derivatives of $G_{k}$ can be found by differentiating (3.15) term by term. The degenerate kernel method using (3.15) for solving the exterior Helmholtz problem is known as the T-matrix method, see Waterman [60].

### 3.1.3 Projection Methods.

The methods most commonly used to solve the exterior acoustic problem are the projection methods. Let $X$ be a linear space and let $X_{n}$ be a sub-space of $X$ spanned by the basis functions $\left\{\psi_{1}, \psi_{2}, \ldots, \psi_{n}\right\}$. Let $\mathcal{P}_{n}$ be a projection operator from $X$ onto $X_{n}$, that is $\mathcal{P}_{n} \phi=\phi$ for all $\phi \subseteq X_{n}$. The projection methods for finding an approximate solution to (3.2) are to solve 9 :

$$
\begin{equation*}
\left(-\lambda I+\mathcal{P}_{n} \mathcal{K}\right) \phi_{n}=\mathcal{P}_{n} f \quad \phi_{n} \equiv X_{n} \tag{3.16}
\end{equation*}
$$

The residual, $r_{n}$, is defined by

$$
\begin{equation*}
r_{n}(p)=(-\lambda I+\mathcal{K}) \phi_{n}(p)-f(p) \tag{3.17}
\end{equation*}
$$

Since $\phi_{n} \in X_{n}$ there are unique constants $\alpha_{i}$ such that $\phi_{n}(p)=\sum_{i=1}^{n} \alpha_{i} \psi_{i}(p)$. The two most commonly used examples of projection methods are the Galerkin and collocation methods.

The Galerkin Method The Galerkin method requires that

$$
\begin{equation*}
\left(r_{n}, \psi_{j}\right)=\int_{S} r_{n}(q) \overline{\psi_{j}(q)} d S_{q}=0 \quad j=1,2, \ldots, n . \tag{3.18}
\end{equation*}
$$

That is, the residual is orthogonal to every element of $X_{n}$ and so has no component in $X_{n}$. It is hoped that this condition will force $\left\|r_{n}\right\|$ to be small, and hence expect that the approximate solution $\phi_{n}$ to be close to the true solution $\phi$. Applying (3.18) to (3.17) gives the linear system

$$
\begin{equation*}
\sum_{i=1}^{n} \alpha_{i} \int_{S}\left(-\lambda \psi_{i}(p) \overline{\psi_{j}(p)}+\left(\mathcal{K} \psi_{i}\right)(p) \overline{\psi_{j}(p)}\right) d S_{p}=\int_{S} f(p) \overline{\psi_{j}(p)} d S_{p} \quad j=1, \ldots, n \tag{3.19}
\end{equation*}
$$

for $\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$. It is easy to show that this is a projection method with $\mathcal{P}_{n} \phi=$ $\sum_{i=1}^{n}\left(\phi, \psi_{i}\right) \psi_{i}$, see At kinson $[9]$ or Baker [12]. The Galerkin method is expensive to use in practise as we need to evaluate a double surface integral for each value of $j$.

The Collocation Method. The collocation method requires us to satisfy the conditions

$$
\begin{equation*}
r_{n}\left(p_{j}\right)=0 \quad j=1, \ldots, n \tag{3.20}
\end{equation*}
$$

for $n$ distinct collocation points $\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$. Now it is hoped that forcing the residual to satisfy (3.20) will make $r_{n}(p)$ small for all $p \in S$. Substituting (3.17) into (3.20) yields

$$
\begin{equation*}
\sum_{i=1}^{n} \alpha_{i}\left\{-\lambda \psi_{i}\left(p_{j}\right)+\int_{S} k\left(p_{j}, q\right) \psi_{i}(q) d S_{q}\right\}=f\left(p_{j}\right) \quad j=1, \ldots, n \tag{3.21}
\end{equation*}
$$

which is a linear system for the coefficients $\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$.
To show that this is a projection method we define $\mathcal{P}_{n}: X \rightarrow X_{n}$ as the element $\phi_{n} \in X_{n}$ which interpolates $\phi \in X$ at the points $\left\{p_{1}, \ldots, p_{n}\right\}$. The function $\phi_{n}$ is of the form $\phi_{n}=\sum_{i=1}^{n} \alpha_{i} \psi_{i}(p)$ where the $\alpha_{i}$ 's are chosen so that

$$
\begin{equation*}
\sum_{i=1}^{n} \alpha_{i} \psi_{i}\left(p_{j}\right)=\phi\left(p_{j}\right) \quad j=1, \ldots, n \tag{3.22}
\end{equation*}
$$

This system is non-singular because the $\psi_{i}$ 's are a basis for $X_{n}$ and so they are linearly independent. It is clear, from (3.22), that $\mathcal{P}_{n} \phi_{n}=\phi_{n}$ for all $\phi_{n} \in X_{n}$ and that $\mathcal{P}_{n} \phi=\mathcal{P}_{n} \psi$ implies that $\phi\left(p_{j}\right)=\psi\left(p_{j}\right)$, for $j=1, \ldots, \pi$. Applying $\mathcal{P}_{n}$ to (3.2) and approximating $\phi$ by $\phi_{n}$ gives (3.21) and hence the collocation method is a projection method. The collocation method is the simplest and consequently most widely used projection method for the exterior acoustic problem, and is the approach we shall use here.

For all projection methods we have the following result concerning the error and the rate of convergence of the approximate solution to the exact solution.

## THEOREM 3.1

Let $X$ be a Banach space, let $\mathcal{K}$ be a bounded linear operator from $X$ onto $X$ and let $\mathcal{P}_{n}$ be a bounded projection operator from $X$ onto the subspace $X_{n}$.

Assume that $(-\lambda I+\mathcal{K})^{-1}$ exists on $X$ and that

$$
\begin{equation*}
\left\|\mathcal{K}-P_{n} \mathcal{K}\right\| \leq \frac{1}{\left\|(-\lambda I+\mathcal{K})^{-1}\right\|} \tag{3.23}
\end{equation*}
$$

Then, $\left(-\lambda I+\mathcal{P}_{n} \mathcal{K}\right)^{-1}$ exists on $X$ with

$$
\begin{equation*}
\left\|\left(-\lambda I+\mathcal{P}_{n} \mathcal{K}\right)^{-1}\right\| \leq \frac{\left.\|(-\lambda I+\mathcal{K})^{-1}\right) \|}{1-\left\|(-\lambda I+\mathcal{K})^{-1}:\right\| \mathcal{K}-\mathcal{P}_{n} \mathcal{K} \|} \tag{3.24}
\end{equation*}
$$

If $(-\lambda I+\mathcal{K}) \phi=f$ and $\left(-\lambda I+\mathcal{P}_{n} \mathcal{K}\right) \phi_{n}=\mathcal{P}_{n} f$ then

$$
\begin{equation*}
\left.\left\|\phi-\phi_{n}\right\| \leq|\lambda|:\left(-\lambda I+\mathcal{P}_{n} \mathcal{K}\right)^{-1}\right)\left\|\left\|\phi-\mathcal{P}_{n} \phi\right\| .\right. \tag{3.25}
\end{equation*}
$$

Proof: See Atkinson [9'.
The last part of the above theorem shows that the error in the approximate solution $\phi_{n}$ is governed by the error in approximating $\phi$ by $\mathcal{P}_{n} \phi$.

### 3.1.4 An Application of the Collocation Method to the Exterior Neumann Problem for the Helmholtz Equation.

The collocation scheme that we have chosen to use is the simplest one to implement, based on a piecewise constant approximation to $\dot{\phi}$. Divide $S$ into $n$ sub-surfaces $S_{i, i}=1, \ldots, n$ and define

$$
\psi_{i}(p)=\left\{\begin{array}{ll}
1 & \text { if } p \in S_{i}  \tag{3.26}\\
0 & \text { otherwise }
\end{array} \quad i=1,2, \ldots, n\right.
$$

The $n$ collocation points $\left\{p_{1}, \ldots, p_{n}\right\}$ are usually chosen so that the point $p_{i}$ is close to the centroid of $S_{\boldsymbol{i}}$. We discuss the appropriate sub-division of $S$ in Section 3.2. The solution $\phi$ is approximated by

$$
\begin{equation*}
\phi_{n}(p)=\sum_{i=1}^{n} \phi_{i} \psi_{i}(p) \tag{3.27}
\end{equation*}
$$

where $\phi_{i}=\phi_{n}\left(p_{i}\right)$. Substituting (3.27) into (3.21) and collocating at the points $p_{i}$ yields the linear system

$$
\begin{equation*}
-\lambda \phi_{i}+\sum_{j=1}^{n} \phi_{j} \int_{S_{j}} k\left(p_{i}, q\right) d S_{q}=f\left(p_{i}\right) \quad i=1, \ldots, n \tag{3.28}
\end{equation*}
$$

To find the rate of convergence of $\phi_{n}$ to $\phi$ let

$$
\begin{equation*}
h=\max _{i}\left[\max _{p, q \in S_{i}}|p-q|\right] \tag{3.29}
\end{equation*}
$$

denote the maximum of the sub-region diameters. By expanding $\phi(p)$ for $p \in S_{i}$ in a Taylor series about the point $p_{i}$ it can be shown that

$$
\begin{equation*}
\left\|\phi-\mathcal{P}_{n} \phi\right\| \leq c h \tag{3.30}
\end{equation*}
$$

where $c$ is a constant, (see Baker [12]). Using (3.30) and Theorem 3.1 we can see that the rate of convergence of the scheme, as $h$ gets smaller, is governed by

$$
\begin{equation*}
\left\|\phi-\phi_{n}\right\|=O(h) \tag{3.31}
\end{equation*}
$$

Using this collocation scheme the matrix approximations to the layer potential operators $L_{k}, M I_{k}$ and $M_{k}^{T}$ are given by

$$
\begin{align*}
\left(\dot{L}_{k}\right)_{i j} & =\int_{S_{j}} G_{k}\left(p_{i}, q\right) d S_{q}  \tag{3.32}\\
\left(\bar{M}_{k}\right)_{i j} & =\int_{S_{j}} \frac{\partial G_{k}\left(p_{i}, q\right)}{\partial n_{q}} d S_{q} \tag{3.33}
\end{align*}
$$

and

$$
\begin{equation*}
\left(\bar{M}_{k}^{T}\right)_{i j}=\int_{S_{j}} \frac{\partial G_{k}\left(p_{i}, q\right)}{\partial n_{p}} d S_{q} \tag{3.34}
\end{equation*}
$$

respectively. We note that the matrix $\tilde{M}_{k}^{T}$ is not the transpose of the matrix $\tilde{M}_{k}$. If $i=j$ then the integrands of all three integrals are weakly singular and we must
take care in evaluating them. Appropriate quadrature rules are discussed in Section

## 3.3.

It follows from (3.32) and (3.33) that the matrix approximation to (2.34) is given by

$$
\begin{equation*}
\left(-\frac{1}{2} I+\bar{M}_{k}\right) \underline{\phi}=\bar{L}_{k} \underline{v} \tag{3.35}
\end{equation*}
$$

where $\underline{\phi}=\left[\phi_{1}, \ldots, \phi_{n}\right]^{T}$ and $\underline{v}=\left[i \omega \cdot \rho_{f} v_{1}, \ldots, i \omega \rho_{f} v_{n}\right]^{T}=\left[\frac{\partial \phi}{\partial n}\left(p_{1}\right), \ldots, \frac{\partial \phi}{\partial n}\left(p_{n}\right)\right]^{T}$.
Since we require a numerical approximation which is valid for all values of the wavenumber $k$, we have chosen to discretise the Burton and Miller formulation (2.86) introduced in Section 2.4.5.

To discretise the regularised Burton and Miller formulation, we approximate the operator $N_{k}-N_{0} \mathrm{by}$

$$
\begin{equation*}
\left(\tilde{N}_{k 0}\right)_{i j}=\int_{S_{j}} \frac{\partial^{2}}{\partial n_{p} \partial n_{q}}\left(G_{k}\left(p_{i}, q\right)-G_{0}\left(p_{i}, q\right)\right) d S_{q} \tag{3.36}
\end{equation*}
$$

The operators $L_{0}$ and $M_{0}$ can be approximated by using (3.32) and (3.33) respectively, with $k=0$. Substituting the approximations into (2.86) yields the matrix equation

$$
\begin{equation*}
\left[\left(-\frac{1}{2} I+\bar{M}_{k}+\lambda\left(\bar{L}_{0} \bar{N}_{k 0}+\bar{M}_{0}^{2}-\frac{1}{4} I\right) \phi=\tilde{L}_{k}+\lambda \bar{L}_{0}\left(\frac{1}{2} I-\tilde{M}_{k}^{T}\right) \underline{\underline{v}}\right.\right. \tag{3.37}
\end{equation*}
$$

The matrix $A$ can either be of the form $A=\alpha I$ corresponding to a constant coupling parameter $\alpha$ or a diagonal matrix whose elements are the values of the function $\alpha(p)$, introduced in Section 2.4.5, at the collocation points.

Applying the projection operator $\mathcal{P}_{n}$ to the direct Burton and Miller formulation
(2.90) gives

$$
\begin{equation*}
\mathcal{P}_{n} N_{k} \phi_{n}\left(p_{i}\right)=\sum_{j=1}^{n}\left(\phi_{j}-\phi_{i}\right) \int_{S_{j}} \frac{\partial^{2} G_{k}\left(p_{i}, q\right)}{\partial n_{p} \partial n_{q}} d S_{q}+\phi_{i} \int_{S} k^{2} n_{p} . n_{q} G_{k}\left(p_{i}, q\right) d S_{q} . \tag{3.38}
\end{equation*}
$$

The second term on the right-hand side of (3.38) is weakly singular, and can be approximated by an appropriate quadrature rule. If $i=j$ in the first term on the right-hand side the corresponding term in the sum is zero, whilst if $i \neq j$ the corresponding integrals are non-singular. Explicitly, the elements of the matrix $\bar{N}_{k}$ are given by

$$
\begin{equation*}
\left(\bar{N}_{k}\right)_{i j}=\int_{S_{j}} \frac{\partial^{2} G_{k}\left(p_{i}, q\right)}{\partial n_{p} \partial n_{q}} d S_{q} \quad i \neq j \tag{3.39}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\tilde{N}_{k}\right)_{i i}=\int_{S} k^{2} n_{p} . n_{q} G_{k}\left(p_{i}, q\right) d S_{q}-\sum_{j=1 . j \neq i}^{n} \int_{S_{j}} \frac{\partial^{2} G_{k}\left(p_{i}, q\right)}{\partial n_{p} \partial n_{q}} d S_{q} . \tag{3.40}
\end{equation*}
$$

This yields the matrix equation

$$
\begin{equation*}
\left(-\frac{1}{2} I+\bar{M}_{k}+\Lambda \bar{N}_{k}\right) \underline{\underline{L}}=\left[\tilde{L}_{k}+\Lambda\left(\frac{1}{2} I+\bar{M}_{k}^{T}\right) \underline{v}\right. \tag{3.41}
\end{equation*}
$$

as an approximation to (2.73). For any of the above numerical formulations, once we have found $\phi$ it is possible to obtain an approximation to $\phi(p)$ for $p \in D_{\odot}$ using

$$
\begin{equation*}
\phi_{n}(p)=\sum_{j=1}^{n} \phi_{j} \int_{S_{j}} \frac{\partial G_{k}(p, q)}{\partial n_{q}} d S_{q}+\sum_{j=1}^{n} v_{j} \int_{S_{j}} G_{k}(p, q) d S_{q} \quad p \subseteq D_{+}, \tag{3.42}
\end{equation*}
$$

a discretisation of (2.33) for $p \in D_{+}$.
To compute the matrix approximations to the operators $L_{k}, M I_{k}, M M_{k}^{T}$ and $N_{k}$, we need to evaluate the normal derivatives of the Green's function. It is easy to show that the first derivatives are given by

$$
\begin{equation*}
\frac{\partial G_{k}(p, q)}{\partial n_{q}}=\frac{d G_{k}}{d r} \frac{\partial r}{\partial n_{q}}=\frac{d G_{k}}{d r} \frac{(q-p) \cdot \underline{n}_{q}}{r} \tag{3.43}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial G_{k}(p, q)}{\partial n_{p}}=\frac{d G_{k}}{d r} \frac{\partial r}{\partial n_{p}}=\frac{d G_{k}}{d r} \frac{(p-q) \cdot \underline{n}_{p}}{r} \tag{3.44}
\end{equation*}
$$

respectively, where

$$
\begin{equation*}
\frac{d G_{k}}{d r}=\epsilon^{i k r} \frac{i k r-1}{4 \pi r^{2}}, \quad r=p-q \tag{3.45}
\end{equation*}
$$

The second derivative is given by

$$
\begin{equation*}
\frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}}=\frac{d^{2} G_{k}}{d r^{2}} \frac{\partial r}{\partial n_{p}} \frac{\partial r}{\partial n_{q}}+\frac{d G_{k}}{d r} \frac{\partial^{2} r}{\partial n_{p} \partial n_{q}} \tag{3.46}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{d^{2} G_{k}}{d r^{2}}=\frac{e^{i k r}}{4 \pi r^{3}}\left[(i k r)^{2}-2 i k r+2\right], \quad r=|p-q| \tag{3.47}
\end{equation*}
$$

It can be shown that the second derivative of $r$ can be expressed in terms of the first derivatives as [16]

$$
\begin{equation*}
\frac{\partial^{2} r}{\partial n_{p} \partial n_{q}}=-\frac{1}{r}\left\{\frac{\partial r}{\partial n_{p}} \frac{\partial r}{\partial n_{q}}+\underline{n}_{p} \cdot \underline{n}_{q}\right\} . \tag{3.48}
\end{equation*}
$$

The computation of the normal vectors $\underline{n}_{p}$ and $\underline{n}_{q}$ will be discussed in the next section.

Once our integral equation has been discretised a direct method, such as Gaussian elimination, will be employed to solve the resulting linear system. However, recent research has shown that it is possible to modify well known iterative techniques, like the conjugate gradient method or the multi-grid method, to solve the final linear system of equations more efficiently, see Amini and Chen $[4,5]$, and Amini et al [6]. We have not employed such methods since the discretisation here forms only part of our solution to the coupled fluid-structure interaction problem
to be discussed in Chapter 4, and it is not clear how iterative methods can be applied to this more general problem.

### 3.2 Surface Representation

In this section we will consider how to represent the surface of an axisymmetric structure and also that of a fully three-dimensional structure. In particular we shall be using the boundary element method for representing the surface, derived using similar principles to those used in the finite element method.

In general, a surface $S \in \mathcal{R}^{3}$ can be represented in terms of two independent parameters $u$ and $v$ in the form

$$
\begin{equation*}
x=x(u, v) \quad y=y(u, v) \quad z=z(u, v) \tag{3.49}
\end{equation*}
$$

where ( $x, y, z$ ) are the usual cartesian coordinates. $u$ and $v$ may be global parameters for the whole surface, or local parameters for each surface element $S_{j}$.

Using the parameterisation (3.49), an integral of the form

$$
\begin{equation*}
I=\int_{S} F(q) d S_{q} \tag{3.50}
\end{equation*}
$$

can be written as

$$
\begin{equation*}
I=\iint_{R} f(u, v) g(u, v) d u d v \tag{3.51}
\end{equation*}
$$

where $g(u, v)$ is the Jacobian of the transformation from the $(x, y, z)$ coordinates to the region $R$ in the $(u, v)$ coordinates. Explicitly

$$
\begin{equation*}
g(u, v)=\sqrt{D_{1}^{2}+D_{2}^{2}+D_{3}^{2}} \tag{3.52}
\end{equation*}
$$

where

$$
D_{1}=\left|\begin{array}{ll}
\frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}  \tag{3.53}\\
\frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}
\end{array}\right| \quad D_{2}=\left|\begin{array}{cc}
\frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\
\frac{\partial z}{\partial v} & \frac{\partial x}{\partial v}
\end{array}\right| \quad D_{3}=\left|\begin{array}{cc}
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}
\end{array}\right| .
$$

In addition, the unit normal to the surface is given by [11]

$$
\begin{equation*}
\underline{n}=\frac{\left[D_{1}, D_{2}, D_{3}\right]^{T}}{g} \tag{3.54}
\end{equation*}
$$

although we need to take care to ensure that $\underline{n}$ is the outward normal.
For an axisymmetric surface it is convenient to work in cylindrical polar coordinates $(r, \theta, z)$, where the surface is generated by rotating a given curve in the ( $r, z$ ) plane about the $z$-axis [17]. If this curve can be easily parameterised as $(r(s), z(s))$, $0 \leq s \leq L$, then

$$
\begin{equation*}
x(s, \theta)=r(s) \cos (\theta) \quad y(s, \theta)=r(s) \sin (\theta) \quad z(s, \theta)=z(s) \tag{3.55}
\end{equation*}
$$

with

$$
\begin{gather*}
D_{1}=-\frac{d z(s)}{d s} r(s) \cos (\theta) \\
D_{2}=-\frac{d z(s)}{d s} r(s) \sin (\theta)  \tag{3.56}\\
D_{3}=\frac{d r(s)}{d s} r(s)
\end{gather*}
$$

and

$$
\begin{equation*}
g(s, \theta)=r(s) \sqrt{\left(\frac{d r}{d s}\right)^{2}+\left(\frac{d z}{d s}\right)^{2}} \tag{3.57}
\end{equation*}
$$

Using this parameterisation the surface $S$ transforms into the rectangle $0 \leq s \leq L$, $0 \leq \theta \leq 2 \pi$. As an example of this representation, consider a sphere of radius $a$ where $r(s)=a \sin s, z(s)=a \cos s$, and $L=\pi$. We now divide the rectangle in the $(s, \theta)$ plane into $N \times M$ smaller rectangles and use these smaller rectangles as our
surface elements $S_{j}$, taking the centroids as our collocation points $p_{j}$. This is the surface representation used in Amini and Harris [7].

Alternatively, if there is no simple parameterisation of $r(s)$ and $z(s)$, we may take $N+1$ distinct data points, including the end points, along the generating curve and interpolate $r(s)$ and $z(s)$ with some appropriate functions. Amini and Wilton [3] used a cubic spline method to interpolate these functions. Here the simpler piecewise linear scheme of connecting the adjoining nodes by a straight line to form $N$ linear, axisymmetric boundary elements has been used.

Let the lines which generate the surface elements $S_{j}, j=1, \ldots N$ have end points $\left(r_{j}, z_{j}\right)$ and $\left(r_{j+1}, z_{j+1}\right)$. Then we approximate $r(s)$ and $z(s)$ by

$$
\begin{array}{ll}
\tilde{r}(s)=s r_{j}+(1-s) r_{j+1} & 0 \leq s \leq 1 \\
\tilde{z}(s)=s z_{j}+(1-s) z_{j+1} & \tag{3.58}
\end{array}
$$

respectively, with

$$
\begin{gather*}
D_{1}=\left(z_{j}-z_{j+1}\right) \tilde{r}(s) \cos \theta \\
D_{2}=\left(z_{j}-z_{j+1}\right) \tilde{r}(s) \sin \theta  \tag{3.59}\\
D_{3}=\left(r_{j+1}-r_{j}\right) \tilde{r}(s)
\end{gather*}
$$

and

$$
\begin{equation*}
g=\left[(1-s) r_{j+1}+s r_{j}\right] \sqrt{\left(r_{j}-r_{j+1}\right)^{2}-\left(z_{j}-z_{j+1}\right)^{2}} \tag{3.60}
\end{equation*}
$$

To ensure that the normal vector given by (3.54) and (3.59) is the outward normal, the points $\left(r_{j}, z_{j}\right)$ should be ordered so that $z_{1} \geq z_{2} \geq \ldots \geq z_{N+1}$. This approximation maps each element into the rectangle $0 \leq s \leq 1,0 \leq \theta \leq 2 \pi$ in the ( $s, \theta$ ) plane. This rectangle is sub-divided using $M$ divisions in the $\theta$ direction to give us
$N \times M$ surface elements $S_{j}$, the centroids of which form our collocation points $p_{j}$. This choice also ensures that the normal vector is well defined at each collocation point.

If we have an axisymmetric boundary condition, it is only necessary to solve the integral equation for one value of $\theta$, which is the same as collocating for only one of the $M$ sub-divisions when using the above elements, since the solution is independent of $\theta$.

For the three-dimensional case we employ a parametric mapping similar to those used in finite element analysis. The surface is divided into $n$ triangular elements $S_{j}$ in the $(x, y, z)$ space, each of which can be mapped into a reference triangle $\Delta$ in the $(u, v)$ plane. This procedure will be illustrated for the three-noded linear and six-noded quadratic elements used in this thesis. The order in which the nodes should be labelled, as seen from the exterior domain, is shown in Figures 3.1 and 3.2 respectively. This ensures that the normal to $S_{j}$ is directed into $D_{\ddagger}$.

The interpolation, or basis, functions are defined to be unity at one node and zero at all the others. The basis functions for the linear element are

$$
\begin{equation*}
N_{1}=1-u-v \quad N_{2}=u \quad N_{3}=v . \tag{3.61}
\end{equation*}
$$

The basis functions for the quadratic element are

$$
\begin{array}{lll}
N_{1}=(1-u-v)(1-2 u-2 v) & N_{2}=4 u(1-u-v) & N_{3}=u(2 u-1) \\
N_{4}=4 u v & N_{5}=v(2 v-1) & N_{6}=4 v(1-u-v) . \tag{3.62}
\end{array}
$$

If the cartesian coordinates of the $i^{\text {th }}$ node are $\left(x_{i}, y_{i}, z_{i}\right)$, then the $(x, y, z)$ coordi-
Figure 3.1. The relationship between the nodes of a general linear


Fimure 3.2. The relationship between the nodes of a general

nates of a point $(u, v)$, in a linear element, are given by

$$
\begin{equation*}
x=\sum_{i=1}^{3} x_{i} N_{i}(u, v) \quad y=\sum_{i=1}^{3} y_{i} N_{i}(u, v) \cdot z=\sum_{i=1}^{3} z_{i} N_{i}(u, v) \tag{3.63}
\end{equation*}
$$

with

$$
\begin{array}{ll}
\frac{\partial x}{\partial u}=x_{2}-x_{1} & \frac{\partial x}{\partial v}=x_{3}-x_{1} \\
\frac{\partial y}{\partial u}=y_{2}-y_{1} & \frac{\partial y}{\partial v}=y_{3}-y_{1}  \tag{3.64}\\
\frac{\partial z}{\partial u}=z_{2}-z_{1} & \frac{\partial z}{\partial v}=z_{3}-z_{1}
\end{array}
$$

from which $D_{1}, D_{2}, D_{3}$ and $g$ can be evaluated.
Similarly, for a quadratic element

$$
\begin{equation*}
x=\sum_{i=1}^{6} x_{i} N_{i}(u, v) \quad y=\sum_{i=1}^{6} y_{i} N_{i}(u, v) \quad z=\sum_{i=1}^{6} z_{i} N_{i}(u, v) \tag{3.65}
\end{equation*}
$$

and

$$
\begin{align*}
& \frac{\partial x}{\partial u}=(4 u+4 v-3) x_{1}+(4-8 u-4 v) x_{2}+(4 u-1) x_{3}+4 v x_{4}-4 v x_{6}  \tag{3.66}\\
& \frac{\partial x}{\partial v}=(4 u+4 v-3) x_{1}-4 u x_{2}+4 v x_{4}+(4 v-1) x_{5}+(4-8 u-4 v) x_{6}
\end{align*}
$$

with similar expressions for the derivatives of $y$ and $z$, which are used to compute $D_{1}, D_{2} . D_{3}$ and $g$.

We see that both types of element map into the same reference triangle $\dot{\Delta}$. All the integrations and interpolations required for the calculation of the collocation solution can be carried out over $\triangle$. The location of the collocation point $p_{j} \equiv S_{j}$ is the point corresponding to $\left(\frac{1}{3}, \frac{1}{3}\right) \subseteq \Delta$, which is the centroid of $\Delta$. With the linear elements this point will be the centroid of $S_{j}$, and with the quadratic elements it will be close to the centroid of $S_{j}$.

### 3.3 Numerical Quadrature

When forming the collocation equations, it is necessary to evaluate integrals of the form

$$
\begin{equation*}
\tilde{K}_{i j}=\int_{S_{j}} k\left(p_{i}, q\right) d S_{q}=\iint_{\Delta} k\left(p_{i}, \tilde{q}(u, v)\right) g(u, v) d u d v \tag{3.67}
\end{equation*}
$$

where $\triangle$ is the reference element and $g(u, v)$ is the Jacobian. Clearly, the integrals defining the layer potentials and their derivatives are non-singular when $i \neq j$. When $i=j$ in (3.28) the kernel functions of $L_{k}, M_{k}$ and $M_{k}^{T}$ have an inverse distance singularity. For the regularised Burton and Miller formulation (2.86) the additional operators will also have this inverse distance singularity. Using the direct Burton and Miller formulation (2.73) together with (2.90) and our choice of basis functions, the required integrals are all weakly singular. However, if we employed a higher order approximation to 0 , such as linear or quadratic, a special quadrature rule to evaluate the first integral on the right hand side of (2.90) would be required.

To find the most efficient quadrature rule for evaluating these integrals we applied a number of rules to the representative test problem of evaluating

$$
\begin{equation*}
\int_{S_{1}} \frac{\cos (5|p-q|)}{|p-q|} d S_{q} \tag{3.68}
\end{equation*}
$$

where $S_{j}$ was taken to be a typical element, and $p$ a typical collocation point. The results of evaluating the test integral are discussed below.

### 3.3.1 Axisymmetric Elements.

We recall from Section 3.2 that each linear axisymmetric boundary element is transformed into a rectangle in the $(s, \theta)$ plane, which is then sub-divided into a number of sub-rectangles in the $\theta$ direction. If the integral over a given subrectangle is non-singular, the integral may be simply and efficiently evaluated using an $n \times n$ product Gauss-Legendre rule (which we will refer to as a Gauss rule).

For the singular integrals, Amini and Wilton [3] used a scheme where the rectangle in the $(s, \theta)$ plane containing the collocation point was divided into four smaller sub-rectangles, each having the collocation point at a vertex. The error function transformation $[3,54]$

$$
\begin{equation*}
x=\operatorname{erf}(t)=\frac{2}{\sqrt{\pi}} \int_{0}^{t} e^{-y^{2}} d y \tag{3.69}
\end{equation*}
$$

yields, in the one-dimensional case,

$$
\begin{equation*}
\int_{-1}^{1} f(x) d x=\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\operatorname{er} f(t)) \epsilon^{-t^{2}} d t \tag{3.70}
\end{equation*}
$$

The integrand on the right hand side of (3.70) is dominated by the $e^{-t^{2}}$ term as $t \rightarrow \pm \infty$. The integral on the right hand side is now truncated at $-P$ and $+Q$ and approximated by an $n$ point trapezoidal rule $[1,54]$. This method of transforming a singular integral is known as the ERF rule. The choice of $P, Q$ and $n$ depends on the strength of the singularity in $f$, see Amini [1]. A product rule based on this scheme is applied in each of the four sub-rectangles.

Alternatively, we can diwide the rectangle into four triangles by linking the diagonals so that the collocation point will be at a vertex of each triangle in the
$(s, \theta)$ plane. To perform the integrations over the triangle with vertices $\left(s_{1}, \theta_{1}\right)$, $\left(s_{2}, \theta_{2}\right)$ and $\left(s_{3}, \theta_{3}\right)$, where the singularity is at $\left(s_{1}, \theta_{1}\right)$, we apply the transformation

$$
\begin{array}{ll}
s=(1-u) s_{1}+(1-v) u s_{2}+u v s_{3} & 0 \leq u, v \leq 1 \\
\theta=(1-u) \theta_{1}+(1-v) u \theta_{2}+u v \theta_{3} & \tag{3.71}
\end{array}
$$

derived from a transformation introduced in Duffy [24] which transforms the triangle into a square in the $(u, v)$ plane. The Jacobian of this transformation is

$$
\begin{equation*}
g=\left|\left(s_{2}-s_{1}\right)\left(\theta_{3}-\theta_{1}\right)-\left(s_{3}-s_{1}\right)\left(\theta_{2}-\theta_{1}\right)\right| u \tag{3.72}
\end{equation*}
$$

If the original integrand has an inverse distance singularity in the $(s, \theta)$ plane, the transformed integral in the $(u, v)$ plane is non-singular and a product Gauss rule can be used to evaluate it. This transformation is similar to the one which we will apply to the three-dimensional elements, and is discussed in more detail in Section 3.3.2.

As the level of the discretisation error in our boundary element method is generally larger than $10^{-2}$, we found that calculating the integrals to an accuracy of $10^{-3}$ sufficient to achieve the desired overall accuracy level. Our investigations have shown that it is more efficient to apply the transformation (3.71) than the ERF rule. For example, applying (3.71) and a $5 \times \overline{5}$ product Causs rule to the test integral (3.68) required 100 integrand evaluations to obtain the required level of accuracy; whereas the ERF rule with $P=Q=3$ and $n=9$ required $32 \cdot 4$ integrand evaluations to achieve the same level of accuracy. For this reason we shall only consider the transformation (3.71) from now on.

Let us now discuss the results of our numerical experiments. All the calculations were carried out in double precision on a Prime 850 processor which has an accuracy of around 14 significant figures. Figure 3.3 shows the $\log$ of the absolute error in evaluating the test integral (3.68) over elements whose centroids are different distances from the collocation point using a number of $n \times n$ product Gauss rules. Where the distance between the collocation point and the centroid of the element that we are integrating over is zero, we have applied the transformation (3.71) before applying the quadrature rule to each sub-triangle. It can be seen from Figure 3.3 that it is possible to employ a $4 \times 4$ product Gauss rule to evaluate the singular integrals. However, the use of a $\mathbf{5} \times 5$ product Gauss rule will make certain that the desired level of accuracy is obtained.

For the non-singular integrals it is clear that a $3 \times 3$ product Gauss rule should be used to evaluate the near singular integrals (those over an element whose centroid is less than 0.4 units from the collocation point), and a $2 \times 2$ product Gauss rule to evaluate the remaining integrals. However, for simplicity, we have chosen to use a $3 \times 3$ product Gauss rule to evaluate all the non-singular integrals. We note our results showed that increasing the order of the integration rules employed without refining the boundary element mesh produced little or no improvement in the overall accuracy of the numerical results. Further, we note that for the singular and near singular integrals increasing the order of the Gauss rules resulted in slow improvement of the accuracy, but whenever the integrand is smooth we obtained fast convergence as expected.

### 3.3.2 Three-Dimensional Elements.

A non-singular integral over the three-dimensional triangular elements considered in Section 3.2 can be efficiently evaluated using a product Gauss rule on a triangle, or the Gauss type rule [23]

$$
\begin{equation*}
\int_{u=0}^{1} \int_{v=0}^{1-u} f(u, v) d v d u \approx \frac{1}{6}\left[f\left(0, \frac{1}{2}\right)+f\left(\frac{1}{2}, \frac{1}{2}\right)+f\left(\frac{1}{2}, 0\right)\right] . \tag{3.73}
\end{equation*}
$$

which has errors of order $h^{2}$ where $h$ is the element diameter given by (3.29).
For both the linear and quadratic elements introduced in Section 3.2 the singularity will occur at $\left(\frac{1}{3}, \frac{1}{3}\right)$ in the $(u, v)$ plane. The reference triangle is divided into three sub-triangles by linking the point $\left(\frac{1}{3}, \frac{1}{3}\right)$ to each of the vertices and the transformation

$$
\begin{array}{ll}
u=(1-s) u_{1}+(1-t) s u_{2}+s t u_{3} \\
v=(1-s) v_{1}-(1-t) s v_{2}+s t v_{3} & 0 \leq s, t \leq 1 \tag{3.74}
\end{array}
$$

(similar to (3.71)) is applied to each sub-triangle, where $\left(u_{1}, v_{1}\right),\left(u_{2}, v_{2}\right)$ and $\left(u_{3}, v_{3}\right)$ are the vertices of the sub-triangle and $\left(u_{1}, v_{1}\right)=\left(\frac{1}{3}, \frac{1}{3}\right)$. From (3.63) or (3.65),

$$
\begin{equation*}
x_{q}-x_{p}=\sum_{i=1}^{m} x_{i} N_{i}\left[(1-s) u_{1} \div(1-t) s u_{2}+s t u_{3},(1-s) v_{1}+(1-t) s v_{2}+s t v_{3}\right]-x_{p} \tag{3.75}
\end{equation*}
$$

with similar expressions for $y_{q}-y_{p}$ and $z_{q}-z_{p}$ and where $m$ is the number of nodes in the element. It follows from (3.75) that provided the nodes of the original surface element are distinct

$$
\begin{equation*}
r^{2}=\left(x_{q}-x_{p}\right)^{2}+\left(y_{q}-y_{p}\right)^{2}+\left(z_{q}-z_{p}\right)^{2}=0 \tag{3.76}
\end{equation*}
$$

if and only if $s=0$. It is easy to show that $N_{i}(u, v)$ can written in the form

$$
\begin{equation*}
N_{i}(u, v)=N_{i}\left(u_{1}, v_{1}\right)+s \bar{N}_{i}(s, t) \quad i=1, \ldots, m \tag{3.77}
\end{equation*}
$$

and hence

$$
\begin{align*}
x_{q}-x_{p} & =\sum_{i=1}^{m} x_{i}\left[N_{i}\left(u_{1}, v_{1}\right)+s \bar{N}_{i}(s, t)\right]-x_{p}  \tag{3.78}\\
& =\quad s \sum_{i=1}^{m} \bar{N}_{i}(s, t) x_{i}
\end{align*}
$$

Thus

$$
\begin{equation*}
r^{2}=s^{2}\left[\left(\sum_{i=1}^{m} x_{i} \bar{N}_{i}(s, t)\right)^{2}+\left(\sum_{i=1}^{m} y_{i} \bar{N}_{i}(s, t)\right)^{2}+\left(\sum_{i=1}^{m} z_{i} \bar{N}_{i}(s, t)\right)^{2}\right] \tag{3.79}
\end{equation*}
$$

where the term in [ ] is non-zero for $0 \leq s, t, \leq 1$. The Jacobian of the mapping from the $(u, v)$ sub-triangle to the $(s, t)$ plane is

$$
\begin{equation*}
g=\left|\left(u_{2}-u_{1}\right)\left(v_{3}-v_{1}\right)-\left(u_{3}-u_{1}\right)\left(v_{2}-v_{1}\right)\right| s . \tag{3.80}
\end{equation*}
$$

It can be seen from (3.79) and (3.80) that if the original integral had a $\frac{1}{r}$ singularity, the transformed integral is non-singular.

We now present the results of applying different $n \times n$ Gauss rules and the three point rule (3.73) to evaluate the test integral over a number of elements whose centroids are at different distances from the collocation point. The log of the absolute error in the computed results using linear element is given in Figure 3.4, whilst the corresponding results using the quadratic element are given in Figure 3.5. When the collocation point and the centroid of the element that we are integrating over coincide we have applied the transformation (3.74) before applying the quadrature rule over each sub-triangle.

It is clear that the results for each type of element are similar. For the singular integrals the results seem to show that a $3 \times 3$ product Gauss rule will yield an accuracy of $10^{-3}$. However, it seems that there is little to choose between the
different rules, and experience has shown that it is preferable to use a $5 \times 5$ product Gauss rule. For the non-singular integrals a $3 \times 3$ product Gauss rule is required to achieve the desired accuracy over elements close to the collocation point, whilst a $2 \times 2$ product Gauss rule, or the 3 -point rule (3.73) will yield sufficiently accurate results for elements further away. However, for simplicity, we have chosen to employ a $3 \times 3$ product Gauss rule to evaluate all the non-singular integrals. Again, we observed using higher order quadrature rules than those discussed above produced little or no improvement in the accuracy of the overall solution (ie the errors are due to the discretisation error in our collocation method and not due to the quadrature error).

$$
\begin{aligned}
& \text { Figure 3.3. A comparison of different Gauss-Legendre rules } \\
& \text { for evaluating the test integral over different elements } \\
& \text { for an axisymmetic structure. }
\end{aligned}
$$


Figure 3.4. A comparison of different Gauss-Legendre rules
for evaluating the test integral over different linear
three dimensional elements.


Figure 3.5. A comparison of different Gauss-Legendre rules


### 3.4 The Choice of the Coupling Parameter

The accuracy of the results obtained from the boundary element discretisation will depend on the conditioning of the integral equation. If the integral operator is ill-conditioned, it will be difficult to accurately approximate the solution of the integral equation. We recall that the Burton and Miller formulation (2.73) contains an arbitrary coupling parameter $\alpha$ and the conditioning of this formulation will depend upon the choice of $\alpha$. In this section the effect of different choices of $\alpha$ on the conditioning of the Burton and Miller formulation for the exterior Neumann problem will be investigated.

The following theorem states a basic stability result in studying the error in the solution due to perturbations in the original operator equation.

## THEOREM 3.2

Let $\mathcal{K}: X \rightarrow Y$ and $\dot{\mathcal{K}}: X \rightarrow Y$ be bounded linear operators with

$$
\begin{equation*}
\mathcal{K} \phi=f \quad \overline{\mathrm{~K}} \dot{\phi}=\dot{f} \tag{3.81}
\end{equation*}
$$

and let $\mathcal{K}^{-1}: Y \rightarrow X$ be a bounded linear operator. If $\|\mathcal{X}-\dot{K}\| \mathcal{K}^{-1: 1}=\boldsymbol{B}<1$ then

$$
\begin{equation*}
\frac{\|\phi-\dot{\phi}\|}{\|\phi\|} \leq \frac{\operatorname{cond}(\mathcal{K})}{1-\beta\|\mathcal{K}\|}\left\{\frac{\|\mathcal{K}-\dot{\mathcal{K}}\|}{\|\mathcal{K}\|}-\frac{\|f-\tilde{f}\|}{\|f\|}\right\} \tag{3.82}
\end{equation*}
$$

where $\operatorname{cond}(\mathcal{K})=\|\mathcal{K}\|\left\|\mathcal{K}^{-1}\right\|$ and the norms are understood in appropriate settings.
ב
The quantity $\operatorname{cond}(\mathcal{K})$ is known as the condition number of the linear operator $\mathcal{K}$, and the proof of the theorem is straight forward, see Atkinson $|10|$.

Working in Hilbert space settings, it can be shown that

$$
\begin{equation*}
\|\mathcal{K}\|_{2}=\sqrt{\rho(\mathcal{K} \cdot \mathcal{K})} \text { and }\left\|\mathcal{K}^{-1}\right\|_{2}=\sqrt{\sigma(\mathcal{K} * \mathcal{K})} \tag{3.83}
\end{equation*}
$$

where $\rho$ and $\sigma$ are the largest and smallest eigenvalue, in absolute value, of $\mathcal{K} * \mathcal{K}$ respectively. In other words, it is possible to compute the condition number of $\mathcal{K}$ without any explicit knowledge of $\mathcal{K}^{-1}$. This expression can be further simplified if the eigenfunctions of $\mathcal{K}$ simultaneously form an orthogonal basis in both the Hilbert spaces $X$ and $Y$.

## THEOREM 3.3

Let $\mathcal{K}$ be a bounded linear operator from the Hilbert space $X$ with inner product $(\cdot, \cdot)_{X}$ to the Hilbert space $Y^{\cdot}$ with inner product $(\cdot, \cdot)_{Y}$, and let $\left(\lambda_{n}, \dot{\psi}_{n}\right)$ be the eigensystem of $\mathcal{K}$. If $\left\{\dot{\psi}_{n}\right\}$ form a complete orthogonal set in $X$ and in $Y$, then

$$
\begin{equation*}
\operatorname{cond}(\mathcal{K})=\frac{\mu_{\max }}{\mu_{\min }} \tag{3.84}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu_{\max }=\sup _{i}\left\{\left|\lambda_{i}\right| \frac{y_{i}}{x_{i}}\right\} \quad \text { and } \quad \mu_{\min }=\inf _{i}\left\{\left|\lambda_{i}\right| \frac{y_{i}}{x_{i}}\right\} \tag{3.85}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{i}^{2}=\left(\leftarrow_{i}, \dot{u}_{i}\right)_{X} \quad \text { and } \quad y_{i}^{2}=\left(\iota_{i}, \iota_{i}^{\prime}\right)_{Y}= \tag{3.86}
\end{equation*}
$$

Proof. See $\lambda$ mini $\{2]$.
We note that if $I \equiv Y^{-}$then (3.8.t) is simply the ratio of the largest eigenvalue of $\mathcal{K}$ to the smallest, in absolute value. Even if $\mathcal{K}$ does not satisfy the conditions of Theorem 3.3, (3.84) is often used as an approximation to the condition number.

In general it is not possible to obtain the eigensystems of the layer potential operators for an arbitrary surface $S$. It is possible, however, to obtain the eigensystems of the layer potential operators in the special case where $S$ is a sphere of radius $a$. Using the results from this simple case, it may be possible to obtain some insight into the conditioning of the Helmholtz potential operators on other surfaces.

Following the analysis of Kress [39, 40], we write the solution to Helmholtz equation in the domain exterior to a sphere of radius $a$ as

$$
\begin{equation*}
\phi(r, \theta, \phi)=\sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{n} h_{n}(k r) S_{n}^{m}(\theta, \phi) \tag{3.87}
\end{equation*}
$$

where the $h_{n}$ 's are the spherical Hankel functions of the first kind and the $S_{n}^{m}$ 's are the spherical harmonics given by (3.14). For the interior problem the solution can be written as

$$
\begin{equation*}
\phi_{i}(r, \theta, \phi)=\sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{n} j_{n}(k r) S_{n}^{m}(\theta, \phi) \tag{3.88}
\end{equation*}
$$

where the $j_{n}$ 's are the spherical Bessel functions. Substituting (3.87) and (3.88) into the appropriate integral equation (2.33) and (2.36) respectively, and noting that $\frac{\partial}{\partial n} \equiv \frac{\partial}{\partial r}$, then for each value of $n$ we obtain

$$
\begin{equation*}
-\frac{1}{2} h_{n}(k a) S_{n}^{m}+h_{n}(k a) M M_{k} S_{n}^{m}=k h_{n}^{\prime}(k a) L_{k} S_{n}^{m} \tag{3.89}
\end{equation*}
$$

and

$$
\begin{equation*}
-\frac{1}{2} j_{n}(k a) S_{n}^{m}+j_{n}(k a) M_{k} S_{n}^{m}=k j_{n}^{\prime}(k a) L_{k} S_{n}^{m} \tag{3.90}
\end{equation*}
$$

where a prime denotes the derivative with respect to the argument. Using the

Wronskian relationship

$$
\begin{equation*}
j_{n}(k a) h_{n}^{\prime}(k a)-h_{n}(k a) j_{n}^{\prime}(k a)=\frac{i}{(k a)^{2}} \tag{3.91}
\end{equation*}
$$

(3.89) and (3.90) can be solved for $L_{k} S_{n}^{m}$ and $M_{k} S_{n}^{m}$ to yield

$$
\begin{equation*}
L_{k} S_{n}^{m}=i k^{2} a^{2} h_{n}(k a) j_{n}(k a) S_{n}^{m} \tag{3.92}
\end{equation*}
$$

and

$$
\begin{equation*}
M I_{k} S_{n}^{m}=\left(\frac{1}{2}+i k^{2} a^{2} h_{n}^{\prime}(k a) j_{n}(k a)\right) S_{n}^{m} \tag{3.93}
\end{equation*}
$$

It follows from (3.93) that the eigenvalues $\lambda_{n}$ of the operator $-\frac{1}{2} I+M I_{k}$ are given by

$$
\begin{equation*}
\lambda_{n}\left(-\frac{1}{2} I+M M_{k}\right)=i k^{2} a^{2} h_{n}^{\prime}(k a) j_{n}(k a) \tag{3.94}
\end{equation*}
$$

with the corresponding $2 n+1$ eigenvectors $S_{n}^{m}, m=-n, \ldots, 0, \ldots n$. Clearly if $k a$ is a zero of $j_{n}$ then $-\frac{1}{2} I+M_{k}$ has a zero eigenvalue and its null-space is of dimension $2 n+1$ (ie the space spanned by $S_{n}^{m}, m=-n, \ldots, 0, \ldots n$ ).

Using the same approach as above, but using the differentiated form of (2.34) and (2.36) we obtain

$$
\begin{equation*}
M_{k}^{T} S_{n}^{m}=\left(\frac{1}{2}+i k^{2} a^{2} h_{n}^{\prime}(k a) j_{n}(k a)\right) S_{n}^{m} \tag{3.95}
\end{equation*}
$$

and

$$
\begin{equation*}
N_{k} S_{n}^{m}=i k^{3} a^{2} h_{n}^{\prime}(k a) j_{n}^{\prime}(k a) S_{n}^{m} \tag{3.96}
\end{equation*}
$$

The equations (3.92), (3.93), (3.95) and (3.96) could be derived using the series expansion (3.15) for $G_{k}(p, q)$, (See Amini $[2$ ).

We know from Theorem 2.9 that the operator $-\frac{1}{2} I+M I_{k}+\alpha N_{k}$ has only a trivial null-space for real $k$, provided the function $\alpha$ satisfies $\operatorname{im}(\alpha(p, k))>0$ for all $p \in S$. For ease of analysis, let $\alpha(p, k)=i \nu(p, k)$ where $\nu(p, k)$ is some positive real-valued function. If $\nu$ is further restricted to be a function of $k$ only, it follows from (3.94) and (3.96) that

$$
\begin{equation*}
\left.\left(-\frac{1}{2} I+M_{k}+i \nu(k) V_{k}\right) S_{n}^{m}=i k^{2} a^{2} h_{n}^{\prime}(k a) j_{n}(k a)+i k \nu(k) j_{n}^{\prime}(k a)\right] S_{n}^{m} \tag{3.97}
\end{equation*}
$$

Since the $S_{n}^{m}$ 's form a complete orthonormal set for both $H^{r}$ and $H^{r-1}$, the conditions of Theorem 3.3 hold, and we can use (3.84) to compute the condition number of our integral equation. We note that because of the presence of the operator $N_{k}$; we have to treat the operator $-\frac{1}{2} I+M_{k}-i \nu(k) N_{k}$ as an operator from $H^{r}(s)$ to $H^{r-1}(s)$ 2. Therefore it is necessary to evaluate $\frac{y_{i}}{x_{i}}$ in (3.85), the relative values of the norm of these basis functions in $H^{r}$ and $H^{r-1}$. It can be shown (see Kress (40]) that

$$
\begin{equation*}
\left\|S_{n}^{m}\right\|_{H^{r}}^{2}=(1-n(n+1))^{r}\left[\frac{1}{2 \pi}\left(n-\frac{1}{2}\right) \frac{(n-m)!}{(n-m)!}\right] \tag{3.98}
\end{equation*}
$$

This yields

$$
\begin{equation*}
\frac{y_{n}}{x_{n}}=(1+n(n+1))^{-\frac{1}{2}} \tag{3.99}
\end{equation*}
$$

which is a measure of the relative norms of the eigenfunctions $S_{n}^{m}$ on $H^{r-1}(S)$ and $H^{r}(S)$. Even though it is not possible to obtain a simple function $\nu(k)$ which minimises the condition number of the operator $-\frac{1}{2} I+M_{k}-i \nu(k) N_{k}$, it is possible to prove the following result.

## THEOREM 3.4

For the direct Burton and Miller formulation (2.73), a near optimal choice of $\nu(k)$ is

$$
\nu(k)= \begin{cases}\frac{1}{k} & \text { if } k \geq \frac{1}{2}  \tag{3.100}\\ 2 & \text { if } k<\frac{1}{2}\end{cases}
$$

For the regularised Burton and Miller formulation (2.86), the near optimal choice of $\nu(k)$ is

$$
\begin{equation*}
\nu(k)=1 \tag{3.101}
\end{equation*}
$$

Proof. See Amini [2]. $=$
It has been shown in Amini and Harris [7] that the choice of $\nu(k)$ as given in (3.100) can considerably improve the accuracy of the everall results, as compared to the commonly used choice $\nu(k)=1$.

One of our aims is to aroid computing $N_{k} \phi$ or $\left(\Lambda_{z}-N_{0}\right) \phi$ for all $p \in S$ by letting $\nu(p, k)$ be so small that it can regarded a zero over some large section of $S$. This is not possible with the regularised Burton and Miller formulation because we have to pre-multiply $N_{k}-V_{0}$ by $L_{0}$. which means it is necessary to compute our approximation to $N_{k}-N_{0}$ for all the collocation points. For this reason, we will now restrict our discussion to the direct Burton and Miller formulation (2.73).

For the model problem of a sphere, we now consider a general $\nu(p, k)$ and expand $\frac{\partial^{2} G_{k}}{\partial n_{p} \partial n_{q}}$ using the series given in (3.15) to obtain

$$
\begin{equation*}
\nu(p, k) \frac{\partial^{2} G_{k}(p, q)}{\partial n_{p} \partial n_{q}}=i k^{3} a^{2} \nu(p, k) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_{n}^{\prime}(k a) h_{n}^{\prime}(k a) S_{n}^{m}\left(\theta_{p}, \phi_{p}\right) \overline{S_{n}^{m}\left(\theta_{q}, \phi_{q}\right)} . \tag{3.102}
\end{equation*}
$$

For each $(n, m)$ we expand $\nu(p, k) S_{n}^{m}\left(\theta_{p}, \phi_{p}\right)$ as

$$
\begin{equation*}
\nu(p, k) S_{n}^{m}\left(\theta_{p}, \phi_{p}\right)=\sum_{n^{\prime}=0 m^{\prime}=-n^{\prime}}^{\infty} a_{n m}^{n^{\prime}}{ }^{n^{\prime} m^{\prime}} S_{n^{\prime}}^{m^{\prime}}\left(\theta_{p}, \phi_{p}\right) \tag{3.103}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{n m}^{n^{\prime} m^{\prime}}=\int_{S} \nu(p, k) S_{n}^{m}\left(\theta_{p}, \phi_{p}\right) \overline{S_{n^{\prime}}^{m^{\prime}}\left(\theta_{p}, \phi_{p}\right)} d S_{p} \tag{3.104}
\end{equation*}
$$

as the $S_{n}^{m}$ 's are orthogonal. Since the functions $S_{n}^{m}$ form a complete set in $H^{r}(S)$, the eigenfunctions of the integral operator $-\frac{1}{2} I+M_{k}+i \nu(p, k) N_{k}$ can be expressed in the form $\phi_{e}=\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n m} S_{n}^{m}$ to give

$$
\begin{gather*}
i k^{2} a^{2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n m} h_{n}^{\prime}(k a)\left[\sum_{n^{\prime}=0}^{\infty} \sum_{m^{\prime}=-n^{\prime}}^{n^{\prime}} \delta_{n n^{\prime}} \delta_{m m^{\prime}} j_{n}(k a)+i k a_{n m}^{n^{\prime} m^{\prime}} j_{n}^{\prime}(k a)\right] S_{n}^{m}\left(\theta_{p}, \phi_{p}\right) \\
=\lambda \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n m} S_{n}^{m}\left(\theta_{p}, \phi_{p}\right) \tag{3.105}
\end{gather*}
$$

where $\delta_{n n^{\prime}}$ is the Krondecker delta and we have expanded $\frac{\partial G_{k}}{\partial n_{g}}$ in a series using (3.15). Taking the inner product of both sides of (3.105) with $S_{\bar{n}}^{\bar{m}}$ gives

$$
\begin{equation*}
i k^{2} a^{2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n m} h_{n}^{\prime}(k a)_{i} \delta_{n \bar{n}} \delta_{m \bar{m}} j_{n}(k a)-i k a_{n m}^{\bar{n} \dot{m}} j_{n}^{\prime}(k a)!=\lambda \alpha_{\bar{n} \bar{m}} \tag{3.106}
\end{equation*}
$$

for $\vec{n}=0,1, \ldots$ and $\bar{m}=-\bar{n}, \ldots, 0, \ldots, \bar{n}$. Truncating the infinite sums after $N+1$ terms of the outer sum leads to the standard matrix eigenvalue probiem

$$
\begin{equation*}
A \underline{\alpha}=\lambda \underline{\alpha} \tag{3.107}
\end{equation*}
$$

where $\underline{a}=\left\{\alpha_{00}, \alpha_{1-1}, \alpha_{10}, \ldots,\left.\alpha_{N N}\right|^{T}\right.$ and

$$
\begin{equation*}
A_{(\tilde{n} \dot{m})(n m)}=i k^{2} a^{2} h_{n}^{\prime}(k a)\left[\delta_{n \bar{n}} \delta_{m \dot{m}} j_{n}(k a) \div i k a_{n m}^{n} j_{n}^{n \dot{n}} j_{n}^{\prime}(k a)\right] \tag{3.108}
\end{equation*}
$$

$A_{(\bar{n} \bar{m})(n m)}$ is the element of $A$ in the row corresponding to $(\bar{n}, \bar{m})$ and the column corresponding to $(n, m)$.

The solution of (3.107) will only give an approximation to the eigenvalues of the operator $-\frac{1}{2} I+M_{k}+i \nu N_{k}$ since the infinite sums have been truncated. Also, in general, the eigenfunctions will not be orthogonal and so the conditions of Theorem 3.3 will not hold. However, it is hoped that this method will at least give us some valuable insight into the condition number of the direct Burton and Miller formulation for a wide choice of $\nu(p, k)$. In principle, this method can be used for other shaped surfaces provided we can obtain appropriate series expansions for the Green's function $G_{k}$ and its derivatives. However, there are practical problems in implementing this method for finding the approximate condition number. Unless $\nu(p, k)$ is a continuous function of $p$, it will be necessary to take a large number of terms in the series (3.103) in order to obtain an accurate approximation. In general, for a given $\nu(p, k)$ the coefficients $a_{n m}^{n^{\prime} m^{\prime}}$ will have to be computed numerically. As $n$ increases, the associated Legendre functions $P_{n}^{m}$ become more oscillatory which makes the integrals in (3.104) more difficult to evaluate accurately.

We now consider an alternative method for estimating the condition number using the equations derived from the piecewise constant collocation method discussed in Section 3.1.4. Consider an integral operator $K: X \rightarrow Y$. It is possible to view the collocation equations as being equivalent to some approximate operator $\mathcal{K}_{n}: X_{n} \rightarrow Y_{n}$, where $X_{n}$ and $Y_{n}$ are subspaces of $X$ and $Y$.

Clearly any $\phi_{n} \in X_{n}$ can be written in the form

$$
\begin{equation*}
\phi_{n}(p)=\sum_{i=1}^{n} \phi_{i} \psi_{i}(p) \quad p \in S \tag{3.109}
\end{equation*}
$$

where the $\psi_{i}$ 's are the piecewise constant basis functions given in (3.26). Hence, for $\phi_{n} \in X_{n}$

$$
\begin{equation*}
f(p)=\mathcal{K} \phi_{n}=\sum_{i=1}^{n} \phi_{i} \int_{S_{i}} k(p, q) d S_{q} . \tag{3.110}
\end{equation*}
$$

We now approximate $f$ by $f_{n} \in Y_{n}$ by applying the projection operator $\mathcal{P}_{n}: Y \rightarrow Y_{n}$ defined by (3.22). Define $\mathcal{K}_{n}=\mathcal{P}_{n} \mathcal{K}: X_{n} \rightarrow Y_{n}$ to obtain

$$
\begin{equation*}
f_{n}(p)=\sum_{i=1}^{n} \phi_{i} \int_{\mathcal{S}_{i}} \mathcal{P}_{n}(k(p, q)) d S_{q}\left(=\mathcal{K}_{n} \phi_{n}\right) \tag{3.111}
\end{equation*}
$$

which can be written as

$$
\begin{equation*}
\sum_{j=1}^{n} f_{j} \dot{w}_{j}(p)=\sum_{j=1}^{n}\left[\sum_{i=1}^{n} \phi_{i} \int_{S_{i}} k\left(p_{j}, q\right) d S_{q}\right] \psi_{j}(p) . \tag{3.112}
\end{equation*}
$$

Since the $\psi_{j}$ 's are linearly independent, the $f_{i}$ 's and $\phi_{i}$ 's are related through the matrix equation

$$
\begin{equation*}
\underline{f}=\bar{K} \underline{\phi} \tag{3.113}
\end{equation*}
$$

where $\underline{f}=\left[f_{1}, \ldots, f_{n}\right]^{T}$ and $\left.\underline{\phi}=\phi_{1}, \ldots, \phi_{n}\right]^{T}$ and

$$
\begin{equation*}
\bar{K}_{i j}=\int_{S_{i}} k\left(p_{j}, q\right) d S_{q} . \tag{3.114}
\end{equation*}
$$

Clearly $\tilde{K}$ is the same matrix as the one in (3.28) for the collocation equations.
The eigenvalues and eigenfunctions of $\mathcal{K}_{n}$ satisfy $\lambda \phi_{n}=\mathcal{K}_{n} \dot{O}_{n}$ or

$$
\begin{equation*}
\lambda \sum_{j=1}^{n} \phi_{j} \dot{w}_{j}(p)=\sum_{j=1}^{n} \sum_{i=1}^{n} \phi_{i} \int_{S_{i}} k\left(p_{j}, q\right) d S_{q} \psi_{j}(p) \tag{3.115}
\end{equation*}
$$

Since the set $\left\{\psi_{j}\right\}$ is linearly independent then

$$
\begin{equation*}
\lambda \phi_{j}=\sum_{i=1}^{n} \phi_{i} \int_{S_{i}} k\left(p_{j}, q\right) d S_{q} \quad j=1, \ldots, n \tag{3.116}
\end{equation*}
$$

or

$$
\begin{equation*}
\lambda \underline{\phi}=\tilde{K} \underline{\phi} \tag{3.117}
\end{equation*}
$$

That is, the eigenvalues of $\mathcal{K}_{n}$ are the same as those of $\bar{K}$. This result is not surprising as the collocation operator $\mathcal{K}_{n}$ is a finite dimensional operator. We can compute the matrices equivalent to $\mathcal{K}_{n}$ and $\mathcal{K}_{n}^{*}$ and use (3.83) to compute $\operatorname{cond}\left(\mathcal{K}_{n}\right)$, which is an approximation to $\operatorname{cond}(\mathcal{K})$. We note that the matrix equivalent to $\mathcal{K}_{n}^{*}$ is not the same as taking the complex conjugate and transpose of $\tilde{K}$.

The following theorem gives some insight into the convergence of the condition number of the approximate operator.

## THEOREM 3.5

If $\left\|\mathcal{K}-\mathcal{K}_{n}\right\| \rightarrow 0$ as $n \rightarrow \infty$ then

$$
\begin{equation*}
\operatorname{cond}\left(\lambda I+\mathcal{K}_{n}\right) \rightarrow \operatorname{cond}(\lambda I+\mathcal{K}) \quad \text { as } \quad n \rightarrow \infty \tag{3.118}
\end{equation*}
$$

Proof. See Baker [12].
This theorem shows that the approximation obtained for the condition number of $-\frac{1}{2} I+M_{k}$ will converge since $M_{k}$ is compact and so our approximation to $M_{k}$ will converge uniformly (see Definition 2.2 and Theorem 2.3). However, it is not possible to guarantee such convergence for the direct Burton and Miller formulation since $N_{k}$ is not compact. It seems reasonable, though, to use this method to obtain an
estimate of the condition number of the integral operator, even though theoretically we can not prove that taking the limit $n \rightarrow \infty$ will yield the exact condition number.

We recall that the coupling parameter in (2.75) is of the form $\alpha(p, k)=i \nu(p, k)$ where $\nu(p, k)$ is a real valued function. Assume that $\nu(p, k)=\sum_{i=1}^{n} \nu_{i}(k) \psi_{i}(p)$ and approximate the operator $-\frac{1}{2} I+M I_{k}+\alpha N_{k}$ by

$$
\begin{equation*}
\dot{K}=-\frac{1}{2} I+\bar{M}_{k}+i . \Lambda \tilde{N}_{k} \tag{3.119}
\end{equation*}
$$

where $\Lambda$ is the diagonal matrix $\operatorname{diag}\left[\nu_{1}, \ldots, \nu_{n}\right]$. The matrix approximation to the adjoint operator is given by

$$
\begin{equation*}
\bar{K}^{*}=-\frac{1}{2} I+\overline{\left(\tilde{M}_{k}^{T}\right)}-\overline{i\left(\bar{N}_{k}\right)} \Lambda \tag{3.120}
\end{equation*}
$$

The approximate condition number can now be found using (3.83).
We can treat the condition number computed from (3.119) and (3.120) as a function of the $n$ parameters $\left\{\nu_{1}, \ldots, \nu_{n}\right\}$ and try to find the ralues which minimise $\operatorname{cond}\left(\mathcal{K}_{n}\right)$. However, there is no simple method for finding $\left\{\nu_{1}, \ldots \nu_{n}\right\}$ which minimise $\operatorname{cond}\left(\mathcal{K}_{n}\right)$ and it is necessary to employ a numerical technique such as the simplex method, the details of which are given in Fletcher 26 . It is not possible to easily use other methods, such as conjugate gradients, since there are both theoretical and practical difficulties in computing the derivatives of $\operatorname{cond}\left(\mathcal{K}_{n}\right)$ with respect to $\left\{\nu_{1} \ldots \nu_{n}\right\}$.

The numerical results presented here are for surfaces with the same typical dimension $d$. For a sphere $d$ is simply the radius, whilst for a cylinder of radius $r$ and height $h, d$ is given by $d=\frac{2 r+h}{3}$. For a spheroid of the form $\left(\frac{x}{a}\right)^{2}+\left(\frac{y}{a}\right)^{2}+\left(\frac{z}{b}\right)^{2}=1$,
$d$ is given by $d=\frac{2 a+b}{3}$. The particular surfaces we shall consider are a unit sphere, a cylinder of radius 0.6 and height 1.8 and a spheroid with $a=0.75$ and $b=1.50$. For all of these surfaces $d=1$. Each of these surfaces was modelled using 15 linear axisymmetric boundary elements.

Estimates of the condition number of $-\frac{1}{2} I+M_{k}+i \nu N_{k}$ for the sphere, cylinder and spheroid are shown in Figures 3.6, 3.7 and 3.8 respectively, where $\nu(p, k)=$ OPT denotes the choice of $\left\{\nu_{1}, \ldots, \nu_{n}\right\}$ which minimises the condition number. It is clear from these results that the near optimal choice is $\nu(p, k)=\frac{1}{k}$ for larger values of $k$, which is in agreement with the results of Amini [2] and Kress [40]. For smaller values of $k$, then the appropriate values of $\nu$, as given in Theorem 3.4, should be used.

One of the objectives of our analysis in this section was to see if we could still obtain a well conditioned integral operator by choosing $\nu(p, k)=0$ over a large section of the surface $S$, and thereby avoid having to compute an approximation to the differentiated equation for most of the collocation points $p_{i}$. Here we have chosen the simple scheme of taking $\nu(p, k)=0$ over the first $m$ elements from one pole of the surface, and taking $\nu(p, k)=\frac{1}{k}$ over the remaining elements.

Figures 3.9 and 3.10 show the estimated condition number of the operator $-\frac{1}{2} I+M_{k}+i \nu N_{k}$ for the sphere and the cylinder respectively, at their first three characteristic $k$. For clarity, we have not shown results when $\nu(p, k)$ is zero over 14 or 15 elements since these values of $\operatorname{cond}\left(\mathcal{K}_{n}\right)$ are greater than 30 . The corresponding numerical results are shown in Figures 3.14 and 3.15 respectively.

It is clear from (3.94) that the first characteristic wavenumber, $k_{c}$, of a unit sphere is given by $k_{c}=\pi$. However, because of the way in which we have discretised the sphere using linear axisymmetric elements, the collocation points actually lie on a sphere of radius $\cos \left(\frac{\pi}{30}\right)$. Hence, the first characteristic wavenumber of the sphere is given by $k_{c}=\pi \times \cos \left(\frac{\pi}{30}\right)=3.1589$. There is a similar change in the location of the other characteristic wavenumbers of the sphere. In the case of the cylinder the linear axisymmetric element represent the surface exactly and so there is no change in the location of the characteristic wavenumbers.

It is clear from our results that when $\nu(p, k)=0$ over nearly all the elements then the integral operator is ill-conditioned near characteristic wavenumbers of the structure. However, taking $\nu(p, k)=0$ over half the surface gives an operator which is almost as well conditioned as the operator with $\nu(p, k)=\frac{1}{k}$ over all the surface. As stated before, here we chose $\nu(p, k)=\frac{1}{k}$ over the parts of the surface where $\nu$ is non-zero. However, the choice of $\nu$ over the elements where it is non-zero requires some further investigation, since $\frac{1}{k}$ may no longer be the near optimal choice.

On the basis of the these results, we can make the following recommendations about the choice of $\nu(p, k)$.
(1) In order to obtain a well conditioned integral operator we should choose $\nu(p, k)$ to be non-zero over at least half the surface.
(2) If we choose $\nu(p, k)$ to be non-zero everywhere, then we should choose $\nu$ to be a constant with respect to the space variables, the size of which is given by Theorem 3.4.
Figure 3.6. Condition number of the direct formulation






### 3.5 Numerical Results

Exact solutions in closed form, usually as infinite series of slowly converging wave functions, can only be obtained in special cases [45]. However, exact solutions can be generated for problems which are equivalent to those having acoustic point sources in the interior region $D_{-}[3,31]$. The acoustic pressure at a point $p \in \mathcal{R}^{3}$ generated by a set of point sources at $q_{j} \in D_{-}$with strengths $A_{j}, j=1, \ldots, Q$, in the absence of the structure, is given by

$$
\begin{equation*}
\phi(p)=\sum_{j=1}^{Q} A_{j} \frac{e^{i k\left|p-q_{j}\right|}}{4 \pi\left|p-q_{j}\right|} \tag{3.121}
\end{equation*}
$$

For $p \in S$ we can differentiate (3.121) with respect to $\underline{n}_{p}$ to obtain

$$
\begin{equation*}
\frac{\partial \phi(p)}{\partial n_{p}}=\sum_{j=1}^{Q} A_{j} \frac{e^{i k\left|p-q_{j}\right|}}{4 \pi\left|p-q_{j}\right|^{2}}\left(i k\left|p-q_{j}\right|-1\right) \frac{\left(p-q_{j}\right) \cdot n_{p}}{\left|p-q_{j}\right|} \quad p \sqsubseteq S . \tag{3.122}
\end{equation*}
$$

The problem with this distribution of $\frac{\partial \dot{\phi}}{\partial n}$ on the surface $S$ is equivalent to the point source problem for which the solution (3.121) is known.

The axisymmetric surfaces considered here are a unit sphere and a cylinder of height 1.8 metres and radius 0.6 metre. We also consider a spheroid of the form $\left(\frac{x}{a}\right)^{2}+\left(\frac{y}{a}\right)^{2}-\left(\frac{z}{b}\right)^{2}=1$, with $\mathrm{a}=0.75$ metres and $\mathrm{b}=1.50$ metres. The boundary data for each of the axisymmetric structures is equivalent to a single point source placed at ( $0,0,0,0.5$ ) with strength $2-3 i$. The mesh for each structure has 15 linear. axisymmetric elements.

The three-dimensional surfaces considered are a unit sphere, a cylinder of length 1.8 metres and radius 0.6 metre and a cube with sides length 1 metre. The boundary data for the sphere and the cylinder is equivalent to two point sources placed
at $(0,0,0.5)$ and $(0.25,0.25,0.25)$ with strengths $2+3 i$ and $t-i$ respectively. For the cube, the boundary data is equivalent to two point sources placed at $(0.5,0.5,0.5)$, the centre of the cube, and ( $0.25,0.25,0.25$ ) with strengths $2+3 i$ and $4-i$ respectively. The surfaces of the sphere and cylinder have been modelled using 420 quadratic triangular boundary elements, and the surface of the cube has been modelled using 432 linear triangular boundary elements.

The measure of error that we have chosen to use here is the mean relative error (MRE) given by

$$
\begin{equation*}
M R E=\frac{1}{n} \sum_{j=1}^{n} \frac{\left|\dot{\varphi}\left(p_{j}\right)-\phi_{n}\left(p_{j}\right)\right|}{\left|\phi\left(p_{j}\right)\right|} \tag{3.123}
\end{equation*}
$$

expressed as a percentage, where $\phi\left(p_{j}\right)$ is the exact solution at the point $p_{j}$ and $\phi_{n}\left(p_{j}\right)$ the numerical solution. If any of the $\dot{\phi}\left(p_{j}\right)$ are close to zero. then this measure can exaggerate the error and it may be preferable to use the relative mean error (RME) defined by

$$
\begin{equation*}
R M E=\frac{\sum_{j=1}^{n}\left|\phi\left(p_{j}\right)-\phi_{n}\left(p_{j}\right)\right|}{\sum_{j=1}^{n}\left|\phi_{n}\left(p_{j}\right)\right|} \tag{3.124}
\end{equation*}
$$

or the vector 2 -norm of the crror, although this has not been done here.
The results for the axisymmetric sphere. cylinder and spheroid are shown in Figures $3.11,3.12$ and 3.13 respectively. These results show that, in general, the direct formulation gives somewhat more accurate results than the regularised formulation. This is in contrast to the the results given in Amini and Harris i. However, in 7 the surfaces of the structures were represented exactiy, whereas we have now chosen the more general approach of approximating the surfaces using
linear and quadratic boundary elements.
Figures 3.14 and 3.15 present the results for a sphere and a cylinder at their first three characteristic wavenumbers, where $\nu(p ; k)$ has been taken to be zero over different numbers of elements. Here we have chosen $\nu(p, k)=0$ over the first $m$ elements from one pole of the surface, and $\nu(p, k)=\frac{1}{k}$ over the remaining elements. We recall that the condition numbers of these equations are shown in Figures 3.9 and 3.10 respectively. For clarity, the results have not been shown when the error is greater than $25 \%$. It is clear that taking $\nu(p, k)=0$ over two thirds of the surface elements still yields results which are accurate to within $5 \%$, which is the level of our discretisation error. In fact, there is little or no improvement in the accuracy by taking $\nu(p, k)$ to be non-zero over a larger proportion of the surface.

The results for a three dimensional sphere, cylinder and cube are shown in Figures $3.16,3.17$ and 3.18 respectively. In general the regularised formulation yields somewhat more accurate results than the direct formulation, for a fully three dimensional structure. However, the regularised formulation is considerably more expensive to implement, in terms of both CPU time and computer storage. For - example, it takes 48 CPU seconds to find the pressure on a surface with 112 el ements using the direct formulation, whereas it takes 75 CPU seconds using the regularised formulation, on the Prime 850 processor at Polytechnic South West. These computational considerations make the direct formulation much more attractive for use in the coupled fluid-structure interaction problem, to be considered in the next chapter.

The general aim of this chapter was to devise a practical numerical method for solving the exterior Neumann problem for Helmholtz equation accurately and efficiently. In Section 3.1 we identified the collocation method as being the most practical projection method while Section 3.2 considered both axisymmetric and three-dimensional surface representation. In Section 3.3 we compared some different quadrature rules used to compute the matrix approximations to the integral operators and discussed the choice of rule to give the desired accuracy. In Section 3.4 we found that the near optimal choice of the coupling parameter is $\nu(p, k)=\frac{1}{k}$. However, we also found that is possible to obtain accurate results with $\nu(p, k)=0$ over a large part of the surface. This has the advantage that for elements where $\nu(p, k)=0$ we do not have to compute an approximation to the differentiated equation for the direct formulation.

The numerical results of this section show that the direct formulation and the regularised formulation yield numerical results of similar accuracy. However, the regularised method is more expensive to implement, particularly if we choose $\nu(p, k)=0$ for most of the surface elements when using the direct formulation. Thus it would appear that direct Burton and Miller formulation, with the appropriate choice of the coupling parameter $\nu$, is a good choice for solving the exterior Neumann problem for Helmholtz equation.
Legend
DIRECT
REGULARISED


Figure 3.12. Error of the different formulations used
on an axisymmetric cylinder, $r=0.6, I=1.8$, for different
values of $k$.





| $\begin{array}{c}\text { Figure 3.14. Error using the direct formulation on a } \\ \text { unit sphere with } v(p, k) \text { zero over different numbers of } \\ \text { elements. }\end{array}$ |
| :---: |
| 而 |



$$
\begin{gathered}
\text { Figure } 3.15 \text {. Error using the direct formulation on a } \\
\text { cylinder, } r=0.6,1=1.8 \text {, with } v(p, k) \text { zero over different } \\
\text { numbers of elements. }
\end{gathered}
$$

Legend
k=3.1589
$\square k=4.5182$
$k=5.7922$

Figure 3.16. Error of the different formulations used on a



Figure 3.17. Error of the different formulations used on a
$3-D$ cylinder, $r=0.6,1=1.8$, for different values of $k$.





## 4 THE DYNAMIC FLUID-STRUCTURE IN-

## TERACTION PROBLEM

### 4.1 Introduction

The previous chapters have considered the problem of determining the acoustic field around an arbitrary structure where the normal particle velocities on the surface were assumed to be known. We shall now include the motion of the elastic structure in the analysis in order to predict the resulting sound radiation due to applied forces throughout the structure, and also to model the scattering by elastic structures more accurately. It is now necessary to couple the Helmholtz equation (formulated as an integral equation) in the fluid region $D_{\text {- }}$ with the equations of motion of the structure $D_{-}$at the surface $S$ by ensuring that the normal particle velocity and the normal stress are continuous at $S$.

This problem is of considerable interest in many areas of mathematical physics, including underwater acoustics and aeronautics, where it is required to determine the acoustic fields about an arbitrary three-dimensional structure. However, there only a few simple structures, such as spheres and infinite circular cylinders, for which it is possible to obtain analytical solutions. For this reason the only feasible way of obtaining a solution is to use numerical methods. In Chapter 5 we shall use the the numerical scheme developed in this chapter to model the acoustic fields radiated by some simple sonar transducers.

The analytical solution of this coupled problem for the simple case of a sphere is considered in Goodman and Stern [29] and Gaunaurd and Uberall [28] where the displacements within the structure are represented in terms of two potential functions. By using the classical technique of separation of variables it can be shown that these potentials and the exterior acoustic field can be expressed as series of spherical Bessel and Hankel functions, and associated Legendre functions, the coefficients of which are found by enforcing the continuity conditions on $S$ (see Appendix A).

We now give a brief review of the different numerical schemes that have been proposed for the solution of the coupled fluid-structure interaction problem.

Chen and Schweickert [18] assume that the continuous structure can be represented by a system of interconnected discrete masses. It is possible to derive the equations of motion of these masses to give an approximation to the equations of motion of the structure and couple these equations to an indirect formulation of the exterior acoustic problem. Chen [19] also uses the finite element method to model the motion of the structure and couples this to an indirect boundary integral analysis of the acoustic field.

Hunt et al [34] employ the finite element method to model the motion of the structure and the acoustic field in a sphere surrounding the structure. The acoustic field in the region exterior to the sphere is represented using a partial wave series, which avoids the use of the integral equation formulations and hence the associated non-uniqueness or non-existence problems. However, the fluid region between the
surface of the structure and the surface of the sphere forms an interior problem for Helmholtz equation, and for certain frequencies (wavenumbers) the solution to this problem will be non-unique, although Hunt et al [34] propose a solution method for overcoming this problem.

One of the most commonly employed techniques for analysing the motion of an elastic structure is the finite element method. Wilton 61,62 proposes a number of different methods for coupling a direct integral equation formulation of the exterior acoustic field to a finite element analysis of the motion of the structure, and gives some results for thick elastic shells scattering an incident plane wave. Everstine et al [25] and Mathews [43] employ a similar scheme to model the acoustic field around a thin shell. Mathews [43] also gives a survey of different methods for solving the interaction problem, as do Piaszczyk and Klosner [48.

Seybert et al [51] use a boundary integral formulation of the equations of motion of the structure and couple it to a direct boundary integral formulation of the acoustic field. This scheme has the advantage that all the computations are restricted to the surfaces of the structure. However, it has the disadrantage that if the structure is made of more than one material we must include a separate boundary integral analysis of the displacements within each different material. Also, unlike the finite element method, the boundary element method does not automatically calculate the displacements within the structure which may be of considerable interest.

The method for analysing the motion of the structure employed here is the finite element method. It has the advantage that it can readily deal with complex struc-
plex structures made of more than one material. The finite element methods for analysing the motion of either an axisymmetric or a fully three-dimensional elastic structure are given in Section 4.2. In Section 4.3 the method is coupled with a direct boundary integral formulation of the exterior acoustic field, and different schemes for solving the coupled problem are discussed.

In the previous chapters we have seen that as the wavenumber approaches a characteristic wavenumber the integral operators $-\frac{1}{2} I+M_{k}$ and $L_{k}$, and hence their approximating matrices, become more ill-conditioned. Recently several authors $[19,33,48]$ have indicated that they have observed no great loss of accuracy in the solution to the coupled problem at the characteristic wavenumbers when using the surface Helmholtz equation to represent the exterior field. However, other authors [43, 62] suggest that for the solution to the coupled problem to be unique it is still necessary to use an integral equation formulation which is valid for all wavenumbers. In Section 4.4 we present an argument for resolving this apparent controversy.

### 4.2 Finite Element Analysis of the Structure

Assume that the region $D$. contains an isotropic elastic structure and that the displacements due to any applied forces are sufficiently small for the generalised Hooke's law to apply. In other words, the structure will return to its initial form completely after the removal of all forces: and its elastic properties are the same in all directions. In this section we derive the finite element model of both an
axisymmetric structure and a fully three-dimensional structure.

### 4.2.1 Finite Element Analysis of an Axisymmetric Structure.

Consider an axisymmetric structure $D_{-}$which is formed by rotating a bounded region $A$ in the $(r, z)$ plane (with $r \geq 0$ ) about the $z$ axis. This region is discretised into a finite number of elements $A_{e}$, interconnected at a finite number of nodal points. The displacements $(u, v)$ at the nodes, in the radial $(r)$ and axial $(z)$ directions respectively, are the unknown parameters of the structural part of the analysis. We assume that the displacements can be written as

$$
\begin{equation*}
u(r, z)=\sum_{i=1}^{n} u_{i} \Psi_{i}(r, z) \quad v(r, z)=\sum_{i=1}^{n} v_{i} \Psi_{i}(r, z) \tag{4.1}
\end{equation*}
$$

where $n$ is the number of nodes, $\left(u_{i}, v_{i}\right)$ are the displacements at the $i^{t h}$ node and $\Psi_{i}(r, z)$ is the basis function associated with the $i^{t h}$ node. We also assume that the displacements have harmonic time dependence of the form $e^{-i \omega t}$. The exact form of the basis functions depends on the type of finite element used, but in general $\Psi_{i}$ will be chosen to be unity at the $i^{\text {th }}$ node and zero at all the other nodes.

The linear stress-strain relationship takes the form : 42,64$]$

$$
\begin{equation*}
\underline{\sigma}=D \underline{\epsilon} \tag{4.2}
\end{equation*}
$$

where $\underline{\sigma}$ and $\underline{\epsilon}$ are vectors of the four non-zero components of the symmetric stress
and strain tensors respectively $[42]$. The $4 \times 4$ matrix $D$ is given by

$$
D=\frac{E}{(1+\nu)(1-2 \nu)}\left[\begin{array}{cccc}
1-\nu & \nu & \nu & 0  \tag{4.3}\\
\nu & 1-\nu & \nu & 0 \\
\nu & \nu & 1-\nu & 0 \\
0 & 0 & 0 & \frac{1-2 \nu}{2}
\end{array}\right]
$$

where $E(>0)$ is Young's modulus and $\nu\left(0 \leq \nu<\frac{1}{2}\right)$ is Poisson's ratio. The strain is related to the displacements by

$$
\underline{\epsilon}=\left(\begin{array}{c}
\frac{\partial v}{\partial z}  \tag{4.4}\\
\frac{\partial u}{\partial r} \\
\frac{u}{r} \\
\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}
\end{array}\right)
$$

After substituting (4.1) into (4.4): the strain vector is of the form

$$
\begin{equation*}
\underline{\epsilon}=B \underline{q} \tag{4.5}
\end{equation*}
$$

where $\underline{q}=\left\{u_{1}, v_{1}, u_{2}, v_{2}, \ldots, u_{n}, v_{n}\right]^{T}$ and $B=\left\{B_{1}, B_{2}, \ldots B_{n}{ }^{\}}\right.$with

$$
B_{i}=\left(\begin{array}{cc}
0 & \frac{\partial \Psi_{i}}{\partial z}  \tag{4.6}\\
\frac{\partial \Psi_{i}}{\partial r} & 0 \\
\frac{\Psi_{i}}{r} & 0 \\
\frac{\partial \Psi_{i}}{\partial z} & \frac{\partial \Psi_{i}}{\partial r}
\end{array}\right) .
$$

The strain energy. $l^{i}$, of the structure is given by

$$
\begin{equation*}
U=\frac{1}{2} \int_{D_{-}} \underline{\epsilon}^{T} D \underline{\underline{\epsilon}} d V=\frac{1}{2} \underline{q}^{T} K \underline{q} \tag{4.7}
\end{equation*}
$$

where $K$ is the $2 n \times 2 n$ matrix known as the stiffness matrix of the structure, and is given by

$$
\begin{equation*}
K=\int_{D_{-}} B^{T} D B d V=2 \pi \int_{A} B^{T} D B r d r d z \tag{4.8}
\end{equation*}
$$

The kinetic energy, $T$, of the structure is

$$
\begin{equation*}
T=\frac{1}{2} \int_{D_{-}} \rho_{s}\left[\left(\frac{\partial u}{\partial t}\right)^{2}+\left(\frac{\partial v}{\partial t}\right)^{2}\right] d V \tag{4.9}
\end{equation*}
$$

where $\rho_{s}$ is the density of the structure. For harmonic time dependence, $T$ takes the matrix form

$$
\begin{equation*}
T=-\frac{1}{2} \underline{q}^{T} \omega^{2} M \underline{q} \tag{4.10}
\end{equation*}
$$

where $M$ is the $2 n \times 2 n$ positive definite matrix known as the mass matrix given by

$$
\begin{equation*}
M=2 \pi \iint_{A} \rho_{s} V^{T} N r d r d z \tag{4.11}
\end{equation*}
$$

where $N=\left[N_{1}, V_{2}, \ldots, N_{n}\right]$ and

$$
N_{i}(p)=\left(\begin{array}{ll}
1 & 0  \tag{4.12}\\
0 & 1
\end{array}\right) \Psi_{i}(p) .
$$

If the work done by the system is given by

$$
\begin{equation*}
W=\underline{q}^{T} \underline{f} \tag{4.13}
\end{equation*}
$$

where $\underline{f}$ is the consistent load or force vector, then the equations of motion of the structure, in matrix form, are given by the stationary value of

$$
\begin{equation*}
T+U-W=\frac{1}{2} \underline{q}^{T}\left(K-\omega^{2} M\right) \underline{q}-\underline{q}^{T} \underline{f} . \tag{4.14}
\end{equation*}
$$

That is, when

$$
\begin{equation*}
\left(K-\omega^{2} M I\right) \underline{q}=\underline{f} . \tag{4.15}
\end{equation*}
$$

This system of equations could also be obtained by starting with the linear law of elasticity

$$
\begin{equation*}
\operatorname{div} \underline{\sigma} \div \underline{F}=\rho_{s} \frac{\partial^{2} \underline{u}}{\partial t^{2}} \tag{4.16}
\end{equation*}
$$

where $\underline{\sigma}$ is the symmetric stress tensor, $\underline{F}$ is the vector of applied forces, and $\underline{u}$ is the vector of the displacements, and using the techniques given in 63 .

We have assumed that there is no structural damping. If structural damping is required then a term of the form $i \omega C \underline{q}$ must be included in the left hand side of (4.15) [64].

The basis functions $\Psi_{i}$ are defined to be zero over all the elements not containing the $i^{t h}$ node. If each element has $n^{\prime}$ nodes the element stiffness and mass matrices, denoted by $K^{e}$ and $M^{e}$ respectively, are defined by

$$
\begin{align*}
& K_{i j}^{e}=2 \pi \iint_{A_{e}} B_{i}^{T} D B_{j} r d r d z  \tag{4.17}\\
& M_{i j}^{e}=2 \pi \rho_{s} \iint_{A_{e}} V_{i}^{T} V_{j} r d r d z
\end{align*} \quad i, j=1, \ldots, n^{\prime}
$$

where $i, j$ are the nodes of the element. The contributions from each element are added together to form the global matrices. These global matrices will be real, symmetric and banded. The band-width of the system can be minimised with an appropriate numbering of the nodes.

The type of element employed here is an isoparametric element where the same basis functions are used to interpolate the coordinates of the element and the unknown displacements $[59,64]$. Each element is then mapped into a reference
element in the $(\xi, \eta)$ plane using a transformation of the form

$$
\begin{equation*}
r=\sum_{i=1}^{n^{\prime}} r_{i} \bar{\Psi}_{i}(\xi, \eta) \quad z=\sum_{i=1}^{n^{\prime}} z_{i} \bar{\Psi}_{i}(\xi, \eta) \tag{4.18}
\end{equation*}
$$

where $n^{\prime}$ is the number of nodes in the element, and the unknown displacements are interpolated by

$$
\begin{equation*}
u=\sum_{i=1}^{n^{\prime}} u_{i} \tilde{\Psi}_{i}(\xi, \eta) \quad v=\sum_{i=1}^{n^{\prime}} v_{i} \tilde{\Psi}_{i}(\xi, \eta) \tag{4.19}
\end{equation*}
$$

where $\dot{\Psi}_{i}(\xi, \eta)$ is the basis function $\Psi_{i}$ written in the local coordinates. Since the basis functions in global coordinates and local coordinates are equivalent we will not make any further distinction between them :59].

To evaluate the derivatives of the basis functions with respect to the global variables ( $r, z$ ), needed to compute the stiffness matrix, the chain rule is used to obtain the following matrix equation

$$
\binom{\frac{\partial \Psi_{i}}{\partial \xi}}{\frac{\partial \Psi_{i}}{\partial \eta}}=\left(\begin{array}{cc}
\frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi}  \tag{4.20}\\
\frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta}
\end{array}\right)\binom{\frac{\partial \Psi_{i}}{\partial r}}{\frac{\partial \Psi_{i}}{\partial z}}=J\binom{\frac{\partial \Psi_{i}}{\partial r}}{\frac{\partial \Psi_{i}}{\partial z}} i=1, \ldots n^{\prime}
$$

where the Jacobian matrix $J$ is computed using (4.18). Hence the derivatives with respect to to global variables are given by

$$
\begin{equation*}
\binom{\frac{\partial \Psi_{i}}{\partial r}}{\frac{\partial \Psi_{i}}{\partial z}}=J^{-1}\binom{\frac{\partial \Psi_{i}}{\partial \xi}}{\frac{\partial \Psi_{i}}{\partial \eta}} i=1, \ldots, n^{\prime} . \tag{4.21}
\end{equation*}
$$

Using the transformation (4.18), an integral of the form

$$
\begin{equation*}
\iint_{A_{e}} f(r, z) r \dot{d r} d z \tag{4.22}
\end{equation*}
$$

becomes

$$
\begin{equation*}
\iint_{A_{0}} \tilde{f}(\xi, \eta)\left[\sum_{i=1}^{n^{\prime}} r_{i} \Psi_{i}(\xi, \eta)\right]|\operatorname{det}(J)| d \xi d \eta \tag{4.23}
\end{equation*}
$$

where $|\operatorname{det}(J)|$ is the absolute value of the determinant of the Jacobian matrix and $A_{0}$ is the reference element. We see from (4.4) that for elements with nodes on the axis of symmetry there appears to be a $\frac{1}{r}$ singularity for certain integrals. However, a physical requirement of the problem is that $u=0$ when $r=0$, which cancels out the singularity.

The particular element that is used here is the six noded quadratic triangular element. The basis functions for this element are given by

$$
\begin{array}{ccc}
\Psi_{1}=(1-\xi-\eta)(1-2 \xi-2 \eta) & \Psi_{2}=4 \xi(1-\xi-\eta) & \Psi_{3}=\xi(2 \xi-1) \\
\Psi_{4}=4 \xi \eta & \Psi_{5}=\eta(2 \eta-1) & \Psi_{6}=4 \eta(1-\xi-\eta) \tag{4.24}
\end{array}
$$

which maps the three nodes at the vertices of a general element into the points $(0,0),(0,1)$ and $(1,0)$ in the $(\xi, \eta)$ plane, and the mid-side nodes into the points ( $0, \frac{1}{2}$ ), ( $\frac{1}{2}, 0$ ) and $\left(\frac{1}{2}, \frac{1}{2}\right)$. The relationship between the nodes of a general element and those of the reference element is shown in Figure 3.2.

The elements of the matrices $K$ and $M$ can be found by applying an appropriate quadrature rule over the reference element $A_{0}$. For the matrix $M$, it can be seen from (4.17) and (4.20) that the integrands will, in general, be polynomials of degree 6 in both $\eta$ and $\xi$ which can be integrated exactly by an appropriate quadrature rule. Similarly, it can be seen from (4.17) and (4.20), that some elements of $K$ will be polynomials of degree 4 in both $\eta$ and $\xi$, whilst others will contain a $\frac{1}{\tau}$ term.

From the results of Section 3.3, it seems that the use of a $3 \times 3$ product Gauss rule is a good choice of quadrature rule for evaluating the elements of the matrices $K$ and $M$.

Although these finite elements may have quadratically curved sides, here we have chosen all the finite elements to have straight sides, and that the mid-side nodes should be half-way between the appropriate vertex nodes. Hence, for finite elements with vertex nodes on the surface $S$, the appropriate edge of the element can be used as a linear axisymmetric boundary element for the analysis of the exterior acoustic field.

### 4.2.2 Finite Element Analysis Of A Three-Dimensional Structure.

Here the analysis of the previous section is modified to deal with a fully threedimensional structure. For the analysis of a three-dimensional structure, there are three components of the displacement, $(u, v, w)$ in Cartesian coordinates, assumed to be of the form

$$
\begin{equation*}
u=\sum_{i=1}^{n} u_{i} \Psi_{i}(x, y, z) \quad v=\sum_{i=1}^{n} v_{i} \Psi_{i}(x, y, z) \quad w=\sum_{i=1}^{n} w_{i} \Psi_{i}(x, y, z) \tag{4.25}
\end{equation*}
$$

where ( $u_{i}, v_{i}, w_{i}$ ) are the displacements of the $i^{t h}$ node. The linear stress-strain relationship is of the form (4.2) where $\underline{\sigma}$ and $\underline{\epsilon}$ are the rectors of the six components of the symmetric stress and strain tensors respectively $[42]$ and $D$ is the $6 \times 6$ stress-
strain matrix given by [64]

$$
D=\frac{E}{(1+\nu)(1-2 \nu)}\left(\begin{array}{cccccc}
1 & \nu & \nu & 0 & 0 & 0  \tag{4.26}\\
\nu & 1 & \nu & 0 & 0 & 0 \\
\nu & \nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2 \nu}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2 \nu}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2 \nu}{2}
\end{array}\right) .
$$

The strains are related to the displacements by

$$
\underline{\epsilon}=\left(\begin{array}{ccc}
\frac{\partial u}{\partial x} & 0 & 0  \tag{4.27}\\
0 & \frac{\partial v}{\partial y} & 0 \\
0 & 0 & \frac{\partial w}{\partial z} \\
\frac{\partial u}{\partial y} & \frac{\partial v}{\partial x} & 0 \\
0 & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial y} \\
\frac{\partial u}{\partial z} & 0 & \frac{\partial w}{\partial x}
\end{array}\right) .
$$

On substituting (4.25) into (4.27), the strain vector is of the form (4.5) with $\underline{q}=$ $\left\{u_{1}, v_{1}, w_{1}, \ldots, u_{n}, v_{n}, w_{n}\right\}^{T}$ and

$$
B_{i}=\left(\begin{array}{ccc}
\frac{\partial \Psi_{i}}{\partial x} & 0 & 0  \tag{4.28}\\
0 & \frac{\partial \Psi_{i}}{\partial y} & 0 \\
0 & 0 & \frac{\partial \Psi_{i}}{\partial z} \\
\frac{\partial \Psi_{i}}{\partial y} & \frac{\partial \Psi_{i}}{\partial x} & 0 \\
0 & \frac{\partial \Psi_{i}}{\partial z} & \frac{\partial \Psi_{i}}{\partial y} \\
\frac{\partial \Psi_{i}}{\partial z} & 0 & \frac{\partial \Psi_{i}}{\partial x}
\end{array}\right) .
$$

Following the same analysis as for the axisymmetric case, we obtain the stiffness matrix

$$
\begin{equation*}
K=\int_{D_{-}} B^{T} D B d v \tag{4.29}
\end{equation*}
$$

and the mass matrix

$$
\begin{equation*}
M=\rho_{s} \int_{D_{-}} N^{T} N d v \tag{4.30}
\end{equation*}
$$

where $N=N_{1}, \ldots, N_{n}$ and

$$
V_{i}=\left(\begin{array}{lll}
1 & 0 & 0  \tag{4.31}\\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \Psi_{i}(x, y, z)
$$

If $\underline{f}$ is the consistent load vector of the forces applied to the structure, then the displacements $\underline{q}$ are given by (4.15). as for the axisymmetric case.

We use the isoparametric concept to interpolate the coordinates of an element over a reference element $V^{\prime r}$ in the form

$$
\begin{equation*}
x=\sum_{i=1}^{n^{\prime}} x_{i} \Psi_{i}(\eta, \xi, \zeta) \quad y=\sum_{i=1}^{n^{\prime}} y_{i} \Psi_{i}(\eta, \xi, \zeta) \quad z=\sum_{i=1}^{n^{\prime}} z_{i} \Psi_{i}(\eta, \xi, \zeta) \tag{4.32}
\end{equation*}
$$

where $n^{\prime}$ is the number of nodes in the element and $\left(x_{i}, y_{i}, z_{i}\right)$ are the coordinates of the $i^{\text {th }}$ node of the element. The derivatives of the basis functions with respect to the global coordinates are related to the derivatives with respect to the local coordinates through

$$
\left(\begin{array}{c}
\frac{\partial \Psi_{1}}{\partial x}  \tag{4.33}\\
\frac{\partial \psi_{1}}{\partial y} \\
\frac{\partial \Psi_{1}}{\partial z}
\end{array}\right)=J^{-1}\left(\begin{array}{c}
\frac{\partial \psi_{i}}{\partial \eta} \\
\frac{\partial \psi_{1}}{\partial \xi} \\
\frac{\partial \Psi_{1}}{\partial \zeta}
\end{array}\right) i=1, \ldots, n^{\prime}
$$

where the Jacobian matrix $J$ is given by

$$
J=\left(\begin{array}{lll}
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta}  \tag{4.34}\\
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{array}\right) .
$$

U'sing this transfomation, integrals over a general element $V^{e}$ of the form

$$
\begin{equation*}
\iiint_{V^{\prime} e} f(x, y, z) d x d y d z \tag{4.35}
\end{equation*}
$$

are transformed to

$$
\begin{equation*}
\iiint_{V^{r}} \tilde{f}(\eta, \xi, \zeta)|\operatorname{det}(J)| d \eta d \xi d \zeta \tag{4.36}
\end{equation*}
$$

The element used here is the ten noded quadratic tetrahedron. The reference element is the tetrahedron which has four nodes at its vertices $(0,0,0),(1,0,0)$, ( $0.1,0$ ) and $(0,0,1)$, and the remaining six nodes at the mid-points between the vertices. We label the vertices 1 to 4 and introduce the functions 64

$$
\begin{array}{cc}
L_{1}=1-\eta-\xi-\zeta & L_{2}=\eta  \tag{4.37}\\
L_{3}=\xi & L_{4}=\zeta
\end{array}
$$

The basis function for each vertex node is of the form

$$
\begin{equation*}
\Psi_{i}(\eta, \xi, \zeta)=\left(2 L_{i}-1\right) L_{i} \tag{+4.38}
\end{equation*}
$$

and for a node $k$ between vertices $i$ and $j$,

$$
\begin{equation*}
\Psi_{k}(\eta, \xi, \zeta)=4 L_{i} L_{j} \quad i=1,2,3 \quad i<j \leq 4 \tag{4.39}
\end{equation*}
$$

We can see from the definition of these basis functions, (4.32) and (4.33) that the integrands of the integrals defining the elements of $K$ and $M$ will be polynomials
of up to degree seven in each of the variables. Applying a $4: 4 \times 4$ product Gauss rule should yield exact results. Computationally it would have been simpler to use the four noded linear tetrahedral element, but this represents the stresses in each element as constant and consequently does not give an accurate model [64].

For elements with nodes on the surface $S$, the appropriate faces of these elements form a quadratic boundary element which we can use for the analysis of the exterior acoustic field.

It is not essential that the elastic structure be homogenevus or isotropic. The elastic parameters may vary between elements, although they must be constant within each element, and for a non-isotropic body the elements of the matrix $D$ (the stress-strain matrix) merely take on the appropriate values $64!$. This allows this method to model the displacements in a structure made of more than one material.

### 4.3 The Coupled System of Equations

The finite element analysis of the structure in $D_{\text {_ }}$ and the beundary element analysis of the acoustic field in $D_{\mp}$ are coupled together by ensuring that the normal particle velocity is continuous at $S$, and that the surface force due to the acoustic pressure equals the stress in the structure at the surface. If $\underline{g}$ denotes the force acting on the surface, $\underline{g}(p)=-\phi(p) \underline{n}_{p}$ where $\phi(p)$ is the pressure and $-\underline{n}_{p}$ is the unit normal directed into $D_{-}$.

The consistent load vector $\underline{f}$ can be written as

$$
\begin{equation*}
\underline{f}=\underline{f}^{k}+\underline{f}^{s} \tag{4.40}
\end{equation*}
$$

where $\underline{f}^{k}$ is the consistent load vector of known applied forces, and $\underline{f}^{S}$ is the consistent load vector of surface forces. We can write $\underline{f}^{\mathcal{S}}=\left[\underline{f}_{1}^{\mathcal{S}}, \ldots, \underline{f}_{n}^{S}\right]$ where each $\underline{f}_{i}^{S}$ is a vector given by

$$
\begin{equation*}
\underline{f}_{i}^{S}=\int_{S} \underline{g}(q) \Psi_{i}(q) d S_{q}=-\int_{S} \phi(q) \Psi_{i}(q) \underline{n}_{q} d S_{q} . \tag{4.41}
\end{equation*}
$$

In Chapter 3 it was assumed that the surface pressures are approximated by

$$
\begin{equation*}
\phi_{m}(q)=\sum_{i=1}^{m} \phi_{i} \psi_{i}^{\prime}(q) \tag{4.42}
\end{equation*}
$$

where $m$ is the total number of collocation points for the boundary element analysis. From now on we shall refer to these as the fluid nodes, and the nodes of the finite element analysis of the structure as the structure nodes. Substituting (4.42) as an approximation for $\phi$ into (4.41) gives

$$
\begin{equation*}
\underline{f}_{i}^{S}=-\int_{S_{j=1}}^{m} \phi_{j} \psi_{j}(q) \Psi_{i}(q) \underline{n}_{q} d S_{q} . \tag{4.43}
\end{equation*}
$$

This can be written as

$$
\begin{equation*}
\underline{f}^{\mathcal{S}}=-L \underline{\phi} \tag{4.44}
\end{equation*}
$$

where each $L_{i j}$ is a vector given by

$$
\begin{equation*}
L_{i j}=\int_{S} \dot{\psi}_{j}^{\prime}(q) \Psi_{i}(q) \underline{n}_{q} d S_{q} . \tag{4.45}
\end{equation*}
$$

For an axisymmetric structure the normal $\underline{n}$ has only two independent components, in the $r$ and $z$ directions respectively, and so each $L_{i j}$ is a $2 \times 1$ vector. If the
line generating the linear axisymmetric surface elements $S_{j}$ has end points $\left(r_{1}, z_{1}\right)$ and $\left(r_{2}, z_{2}\right)$ and mid-point $\left(r_{c}, z_{c}\right)$ then

$$
L_{i j}=\left\{\begin{array}{cc}
\pi \underline{n} \frac{r_{1} l}{3} & \text { if } i \text { corresponds to }\left(r_{1}, z_{1}\right)  \tag{4.46}\\
\pi \underline{n} \frac{2\left(r_{1}+r_{2}\right) l}{3} & \text { if } i \text { corresponds to }\left(r_{c}, z_{c}\right) \\
\pi \underline{n} \frac{r_{2} l}{3} & \text { if } i \text { corresponds to }\left(r_{2}, z_{2}\right) \\
0 & \text { otherwise }
\end{array}\right.
$$

where $l=\sqrt{\left(r_{2}-r_{1}\right)^{2}+\left(z_{2}-z_{1}\right)^{2}}$ and $\underline{n}$ is the constant normal to the $j^{\text {th }}$ boundary element in the $(r, z)$ plane.

For a general three-dimensional structure $\underline{n}$ has three components, in the $x$, $y$ and $z$ directions respectively, and each $L_{i j}$ is a $3 \times 1$ vector. Clearly, from the definitions of the basis functions $\psi_{j}^{\prime}$ and $\Psi_{i}$

$$
\begin{equation*}
L_{i j}=\underline{0} \tag{4.47}
\end{equation*}
$$

if the $i^{\text {th }}$ structure node is not a node of the $j^{\text {th }}$ boundary element. In this case it is necessary to evaluate the non-zero elements of $L$ numerically as the normal is not a constant over the surface element $S_{j}$. Since these integrals are non-singular, it seems reasonable from the results of Section 3.3 that a $3 \times 3$ product Gauss rule is a suitable choice to evaluate the elements of $L$ accurately. Equation (4.15) can be now be written as

$$
\begin{equation*}
\left(K-\omega^{2} M\right) \underline{q}=\underline{f}^{k}-L \underline{\phi} . \tag{4.48}
\end{equation*}
$$

The velocities $\underline{V}$ of the structure nodes are related to the displacements through

$$
\begin{equation*}
\underline{V}=-i \omega \underline{q} . \tag{4.49}
\end{equation*}
$$

The velocity at the fluid nodes $p_{1}, \ldots, p_{m}$ is found by interpolating the velocity at structural nodes to give

$$
\begin{equation*}
\underline{V}\left(p_{i}\right)=-i \omega V\left(p_{i}\right) \underline{q} . \tag{4.50}
\end{equation*}
$$

The normal surface velocity $v$ at the fluid nodes is then obtained from

$$
\begin{equation*}
v\left(p_{i}\right)=-i \omega \underline{n}_{p_{i}}^{T} N\left(p_{i}\right) \underline{q} . \tag{4.51}
\end{equation*}
$$

Hence the vector of the normal surface velocities at the fluid nodes, $\underline{v}$ is of the form

$$
\begin{equation*}
\underline{v}=-i w \cdot L^{\prime} \underline{q} \tag{4.52}
\end{equation*}
$$

where

$$
\begin{equation*}
L_{i j}^{\prime}=\underline{n}_{p_{i}}^{T} N_{j}\left(p_{i}\right)=\underline{n}_{p_{i}}^{T} \Psi_{j}\left(p_{i}\right) . \tag{4.53}
\end{equation*}
$$

For the axisymmetric problem each $L_{i j}^{\prime}$ is a $1 \times 2$ vector and for three-dimensional problems each $L_{i j}^{\prime}$ is a $1 \times 3$ vector.

Finally, the normal surface velocities at the fluid nodes are related to the acoustic pressures at the fluid nodes by

$$
\begin{equation*}
\underline{v}=-i \omega L^{\prime}\left(K-\omega^{2} \cdot I\right)^{-1}\left(\underline{f}^{k}-L \underline{\phi}\right) . \tag{4.54}
\end{equation*}
$$

From the boundary element analysis of the exterior acoustic field, we obtain a matrix equation of the form

$$
\begin{equation*}
A \underline{\phi}=i \omega \rho_{f} B \underline{v}-\underline{c} \tag{4.55}
\end{equation*}
$$

where $A$ and $B$ are the matrices obtained from the discretisation of the direct Burton and Miller formulation (2.73), and are given by

$$
\begin{equation*}
A=-\frac{1}{2} I+\bar{M}_{k}+i \nu \bar{N}_{k} \text { and } B=\tilde{L}_{k}-i \nu: \frac{1}{2} I+\bar{M}_{k}^{T} \tag{4.56}
\end{equation*}
$$

respectively. The vector $\underline{c}$ represents the pressure due to an incident wave (see (2.37)), and is given by

$$
\begin{equation*}
c_{i}=\phi_{i n c}\left(p_{i}\right)+i \nu \frac{\partial \phi_{i n c}\left(p_{i}\right)}{\partial n} . \tag{4.57}
\end{equation*}
$$

Here it is assumed that the coupling parameter $\nu$ is a constant. Combining ( 4.54 ) and (4.55) we obtain a block matrix equation of the form

$$
\left(\begin{array}{cc}
A & -i \omega \rho_{f} B  \tag{4.58}\\
-i \omega L^{\prime}\left(K-\omega^{2} M\right)^{-1} L & I
\end{array}\right)\binom{\underline{\phi}}{\underline{v}}=\binom{\underline{c}}{-i \omega L^{\prime}\left(K-\omega^{2} M\right)^{-1} \underline{f}^{k}}
$$

There are various options (schemes) for solving (4.58) for $\underline{\phi}$ and $\underline{v}$ and we shall briefly compare these schemes.

Solution scheme I: Solve the $2 m \times 2 m$ system (4.58) for $\underline{\underline{\phi}}$ and $\underline{v}$ directly.
Solution scheme II: Eliminate $\underline{v}$ by substituting (4.54) into (4.55) to obtain

$$
\begin{equation*}
\left(A+\omega^{2} \rho_{f} B L^{\prime}\left(K-\omega^{2} M\right)^{-1} L\right) \underline{\phi}=\underline{c}+\omega^{2} \rho_{f} B L^{\prime}\left(K-w^{2} M\right)^{-1} \underline{f}^{k} \tag{4.59}
\end{equation*}
$$

After solving (4.59) for $\underline{\phi}, \underline{q}$ or $\underline{v}$ can be found from (4.48) or (4.54) respectively. It is clear that scheme II is obtained by carrying out a block solution of scheme I.

Solution scheme III: Substitute (4.55) into (4.54) and eliminate $\underline{\phi}$ to obtain

$$
\begin{equation*}
\left(I+\omega^{2} \rho_{f} L^{\prime}\left(K-\omega^{2} . I\right)^{-1} L \cdot \mathcal{A}^{-1} B\right) \underline{v}=-i \omega L^{\prime}\left(K-\omega^{2} M\right)^{-1}\left(f^{k}-L \cdot A^{-1} \underline{c}\right) \tag{4.60}
\end{equation*}
$$

After solving (4.60) for $\underline{v}$ we can compute $\underline{\phi}$ from (4.55). Since this scheme requires three matrix inversions instead of two we shall not consider it further.

Solution scheme IV: This scheme is not based on a block solution of (4.58).

Substitute (4.55) into (4.48) and eliminate $\phi$ to obtain

$$
\begin{equation*}
\left(K-\omega^{2} M I+\omega^{2} \rho_{f} L A A^{-1} B L^{\prime}\right) \underline{q}=\underline{f}^{k}-L \cdot A^{-1} \underline{c} \tag{4.61}
\end{equation*}
$$

which can be solved for $\underline{q}$. We then use $(4.52)$ to compute $\underline{v}$ followed by (4.55) to compute $\underline{\phi}$. A drawback of this scheme is that the matrix $K-\omega^{2} M$ is a real, symmetric and banded matrix, but the matrix on the left hand side of (4.61) will not, in general, have these properties. Also, for three-dimensional problems, the matrix on the left hand side of (4.61) will be large and require excessive amounts of storage, as pointed out by Wilton 62 :.

Clearly each solution scheme should give the same results since they are all based on the same two simultaneous matrix equations. The adrantages to be gained in using a particular solution scheme are in terms of efficiency and computer storage. It is clear that solution scheme II requires less computer storage than either scheme I or scheme IV. It is also clear, from the comments above, that both schemes I and II are more efficient than scheme IV. However, it is not so clear whether scheme I or scheme II is more efficient. Here we shall employ scheme II since it requires the least storage, a critical factor in the analysis of a three-dimensional problem.

Regardless of the solution scheme employed, once $\underline{\phi}$ and $\underline{y}$ are known a suitable discretisation of $(2.33)$ for $p \in D_{+}$can be used to find the pressure in the exterior domain (see (3.42)).

### 4.4 The Conditioning of the Coupled Equations

Here an argument is presented to show that if the SHE (2.34) is used to relate the surface pressure to the normal surface velocity in the fluid, the coupled boundary element and finite element analysis would also suffer from the non-uniqueness problems associated with the SHE at the characteristic frequencies. It is possible to view the solution to the linear elasticity problem as a Robin type boundary condition, on $S$, for the exterior Helmholtz equation. Assume that we can write

$$
\begin{equation*}
\frac{\partial \phi}{\partial n}=\mathcal{F} \phi+f \tag{4.62}
\end{equation*}
$$

where $\mathcal{F}$ is a linear operator, derived from the linear law of elasticity, which relates the surface pressures to the normal surface velocities. The function $f$ is due to any other forces applied throughout the structure. Clearly $\mathcal{F}$ must satisfy $\mathcal{F} 0=0$. The homogeneous form of (4.62) is $\frac{\partial \phi}{\partial n}=\mathcal{F} \phi$.

Using the indirect integral formulation we recall that

$$
\begin{equation*}
\left(-\frac{1}{2} I+M_{k}^{T}\right) \sigma=\frac{\partial \phi}{\partial n} \quad p \in S \tag{4.63}
\end{equation*}
$$

where $\phi=L_{k} \sigma$, which, after applying the homogeneous boundary condition gives

$$
\begin{equation*}
\left(-\frac{1}{2} T+M M_{k}^{T}-\mathcal{F} L_{k}\right) \sigma=0 \tag{4.64}
\end{equation*}
$$

It follows from Section 2.3 and [38] that there exists a non-trivial function $\sigma_{e}$ such that $\left(-\frac{1}{2} I+M_{k}^{T}\right) \sigma_{e}=0$ if and only if $L_{k} \sigma_{e}=0$. Consequently, $\sigma_{e}$ will satisfy (4.64). Hence if $k \in I_{D}$ there is a non-trivial solution to (4.64), and if a solution to the coupled problem using an indirect formulation exists for $k \in I_{D}$, it will be non-unique.

In Section 2.3 it was shown that the direct integral equation formulation, from Green's Theorem, of the exterior Neumann Helmholtz problem does not have a unique solution for $k \in I_{D}$. Applying the boundary condition (4.62), the homogeneous equation is

$$
\begin{equation*}
\left(-\frac{1}{2} I+M I_{k}-L_{k} \mathcal{F}\right) \phi=0 \tag{4.65}
\end{equation*}
$$

Taking the transpose gives

$$
\begin{equation*}
\left(-\frac{1}{2} I+M_{k}^{T}-\mathcal{F}^{T} L_{k}\right) \phi=0 \tag{4.66}
\end{equation*}
$$

where $\mathcal{F}^{T}$ is the transpose of $\mathcal{F}$. Clearly if $k \in I_{D}$. then there is a non-trivial solution to (4.66) and hence by Theorem 2.7 there is a non-trivial solution to (4.65). It is clear from this analysis that it is necessary to employ an integral equation formulation that is valid for all wavenumbers in order to obtain an unique solution to the coupled problem.

Here we shall employ the direct Burton and Miller formulation discussed in Chapter 2 to overcome this non-uniqueness problem. Clearly, the conditioning of the linear system (4.59) will depend on the choice of the coupling parametr $\nu$. It was shown in Section 3.4 that the near optimal choice is $\nu=\frac{1}{k}$ when $\nu$ is a constant. It does not follow, however, that this is a good choice for the coupled problem since the condition number will depend on the physical parameters of the structure and the fluid.

The conditioning of the linear system (4.59) will not just depend on the parameters of the exterior acoustic problem. There are certain values of $\omega$ for which the
homogeneous equation

$$
\begin{equation*}
\left(K-\omega^{2} M\right) \underline{q}=\underline{0} \tag{4.67}
\end{equation*}
$$

has non-trivial solutions. These values of $\omega$ are known as the natural frequencies of the structure. These natural frequencies can be found by solving the generalised eigenvalue problem

$$
\begin{equation*}
K \underline{x}=\omega^{2} M \underline{x} . \tag{4.68}
\end{equation*}
$$

Clearly, if $\omega$ is close to one of the natural frequencies, $K-\omega^{2} M$ will, in general, have a large condition number. In the physical situation, as $\omega$ approaches a natural frequency the response of the structure increases and the phenomenon of resonance occurs.

The physical parameters of the different materials considered in this chapter and in Chapter 5 are given in Table 4.1, where a blank entry means that particular parameter is not required. Table 4.2 presents the results of computing the condition number of the final linear system (4.59), where the structure is a spherical shell of thickness 0.5 m and outer radius 1 m . Here the Burton and Miller coupling parameter $\nu(p, k)$ has been taken to be a constant with respect to the space variables. The condition number of the final linear system ( 4.59 ) for the near optimal choice given in Theorem 3.4 is compared with the minimum condition number obtained from the computed optimum value of $\nu$. Table 4.3 presents the corresponding results for a spherical shell of thickness 0.25 m and outer radius 1 m . In both cases, the structural displacements were modelled using 200 axisymmetric finite elements and the acoustic field with 25 axisymmetric boundary elements.

It is clear that the choice $\nu(p, k)=\frac{1}{k}$ as recommended in Section 3.4 leads to a relatively well conditioned set of equations. However, this may not be the near optimal choice in this case, and it requires further research in order to establish the optimal choice.

These result also show that the condition number depends on both the geometry of the structure and the structural properties in addition to the frequency. This is in contrast to the purely acoustic problem where the conditioning depends only upon the geometry of the fluid-structure surface and the frequency. The system (4.59) becomes ill-conditioned as the frequency approaches one of the natural frequencies of the structure. The first seven non-zero natural frequencies of each structure are presented in Table 4.4, where it is seen that the thinner shell has more natural frequencies in the range $0-15000$ rads $/ \mathrm{sec}$ than the thicker shell. This implies that thin shells have more natural frequencies in the frequency range of interest and hence the coupled fluid-structure interaction problem may be more susceptible to ill-conditioning for thin shells. It may be possible to overcome this by the use of special shell type finite elements, although this has not been carried out here.

It is clear from the analysis of this section that it is necessary to employ an integral equation formulation of the exterior Helmholtz problem which has a unique solution for all frequencies. It is also apparent that the coupled system of equations become ill-conditioned at frequencies which are close to a natural frequency of the structure. Since this corresponds to the physical phenomenon of resonance it may be a more difficult problem to overcome.

| Material | Young's Modulus | Poission's Ratio | Density | Speed of Sound |
| :---: | :---: | :---: | :---: | :---: |
| $\mathrm{Nm}^{-2}$ |  | $\mathrm{Kgm}^{-3}$ | $\mathrm{~ms}^{-1}$ |  |
| Steel | 209 E 9 | 0.30 | 7800 | - |
| Aluminium | 68.5 E 9 | 0.33 | 2695 | - |
| ceramic | 80.0 E 9 | 0.285 | 7500 | - |
| Water | - | - | 1000 | 1500 |

Table 4.1: The physical parameters of the materials considered.

| $\omega$ |  | Steel |  | Aluminium |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $k$ | $\nu=\frac{1}{k}$ | $\nu=O P T$ | $\nu=\frac{1}{k}$ | $\nu=O P T$ |
| 1500 | 1 | 9.7286 | 2.3231 | 8.9098 | 2.3372 |
| 3000 | 2 | 6.7581 | 2.3176 | 7.4161 | 2.5431 |
| 4500 | 3 | 4.9742 | 2.8254 | 5.4702 | 3.1050 |
| 6000 | 4 | 4.3862 | 3.4869 | 147.6597 | 121.3771 |
| 7500 | 5 | 3.3963 | 2.7015 | 3.6783 | 3.3256 |
| 9000 | 6 | 2.9675 | 2.6769 | 3.7099 | 3.0351 |
| 10500 | 7 | 3.4626 | 3.0363 | 4.7253 | 3.4724 |
| 12000 | 8 | 4.6494 | 4.3754 | 8.2889 | 7.3712 |
| 13500 | 9 | 3.1301 | 3.1251 | 10.2334 | 9.8895 |
| 15000 | 10 | 3.0793 | 2.9565 | 4.0851 | 3.9047 |

Table 4.2: The condition number of the linear system (4.59) spherical shell, thickness 0.5 m .

| $\omega$ | $k$ | Steel |  | Aluminium |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $\nu=\frac{1}{k}$ | $\nu=O P T$ | $\nu=\frac{1}{k}$ | $\nu=O P T$ |
| 1500 | 1 | 9.3316 | 2.3395 | 7.2164 | 2.3128 |
| 3000 | 2 | 7.6303 | 2.4945 | 11.6822 | 4.2753 |
| 4500 | 3 | 7.2126 | 7.1244 | 288.7537 | 96.7130 |
| 6000 | 4 | 4.9738 | 3.8562 | $13.2 \bar{i} \bar{i}$ | 10.6735 |
| 7500 | 5 | 3.7685 | 3.7358 | 7.4195 | 6.4217 |
| 9000 | 6 | 85.8567 | 51.2289 | 25.8885 | 14.2825 |
| 10500 | 7 | 3.7705 | 3.6598 | 8.9165 | 7.8988 |
| 12000 | 8 | 79.5560 | 48.4989 | 47.1273 | $\underline{26.9341}$ |
| 13500 | 9 | 3.1475 | 3.1475 | 5.3062 | 4.8123 |
| 15000 | 10 | 12.2290 | 7.1426 | 19.2692 | 12.5256 |

Table 4.3: The condition number of the linear system (4.59) spherical shell, thickness 0.25 m .

| 0.5 m Thick Shell | 0.25 m Thick Shell |  |  |
| :---: | :---: | :---: | :---: |
| Steel | Aluminium | Steel | Aluminium |
| 6215 | 6015 | 4635 | 4485 |
| 10114 | 9812 | 6581 | 6397 |
| 12152 | 12225 | 8978 | 8748 |
| 12450 | 12339 | 10094 | 10066 |
| 14201 | 15648 | 11879 | 11582 |
| 15970 | 17665 | 12008 | 11956 |
| 18222 | 20158 | 15118 | 14738 |

Table 4.4: The first seven non-zero natural frequencies of the steel and aluminium spherical shells.

### 4.5 Numerical Results

In this section we present the results of applying the numerical methods developed in this chapter to some different test problems. In particular, we shall consider the problems of acoustic scattering and radiation from an elastic spherical shell and scattering by a solid elastic sphere. The geometry is necessarily simple in order to allow us to obtain an analytical solution. This will enable us to assess the accuracy of our numerical scheme in these particular situations and hence gain some insight into the error in the numerical solution of more complicated practical problems. To demonstrate that our numerical method yields accurate results for other shaped structures, we also consider the problem of a cylinder radiating an acoustic field. For all the problems considered here the structure is made of either steel or aluminium and is immersed in water. The physical parameters are all given in Table 4.1.

The measure of the error that we have chosen here is the relative mean error, given by

$$
\begin{equation*}
R M E=\frac{\sum_{i=1}^{n}\left|\phi\left(p_{i}\right)-\phi_{n}\left(p_{i}\right)\right|}{\sum_{i=1}^{n}\left|\phi\left(p_{i}\right)\right|} \tag{4.69}
\end{equation*}
$$

where $\phi\left(p_{i}\right)$ and $\phi_{n}\left(p_{i}\right)$ are the exact and numerical solution respectively, at the collocation points $p_{1}, \ldots, p_{n}$. We have chosen this measure of the error rather than the mean relative error given by (3.123) since, for certain points, $\phi(p)$ is close to zero, and (3.123) may give a distorted measure of the error at these points.

In Section 4.4 it was shown that the solution to the coupled problern would
not be unique at the characteristic frequencies if we use ${ }_{a}^{d}$ the surface Helmholtz equation (2.34) to model the acoustic field. Figure 4.1 shows the computed surface pressure for a solid steel sphere scattering an incident plane wave using both the surface Helmholtz equation (denoted SHE on the figure) and the direct Burton and Miller formulation (denoted DIRECT). REAL and IMAG are used to denote the real and imaginary parts of the solution. Figure 4.2 shows the corresponding surface velocity. The sphere was modelled using 147 axisymmetric finite elements and 21 axisymmetric boundary elements. Clearly employing the surface Helmholtz equation has failed to yield the required solution, whilst the direct formulation gives accurate results. It is obvious from these results that it is necessary to employ an integral equation formulation of the exterior Helmholtz problem which is valid for all wavenumbers.

The axisymmetric structures considered in the following results are spherical shells of thickness 0.5 m and 0.25 m respectively. Both of the shells have an outer radius of 1 m . The displacements within the shells were modelled using 200 axisymmetric finite elements and the acoustic field was modelled using 25 axisymmetric boundary elements. We will also consider a_thin spherical shell of thickness 0.05 m and outer radius 1 m , where we modelled the displacements using 150 axisymmetric finite elements and the acoustic field with 25 axisymmetric boundary elements.

Figure 4.3 gives a comparison between the exact surface pressures (denoted ELASTIC on the figure) and the surface pressures computed using our numerical scheme (denoted NUMER) for an aluminium shell of thickness 0.25 m scattering
a plane wave at an angular frequency of $7500 \mathrm{rads} / \mathrm{sec}$. The exact solution for a rigid sphere of radius 1 m (denoted RIGID) scattering the same incident plane wave is also shown in order to demonstrate the necessity of taking the structural motion into account. In Figure 4.4 we give the corresponding results for a steel shell of thickness 0.05 m scattering an incident plane wave at an angular frequency of 9000 rads $/ \mathrm{sec}$. In both cases our numerical scheme yields an accurate solution to coupled fluid-structure interaction problem which is different from the solution to corresponding rigid body scattering problem.

Figures 4.5 and 4.6 show the mean relative error in the surface pressures and surface velocities respectively, for the spherical shell of thickness 0.5 m scattering an incident plane wave at different angular frequencies, (equivalent to wavenumbers $k=1,2, \ldots, 10)$. Figures 4.7 and 4.8 show the corresponding results for the shell of thickness 0.25 m . It is clear from these results that the error is increasing as the frequency increases, which is due to the more oscillatory behaviour of the solution at higher frequencies.

Figures 4.9 and 4.10 show the relative mean error for the surface pressures and the surface velocities respectively, for the spherical shell of thickness 0.5 m , with a known pressure distribution on the inner surface of the shell. The corresponding results for a shell of thickness 0.25 m are given in Figures 4.11 and 4.12 . Unlike the scattering problem, there is not the deterioration in the accuracy of the results as the frequency increases. However here the solution is less oscillatory than the solution to the scattering problem at high frequencies.

For both problems we notice that for certain frequencies the errors are significantly larger than one might expect, particularly for the steel sphere. From Table 4.4, we see that these peaks in the error correspond to frequencies which are close to the natural frequencies of the structure in vacuo. Similarly, we see from Table 4.3 that the condition number of the final linear system, for the shell of thickness 0.25 m , is larger if $\omega$ is close to a natural frequency. However, a large condition number does not necessarily imply poor results. For example, the results for an aluminium shell of thickness 0.25 m with $\omega=4500$ radians per second are not noticabley less accurate than the results for other values of $\omega$, although the condition number is considerably larger.

Figures 4.13 and 4.14 we show the mean relative error in the computed solution for a three-dimensional spherical shell, thickness 0.25 m and outer radius 1 m , with a known pressure distribution on the inner surface. The shell was modelled using 750 finite elements and the acoustic field was modelled with 112 boundary elements. It can be seen that the results are similar to those for the axisymmetric model of the problem. In particular, we see that this three dimensional model has the same pattern of large errors due to the natural frequencies of the structure.

Finally, in this section, we shall consider the problem of a cylinder, radius 0.6 m and length 1.8 m radiating an acoustic field due to a uniform body force in the positive $z$ direction. The consistent load vector due to known body forces $\underline{E}$ is given by

$$
\begin{equation*}
\underline{f}^{k}=\int_{D_{-}} N^{T}(p) \underline{F}(p) d V_{P}^{\prime} \tag{4.70}
\end{equation*}
$$

where $N(p)$ is given by (4.12) for axisymmetric structures or (4.31) for threedimensional structures. Although there is no analytical solution to this axisymmetric problem, it is possible to compare the results produced by an axisymmetric model to those produced by a three-dimensional model. The solution should also be computed on refined finite element and boundary element meshes to ensure that the numerical scheme is converging. The axisymmetric cylinder was modelled using a coarse mesh of 54 quadratic finite elements and 15 linear boundary elements, and a fine mesh of 320 quadratic finite elements and 30 linear boundary elements. The three-dimensional cylinder was modelled using 320 quadratic finite elements and 128 quadratic boundary elements. In this case it was not possible to refine the mesh due to the large amount of computer storage required.

Figure 4.15 gives a comparison of the pressures computed in the exterior field using the different axisymmetric models at an angular frequency of 1500 rads 'sec. COARSE and FINE denote the solution computed using the coarse and fine axisymmetric models respectively. A comparison between the computed pressure in the exterior field using the fine axisymmetric model and the three-dimensional _model (denoted 3D) at an angular frequency of 1500 rads,'sec is given in Figure 4.16. The points at which the exterior field pressures have been computed lie on a sphere of radius 5 m , where PHI denotes the $\phi$ coordinate of the spherical polar coordinates $(r, \theta, \phi)$. The corresponding results for an angular frequency of 7500 $\mathrm{rad} / \mathrm{sec}$ are given in Figures 4.17 and 4.18 respectively. Since there is good agreement between the results using the two axisymmetric models, and in light of the
results for spheres given above, it seems that our computed solution to this problem is accurate. Similarly, as the computed solution using the three-dimensional model is in fairly good agreement with the solution using the fine axisymmetric model, we conclude that the results produced using the three-dimensional model are fairly accurate.

In conclusion, it would seem that we can use this technique to effectively model the acoustic field around an arbitrary-shaped elastic structure. It is also clear from our results that it is not feasible to assume that the structure is rigid, especially when the structure is a thin shell.

We have also established in this chapter that it is essential to employ an integral equation formulation of the exterior acoustic problem which is valid for all frequencies, if we are to obtain meaningful results. Howerer, the method is not as accurate if the frequency is close to a natural frequency of the structure. It is possible that there are frequencies for which the coupled system may not possess a unique solution, but it requires more research to establish whether or not such frequencies exist. In the next chapter we shall demonstrate the practical use of our numerical methods by analysing the acoustic field radiated by a sonar transducer immersed in water.
Figure 4.1. The surface pressure on the surface on a steel
sphere of radius 1 , at the first characteristic $k$, using
different formulations.

(0d) $38 n S S 3 y d$



Figure 4.5. Relative mean error of the surface pressures
on a spherical shell, thickness 0.5m, scattering a
plane wave at different angular frequencies.


Figure 4.6. Relative mean error of the surface velocities
on a spherical shell, thickness 0.5 m , scattering a
plane wave at different angular frequencies.


Figure 4.8. Relative mean error of the surface velocities on a spherical shell, thickness 0.25 m , scattering a
plane wave at different angular frequencies.



Figure 4.9. Relative mean error of the surface pressures
for a spherical shell, thickness 0.5 m , radiating a field
at different angular frequencies.

16000

Figure 4.10. Relative mean error of the surface velocities
for a spherical shell, thickness 0.5 m , radiating a field
at different angular frequencies.



Figure 4.11. Relative mean error of the surface pressures
for a spherical shell, thickness 0.25 m , radiating a field
at different angular frequencies.

| 01 |  |  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | 2000 | 1000 | 6000 | 8000 | 10000 | 12000 | 14000 | 16000 |



Figure 4.13. Relative mean error of the surface pressures
for a three dimensional spherical shell, thickness 0.25 m ,
radiating a field at different angular frequencies.

(\%) צ0४У N N
Figure 4.14. Relative mean error of the surface velocities
for a three dimensional spherical shell, thickness 0.25 m ,
radiating a field at different angular frequencies.



Figure 4.15. A comparison of the exterior field pressure
around a cylinder vibrating at $w=1500$ rad $/ \mathrm{sec}$ using both
a coarse and a fine axisymmetric model.

(0d) 3 \#ñS3yd





## 5 THE DETERMINATION OF THE RESPONSE FROM SONAR TRANSDUCERS

### 5.1 An Introduction to Piezoelectric Sonar Transducers

In this chapter we shall apply the methods developed and analysed in the previous chapters to determine the sound field radiated by a piezoelectric sonar transducer. In particular, we wish to determine the frequency at which we obtain the maximum response.

Before describing the transducers that are considered here, it is necessary to give a brief description of the piezoelectric effect. In general, the piezoelectric materials used in sonar transducers are certain types of ceramic and for them to exhibit any piezoelectric properties they have to be polarised. This is achieved by heating the ceramic to a certain temperature and allowing it to cool in the presence of a strong electric field. This causes the electric dipoles within the crystals of the ceramic to align in the direction of the electric field [58;. If a force is applied to the ceramic in the direction of the polarisation, this will cause a potential difference, or voltage, across the surface of the ceramic. Conversely, applying a potential difference to the ceramic in the direction of the polarisation will cause a deformation of the ceramic material [58]. It is this relationship between the applied voltage and the deformation that is known as the piezoelectric effect. If the applied electric potential has harmonic time dependence of the form $\epsilon^{-i\lrcorner t}$, so will the displacements
within the ceramic.
The types of transducers considered here are the ring transducers which consist of a ring of ceramic material, rectangular in cross section, which may be surrounded by an outer ring made of a metal such as aluminium, (see Figure 5.1). For the three transducers considered here, denoted $A, B$ and $C$, the inner radius of the ring is 50.8 mm , the height 28 mm and the thickness of the ceramic material is 6.35 mm . Transducer $A$ does not have an outer ring of metal. Transducers B and C have outer rings of aluminium of thickness 1.5875 mm and 4.7625 mm respectively [27]. The elastic constants of the ceramic material and the aluminium are given in Table 4.1. It can be seen from Figure 5.1 that these transducers are axisymmetric about the central axis. It is the case that many practical sonar transducers are axisymmetric in design. The transducer is excited by an alternating voltage applied across the inner and outer surfaces of the ceramic ring, which are coated with a conducting paint to ensure that the potential is evenly distributed.

### 5.2 Loading on the Structure

In order to compute the frequency at which a peak response is obtained it is necessary to model the stresses within the ceramic part of the transducer due to the piezoelectric effect. We can write the linear modified stress-strain relationship (4.2) as 52

$$
\begin{equation*}
\underline{\sigma}=D \underline{\epsilon}-\underline{\sigma_{0}} \tag{5.1}
\end{equation*}
$$



Figure 3.1: Atypical ring type sonar transducer.
where $\underline{\sigma_{0}}$ is the stress due to the piezoelectric terms. Explicitly

$$
\begin{equation*}
\underline{\sigma_{0}}=e_{p} \underline{E} \tag{5.2}
\end{equation*}
$$

where $\underline{E}$ is the electric field strength and $e_{p}$ is the tensor of piezoelectric parameters. The electric field strength is related to the electric potential $V$ through

$$
\begin{equation*}
\underline{E}=\nabla V . \tag{5.3}
\end{equation*}
$$

If the potential on the inner and outer surfaces of the ceramic ring is $V_{i}$ and $V_{o}$ respectively, then $\underline{E}$ is given by

$$
\underline{E}=\frac{V_{i}-V_{o}}{a}\left(\begin{array}{c}
\cos \theta  \tag{5.4}\\
\sin \theta \\
0
\end{array}\right)
$$

where $a$ is the thickness of the ceramic ring, and $(r, \theta, z)$ are cylindrical polar coordinates whose origin is the centroid of the transducer.

Because of the way in which the ceramic material is polarised, and the way in which the electric field is applied across the ceramic material, the only stress due to the piezoelectric effect will be in the radial direction. In an axisymmetric formulation $\underline{\sigma}_{0}$ will be of the form [64]

$$
\underline{\sigma_{0}}=\left(\begin{array}{c}
\sigma_{r}  \tag{5.5}\\
0 \\
0 \\
0
\end{array}\right)
$$

where $\sigma_{r}$ is the radial stress. In a three-dimensional formulation $\underline{\sigma_{0}}$ is given by

$$
\underline{\sigma_{0}}=\left(\begin{array}{c}
\sigma_{r} \cos \theta  \tag{5.6}\\
\sigma_{r} \sin \theta \\
0 \\
0 \\
0 \\
0
\end{array}\right) .
$$

The magnitude of $\sigma_{T}$ depends on the piezoelectric parameters of the ceramic material and the strength of the electric field applied to the ceramic ring. In the examples considered here the exact values of the piezoelectric parameters are not known, but it is clear from (5.2) and (5.4) that $\sigma_{r}$ is a constant. It is possible to assume an arbitrary value for $\sigma_{r}$ and use this to determine how the response changes with frequency and then compare the shape of this response function to the one obtained experimentally.

By treating the $\sigma_{0}$ term as a pre-stress term in the finite element model of the transducer, we obtain the consistent load vector $\{64]$

$$
\begin{equation*}
\underline{f}^{k}=\int_{V} B^{T} \underline{\sigma_{0}} d V \tag{5.7}
\end{equation*}
$$

where $B$ is the elastic strain-displacement matrix, given by (4.6) for an axisymmetric structure, or (4.28) for a three dimensional structure.

### 5.3 Experimental Determination of the Response

In order to validate our numerical method for predicting the frequencies at which the maximum response occurs, we make a comparison between our computed results and some experimental results. The experimental results for each of the three transducers described in Section 5.1 were obtained using the procedure described below.
$A s$ an initial guess to the resonant frequency in water, we determine the resonant frequency in air by performing a loop test on the transducer as follows. A known voltage is applied to the transducer at different frequencies and the impedance of the transducer is measured at each frequency. The frequency which minimises the impedance is the resonant frequency of the transducer. The resonant frequency in air is now used as a starting point for finding the resonant frequency in water.

Each transducer was immersed in water in a test tank which was approximately 2.3 metres wide by 5.25 metres long and 2 metres deep. The transducer was excited by a unit alternating voltage applied across the electrodes at different frequencies, and the acoustic pressure was measured one metre from the transducer. Clearly this situation was not ideal since there was a strong possibility of reflections from the sides and the bottom of the tank and, to a lesser extent, from the surface of the water, although sides of the tank were covered in a material designed to minimise any reflections. To further complicate the situation, the transducer had to be put in a bag of castor oil, since the water would short circuit the terminals of the ceramic ring. Castor oil was used since it has almost identical acoustic properties to water

| Transducer | Finite Elements | Boundary Elements |
| :---: | :---: | :---: |
| A | 40 | 24 |
| B | 60 | 26 |
| C | 80 | 28 |

Table 5.1: The number of finite elements and boundary elements used to model each transducer.
but is a poor conductor. We assumed any effects the bag had on the acoustic field were negligible. The full experimental set-up is shown in Figure 5.2.

This set-up is likely to introduce a number of sources of experimental error into the results that we obtain. To minimise the effect of these errors, the experiment was repeated a number of times in different positions in the tank, and the results averaged.

### 5.4 Results and Conclusions

The numbers of quadratic, axisymmetric finite elements and linear, axisymmetric boundary elements used to model each transducer are shown in Table 5.1.

To find the peak response the problem must be solved for a number of different frequencies. However this is expensive since the boundary element matrices have to be re-computed for each new frequency. We can obtain an initial guess for the frequency which gives the peak response by finding the natural frequency of


Figure 5.2: The experimental set-up for determining the peak response of the sonar transducer.

| Transducer | Frequency | Computed Natural <br> Frequency in Air |
| :---: | :---: | :---: |
| A | 9634 | 9755 |
| B | 10058 | 10225 |
| C | 10593 | 10770 |

Table 5.2: The appropriate natural frequency, in Hz , of the sonar transducers considered.
the structure, in a vacuum, which has an eigenvector, or mode shape, similar to the displacements which we are inducing in the transducer, and hopefully the frequency giving the peak response is close to this natural frequency. Table 5.2 gives the appropriate computed natural frequency in vacuo (in hertz) for each type of transducer with the experimental resonant frequency in air. There is close agreement between the computed and experimental natural frequency, which gives us some confidence in our measuring instruments. Since all three transducers have a natural frequency at about 10 Khz , we shall study the response in the frequency range $5-15 \mathrm{KHz}$. We note that it is possible to use these high frequencies, and hence high wavenumbers, since the dimensions of the transducers are relatively small. For example, using the largest transducer, C , the maximum value of $d=!p-q \mid$ is 0.12695 m . For $f=15 \mathrm{KHz}$ we have $\omega=94247.78$ and $k=62.8319$, and hence the maximum value of $k d$ is 7.9766 .

The results presented here are for the absolute value of the acoustic pressure
at one metre from the transducer in both the radial and the axial directions. The numerical results have been scaled to give pressures of the same magnitude as the experimental results. This is allowed since we have chosen the value of $\underline{\sigma}_{r}$ arbitrarily.

Figures 5.3, 5.4 and 5.5 show the acoustic pressure one metre from transducers $A, B$ and $C$ respectively in the radial direction, over the entire frequency range considered.

Figures 5.6, 5.7 and 5.8 show the acoustic pressure one metre from transducers $\mathrm{A}, \mathrm{B}$ and C respectively in the radial direction, around the frequency giving the peak response. It can be seen that there is a good agreement between the computed results and the experimental results for transducers $A$ and $B$. The results for transducer C do not agree so well, but the measured pressure for transducer C is smaller than that for $A$ and $B$ and so could be more susceptible to experimental error due to reflections and other spurious acoustic waves in the tank.

Figures 5.9, 5.10 and 5.11 show the results for measuring the pressure one metre from the transducers $A, B$ and $C$ respectively in the axial direction. We note that there is a peak in the response in the axial direction at the same frequency as for the radial direction, but this peak is not as big. In this case the numerical solution is increasing for the higher frequencies, but only the experimental results for transducer $C$ seem to show this trend.

It is clear that this numerical technique can be used to predict the response patterns of sonar transducers. If accurate data is available on the piezoelectric
properties of the ceramic part, then it is feasible to extend the finite element method to obtain an accurate model to predict the exact response [52].

On the experimental side, it would seem that we need to obtain more accurate results when we are trying to measure the smaller pressures. However, this would probably require us to use a bigger test tank to reduce the interference from any echoes.





Figure 5.8. A comparison of the experimental and computed
results one metre from transducer C in the radial direction,
in the frequency range $7-8.5 \mathrm{Khz}$.
3.57

Figure 5.9. A comparison of the experimental and computed
results one metre from transducer. A in the axial direction,
in the frequency range $5-15 \mathrm{Khz}$.


Figure 5.11. A comparison of the experimental and computed
results one metre from transducer C in the axial direction,
in the frequency range $5-15 \mathrm{Khz}$.



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## A The Analytical Solution for a Hollow Elastic

## Sphere in an Acoustic Medium.

We now present an analytical solution to the problem of acoustic scattering and radiation from a spherical isotropic elastic shell, outer radius $a$ and inner radius b. The displacements in the shell can be represented in terms of $t$ wo potentials $\psi s$ and $\dot{\psi}_{d}$ which satisfy $[28,29]$

$$
\begin{equation*}
\left(\Gamma^{2}+k_{s}^{2}\right) \psi_{s}=0 \quad\left(\Gamma^{2}+k_{d}^{2}\right) \psi_{d}=0 \tag{A.1}
\end{equation*}
$$

where $k_{s}=\frac{\omega}{c_{s}}$ and $k_{d}=\frac{\omega}{c_{d}}$ and $c_{s}$ and $c_{d}$ are the speeds of shear and dialational waves in the shell. These are related to the elastic parameters of the shell through

$$
\begin{equation*}
c_{s}^{2}=\frac{\mu}{\rho_{s}} \quad c_{d}^{2}=\frac{\lambda+2 \mu}{\rho_{s}} \tag{A.2}
\end{equation*}
$$

where $\mu$ and $\lambda$ are the Lame elastic parameters, which in turn are related to Young's modulus and Poission's ratio through

$$
\begin{equation*}
\lambda=\frac{E \nu}{(1-\nu)(1-2 \nu)}, \quad \mu=\frac{E}{2(1+\nu)} . \tag{A.3}
\end{equation*}
$$

Let $(r, \theta, \phi)$ be spherical polar coordinates whose origin is at the centre of the sphere, and assume axial symmetry about the $z$ axis, so the solution is independent of $\phi$. Then, the normal displacements $u_{n}$ are given by

$$
\begin{equation*}
u_{n}=\frac{\partial}{\partial r}\left[\psi_{d}+\frac{\partial}{\partial r}\left(r \psi_{s}^{\prime}\right)\right]+r k_{s}^{2} \psi_{s}^{\prime} . \tag{A.4}
\end{equation*}
$$

The three boundary conditions on the outer surface $r=a$ are

$$
\begin{equation*}
\tau_{r r}=-\phi_{t} \quad u_{n}=\frac{-i}{\omega \rho \rho} \frac{\partial \phi_{t}}{\partial r} \quad \tau_{r \theta}=0 \tag{A.5}
\end{equation*}
$$

where $\phi_{t}$ is the total acoustic pressure in the exterior field. The normal stress $\tau_{r r}$ is given by [28]

$$
\begin{equation*}
\tau_{r r}=-\lambda k_{d}^{2} \psi_{d}+2 \mu\left\{\frac{\partial^{2}}{\partial r^{2}}\left[\psi_{d}+\frac{\partial}{\partial r}\left(r \psi_{s}\right)\right]+k_{s} \frac{\partial}{\partial r}\left(r \psi_{s}\right)\right\} \tag{A.6}
\end{equation*}
$$

and the shear stress $\tau_{r \theta}$ is given by

$$
\begin{equation*}
\tau_{r \theta}=\mu\left\{2 \frac{\partial}{\partial r}\left[\frac{1}{r} \frac{\partial}{\partial \theta}\left(\psi_{d}+\frac{\partial}{\partial r}\left(r \psi_{s}\right)\right)\right]+k_{s}^{2} \frac{\partial \psi_{s}}{\partial \theta}\right\} . \tag{A.7}
\end{equation*}
$$

On the inner surface $r=b$ we have the boundary conditions

$$
\begin{equation*}
\tau_{r r}=\phi_{i n t} \quad \tau_{r \theta}=0 \tag{A.8}
\end{equation*}
$$

where $\phi_{\text {int }}$ is some known pressure distribution on the inner surface. If the interior of the shell is a vacuum then $\phi_{i n t}=0$. We assume that $\phi_{i n t}$ can be written in the form

$$
\begin{equation*}
\phi_{i n t}=e^{i \omega t} \sum_{n=0}^{\infty} z_{n} P_{n}(\cos \theta) . \tag{A.9}
\end{equation*}
$$

It can be shown that the potentials can be written in the form [28, 29]

$$
\begin{equation*}
\psi_{s}=e^{i \omega t} \sum_{n=0}^{\infty} P_{n}(\cos \theta)\left[b_{n} j_{n}\left(k_{s} r\right)+d_{n} y_{n}\left(k_{s} r\right) . \quad a \leq r \leq b\right. \tag{A.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi_{d}=\epsilon^{i \omega t} \sum_{n=0}^{\infty} P_{n}(\cos \theta)\left[c_{n} j_{n}\left(k_{s} r\right)+e_{n} y_{n}\left(k_{s} r\right)_{:} \quad a \leq r \leq b\right. \tag{A.11}
\end{equation*}
$$

where $j_{n}$ and $y_{n}$ are the spherical Bessel and Neumann functions respectively, and $P_{n}$ are the Legendre polynomials. In the exterior acoustic field we have $\phi_{t}=\phi_{s}+\phi_{i}$ where $\phi_{i}$ and $\phi_{s}$ are the incident and scattered (radiated) sound fields, given in the form

$$
\begin{equation*}
\phi_{i}=e^{i \omega t} \sum_{n=0}^{\infty} x_{n} P_{n}(\cos \theta) j_{n}(k r) \tag{A.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\phi_{s}=e^{i \omega t} \sum_{n=0}^{\infty} a_{n} P_{n}(\cos \theta) h_{n}(k r) \tag{A.13}
\end{equation*}
$$

where $h_{n}$ are the spherical Hankel functions of the first kind.

Substituting these into the boundary conditions yields, for each value of $n$, the linear system

$$
\begin{equation*}
D \underline{\alpha}=\underline{r} \tag{A.14}
\end{equation*}
$$

where $\underline{\alpha}=\left[a_{n}, b_{n}, c_{n}, d_{n}, \epsilon_{n}\right]^{T}$ and

$$
\underline{r}=\left(\begin{array}{c}
-j_{n}(k a) k_{s}^{2} a^{2} \frac{\rho_{f}}{\rho_{s}} x_{n}  \tag{A.15}\\
k a j_{n}^{\prime}(k a) x_{n} \\
0 \\
k_{s}^{2} b^{2} \frac{\rho_{f}}{\rho_{s}} z_{n} \\
0
\end{array}\right) .
$$

The elements of the $5 \times 5$ matrix $D$ are given by

$$
\begin{gathered}
D_{11}=k_{s}^{2} a^{2} h_{n}(k a) \frac{\rho_{f}}{\rho_{s}} \\
D_{21}=-k a h_{n}^{\prime}(k a) \\
D_{31}=0 \quad D_{41}=0 \quad D_{51}=0 \\
D_{12}=\left(2 n(n+1)-k_{s}^{2} a^{2}\right) j_{n}\left(k_{d} a\right)-4 k_{d} a j_{n}^{\prime}\left(k_{d} a\right) \\
D_{22}=k_{d} a j_{n}^{\prime}\left(k_{d} a\right) \\
D_{32}=2\left(j_{n}\left(k_{d} a\right)-k_{d} a j_{n}\left(k_{d} a\right)\right) \\
D_{42}=\left(2 n(n+1)-k_{s}^{2} b^{2}\right) j_{n}\left(k_{d} b\right)-4 k_{d} b j_{n}^{\prime}\left(k_{d} b\right)
\end{gathered}
$$

$$
\begin{gathered}
D_{52}=2\left(j_{n}\left(k_{d} b\right)-k_{d} b j_{n}\left(k_{d} b\right)\right) \\
D_{13}=2 n(n+1)\left(k_{s} a j_{n}^{\prime}\left(k_{s} a\right)-j_{n}\left(k_{s} a\right)\right) \\
D_{23}=n(n+1) j_{n}\left(k_{s} a\right) \\
D_{33}=2 k_{s} a j_{n}\left(k_{s} a\right)+\left(k_{s}^{2} a^{2}-2 n(n+1)+2\right) j_{n}\left(k_{s} a\right) \\
D_{43}=2 n(n+1)\left(k_{s} b j_{n}^{\prime}\left(k_{s} b\right)-j_{n}\left(k_{s} b\right)\right) \\
D_{53}=2 k_{s} b j_{n}\left(k_{s} b\right)+\left(k_{s}^{2} b^{2}-2 n(n+1)+2\right) j_{n}\left(k_{s} b\right) \\
D_{14}=\left(2 n(n+1)-k_{s}^{2} a^{2}\right) y_{n}\left(k_{d} a\right)-4 k_{d} a y_{n}^{\prime}\left(k_{d} a\right) \\
D_{24}=k_{d} a y_{n}^{\prime}\left(k_{d} a\right) \\
D_{34}=2\left(y_{n}\left(k_{d} a\right)-k_{d} a y_{n}\left(k_{d} a\right)\right) \\
D_{44}=\left(2 n(n+1)-k_{s}^{2} b^{2}\right) y_{n}\left(k_{d} b\right)-4 k_{d} b y_{n}^{\prime}\left(k_{d} b\right) \\
D_{54}=2\left(y_{n}\left(k_{d} b\right)-k_{d} b y_{n}\left(k_{d} b\right)\right) \\
D_{55}=2 k_{s} b y_{n}\left(k_{s} b\right)+\left(k_{s}^{2} b^{2}-2 n(n+1)+2\right) y_{n}\left(k_{s} b\right) .
\end{gathered}
$$

If the incident wave is a plane wave of amplitude $\phi_{0}$ then the coefficients $x_{n}$ are given by

$$
\begin{equation*}
x_{n}=\phi_{0} i^{n}(2 n+1) \tag{A.16}
\end{equation*}
$$

If the structure is a solid sphere, we set $b=0$, and see that the $d_{n}$ and $e_{n}$ must be zero since the solution must be finite at $r=0$. Since there no inner surface, we omit the boundary conditions there, and hence the last $t$ wo equations in the above system, to obtain a $3 \times 3$ system for the remaining coefficients.

Once we have found the coefficient $a_{n}$ we can use (A.12) and (A.13) to find the total pressure on the surface or in the external domain. The normal velocity on the surface can be found using

$$
\begin{equation*}
v_{n}=\frac{i}{\omega \rho_{f}} \frac{\partial \phi_{t}}{\partial r} . \tag{A.17}
\end{equation*}
$$

## A comparison between various

boundary integral formulations
of the exterior acoustic problem

by<br>S Amini \& P J Harris<br>Research Report MSOR-88-01

## Figure Captions

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Figure 1: Condition number of Burton and Miller's formulae (2.6) for
    a unit sphere.
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Figure 2: Condition number of Burton and Miller's regularised formulae (2.12) for a unit sphere.

Figure 3: Formulation (2.6) for a unit sphere, using a $40 \times 40$ subdivision.

Figure 4: For a unit sphere, using a $20 \times 20$ subdivision.

Figure 5: For an axisymmetric ellipsoidwith b/a $=3$, using a $20 \times 20$ subdivision.

Figure 6: CHIEF method for a unit sphere, using a $40 \times 40$ subdivision.

Figure 7: For a family of right circular cylinders at $k=1$, using a $20 \times 20$ subdivision.


#### Abstract

Classical boundary integral formulations of the Helmholtz equation in the exterior domain either via the Helmholtz formulae (leading to direct formulations) or via layer potential representation of the solution (leading to indirect formulations) fail to have a unique solution for certain real values of the wavenumber. In this paper we consider three of the most commonly used formulations which are valid for all wavenumbers. We carry out a systematic comparison between these formulations, investigating their suitability for use in a general boundary element code. We study the effect of parameters in these formulations on the stability and the attainable accuracy of the discretised equations. The computational aspects of our boundary element methods, such as the numerical evaluation of singular integrals and the efficient solution of the resulting linear systems are also discussed.


## 1. Introduction

In recent years many boundary value problems of mathematical physics and engineering have been formulated as integral equations over the finite boundary of the domain of interest and subsequently solved by finite element type methods $[8,22,40]$. Here we are interested in the solution of time harmonic acoustic or electromagnetic radiation or scattering, by a three dimensional body whose bounded surface $S$ belongs to the class $C^{2}$. We denote by $D_{-}$the interior of $S$, by $D_{+}$the unbounded domain exterior to $S$, whilst $n$ denotes the unit normal to $S$ directed into $D_{+}$. The governing equation is the Helmholtz (reduced wave) equation defined by

$$
\begin{equation*}
\left(\nabla^{2}+k^{2}\right) \varphi(p)=0 \quad p \in D_{+}, \operatorname{Re}(k), \operatorname{Im}(k) \geqslant 0 \tag{1.1}
\end{equation*}
$$

with either of the boundary conditions

$$
\begin{equation*}
\varphi(p)=f(p) \quad p \in S \tag{1.1}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{\partial \varphi}{\partial n}(p)+h(p) \varphi(p)=f(p) \quad p \in S, \operatorname{Re}(h) \geqslant 0 \tag{1.1}
\end{equation*}
$$

and $\varphi$ also satisfying the Sommerfeld radiation condition:

$$
\lim _{i p i=r \rightarrow \infty} r\left[\frac{\partial \varphi}{\partial r}(p)-i k \varphi(p)\right]=0
$$

uniformly for all directions p/Ipl.

In this paper we study various direct boundary integral formulations of the above problem. The indirect formulations, based on layer potential
representations of the unknown function, $[9,13,32]$, yield integral equations which are similar to those obtained by the direct formulation. Furthermore we concentrate mainly on the radiation problem (the Neumann problem); that is we seek a "radiating wave-function" (solution of (1.1) a and (1.1)d) also satisfying (1.1)c with $h(p) \geqslant 0$. The starting point for the direct formulation of the exterior acoustic problem is the well-known Helmholtz formulae

$$
\int_{S}\left\{\varphi(q) \frac{\partial G_{k}}{\partial n_{q}}(p, q)-\frac{\partial \rho}{\partial n_{q}}(q) G_{k}(p, q)\right\} d S_{q}= \begin{cases}\varphi(p) & p \in D_{+}  \tag{1.2}\\ \frac{1}{2} \varphi(p) & p \in S \\ 0 & p \in D_{-}\end{cases}
$$

where

$$
G_{k}(p, q)=e^{i k|p-q|} / 4 \pi|p-q|
$$

is the fundamental solution or the free-space Green's function for (1.1)a. We note here that if $S$ is non-smooth, that is at some points $p \in S$, the tangent to $S$ at $p$ is non-unique, the term on the right hand side of (1.2) has to be suitably modified [9].

Let us define similar to the treatment of the potential problems [22], the following radiating wave functions known as Helmholtz single and double layer potentials respectively:

$$
\begin{align*}
& \left(L_{k} \sigma\right)(p)=\int_{S} \sigma(q) G_{k}(p, q) d S_{q}  \tag{1.3}\\
& \left(M_{k} \sigma\right)(p)=\int_{S} \sigma(q) \frac{\partial G_{k}}{\partial n_{q}}(p, q) d S_{q} ; \tag{1.4}
\end{align*}
$$

where $\sigma$ is a surface density function. Let us also denote the derivative of the operators $L_{k}$ and $M_{k}$ in the direction of $n_{p}$, the normal to $S$ at $p \in S$, by $M_{k} T$ and $N_{k}$ respectively $[9,13]$. It can be shown that the operators $L_{k}, M_{k}, M_{k} T: H^{r}(S) \rightarrow H^{r}(S)$ are compact, where $H^{r}(S)$ denotes the Sobolev space of $r$ times continuously differentiable functions in the sense of distributions; see also $[13,27,31]$ for smoothness results for these operators over other function spaces. Furthermore the operator $N_{k}: H^{r}(S) \rightarrow H^{r}(S)$ is an unbounded operator, though $\mathrm{N}_{\mathrm{k}}: \mathrm{H}^{r}(\mathrm{~S}) \rightarrow \mathrm{H}^{(\mathrm{r}-1)}(\mathrm{S})$ is a bounded linear operator, but not compact, [13].

From (1.2), the boundary integral equation for the Neumann problem can be written in the form

$$
\begin{equation*}
\left[-\frac{1}{2}+M_{k}\right] \varphi(p)=\left[L_{k} \frac{\partial \rho}{\partial n}\right](p)=\left(L_{k} f\right)(p) \quad p \in S \tag{1.5}
\end{equation*}
$$

known as the Surface Helmholtz Equation (SHE). Once $\varphi(\mathrm{p})$ is known on the boundary $S$, we use (1.2) to obtain $\varphi(p)$ for $p \in D_{+}$in the form $\varphi(p)=M_{k} \varphi(p)-L_{k} \frac{\partial \varphi}{\partial n}(p)$. The kernel of the operator $M_{k}$ in (1.5) has only a weak singularity of the form $1 /|p-q|$ and the Fredholm-Riesz theory [13] can be used to deduce that for a countable set of values of $k$, say $I_{S}$, the operator $M_{k}$ will have $\frac{1}{2}$ as an eigenvalue and hence for such values of $k$, the equation (1.5) should not be used as ( $-\frac{1}{2}+M_{k}$ ) is not invertiable. Clearly for a given value of $k=k^{*}$ say, the conditioning of the integral equation (1.5) is inversely proportional to the distance of $k *$ from $I_{S}$. If dist $\left(k^{*}, I_{S}\right)$ is small then the formulation (1.5) is ill-conditioned and hence the results from a boundary element solution of (1.5) are likely to bear little or no resemblence to the exact solution $\varphi(\mathrm{p})$. It is well-known [9] that asymptotically (as $k \rightarrow \infty$ ) the number of terms in $l^{\prime}$ less than a given value of
$k$ is proportional to $k^{3}$. Clearly as $k$ increases the dist ( $k, I S$ is likely to be small for all values of $k$. For example, assuming that $S$ is anit sphere, it can be shown [4] that the eigenvalues of the operator ( $-\frac{1}{2}+M_{k}$ ) are given by

$$
\begin{equation*}
\lambda_{n}\left(-\frac{1}{2}+M_{k}\right)=i k^{2} j_{n}(k) h_{n}^{\prime}(k) \quad n=0,1,2, \ldots \tag{1.6}
\end{equation*}
$$

where $j_{n}$ denotes the spherical Bessel functions and $h_{n}$ the spherical Hankel functions of the first kind. It follows from (1.6) that, [1,4], $I_{S}=\left\{k \mid j_{n}(k)=0, n=0,1, \ldots\right\}$. Here $I_{S}=\{\ldots, 20.12,20.18,20.20,20.37,20.54,20.98, \ldots\}$, indicating the density of the eigenvalues; (eigenvalues given correct to 4 significant figures only). Therefore in general there is a need for formulations which do not suffer from non-uniqueness at these (irregular) frequencies.

## 2. Formulations Valid for all Wave-numbers

Over the past twenty years many formulations have been suggested in order to overcome the non-uniqueness at irregular frequencies associated with (1.5); see for example $[9,10,23,24,38,39]$. Here we shall concentrate on 3 such formulations which are used most commonly in practice, at present.

### 2.1 Method of Schenck (1968)

Schenck [35] shows that the equation (1.5) has a solution for all wavenumbers, but when $k \in I S$ the solution is not unique. This follows from the Fredholm alternative theorem by establishing that for $k \in I$, the function $L_{k} \partial \rho / \partial n$, the right hand side of (1.5), is orthogonal to all the
eigenfunctions $\psi_{i, k}(p), i=1,2, \ldots, l$ of the operator $\mu_{k}^{k}$, the adjoint of $M_{k}$.

On the other hand, the interior Helmholtz functional relation, (1.2) with $p \in D_{-}$,

$$
\begin{equation*}
\left(M_{k} \varphi\right)(p)=\left[L_{k} \frac{\partial \varphi}{\partial n}\right](p) \quad p \in D_{-} \tag{2.1}
\end{equation*}
$$

always has a unique solution $\varphi(p), p \in S$, which is however difficult to compute numerically [14,35]. Schenck [35] suggests the following numerical procedure for obtaining an approximation to the common solution of (1.5) and (2.1) (which would be the required solution to our problem). Let us assume that on discretising (1.5) using a boundary element method (eg. a collocation method with the collocation points $p_{1}, p_{2}, \ldots, p_{N} \epsilon S$ ) see section 3 , we obtain an $N x N$ linear system of equations in the form

$$
\begin{equation*}
\left(-\frac{1}{2} I_{N}-\hat{M}_{k}\right)_{Q}=\underline{b} . \tag{2.2}
\end{equation*}
$$

We then choose $M=M(k) \ll N=N(k)$ points $P_{N+1}, \ldots P_{N}+M_{C} D_{-}$and discretise (2.1) similarly, to obtain an $M x N$ linear system in the form

$$
\begin{equation*}
\overline{\tilde{M}}_{k \ell}=\underline{c} . \tag{2.3}
\end{equation*}
$$

Schenck suggests combining (2.2) and (2.3) to obtain (M+N) equations for the $N$ unknowns $\wp$, which would then be solved in a least square sense. This method, based on the Combined Helmholtz Integral Equation Formalation is commonly referred to by the acronym CHIEF. The success of CHIEF in resolving the non-uniqueness of (1.5) for $k \in I S$ depends on the choice of the interior collocation points $p_{N+1}, \ldots P_{N+M} \in D_{-}$. It is possible that if all these points
are chosen sufficiently close to the nodal surfaces of the interior Dirichlet eigenfunction $[9,35]$ this method may fall to adequately resolve the non-uniqueness of the surface Helmholtz equation. To be precise if $I_{S}=\left\{k_{0}, k_{1}, \ldots\right\}$ with $k_{i}>k_{i+1}, i=0,1, \ldots$ and if $k \approx k_{j} \in I_{S}$, then provided $\min \left(k_{j}-k_{j-1}, k_{j+1}-k_{j}\right)$ is sufficiently large, relative to the discretisation error in our method, we expect Schenck's method to work adequately provided at least one of the $M$ interior points is sufficiently removed from the nodal surfaces corresponding to $k_{j}$. For large $k$ however, it is possible that several of the values of $k_{i} \in I_{S}$, say $k_{\ell}, k_{\ell+1}, \ldots, k_{\ell+m}$ are sufficiently close to $k$. In this case we may expect that at least $m+l+1$ of the $M$ interior collocation points should avoid the corresponding nodal surfaces, see Figure 4. (We must emphasise that perhaps at such high frequencies other methods may be more suitable [12]).

In general however, neither the critical wavenumbers $I_{S}$ nor their corresponding interior eigenfunctions are known. Therefore, we need to choose $M$ sufficiently large in the hope that the rank of the ( $N+M$ ) by $N$ matrix, (2.2) $+(2.3)$, is equal to $N$ and furthermore its $N$ singular values [16] are sufficiently away from zero to ensure that the non-uniqueness problem has been adequately resolved; see Figure 6 .

### 2.2 Method of Burton and Miller (1971)

This method is closely related to the indirect formulation of Panich [32]. On differentiating (1.5) in the direction of $n_{p}$, the normal to $S$ at $p$ towards $D_{+}$, we obtain, using appropriate jump conditions of the layer potentials $[9,13]$, the first kind equation

$$
\begin{equation*}
N_{k} \varphi(p)=\left(\frac{1}{2}+M_{k} T\right) \frac{\partial \varphi}{\partial n}(p) \quad p \in S \tag{2.4}
\end{equation*}
$$

This equation is also singular for $k \in J_{S}$, where $J_{S}$ is the countable set of the eigenvalues of the interior Neumann problem. For example, if $S$ is a unit sphere, it can be shown [4] that the eigenvalues of the operator $N_{k}$ are defined by

$$
\begin{equation*}
\lambda_{n}\left(N_{k}\right)=i k^{3} h_{n}^{\prime}(k) j_{n}^{\prime}(k) \quad n=0,1,2, \ldots \tag{2.5}
\end{equation*}
$$

and hence $J_{S}=\left\{k \mid j_{n}^{\prime}(k)=0, n=0,1, \ldots\right\}$. In fact for this case $J_{S}=(\ldots, 20.22,20.37,20.52,20.56,20.68,20.81,,$,$\} ; indicating the density of$ the eigenvalues; (eigenvalues given correct to 4 significant figures).

Burton and Miller [11] suggest coupling (2.4) with (1.5) to obtain the following second kind integral equation

$$
\begin{equation*}
\left\{\left(-\frac{1}{2}+M_{k}\right)+i \nu N_{k}\right\} \varphi(p)=\left\{L_{k}+i \nu\left(\frac{1}{2}+M_{k} T\right)\right\} \frac{\partial \rho}{\partial n}(p) \quad p \in S \tag{2.6}
\end{equation*}
$$

where $i^{2}=-1$ and $\nu$ is an arbitrary positive coupling parameter. It can be shown that the formulation (2.6) is valid for all wavenumbers and can be used for the Dirichlet problem as well as the Robin problem. In the latter case we replace $\partial \rho / \partial n$ by $f(p)-h(p) \varphi(p)$; see (1.1)c.

Before the formulation (2.6) can be used in a numerical calculation we need to address the following two outstanding issues, namely (i) the effect of $\nu$ on the conditioning of the problem and (ii) the interpretation of the hyper-singular operator $N_{k}$, suitable for use in numerical calculations.

### 2.2.1 Choice of the Coupling Parameter

Although in theory equation (2.6) is valid for all wavenumbers provided $\nu>0$, the choice of $\nu$ greatly affects the conditioning of this integral equation. For a given surface $S$ and a value of $r$ the eigenvalues of the integral operators in (2.6), though never zero, can become close to it for certain values of $k$. Ideally we should choose $r=r(S, k)$ so as to minimise the condition number of the integral operators (ie. to avoid ill-conditioning of the formulation). Restricting $S$ to be a unit sphere it is possible to obtain the eigensystems for all the integral operators involved [4] and hence choose $\nu=\nu(\mathrm{S}, \mathrm{k})$ so as to minimise the condition number of the integral equation (2.6); see $[3,4,25]$. In $[4,25]$ the choice $r(S, k)=1 / k$ was shown to be "almost optimal" for $S$ a sphere. In general provided the surface is not too thin or elongated, the same choice can be shown to yield well-conditioned integral operators [4].

The choice of $v(S, k) \equiv 1$ has often been used in practice. In Figure 1 , values of the condition number of the operator $\left(-\frac{1}{2}+M_{k}\right)+i \nu N_{k}$ in (2.6) are plotted against $(y, k) \in[0.1,10] \times[0,10]$. It can be seen that the choice of coupling parameter can greatly affect the conditioning of the equation (2.6) and, as can be seen in Figure 3, influence the attainable accuracy by a numerical method; see also [4] for more details.

### 2.2.2 Interoretation of the Operator $N_{k}$

It can be shown that

$$
\begin{equation*}
\frac{\partial^{2} G_{k}}{\partial n_{p} \partial n_{q}}(p, q)=0\left(|p-q|^{-3}\right) \text { as } p \rightarrow q ; p, q \in S \text {. } \tag{2.7}
\end{equation*}
$$

Let us recall that

$$
\begin{equation*}
N_{k} \varphi(p)=\frac{\partial}{\partial n_{p}} \int_{S} \varphi(q) \frac{\partial}{\partial n_{q}} G_{k}(p, q) d S_{q} \tag{2.8}
\end{equation*}
$$

In order to view $\mathrm{N}_{\mathrm{k}}$ as an integral operator with the kernel $\partial^{2} \mathrm{G}_{\mathrm{k}} / \partial \mathrm{n}_{\mathrm{p}} \partial \mathrm{n}_{\mathrm{q}}$, (which is ideal for numerical calculations) we need to change the order or differentiation and integration in (2.8). Because of (2.7) however, this change of order is not strictly valid as the resulting integral operator will have non-integrable singularity. We are however allowed to change the order of integration and differentiation (which is always done in practice) provided the resulting hypersingular integral is interpreted in the sense of Hadamard finite part [18]. Clearly one needs to carry out some mathematical analysis on $N_{k} \varphi$ before employing a numerical quadrature to approximately evaluate such integrals. Various methods $[10,37]$ have been proposed and here we employ the formulation used in [29] which follows from the pointwise variational principle introduced in [36]:

$$
\begin{align*}
\int_{S} \varphi(q) \frac{\partial^{2} G_{k}}{\partial n_{p} \partial n_{q}}(p, q) d S_{q} & =k^{2} \varphi(p) \int_{S} n_{p} \cdot n_{q} G_{k}(p, q) d S_{q}+ \\
& \int_{S}\{\varphi(q)-\varphi(p)\} \frac{\partial^{2} G_{k}}{\partial n_{p} \partial n_{q}}(p, q) d S_{q} . \tag{2.9}
\end{align*}
$$

The second integral on the right hand side of (2.9) still has $0\left(|p-q|^{-2}\right)$ signularity and care must be taken in numerically evaluating it whenever the collocation point is in or near the element over which integration is carried out; see section 3 .

### 2.3 Regularised Burton and Miller Formulation

In an at tempt to avoid the hyper-singular operator $N_{k}$ in the formulation (2.6), Burton [10], suggested, following the work of Panich [32], to regularise the operator $N_{k}$. By applying Green's second theorem in the interior domain $D_{-}$, to the function $G_{k}(p, q)$ and $M_{k} \sigma(p)$, it can be shown that [10]

$$
\begin{equation*}
L_{k} N_{k} \sigma(p)=\left(M_{k}+\frac{1}{2}\right)\left(M_{k}-\frac{1}{2}\right) \sigma(p) \quad p \in S \tag{2.10}
\end{equation*}
$$

Burton [10] suggests premultiplying the equation (2.4) by $L_{o}$, using (2.10) with $k=0$ to obtain in place of (2.4) the equation

$$
\begin{equation*}
\left(L_{0}\left(N_{k}-N_{0}\right)+M_{o}^{2}-\frac{1}{4}\right) \varphi(p)=L_{o}\left(\frac{1}{2}+M_{k} T\right) \frac{\partial \varphi}{\partial n}(p) \quad p \in S \tag{2.11}
\end{equation*}
$$

This equation is now coupled with (1.5) to yield the regularised version of (2.6) in the form

$$
\begin{aligned}
& \left\{\left(-\frac{1}{2}+M_{k}\right)+i v\left[L_{o}\left(N_{k}-N_{o}\right)+M_{o}^{2}-\frac{1}{4}\right]\right) \varphi(p)=\left[L_{k}+i \nu L_{o}\left(\frac{1}{2}+M_{k} T\right)\right] \frac{\partial \rho}{\partial n}(p), p \in S . \text { (2.12) } \\
& \text { It can be shown that }
\end{aligned}
$$

$$
\begin{equation*}
\frac{\partial^{2}}{\partial n_{p} \partial n_{q}}\left(G_{k}(p, q)-G_{o}(p, q)\right)-0\left(|p-q|^{-1}\right)+0(1) \tag{2.13}
\end{equation*}
$$

and hence all the integral operators in the regularised formulation (2.12) are compact and perhaps more amenable to accurate numerical solutions [6]. Clearly the formulation (2.12) is more complicated (and hence more expensive in numerical computation) than that of (2.6), however it is possible to reduce this extra computational cost if we were to store the matrix approximations to $L_{0}$ and $M_{0}^{2}$ and use them to solve the problem for a range of values of $k ;[3]$.

The work of Amini [4] on the conditioning of the regularised formulation (2.12) indicates that the equation (2.12) has a smaller condition number than that of (2.6) for all values of $k$ and $v$. In Figure 2 values of the condition number of the integral operator (2.12) for a unit sphere are plotted against $(\nu, k) \in[0,1,10] \times[0,10]$. It can be seen that there are peaks in the value of the condition number which cannot be avoided by simple variation in $v$; see [4]. In general the choice of $\nu(S, k) \equiv 1$ ensures that the formulation (2.12) is reasonably well conditioned.

## 3. Numerical Methods

### 3.1 General Remarks

The formulations (1.5), (2.6) and (2.12) are boundary integral equations of the second kind which may be represented in a common operator notation by

$$
\begin{equation*}
(\lambda *-\kappa) \varphi(p)=g(p) \quad p \in S, \tag{3.1}
\end{equation*}
$$

where $\lambda \star \epsilon[$ is some complex constant and $\kappa: X \rightarrow Y$ is a bounded integral equation from some Banach space $X$ to a Banach space $Y$, with $S$ as its domain of integration. The most commonly used numerical methods for solving (3.1) are the weighted residual methods which are based upon seeking a solution $\varphi_{N}(p) \approx \varphi(p)$, from a finite dimensional subspace $X_{N} \subset X$, spanned by some linearly independent basis function $\left\{\psi_{1}, \psi_{2}, \ldots, \psi_{N}\right\}$. In general $\psi_{i}$ has "local support"; that is $\psi_{i}(p)$ is non-zero only over a small portion of $S$; for $i=1,2, \ldots, N$. We have

$$
\begin{equation*}
\varphi_{N}(p)=\sum_{i=1}^{N} a_{i} \psi_{i}(p) \quad p \in S \tag{3.2}
\end{equation*}
$$

Replacing (3.2) into (3.1) and attempting to minimise ${ }^{n}$ in some sensen the residual function $r_{N}(p)-g(p)-(\lambda *-K) \varphi_{N}(p)$, results in a linear system of algebraic equations of the form

$$
\begin{equation*}
\left(\lambda * \mathrm{I}_{N}-\mathrm{K}_{\mathrm{N}}\right) \underline{a}_{N}=\mathrm{g}_{\mathrm{N}}, \tag{3.3}
\end{equation*}
$$

to be solved for the $N$ unknowns $a_{1}, a_{2}, \ldots, a_{N}$. Once $a_{N}$ is known, the equation (3.2) may be used to find $\varphi_{N}(p) \approx \varphi(p)$ for $p \in S$.

The most commonly used weighted residual method for $3-\mathrm{D}$ problems is the collocation method; (see also [17] and references therein for a variational approach). Here in order to obtain the linear system (3.3) we choose $N$ collocation points $p_{1}, p_{2}, \ldots, p_{N}$ on $S$, appropriately and impose the condition $r_{N}\left(p_{i}\right)=0$ for $i=1,2, \ldots, N$. The results presented in this paper are based on the collucation method with the simplest choice of the basis
functions, namely piecewise constant functions. The surface $S$ is divided into $N$ disjoint subsets $S_{i}$ where $\underset{i=1}{\mathbb{U}} S_{i}=S$ and $\left\{\psi_{i}\right\}$ are chosen as follows:

$$
\psi_{i}(p)=\left\{\begin{array}{ll}
1 & \text { if } p \in S_{i}  \tag{3.4}\\
0 & \text { otherwise }
\end{array} \quad i=1,2, \ldots, N .\right.
$$

The collocation points are chosen with $\mathrm{p}_{\mathrm{i}} \in \mathrm{S}_{\mathrm{i}}$ and whenev possible close to the centroids of $S_{i}$. In this case the unknowns $a_{i}$ in (3.3) satisfy $a_{i}=\varphi_{N}\left(p_{i}\right) \simeq \varphi\left(p_{i}\right) ; i=1,2, \ldots, N$. The elements of the $N \times N$ complex values matrix $K_{N}$ are given by

$$
\begin{equation*}
\left(K_{N}\right)_{i j}=\int_{S_{j}} K\left(p_{i}, q\right) d S_{q} \quad i, j=1,2, \ldots, N \tag{3.5}
\end{equation*}
$$

which in general have to be evaluated numerically. Collocation methods where the integrals (3.5) are calculated numerically are referred to as "discrete collocation methods". We would expect that provided the quadrature rules are sufficiently accurate any error estimate and analysis for the collocation method [7] to be also valid for the discrete collocation method.

In practice we are often forced to make a further approximation, namely
the surface elements $S_{i}$ 's are approximated by $\hat{S}_{i}$, where in general $\hat{S}_{i}$ is a linear or quadratic curved element approximating $S_{i}$. The results of [28] indicate the importance of a good approximation $\hat{S}={\underset{U}{U}=1}_{N}^{S_{i}}$ for $S$. The surfaces considered for the results in this paper are all axisymmetric, which lend themselves to accurate and cheap parametrisation by cubic splines [5]. The cylindrical polar coordinates $(\rho, \theta, z)$ is a convenient system for axisymmetric structures with $z$ as the axis of symmetry. We consider the parametric representation of the surface $S$ in the form

$$
\begin{cases}x=x(s, \theta)=\rho(s) \cos (\theta) &  \tag{3.6}\\ y=y(s, \theta)=\rho(s) \sin (\theta) & 0<s<L \\ z=z(s) & 0<\theta<2 \pi\end{cases}
$$

where $s$ is the arc-length from one end of a generating curve (with $\theta=0$ say) with total length $L$. The surface $S$ is now globally transformed to a rectangle in s- $\theta$ plane with the Jacobian of the transformation given by $\rho(s)$;
[5]. The integral operator in (3.1) is transformed to

$$
\begin{equation*}
\kappa \varphi(p)=\int_{0}^{2 \pi} \int_{0}^{L} K\left(\left(s^{*}, \theta^{*}\right),(s, \theta)\right) \varphi(s, \theta) \rho(s) d s d \theta \tag{3.7}
\end{equation*}
$$

where $(s *, \theta *)$ and $(s, \theta)$ refer to the surface points $p$ (collocation) and $q$ (integration) respectively. The rectangle in $s-\theta$ plane is then subdivided into $m n=N$ subrectangles with $0=s_{0}<s_{1}<\ldots<s_{m}=L$ and $0=\theta_{0}<\theta_{1}<\ldots<\theta_{n}=2 \pi$. The collocation points $P \ell \equiv\left(s_{i}^{*}(\ell), \theta_{j}^{*}(\ell)\right)$, $Q=1,2, \ldots, N$ are chosen as the centroids of these rectangles.

The general boundary element method consists of the following two steps:
(i) setting up the coefficient matrix in (3.3), ie. the evaluation of $N^{2}$
double integrals of the form (3.5), (ii) solving the $N \times N$ complex linear system (3.3). For moderate values of $N$, the computational cost of setting up the matrix forms a major part of the total cost. If the system (3.3) is solved by Gaussian elimination type methods, as $N$ gets larger the solution time will begin to dominate the total computing time. Precisely how large $N$ has to be in order for this to be true depends on the kernel $K(p, q)$ as well as the specific machine used (ie. sequential or parallel processors).

Let us now briefly discuss our numerical method of dealing with the above two steps.

### 3.2.1 Numerical Integration Scheme

Here all the calculations are carried out on the rectangles in the $s-\theta$ plane. If the collocation point is outside the region of integration, the integrands are non-singular and a $3 \times 3$ product Gauss-Legendre rule is employed. Referring back to (3.5), we must point out that if $p_{i} S_{j}$, but dist ( $P_{i}, S_{j}$ ) is small, it may be necessary to use a more accurate integration scheme even though the integrand is non-singular; see [19] and references therein.

The diagonal elements of $K_{N}$ involve the integration of singular integrals, where $p_{i} \in S_{j}$ (ie. here $i=j$ ). A very effective method of dealing with weakly singular integrands, based on the error function transformation, was employed in [5] and analysed in [2]. Here we shall employ a transformation method similar to $[15,26]$ which is equivalent to a polar coordinate transformation in order to annihilate the $0\left(|p-q|^{-1}\right)$
singularities and subsequently employ a product Causs-Legendre rule to approximate the non-singular integrals. Let us refer back to the rectangles in the $s-\theta$ plane, with the collocation points $\left(s_{i}^{\star}, \theta_{j}^{*}\right)-\left[\frac{s_{i-1}+s_{i}}{2}, \frac{\theta_{j-1}+\theta_{j}}{2}\right]$ for $i=1,2, \ldots, m$ and $j=1,2, \ldots, n$. We now divide the rectangle containing the collocation points into 4 triangles each having the collocation point as a vertex, by joining the two diagonals. If we are integrating over a triangle in $s-\theta$ plane with vertices $(s 1, \theta 1),(s 2, \theta 2)$ and $(s 3, \theta 3)$ say, where the singularity in the integrand is at vertex ( $s 1, \theta 1$ ), we use the following transformation

$$
\begin{array}{ll}
s=(1-\eta) s 1+(1-\xi) \eta s 2+\xi \eta s 3 & 0<\xi, \eta<1 \\
\theta=(1-\eta) \theta 1+(1-\xi) \eta \theta 2+\xi \eta \theta 3 &
\end{array}
$$

to map the triangle into a unit square in $\xi-\eta$ plane. The Jacobian of this transformation is

$$
\begin{equation*}
\mathrm{J}=|(\mathrm{s} 2-\mathrm{s} 1)(\theta 3-\theta 1)-(s 3-s 1)(\theta 2-\theta 1)| \eta . \tag{3.9}
\end{equation*}
$$

If the original integrand had an inverse distance singularity in the $s-\theta$ plane the transformed integral in $\xi-\eta$ plane is non-singular.

Let us remark here that although on the original curved surface $S$ the singularity is of the form $0\left(|p-q|^{-1}\right)$ after the transformation to the $s-\theta$ plane followed by the $\xi-\eta$ transformation the resulting integrand though non-singular, is not as smooth as the integrands for the calculation of the off-diagonal elements of $K_{N}$. For this reason we employ a $5 \times 5$ product Gauss-rule in $\xi-\eta$ plane. Furthermore, as piecewise constant function approximation is employed here, the evaluation of the second integral on the
right hand side of (2.9) causes no difficulty as $\varphi(p)$ and $\varphi(q)$ are assumed to be equal whenever $p$ and $q$ are on the same element. If higher order function approximation is employed then careful analysis is required in evaluating those singular integrals; see $[19,37]$.

### 3.2.2 Solution of the Linear System (3.3)

The direct solution of a full NxN simultaneous linear equations by Gaussian elimination type method requires $0\left(N^{3}\right)$ floating point operations. As the number of collocation points increases the cost of solving the full non-hermitian complex matrix equation (3.3), by direct methods will begin to dominate the total computing costs.

The work of [6] shows that for integral equations of the second kind with compact operators the conjugate gradient method $[16,21]$ can yield accurate results, to the level of the truncation error of the underlying collocation method, very efficiently. Other iterative procedures such as the multigrid methods [33] or the 2 -grid method of Atkinson [7], can be used for solving (3.3) with $0\left(N^{2}\right)$ operations and should be implemented for large problems; see also [3]. For arbitrary shaped 3-dimensional structures with edges the underlying integral operators will be non-compact and in general the problem will have non-smooth solutions. In such cases unless special care is taken (in splitting the operator into a smooth and a non-smooth part [33]) the iterative techniques are unlikely to work efficiently and hence the direct solution of the linear system (3.3) may be the only viable alternative.

## 4. Numerical Results

### 4.1 Test Problems

Exact solutions in closed form, usually as infinite series of slowly converging wave functions, can only be obtained in special cases [29].

However, exact solutions for any geometry can be generated for problems which are equivalent to those having acoustic point sources in the interior region $D_{-}[5,20]$. The field at any point $p \epsilon R^{3}$, generated by a set of point sources at $q_{j} \in D_{\mathcal{L}}$ with strengths $A_{j}, j=1,2, \ldots, Q$, in the absence of the structure is given by

$$
\begin{equation*}
\varphi(p)=\sum_{i=1}^{Q} A_{j} \frac{e^{i k\left|p-q_{j}\right|}}{\left|p-q_{j}\right|} \tag{4.1}
\end{equation*}
$$

For $p \in S, \partial \rho / \partial n_{p}=\nabla \varphi . n_{p}$ can be easily calculated from (4.1) by re-introducing the body. A problem with this distribution of $\partial_{\rho} / \partial n_{p}$ on the structure is then equivalent to the point source problem for which the solution (4.1) is known.

Here we consider a family of axisymmetric ellipsoids, characterised by $(x / a)^{2}+(y / a)^{2}+(z / b)^{2}-1$, with the major axis $b$ and minor axis $a$, normalised so that $d=(a+a+b) / 3$, the typical length of the surface is unity. We also consider a family of right rectangular cylinders with radius $r$ and heigh $2 h$. Here again we choose $r$ and $h$ so that $d=(r+r+h) / 3=1$.

The boundary data considered for the family of ellipsoids is equivalent to that of two point sources placed at $(0,0,0)$ and $(0,0,0.5)$ with strengths $(-2+3 i)$ and $(2+i)$ respectively. For the family of cylinders, we consider the
problem equivalent to having 1 point source placed at ( $0,0,0$ ) with strength ( $2+3 i$ ).

### 4.2 Results and Discussion

Here the measure of the error is the Mean Relative Error (MRE) expressed as a percentage, where

$$
\begin{equation*}
\operatorname{MRE}=\frac{1}{N} \sum_{i=1}^{N} \frac{\left|\varphi\left(p_{i}\right)-\hat{\varphi}\left(p_{i}\right)\right|}{\left|\varphi\left(p_{i}\right)\right|} \tag{4.2}
\end{equation*}
$$

The points $p_{i} \in S, i=1,2, \ldots, N$, are the collocation points and $\hat{\varphi}\left(p_{i}\right)$ is the computed approximation to the exact value $\varphi\left(p_{i}\right)$. Clearly if $\left|\varphi\left(p_{i}\right)\right|$ is small for some $i=1,2, \ldots, N$, the measure (4.2) may exaggerate the error and it may be preferable to use the Relative Mean Error (RME) defined by

$$
\begin{equation*}
\operatorname{RME}=\sum_{i=1}^{N}\left|\varphi\left(p_{i}\right)-\hat{\varphi}\left(p_{i}\right)\right| / \sum_{i=1}^{N}\left|\varphi\left(p_{i}\right)\right| ; \tag{4.3}
\end{equation*}
$$

though this has not been used here.

Figure 1 is the plot of the condition number of the operator $\left(-\frac{1}{2}+M_{k}\right)+i r N_{k}$ in the Burton and Miller formulation (2.6), for a unit sphere over the range of values of $(\nu, k) \in[0.1,10] \times[0,10]$. Clearly for a given value of the wavenumber $k$, the choice of the coupling parameter $v$ can seriously affect the conditioning of (2.6). Furthermore the choice of $\nu=1 / k$ (with $\nu=2$ for $k<\frac{1}{2}$ ) does appear to be an "almost optimal" choice; see [4,25] for more details.

Similarly, Figure 2 is the plot of the condition number of the regularised formulation (2.12) for a unit sphere over the same range of values of ( $\nu, k$ ) as in Figure 1. Clearly, the condition number of (2.12) is generally smaller than (2.6). However, a simple relation for an "almost optimal" choice of $v$ as a function of $k$ does not seem possible. The choice of $r \equiv 1$, appears to be quite satisfactory for a large range of values of $k$.

In Figure 3 we present the mean relative error obtained for a unit sphere with the formulation (2.6), (here on referred to as the direct formulation due to the direct evaluation of the hyper-singular operator $N_{k}$ ) for a range of values of $k$ and with 2 choices of values of $\nu$, namely the commonly used choice of $\nu \equiv 1$ and the almost optimal choice of $\nu \equiv 1 / k$ (with $\nu=2$ if $k<\frac{1}{2}$ ). The number of subdivisions in s- $\theta$ plane is $40 \times 40$, but due to the axisymmetric nature of the problem we only collocate at the 40 points in s-direction (for a fixed value of $\theta$ ). Clearly as $k$ increases, with a fixed number of collocation points, we would expect the errors to increase, however for the case $v \equiv 1$, near $k=10$ the (relative) ill-conditioning of the equation (2.6), (see Figure 1) has significant effect on the accuracy of the result, emphasising the importance of the choice of $v$. From here on the almost optimal choice of $\nu$ is used for the direct formulation and $\nu \equiv 1$ for the regularised formulation (2.12).

In Figure 4 we present a comparison between the three methods, on a unit sphere with a $20 \times 20$ subdivisions. For the CHIEF formulation we have used 2 interior points. Both the regularised and the CHIEF method yield somewhat more accurate results than the direct formulation, though the regularised formulation is computationally the most expensive and CHIEF is the least
expensive of the three.

Figure 5 , is similar to that of Figure 4 except that here the region of interest is an axisymmetric ellipsoid with the ratio of major to minor axis b:a as 3:1. The errors are somewhat larger than that for anit sphere, though in this case all three methods yield results with similar accuracy.

In Figure 6, we present results for the CHIEF method on a unit sphere at 2 wave numbers, namely $k=20$ and $k=20.18$, on a $40 \times 40$ mesh with various number of interior points (ie. CHIEF points). The wavenumber $k=20$ is not an eigenvalue of the problem and hence the Surface Helmholtz Equation (1.5) (ie. the method of CHIEF with no interior points) does yield a reasonable result, though two or three CHIEF points seem to annihilate any ill-conditioning due to the eigenvalues of the (1.5) close to $k=20$. On the other hand $k=20.18$ is an eigenvalue of (1.5) and hence in theory we require one good CHIEF point, see [34], to overcome the non-uniqueness at this wavenumber. In practice however, because of the proximity of 20.18 to other eigenvalues of (1.5), namely $20.12,20.20$ and 20.37 we require at least 4 good CHIEF points to annihilate their effect on the conditioning of the equation.

Finally, in Figure 7 we present a comparison of the three methods on a right circular cylinder at $k=1$, for various ratios of height/radius, using a $20 \times 20$ subdivision. Both the regularised and the CHIEF method (with 2 interior points) are considerably more accurate than the direct method, though the results become less accurate as the ratio height/radius decrease.

### 4.3 Conclusion

Here we have presented some numerical results based on the collocation method as applied to three of the most commonly used formulations of the exterior acoustic problem which are designed to overcome the non-uniqueness of the classical boundary integral formulations. All three methods are shown to resolve the non-uniqueness problem adequately and yield reasonable results for a large class of test problems. The CHIEF method is much the easiest formulation though sufficient number of interior points should be taken to ensure that the effect of eigenvalues of the surface Helmholtz equation (1.5) near the wavenumber of interest has been annihilated. Furthermore, the choice of the coupling parameter $\nu$ plays an important role in the conditioning of (2.6). The regularised formulation (2.12) on the other hand is the most complicated formulation amongst the three, though it appears to be well-conditioned and does not require any special consideration to yield accurate results. Indeed the integral operators in (2.12), having weakly singular kernels are compact, making the numerical evaluation of the integrals more amenable to accurate approximation. Furthermore, in this case the final linear system of equations can be solved efficiently by iterative methods [6].

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MEAN RELATIVE ERROR (\%)


Figure 4: For a unit sphere using a $20 \times 20$ subdivision.

MEAN RELATIVE ERROR (\%)


MEAN RELATIVE ERROR (\%)



# Boundary element and Finite element methods for the coupled fluid-structure interaction problem 

by<br>S Amini \& P J Harris<br>Research Report MSOR-88-04

Boundary Element and Finite Element Methods for the Coupled Fluid-Structure Interaction Problem
$\mathbf{S}$ Amini and $\mathbf{P} \mathbf{J}$ Harris
Department of Mathematics and Statistics, Plymouth Polytechnic, Plymouth, UK

## 1. INTRODUCTION

In this paper we are concerned with the numerical solution of the dynamic fluid-structure interaction of a finite elastic structure immersed in an infinite homogeneous acoustic medium. The equations to be solved are of the same form as those in wave scattering, electrostatics, wave forces on structures and so on. In our area of interest, the field of underwater acoustics, it is desirable to determine the acoustic field, both radiated by a submerged vibrating structure and also scattered by a submerged elastic structure $6,9,10,14,18,19$. Here the impedance mismatch is much less than that between a structure and air ${ }^{11}$ and hence it is not feasible to assume perfect rigidity of the structure.

Let us denote the surface of an arbitrary shaped structure by $S$ where $S$ belongs to the class $C^{2}$. We denote by $D_{\text {_ }}$ the interior of $S$ and by $D_{+}$the unbounded domain exterior to $S$, whilst $n$ denotes the unit normal to $S$ directed into $D_{+}$Small amplitude acoustic waves propagate through an ideal homogeneous medium according to the linear wave equation

$$
\begin{equation*}
\nabla^{2} \Phi(p, t)-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \Phi(p, t)=0 \tag{1}
\end{equation*}
$$

where $c$ is the speed of sound in the medium and $\Phi(p, t)$ represents the excess acoustic pressure at a point $p$ and at time $t$. The sound pressure is related to the particle velocity through

$$
\begin{equation*}
\nabla \Phi(p, t)=-\rho_{f} \frac{\partial}{\partial t} \underline{v}(p, t) \tag{2}
\end{equation*}
$$

where $\rho_{f}$ represents the density of the fluid.
The motion of the elastic structure contained in $S$ is
governed by the linear equation of elasticity

$$
\begin{equation*}
\nabla \cdot \sigma(p, t)+\underline{E}(p, t)=\rho_{s} \frac{\partial^{2}}{\partial t^{2}} \underline{U}(p, t) \tag{3}
\end{equation*}
$$

where $\sigma(p, t)$ is the stress tensor, $E$ represents the external forces, $\rho_{s}$ is the structure density and $\underline{U}(p, t)$ is the particle displacement.

The natural numerical method for solving this coupled fluid-structure interaction problem, equations (1) and (3), is that of matching a finite element analysis of the vibrational motion of the structure, governed by equation (3), with a boundary element analysis of the exterior acoustic problem, governed by (1), at the structure-fluid interface $19,6,9,14$. In section 2 we discuss various aspects of integral equation formulations of the acoustic problem (1). For any surface S, it can be shown that the classical direct boundary integral formulation for the time harmonic solution of (1) via Green's second theorem suffers from non-uniqueness at a countable set of frequencies which depends upon $S$. We consider a formulation which is valid and well-conditioned for all frequencies. In section 3 we discuss the finite element analysis of the structural equation (3), whilst in section 4 the coupled problem is considered. A major objective in this paper is to investigate whether for the coupled problem it is still necessary to use integral formulations valid for all wavenumbers 9,14 , or can the elasticity of the vibrating structure modify the acoustic equations to an extent that the simple (classical) integral formulation is adequate and no longer suffers from ill-conditioning near characteristic wavenumbers $7,9,16$. These points will be investigated in section 5 by considering some scattering and radiation problems near their characteristic wavenumbers.

## 2. BOUNDARY INTEGRAL FOPMULATIONS OF THE ACOUSTIC PROBLEM

Assuming time harmonic time dependence of the form $e^{-\mathrm{i} \omega t}$, the linear wave equation (1) reduces to the Helmholtz equation

$$
\begin{equation*}
\left(\nabla^{2}+\mathrm{k}^{2}\right) \varphi(\mathrm{p})=0 \quad \mathrm{p} \in \mathrm{D}_{+} \tag{4}
\end{equation*}
$$

where $\Phi(p, t)=e^{-i \omega t} \varphi(p)$ is the excess pressure and $k=\omega / c$ is the acoustic wavenumber, $\omega$ being the angular frequency of the sound source. It now follows from (2) that the normal particle velocity $v(p)$ is related to $\varphi$ through

$$
\begin{equation*}
\frac{\partial \varphi}{\partial n}(p)=i \omega \rho_{f} v(p) \cdot \quad p \epsilon S \tag{5}
\end{equation*}
$$

For the existence of a unique solution of equation (4), as well as the Neumann boundary condition (5) (assuming $v(p)$ is known for the moment), we also need to impose on $\varphi$ the Sommerfeld radiation condition (or similar conditions)

$$
\begin{equation*}
\operatorname{limit}_{|p|=r \rightarrow \infty} r\left[\frac{\partial \varphi}{\partial r}(p)-i k \varphi(p)\right]=0 \tag{6}
\end{equation*}
$$

uniformly for all directions $p /|p|$. The radiation condition (6) ensures that the scattered and radiated waves are outgoing at infinity.

Various direct as well as indirect boundary integral formulations of (4)-(6) have been proposed in the literature; see Burton ${ }^{4}$ for an excellent survey of these formulations. Assuming that an incident wave $\varphi_{\text {inc }}$ on $S$ results in a scattered wave $\varphi_{\text {scatt }}$, it follows from Green's second theorem that the total acoustic pressure $\varphi=\varphi_{\text {inc }}+\varphi_{\text {Scatt }}$ satisfies the Helmholtz integral formula ${ }^{4}$
$\int_{S}\left\{\varphi(q) \frac{\partial G_{k}(p, q)}{\partial n_{q}}-G_{k}(p, q) \frac{\partial \varphi}{\partial n_{q}}(q)\right\} d S_{q}=\left\{\begin{array}{rr}\varphi(p)-\varphi_{i n c}(p) & p \in D_{+} \\ \frac{1}{2} \varphi(p)-\varphi_{i n c}(p) & p \in S \\ -\varphi_{i n c}(p) & p \in D_{-}\end{array}\right.$
where $G_{k}(p, q)$ is the fundamental solution or the free space Green's function for the Helmholtz equation (4) and is given by

$$
\begin{equation*}
G_{k}(p, q)=\frac{e^{i k|p-q|}}{4 \pi|p-q|} \tag{8}
\end{equation*}
$$

Equation $7(\mathrm{~b})$ is a second kind integral equation for $\varphi(\mathrm{p})$, $p \in S$, and is known as the Surface Helmholtz Equation (SHE) which, in operator notation, may be written as

$$
\begin{equation*}
\left[-\frac{1}{2} I+M_{k}\right] \varphi=\left[L_{k} \frac{\partial \varphi}{\partial n}\right](p)-\varphi_{i n c}(p) \quad p \in S \tag{9}
\end{equation*}
$$

where, similar to the treatment of the potential problem ${ }^{12}$, we define

$$
\begin{equation*}
\left(L_{k} \sigma\right)(p)=\int_{S} \sigma(q) G_{k}(p, q) d S_{q} \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(M_{k} \sigma\right)(p)=\int_{S} \sigma(q) \frac{\partial}{\partial n_{q}} G_{k}(p, q) d S_{q} \tag{11}
\end{equation*}
$$

which are known as the Helmholtz single and double layer
potentials respectively ${ }^{4}, 7$. Once both $\varphi$ and $\partial \varphi / \partial n$ are known on $S$ we may use (7a) to calculate $\varphi(p)$ for $p \in D_{+}$.

In our formulations below we also require the derivative of the operators $L_{k}$ and $M_{k}$ in the direction of $n_{p}$, the normal to $S$ at $p \in S$, which we denote by $M_{k}{ }^{T}$ and $N_{k}$ respectively ${ }^{4}, 7$. It can be shown that the operators $\mathrm{L}_{\mathrm{k}}, \mathrm{M}_{\mathrm{k}}, \mathrm{M}_{\mathrm{k}} \mathrm{T}: \mathrm{H}^{\mathrm{r}}(\mathrm{S}) \rightarrow \mathrm{H}^{\mathrm{r}}(\mathrm{S})$ are compact, where $\mathrm{H}^{\mathrm{r}}(\mathrm{S})$ denotes the Sobolev space of r times continuously differentiable functions in the sense of distribution ${ }^{7}$. Furthermore, the operator $N_{k}$, the derivative of the double layer potential, is a hypersingular operator and it can be shown that $\mathrm{N}_{\mathrm{k}}: \mathrm{H}^{\mathrm{r}}(\mathrm{S}) \rightarrow \mathrm{H}^{\mathrm{r}-1}(\mathrm{~S})$ is bounded but not compact.

Let us denote the spectrum of the operator $M_{k}$ by $\Sigma\left(M_{k}\right)$. For the second kind equation (9) the Fredholm-Riesz theory can be applied to establish existence of a unique solution provided $\frac{1}{2} \not \& \sum\left(M_{k}\right)$; that is provided $\left(-\frac{1}{2} I+M_{k}\right)^{-1}$ exist. Clearly, even if $\frac{1}{2} \neq \Sigma\left(M_{k}\right)$ but dist ( $\frac{1}{2}, \Sigma\left(M_{k}\right)$ ) is small the equation (9) will suffer from ill-conditioning and should not be used as a basis for numerical approximations. In fact in general it can be shown that for a countable set of values of the wavenumber $k$, say $I_{S}$, the operator $M_{k}$ will have $\frac{1}{2}$ as an eigenvalue. Furthermore it can be shown that asymptotically (as $k \rightarrow \infty$ ) the number of elements in $I_{S}$ less than $k$ is proportional to $k^{3}$. For example, taking $S$ to be a sphere of radius a we can show that ${ }^{1,13}$

$$
\begin{equation*}
I_{S}=\left(k \mid j_{n}(k a)=0, \quad n=0,1,2, \ldots\right) \tag{12}
\end{equation*}
$$

where $j_{n}$ denotes the spherical Bessel function.
Clearly therefore there is a real need for formulations which unlike (9) do not break down at characteristic frequencies and are well conditioned for all values of $k$.

### 2.1 Formulations valid for all wavenumbers

Over the past two decades many formulations have been proposed to overcome the breakdown of (9) at irregular
 have evaluated several such formulations. The specific direct boundary integral formulation that we present and employ here is due to Burton and Miller ${ }^{5}$ (and Burton ${ }^{4}$ in regularised form) which are closely related to the earlier indirect formulation of Panich ${ }^{15}$. On differentiating (9) in the direction of $n_{p}$, we obtain the first kind integral equation

$$
\begin{equation*}
\left(N_{k} \varphi\right)(p)=\left(\frac{1}{2} I+M_{k} T\right) \frac{\partial \varphi}{\partial n}(p)-\frac{\partial \rho_{i n c}}{\partial n}(p) \quad p \in S \tag{13}
\end{equation*}
$$

which itself suffers from non-uniqueness at a countable set of values of $k$, say $J_{S}$. However, it can be shown that the formulation obtained by taking a linear combination of (9) and (13) in the form

$$
\begin{equation*}
\left\{\left(-\frac{1}{2} \mathrm{I}+\mathrm{M}_{\mathrm{k}}\right)+\mathrm{i} \nu \mathrm{~N}_{\mathrm{k}}\right\} \varphi=\left\{\mathrm{L}_{\mathrm{k}}+\mathrm{i} \nu\left(\frac{1}{2} \mathrm{I}+\mathrm{M}_{\mathrm{k}} \mathrm{~T}\right)\right\} \frac{\partial \varphi}{\partial \mathrm{n}}-\left[\varphi_{\mathrm{inc}}+\mathrm{i} r \frac{\partial \varphi_{i n}}{\partial \mathrm{n}} \mathrm{nc}\right] \tag{14}
\end{equation*}
$$

with $\mathrm{i}^{2}=-1$ and $\nu$ an arbitrary positive parameter, is valid for all wavenumbers $k$ with $\operatorname{Re}(k)>0$. To avoid discretising the hypersingular operator $\mathrm{N}_{\mathrm{k}}$ in (14) we regularise equation (13) before coupling it with equation (9). It can be shown that 15,4

$$
\begin{equation*}
\left(L_{k} N_{k}\right) \sigma(p)=\left(M_{k}+\frac{1}{2} I\right)\left(M_{k}-\frac{1}{2} I\right) \sigma(p) \quad p \in S . \tag{15}
\end{equation*}
$$

By premultiplying (13) by $L_{o}$ (ie. $L_{k}$ with $k=0$ ) we obtain in place of (13) the equation

$$
\begin{equation*}
\left\{L_{o}\left(N_{k}-N_{o}\right)+M_{0}^{2}-\frac{1}{4} I\right) \varphi(p)=L_{o}\left(\frac{1}{2} I+M_{k} T\right) \frac{\partial \varphi}{\partial n}(p)-L_{o} \frac{\partial \varphi_{i n c}}{\partial n}(p) p \in S \tag{16}
\end{equation*}
$$

which when coupled with (9), similar to (14), yields the regularised integral equation

$$
\begin{gather*}
\left\{\left(-\frac{1}{2} I+M_{k}\right)+i \nu\left[L_{o}\left(N_{k}-N_{o}\right)+M_{0}^{2}-\frac{1}{4} I\right]\right) \varphi=\left[L_{k}+i \nu L_{o}\left(\frac{1}{2} I+M_{k} T\right)\right] \frac{\partial \rho}{\partial n} \\
-\left(\varphi_{i n c}+i \nu L_{o} \frac{\partial \varphi_{i n c}}{\partial n}\right) . \tag{17}
\end{gather*}
$$

Regarding the operator $\left(N_{k}-N_{o}\right)$ it can be shown that

$$
\begin{equation*}
\frac{\partial^{2}}{\partial n_{p} \partial n_{q}}\left(G_{k}(p, q)-G_{o}(p, q)\right)=0\left(|p-q|^{-1}\right)+0(1) \tag{18}
\end{equation*}
$$

and hence all the operators in (17) are compact and perhaps more amenable to accurate numerical evaluation ${ }^{2,3}$. Furthermore the work of Amini ${ }^{1}$, has shown that equation (17) is generally well conditioned and unlike the formulation (14) the choice of the coupling parameter $\nu=1$ is satisfactory for a large range of values of $k$ and surfaces $S$.

### 2.2 Discretisation of the Integral Equations

Here the integral equations (9) or (17) are discretised using the collocation method, based on piecewise linear approximation of both the surface $S$ as well as the funtions $\varphi$ and $\partial \rho(p) / \partial n=i \omega \rho_{f} v(p)$. For moderate number of degrees of freedom (collocation points), the most crucial and time consuming aspect of a 3-D boundary element analysis is that of accurate and efficient evaluation of the elements of the matrices approximating the integral operators ${ }^{3,2}$. Here the non-singular integrals (the off-diagonal elements) are evaluated using Gauss-Legendre rules whilst for the evaluation of the diagonal elements the simple transformation introduced by Duffy ${ }^{8}$, designed to deal with vertex singularities, is used. For more details on this, the choice of the coupling parameter $\nu$, and other issues such as efficient solution of
the linear equations for large number of collocation points we refer the reader to Amini ${ }^{1}$ and Amini and Harris ${ }^{2}$.

Using $n$ collocation points, $p_{1}, p_{2}, \ldots, p_{n} \in S$, discretising (9) or (17) we obtain a matrix equation in the form

$$
\begin{equation*}
\mathrm{A} \mathscr{e}_{\mathrm{n}}=i \omega \rho \rho_{f} B \underline{v}_{n}+\underline{c}_{n} \tag{19}
\end{equation*}
$$

where $\varphi_{\mathrm{n}}=\left[\varphi_{1}, \ldots, \varphi_{\mathrm{n}}\right]^{\mathrm{T}} \approx\left[\varphi\left(\mathrm{p}_{1}\right), \varphi\left(\mathrm{p}_{2}\right), \ldots, \varphi\left(\mathrm{p}_{\mathrm{n}}\right)\right]^{\mathrm{T}}$ and similarly $\left(\underline{v}_{n}\right)_{i} \approx v\left(p_{i}\right)$ with obvious definitions for the $n \times n$ matrices $A$ and $B$ and the $n$-vector $\varrho_{n}$.

In the next section we obtain a similar set of equations from the structural analysis which we then have to solve together with (19) to obtain approximations to $\varphi(p)$ and $v(p)$ for $\mathrm{p} \in \mathrm{S}$.

## 3. STRUCTURAL ANALYSIS

The discretisation of the linear equations of elasticity (3) by finite element methods is now a familiar technique ${ }^{20}$ and any of the commercially available packages may be used. The results presented in this paper are based on our own finite element program, as most commercial codes do not easily allow the extraction of relevant matrices which we need to use in conjunction with the boundary element equation (19). Assuming harmonic time dependence of the form $e^{-i \omega t}$ the finite element equations for the structural analysis, in the absence of structure damping, can be written in the form ${ }^{20}$

$$
\begin{equation*}
\left(K-\omega^{2} M\right) g_{m}=\underline{f}_{m}^{(k)}+\underline{f}_{m}^{(p)} \tag{20}
\end{equation*}
$$

where $K$ and $M$ represent the stiffness and mass matrices respectively and $g_{m}$ is the vector approximating the displacement $\underline{\underline{u}}(p)$ at nodal points where $\underline{U}(p, t)=e^{-i \omega t} \underline{u}(p)$. In equation (20) $f_{m}^{(k)}$ is a consistent load vector resulting from known applied forces and $\mathrm{f}_{\mathrm{m}}(\mathrm{p})$ is a consistent load vector representing the interaction forces generated by the acoustic fluid acting on the fluid-structure boundary. The interaction force vector $f_{m}(p)$ can be related through a coupling matrix $L$ (of dimensions $m \times n$ with $m \gg n$ ) to the nodal pressure distribution $\varphi_{n}$ where

$$
\begin{equation*}
\underline{f}_{\mathrm{m}}^{(\mathrm{p})}=-\mathrm{L} \varphi_{\mathrm{n}} . \tag{21}
\end{equation*}
$$

From (20) and (21) it follows that the structural equations relate the structural displacements $g_{m}$ to the acoustic pressure $\ell_{n}$ at the fluid-structure interface; see Wilton ${ }^{19}$ for more details.

## 4. COUPLED EQUATIONS OF MOTION

To obtain the solution of the interaction problem we need to combine the acoustic equation (17) with the structural equation (20). Various procedures for the coupling of these $t$ wo equations have been proposed ${ }^{19}$ and examined ${ }^{14}$. Here we use the preferred fluid variable method first given by Wilton ${ }^{19}$. In this approach the structural equation is substituted into the acoustic equation to yield a full complex-valued linear system of equations similar to (19) for the surface pressure $\varphi_{\mathrm{n}}$. Briefly, it follows from (20) and (21) that

$$
\begin{equation*}
g_{m}=\left(K-\omega^{2} M\right)^{-1}\left(f_{m}^{(k)}-L_{\ell_{n}}\right) \tag{22}
\end{equation*}
$$

Now, assuming that the fluid and structure meshes coincide at the boundary, we can relate the structural displacement to the normal particle velocity on $S$ in the form

$$
\begin{equation*}
\underline{v}_{\mathrm{n}}=-\mathrm{i} \omega L^{\prime} \mathbf{g}_{\mathrm{m}} \tag{23}
\end{equation*}
$$

where elements of the $n \times m$ matrix $L^{\prime}$ are simply the where elements of the outward normals at the surface nodes ${ }^{19}$. Finally, using the expression for $g_{m}$ from (22) into (23) and then substituting the resulting expression for $\underline{v}_{n}$ into the acoustic equation (19) we obtain

$$
\begin{equation*}
(\mathrm{A}+\mathrm{DL}) \varphi_{\mathrm{n}}=\mathrm{D} \mathrm{f}_{\mathrm{m}}^{(\mathrm{k})}+{\underline{c_{n}}} \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{D}=\omega^{2} \rho_{\mathrm{f}} \mathrm{BL}^{\prime}\left(\mathrm{K}-\omega^{2} \mathrm{M}\right)^{-1} . \tag{25}
\end{equation*}
$$

Once (24) is solved for $\varepsilon_{n}$, we obtain the displacement $g_{m}$ from (22), the normal particle velocity from (23) giving nodal values of $\partial \rho / \partial n$ through (5) and hence equation (7a) can be used to yield an approximation to $\varphi(p)$ for $p \in D_{+}$.

If the simpler integral equation (9) is discretised in place of (17) to yield the matrix equation (19) then for $k$ close to IS, the set of characteristic wavenumbers, the operator ( $-\frac{1}{2} 1+M_{k}$ ) and hence its matrix approximation $A$ in (20) will be ill-conditioned. Recently several authors $6,9,16$ have indicated that by coupling the structural equations (20), along the lines suggested here, with the discretisation of (9) they have observed no great loss of accuracy at characteristic wavenumbers. However, other authors ${ }^{19,14}$ suggest that in general for equation (24) to have a unique solution, it is necessary to use an integral formulation such as (14) or (17) which is valid for all wavenumbers.

In the next section we shall attempt to resolve this apparent controversy by testing both approaches on various
scattering and radiation problems.

## 5. NUMERICAL RESULTS AND DISCUSSION

Here we present some numerical results for a vibrating hollow sphere and a plane wave scattered by both a solid sphere and a solid cylinder. The fluid and structure parameters are as follows:

| Youngs modulus, | $E=2 \times 10^{12}$ dynes $/ \mathrm{cm}^{2}$ |
| :--- | :--- |
| Poissons ratio, | $\eta=0.25$ |
| Structure density, | $\rho_{\mathrm{s}}=8 \mathrm{grms} / \mathrm{cm}^{3}$ |
| Fluid density, | $\rho_{\mathrm{f}}=1 \mathrm{grm} / \mathrm{cm}^{3}$ |
| Speed of sound, | $\mathrm{c}=1.5 \times 10^{5} \mathrm{~cm} / \mathrm{sec}$ |

This data corresponds to a steel structure immersed in water.
The measure of error that we have chosen to use is the mean relative error, expressed as a percentage, given by

$$
\begin{equation*}
M R E=\frac{1}{n} \sum_{i=1}^{n} \frac{\left|\varphi\left(p_{j}\right)-\tilde{\varphi}\left(p_{i}\right)\right|}{\left|\varphi\left(p_{i}\right)\right|} \times 100 \tag{26}
\end{equation*}
$$

where $\tilde{\varphi}\left(p_{\mathfrak{i}}\right)$ represents the computed value and $\varphi\left(p_{\mathfrak{i}}\right)$ the exact value at the collocation point $p_{i}$.

In the Figures below we refer to the method using the Surface Helmholtz Equation (9) in conjunction with the structural equation (20) as the SHE method and the method using equation (17) with (20) as the Regularised method. Figure 1 shows a typical mesh for a domain which generates a solid sphere when rotated. We used the mesh in Figure 1 with 21 linear axisymmetric surface elements to approximate a solid sphere of radius 1 cm . Taking the midpoint of the surface elements as the collocation points for the discretisation of the boundary integral formulations (9) or (17), it is easy to see that these points lie on a sphere of radius $a=\cos (x / 42)$. It now follows from (12) that $k \approx 3.15$ (ie. ka $=x$ ) is a characteristic wavenumber for this surface. Figure 2 shows the surface pressure on a solid sphere of radius 1 cm which is scattering an incident plane wave at $k=3.15$. It can be seen that the computed results using the regularised formulation are very accurate whilst the results based on the equation (9) are in general inaccurate.

In Figure 3 we emphasise that the accuracy of the numerical results based on equation (9) deteriorates sharply whenever $k$ is in a neighbourhood of a characteristic wavenumber. This is demonstrated by computing the pressures on the surface of a vibrating hollow sphere, of inner radius 0.75 cm and outer radius 1 cm , for a range of values of $k$ around 3.15. The results are based on a discretisation with 21

## linear axisymmetric elements, similar to that in Figure 1.

Figure 4 shows the exact and the computed surface pressures for a solid cylinder of radius 1 cm and height 2 cm , scattering an incident plane wave at $k=3.957$ which is one of its characteristic wavenumbers. The results based on a subdivision with 20 linear axisymmetric surface elements clearly demonstrates the inadequacy of the SHE method.

## 6. CONCLUSIONS

We have shown that it is possible to obtain accurate results for the dynamic fluid-structure interaction problem by combining a finite element analysis of the structural equations with a boundary element analysis of the exterior acoustic field. Furthermore, our results overwhelmingly demonstrate the need for using a boundary element formulation which is valid for all wavenumbers if we are to overcome the ill-conditioning at characteristic frequencies.

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Figure 1: A typical finite element mesh for a solid sphere.


Figure 2: The surface pressure for a unit sphere scattering a plane wave at $k=3.15$.


Figure 3: The percentage mean relative error in the computed solutions for a hollow sphere near $k=3.15$.


Figure 4: The surface pressure for a solid cylinder scattering a plane wave at $k=3.957$.

## Iterative Solution of Boundary Element Equations

 for the Exterior Acoustic Problem.
## by

S Amini, Chen Ke and P J Harris

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# Iterative Solution of Boundary Element Equations 

## for the Exterior Acoustic Problem

S. Amini<br>Chen Ke<br>P.J. Harris<br>Department of Mathematics and Statistics, Plymouth Polytechnic, Plymouth, Devon, U.K.


#### Abstract

In this paper we study an efficient Boundary Element Method for the determination of the acoustic field around arbitrary-shaped finite structures immersed in an infinite homogeneous acoustic medium.

The direct boundary integral equation due to Burton and Miller is used to overcome the non-uniqueness and non-existence of solution associated with the classical boundary integral formulations of the exterior Helmholtz equation. The choice of the coupling parameter in the Burton and Miller formulation is discussed in order to minimise the condition number of the boundary integral equation.

Efficient numerical quadrature rules, based on suitable variable transformations, are studied for the evaluation of the singular and nearly-singular elements :n the discrete collocation equations.

The large full non-hermitian linear systems arising form the discretisation of the integral equations are then solved by the multi-grid method.

Numerical results are presented for the problem of acoustic radiation from several structures of practical interest.


## 1 Introduction

Since the early 1970 s many linear boundary value problems of mathematical physics and engineering, traditionally solved by domain methods, have been increasingly reformulated as integral equations over the boundary of the domain of interest $8,9,10,11,16]$ In cases where the domain of interest is of infinite extent boundary integral equations and their subsequent discretisation by finite element type methods, known as boundary element methods, offer an obvious advantage over the domain methods, as the dimensionality of the problem is effectively reduced by one.

In this paper we are concerned with the determination of the acoustic field radiated (or scattered) by an arbitrary shaped, finite, vibrating structure immersed in an infinite homogeneous acoustic medium. Such a problem is of considerable interest in many areas including underwater acoustics and aeronautics. In the field of underwater acoustics where the impedance mismatch between the structure and water is much less than that between a structure and air, the assumption of perfect rigidity of the structure is no longer valid. The approach considered in this paper can provide an impedance relationship at the structure surface, to be matched with a similar relationship from a finite element or boundary element analysis of the structure dynamics $\left[5,28,31^{\circ}\right.$.

In section 2 we consider the "direct" boundary integral formulation of the exterior Neumann problem for the Helmholtz equation. We discuss briefly the problem of existence and uniqueness of solutions of various formulations and also consider the conditioning of these equations. In section 3 we present an overview of our numerical methods based on piecewise constant collocation over linear triangular elements.

Since boundary element methods yield large systems of full non-hermitian linear equations, whose solution, for many practical problems, dominate the computational cost, iterative methods are naturally used for efficiency. In section $f$ we introduce a modified multi-grid method which allows us to employ these highly efficient iterative methods, even when dealing with "hyper-singular" operators, namely the derivative of the double layer potential. Finally in section 5 we present the results of our numerical methods for the acoustic radiation from a sphere, an ellipsiod and a cylinder.

## 2 Formulation

Let $S$ be a closed, bounded, piecewise smooth surface is $\mathcal{R}^{3}$. We denote the interior and exterior of $S$ by $D_{-}$and $D_{+}$respectively: The propagation of small amplitude acoustic waves through a homogeneous medium is modelled by the linear wave equation

$$
\begin{equation*}
\Gamma^{2} \Phi(p, t)-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \Phi(p, t)=0 \quad p \in D_{+} \tag{2.1}
\end{equation*}
$$

where the scalar complex-valued function $\Phi(p, t)$ may be the acoustic pressure or the velocity potential at time $t$ and $c$ is the speed of sound in the medium. Sound pressure is related to particle velocity through

$$
\begin{equation*}
\Gamma \Phi(p, t)=-\rho \frac{\partial v}{\partial t}(p, t) \tag{2.2}
\end{equation*}
$$

where $\rho$ denotes the density of the medium. If we assume a harmonic time dependence of the form $e^{-i \omega t}$ : where $\omega$ is the angular frequency of the sound source, equation (2.1) becomes the Helmholtz (reduced wave) equation

$$
\begin{equation*}
\left(\Gamma^{2}+k^{2}\right) \phi(p)=0 \quad p \in D_{+} \tag{2.3}
\end{equation*}
$$

where $\Phi(p, t)=e^{-i \nu t} \phi(p)$ and $k=\omega / c$ is the acoustic wavenumber. It now follows from (2.2) that

$$
\begin{equation*}
\frac{\partial \phi(p)}{\partial n}=i \omega \rho v_{n}(p) \quad p \in S \tag{2.4}
\end{equation*}
$$

where $n$ denotes the unit outward normal to $S$ and $v_{n}(p)$ denotes the normal particle velocity, which we assume is known. To ensure the existence of a unique solution to (2.3) with the Neumann boundary condition (2.4) on the finite boundary, we also need to impose on $\phi$ the Sommerfeld radiation condition (or a similar condition)

$$
\begin{equation*}
\lim _{r_{p} \rightarrow \infty} r_{p}\left(\frac{\partial \phi(p)}{\partial r_{p}}-i k \dot{(p)}:=0\right. \tag{?.5}
\end{equation*}
$$

where $p=\left(r_{p}, \theta_{p}, \phi_{p}\right)=R^{3}$.
Here we consider a direct boundary integral formulation of equations (2.3). It follows from Green's second identity that $\phi(p)$ a radiating wave function, (i.e. one satisfying (2.3) and (2.5)), satisfies the so called Helmholtz integral formulae for the exterior region, given by

$$
\int_{S}\left\{\phi(q) \frac{\partial G_{k}}{\partial n_{q}}(p, q) \cdots G_{k}(p, q) \frac{\partial \phi}{\partial n_{q}}(q)\right\} d S_{q}= \begin{cases}\phi(p) & p \in D_{+}  \tag{2.6}\\ \frac{\mathrm{a}(p)}{4 \pi} \phi(p) & p \in S \\ 0 & p \in D_{-}\end{cases}
$$

where $\alpha(p)$ is the solid angle subtended by $D_{+}$at $p \in S$ (for $p$ on a smooth part of the surface $\alpha(p) \equiv 2 \pi)$, and

$$
\begin{equation*}
G_{k}(p, q)=\frac{e^{i k r}}{4 \pi r}, \quad r=|p-q| \tag{2.7}
\end{equation*}
$$

is the free-space Green's function, or the fundamental solution for (2.3). Assuming for the moment that S is a smooth surface, the equation (2.6) for $p \equiv S$ : known as the surface Helmholtz equation, can be written in the form

$$
\begin{equation*}
\left(-\frac{1}{2} I-M_{k}\right) \phi(p)=\left(L_{k} \frac{\partial \phi}{\partial n}\right)(p) \quad p \in S \tag{2.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(L_{k} \sigma\right)(p)=\int_{S} \sigma(q) G_{k}(p, q) d S_{q} \tag{2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(M M_{k} \sigma\right)(p)=\int_{S} \sigma(q) \frac{\partial G_{k}}{\partial n_{q}}(p, q) d S_{q} \tag{2.10}
\end{equation*}
$$

are the single layer and double layer Helmholtz potentials respectively. If the second kind integral equation (2.8) could be solved for $\phi(p), p \in S$, then it follows from (2.6) for $p \in D_{+}$ that the solution of (2.3) is given by

$$
\begin{equation*}
\phi(p)=\left(M_{k} \phi\right)(p)-\left(L_{k} \frac{\partial \phi}{\partial n}\right)(p) \quad p \in D_{+} \tag{2.11}
\end{equation*}
$$

which involves a simple integration.
It is well known, however, that for a countable set, $I_{1}(S)$, of values of $k$ referred to as 'irregular frequencies', the operator $-\frac{1}{2} I-M_{k}$ is singular. Many formulations valid for all wavenumbers have been proposed since 1965, viz $10,17,21,24 \mid$. Here we employ one such formulation, due to Burton and Miller :10]. Briefly, differentiating (2.8) in the direction of $n_{p}$, the outward normal at $p$, we obtain

$$
\begin{equation*}
\left(. V_{k} \phi\right)(p)=\left(\frac{1}{2} I+M_{k}^{T}\right) \frac{\partial \phi}{\partial n} \tag{2.12}
\end{equation*}
$$

where $N_{k}=\frac{\partial}{\partial n} M_{k}$ and $M_{k}^{T}=\frac{\partial}{\partial n} L_{k}$. The operator $N_{k}$ is also singular at a countable set, $I_{2}(S)$, of values of $k$. If we couple equations (2.8) and (2.12), we obtain

$$
\begin{equation*}
\left\{\left(-\frac{1}{2} I-M I_{k}\right)+i \eta N_{k}\right\} \phi(p)=\left\{L_{k}+i \eta\left(\frac{1}{2} I+M I_{k}^{T}\right)\right\} \frac{\partial \phi}{\partial n} \tag{2.13}
\end{equation*}
$$

with $i^{2}=-1$, which can be shown to possess a unique solution for all wavenumbers $k$; provided the coupling parameter $\eta$ satisfies $\operatorname{Re}(k) \eta>0$.

The operator $N_{k}$, the derivative of the double layer potential

$$
\begin{equation*}
\left(N_{k} \sigma\right)(p)=\frac{\partial}{\partial n_{p}} \int_{S} \sigma(q) \frac{\partial}{\partial n_{q}} G_{k}(p, q) d S_{q} \tag{2.14}
\end{equation*}
$$

is a hyper-singular operator. In general some mathematical analysis should be carried out before employing a numerical quadrature rule to approximate this operator : $19: 30$ ]. Here we adopt the formulation of $V_{k}$ used in [19] which follows from the pointwise variational principle introduced in [29]

$$
\begin{equation*}
\int_{S} \sigma(q) \frac{\partial^{2} G_{k}}{\partial n_{p} \partial n_{q}}(p, q) d S_{q}=k^{2} \sigma(p) \int_{S} n_{p} \cdot n_{q} G_{k}(p, q) d S_{q}+\int_{S}\{\sigma(q)-\sigma(p)\} \frac{\partial^{2} G_{k}}{\partial n_{p} \partial n_{q}}(p, q) d S_{q} \tag{2.15}
\end{equation*}
$$

The operators on the right hand side of (2.15) are now more amenable to numerical approximation; see section 3 .

It is possible to regularise the operator $N_{k}$ to yield a modified equation (2.10) before coupling it with (2.8). It can be shown that $[9,21]$

$$
\begin{equation*}
\left(L_{k} V_{k}\right) \sigma(p)=\left(M_{k}+\frac{1}{2} I\right)\left(M I_{k}-\frac{1}{2} I\right) \sigma(p) \quad p \subseteq S \tag{2.16}
\end{equation*}
$$

By premultiplying equation (2.12) by $L_{0}$, (i.e. $L_{k}$ with $k=0$ ) we obtain

$$
\begin{equation*}
\left.\left\{L_{0}\left(N_{k}-N_{0}\right)-M_{0}^{2}-\frac{1}{4} I\right)\right\} \phi(p)=L_{0}\left(\frac{1}{2} I+M_{k}^{T}\right)^{\frac{\partial \phi}{\partial n}} \quad p \in S \tag{2.17}
\end{equation*}
$$

which when coupled with (2.8), similar to (2.13), yields the regularised Burton and Miller formulation

$$
\begin{equation*}
\left.\left\{\left(-\frac{1}{2} I+M M_{k}\right)-i \eta L_{0}\left(N_{k}-N_{0}\right)+M M_{0}^{2}-\frac{1}{4} I\right\}\right\} \phi=\left[\left.L_{k}+i \eta L_{0}\left(\frac{1}{2} I+M M_{k}^{T}\right) \right\rvert\, \frac{\partial \phi}{\partial n}\right. \tag{2.18}
\end{equation*}
$$

All the operators in (2.18) are compact and in general more amenable for accurate discretisations.

We now have a choice of 2 boundary integral formulations for (2.3)-(2.5) which are valid for all wavenumbers. Before attenpting to discretise them let us discuss briefly the choice of the coupling parameter $\eta$. Ideally one should choose $\eta=\eta(S ; k)$ so as minimise the condition number of the integral operators in (2.13) and (2.18). For a general surface $S$, analytical evaluation of the condition number of an integral operator is not possible. However to gain some insight, the case of $S$ being a sphere of diameter $2 a$ has been considered 11,18 , where it was found that the choice of $\eta=\frac{1}{k a}$ is an "almost optimal" choice for (2.13) whilst $\eta=\frac{1}{a}$
was found to be satisfactory [1] for a large range of values of $k$ for (2.18), as in this case the equations are naturally well-conditioned. For a non-spherical surface $S$, which is not too elongated, it can be argued that $\eta=\frac{2}{k d}$ and $\frac{2}{d}$ may be chosen in (2.13) and (2.18) respectively, where $d=\max |p-q|, p, q \in S$ is the "diameter" of $S$.

## 3 Numerical Methods

The numerical method used in this paper for discretising (2.13) and (2.18) is the collocation method based on piecewise constant approximations over linear triangular elements. Consider the second kind integral equation

$$
\begin{equation*}
\phi(p)-\int_{S} K(p: q) \phi(q) d S_{q}=f(p) \quad p \in S \tag{3.1}
\end{equation*}
$$

where $S$ is a piecewise smooth surface in $\mathcal{R}^{3}$. To be precise $S=S_{1}-S_{2} \cup \cdots-S_{I}$, where each $S_{i}$ is a smooth surface in $\mathcal{R}^{3}$. We then divide (and approximate) each $S_{i}$ into a number of flat triangles $S_{i j}, j=1,2, \ldots, n_{i}$ having their corners on $S_{i}$. Hence

$$
\begin{equation*}
S \approx \bigcup_{i=1}^{l} \bigcup_{j=1}^{n_{2}} S_{i j}=\bigcup_{i=1}^{n} \Delta_{i}=S_{n} \tag{3.2}
\end{equation*}
$$

where $n=\sum_{i=1}^{n} n_{i}$ and the $\Delta_{i}^{\prime} s$ are simply a renumbering of the $S_{i j}^{\prime} s$. Assuming a piecewise constant approximation for $\phi(p)$, the equation (3.1) can be approximated by

$$
\begin{equation*}
\phi_{n}(p)-\sum_{j=1}^{n} \phi_{j} \int_{\dot{\Delta}} k(p, q) d S_{q}=f(p) \quad p \in S_{n} \tag{3.3}
\end{equation*}
$$

It is convenient to write (3.3) in the operator form

$$
\begin{equation*}
\left(I \cdots \kappa_{n}\right) \phi_{n}=f \tag{3.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(\kappa_{n} \sigma\right)(p)=\sum_{j=1}^{n} \sigma\left(p_{j}\right) \int_{\Delta_{j}} K\left(\rho_{;} q\right) d S_{q} \tag{3.5}
\end{equation*}
$$

where $\phi_{n}(p) \approx \dot{\varphi}(p)$ and $\phi_{j} \approx \phi(p), p \approx \hat{\Delta}_{j}$. Now choose n collocation points $p_{i}, i=1,2, \ldots, n$ where each $p_{i}$ is the centroid of $\hat{\Delta}_{i}$. By insisting that $\phi_{n}\left(p_{i}\right)=\phi_{i}$, we obtain the usual piecewise constant collocation equations

$$
\begin{equation*}
\phi_{i}-\sum_{j=1}^{n} \phi_{j} \int_{\triangle,} K\left(p_{i}, q\right) d S_{q}=f\left(p_{i}\right) \quad i=1,2, \ldots, n \tag{3.6}
\end{equation*}
$$

which is a full system of linear equations to be solved for $\phi=\left[\phi_{1}, \phi_{2}, \ldots, \phi_{n}\right]^{T}$. The equation (3.3) is of considerable importance in our analysis. Firstly, once (3.6) has been solved to yield $\phi_{1}, \phi_{2}, \ldots, \phi_{n}$, we may use (3.3) to obtain the iterated collocation solution

$$
\begin{equation*}
\phi_{n}(p)=\sum_{j=1}^{n} \phi_{j} \int_{\Delta}, K(p, q) d S_{q}+f(p) \quad p \in S_{n} \tag{3.7}
\end{equation*}
$$

which is, in general, a continuous function and known to be globally a more accurate approximation to $\phi(p)$ than the collocation solution (though $\phi_{n}\left(p_{i}\right)=\phi_{i}$ ) :27]. Secondly (3.7), similar to the Nystrom extension, allows us to interpolate (and smooth at the same time) values at non-collocation points, which is to be exploited in the multi-grid solution of (3.6).

The computational complexity of a boundary element method can be divided into 2 major parts, (i) the set up stage, that of evaluating the elements of the $n \times n$ matrix in (3.6), and (ii) the solution stage, that of solving the system (3.6). In general the set up time is of the form $c_{1} n^{2}$ where $c_{1}$ is a very large positive constant, as the kernel $K(p, q)$ is rather costly to evaluate. Efficiency of the integration rules play a major role in reducing this cost. The solution time by direct methods is of the form $c_{2} n^{3}$ where $c_{2}$ is in general a very small number compared with $c_{1}$. Therefore, for small values of $n$, the total cost is dominated by the set-up time. For practical 3-D problems, the value of $n$ could be large and therefore the solution time by direct methods would dominate the computation time. In cases where the integral operators are compact, having desirable smoothing and spectral properties (zero as the only point of accumulation of their countable set of eigenvalues [20] , their discrete approximations possess similar properties which can be exploited to obtain an efficient iterative scheme such as the conjugate gradient method $\{4,3 ; 23$ and the multi-grid method $[3,22,25,26]$. Such schemes can yield solutions to (3.6) to within the level of the discretisation error in $c_{3} n^{2}$ operations, where $c_{3}$ is a moderately small constant. In this case therefore the substantial part of the computation is in the set-up stage. Much recent work has been carried out to reduce this time -13,23!

Let us briefly consider the numerical integration of the collocation equations. The integrals in (3.6) are of the form

$$
\begin{equation*}
K_{i j}=\int_{\Delta,} K\left(p_{i}, q\right) d S_{q}=\mu_{j} \int_{\Delta} K\left(p_{i}(u, v), q(u, v)\right) d u d v \tag{3.8}
\end{equation*}
$$

where $\Delta$ is the reference right angled unit triangle in the $(u, v)$ plane with corners at $(u, v)=$ $(0,1),(0,0)$ and $(1,0)$. The constant $\mu_{j}$ is the Jacobian of the linear transformation taking $\dot{A}_{j}$
to $\Delta$, which is twice the area of $\Delta_{j}$. Here the integrals are non-singular if $i \neq j$ and singular if $i=j$. In the former case however, the closer the collocation point is to the integration region $\Delta_{j}$, the more peaked, ("nearly singular") the integrand $K\left(p_{i}, q\right)$ will be for $q$ near to $p_{i}$. To retain a given level of accuracy in numerical integration, more accurate rules should be employed for "near-singular" integrals, see 2,7 .

Here, away form the singularity, we use the 3 -point rule

$$
\begin{equation*}
\int_{\Delta} f(u, v) d u d v \approx \frac{1}{6}\left[f\left(\frac{1}{2}, 0\right)+f\left(\frac{1}{2}, \frac{1}{2}\right)+f\left(0, \frac{1}{2}\right)\right. \tag{3.9}
\end{equation*}
$$

with degree of precision 2, whilst for the near-singular integrals we employ the product 3 point Gauss rule (on the triangle). For the evaluation of the singular integrals $K_{i i}$, similar to Atkinson [7], we divide the region of integration $\dot{\Delta}_{j}$ into three smaller triangles by joining the collocation point $p_{i}$ to each corner of $\Delta_{j}$. Each smaller triangle is then transformed to the reference triangle with the singular point $p_{i}$ transformed to the $(0,0)$ corner in each case. We are now faced with the evaluation of three integrals of the form

$$
\begin{equation*}
I=\int_{\Delta} f(u, v) d u d v \tag{3.10}
\end{equation*}
$$

with a singularity at $(0,0)$. Introducing the change of variable $[7,12]$

$$
\begin{equation*}
u=(1-s) t, \quad v=s t \quad 0 \leq s, t \leq 1 \tag{3.11}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
I=\int_{0}^{1} \int_{0}^{1} t f((1-s) t: s t) d s d t \tag{3.12}
\end{equation*}
$$

The integrand in (3.12) can be shown to be no longer singular in the case of potential type Green functions. The integral (3.12) can be adequately approximated by a product m-point Gauss rule. Here we choose $m=5$. We point out here that other transformations such as polar coordinate or error function transformations can be used ? to sield similar results.

Finally we note in passing that in our case of piecewise constant approximation, the second integrand in (2.15) is everywhere non-singular as $\phi(q)-\phi(p)$ is appoximated by 0 for $p, q \in \hat{\Delta}_{i}$. However for higher order approximations, special quadrature rules need to be employed to deal with Cauchy type singularities.

## 4 Iterative Solution of Linear Systems

The numerical solution of the full linear discretized system by iterative techniques has been previously studied for 2D acoustic problems ( 3$],[4]$ ) and potential problems ([23]). In [3], both the conjugate gradient method and the multi-grid type methods are applied to equations with compact operators. In such cases the numerical analysis can be carried out for these methods using the well-known theory of compact operators : $6^{\circ}$. From our experience we concluded that the two grid methods are more efficient than the conjugate gradient method and nearly as efficient as multigrid methods, yet being much easier than the latter to implement; (see [3; and [22]). For a second kind integral equation

$$
\begin{equation*}
(\lambda I-\kappa) u=f \tag{4.1}
\end{equation*}
$$

with the compact operator $\kappa$, discretizing it as in $\S 3$ on a coarse grid $G[n]$ and a fine grid $\left.G^{\prime} m\right]$ with $n$ and $m$ subdivisions respectively, we obtain two discrete collocation equations similar to (3.6) in the form

$$
\begin{equation*}
\left(\lambda I+K_{l}\right) \underline{u}_{l}=\underline{f}_{l}, \quad \text { with } l=n, m . \tag{4.2}
\end{equation*}
$$

The idea is to solve the equation (4.2) with $l=m$, using the information from the solution of (4.2) with $l=n$. To implement the $t$ wo grid method, we require the following quantities :

$$
\begin{aligned}
& K_{m}: m \times m \text { matrix with }\left(K_{m}\right)_{i j}=\int_{\Delta_{m},} K\left(p_{i}, q\right) d S_{q} ; \\
& K_{m n}: m \times n \text { matrix with }\left(K_{m n}\right)_{i j^{\prime}}=\int_{\Delta_{n},} K\left(p_{i}, q\right) d S_{q} ; \\
& K_{n}: n \times n \text { matrix with }\left(K_{n}\right)_{i^{\prime} j^{\prime}}=\int_{\Delta_{n},} K\left(p_{i^{\prime}}, q\right) d S_{q} ;
\end{aligned}
$$

and vectors $f_{m}, f_{n}$ with $\left(f_{m}\right)_{j}=f\left(p_{j}\right)$ and $\left(f_{n}\right)_{j^{\prime}}=f\left(p_{j^{\prime}}\right)$, for $i, j=1, \cdots, m$ and $i^{\prime}, j^{\prime}=$ $1, \cdots, n$. Then starting from an initial guess $u_{m}=0$, we can describe the two-grid algorithm as follows(writing $x$ for a vector $\underline{x}$ from here onwards):

1) Set $r_{m}=f_{m}, u_{m}=0$ and go to step 3);
2) Find the residual on $G \|:=r_{m}=f_{m}-\lambda u_{m}-K_{m} u_{m}$;
3) Perform smoothing on $G!m \mid, u_{m}=u_{m}-r_{m} / \lambda$ and $r_{m}=-K_{m} r_{m} / \lambda$; (optional step)
4) Restrict the residual $r_{m}$ to $G\left[n, r_{n}=R_{m}^{n} r_{m}\right.$ :
5) Solve exactly on $G_{i}^{\prime} n \mid,\left(\lambda I \div K_{n}\right) v_{n}=r_{n}$;
6) Interpolate $v_{n}$ to obtain $v_{m}$ on $G|m|, v_{m}=\left(r_{m}-K_{m n} v_{n}\right) / \lambda$;
7) Add on the correction $v_{m}$ to $u_{m}, u_{m}=u_{m}+v_{m}$;
8) If $!v_{m}!\leq T O L$ exit with solution in $u_{m}$; otherwise go to step 1).

Here $R_{m}^{n}$ is a restriction operator which can be taken to be the so-called injection operator, provided that we choose the nodes of $G\left[n_{i}^{\prime}\right.$ to coincide with those of $\left.G: m\right]$. Note that step 4) is in general carried out by a direct solver where the $L U$ decomposition is kept for use in each iteration. The algorithm given above is similar to Atkinson's method $I I$ ( based on the residual correction principle, see $[3]$ ). Missing out the smoothing step 2 ) will give rise to the corresponding method $I$ of Atkinson. In general, method I may be preferable if the operator $\kappa$ is sufficiently smooth, since each iteration of this version requires roughly half the number of operations compared with method $I I$. For example, it is easy to show that the convergence of the iterative scheme above is governed by the quantity ( $\left[_{6} 6^{\circ}\right.$ )

$$
\begin{equation*}
\mathcal{B}=\left\|\left(\lambda I+\kappa_{n}\right)^{-1}\left(\kappa_{m}-\kappa_{n}\right) \kappa_{m}\right\| \tag{4.3}
\end{equation*}
$$

If $\kappa$ is a compact operator, it can shown that $\beta \rightarrow 0$ as $n \rightarrow \infty$.
Recall from $\S 2$ that the regularised Burton and Miller's formulation (2.18) involves compact operators only and therefore the above algorithm can be readily applied to it. But the major disadvantage of the regularised formulation, from the practical point of view, is the excessive computational cost and complexity of setting up the various discrete operators $15 \%$. On the other hand, due to the presence of the non-compact operator $N_{k}$, the direct application of the above iterative method to the formulation (2.13) simply does not work. As far as we are aware, no efficient techniques have been developed to deal with the discretization of (2.13) and therefore the direct solution has been used (19.).

The equation (2.13) is of the form (4.1) with $\kappa=M+V$, where the subscripts $k$ have been dropped and $N$ denotes in $N$ for ease of notation. Roughly speaking, the problem here is that because of the non-compact operator $N$ : the quantity $\|\left(\kappa_{m}-\kappa_{n}\right) \kappa_{m}$ can not be shown to be small for all $m>n$ and $n$ sufficiently large; a crucial factor in the convergence analysis of the 2 -grid methods. We shall show that by modifying steps 2 ) and 5) of the procedure above, to reduce the non-smoothing effect of $N$, we are able to restore the fast convergence of the algorithm for (2.13).

To explain our modifications, which are in a sense similar to the operator splitting idea of 26 . and [25], let us rewrite the step 2) of the algorithm explicitly, in a form suitable for analysis

$$
\begin{equation*}
\bar{u}_{m}=u_{m}+r_{m} / \lambda \text { and } \bar{r}_{m}=-K_{m} r_{m} / \lambda, \tag{4.4}
\end{equation*}
$$

or eqivalently

$$
\bar{u}_{m}=\left(f_{m}+K_{m} u_{m}\right) / \lambda \text { and } \bar{r}_{m}=f_{m}-\lambda \bar{u}_{m}-K_{m} \bar{u}_{m}
$$

Concentrating on the second term in (4.4), the residual smoothing step, we have (with $\lambda=$ $-1!2$ and $\left.K_{m}=M_{m}+N_{m}\right)$

$$
\begin{align*}
\bar{r}_{m} & =2\left(M M_{m}+N_{m}\right) r_{m} \\
& =2\left(M M_{m}+\bar{N}_{m}-D_{m}\right) r_{m} \tag{4.5}
\end{align*}
$$

where $D_{m}$ is the diagonal matrix of diagonal elements of $N_{m}$ and $\bar{V}_{m}=N_{m}-D_{m}$. We propose to modify the smoothing step 2) given by (4.5) in the form

$$
\begin{equation*}
\left(I-2 D_{m}\right) \dot{r}_{m}=2\left(M_{m}+\bar{N}_{m}\right) r_{m} \tag{4.6}
\end{equation*}
$$

or

$$
\dot{r}_{m}=2\left(I-2 D_{m}\right)^{-1}\left(M_{m}+\bar{N}_{m}\right) r_{m}
$$

Observe that this step increases the computational cost of the original step 2) by $m$ divisions only. Similar splitting of the operator $N_{m}$ should be carried out for the interpolation step 5). We shall see in the next section that the above modification of the 2 -grid method, based on diagonal splitting of the $N_{k}$ operator, results in the convergence of the scheme as applied to the discretisation of (2.13).

## 5 Numerical Experiments

Here we present some numerical results for the iterative solution of the boundary integral equations for the exterior acoustic problem. We consider the problem of acoustic radiation from three structures: (S1) a unit sphere. (S2) an axisymmetric ellipsoid ( $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{a^{2}}+\frac{z^{2}}{b^{2}}=1$ ) with the ratio of major to minor axis $b / a=2$; (choosing $a=3 / 4, b=6 / 4$ so that the average dimension $(a+a+b) / 3=1)$. $(S 3)$ a right circular cylinder with height $=2$ and base radius $=1$.

In each case the surface of the structure is approximated by $n=112$ linear triangular elements. Each triangle is then subdivided into 4 smaller triangles by joining the mid-points of its sides, giving $m=4 n=448$. The test problem considered in all cases is the radiation problem equivalent to that generated by placing 2 point sources inside the structure at locations $q_{1}=(0.0,0.0,0.5)$ and $q_{2}=(0.25,0.25,0.25)$ with the strengths $A_{1}=2+3 i$ and $A_{2}=4-i$ respectively: $\{2,1+!$. The field generated is then given by

$$
\begin{equation*}
\phi(p)=A_{1} \frac{e^{i k\left|p-q_{1}\right|}}{\left|p-q_{1}\right|}+A_{2} \frac{e^{i k ; p-q_{2}!}}{\left|p-q_{2}\right|} \tag{5.1}
\end{equation*}
$$

from which $\frac{\partial \phi}{\partial n}(p)$, for $p \in S$ can be calculated. The measure of the error chosen here is the mean relative error (MRE) defined as

$$
\begin{equation*}
M R E=\frac{1}{m} \sum_{i=1}^{m} \frac{\left|\phi\left(p_{i}\right)-\dot{\phi}\left(p_{i}\right)\right|}{\dot{\phi\left(p_{i}\right) \mid}} \tag{5.2}
\end{equation*}
$$

where $p_{i}$ 's are the collocation points and $\dot{\phi}\left(p_{i}\right)$ represents the computed approximation to $\phi\left(p_{i}\right)$.

In table 1 we present the results of discretising the regularised equation (2.18) for the case of structures (S1) and (S3) near their first characteristic wavenumbers $\mathrm{k}=3.142$ and $\mathrm{k}=3.04$ respectively. The coupling parameter $\eta \equiv 1$ is chosen in both cases. The results are for (method I) the iterative algorithm in section 4 without the smoothing step 2 . We observe that the convergence of the 2 -grid method is rapid. Indeed, for the iterative solution, the total compuational time is greatly reduced. We point out here that in table 1 the difference in set-up time is due to the fact that for the direct solution, the matrix multiplications corresponding to product operators $L_{0}\left(N_{k}-N_{0}\right), M_{0}^{2}$, and $L_{0}\left(\frac{1}{2} I+M M_{k}^{T}\right)$ have to be carried out explicitly, whilst in the latter case this is not required (since $\underline{y}=A B \underline{x}$ can be carried out as $\underline{\underline{y}}=B \underline{x}$ followed by $\underline{y}=. \operatorname{lv}$ ). (All programs were run on Prime 750 at Plymouth Polytechnic and the value of $T() L$ in our iterative algorithm was set to $10^{-3}$ ).

We shall now concentrate on the computationally more attractive formulation (2.13) where the suggested modification to the 2 -grid algorithm is essential for its convergence. We also found the smoothing step 2 to improve the convergence rate of the scheme. The results in tables 2-6 below are therefore for the modified algorithm with smoothing step 2 included. In tables 2 , 4 and 6 the set-up time is around 600 CPU seconds and the direct solution time is around 920 CPU seconds. We observe that in all cases the modified 2 -grid algorithm substantially reduces

Table 1: Regularised Burton and Miller, equation (2.18).

|  | Direct Method |  | Iterative Method |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | set up | solution | set up | number of | solution | MRE |
| Structure | time | time | time | iterations | time | $(\%)$ |
| S1 (k=3.142) | 7000 | 920 | 940 | 4 | 330 | 3.8 |
| S3 (k=3.04) | 7000 | 920 | 940 | 4 | 330 | 4.0 |

Table 2: (S1) Sphere, Formulation (2.13)

|  | number of | 2-Grid |  |
| :---: | :---: | :---: | :---: |
| k | Iterations | Solution Time | MRE (\%) |
| 3.142 | 3 | 64 | 4.2 |
| 5 | 6 | 112 | 5.4 |

the solution time. In tables 3 and 5 we present results of the sound-field calculations in the exterior domain using (2.11). It is well known that the exterior field solution are in general more accurate than the surface values, as a result of the integration (which is a smoothing operator).

## 6 Conclusion

Efficient solution of the discrete boundary element equations are of paramount importance in practical applications. We have shown that the modified 2-grid method can be efficient for the non-compact integral equations arising in the field of acoustics. (The superiority of the iterative methods over direct solution methods can only be fully appreciated when the number of equations is large).

Table 3: (S1) Sphere. Error in the exterior field.

| k | \% Error <br> at $(0,0,2)$ | \% Error <br> at $(0,0,3)$ | $\%$ Error <br> at $(0,0,5)$ |
| :---: | :---: | :---: | :---: |
| 3.142 | 2.1 | 2.2 | 2.2 |
| 5 | 2.0 | 2.2 | 2.2 |

Table 4: (S2) Ellipsoid. Formulation (2.13)

|  | number of | 2-Grid |  |
| :---: | :---: | :---: | :---: |
| $k$ | Iterations | Solution Time | $\operatorname{MRE}(\%)$ |
| 3 | 3 | 66 | 5.9 |
| 5 | 4 | 82 | 6.2 |

Table 5: (S2) Ellipsoid. Error in the exterior field.

|  | \% Error <br> at $(0,0,2)$ | \% Error <br> at $(0,0,3)$ | $\%$ Error <br> at $(0,0,5)$ |
| :---: | :---: | :---: | :---: |
| 3 | 2.4 | 2.4 | 2.4 |
| 5 | 4.3 | 3.9 | 3.5 |

Table 6: (S3) C.vlinder. Furmulation (2.13)

|  | number of <br> k | 2-Grid <br> Iterations | Solution Time |
| :---: | :---: | :---: | :---: | MRE (\%) | 3.04 | 4 | 82 |
| :---: | :---: | :---: |
| 5 | 6 | 111 |

If the non-smoothness of the integral operator is due to the non-smoothness of the surface $S$, then the iterative techniques will need to be suitably modified $[25,26]$. The numerical analysis of iterative solution of boundary integral equations with non-compact operators is an area of much current interest by many researchers.

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