# Flow of power law fluids with application to oil drilling 

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# FLOW OF POWER LAW FLUIDS 

 WITH APPLICATION TO OIL DRILLING
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TO MY PARENTS

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## FLOW OF POWER LAW FLUIDS

WITH APPLICATION TO OII DRILLING

E.J. CARTER

## ABSTRACT

The thesis is concerned with a theoretical study of the flow behaviour of inelastic power law fluids in two different types of flow situation. These are:

1. The creeping motion of a sphere moving through an expanse of liquid.
2. The combined steady and oscillatory flow of a liquid through a straight tube of circular cross section.

The first part of the work is devoted to the prediction of the drag correction factor for a sphere falling slowly through a bounded inelastic power law fluid. The analysis is carried out for the case when the outer spherical boundary has a finite or infinite radius. A perturbation technique is used to produce the resulting equations for a slightly power law fluid which are solved using the finite element method. An asymptotic expansion is used to provide an analytical far field solution for the infinite outer sphere case.

The second part considers the combined steady and oscillatory flow of an inelastic power law liquid in a tube. The analysis is carried. out for the case when both the steady flow rate and the oscillatory flow rate are known. An expression for the pressure gradient reduction in the tube is then derived. The resulting partial differential equation is solved by finite difference techniques. An analytical solution for the pressure gradient is also obtained using a perturbation analysis for the case when the fluid inertial effects are small.

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## CHAPTER 1

## INTRODUCTION

Since ancient times the study of the deformation and flow of matter has been both a subject of speculation and of practical application. A discipline, drawing its inspiration from mathematics and engineering and epitomised by such a person as Archimedes ( $285-215 \mathrm{BC}$ ), who was one of the early workers in this field. The modern mathematical foundations were laid down in the 17th Century when Robert Hooke presented his law relating the stress and strain in an elastic solid and Isaac Newton formulated his law of friction for viscous fluids. At this time materials were classified either as Hookean solids or as Newtonian fluids. Newton, however, had considered only situations involving simple shear in the fluid and the generalisation of his work to include arbitrary types of flow was carried out by G.G. Stokes. In 1849 Stokes derived the now celebrated Navier-Stokes equations. Between the work of Hooke and Newton and that of Stokes and Navier lay all the development of the science of fluid mechanics created by such mathematicians as D'Alembert, Lagrange and Johannes and Daniel Bernouili by developing and extending the fundamental work of Euler. The subject was then further developed by such scientists as Helmholtz, Kelvin and Rayleigh during the 19th Century.

With the coming of the industrial revolution predicting and using the behaviour of fluid flows became very important. Understanding played a subservient role to application and empirical relationships were widely used. Equally important, new materials were being produced with properties very different from those previously considered. It was this development in material technology which finally demonstrated the inadequacy of the old way of classifying substances. Experimental work by Bingham and Green (1919) on oil paints and, later, German research on the flow of colloidal solutions, showed that a whole class of fluids existed whose behaviour could not be described by Newton's law of friction. The oil paints exhibited a yield stress, the colloidal solutions a shear rate dependent viscosity. In 1924 Ostwald considered the flow
of a pseudoplastic fluid over a wide range of shear rates. In order to account for this behaviour Ostwald proposed the idea of a structural viscosity. This interest in the underlying structural foundations of flow phenomena has continued to the present day. Molecular theories have recently attracted a lot of interest and stand as alternatives to the more usual continuum mechanics approach (Bird ${ }^{1}$ (1977)).

Materials were also examined which were found to possess both viscous and elastic behaviour. These materials were classified according to their response to small applied stresses. This classification is still used today. Materials not continually changing their shape when subjected to small applied stresses were classified as elastico-viscous solids. Materials which did change shape continually when subjected to a small applied stress (however small) were classified as visco-elastic fluids. These fluids have a 'memory' of past events and the ability to store energy when work is done on them. Elastic properties are exhibited by many different types of fluid including emulsions, suspensions and polymer solutions.

This growth of knowledge marked the beginning of a new science - rheology - and in 1929 the (American) Society of Rheology was founded.

In Chapter 2, in order to facilitate the understanding of the studies in this thesis, and to put them into their rheological perspective, some general aspects of the theory of deformation and flow of a homogeneous and continuous material are considered. In particular the important work of J.G. Oldroyd concerning the characterisation and formulation of rheological equations of state is discussed. The generalised Newtonian fluids and the more general Oldroyd four constant fluids, the simple fluid of Coleman and Noll and the Rivlin-Ericksen fluids are considered. In the second part of the chapter the applicability of the generalised Newtonian fluids in the oil drilling industry is discussed. It is established that, in the flow regime in which drill cuttings are transported to the surface, or when they settle after drilling is halted, the power law fluid accurately models many modern drilling fluids.

In order to introduce the numerical techniques used in Chapter 5 the Galerkin formulation for the finite element method is discussed in Chapter 3. The finite element method is a discretization technique which enables the approximate solution of a partial differential equation to be obtained at discrete points in its domain of definition. This is achieved by solving a set of simultaneous linear equations constrained by the relevant boundary conditions. The Galerkin formulation for the finite element method was chosen because of its proven success in fluid flow problems (Refer T.J. Chung2 (1978)).

In Chapter 4 an analysis of the creeping flow of a power law fluid past a sphere is presented. The analysis is carried through for both a finite and an infinite expanse of fluid. This fluid is considered to be slightly power law (i.e. to have a power law index near 1). Its governing equation is derived using a perturbation analysis about $n=1$. An expression is derived for the drag force on the sphere, first in a finite, and then in an infinite, expanse of fluid. Using the latter expression for the drag force on the sphere the drag correction factor is shown to be a linear relation of the power law index. This linear relation is known to within a constant which is dependent on derivatives of the stream function of the power law fluid. Thus, in order to determine the drag correction factor the governing equation of the fluid must be solved, numerically, to obtain the stream function. The outer boundary condition required in the case of an infinite expanse of fluid was obtained by first deriving an expression for the asymptotic far field stream function and then using the field matching technique.

In Chapter 5, the numerical solution for the stream function was obtained by decoupling the governing equation into a system of two second order partial differential equations. These equations were then solved using the Galerkin formulation of the finite element method. In this chapter the full numerical procedure for obtaining the drag correction factor for a sphere falling slowly in a slightly power law fluid is presented. In addition the validation of the program and the numerical procedures used to obtain intermediate results are discussed. The work of other researchers in this area is commented upon and the result obtained in this chapter
is compared with their predictions.
In Chapter 6, a theoretical study of the pulsatile flow of a power law fluid through a long, straight horizontal pipe of circular cross section is presented. Experimental work has shown that when a sinusoidal pressure gradient is superimposed onto the steady flow of visco-elastic fluid in a straight tube the mean flow rate is increased for a given mean pressure gradient. This phenomenon has been called 'flow enhancement' and is of considerable industrial interest. In the conventional pulsatile flow apparatus the mean pressure gradient is controlled and the mean flow rate measured. In our study the mean flow rate and the pulsatile flow rate are assumed known and the mean pressure gradient is evaluated. The relevant equation of motion is solved numerically by using a finite difference scheme. A perturbation analysis was also carried out for the case when fluid inertia is small and the results obtained are compared with the corresponding numerical results.

## CHAPTER 2

## RHEOLOGICAL EQUATIONS OF STATE

### 2.1 Introduction

This thesis is concerned with two flow situations:
(i) a sphere falling slowly through an infinite expanse of power law fluid.
(ii) the pumping of a power law fluid through a long horizontal pipe, of circular cross section, under the action of a pulsatile pressure gradient with a non-zero mean.

Both of these flow situations have industrial significance. The first flow situation is of particular importance to the oil drilling industry since many oil drilling fluids conform to the power law model and drill cuttings may be modelled, to a first approximation by spheres. The industrial significance of the second flow situation is self evident and is discussed in Chapter 6.

To facilitate the understanding of the studies in this thesis and to put them into their rheological perspective some general aspects of the theory of deformation and flow of a homogeneous and continuous material (fluid) are considered. First, some basic principles are stated and discussed and then some equations of state are considered. These equations of state are divided into two classes, those describing the generalised Newtonian fluids and those describing more general fluids - namely the Oldroyd, Simple and Rivlin-Ericksen fluids. The chapter concludes with a short discussion of the use of the generalised Newtonian fluids to the oil drilling industry.

Firstly, equations of state are obtained which specify the rheological properties of an arbitrary element of the fluid.

Secondly, the behaviour of the fluid in bulk, under the given initial and boundary conditions and the body forces operating, is predicted. This is achieved by solving, the equation of motion in conjunction with the equation of continuity.

In this thesis the component notation is used throughout. According to this approach a tensor is synonymous with its set of components, this set transforming covariantly, contravariantly or in mixed mode under a transformation of the coordinate system used. A tensor may accordingly be written as $T_{i j}$, $T^{i j}, T_{j}^{i}$ or $T_{i}{ }^{j}$.

The simplest model of a fluid is the Newtonian fluid given by

$$
\begin{equation*}
e_{i i}^{(1)}=0 \quad \text { and } \quad p_{i k}=-p g_{i k}+p_{i k}^{\prime} \tag{2.1}
\end{equation*}
$$

where

$$
\begin{aligned}
& p_{i k}^{\prime}=2 \eta e_{i k}^{(1)} \\
& e_{i k}^{(1)}=\text { rate of strain tensor } \\
& p_{i k}=\text { Cauchy stress tensor } \\
& p=\text { an arbitrary isotropic pressure } \\
& g_{i k}=\text { the metric tensor of a fixed coordinate system } x^{i} \\
& \eta_{i}=\text { a scalar constant (the fluid viscosity) } \\
& p_{i k}^{\prime}=\text { extra stress tensor }
\end{aligned}
$$

The extra stress tensor is that part of the stress tensor associated with changes in shape of the fluid element. In terms of the velocity vector $v_{i}$, the rate of strain tensor is given by:

$$
e_{i k}^{(1)}=\frac{1}{2}\left(v_{i, k}+v_{k, i}\right)
$$

The equation of motion and the equation of continuity are given respectively by:

$$
\begin{equation*}
p^{k i}, i+p F^{i}=p \frac{D v^{i}}{D t} \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{D p}{D t}+\rho v^{i} i=0 \tag{2.5}
\end{equation*}
$$

where $\rho$ is the fluid density, $F$ is the body force per unit mass, $v$ is the fluid element velocity and D/Dt is the material derivative. For an incompressible fluid $D \rho / D t=0$.

A Newtonian fluid is quite inadequate for describing the complex rheological behaviour of many fluids. The most interesting fluids are the visco-elastic fluids. Under a shearing deformation these fluids behave neither as a perfectly elastic solid nor as a perfectly viscous fluid. The work done in a shearing deformation of such a visco-elastic fluid is neither totally dissipated as heat nor is it totally conserved. The conservation of work done manifests itself in many interesting phenomena including the die-swell effect, Weissenberg rod climbing and various strainrecovery effects. Visco-elastic fluid behaviour is still not fully understood, a fact that is witnessed by the numerous modern papers on the subject.

Very important developments took place in continuuum theoretical rheology during the 1950's and onwards. Progress in this country was largely due to the pioneering work of the late J.G. $01 d_{r o y d}{ }^{3}$ and was inaugurated by him in his paper of 1950. This paper comprehensively discusses the general principles for the formation of equations of state and the application of the general theory through the construction of models.

The principles to be satisfied in constructing a model were stated by 0ldroyd as follows:
(1) the equation must describe fluid properties independently of the frame of reference.
(2) the behaviour of any fluid element is to be dependent only upon its own deformation history.

Note (1) The usual convention of summing over repeated suffixes is complied with.
Note (2) A suffix following a comma indicates a covariant derivative
(3) the behaviour of any fluid element is to be independent of any translatory or rotational motion of the fluid as a whole.

The system of co-ordinates used by Oldroyd was that proposed by H. Hencky ${ }^{4}$ in 1925, namely the system of convected coordinates. These co-ordinates are related to surfaces in the fluid which transform continuously with the fluid during its motion. They are independent of time and label the material points of the fluid. More significantly they allow the principles for constructing fluid models to be satisfied.

The visco-elastic behaviour of fluids can be modelled by mechanical analogy. Dashpots and springs may be taken as perfectly viscous and perfectly elastic elements respectively and various parallel and series combinations can be constructed. In many visco-elastic fluids memory effects are important and the superposition principle of L Boltzmann5 (1876) becomes applicable. Oldroyd ${ }^{6}$ (1958) discusses a material which in simple shearing flow may be rheologically modelled by a viscous and an elastic element in series where the latter has the elastic after-effect properties consistent with Boltzmann's principle.

Oldroyd's work provides a constructive framework and guiding principles for the formulation of equations of state. Other workers, notably Riviin and Ericksen 7 (1955), Green and Riviin 8,9 (1957, 1960) and Coleman and Noll $10-12$ (1960, 1961, 1961) have approached the problem of formulation in a different way from Oldroyd.

A significant simplification is achieved if only those motions are considered in which non-linear effects in the elastic and viscous deformation process may be ignored. These motions are those in which small deformations (solids) or small rates of strain (fluids) only occur. Such circumstances arise in experiments widely used by physical chemists to study the molecular structure of fluids and in rheological experiments when small amplitude oscillatory flows are considered.

The general linearized equations of state for an incompressible fluid are given by K. Walters ${ }^{13}$ (1960) as

$$
\begin{equation*}
p_{i k}=-p g_{i k}+p_{i k}^{\prime} \tag{2.6}
\end{equation*}
$$

where

$$
\begin{align*}
& p_{i k}^{\prime}=2 \int_{-\infty}^{t} \Psi\left(t-t^{\prime}\right) e_{i k}^{(\prime)}\left(x, t^{\prime}\right) d t^{\prime}  \tag{2.7}\\
& \Psi\left(t-t^{\prime}\right)=\int_{0}^{-\infty} \frac{N(\tau)}{\tau} \exp \left(\frac{t^{\prime}-t}{\tau}\right) d \tau \tag{2.8}
\end{align*}
$$

and

$$
\begin{aligned}
\tau & \equiv \text { the relaxation time } \\
N(\tau) & \equiv \text { the relaxation spectrum over } \tau
\end{aligned}
$$

A brief description of some equations of state now follows.

### 2.3 Equations of State

### 2.3.1 <br> The Generalised Newtonian Fluids

The models discussed here are empirical and formulated by allowing the viscosity to be dependent either upon the rate of strain or upon the extra stress tensor.

The viscosity of a fluid is a scalar quantity and, if made dependent upon the rate of strain, it must be a function of the invariants of the rate of strain tensor e (ilk. These invariants are I, II and III. For an incompressible fluid $I=\operatorname{Div} \nu=0$, while for a shearing flow III $=0$ (see R.B. Bird ${ }^{14}$ 1977, page 207). Thus the viscosity $\eta$ may be written as:
where

$$
\eta=\eta(\dot{\gamma})
$$

$$
2 \dot{\gamma}^{2}=I I
$$

and

$$
\dot{\gamma}=\text { shear rate }
$$

Thus the Newtonian fluid may be generalised to give

$$
p_{i k}^{\prime}=\eta(\dot{\gamma}) e_{i k}^{(1)}
$$

particular fluid models being formed by taking different functions $\eta(\dot{\gamma})$.

The models considered here are:
the Bingham fluid, the power law fluid, the Herschel-Bulkley fluid, Robertson-Stiff fluid, The Casson fluid.

The Ellis model is included as an example where the viscosity is considered to be a function of the extra stress.

The Bingham Fluid
In this model

$$
\begin{array}{ll}
\eta=\infty & \tau \leqslant \tau_{0} \\
\eta=\mu_{0}+\frac{\tau_{0}}{\dot{\delta}} & \tau>\tau_{0} \tag{2.13}
\end{array}
$$

where

$$
\begin{aligned}
& \tau \equiv \text { magnitude of } p_{i k}^{\prime} \\
& \dot{\gamma} \equiv \text { shear rate }
\end{aligned}
$$

and $\tau_{0}, \mu_{0}$ are two parameters, the first one being identifiable as the yield stress of the fluid. Since, for a generalised Newtonian fluid,
then

$$
\begin{equation*}
p_{i k}^{\prime}=\eta(\dot{\gamma}) e_{i k}^{(1)} \tag{2.14}
\end{equation*}
$$

$$
\tau=\eta(\dot{\gamma}) \dot{\gamma}
$$

and

$$
\tau-\tau_{0}=\mu_{0} \dot{\gamma}
$$

## The Power Law Fluid

In this model $\eta(\dot{\gamma})=k \dot{\gamma}^{n-1}$ where $K$ and $n$ are parameters. $K$ is called the consistency factor and $n$ the power law index.
But

$$
\begin{aligned}
& \tau=\eta(\dot{\gamma}) \dot{\gamma} \\
& \tau=k \dot{\gamma}^{n}
\end{aligned}
$$

$K$ is the viscosity when the shear rate is $1 s^{-1}$, effectively it indicates how viscous a fluid is at rest. When $n=1$ the model reduces to a Newtonian fluid.

## The Herschel-Bulkley Fluid

This is a three parameter model given by:

$$
\begin{equation*}
\tau=\tau_{0}+k \dot{\gamma}^{n} \tag{2.19}
\end{equation*}
$$

It allows for a yield stress and a power law relationship between shear stress and shear rate when this yield stress is exceeded.

## The Robertson-Stiff Fluid

This is also a three parameter model and is given by:

$$
\begin{equation*}
\tau=A(\dot{\gamma}+C)^{B} \tag{2.20}
\end{equation*}
$$

The Newtonian fluid ( $B=1, C=0$ ), the Bingham fluid ( $B=1, C \neq 0$ ) and the power law fluid ( $B \neq 1, C=0$ ) are all special cases of this fluid.

## The Casson Fluid

This model was first used in 1959 to describe the flow of pigment-oil suspensions in shearing flows. It is given by:

$$
\begin{equation*}
\eta^{1 / 2}=\eta_{\infty}^{1 / 2}+C \dot{\gamma}^{-1 / 2} \tag{2.21}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau=\eta \dot{\gamma} \tag{2.22}
\end{equation*}
$$

When $\dot{\gamma}$ is very large $\eta \simeq \eta \infty$ (the viscosity at infinite shear rate). The degree of shear thinning is given by $C$, the Casson slope.

## The Ellis Fluid

In this case

$$
\begin{equation*}
\frac{\eta_{0}}{\eta}=1+\left(\frac{\tau}{\tau_{1 / 2}}\right)^{\alpha-1} \tag{2.23}
\end{equation*}
$$

where

$$
\begin{aligned}
& \eta_{0} \equiv \text { zero shear-rate viscosity } \\
& \tau_{\frac{1}{2}} \equiv \text { shear stress at which } \eta=\frac{1}{2} \eta_{0}
\end{aligned}
$$

### 2.3.2 More General Fluids

## The Oldroyd Fluids

This class of fluid arose from the general discussion presented by J.G. Oldroyd ${ }^{6}$ in his 1958 paper. The 4 constant model
equation of state is given by:
where the material constants $\mu_{0} \eta_{0} \eta_{0} \lambda_{1}$ and $\lambda_{2}$ satisfy the inequalities

$$
\begin{gather*}
\eta_{0}>0  \tag{2.25}\\
9 \lambda_{2}>\lambda_{1} \geqslant \lambda_{2}>0 \tag{2.26}
\end{gather*}
$$

and $\vartheta / \mathcal{V}$ is a time derivative which does not imply any dependence on absolute motion in space.

This model produces realistic values for shear stress and first normal stress difference over a fairly wide range of shear rates. However, a zero second normal stress difference is predicted in simple shear flows.

## The Simple Fluid of Coleman and Noil

This fluid was proposed by B. Coleman and W. Noil in
1961. The equation of state was given by:

$$
\begin{equation*}
p_{i k}(x, t)=p_{i k}^{\prime}-p g_{i k} \tag{2.27}
\end{equation*}
$$

where

$$
\begin{equation*}
p_{i k}^{\prime}(x, t)={\underset{-\infty}{F}}_{F}\left[C_{i k}\left(x, t, t^{\prime}\right)\right] \tag{2.28}
\end{equation*}
$$

and $F$ is a functional of the Cauchy deformation tensor $C_{i k}$. The equation of state has been given with respect to a fixed coordinate system.

Since a visco-elastic fluid necessarily has a fading memory this behaviour must be incorporated in the model through the functional F. Coleman and Noil specified an influence function $h(t-$ $t^{\prime}$ ) and used it to define a norm, $\left\|C_{i k}\right\|$, of the strain history, where

$$
\begin{equation*}
\left\|C_{i k}\right\|^{2}=\int_{0}^{\infty} C_{i k}^{2}(s) h^{2}(s) d s \tag{2.29}
\end{equation*}
$$

Here $s=t-t^{\prime}$, where $t$ is the current time and $t^{\prime}$ an earlier time.

Finally they were able to approximate their functional by:

$$
\begin{align*}
& \int_{-\infty}^{t} M_{1}\left(t-t^{\prime}\right) C_{i k}\left(t^{\prime}\right) d t^{\prime} \\
& +\int_{-\infty}^{t} \int_{-\infty}^{t} M_{2}\left(t-t^{\prime}, t-t^{\prime \prime}\right) C_{i j}\left(t^{\prime}\right) C_{i j}\left(t^{n}\right) d t^{\prime} d t^{\prime \prime} \\
& +O R D E R\left\|C_{i k}\right\|^{3} \tag{2.30}
\end{align*}
$$

To a first order approximation the equation of state may be written as:

$$
\begin{equation*}
p_{i k}=p_{i k}^{\prime}-p g_{i k} \tag{2.31}
\end{equation*}
$$

where

$$
\begin{equation*}
p_{i k}^{\prime}=\int^{t} M_{1}\left(t-t^{\prime}\right) C_{i k}\left(t^{\prime}\right) d t^{\prime} \tag{2.32}
\end{equation*}
$$

These equations are called the equations of finite linear elasticity. In 1966 A.S. Lodge ${ }^{15}$ showed the simple fluids to be a special case of Oldroyd's visco-elastic fluids.

## The Rivlin-Ericksen Fluids

These fluids may be considered to be approximations to the simple fluids. The models are valid for retarded motion or rapidly fading memory (K. Walters ${ }^{16} 1970$ ). The equations of state are constructed on the assumption that the stress is a function of the velocity gradients and the acceleration gradients up to the ( $n$ 1) th. The stress is deduced to be a function of the first $n$ RivenEricksen tensors $A\{R)$, where $A\{n)=2 e\{(\mathbb{K})$.

The lower order Rivlin-Ericksen fluids can be shown to be equivalent to

$$
\begin{aligned}
p_{i k}^{\prime}= & 2 \alpha_{1} e_{i k}^{(1)} \\
p_{i k}^{\prime}= & 2 \alpha_{1} e_{i k}^{(1)}+2 \alpha_{2} e^{(2)} \\
& +4 \alpha_{3} e_{i k}^{(1)} e^{(1)} j k
\end{aligned}
$$

and

$$
\begin{align*}
p_{i k}^{\prime}= & 2 \alpha_{1} e_{i k}^{(1)}+2 \alpha_{2} e_{i k}^{(2)}+4 \alpha_{3} e_{i}^{(1)} e_{j k}^{(1)} \\
& +8 \beta_{1} e_{j}^{(1)} \ell e^{(1)} j e_{i k}^{(1)}+2 \beta_{2} e^{(3)} \\
& +4 \alpha_{5}\left(e_{i}^{(1)} j e_{j k}^{(2)}+e_{i}^{(2)} j e_{j k}^{(1)}\right) \tag{2.35}
\end{align*}
$$

(Third order)
where $\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{5}$ and $\beta_{1}, \beta_{2}$ are all material constants.

### 2.4 Some Industrial Applications of the Generalised Newtonian Fluids

2.4.1 Introduction to Oil Drilling Fluids

The oil drilling fluids used today are both rheologically complex and highly developed to perform many different tasks. A far from exhaustive list of these tasks would include the following:

Cleaning of the hole.
Cooling and lubricating the bit and drill string.
Preventing cave-in of the drilled formation.
Suspending drill cuttings when drilling has stopped (e.g. to change the bit).

Transporting cuttings to the surface during drilling.

With the highly deviated boreholes currently being drilled, rheological studies of oil drilling fluids are of more interest to the drilling industry than at any previous time. This was the stimulus that prompted the studies of Chapters 4-6 of this thesis.

The base material for a drilling fluid is either a clay/water dispersion, an oil/water emulsion or a polymer. A very wide range of additives has been developed and a drilling fluid is a mixture of its base fluid and many of these additives. The drilling situation determines which formulation is most suitable for the drilling fluid to be used and practical field experience is vital at this stage.

The most commonly used drilling fluids are the
water/clay based ones since oil based drilling fluids are expensive and polymer based drilling fluids degrade, both biologically and under high temperatures. To the water/clay base barium sulphate (barite) and a suitable polymer are usually added. The barite is a weight material ensuring adequate pressure at the bit, the polymer is a viscosity modifier.

Although very varied, drilling fluids are, as a rule, non-Newtonian, pseudoplastic and thixotropic.

A detailed reference for the formulation, nature and function of oil drilling fluids is the book 'Drilling and Drilling Fluids' by G.V. Chilingarian and P. Vorabutr 17 (1981).

### 2.4.2 The Field Application of Model Fluids

A fluid model is required which is easily applied at the well-site. No model is valid over the whole of any of the flow regimes and engineering approximations are resorted to. From this it can be seen that the use of sophisticated models is unsuitable in the field and major efforts have been focused instead on the use of simple models allied to computer facilities at the well-site.

The Bingham fluid has long been used as the standard model but has recently been partly replaced by the power law model which still continues to gain favour. The Herschel-Buikley and Robertson-Stiff fluids are also beginning to achieve proper acceptance in the field while in the last five years the Casson model has also been used.

Figure 1 (after F. Bagshaw, Company Report of Kelco Oil Field Products) shows the shear stress - shear rate relationship for one of their commercial drilling fluids. The instrument used to achieve this being a Fann viscometer. A Bingham fluid and a power law fluid were constructed from the data to represent the real drilling fluid. The shear stress - shear rate behaviour of these fluids is also given on the figure to allow comparison. The Kelco drilling fluid may be taken as fairly typical. The power law fluid provides a better representation than the Bingham fluid over the annular shear-rate range. This is particularly true over the lower end of that range. At shear rates above $300 \mathrm{~s}^{-1}$ the simpler Bingham fluid provides an equally good representation of the behaviour of the
oil drilling fluid. In the field the parameters of the Bingham fluid are fixed by readings taken at just two shear rates, namely $400 \mathrm{~s}^{-1}$ and $900 \mathrm{~s}^{-1}$.

The power law fluid has enjoyed wide usage in the field only in the last ten years or so. It usefully describes the upward flow regime of many modern drilling fluids. However, neither the Bingham fluid nor the power law fluid properly describes the flow situation at the bit where the very high shear rates are experienced (R. Lauzon, K. Reid ${ }^{18}$ (1979)).

The more simple clay/water mixtures can approach plastic behaviour while many polymer based drilling fluids exhibit power law behaviour. However most drilling fluids exhibit both a yield stress and a non-linear shear stress - shear rate relationship when the yield stress has been exceeded (M. Zamora and R. Bleier ${ }^{19}$ (1976)).

The Herschel-Bulkely and Robertson-Stiff fluids are suitable in this case and were used by the oil drilling industry for the first time in 1976. Robertston and Stiff 20 (1976) showed their model to be significantly better than the power law fluid, for the three different drilling fluids they considered, at describing the shear stress - shear rate behaviour over a shear rate range of $5 \mathrm{~s}^{-1}$ to $1000 \mathrm{~s}^{-1}$. However, obtaining the parameters for this model is not without its difficulties (Zamora, Bleier ${ }^{19}$ (1976)).

A drilling fluid may best be represented, over a wide shear rate range, by several power-law regions. Such composite power-law behaviour can be described by the Casson fluid, used by the oil-drilling industry since 1979. Its main advantages (R. Lauzon, K. Reid ${ }^{18}$ (1979)) are that it is valid from very low to very high shear rates and is easily implemented in the field.
D. McEachern ${ }^{21}$ (1966) used the Ellis model to describe his experimental results for the laminar flow of dilute polymer solutions in an annulus and found it to be superior to the power-law fluid. The polymer solutions used were of a type relevant to the oil-drilling industry but as yet the Ellis fluid has been little, if at all, used in the field.

A further discussion of the types of rheological behaviour found in drilling fluids is given by M. Slawomirski ${ }^{22}$ (1975).

# THE GALERKIN FORMULATION FOR THE FINITE ELEMENT METHOD 

### 3.1 Introduction

In this chapter the Galerkin formulation of the finite element method is described in sufficient detall to explain its application in Chapter 5. The modern use of this method started in the 1940's when the techniques were applied to the structural design of aircraft. From these beginnings the method has developed to the point where today it is applied extensively by engineers to solve a wide range of problems.

In Chapter 5 the numerical procedures used to obtain the drag correction factor for a sphere falling slowly through an infinite expanse of power law fluid are presented. These procedures require the solution of the governing equation of a power law fluid. This is a fourth order partial diffential equation. The solution is obtained by decoupling it into a pair of second order partial differential equations and applying the finite element method within an iterative routine. Due to its past use in fluid dynamics (Chung (1978)) the Galerkin formulation of the method was the one implemented. This formulation, due to both Bubnov and Galerkin, is of ten called the method of weighted residuals.

## 3.2

The Galerkin Formulation

Let the equation to be solved approximately by the Galerkin method be

$$
A u=f
$$

where $A$ is a differential operator. Equation (3.1) is defined over a physical region, $\Omega$, with boundary $\Gamma$, and is solved subject to the relevant boundary conditions on $\Gamma$. The approximate solution, $\hat{u}$, is expressed as a linear combination of
a set of $n$ linearly independent functions, $\phi_{i}$, which satisfy the boundary conditions. i.e.

$$
\hat{u}=\sum_{i=1}^{n} c_{i} \phi_{i}
$$

where $c_{i}: i=1, \ldots, n$ are constants.
The functions $\phi_{i}: i=1, \ldots, n$ are called the basis, or shape, functions and $\hat{u}$ is called the nth trial function. The coefficients $c_{i}$ must be such that, as $n$ increases, the resultant sequence of trial functions is convergent to the true solution. The Galerkin formulation is a particular method for deriving the coefficients $c_{i}$ • Since $\hat{u}$ is an approximation to the true solution $u$

$$
\begin{equation*}
A \hat{u}-f=r \tag{3.3}
\end{equation*}
$$

where $r$ is called the residual and equals the error incurred when $\hat{u}$, and not $u$, occurs in equation (3.1). An inner product $\left\langle r, w_{1}\right\rangle$ is now constructed where $w_{i}: i=1, \ldots, n$ is a set of weighting functions to be defined. The residual, $r$, is now forced to zero, in an average sense, over $\Omega$ by setting

$$
\left\langle r, w_{i}\right\rangle=0 \quad i=1, \ldots, n
$$

The weighting functions may be chosen in different ways and this has led to four different methods: the method of moments, the method of least squares, the collocation method and the Galerkin method. In the Galerkin method we set
and

$$
w_{i}=\phi_{i}
$$

Thus

$$
\begin{equation*}
\left\langle r, w_{i}\right\rangle=\int_{\Omega} r w_{i} d \Omega \quad i=1, \ldots, n \tag{3.6}
\end{equation*}
$$

$$
\begin{equation*}
\int_{\Omega} r \phi_{i} d \Omega=0 \quad i=1, \ldots, n \tag{3.7}
\end{equation*}
$$

Equations (3.7) are solved simultaneously for $c_{i}: i=1, \ldots, n$ and these values are then substituted into equation (3.2) to give the approximate solution of the given differential equation.

In order to implement the work above, the basis functions $\emptyset_{i}: i=1, \ldots, n$ must be known. The construction of these
functions is described in the next section.

## 3.3 <br> Constructing the Finite Element Mesh and the Local Basis Functions

Due to the symmetry of the flow situation considered in Chapter 5, our problem may be considered to be a two dimensional one from the outset. Consequently, the implementation of the finite element method is discussed only over plane regions $\Omega$.

First the plane region is subdivided into an equivalent set of finite elements. The simplest example would be the subdivision of a rectangular area, by means of a grid, into a collection of squares or rectangles. These squares or rectangles form the finite elements and the grid forms the finite element mesh. If the plane area has a curved boundary then the collection of squares and rectangles will not coincide exactly with it. A better approximation will be produced if a mesh of quadilateral elements is used.: In practice, many hundreds (or thousands) of elements may be used and a close approximation to the physical region of interest can be obtained.

Nodes are now allocated to each element in the mesh. A node is always allocated to each element vertex and may also be allocated to the element mid-side or its interior. Eight noded and four noded quadrilateral elements are often used.

It is necessary to construct the finite elements and the mesh because the basis functions $\phi_{i}: i=1, \ldots, n$ are difficult to define as functions over the whole of the physical area. However, by means of a local coordinate system attached to each finite element, local basis functions may be easily defined. Using these local basis functions, equations equivalent to (3.7) but valid only over each of the finite elements separately can be obtained. When these equations have been obtained for every one of the finite elements, the results may be combined to obtain the required equation (3.7).

Some further aspects of this procedure are now discussed. Four noded plane quadrilateral elements have local basis functions defined over them which allow for a bilinear variation with respect to the local coordinate system. This local coordinate system
is defined in Fig. 2(a). The nodes have been labelled locally, that is with respect to the element and not with respect to the grid. Numbering nodes with respect to the grid is called global node numbering. The local coordinates of the vertices of the finite element, and a typical interior point, are given in Fig. 2(b). In Fig. 2(c), it is shown that every finite element can be represented by a square with centre the origin, if the local coordinate system is referred to normal rectilinear axes with equal scales. The local shape functions are given by

$$
\begin{align*}
& \phi_{1}^{(e)}=\frac{1}{4}(1-u)(1-v) \\
& \phi_{2}^{(e)}=\frac{1}{4}(1-u)(1+v) \\
& \phi_{3}^{(e)}=\frac{1}{4}(1+u)(1+v) \\
& \phi_{4}^{(e)}=\frac{1}{4}(1+u)(1-v)
\end{align*}
$$

for all points within the element and on its boundary, and zero at all exterior points. We have

$$
\begin{align*}
\phi_{i}^{(e)} & =1 & & \text { at node } i  \tag{3.9}\\
& =0 & & \text { at the other nodes }
\end{align*}
$$

Also

$$
\begin{equation*}
\hat{u}^{(e)}=\sum_{i=1}^{4} c_{i}^{(e)} \phi_{i}^{(e)} \tag{3.10}
\end{equation*}
$$

over the element (e). From equation (3.9) and (3.10) we have

$$
\begin{equation*}
c_{i}^{(e)}=\hat{u}_{i}^{(e)} \tag{3.11}
\end{equation*}
$$

where $\hat{u}_{i}(e)$ is the approximate value of $u$ at the $i$ th node of the element (e). Thus

$$
\begin{equation*}
\hat{u}^{(e)}=\sum_{i=1}^{4} \hat{u}_{i}^{(e)} \phi_{i}^{(e)} \tag{3.12}
\end{equation*}
$$

In Chapter 5, the finite elements used are isoparametric. That is we assume equations (3.12) to be true for the global coordinates $(U(e), V(e)$ of a point in element (e).
i.e.

$$
\begin{equation*}
U^{(e)}=\sum_{i=1}^{4} U_{i}^{(e)} \phi_{i}^{(e)} \tag{3.13}
\end{equation*}
$$

and

$$
\begin{equation*}
V^{(e)}=\sum_{i=1}^{4} V_{i}^{(e)} \phi_{i}^{(e)} \tag{3.14}
\end{equation*}
$$

where ( $u_{1}(e), v_{1}(e)$ are the global coordinates of the th local node of element (e).
3.4

Constructing the Stiffness Equation
3.4.1 The General Method

Green's theorem in the plane is applied to equations (3.7) and the resulting equations are applied to each of the finite elements in turn. After using equation (3.12), implementing the necessary changes of variable and carrying out the numerical integration we obtain, for each element (e)

$$
\begin{equation*}
S^{(e)} \underline{\hat{u}}^{(e)}=\underline{\ell}^{(e)} \tag{3.15}
\end{equation*}
$$

where

$$
\begin{align*}
& \underline{\underline{u}}^{(e)}=\left[u_{i}^{(e)}: i=1,2,3,4\right]  \tag{3.16}\\
& \underline{\ell}^{(e)}=\left[\ell_{i}^{(e)}: i=1,2,3,4\right] \tag{3.17}
\end{align*}
$$

$s(e)$ is the element stiffness matrix, $\mathcal{l}^{(e)}$ the element load vector. The element stiffness equations are used to construct the full stiffness equation given by
where

$$
\begin{equation*}
s \hat{u}=\underline{\ell} \tag{3.18}
\end{equation*}
$$

$$
\underline{\hat{u}}=\left[\hat{u}_{i}: \quad i=1, \cdots, N\right]
$$

$N=$ total number of nodes in mesh
and
$\hat{u}_{i}$ is the approximate value of $u$ at the fth globally numbered node of the mesh.

The following example shows how equation (3.18) is derived from equations (3.15). We consider the very simple two element mesh given in Fig. 2(d). In this figure the global labels of the nodes are shown on the outside of the mesh, the locai labels are shown on the inside. The element stiffness matrices are $S^{(1)}$ and $s^{(2)}$ and the full stiffness matrix is $S$. The matrix $s^{(e)}$ contributes $S_{i, j}^{(e)}$, to $S_{G}^{(e)} G_{j}^{(e)}$ where $G_{i}^{(e)}$ and $G(e)$ are the global node numbers corresponding to local node numbers $i$ and $j$ in element (e). The elements in $S$ are obtained by adding together the contributions made by $S^{(1)}$ and $S^{(2)}$ separately. For example

$$
\begin{equation*}
s_{2,3}=s_{1,3}^{(1)}+s_{2,1}^{(2)} \tag{3.20}
\end{equation*}
$$

If neither element has a contribution to make then that element in $S$ is zero. For example

$$
\begin{equation*}
s_{4,1}=0 \tag{3.21}
\end{equation*}
$$

Two important points to note are
(i) If the element stiffness matrices are symmetric then it follows that the full stiffness matrix is also symmetric.
(ii) The full stiffness matrix will be banded (i.e. its non-zero elements will be grouped about the leading diagonal). The width of the band depends upon the way in which the global numbering of the nodes is carried out.

The stiffness matrices used in the numerical work presented in Chapter 5 were, in some cases, of such a high order that techniques were developed to modify them so that only the non-zero elements were considered.

Details of the boundary conditions and their implementation are given in Chapter 5.

## CHAPTER 4

## THEORY OF THE MOTION OF A SPHERE IN A PONER LAN FLUID

### 4.1 Introduction

The ability of a drilling fluid to convey cuttings from a well is not fully understood and this is particularly so in highly deviated wells where difficulties are frequently experienced in cleaning the hole. The cuttings travel with a lower velocity than the drilling fluids and these cuttings can accumulate in the well bore. If this is not kept to a minimum then it can lead either to the degradation of the cuttings or the drill string may get lodged in the hole. Experimental work by P. Reynolds 23 (1982) has shown that, in a power law fluid, the settling velocity of an irregular shaped particle may be adequately represented by the settling velocity of a sphere having the same volume and density. In the case of the fluid model, it has been well established by the oil drilling industry that, in the shear rate range of interest, many modern drilling fluids exhibit power law behaviour. Therefore, in this thesis we have studied the problem of the flow of a power law fluid past a stationary sphere. The effect of fluid inertia is ignored and the fluid is considered to be slightly power law. That is its power law index is near unity.

The theoretical studies of this problem have, in the past, been based mainly on perturbation and variational methods (Acharya et al 24 (1976)). Wasserman and Slattery 25 (1964) used variational principles applied to inelastic power law fluids to obtain upper and lower bounds of the drag correction factor as the power law index varied from 1.0 to 0.2 . These bounds have been improved by Mohan and Venkateshwarlu ${ }^{26}$ (1976), by Yoshioka and Adachi 27 (1973) and by Cho and Hartnett ${ }^{28}$ (1983). Chhabra 29 (1983) has commented on the results of Cho and Hartnett and has provided a comparison of their results with experimental values.

More recently Crochet et al ${ }^{30}$ (1984) and Gu Dazhi and Tanner ${ }^{31}$ (1985) have used a finite element analysis. Crochet et al
considered a sphere in a cylinder geometry with the ratio of the radius of the cylinder to the radius of the sphere being equal to 50 . Gu Dazhi and Tanner used a sphere within a sphere configuration and an extrapolation technique to obtain the drag force on the falling sphere when the fluid is infinite in extent. They note that wall effects are important for a slightly power law fluld but for a power law index less than, or equal to 0.5 they are negligible. The analysis in this chapter, although valid only when the power law index is near unity, uses a field matching technique to solve the equations when the fluid is unbounded. The use of this technique was influenced by a paper by Tiefenbruck and Leal 32 (1979). Other workers who have used discretization techniques include Bush and Phan-Thien 33 (1985). In their case the drag force on a sphere in creeping motion through a Carreau fluid was calculated.

## 42 The Governing Equations for a Slightly Power Law Eluid

We consider the slow flow of a power law fluid between two concentric spheres. The inner sphere is fixed and the outer sphere is given an instantaneous velocity $U$ in the positive $z$ direction. A spherical polar coordinate system ( $r, \theta, \phi$ ) is used with its origin at the centre of the fixed sphere. Since the flow situation is axi-symmetrical a velocity distribution of the form

$$
V_{r}=V_{r}(r, \theta), \quad V_{\theta}=V_{\theta}(r, \theta), \quad V_{\phi}=0
$$

may be assumed. We consider first the case when the outer sphere has a finite radius and later extend this work to the case of an outer sphere with an 'infinite' radius. The schematic diagram of the flow situation used when the outer sphere has a finite radius is given in Fig. 3.

In spherical polar coordinates the equation of continuity for axi-symmetrical flow is given by

$$
\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} V_{r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(V_{\theta} \sin \theta\right)=0
$$

We may therefore define a stream function $\psi$ by the equations

$$
\begin{align*}
& V_{r}=\frac{-1}{r^{2} \sin \theta} \frac{\partial \psi}{\partial \theta} \\
& V_{\theta}=\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}
\end{align*}
$$

(4.4)
where $V_{r}$ and $V_{\theta}$ satisfy the equation of continuity. When inertial terms are neglected the equations of motion become

$$
\begin{align*}
-\frac{\partial p}{\partial r}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \tau_{r r}\right) & +\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\tau_{\theta r} \sin \theta\right) \\
& -\frac{1}{r}\left(\tau_{\theta \theta}+\tau_{\phi \phi}\right)
\end{align*}
$$

and

$$
-\frac{1}{r} \frac{\partial p}{\partial \theta}=\frac{1}{r^{3}} \frac{\partial}{\partial r}\left(r^{3} \tau_{r \theta}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\tau_{\theta \theta} \sin \theta\right)-\frac{\tau \phi \phi}{r} \cot \theta
$$

(4.6)
where $\tau_{i k}$ is the stress tensor. The equation of state is

$$
\begin{equation*}
\tau_{i k}=-\eta(\dot{\gamma}) \dot{\gamma}_{i k} \tag{4.7}
\end{equation*}
$$

where $\dot{\gamma}_{i k}$ is the rate of strain tensor with magnitude $\dot{\gamma}$ and viscosity

$$
\begin{equation*}
\eta(\dot{\gamma})=K \dot{\gamma}^{n-1} \tag{4.8}
\end{equation*}
$$

where $K$ is a constant and $n$ is the power law index. When $n=1$ the equation of state refers to a Newtonian fluid and

$$
\tau_{i k}=-K \dot{\gamma}_{i k}
$$

The governing equation for a Newtonian fluid is derived from equations (4.2) - (4.6) and (4.9). Full details of the derivation are given in Appendix 1. The resulting equation is

$$
\begin{equation*}
E^{4} \psi=0 \tag{4.10}
\end{equation*}
$$

where

$$
\begin{equation*}
E^{4} \psi=E^{2}\left(E^{2} \psi\right) \tag{4.11}
\end{equation*}
$$

and

$$
E^{2} \psi \equiv \frac{\partial^{2}}{\partial r^{2}}+\frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta}\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\right]
$$

The governing equation for a power law fluid is derived from equations (4.2) - (4.7) and (4.10). Equation (4.7) can be rewritten as

$$
\begin{equation*}
\tau_{i_{k}}=-K\left(\frac{I I}{2}\right)^{\frac{n-1}{2}} \dot{\gamma}_{i k} \tag{4.13}
\end{equation*}
$$

where II is the second invariant of $\dot{\gamma}_{i k}$ and is given by

$$
\begin{equation*}
\text { II }=\dot{\gamma}_{r r}^{2}+\dot{\gamma}_{\theta \theta}^{2}+\dot{\gamma}_{\phi \phi}^{2}+2 \dot{\gamma}_{r \theta}^{2} \tag{4.14}
\end{equation*}
$$

Substituting the components of the rate of strain tensor from equation (4.13) into the equations of motion (4.5) and (4.6)

$$
\begin{aligned}
\frac{\partial p}{\partial r}= & k\left(\frac{\Pi}{2}\right)^{\frac{n-1}{2}}\left[\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \dot{\gamma}_{r r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \dot{\gamma}_{r \theta}\right)\right. \\
& \left.-\frac{1}{r}\left(\dot{\gamma}_{\theta \theta}-\dot{\gamma}_{\phi \phi}\right)+\left(\frac{n-1}{2}\right)\left(\frac{\dot{\gamma}_{r r}}{\Pi} \frac{\partial \Pi}{\partial r}+\frac{\dot{\gamma}_{r \theta}}{r \Pi} \frac{\partial I}{\partial \theta}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{r} \frac{\partial b}{\partial \theta}= & k\left(\frac{\pi}{2}\right)^{\frac{n-1}{2}}\left[\frac{1}{r^{3}} \frac{\partial}{\partial r}\left(r^{3} \dot{\gamma}_{r \theta}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\dot{\gamma}_{\theta \theta} \sin \theta\right)\right. \\
& \left.-\frac{\dot{\gamma} \phi \phi}{r} \cot \theta+\left(\frac{n-1}{2}\right)\left(\frac{\dot{\gamma} r \theta}{\Pi} \frac{\partial \pi}{\partial r}+\frac{\dot{\gamma} \theta \theta}{r \pi} \frac{\partial \pi}{\partial \theta}\right)\right]
\end{aligned}
$$

Since a slightly power law fluid is considered then we can ignore terms of order $(n-1)^{2}$ and higher in our analysis.
Using

$$
\begin{equation*}
\frac{\partial}{\partial \theta}\left(\frac{\partial p}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{\partial p}{\partial \theta}\right) \tag{4.17}
\end{equation*}
$$

from equations (4.15), (4.16) and (4.17) we obtain

$$
\begin{gathered}
\frac{\partial}{\partial \theta}\left[\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \dot{\gamma}_{r r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \dot{\gamma}_{r \theta}\right)-\frac{1}{r}\left(\dot{\gamma}_{\theta \theta}+\dot{\gamma}_{\phi \phi}\right)\right] \\
+\left(\frac{n-1}{2}\right) \frac{\partial}{\partial \theta}\left[\frac{\dot{\gamma}_{r r}}{\pi} \frac{\partial \pi}{\partial r}+\frac{\dot{\gamma}_{r \theta}}{r \pi} \frac{\partial \Pi}{\partial \theta}\right] \\
+\left(\frac{n-1}{2}\right) \frac{\partial \pi}{\partial \theta}\left[\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \dot{\gamma}_{r r}\right)+r \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \dot{\gamma}_{r \theta}\right)-\left(\frac{\dot{\gamma}_{\theta \theta}+\dot{\gamma}_{\phi \phi \phi}}{r}\right)\right] \\
-26-
\end{gathered}
$$

$$
\begin{align*}
= & \frac{\partial}{\partial r}\left[\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{3} \dot{\gamma}_{r \theta}\right)+\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\dot{\gamma}_{\theta \theta} \sin \theta\right)-\dot{\gamma}_{\phi \phi} \cot \theta\right] \\
& +\frac{n-1}{2} \frac{\partial}{\partial r}\left[r \frac{\dot{\gamma} r \theta}{I I} \frac{\partial I I}{\partial r}+\frac{\dot{\gamma}_{\theta \theta}}{I I} \frac{\partial I I}{\partial \theta}\right] \\
& +\frac{n-1}{2} \frac{1}{\text { III }} \frac{\partial I I}{\partial r}\left[\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{3} \dot{\gamma}_{r \theta}\right)+\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\dot{\gamma}_{\theta \theta} \sin \theta\right)\right. \\
& \left.-\dot{\gamma}_{\phi \phi} \cot \theta\right] \tag{4.18}
\end{align*}
$$

Which may be written in the form

$$
\frac{\partial}{\partial r} A_{1}-\frac{\partial}{\partial \theta} B_{1}=\frac{n-1}{2}\left[\frac{\partial}{\partial \theta} C_{1}-\frac{\partial}{\partial r} D_{1}-\frac{1}{I I} \frac{\partial \pi}{\partial r} A_{1}+\frac{1}{I I} \frac{\partial I}{\partial \theta} B_{1}\right] \text { (4.19) }
$$

where

$$
\begin{align*}
& A_{1}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{3} \dot{\gamma}_{r \theta}\right)+\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\dot{\gamma}_{\theta \theta} \sin \theta\right)-\dot{\gamma}_{\phi \phi} \cot \theta \\
& B_{1}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \dot{\gamma}_{r r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\dot{\gamma}_{r \theta} \sin \theta\right)-\frac{\dot{\gamma}_{\theta \theta}+\dot{\gamma}_{\phi \phi}}{r} \quad(4.21) \\
& C_{1}=\frac{\dot{\gamma}_{r r}}{I I} \frac{\partial I I}{\partial r}+\frac{\dot{\gamma}_{r \theta}}{r I I} \frac{\partial I I}{\partial \theta} \tag{4.22}
\end{align*}
$$

and

$$
\begin{equation*}
D_{1}=\frac{r \dot{\gamma}_{r \theta}}{I I} \frac{\partial I I}{\partial r}+\frac{\dot{\gamma}_{\theta \theta}}{I I} \frac{\partial I}{\partial \theta} \tag{4.23}
\end{equation*}
$$

Following closely the analysis for a Newtonian fluid (see Appendix 1) equation (4.19) may be written as

$$
\begin{equation*}
E^{\psi} \psi=\left(\frac{n-1}{2}\right) F(\psi) \sin \theta \tag{4.24}
\end{equation*}
$$

where

$$
\begin{equation*}
F(\psi)=\frac{\partial}{\partial \theta} C_{1}-\frac{\partial}{\partial r} D_{1}-\frac{1}{\pi} \frac{\partial I}{\partial r} A_{1}+\frac{1}{I} \frac{\partial \pi}{\partial \theta} B_{1} \tag{4.25}
\end{equation*}
$$

Since $n$ is near unity we may use a perturbation analysis to solve equation (4.24). The stream function, therefore, is given by

$$
\begin{equation*}
\psi=\psi_{N}+(n-1) \psi_{N N} \tag{4.26}
\end{equation*}
$$

where
$\psi_{N}$ is the Newtonian stream function
and $\psi_{N N}$ is the non-Newtonian stream function contribution

Substituting equation (4.26) into equation (4.24) and equating powers of $(n-1)$ we have, on ignoring terms of order $(n-1)^{2}$ and higher and using equation (4.10)

$$
E^{4} \psi_{H N}=\frac{1}{2} F\left(\psi_{N}\right) \sin \theta
$$

The boundary conditions for $\psi_{N N}$ are similar to those for the Newtonian case (as shown in Appendix'2) and are given by

$$
\begin{align*}
& \left.\psi_{N N}\right|_{r=a}=\left.\psi_{N N}\right|_{r=b}=0 \\
& \left.\frac{\partial \psi_{N N}}{\partial r}\right|_{r=a}=\left.\frac{\partial \psi_{N N}}{\partial r}\right|_{r=b}=0 \\
& \left.\frac{\partial \psi_{N N}}{\partial \theta}\right|_{\theta=0}=\left.\frac{\partial \psi_{N N}}{\partial \theta}\right|_{\theta=\pi}=0
\end{align*}
$$

In the infinite outer sphere case the boundary conditions for $\psi_{N N}$ are discussed in the asymptotic far field section.
We shall now derive $F\left(\Psi_{N}\right)$ which is required in equation (4.27). The following expressions are for a Newtonian velocity field

$$
\begin{array}{ll}
\dot{\gamma}_{r r}=2 \frac{\partial V_{r}}{\partial r} \\
\dot{\gamma}_{\theta \theta}=2\left(\frac{1}{r} \frac{\partial V_{\theta}}{\partial \theta}+\frac{V_{r}}{r}\right) & (4.31) \\
\dot{\gamma}_{\phi \phi}=2\left(\frac{V_{r}}{r}+V_{\theta} \frac{\cot \theta}{r}\right) & (4.33)  \tag{4.33}\\
\dot{\gamma}_{r \theta}=r \frac{\partial}{\partial r}\left(\frac{V}{r}+\frac{1}{r} \frac{\partial V_{r}}{\partial \theta}\right) & (4.34)
\end{array}
$$

and

From Appendix 3 equation (A3.11)

$$
\begin{equation*}
\psi_{N}=-\frac{1}{2} U a^{2}\left(A \frac{a}{r}+B \frac{r}{a}+C \frac{r^{2}}{a^{2}}+D \frac{r^{4}}{a^{4}}\right) \sin ^{2} \theta \tag{4.35}
\end{equation*}
$$

and $\Psi_{N}$ is given in terms of $V_{r}$ and $V_{\theta}$ by equations (4.3) and (4.4). Substituting these expressions into equations (4.20) and (4.21) we have

$$
\begin{align*}
& A_{1}=-\frac{u}{a}\left(10 D \frac{r}{a}+B \frac{a^{2}}{r^{2}}\right) \sin \theta \\
& B_{1}=\frac{2 U}{r^{2}}\left(5 D \frac{r^{2}}{a^{2}}-B \frac{a}{r}\right) \cos \theta
\end{align*}
$$

We now need to define the following

$$
\begin{align*}
& G_{1}=\frac{2 U}{a}\left(-3 A \frac{a^{4}}{r^{4}}-B \frac{a^{2}}{r^{2}}+2 D \frac{r}{a}\right) \cos \theta  \tag{4.38}\\
& G_{2}=\dot{\gamma}_{\theta \theta}=-\frac{1}{2} G_{1}  \tag{4.39}\\
& G_{3}=\dot{\gamma}_{\phi \phi}=G_{2} \\
& G_{4}=\dot{\gamma}_{r \theta}=\frac{3 U}{a}\left(A \frac{a^{4}}{r^{4}}+D \frac{r}{a}\right) \sin \theta \\
& P_{1}=I I=\frac{3}{2} G_{1}^{2}+2 G_{4}^{2}  \tag{4.42}\\
& G_{5}=\frac{\partial \dot{\gamma}_{r r}}{\partial r}=\frac{2 U}{a^{2}}\left(12 A \frac{a^{5}}{r^{5}}+2 B \frac{a^{2}}{r^{2}}+2 D\right) \cos \theta(4.40) \\
& G_{6}=\frac{\partial \dot{\gamma}_{r r}}{\partial \theta}=\frac{2 U}{a}\left(-3 A \frac{a^{4}}{r^{4}}-B \frac{a^{2}}{r^{2}}+2 D \frac{r}{a}\right) \sin \theta(4.44) \\
& G_{7}=\frac{\partial \dot{\gamma}_{r \theta}}{\partial r}=-3 \frac{U}{a^{2}}\left(-4 A \frac{a^{5}}{r^{5}}+D\right) \sin \theta \\
& G_{8}=\frac{\partial \dot{\gamma}_{r \theta}}{\partial \theta}=-3 \frac{U}{a}\left(A \frac{a^{4}}{r^{4}}+D \frac{r}{a}\right) \cos \theta \\
& S \\
& S_{1}=\frac{\partial^{2} \dot{\gamma}_{r r}}{\partial r^{2}}=2 \frac{U}{a^{3}}\left(-60 A \frac{a^{6}}{r^{6}}-6 B \frac{a^{4}}{r^{4}}\right) \cos \theta(4.47) \\
& \text { (4.46) } \\
& \text { (4.46) }
\end{align*}
$$

$$
\begin{align*}
& S_{2}=\frac{\partial^{2} \dot{\dot{r}}_{r r}}{\partial r \partial \theta}=-\frac{2 U}{a^{2}}\left(12 A \frac{a^{5}}{r^{5}}+2 B \frac{a^{3}}{r^{3}}+2 D\right) \begin{array}{l}
\sin \theta \\
(4.48) \\
(4.49)
\end{array} \\
& S_{3}=\frac{\partial^{2} \dot{\gamma}_{r r}}{\partial \theta^{2}}=-G_{1} \\
& S_{4}=\frac{\partial^{2} \dot{\gamma}_{r \theta}}{\partial r^{2}}=-3 \frac{U}{a^{3}}\left(20 A \frac{a^{6}}{r^{6}}\right) \sin \theta \\
& S_{5}=\frac{\partial^{2} \dot{\gamma}_{r \theta}}{\partial r \partial \theta}=-\frac{3 U}{a^{2}}\left(-4 A \frac{a^{5}}{r^{5}}+D\right) \cos \theta \quad \text { (4.50) }  \tag{4.50}\\
& S_{6}=\frac{\partial^{2} \dot{\gamma}_{r \theta}}{\partial \theta^{2}}=-G_{4} \\
& P_{2}=\frac{\partial I I}{\partial r}=3 G_{1} G_{5}+4 G_{4} G_{7}
\end{align*}
$$

Let

$$
\begin{align*}
& P_{3}=\frac{\partial I I}{\partial \theta}=3 G_{1} G_{6}+4 G_{4} G_{8} \\
& P_{4}=\frac{\partial^{2} I I}{\partial r^{2}}=3 G_{1} S_{1}+3 G_{5}^{2}+4 G_{4} S_{4}+4 G_{7}^{2}  \tag{4.55}\\
& P_{5}=\frac{\partial^{2} I I}{\partial r \partial \theta}=3 G_{1} S_{2}+3 G_{6} G_{5}+4 G_{4} S_{5}+4 G_{8} G_{7} \tag{4.56}
\end{align*}
$$

Let

$$
\begin{align*}
& P_{6}=\frac{\partial^{2} I I}{\partial \theta^{2}}=3 G_{1} S_{3}+3 G_{6}^{2}+4 G_{4} S_{6}+4 G_{8}^{2} \text { (4.57) } \\
& C_{1}^{*}=\frac{\partial C_{1}}{\partial \theta}, \quad D_{1}^{*}=\frac{\partial D_{1}}{\partial r} \tag{4.58}
\end{align*}
$$

Finally we have

$$
F\left(\Psi_{N}\right)=C_{1}^{*}-D_{1}^{*}-\frac{P_{2} A_{1}}{P_{1}}+\frac{P_{3} B_{1}}{P_{1}} \quad(4.59)
$$

Having obtained an analytical expression for $F\left(\Psi_{N}\right)$ we may now solve equation (4.27) and obtain the non-Newtonian perturbation stream
function, $\Psi_{N N}$, subject to the boundary conditions of equations (4.28)-(4.30)
4.3 $\begin{aligned} & \text { Derivation of the Drag Force on a Sphere Moving } \\ & \text { Slowly Through a Slightly Power Law Fluid }\end{aligned}$

### 4.3.1 <br> Finite Outer Sphere Drag Force Calculation

The orientation of the components of the total stress tensor at the surface of the sphere are given in Fig. 4. The total stress tensor, $\Pi_{i k}$, is given by

$$
\begin{equation*}
\pi_{i k}=p \delta_{i k}+\tau_{i k} \tag{4.60}
\end{equation*}
$$

where

$$
\begin{aligned}
& \tau_{i k} \text { is the extra stress tensor } \\
& \delta_{\text {ic }} \text { is the unit tensor }
\end{aligned}
$$

The drag force, $\mathrm{F}_{\mathrm{D}}$, acting on the inner sphere is given by

$$
F_{D}=\left.2 \pi a^{2} \int_{0}^{\pi}\left(\pi_{r \theta} \sin ^{2} \theta-\pi_{r r} \sin \theta \cos \theta\right)\right|_{r=a} d \theta \text { (4.61) }
$$

Integrating by parts we have

$$
\begin{equation*}
F_{D}=\left.2 \pi a^{2} \int_{0}^{\pi}\left(\pi_{r \theta}+\frac{1}{2} \frac{\partial \pi_{r r}}{\partial \theta}\right)\right|_{r=a} \sin ^{2} \theta d \theta \tag{4.62}
\end{equation*}
$$

From equations (4.3), (4.4) and the no slip condition on $r=a$ we obtain

$$
\left.\frac{\partial \psi}{\partial \theta}\right|_{r=a}=\left.\frac{\partial \psi}{\partial r}\right|_{r=a}=0
$$

Substituting equation (4.3) into equation (4.31) gives

$$
\left.\dot{\gamma}_{r r}\right|_{r=a}=\frac{-2}{\sin \theta}\left[-\frac{2}{r^{3}} \frac{\partial \psi}{\partial \theta}+\frac{1}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{\partial \psi}{\partial r}\right)\right]_{r=0}=0 \quad \text { (4.64) }
$$

Therefore

$$
\begin{equation*}
\left.\pi_{r r}\right|_{r=a}=p \tag{4.65}
\end{equation*}
$$

The drag force, $F_{D}$, on the inner sphere is now given by

$$
F_{D}=\left.2 \pi a^{2} \int_{0}^{\pi}\left(\tau_{r \theta}+\frac{1}{2} \frac{\partial p}{\partial \theta}\right)\right|_{r=a} \sin ^{2} \theta d \theta \quad(4.66)
$$

Similarly, from equations (4.32) and (4.33)

$$
\begin{equation*}
\left.\dot{\gamma}_{\theta \theta}\right|_{r=a}=\left.\dot{\gamma}_{\phi \phi}\right|_{r=a}=0 \tag{4.67}
\end{equation*}
$$

Therefore, on substituting equations (4.64) and (4.67) into equation (4.14), we have

$$
\begin{equation*}
\left.I I\right|_{r=a}=2 \dot{\gamma}_{r \theta}^{2} \tag{4.68}
\end{equation*}
$$

From equations (4.3), (4.4) and (4.34), using the no slip conditions on $r=a$ we obtain

$$
\left.\dot{\gamma}_{r \theta}\right|_{r=a}=\left.\frac{1}{a \sin \theta} \frac{\partial^{2} \psi}{\partial r^{2}}\right|_{r=a} \quad(4.69)
$$

From equation (4.6) we have

$$
-\left.\frac{1}{r} \frac{\partial p}{\partial \theta}\right|_{r=a}=\left.\frac{1}{r^{3}} \frac{\partial}{\partial r}\left(\tau_{r \theta} r^{3}\right)\right|_{r=a} \quad(4.70)
$$

Substituting equations (4.13) and (4.69) into equation (4.70) we obtain

$$
\begin{equation*}
\left.\frac{\partial p}{\partial \theta}\right|_{r=a}=\left.\left(\frac{\pi}{2}\right)^{n-1}\right|_{r=a} \frac{k}{\sin \theta}\left[\frac{\partial^{3} \psi}{\partial r^{3}}+(n-1)\left(\frac{\partial^{3} \psi}{\partial r^{3}}-\frac{3}{a} \frac{\partial^{2} \psi}{\partial r^{2}}\right)\right]_{r=a} \tag{4.71}
\end{equation*}
$$

From equations (4.13), (4.66) and (4.68) - (4.71) we have

$$
\begin{align*}
F_{D}= & 2 \pi k a^{2} \int_{0}^{\pi}\left[\frac{1}{a^{2} \sin ^{2} \theta}\left(\frac{\partial^{2} \psi}{\partial r^{2}}\right)^{2}\right]_{r=a}^{\frac{n-1}{2}}\left[\frac{1}{2} \frac{\partial^{3} \psi}{\partial r^{3}}-\frac{1}{a} \frac{\partial^{2} \psi}{\partial r^{2}}\right] \\
& +\left(\frac{n-1}{2}\right)\left[\frac{\partial^{3} \psi}{\partial r^{3}}-\frac{3}{a} \frac{\partial^{2} \psi}{\partial r^{2}}\right] \sin _{r=a} \theta d \theta \tag{4.72}
\end{align*}
$$

We now define the non-dimensional variables $\alpha_{2}, \alpha_{3}, \beta_{2}$ and $\beta_{3}$ by the equations

$$
\begin{align*}
& U \alpha_{2}=\left.\frac{\partial^{2} \psi_{N}}{\partial r^{2}}\right|_{r=a}  \tag{4.73}\\
& \frac{U}{a} \alpha_{3}=\left.\frac{\partial^{3} \psi_{N}}{\partial r^{3}}\right|_{r=a}  \tag{4.74}\\
& U \beta_{2}=\left.\frac{\partial^{2} \psi N N}{\partial r^{2}}\right|_{r=a}  \tag{4.75}\\
& \frac{U}{a} \beta_{3}=\left.\frac{\partial^{3} \psi_{N N}}{\partial r^{3}}\right|_{r=a} \tag{4.76}
\end{align*}
$$

In equation (4.72) we require the evaluation of the term

$$
\begin{equation*}
\left[\left(\frac{\partial^{2} \psi}{\partial r^{2}}\right)^{2}\right]_{r=a}^{\frac{n-1}{2}}=\left|\frac{\partial^{2} \psi}{\partial r^{2}}\right|_{r=a}^{n-1} \tag{4.77}
\end{equation*}
$$

From equations (4.73), (4.75) and (4.26)

$$
\begin{equation*}
\left|\frac{\partial^{2} \psi}{\partial r^{2}}\right|_{r=a}^{n-1}=U^{n-1}\left|\alpha_{2}+(n-1) \beta_{2}\right|^{n-1} \tag{4.78}
\end{equation*}
$$

Using Taylor's expansion

$$
\begin{equation*}
\left|\frac{\partial^{2} \psi}{\partial r^{2}}\right|_{r=a}^{n-1}=U^{n-1}\left(\left|\alpha_{2}\right|^{n-1}(n-1)^{2} \beta_{2} \alpha_{2}\left|\alpha_{2}\right|^{n-3} \cdots\right) \tag{4.79}
\end{equation*}
$$

By ignoring terms of order $(n-1)^{2}$ and higher we obtain

$$
\left.\frac{\partial^{2} \psi}{\partial r^{2}}\right|_{\Gamma=a} ^{n-1}=U^{n-1}\left|\alpha_{2}\right|^{n-1}
$$

Using equation (4.35), equation (4.73) becomes

$$
\begin{equation*}
U \alpha_{2}=\left.\frac{\partial^{2} \psi_{N}}{\partial r^{2}}\right|_{r=a}-\frac{1}{2} U a^{2} \frac{\partial^{2}}{\partial r^{2}}\left[A \frac{a}{r}+B \frac{r}{a}+C \frac{r^{2}}{a^{2}}+D \frac{r^{4}}{a^{4}}\right]_{r=a} \sin ^{2} \theta \tag{4.81}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\alpha_{2}=-[A+C+6 D] \sin ^{2} \theta \tag{4.82}
\end{equation*}
$$

The coefficients $A, C$ and $D$ are expressed in terms of $d$, the ratio of the diameter of the outer sphere to the inner sphere. These expressions are derived in Appendix 3 and are given in equations (A3.31), (A3.29) and (A3.28). Using these equations we obtain

$$
\begin{equation*}
\alpha_{2}=-\frac{3 d\left(2 d^{4}+2 d^{3}+2 d^{2}-3 d-3\right)}{(d-1)^{3}\left(4+7 d+4 d^{2}\right)} \sin ^{2} \theta \tag{4.83}
\end{equation*}
$$

By definition $d>1$ and $\alpha_{2}$ is negative. Thus equation (4.80) becomes

$$
\begin{equation*}
\left|\frac{\partial^{2} \psi}{\partial r^{2}}\right|_{r=a}^{n-1}=U^{n-1}\left(-\alpha_{2}\right)^{n-1} \tag{4.84}
\end{equation*}
$$

Equation (4.72) may, therefore, be rewritten as

$$
\begin{align*}
F_{D}=\pi k a^{2}\left(\frac{U}{a}\right)^{n} & {\left[\int _ { 0 } ^ { \pi } ( - \alpha _ { 2 } ) ^ { n - 1 } \left(\alpha_{3}-2 \alpha_{2}+(n-1)\left(\alpha_{3}-3 \alpha_{2}\right) \sin ^{1-n} \theta \sin \theta d \theta\right.\right.} \\
& \left.+(n-1) \int_{0}^{\pi}\left(-\alpha_{2}\right)^{n-1}\left(\beta_{3}-2 \beta_{2}\right) \sin ^{1-n} \theta \sin \theta d \theta\right] \tag{4.85}
\end{align*}
$$

Since, for a slightly power law fluid, $n-1$ is near zero we may use the approximations

$$
\begin{equation*}
\left(-\alpha_{2}\right)^{n-1}=1+(n-1) \log \left(-\alpha_{2}\right) \tag{4.86}
\end{equation*}
$$

and

$$
\begin{equation*}
(\sin \theta)^{1-n}=1+(1-n) \log (\sin \theta) \tag{4.87}
\end{equation*}
$$

It is convenient to let

$$
\begin{align*}
& \alpha_{2}=\alpha_{2}^{*} \sin ^{2} \theta \\
& \alpha_{3}=\alpha_{3}^{*} \sin ^{2} \theta
\end{align*}
$$

Substituting equations (4.86) - (4.89) into equation (4.85) we obtain

$$
\begin{align*}
F_{D}= & \pi k a^{2}\left(\frac{u}{a}\right)^{n} \int_{0}^{\pi}\left[\left(\alpha_{3}^{*}-2 \alpha_{2}^{*}\right) \sin ^{3} \theta\right. \\
& +(n-1)\left(\alpha_{3}^{*}-3 \alpha_{2}^{*}\right) \sin ^{3} \theta  \tag{4.90}\\
& \left.+(n-1)\left(\alpha_{3}^{*}-2 \alpha_{2}^{*}\right)\left(\log \left(-\alpha_{2}^{*}\right)+\log (\sin \theta)\right) \sin ^{3} \theta\right] d \theta \\
& +\pi k a^{2}\left(\frac{u}{a}\right)^{n}(n-1) \int_{0}^{\pi}\left(\beta_{3}-2 \beta_{2}\right) \sin \theta d \theta
\end{align*}
$$

From equations (4.35), (4.74) and (4.89) we obtain

$$
\frac{U}{a} \alpha_{3}^{*}=-\frac{1}{2} U a^{2} \frac{\partial^{3}}{\partial r^{3}}\left[A \frac{a}{r}+B \frac{r}{a}+C \frac{r^{2}}{a^{2}}+D \frac{r^{4}}{a^{4}}\right]_{r=a}^{(4.91)}
$$

Therefore

$$
\begin{equation*}
\alpha_{3}^{*}=3 A-12 D \tag{4.92}
\end{equation*}
$$

From equations (4.82) and (4.88)

$$
\begin{equation*}
\alpha_{2}^{*}=-A-C-6 D \tag{4.93}
\end{equation*}
$$

In equation (4.90) we require the evaluation of the integrals $I_{1}$ and I2 given by

$$
I_{1}=\int_{0}^{\pi} \sin ^{3} \theta d \theta=\frac{4}{3}
$$

and
(4.94)

$$
I_{2}=\int_{0}^{\pi} \sin ^{3} \theta \log \left(\sin ^{2} \theta\right) d \theta
$$

From Appendix 4, equation (A4.13)

$$
\begin{equation*}
I_{2}=-\frac{20}{9}+\frac{8}{3} \log 2 \tag{4.95}
\end{equation*}
$$

Substituting equations (4.92) - (4.95) into equation (4.90) we obtain

$$
\begin{align*}
F_{D}= & K a^{2}\left(\frac{u}{a}\right)^{n}\left[\frac{4 \pi}{3}(5 A+2 C)\right. \\
& +2 \pi(n-1)\left[(5 A+2 C)\left(\frac{2}{3} \log (2 A+2 C+12 D)-\frac{5}{9}\right)\right. \\
& +4 A+2 C+4 D] \\
& \left.+\pi(n-1) \int_{0}^{\pi}\left(\beta_{3}-2 \beta_{2}\right) \sin \theta d \theta\right] \quad(4.9 \tag{4.96}
\end{align*}
$$

where $\beta_{2}$ and $\beta_{3}$ are defined by equations (4.75) and (4.76) and are evaluated numerically by the method described in Chapter 5.

It should be noted that when the fluid is Newtonian (ie. when $n=1$ ) the coefficients $A$ and $C$ are given by equations (A3.31) and (A3.29) and the drag force on the sphere becomes

$$
\begin{equation*}
F_{D}=24 \pi \cup \eta_{0} a \frac{d\left(1+d+d^{2}+d^{3}+d^{4}\right)}{(d-1)^{3}\left(4+7 d+4 d^{2}\right)} \tag{4.97}
\end{equation*}
$$

where $K=\eta_{0}$ is the Newtonian viscosity.

### 4.3.2 The Drag Force for the Case of an Infinite Outer Sphere

The drag force for the infinite outer sphere problem can be obtained from the finite outer sphere case by letting $d=b / a$ tend to infinity. In this case the constants $A, C$ and $D$ have been derived in Appendix 3 and are given by

$$
\begin{align*}
& A=1 / 2  \tag{4.98}\\
& C=1  \tag{4.99}\\
& D=0
\end{align*}
$$

(4.100)

By substitution into equation (4.96) the drag force becomes

$$
\begin{aligned}
F_{D}=6 \pi a^{2} k & \left(\frac{u}{a}\right)^{n}\left[1+(n-1)\left(\log 3+\frac{1}{2}\right)\right. \\
& \left.+\frac{n-1}{6} \int_{0}^{\pi}\left(\beta_{3}-2 \beta_{2}\right) \sin \theta d \theta\right] \quad(4.101)
\end{aligned}
$$

which may be written as

$$
F_{D}=6 \pi a^{2} k\left(\frac{v}{a}\right)^{n}[1+(n-1) \alpha] \quad \text { (4.102) }
$$

where

$$
\alpha=\log 3+\frac{1}{2}+\frac{1}{6} \int_{0}^{\pi}\left(\beta_{3}-2 \beta_{2}\right) \sin \theta d \theta \quad(4.103)
$$

However the drag force for a general power law fluid is usually expressed in the form

$$
F_{D}=12 \pi a^{2} k\left(\frac{U}{2 a}\right)^{n} F(n)
$$

(4.104)
where $F(n)$ is a function of $n$ only and independent of the geometrical constants. On comparison with our expression for $F_{D}$ in equation (4.102) we have

$$
\frac{F(n)}{2^{n-1}}=1+(n-1) \alpha \quad(4.105)
$$

ie.

$$
F(n)=[1+(n-1) \alpha] 2^{n-1} \quad(4.106)
$$

Ignoring terms of order $(n-1)^{2}$ and higher we have

$$
2^{n-1}=1-(n-1) \log 2 \quad(4 \cdot 107)
$$

and

$$
F(n)=[1+(n-1)(\alpha-\log 2)](4 \cdot 108)
$$

Finally we have

$$
\begin{equation*}
F(n)=1+(n-1) \beta \tag{4.109}
\end{equation*}
$$

where, on using equation (4.106), we have

$$
\begin{equation*}
\beta=\log \frac{3}{2}+\frac{1}{2}+\frac{1}{6} \int_{0}^{\pi}\left(\beta_{3}-2 \beta_{2}\right) \sin \theta d \theta \tag{4.110}
\end{equation*}
$$

where it should be noted that $\beta_{2}$ and $\beta_{3}$ are as defined in equations (4.75) and (4.76) and are functions of $\theta$ only.
$\beta$ is a constant which is related to $\beta_{2}$ and $\beta_{3}$ and can therefore be obtained by solving equation (4.27) numerically and using equations (4.75) and (4.76). The outer boundary condition for equation (4.27) was found using a matching technique over a suitable imaginary sphere placed at a distance of ten or twenty times the radius of the inner sphere. However, to implement this technique we require the far field solution for the stream function.
4.4 The Asymptotic Far Field Stream
Function for an Infinite Outer Sphere

The governing equation for a slightly power law fluid is given by equations (4.27) and (4.59). The asymptotic far field region is valid when $r$ is large. In this region the stream function is given by

$$
\begin{equation*}
\psi_{N}=-U \frac{a^{2}}{2}\left[\frac{1}{2} \frac{a}{r}-\frac{3}{2} \frac{r}{a}+\frac{r^{2}}{a^{2}}\right] \sin ^{2} \theta \tag{4.111}
\end{equation*}
$$

From equations (4.36), (4.37) and (4.58) we obtain

$$
\begin{align*}
& A_{1}=\frac{3 U a}{2 r^{2}} \sin \theta  \tag{4.112}\\
& B_{1}=\frac{3 U a}{r^{3}} \cos \theta
\end{align*}
$$

$$
\begin{align*}
& C_{1}^{*}=\frac{12 \cup a}{r^{3}} \sin \theta \\
& D_{1}^{*}=-\frac{6 \cup a}{r^{3}} \sin \theta
\end{align*}
$$

and ignoring terms of order $1 / r^{m}$ where $m>4$ then equation (4.42) reduces to

$$
\begin{equation*}
I I=\frac{27 v^{2} a^{2}}{2 r^{4}} \cos ^{2} \theta \tag{4.116}
\end{equation*}
$$

By substituting these expressions into equation (4.59) and using equation (4.27)

$$
E^{4} \psi_{N N}=\frac{9 v a_{a}}{r^{3}} \sin ^{2} \theta \quad(4 \cdot 117)
$$

This equation was solved analytically by using a separable solution of the form

$$
\begin{equation*}
\psi_{H N}=g(r) \sin ^{2} \theta \tag{4.118}
\end{equation*}
$$

Substituting into equation (4.117) we obtain

$$
\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r^{2}}\right)\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r^{2}}\right) g(r)=\frac{9 U a}{r^{3}} \quad(4.119)
$$

Let

$$
\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r^{2}}\right) g(r)=g^{*}(r) \quad(4 \cdot 120)
$$

By inspection, a particular integral of equation (4.119) is given by

$$
g^{*}(r)=-\frac{3 U_{a} \log r}{r} \quad(4.121)
$$

and equation (4.121) may be written as

$$
\begin{gathered}
\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r^{2}}\right) g(r)=-\frac{3 \cup a \log r}{r} \quad(4.122) \\
\cdot-39-
\end{gathered}
$$

By inspection a particular integral of equation (4.122) is given by

$$
\begin{equation*}
g(r)=m r \log r+n r \tag{4.123}
\end{equation*}
$$

Comparing terms in equation (4.122) we obtain

$$
\psi_{p}=\frac{3 \cup a}{2}\left(r \log r+\frac{r}{2}\right) \sin ^{2} \theta \quad(4.124)
$$

as a particular integral of equation (4.119). The complementary function, which satisfies $E^{4} \Psi=0$ has the form

$$
\begin{equation*}
\psi_{c}=\left(\frac{A^{\prime}}{r}+B^{\prime} r+C^{\prime} r^{2}+D^{\prime} r^{4}\right) \sin ^{2} \theta \tag{4.125}
\end{equation*}
$$

where $A^{\prime}, B^{\prime}, C^{\prime}$ and $D^{\prime}$ are constants to be determined since

$$
\begin{equation*}
\psi_{N N}=\psi_{C}+\psi_{P} \tag{4.126}
\end{equation*}
$$

Then

$$
\begin{equation*}
\psi_{N N}=\left(\frac{A^{\prime}}{r}+B^{\prime} r+C^{\prime} r^{2}+D^{\prime} r^{4}+\frac{3 v a r}{4}+\frac{3 v a r \log r)}{2} \sin ^{2} \theta\right. \tag{4.127}
\end{equation*}
$$

From equations (4.111) and (4.127) the asymptotic far field value for $\psi$ is given by .

$$
\begin{gathered}
\psi=-\frac{U a^{2}}{2}\left(\frac{a}{2 r}-\frac{3 r}{2 a}+\frac{r^{2}}{a^{2}}\right) \sin ^{2} \theta \\
+(n-1)\left[\frac{A^{\prime}}{r}+B^{\prime} r+C^{\prime} r^{2}+D^{\prime} r^{4}+\frac{3 v a r}{4}+\frac{3 v a r \log r}{2}\right] \sin ^{2} \theta
\end{gathered}
$$

(4.128)

As $r \rightarrow \infty$ we require that $\psi \rightarrow-\frac{U r^{2}}{2} \sin ^{2} \theta$, equation (4.128) may be written as

$$
\begin{gathered}
\psi=-\frac{1}{2} U r^{2} \sin ^{2} \theta \cdot\left[1+\frac{a^{3}}{2 r^{3}}-\frac{3 a}{2 r}\right. \\
\left.-(n-1) \frac{1}{U}\left(\frac{A^{\prime}}{r}{ }_{3}+\frac{B^{\prime}}{r}+C^{\prime}+D^{\prime} r^{2}+\frac{3 U a}{2 r}+\frac{3 V a \log r}{r}\right)\right]
\end{gathered}
$$

(4.129)

For $\psi$ to tend to the required limit

$$
C^{\prime}=D^{\prime}=0
$$

(4.130)
and $\psi_{\mathrm{NN}}$ is given by
$\psi_{N N}=\frac{3}{4}(n-1)\left(\frac{A^{*}}{r}+B^{*} r+\cup a r+2 v a r \log r\right) \sin ^{2} \theta \quad(4.131)$
where

$$
\begin{align*}
& A^{*}=\frac{4}{3} A^{\prime}  \tag{4.132}\\
& B^{*}=\frac{4}{3} B^{\prime}
\end{align*}
$$

and

The asymptotic far field is known to within the value of two unknown constants $A^{*}$ and $\mathrm{B}^{*}$.

The paper by Tiefenbruck and Leal 32 (1982) has shown that, when the outer sphere has an infinite radius, reasonable results for the drag force on the inner sphere can be obtained by matching the asymptotic far field into an imaginary spherical surface concentric with the inner sphere. The radius of this imaginary sphere must be greater than 10 times that of the inner sphere. The drag force over its surface is found using equation (4.96). The term $A^{*} / r$ was assumed small compared with $B^{* / r}$ for the matching radius used and was ignored. This assumption was justified by the fact that when the term was removed from the numerical routines the calculated values for the velocity field were unaffected.

The detailed method for obtaining the drag correction factor numerically, and the results achieved, are given in the next chapter.

## CHAPTER 5

## THE NUMERICAL SOLUTION FOR THE MOTION OF A SPHERE FALLING THROUGH AN INFINITE POWER LAN FLUID

## 5.1 <br> Introduction

This chapter describes the numerical method that is used to solve the fourth order partial differential equation (4.27) which is presented in Chapter 4. This equation will be solved for both the finite outer sphere case and the infinite outer sphere case.

For the numerical solution we rewrite the fourth order partial differential equation in the following way

$$
\begin{equation*}
E^{2} \psi_{N N}=w_{N N} \tag{5.1}
\end{equation*}
$$

and

$$
\begin{equation*}
E^{2} w_{N N}=\frac{1}{2} F\left(\psi_{N}\right) \sin \theta \tag{5.2}
\end{equation*}
$$

where $F\left(\psi_{N}\right)$ is given by equation (4.59). It should be noted that $\omega_{N N}$ does not represent the non-Newtonian vorticity vector.

### 5.2 Boundary Conditions

We shall now require the boundary conditions for $\psi_{\text {NN. }}$. Some of the boundary conditions for $\psi_{\mathrm{NN}}$ have already been presented in Chapter 4 but, for completeness, we shall also present the full boundary conditions.

### 5.2.1 Einite Outer Sphere Case

From equation (5.1) we have

$$
\begin{equation*}
\omega_{N N}=\frac{\partial^{2} \psi_{N N}}{\partial r^{2}}+\frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi_{N N}}{\partial \theta}\right) \tag{5.3}
\end{equation*}
$$

From equation (4.28) we have seen that $\left.\psi_{N N}\right|_{r=a}=0$. Therefore the first and second derivatives of $\psi \mathrm{NN}$ with respect to $\theta$ evaluated on $r=a$ are zero. Hence the boundary condition for $\boldsymbol{w}_{N N}$ on $r=a$ is given by

$$
\begin{equation*}
\left.\omega_{N N}\right|_{r=a}=\left.\frac{\partial^{2} \psi_{N N}}{\partial r^{2}}\right|_{r=a} \tag{5.4}
\end{equation*}
$$

Using a central finite difference scheme, we may approximate $\left.\omega_{\text {NN }}\right|_{r=a}$ by the formula

$$
\begin{equation*}
\left.w_{N N}\right|_{r=a}=\frac{\psi_{N N}(a+d r, \theta)-2 \psi_{N N}(a, \theta)+\psi_{N N}(a-d r, \theta)}{(d r)^{2}} \tag{5.5}
\end{equation*}
$$

However, since $\partial \psi_{\mathrm{NN}} /\left.\partial r\right|_{r=a}=0$ then

$$
\begin{equation*}
\psi_{N N}(a+d r, \theta) \simeq \psi(a-d r, \theta) \tag{5.6}
\end{equation*}
$$

Therefore, on using the result $\psi_{\text {NN }} i_{r=a}=0$ equation (5.5) simplifies to

$$
\begin{equation*}
\left.w_{N N}\right|_{r=a}=\frac{2 \psi_{N N}(a+d r, \theta)}{(d r)^{2}} \tag{5.7}
\end{equation*}
$$

Similarly, the boundary condition for $\mathcal{W}_{N N}$ on $r=b$ is given by

$$
\begin{equation*}
\left.w_{N N}\right|_{r=b}=2 \psi_{N N}(b-d r, \theta) /(d r)^{2} \tag{5.8}
\end{equation*}
$$

To determine the boundary condition for $\omega_{N N}$ on $\theta=0$ we have to express $\boldsymbol{w}_{\mathrm{NN}}$ in a different form. Using equations (4.3) and (4.4) equation (5.3) may be rewritten in the form

$$
\begin{equation*}
w_{N N}=\frac{\partial^{2} \psi_{N N}}{\partial r^{2}}+\sin \theta \frac{\partial}{\partial \theta}\left(V_{r_{N N}}\right) \tag{5.9}
\end{equation*}
$$

where $V_{r_{N N}}$ is the non-Newtonian radial velocity component. By symmetry about $\theta=0$ we have

$$
\begin{equation*}
\partial V_{r_{N N}} / \partial \theta=0 \tag{5.10}
\end{equation*}
$$

From equation (4.28) we also have $\left.\psi_{N N}\right|_{r=a}=0$ which implies that the first and second derivatives of $\psi \mathrm{NN}$ with respect to $r$ on $\theta=0$ are both zero. Therefore, we have

$$
\begin{equation*}
\left.\omega_{N H}\right|_{\theta=0}=0 \tag{5.11}
\end{equation*}
$$

In a similar way, we may show that

$$
\begin{equation*}
\left.w_{N N}\right|_{\theta=\pi}=0 \tag{5.12}
\end{equation*}
$$

The complete boundary conditions for solving the partial differential equations (5.1) and (5.2) are therefore given by

$$
\begin{aligned}
& \left.\psi_{N N}\right|_{r=a}=\left.\psi_{N N}\right|_{r=b}=0 \\
& \left.\psi_{N N}\right|_{\theta=0}=\left.\psi_{N N}\right|_{\theta=\pi}=0 \\
& \left.w_{N N}\right|_{\theta=0}=\left.\omega_{N N}\right|_{\theta=\pi}=0 \\
& \left.w_{N N}\right|_{r=a}=\frac{2 \psi_{N N}(a+d r, \theta)}{(d r)^{2}} \\
& \left.w_{N N}\right|_{r=b}=\frac{2 \psi_{N N}(b-d r, \theta)}{(d r)^{2}}
\end{aligned}
$$

5.2.2 Infinite Outer Sphere Case

In this case the boundary conditions for $\psi_{\mathrm{NN}}$ and $\omega$ NN are similar to those in equations (5.13). However, the boundary conditions on $r=b$ are no longer required since $\psi_{N N}$ and $\omega_{N N}$ are known for large $r$ to within a constant $B^{*}$. The new boundary conditions for $\psi N N$ and $\omega_{N N}$ are discussed in Section 5.3 and are given by .

$$
\left.\Psi_{N N}\right|_{r=R}=\frac{3}{4}\left[B^{*} R+R+2 R \log R\right] \sin ^{2} \theta \quad(5.14)
$$

and

$$
\begin{equation*}
\left.\omega_{N N}\right|_{r=R}=-\frac{3}{2 R}\left[B^{*}+2 \log R\right] \sin ^{2} \theta \tag{5.15}
\end{equation*}
$$

where $R$ is the radius of the matching spherical surface and $B^{*}$ is a constant which has to be determined by the iterative technique discussed in Section 5.3.
5.3

The Numerical Method
5.3.1

The Galerkin Formulation
From equation (4.12) equations (5.1) and (5.2) may be written as

$$
\begin{equation*}
\frac{\partial^{2} \psi_{N N}}{\partial r^{2}}+\frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi_{N N}}{\partial \theta}\right)=w_{N N} \tag{5.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial^{2} \omega_{N N}}{\partial r^{2}}+\frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \omega_{N N}}{\partial \theta}\right)=\frac{F\left(\psi_{N}\right)}{2} \sin \theta \tag{5.17}
\end{equation*}
$$

These equations have to be solved subject to the boundary conditions given in equations (5.13) - (5.15). The domain in the ( $r, \theta$ ) plane over which the finite element mesh is to be constructed is defined by $0 \leqslant \theta \leqslant \pi$ and $a \leqslant r \leqslant R$, where $a$ is the radius of the inner sphere and R is the radius of the imaginary spherical matching surface (Fig. 5).

In order to remove the apparent numerical singularity on the boundaries $\theta=0$ and $\theta=\pi$ equations (5.16) and (5.17) were transformed using the change of variables

$$
\begin{align*}
& r=e^{z}  \tag{5.18}\\
& x=-\cos \theta \tag{5.19}
\end{align*}
$$

The transformed equations are

$$
\begin{equation*}
\frac{\partial^{2} \psi_{N N}}{\partial z^{2}}-\frac{\partial \psi}{\partial z}+\left(1-x^{2}\right) \frac{\partial^{2} \psi_{N N}}{\partial x^{2}}=e^{2 z} w_{N N} \tag{5.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial^{2} w_{N N}}{\partial z^{2}}-\frac{\partial w_{N N}}{\partial z}+\left(1-x^{2}\right) \frac{\partial^{2} w_{N N}}{\partial x^{2}}=e^{2 z} f\left(e^{2}, x\right) \tag{5.21}
\end{equation*}
$$

The finite element mesh is bounded by $Z=\log a, Z=\log R, X=-1$ and $X=1$. A regular finite element mesh was constructed over the ( $2, \mathrm{X}$ ) plane (refer Program 1). From equation (5.18) it follows that in the $(r, \theta)$ plane the mesh has elements which, in the $r$ direction, increase in size away from the inner sphere where both $\psi_{\mathrm{NN}}$ and $\omega_{\mathrm{NN}}$ have their least variation. A typical mesh is shown in Fig. 6.

Let the finite element mesh constructed have $N$ nodes and let $\left\{\phi_{i}: i=1, \cdots, N\right\}$ be a set of global shape functions (refer Section 3.2). Then

$$
\frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z} N \phi_{i}\right)=\frac{\partial^{2} \psi_{N N}}{\partial z^{2}} \phi_{i}+\frac{\partial \psi_{N N}}{\partial z} \frac{\partial \phi_{i}}{\partial z} \text { (5.22) }
$$

and

$$
\begin{aligned}
& \frac{\partial}{\partial x}\left[\left(1-x^{2}\right) \frac{\partial \psi_{N N}}{\partial x} \phi_{i}\right]=\left(1-x^{2}\right) \frac{\partial^{2} \psi_{N N}}{\partial x^{2}} \phi_{i} \\
&+\left(1-x^{2}\right) \frac{\partial \psi_{N N}}{\partial x} \frac{\partial \phi_{i}}{\partial x}-2 x \frac{\partial \psi_{N N}}{\partial x} \phi_{i} \quad \text { (5.23) }
\end{aligned}
$$

By Green's theorem in the plane

$$
\int_{\Omega} \frac{\partial}{\partial z}\left(\frac{\partial \psi_{N N}}{\partial z} \phi_{i}\right) d \Omega=\int_{\Gamma} \phi_{i} \frac{\partial \psi_{N N} n_{z} d \Gamma \quad \text { (5.24) }}{\partial z}
$$

and

$$
\int_{\Omega} \frac{\partial}{\partial x}\left(\left(1-x^{2}\right) \frac{\partial \psi_{N N}}{\partial x} \phi_{i}\right) d \Omega=\int_{\Gamma} \phi_{i}\left(1-x^{2}\right) \frac{\partial \psi_{N N}}{\partial x} \underset{(5.25)}{ } d \Gamma
$$

where $\Omega$ is the area of the finite element mesh, $\Gamma$ is its boundary and $\left(\partial \psi_{N N} / \partial z\right) n_{2}$ and $\left(\partial \psi_{N N} / \partial X\right) n_{X}$ are normal derivatives of $\psi_{N N}$ on $\Gamma$ (ie. $\partial \psi_{N N} / \partial_{n}$ on $\Gamma$ ).

The boundary conditions for $\partial \psi_{\mathrm{NN}} / \partial \mathrm{n}$ (and $\partial \omega_{\mathrm{NN}} / \partial n$ ) on $\Gamma$ are automatically satisfied when the boundary conditions for $\psi_{\mathrm{NN}}$ (and $\omega_{\mathrm{NN}}$ ) are satisfied. The test for this condition is given by Stang and Fix ${ }^{34}$ (1973). Following A.J. Davies 35 (1980) the functions $\phi_{i}$ were chosen to be zero at all nodes on the boundary $\Gamma$.

Therefore

$$
\int_{\Omega} \frac{\partial}{\partial z}\left(\frac{\partial \psi_{N N}}{\partial z} \phi_{i}\right) d \Omega=0
$$

and

$$
\begin{equation*}
\int_{\Omega} \frac{\partial}{\partial X}\left[\left(1-X^{2}\right) \frac{\partial \psi_{N N}}{\partial \dot{X}}\right] d \Omega=0 \tag{5.27}
\end{equation*}
$$

On multiplying equation (5.20) through by $\phi_{i}$, integrating over $\Omega$ and using equations (5.26) and (5.27) we obtain

$$
\begin{align*}
& \int_{\Omega}\left[-\frac{\partial \psi_{N N}}{\partial z} \frac{\partial \phi_{i}}{\partial z}-\left(1-x^{2}\right) \frac{\partial \psi_{N N}}{\partial x} \frac{\partial \phi_{i}}{\partial x}+2 x \frac{\partial \psi_{N N}}{\partial x} \phi_{i}\right. \\
& \left.-\frac{\partial \psi_{N N}}{\partial z} \phi_{i}\right] d \Omega=\int_{\Omega} e^{2 z} \omega_{N N} \phi_{i} d \Omega \tag{5.28}
\end{align*}
$$

for $i=1,2, \ldots, N$.

Similarly, we obtain

$$
\begin{aligned}
& \int_{\Omega}\left[-\frac{\partial w_{N N}}{\partial z} \frac{\partial \phi_{i}}{\partial z}-\left(1-x^{2}\right) \frac{\partial w_{N N}}{\partial x} \frac{\partial \phi_{i}}{\partial x}+2 x \frac{\partial w_{N N}}{\partial x} \phi_{i}\right. \\
& \left.-\frac{\partial w_{N N}}{\partial z} \phi_{i}\right] d \Omega=\int_{\Omega}^{e^{2 z} f\left(e^{z}, x\right) \phi_{i} d \Omega} \text { (5.29) }
\end{aligned}
$$

for $1=1, \ldots, N$

Equations (5.28) and (5.29) were applied over each element in turn. The global shape functions were replaced by the local shape functions $\left\{\phi_{j}^{e}: j=1,2,3,4\right\}$ and the approximations

$$
\begin{array}{ll}
\psi_{N N}=\psi_{N N}^{j} \phi_{j}^{e} & j=1,2,3,4 \\
w_{N N}=w_{N N}^{j} \phi_{j}^{e} & j=1,2,3,4 \tag{5.31}
\end{array}
$$

and
were substituted into equations (5.28) and (5.29). For boundary elements the local shape functions associated with nodes on $\Gamma$ were set to zero. The integrals over the elements were approximated by
the four point Gaussian quadrature rule. The resulting local stiffness matrices were combined, as described in Section 3.4, to form the global stiffness matrix, S. The element load vectors were similarly combined to form the global load vectors $L \psi_{\mathrm{NN}}$ and $\mathrm{L} \omega_{\mathrm{NN}}$. The matrix equations obtained were

$$
\begin{equation*}
S \underline{\psi}_{N N}=L \psi_{N N} \tag{5.32}
\end{equation*}
$$

and

$$
\begin{equation*}
S \underline{w}_{N N}=L w_{N N} \tag{5.33}
\end{equation*}
$$

where $\underline{\Psi}_{\mathrm{NN}}$ is the column vector $\left[\psi_{\mathrm{NN}}^{\prime}, \ldots, \psi_{\mathrm{NN}}^{\mathrm{N}}\right]$
and $\underline{\mathcal{F}}_{\mathrm{NN}}$ is the column vector $\left[\omega_{\mathrm{NN}}^{\prime}, \ldots, \omega_{\mathrm{NN}}^{N}\right]$
Here $\psi_{\mathrm{NN}}^{i}$ and $\omega_{\mathrm{NN}}^{i}$ are the values of $\psi_{\mathrm{NN}}$ and $w_{\mathrm{NN}}$, respectively, at the 1 th node of the mesh.

In order to show how the boundary conditions $\psi_{\mathrm{NN}}$ are implemented numerically, equation (5.32) is written as

$$
\left[\begin{array}{c}
s_{11} \ldots \ldots  \tag{5.34}\\
\cdots \\
s_{i 1}
\end{array} \ldots .\right.
$$

Let the it node of the mesh lie on $\Gamma$ and let

$$
\begin{equation*}
\psi_{N N}^{i}=V \tag{5.35}
\end{equation*}
$$

where the value of V is given by equation (5.13) or (5.14). Equation (5.34) was modified in the following way

where $K$ is chosen to be $10^{24}$ (a suitably large number). Therefore

$$
\begin{equation*}
s_{i 1} \psi_{N N}^{\prime}+\cdots+K s_{i i} \psi_{N N}^{i}+\cdots+s_{i N} \psi_{N N}^{N}=K s_{i i} V \tag{5.37}
\end{equation*}
$$

Since the terms containing $K$ are dominant then equation (5.37) becomes

$$
\begin{equation*}
K s_{i i} \psi_{N N}^{i} \simeq K s_{i i} V \tag{5.38}
\end{equation*}
$$

But $S_{i i} \neq 0$ and hence

$$
\begin{equation*}
\psi_{N N}^{i}=V \tag{5.39}
\end{equation*}
$$

Therefore when equations (5.36) are solved simultaneously, the solution is constrained by the boundary conditions as required. The boundary conditions for $\omega_{\text {NS }}$ were implemented in exactly the same way as those for $\psi \mathrm{NN}^{\circ}$
5.3.2 The Matching Technique

We have already seen from equation (4.131) that
$\psi_{N N}$ for large $r$ is known to within a constant $B^{*}$, when the term in $A^{*}$ is ignored, and is given by

$$
\psi_{N N}=\frac{3}{4}\left(B^{*} r+r+2 r \log r\right) \sin ^{2} \theta(5.40)
$$

Substituting $\psi_{N N}$ from equation (5.40) into equation (5.3) then $w_{N N}$ is also known for large $r$ in terms of the same constant $B^{*}$ and is given by

$$
\begin{equation*}
\omega_{N N}=-\frac{3}{2 r}\left(B^{*}+2 \log r\right) \sin ^{2} \theta \tag{5.41}
\end{equation*}
$$

The iterative technique that is used to determine the unknown constant $B^{*}$ is now discussed.

The value of $B^{*}$ can be obtained in one of two ways.
(i) We consider $\partial \psi_{N N} / \partial r$ to be continuous across some imaginary spherical surface $r=R$

$$
\text { i.e. }\left.\quad \frac{\partial \psi}{\partial r} N\right|_{r=R-}=\left.\frac{\partial \psi_{N N}}{\partial r}\right|_{r=R+} \quad(5.42)
$$

(ii) We consider $\partial \omega_{N N} / \partial r$ to be continuous across the imaginary boundary $r=R$

$$
\text { i.e. }\left.\quad \frac{\partial \omega_{N N}}{\partial r}\right|_{r=R_{-}}=\left.\frac{\partial \omega_{N N}}{\partial r}\right|_{r=R+}(5.43)
$$

The value of $B^{*}$ can be obtained either by matching $\partial \psi_{N N} / \partial r$ or by matching $\partial \omega_{N N} / \partial r$ across the imaginary spherical boundary $r=R$. Obviously we can only satisfy the derivatives of the stream function for a particular value of $\theta$. In our case, this was chosen to be $\theta=\pi / 2$. We shall now discuss these two methods in detail.
(i) We consider

$$
\left.\frac{\partial \psi_{N N}}{\partial r}\right|_{\theta=R-}=\left.\frac{\partial \psi_{N N}}{\partial r}\right|_{r=R+} \quad(5.44)
$$

Now $\left.\Psi_{\text {KN }}\right|_{r=R_{+}}$is given by equation (5.40). Hence, on differentiating equation (5.40) with respect to $r$ and substituting into equation (5.42) we obtain

$$
\left.\frac{\partial \psi N N}{\partial r}\right|_{r=R-}=\frac{3}{4}\left(B^{*}+3+2 \log r\right)(5.45)
$$

By rearranging this equation we may express $B^{*}$ as

$$
B^{*}=\left.\frac{4}{3} \frac{\partial \psi_{N N}}{\partial r}\right|_{\substack{r=R_{-}-3-2 \log r \\ \theta=\pi / 2}}(5.46)
$$

It should be noted that $\partial \psi_{N N} /\left.\partial r\right|_{r=R}$ must be determined from the numerical stream function using a forward finite difference approximation.
(ii) $\quad W e$ consider $\partial \omega_{\mathrm{NN}} / \partial r$ to be continuous across the matching surface $r=R_{\text {. }}$

$$
\text { i.e. }\left.\quad \frac{\partial w}{\partial r} N\right|_{\substack{r=R-\\ \theta=\pi / 2}}=\left.\frac{\partial w}{\partial r} N N\right|_{\theta=R+2} ^{r=R+} \quad \text { (5.47) }
$$

We can determine $\partial \omega_{N N} / \partial r \left\lvert\, \begin{gathered}r=R+ \\ \theta=\pi / 2\end{gathered}\right.$ by differentiating equation (5.41). On substituting into equation (5.43) we obtain

$$
\left.\frac{\partial w_{N N}}{\partial r}\right|_{r=R-} ^{\theta=\pi / 2}=\frac{3}{2 r^{2}}\left(B^{*}-2+2 \log r\right)(5.48)
$$

On rearranging this equation we may express $B^{*}$ as

$$
\begin{equation*}
B^{*}=\left.\frac{2}{3} r^{2} \frac{\partial \omega}{\partial r} N N\right|_{\substack{r=R-\\ \theta=\pi / 2}}+2-2 \log r \tag{5.49}
\end{equation*}
$$

We again note that $\partial \omega_{N N} / \partial r \left\lvert\, \begin{gathered}r=R- \\ \theta=\pi / 2\end{gathered}\right.$ is the derivative of the numerical solution for $\omega_{N N}$ and can be evaluated by a forward difference scheme.

Equations (5.46) and (5.49) offer two possible methods for evaluating the constant $\mathrm{B}^{*}$. The constant, however, can only be determined by an iterative procedure with an initial guess being made for $B^{*}$. We chose $B^{*}=-1.5$ as our initial guess, which is based on the Newtonian solution. The exact iterative procedure which was used to solve the differential equation (4.27) is described in Fig. 7. The reference to 'update ' in this figure refers to the use of equation (5.46) or (5.49). However, we were not able to obtain convergence from either of these two methods for calculating $B^{*}$. It was found that the average of these two equations gave rise to a convergent solution - i.e. we used

$$
B^{*}=\left.\frac{2}{3} \frac{\partial \psi}{\partial r}\right|_{\substack{ \\\theta=\pi / 2}} ^{r=R-}+\left.\frac{r^{2}}{3} \frac{\partial \omega}{\partial r}\right|_{\theta=\pi / 2} ^{r=R-}-\frac{1}{2}-2 \log r(5.50)
$$

to calculate $B^{*}$ iteratively.
When the solution has converged the numerical values for $\psi_{N N}, W_{N N}$ and their derivatives are known for all $r$. In particular, at $r=a$. The non-Newtonian drag correction factor can therefore be evaluated from equations (4.109) and (4.110) where the constants $\beta_{2}$ and $\beta_{3}$ are defined by equations (4.75) and (4.76). It
appears therefore, that, in order to calculate the drag force, we require the second and third derivative of the stream function at $r=a$. However, from equation (5.3) we have

$$
\begin{equation*}
\left.W_{N N}\right|_{r=a}=\left.\frac{\partial^{2} \psi_{N N}}{\partial r^{2}}\right|_{r=a} \tag{5.51}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\frac{\partial \omega_{N N}}{\partial r}\right|_{r=a}=\left.\frac{\partial^{3} \psi N N}{\partial r^{3}}\right|_{r=a} \tag{5.52}
\end{equation*}
$$

Therefore, the drag coefficient can be evaluated in terms of $\omega_{\mathrm{NN}} \mid r=a$ and $\partial \omega_{N N} / \partial r{ }_{i r=a}$

### 5.4 Results

### 5.4.1 Results for the Case when the Outer Sphere has a Einite Radius

In order to test the program validity, the drag force on the inner sphere was produced for the case of a Newtonian fluid when the outer sphere had a finite radius. This drag force is given theoretically by equation (4.97) where $d$ is defined to be the ratio of the radii of the inner and outer spheres. For values of $d$ between 2 and 100, the numerical value obtained for the drag force on the inner sphere agreed to within $0.5 \%$ with the value calculated using equation (4.97). In all cases, this accuracy could be achieved using a finite element mesh having 20 elements in the $\theta$ direction and 40 elements radially.

The program was modified for the case of a slightly power law fluid when the outer sphere had a finite radius. In this case, the drag correction factor $F(n, d)$ was calculated numerically for $d=2,10,40$ and 100. When $d$ was 100, a finite element mesh with 20 elements in the $\theta$ direction and 100 elements radially was used to obtain a stable value for $F(n, d)$. The values are given in Fig. 8 and Table 1. In Fig. 8 the results of Lockyer, Davies and Jones 37 (1980) are also presented. These results have been obtained by means of a finite difference method. In order to approximate the flow situation
when the outer sphere has an infinite radius, a value of $d$ significantly greater than 100 would need to be used. In both the finite element and the finite difference method, this is impractical due to rounding errors, demands on the computer memory and running costs. Therefore to overcome these difficulties, a matching technique was developed.

### 5.4.2 Results for the Case when the Outer Sphere has an Infinite Radius

In order to check our program, the drag force on the inner sphere, when the fluid was Newtonian and the outer sphere had an infinite radius, was obtained numerically. When a finite element mesh having 20 elements in the direction and 140 elements radially was used this value was within $1 \%$ of the drag force value calculated using Stoke's law. Approximately 4000 iterations were required for a convergent solution.

The drag correction factor $F(n)$ was then calculated for the case when the fluid was slightly power law and the outer sphere had an infinite radius. Results were obtained for the case when the matching surface had a radius of 10 times and 20 times the radius of the inner sphere and are presented in Figs. 9 and 10 and Tables 2 and 3.

For a grid having 20 elements in the $\theta$ direction and 100 elements in the radial direction and using a matching radius of 10 times that of the inner sphere we obtained a value of

$$
F^{\prime}(n)=-1.49911
$$

and hence $F(n)$ is given by equation (4.109) as

$$
F(n)=1-1.49911(n-1)
$$

It is evident from Fig. 9 that the value for $F^{\prime}(n)$ has not reached equilibrium and it is likely that more radial elements are required. It appears from Fig. 9 that the value of $F^{\prime}(n)$ for a mesh having 20 elements in the $\theta$ direction, 70 radial elements and a matching radius

10 times that of the inner sphere differs very little from the value $F^{\prime}(n)$ for a mesh having 20 elements in the $\theta$ direction, 140 radial elements and a matching radius 20 times the radius of the inner sphere. This implies that $F^{\prime}(n)$ is independent of the matching radius chosen for a given density of the finite elements.

### 5.5 Discussion

In Fig. 11 we present a comparison of our expression in equation (5.54) with the drag correction factor predictions of other workers (Chhrabra36 (1985)). Most of the theoretical curves 17 were produced using a variational principle method. In 1980 Lockyer et al 37 solved the slow flow of a sphere moving through a power law fluid using numerical techniques. However, the analysis was restricted to power law constants near unity and the use of a finite outer sphere. This result is presented in curve 8. Using finite element techniques Tanner31 (1985) extended the work of Lockyer to the case where the theory was valid for any power law index $n$ (see curve 9). However, the work was carried out for a finite outer sphere with an outer to inner diameter ratio of 20 to 1. A similar approach was used by Crochet 30 (1984) who considered the solution for a sphere moving inside a cylinder with a diameter ratio of 50 to 1 (see curve 10). Although the work of Tanner and Crochet was valid for any power law index $n$ it was, however, restricted to the use of a finite outer boundary and a direct comparison is not possible with curves 1 to 7 which were produced for the flow of a sphere in an infinite expanse of liquid. The work considered in this chapter, is valid for the case of an infinite outer sphere and is presented in curve 11. We also present the drag correction factor for a finite outer sphere, $d=100$, obtained by using the finite element tecnnique (curve 12) and good agreement was obtained with the result of Lockyer in curve 8.

Since our analysis is only valid for a power law index near unity, then we can also make a comparison with the work of Tanner and Crochet, when $n$ is near unity. Assuming that the drag coefficient is linear with $n$ in the region 0.9 to 1.0 then $F^{\prime}(n)$ produced by Tanner has a value of -1.4 , which is of lower magnitude
than our infinite outer sphere value of -1.4991 as expected. We also note that for all values of $n$, the curve produced by Crochet lies above that produced by Tanner. The result is to be expected since Crochet has chosen an outer cylinder to inner sphere diameter ratio of 50 to 1, in comparison with the outer sphere to inner sphere diameter ratio of 20 to 1 used by Tanner.

The oil drilling industry relies heavily on the ability of an oil drilling fluid to transport rock cuttings from the bottom of an oil well to the surface. This is obviously important for good hole cleaning properties, since a high settling velocity can lead to drilling problems. It is therefore important to be able to accurately predict the settling velocity of a particle moving through an oil drilling fluid. It has been shown (Reynolds 23 (1982) private communication), that the settling velocity of spherical particles provide an adequate prediction of the settling velocity of rock particles of the same mass and volume. The work considered in this chapter will obviously be a useful contribution to the understanding of the mechanism necessary for the transport of rock cuttings.

# PULSATILE FLUID FLOW THROUGH A STRAIGHT HORIZONTAL PIPE OF CIRCULAR CROSS SECTION 

## 6.1

## Introduction

An experimental investigation carried out by Barnes et $a^{38}$ (1971) has shown that when a sinusoidal pressure gradient is superimposed on a steady flow of visco-elastic liquid in a straight tube of circular cross section the mean flow rate is increased for a given mean pressure gradient. That is flow enhancement occurs during the pulsatile flow of a visco-elastic liquid through a straight tube.

Flow enhancement associated with pulsatile flow is of considerable industrial interest. For example, some mechanical pumps produce a pulsatile gradient as a side effect. These fluctuations are usually deliberately smoothed out and the liquid is, in effect, pumped under a steady pressure gradient. If, however, flow enhancement occurs during the pumping process it may well be advantageous to dispense with smoothing the flow.

The possibility of deliberately increasing flow rates by pulsing is attractive only if the advantages gained are not offset by the increase in power required to maintain the pulsations. This question has been addressed by Edwards et al 39 (1972) and Phan-Thien and Dudek 40 (1982). These workers have shown that when a power law fluid is used, then there is no economic advantage to be gained in pulsating the flow.

An analysis of the pulsatile flow behaviour has been carried out for several different visco-elastic fluids. These include the second, third and fourth order Oldroyd fluids (Walters and Townsend 41 (1968), Barnes et al (1971)), the Goddard-Miller fluids (Davies et $1^{42}$ (1978)) and the generalised Maxwell fluids (Phan-Thien ${ }^{43,44}(1978,1980)$ ). The experimental results of Barnes et al show that the mean flow rate increased with pulsatile frequency when the ratio of the pulsatile pressure amplitude to the steady pressure gradient was held constant. These results are in contradiction with the theoretical predictions obtained by Walters and Townsend and Davies et al. They are, however, in some agreement
with the theoretical predictions of Phan-Thien.
A conventional apparatus was used by Barnes et al in which the mean pressure gradient was controlled and the mean flow rate was measured. With this arrangement whenever a comparison of the experimental results is made with theory the amplitude of the pulsatile pressure gradient must be measured. This measurement is difficult to obtain accurately and can, therefore, be subject to error. This is particularly true at high frequencies due to the greater effect of inertia in the transducer connecting tube. An alternative apparatus has been constructed by Chakrabati and Davies 45 (1980) who have carried out experimental work in which the mean flow rate and the pulsatile flow rate were controlled and the pressure gradient was measured. Their apparatus is shown, schematically, in Fig. 12.

The following points regarding their experiment are worth noting:
(1) a known steady flow rate was controlled by the constant movement of a piston in a resevoir of an Instron machine and so no smoothing was necessary.
(2) the oscillatory device took the form of a moving oscillatory piston in a sealed chamber and therefore the oscillatory flow rate was known exactly.
(3) the pressure transducer was activated by the pressure of a light, low viscosity oil in the connecting tubes. This light oil was able to faithfully follow the pressure variations in the straight tube. The pressure amplitude measurement, however, is not required for the comparison of experimental results with theoretical predictions but can be used as a check for the consistency of data.

It has been shown by Davies et al ${ }^{42}$ (1978) that flow enhancement is strongly dependent on the shape of the viscosity shear rate curve. Therefore it is important to choose a model which has the correct viscosity behaviour. The Goddard-Miller model used by these authors had a power law viscosity function to predict flow enhancement behaviour but the theory was restricted to small oscillatory amplitudes. Townsend 46 (1973) has carried out his analysis using an Oldroyd four constant model, the theory being valid for any oscillatory pressure amplitude. The Oldroyd model, however,
does not give a realistic viscosity function. In this Chapter we shall be concerned with an inelastic power law fluid analysis which is not restricted to small amplitudes and which also includes the effects of fluid inertia. The relevant equations of motion will be solved numerically. A perturbation analysis was also carried to provide analytical expressions for the flow enhancement when the effects of fluid inertia are small.
6.2

Pulsatile Flow Analysis
6.2.1 Basic Equations

The numerical analysis given here initially follows the work of Edwards et al 39 (1972). As a first check against the validity of the numerical procedures we rederived the numerical results of these authors. The program was then modified to allow for the case when the pulsed flow rate behaviour is known.

A cylindrical polar coordinate system $(r, \theta, z)$ was used with the positive z-axis in the direction of flow. It was assumed that the length of the tube was large compared to its radius and, consequently, edge effects were ignored.

A velocity field of the form
$V_{r}=0, \quad V_{\theta}=0, \quad V_{z}=w(r, t)$
was assumed. This velocity field automatically satisfies the equation of continuity. When body forces are omitted the equations of motion become

$$
\begin{align*}
\rho \frac{\partial w}{\partial t} & =-\frac{\partial p}{\partial z}-\left[\frac{1}{r} \frac{\partial}{\partial r}\left(r \tau_{r z}\right)\right] \\
\frac{\partial p}{\partial r} & =0 \tag{6.2}
\end{align*}
$$

$$
\frac{\partial p}{\partial \theta}=0
$$

where

$$
\begin{equation*}
\tau_{r z}=-k \dot{\gamma}^{n-1} \dot{\gamma}_{r z} \tag{6.3}
\end{equation*}
$$

Here $\dot{\gamma}_{i j}$ is the rate of strain tensor and $\tau_{i j}$ is the stress tensor. From equations (6.1) and (6.3) we obtain

$$
\begin{equation*}
\tau_{r z}=-k\left|\frac{\partial w}{\partial r}\right|^{n-1} \frac{\partial w}{\partial r} \tag{6.4}
\end{equation*}
$$

and the equation of motion becomes

$$
\begin{equation*}
\rho \frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+\frac{1}{r} \frac{\partial}{\partial r}\left(r k\left|\frac{\partial w}{\partial r}\right|^{n-1} \frac{\partial w}{\partial r}\right) \tag{6.5}
\end{equation*}
$$

where $\partial p / \partial z$ is a function of $t$ only.

### 6.2.2 The Case for Steady Flow

When the flow is steady $w=w(r)$ only and $\partial w / \partial t=0$.
The equation of motion becomes

$$
\begin{equation*}
r \frac{\partial p}{\partial z}=k \frac{\partial}{\partial r}\left(r\left(-\frac{\partial w}{\partial r}\right)^{n-1} \frac{\partial w}{\partial r}\right) \tag{6.6}
\end{equation*}
$$

Integrating equation (6.6) with respect to $r$ we obtain

$$
\begin{equation*}
\frac{r}{2} \frac{\partial p}{\partial z}=-k\left(-\frac{\partial w}{\partial r}\right)^{n} \tag{6.7}
\end{equation*}
$$

where $-\partial w / \partial r$ is positive. The mean flow rate, $Q$, is given by

$$
\begin{equation*}
Q=\pi a^{2} \bar{u}=\int_{0}^{a} 2 \pi r w d r \tag{6.8}
\end{equation*}
$$

where $\bar{u}$ is the mean velocity of the fluid and a is the radius of the circular pipe. Integrating by parts and using the no slip condition at the pipe walls we obtain

$$
\begin{equation*}
a^{2} \bar{u}=\int_{0}^{a} r^{2}\left(-\frac{\partial w}{\partial r}\right) d r \tag{6.9}
\end{equation*}
$$

From equations (6.7) and (6.9) using $\partial p / \partial r=0$

$$
\begin{equation*}
\bar{u}=\frac{1}{a^{2}}\left(\frac{1}{2 k}\right)^{\frac{1}{n}}\left(-\frac{\partial p}{\partial z}\right)^{\frac{1}{n}} \int_{0}^{a} r^{2+\frac{1}{n}} d r \tag{6.10}
\end{equation*}
$$

and so

$$
\begin{equation*}
\frac{\partial p}{\partial z}=\frac{-\bar{u} 2 k}{a^{n+1}}\left(\frac{3 n+1}{n}\right)^{n} \tag{6.11}
\end{equation*}
$$

we shall denote the steady pressure gradient by $-p_{s}$ and so

$$
\begin{equation*}
P_{s}=\frac{2 \bar{u}^{n} k}{a^{n+1}}\left(\frac{3 n+1}{n}\right)^{n} \tag{6.12}
\end{equation*}
$$

### 6.2.3 <br> The Case for Pusatile Flow

In the case of pulsatile flow we let

$$
\begin{equation*}
\frac{\partial p}{\partial z}=-p_{s}(1+\epsilon \cos \omega t) \tag{6.13}
\end{equation*}
$$

where $\mathcal{\sigma}$ is the frequency in radians per second and $\epsilon$ is the ratio between the pulsed pressure amplitude and the steady pressure gradient amplitude. The pressure gradient is now a function of time i.e.

$$
-\frac{\partial p}{\partial z}=P(t)
$$

and the equation of motion becomes

$$
\begin{equation*}
\rho \frac{\partial W}{\partial t}=P(t)-\frac{1}{r} \frac{\partial}{\partial r}\left(r K\left(\frac{\partial W}{\partial r}\right)\left|\frac{\partial W}{\partial r}\right|^{n-1}\right) \tag{6.15}
\end{equation*}
$$

The equation can now be non-dimensionalised using the following changes of variable

$$
T=\frac{w t}{2 \pi}, \quad R=\frac{r}{a}, \quad V=\frac{w}{\bar{u}} \quad(6.16)
$$

to give

$$
\bar{u} \rho \frac{W}{2 \pi} \frac{\partial V}{\partial T}=P(T)+\frac{K \bar{u}^{n}}{a^{n+1}} \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \text { (6.17) }
$$

where

$$
\begin{align*}
& P(T)=\frac{\bar{u}^{n}}{a^{n+1}} 2 k \alpha P^{*}(T) \\
& P^{*}(T)=1+\epsilon \cos 2 \pi T \tag{6.19}
\end{align*}
$$

and

Thus

$$
\begin{equation*}
\alpha=\left(\frac{3 n+1}{n}\right)^{n} \tag{6.20}
\end{equation*}
$$

$$
\begin{equation*}
\left(\frac{a^{n+1}}{\bar{u}^{n} k} w \rho \frac{\bar{u}}{2 \pi}\right) \frac{\partial V}{\partial T}=2 \alpha \rho^{*}+\frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \tag{6.21}
\end{equation*}
$$

Defining a Reynold's number by

$$
\begin{equation*}
\operatorname{Re}=8\left(\frac{n}{3 n+1}\right) \rho \frac{\bar{u}^{2-n} a^{n}}{k} \tag{6.22}
\end{equation*}
$$

we have

$$
\frac{F \alpha}{8} \frac{\partial V}{\partial T}=2 \alpha P^{*}(T)+\frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \quad \text { (6.23) }
$$

where $F=\frac{a \omega R_{e}}{2 \pi \bar{u}}$ is dimensionless
Equation (6.23) has to be solved subject to the boundary conditions
(i) $V(1, T)=0 \quad T \geqslant 0$
$\left.\begin{array}{cc}\text { (ii) } & \left.\frac{\partial V}{\partial R}\right|_{R=1}=0\end{array}\right] \geqslant 0$.

These boundary conditions correspond to the physical conditions
(i) there is no slip at the pipe wall
(ii) the fluid shear rate along the tube axis is zero
(iii) the fluid is considered, initially, to be at rest

## 6.2 .4 <br> The Numerical Method for the Case when the Pressure Gradient is known

Equation (6.23) was solved using the explicit difference scheme employed by Edwards et al (1972). The domain was divided by intervals $\Delta R$ and $\Delta T$, the velocity at any point being $V_{I}$, J where $I$ indicates the radial position and $J$ the time step. The radius, $R=0$ to $R=1$, was divided into $N-1$ intervals giving $\Delta R=1 /(N-1)$.

In order to improve the numerical accuracy of the scheme we follow Edwards et al and use forward and backward differences as follows

$$
\begin{aligned}
& \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \\
& \simeq \frac{1}{R \Delta R}\left[\left.(R+\Delta R) \frac{\partial V}{\partial R}\right|_{R+\Delta R}\left|\frac{\partial V}{\partial R}\right|_{R+\Delta R}^{n-1}-\left.R \frac{\partial V}{\partial R}\right|_{R}\left|\frac{\partial V}{\partial R}\right|_{R}^{n-1}\right]
\end{aligned}
$$

And

$$
\begin{align*}
& \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right)  \tag{6.26}\\
& \simeq \frac{1}{R \Delta R}\left[\left.R \frac{\partial V}{\partial R}\right|_{R}\left|\frac{\partial V}{\partial R}\right|_{R}^{n-1}-\left.(R-\Delta R) \frac{\partial V}{\partial R}\right|_{R-\Delta R}\left|\frac{\partial V}{\partial R}\right|_{R-\Delta R}^{n-1}\right] \tag{6.27}
\end{align*}
$$

Equation (6.26) was derived using a backward difference approximation for $\partial V / \partial R$ while equation (6.27) was derived using the corresponding forward difference aproximation. The two approximations given by the equations were then averaged to give

$$
\begin{aligned}
& \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \\
& \simeq \frac{1}{R(\Delta R)^{n-1}}\left[\left(R+\frac{\Delta R}{2}\right)\left(V_{I+1, J}-V_{I, J}\right)\left|V_{I+1, J}-V_{I, J}\right|^{n-1}\right. \\
& -\left(R-\frac{\Delta R}{2}\right)\left(V_{I, J}-V_{I-1, J}\right)\left|V_{I, J}-V_{I-1, J}\right|^{n-1} \text { (6.28) }
\end{aligned}
$$

A forward difference was used for $\partial \mathrm{V} / \partial \mathrm{T}$

$$
\left(\frac{\partial V}{\partial T}\right)_{I, J}=\frac{1}{\Delta T}\left(V_{I, J+1}-V_{I, J}\right) \quad(6.29)
$$

Substituting equations (6.28) and (6.29) into equation (6.23) we have an explicit finite difference scheme for $V_{I, J+1}$ in terms of $V_{I-1}, J$, $V_{I}, J$ and $V_{I+1}$,, valid except when $I=1$ and $I=N$. These values of I correspond to the pipe centreline $(R=0)$ and the pipe wall ( $R=1$ ) On the pipe wall, by boundary condition (i) of equation (6.25)

$$
Y_{N, J}=0 \quad J \geqslant 1 \quad(6.30)
$$

On the pipe centreline equation (6.28) has a singularity since $R=0$. This singularity is removed using L'Hopitals theorem. By this theorem

$$
\begin{equation*}
\operatorname{Lim}_{R \rightarrow 0} \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right)=\left.2 \frac{\partial}{\partial R}\left(\frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right)\right|_{R=0} \tag{6.31}
\end{equation*}
$$

Using a forward difference approximation for $\partial V / \partial R|\partial V / \partial R|^{n-1}$ and a backward difference approximation for $2 \partial / \partial R$ we have

$$
\begin{align*}
\operatorname{Lim}_{R \rightarrow 0} & \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \\
& \simeq \frac{2}{(\Delta R)^{n+1}}\left[\left(V_{I+1, J}-V_{I, J}\right)\left|V_{I+1, J}-V_{I, J}\right|^{n-1}\right. \\
& \left.-\left(V_{I, J}-V_{I-1, J}\right)\left|V_{I, J}-V_{I-1, J}\right|^{n-1}\right] \tag{6.32}
\end{align*}
$$

From boundary condition (ii) of equation (6.25)

$$
\begin{equation*}
V_{I+1, J}=V_{I-1, J} \tag{6.33}
\end{equation*}
$$

when $I=1$. Substituting equation (6.33) into equation (6.32) we obtain

$$
\begin{equation*}
\operatorname{Lim}_{R \rightarrow 0} \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \simeq \frac{4}{(\Delta R)^{n+1}}\left(V_{2, J}-V_{1, J}\right)\left|V_{2, J}-V_{1, J}\right|^{n-1} \tag{6.34}
\end{equation*}
$$

Thus $V_{I, J}$ can now be approximated numerically for $1 \leqslant I \leqslant N$ and $1 \leqslant J \leqslant U$ where $U$ is some fixed upper limit for $J$.

To confirm the correctness of the program the results of Edwards et al were rederived.

## 6.2 .5 <br> Modifying the Numerical Method for the Case when the Flow Rate is known

Equation (6.23) was now modified and expressed in terms of the flow rate rather than the pressure gradient since in our case it is the flow rate, and not the pressure gradient, which is known as a function of time. Using equation (6.16) the flow rate from equation (6.8) can be expressed as

$$
\begin{equation*}
Q=\int_{0}^{1} 2 \pi a^{2} \bar{u} V R d R \tag{6.35}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\frac{\partial Q}{\partial T}=2 \pi a^{2} \bar{u} \int_{0}^{1} \frac{\partial V}{\partial T} R d R \tag{6.36}
\end{equation*}
$$

Integrating equation (6.23) with respect to $R$ and using equation (6.36) we have

$$
\begin{align*}
\frac{F \alpha}{8} \frac{\partial Q}{\partial T}= & 4 \pi \alpha a^{2} \bar{u}\left[\frac{R^{2}}{2}\right]_{0}^{1} P^{*}(T) \\
& +2 \pi a^{2} \bar{u}\left[R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right]_{0}^{1} \tag{6.37}
\end{align*}
$$

We define a non-dimensional flow rate $Q^{*}$ by

$$
Q=\pi a^{2} \bar{u} Q^{*}
$$

Substituting equation (6.38) into equation (6.37) we obtain

$$
\begin{equation*}
\frac{F}{16} \frac{\partial Q^{*}}{\partial T}=P^{*}(T)+\frac{1}{\alpha}\left(\frac{\partial V}{\partial R}\right)_{R=1}\left|\frac{\partial V}{\partial R}\right|_{R=1}^{n-1} \tag{6.39}
\end{equation*}
$$

Eliminating the unknown pressure gradient from equations (6.39) and (6.23) we obtain

$$
\begin{align*}
\frac{F \alpha}{8} \frac{\partial V}{\partial T}= & \frac{F \alpha}{8} \frac{\partial Q^{*}}{\partial T}-2\left(\frac{\partial V}{\partial R}\right)_{R=1}\left|\frac{\partial V}{\partial R}\right|_{R=1}^{n-1} \\
& +\frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial V}{\partial R}\left|\frac{\partial V}{\partial R}\right|^{n-1}\right) \tag{6.40}
\end{align*}
$$

where experimentally it is known that

$$
\begin{equation*}
Q^{*}=1+\epsilon \cos 2 \pi T \tag{6.41}
\end{equation*}
$$

and so

$$
\begin{equation*}
\frac{\partial Q^{*}}{\partial T}=-2 \pi \epsilon \sin 2 \pi T \tag{6.42}
\end{equation*}
$$

Equation (6.42) when substituted into (6.40) gives an equation which can be solved to give the velocity-profile at time T. Hence we can numerically determine $(\partial V / \partial R)_{R=1} \mid \partial V / \partial R_{R=1}^{n-1}$ which on substitution into equation (6.39) will give the value of $P *(T)$.

We used the same finite difference scheme to solve equation (6.40) as we used to solve equation (6.23). However equation (6.40) is insufficient to determine the V-profile uniquely since the steady flow rate component is not considered. To overcome this the $V$-profile was adjusted at the end of each cycle to give a new $V$-profile consistent with $Q^{*}$. The new $V$-profile was expressed in terms of the original V-profile by

$$
\begin{equation*}
V(R)_{\text {NEW }}=V(R)_{\text {ORIG }}+C(R) \tag{6.43}
\end{equation*}
$$

where $C(R)$ is a correction term. At the pipe wall, ie. when $R=1$,
the correction term must be zero since $V$ is held zero here by the no slip condition. A maximum error of $\lambda$ was assumed at the central axis of the pipe, i.e. when $R=0$, and $C(R)$ was taken to be linear in R. Thus

$$
C(R)=\lambda(1-R)
$$

(6. 44 )

Substituting equation (6.38) into equation (6.35) gives

$$
\begin{equation*}
Q^{*}=\int_{0}^{1} 2 V(R) R d R \tag{6.45}
\end{equation*}
$$

Thus, at the end of each cycle we require that

$$
\begin{equation*}
Q^{*}=\int_{0}^{1} 2 V(R)_{N E N} R d R \tag{6.46}
\end{equation*}
$$

On substituting equation (6.43) and (6.44) into equation (6.46) we obtain

$$
\begin{equation*}
Q^{*}=\int_{0}^{1} 2 V(R) R d R+\frac{\lambda}{3} \tag{6.47}
\end{equation*}
$$

which may be written more concisely as

$$
\begin{equation*}
Q^{*}=Q_{N U M}+\frac{\lambda}{3} \tag{6.48}
\end{equation*}
$$

giving

$$
\begin{equation*}
\lambda=3\left(Q^{*}-Q_{\text {MUM }}\right) \tag{6.49}
\end{equation*}
$$

The adjustment of the velocity-profile was carried out at a number of different time steps, from one time step to the full cycle.

When using equation (6.39) the normalised pressure gradient $P^{*}(T)$ was calculated for each $T$ step and it was found that, under certain circumstances, numerical instabilities were present. These instabilities gave rise to alternating sign changes in the pressure values. A routine was incorporated into the program to count the number of sign alterations in each cycle. It was found that repeatable data could be obtained provided that not more than $5 \%$ of the data suffered sign changes. In most cases, the instability could either be removed or reduced to give repeatable results by decreasing $\Delta T$. For some combinations of $N, \in$ and $F$, however, it was not possible to obtain convergence of the solution. Such cases
are indicated in Tables 4 and 5 It was found that for stable, or sufficiently stable, solutions that no more than three cycles were required before flow became fully developed and independent of the starting condition.

Finally, a Fourier analysis was carried out over the values of $P *(T)$ in the third cycle. The fundamental amplitude and higher harmonics were calculated.

The following results were plotted
(i) The variation of the normalised mean pressure gradient with F with N in the range 0.3 to 1.0 for given $\in$. The value of $\epsilon$ being $0.5,1,5$ or 10 . The results are plotted in Figs. 13-16 and the values are tabulated in Tables 4-7.
(ii) As for (i) but the phase angle variation was considered. The results are plotted in Figs. 17-20 and the values are tabulated in Tables 13-16.
(iii) As for (i) but the amplitude variation of the pulsed flow was considered. The results are plotted in Figs. 21-24 and the values are tabulated in Tables $17-20$.
(iv) The normalised mean pressure gradient against $\epsilon$ was considered with F in the range 5 to $100, \mathrm{~N}$ being held constant at 0.5. The results are plotted in Fig. 25 and the values are tabulated in Table 21.

In order to check the correctness of the pressure gradient data, we substituted these values as input data to our original program. The flow rate output data gave good agreement with the known flow rate waveform. In order to check the case when $F$ is small (i.e. when fluid inertia is small) a perturbation analysis was carried out which we shall now consider.

### 6.3 Pulsatile Analysis for Small Fluid Inertia

### 6.3.1 Expressing the Elow Rate in Terms of the Pressure Gradient

## A velocity distribution of the form

$$
w(r, t)=w_{0}(r, t)+\rho w_{1}(r, t)+\rho^{2} w_{2}(r, t)+\cdots \quad(6.50)
$$

was assumed. The boundary conditions which must be satisfied are

$$
\begin{equation*}
w_{0}(a, t)=w_{1}(a, t)=w_{2}(a, t)=0, t \geqslant 0 \tag{6.51}
\end{equation*}
$$

and

$$
\frac{\partial W_{0}}{\partial r}(0, t)=\frac{\partial W_{1}}{\partial r}(a, t)=\frac{\partial W_{2}}{\partial r}(0, t)=0, t \geqslant 0 \quad \text { (6.52) }
$$

Substituting equation (6.50) into equation (6.8) gives

$$
\begin{equation*}
Q(t)=Q_{0}(t)+p Q_{1}(t)+p^{2} Q_{2}(t)+\cdots \tag{6.53}
\end{equation*}
$$

Throughout analysis terms of order $\rho^{3}$ and higher will be ignored.
We have from equation (6.16) that $V=w / \bar{u}$. Substituting this into equation (6.50) gives

$$
V(r, t)=V_{0}(r, t)+\rho V_{1}(r, t)+p^{2} V_{2}(r, t)+\cdots \quad \text { (6.54) }
$$

Substituting $V(r, t)$ from equation (6.54) into equation (6.15) and using Taylor's expansion gives

$$
\begin{align*}
\bar{u}\left(\rho \frac{\partial V_{0}}{\partial t}+\rho^{2} \frac{\partial V_{1}}{\partial t}\right)= & P(t)+\frac{k}{r} \frac{\partial}{\partial r}\left[r | \frac { \partial V _ { 0 } } { \partial r } | ^ { n - 1 } \left(\frac{\partial V_{0}}{\partial r}+\rho n \frac{\partial V_{1}}{\partial r}\right.\right. \\
& \left.\left.+\rho^{2} n\left(\frac{\partial V_{2}}{\partial r}+\frac{n-1}{2} \frac{\left(\partial V_{1} / \partial r\right)^{2}}{\partial V_{0} / \partial r}\right)\right)\right] \bar{u}^{n} \quad(6 \tag{6.55}
\end{align*}
$$

The coefficients of $p^{0}, p^{1}$ and $\rho^{2}$ are now compared as follows: Coefficient of $\rho^{0}$

From equation (6.55)

$$
\begin{equation*}
p(t)+\bar{u}^{n} \frac{k}{r} \frac{\partial}{\partial r}\left(r\left|\frac{\partial V_{0}}{\partial r}\right|^{n-1} \frac{\partial V_{0}}{\partial r}\right)=0 \tag{6.56}
\end{equation*}
$$

Multiplying through by $r$ and integrating with respect to $r$ we obtain

$$
\begin{equation*}
r\left|\frac{\partial V_{0}}{\partial r}\right|^{n-1} \frac{\partial V_{0}}{\partial r}=-\frac{r^{2}}{2 K} \frac{P(t)}{\bar{u}^{n}}+A(t) \tag{6.57}
\end{equation*}
$$

where $A(t)$ is an arbitrary function of time. From equations (6.52) we have

$$
A(t)=0 \quad(6.58)
$$

Rearranging equation (6.57) and using equation (6.58) we obtain

$$
\frac{\partial v_{0}}{\partial r}=\frac{-r^{1 / n}}{(2 k)^{\frac{1}{2}} \bar{u}} p|p|^{\frac{1}{n}-1} \quad(6.5 q)
$$

which allows for both positive and negative values of $P(t)$. Integrating equation (6..59) with respect to $r$ and satisfying the boundary condition

$$
V_{0}(a t)=0, \quad t \geqslant 0 \quad(b .60)
$$

we obtain

$$
V_{0}(r, t)=\frac{n \bar{u}^{-1}}{(2 K)^{1 / n}(n+1)}\left[a^{\frac{n+1}{n}}-r^{\frac{n+1}{n}}\right] P|P|^{\frac{1-n}{n}} \quad(6.61)
$$

On substituting $V=w / \bar{u}$ into equation (6.8) and integrating by parts we obtain

$$
Q(t)=-\pi \int_{0}^{a} \bar{u} r^{2} \frac{\partial v}{\partial r} d r \quad(6.62)
$$

Substituting equation (6.61) into equation (6.62) and integrating we obtain

$$
\left.Q_{0}(t)=\frac{\pi n)^{2 k)^{\prime \prime n}(3 n+1)}}{(3 n+1)} \right\rvert\, 1 \frac{1-n}{n} \quad(6.63)
$$

## Coefficient of $\rho$

Comparing the first order coefficient in $\rho$ in equation (6.55) we obtain

$$
\begin{equation*}
\bar{u} \frac{\partial V_{0}}{\partial t}=\frac{K n}{r} \frac{\partial}{\partial r}\left(r\left|\frac{\partial V_{0}}{\partial r}\right|^{n-1} \frac{\partial V_{1}}{\partial r}\right)^{\bar{u}^{n}} \tag{6.64}
\end{equation*}
$$

Substituting equation (6.61) into equation (6.64) and integrating with respect to $r$ we obtain

$$
\begin{gather*}
\frac{\partial V_{1}}{\partial r}=\frac{2}{(2 k)^{2 / n}(n+1) \bar{\pi}}\left[\frac{r^{1 / n} a^{\frac{1+n}{n}}}{2}-\frac{n r^{1+2 / n}}{3 n+1}\right]|p|^{\frac{1-n}{n}} \frac{d}{d t}\left(P|P|^{\frac{1-n}{n}}\right) \\
+B(t) r^{-2+\frac{1}{n}} \tag{6.65}
\end{gather*}
$$

The boundary condition which must be satisfied is given by equation (6.52)

When $n>1 / 2 B(t)$ must be zero, but when $n<1 / 2$ then $B(t)$ can remain arbitrary and it seems that a further boundary condition is required in order to obtain a unique solution in this case. However, the power low fluid predicts an infinite viscosity when the shear rate is zero. In general at low shear rates power low fluids behave as Newtonian fluids and when $n=1$ (the Newtonian case) then the last term in equation (6.65) is $B(t) / r$ and $B(t)$ must be zero. Hence $B(t)$ is zero for all values that $n$ may take.

On integrating equation (6.65) with respect to $r$ and then satisfying the boundary condition $V_{1}(a, t)=0$ we obtain

$$
\begin{align*}
& V_{1}(r, t)=\frac{n}{u(2 k)^{2 / n}(n+1)^{2}}\left[r^{1+\frac{1}{n}} a^{1+\frac{1}{n}}-\frac{n r^{2+\frac{2}{n}}}{3 n+1}-\left(\frac{2 n+1}{3 n+1}\right) a^{2+\frac{2}{n}}\right] \\
&|p|^{\frac{1-n}{n}} \frac{d}{d t}\left(p|p|^{\frac{1-n}{n}}\right) \tag{6.66}
\end{align*}
$$

On substituting $V_{1}(r, t)$ from equation (6.66) into

$$
\begin{equation*}
Q_{1}(t)=-\pi \int_{0}^{a} \bar{u} r^{2} \frac{\partial V_{1}}{\partial r} d r \tag{6.67}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
Q_{1}(t)=\frac{\pi a^{4+\frac{2}{n}} n}{(2 k)^{2 / n}(3 n+1)(2 n+1)}|P|^{\frac{1}{n}-1} \frac{d}{d t}\left(P|P|^{\frac{1}{n}-1}\right) \tag{6.68}
\end{equation*}
$$

ie.

$$
\begin{equation*}
Q_{1}(t)=\frac{-\pi a^{4+\frac{2}{n}} n}{(2 k)^{2 / n}(3 n+1)(2 n+1)} \frac{|p|^{1 / n}}{p} \frac{d}{d E}\left(|p|^{\frac{1}{n}}\right) \tag{6.69}
\end{equation*}
$$

Coefficient of $p^{2}$
Comparing terms of order $p^{2}$ in equation (6.55) we
have

$$
\frac{\partial V_{1}}{\partial t}=\frac{K n}{r} \frac{\partial}{\partial r}\left[r\left|\frac{\partial V_{0}}{\partial r}\right|^{n-1}\left(\frac{\partial V_{2}}{\partial r}+\frac{n-1}{2} \frac{\left(\partial V_{1} / \partial r\right)^{2}}{\partial V_{0} / \partial r}\right)\right]^{-n-1}(6.70)
$$

Substituting $V_{0}$ from equation (6.61) and $V_{1}$ from equation (6.66) into equation ( 6.70 ) and integrating with respect to $r$ we obtain

$$
\begin{gather*}
\frac{\partial V_{2}}{\partial r}=\frac{2(n-1) \bar{u}^{-1}}{(2 k)^{3 / n}(n+1)^{2}}\left[\frac{r^{1 / n} a^{2+2 / n}}{4}-\frac{n r^{1+2 / n} a^{1+1 / n}}{3 n+1}+\frac{n^{2} r^{2+3 / n}}{(3 n+1)^{2}}\right] \\
\cdot \frac{|P|^{\frac{1}{n}-1}}{P}\left(\frac{d}{d t}\left(P|P|^{\frac{1}{n}-1}\right)\right)^{2} \\
+\frac{2 \bar{u}^{-1}}{(2 k)^{3 / n}(n+1)^{2}(3 n+1)}\left[n r^{1+\frac{2}{n}} a^{1+\frac{1}{n}}-\frac{n^{2} r^{2+3 / n}}{2(2 n+1)}-\frac{a^{2+2 / n} \cdot(2 n+1) r^{\frac{1}{n}}}{2}\right] \\
\cdot|P|^{1 / n} \frac{d}{d t}\left[|P|^{\frac{1}{n}-1} \frac{d}{d t}\left(P|P|^{\frac{1}{n}-1}\right)\right] \tag{ו7.7}
\end{gather*}
$$

where the constant of integration is zero for reasons previously discussed.

The flow rate $Q_{2}(t)$ is given by substituting $\partial v_{2} / \partial r$ from equation (6.71) into

$$
\begin{equation*}
Q_{2}(t)=-\pi \int_{0}^{a} r^{2} \frac{\partial V_{2}}{\partial r} d r \tag{6.72}
\end{equation*}
$$

to give

$$
\begin{align*}
Q_{2}(t)= & \frac{\pi n(8 n+3) a^{\frac{5 n+3}{n}}}{2(2 K)^{3 / n}(3 n+1)(5 n+3)(2 n+1)}\left[-\frac{(n-1)}{P}|P|^{\frac{1}{n}-1}\left(\frac{d}{d t}\left(P|P|^{\frac{1}{n}-1}\right)\right)^{2}\right. \\
& \left.+2|P|^{\frac{1}{n}-1} \frac{d}{d t}\left(|P|^{\frac{1}{n}-1} \frac{d}{d t}\left(P|P|^{\frac{1}{n}-1}\right)\right)\right] \quad(6.73) \tag{6.73}
\end{align*}
$$

Equation (6.73) simplifies to

$$
\begin{gather*}
Q_{2}(t)=\frac{\pi n(8 n+3) a^{\frac{5 n+3}{n}}}{(2 k)^{3 / n}(3 n+1)^{2}(5 n+3)(2 n+1)}\left[-\frac{3}{2}(n-1) \frac{|P|^{\frac{1}{n}-1}}{P}\left(\frac{d}{d t}\left(|P|^{\frac{1}{n}}\right)\right)^{2}\right. \\
\left.+\frac{|P|^{2 / n-1}}{P} \frac{d^{2}}{d t^{2}}\left(|P|^{1 / n}\right)\right] \quad(6.74) \tag{6.74}
\end{gather*}
$$

On carrying out the differentiation it can be seen that when $P(t)=0$ then $Q_{2}(t)$ becomes infinite for $3 / 4<n<1$.

The final expression for $Q(t)$ is obtained by substituting the equation for $Q_{0}, Q_{1}$ and $Q_{2}$ derived above into equation (6.53) to give

$$
\begin{gather*}
Q(t)=\frac{\pi n a^{\frac{3 n+1}{n}}}{(2 k)^{1 / n}(3 n+1)} P|P|^{\frac{1}{n}-1}-\frac{\rho \pi a^{\frac{4 n+2}{n}} n|P|^{1 / n}}{(2 k)^{2 / n}(3 n+1)(2 n+1) P} \frac{d}{d t}\left(|P|^{\frac{1}{n}}\right) \\
+\frac{\rho^{2} \pi n(8 n+3) a^{\frac{5 n+3}{n}}}{(2 k)^{3 / n}(3 n+1)^{2}(5 n+3)(2 n+1)}\left[-\frac{3}{2}(n-1) \frac{|P|^{1 / n-1}}{P}\left(\frac{d}{d t}\left(|P|^{1 / n}\right)\right)^{2}\right. \\
\left.\quad+\frac{|P|^{2 / n-1}}{P} \frac{d^{2}}{d t^{2}}\left(|P|^{1 / n}\right)\right] \tag{6.75}
\end{gather*}
$$

The mean flow rate is given by

$$
\begin{equation*}
\langle Q\rangle=\left\langle Q_{0}\right\rangle+\left\langle Q_{1}\right\rangle+\left\langle Q_{2}\right\rangle \tag{6.76}
\end{equation*}
$$

where < > denotes the time average value, over one cycle of oscillation of the expression contained within the brackets. Hence

$$
\begin{equation*}
\left.\left\langle Q_{0}\right\rangle=\left.\frac{\pi n a^{\frac{3 n+1}{n}}}{(2 K)^{1 / n}(3 n+1)}\langle P| P\right|^{\frac{1}{n}-1}\right\rangle \tag{6,77}
\end{equation*}
$$

$$
\begin{equation*}
\left\langle Q_{1}\right\rangle=\frac{-\rho \pi a^{\frac{4 n+2}{n} n}}{(2 K)^{2 / n}(3 n+1)(2 n+1)}\left\langle\frac{|\rho|^{1 / n}}{P} \frac{d}{d t}\left(|\rho|^{1 / n}\right)\right\rangle \tag{6.78}
\end{equation*}
$$

and

$$
\begin{gather*}
\left\langle Q_{2}\right\rangle= \\
\frac{p^{2} \pi n(8 n+3) a^{\frac{5 n+3}{n}}}{(2 k)^{3 / n}(3 n+1)^{2}(5 n+3)(2 n+1)}\left[\left\langle-\frac{3}{2}(n-1) \frac{|p|^{\frac{1}{n}-1}}{p}\left(\frac{d}{d t}\left(|p|^{\frac{1}{n}}\right)\right)^{2}\right\rangle\right.  \tag{6.79}\\
\left.\quad+\left\langle\frac{|p|^{2 / n-1}}{P} \frac{d^{2}}{d t^{2}}\left(|P|^{1 / n}\right)\right\rangle\right] \quad \text { (6.79) }
\end{gather*}
$$

The expression for $\left\langle Q_{2}\right\rangle$ can now be simplified by using integration by parts on the first term. We obtain

$$
\begin{align*}
\int \frac{|P|^{\frac{1}{n}-1}}{P} \frac{d}{d t}\left(|P|^{1 / n}\right) d t & =\int \frac{|P|^{\frac{1}{n}-1}}{P} \cdot \frac{1}{n} \frac{|P|^{\frac{1}{n}}}{P} \frac{d P}{d t} d t  \tag{6.80}\\
& =\frac{1}{n} \frac{|P|^{\frac{2}{n}-2}}{\left(\frac{2}{n}-2\right)} \frac{|P|}{P} \tag{6.81}
\end{align*}
$$

$$
\begin{equation*}
\text { i.e. } \quad \int \frac{|P|^{\frac{1}{n}-1}}{P} \cdot \frac{d}{d t}\left(|P|^{\frac{1}{n}}\right) d t=\frac{1}{2(1-n)} \frac{|P|^{2 / n-1}}{P} \tag{6.82}
\end{equation*}
$$

Therefore

$$
\begin{gather*}
\left\langle-\frac{3}{2}(n-1) \frac{|P|^{\frac{1}{n}-1}}{P} \frac{d}{d t}\left(|P|^{\frac{1}{n}}\right) \frac{d}{d t}\left(|P|^{1 / n}\right)\right\rangle \\
\quad \equiv \frac{3}{2}(n-1)\left\langle\frac{d^{2}}{d t^{2}}\left(|P|^{\frac{1}{n}}\right) \frac{1}{2(1-n)} \frac{|P|^{\frac{2}{n}-1}}{P}\right\rangle  \tag{6.83}\\
\quad=\frac{3}{4}\left\langle\frac{|P|^{2 / n-1}}{P} \frac{d^{2}}{d t^{2}}\left(|P|^{\frac{1}{n}}\right)\right\rangle \tag{6.84}
\end{gather*}
$$

Adding this to the second component gives

$$
\left\langle Q_{2}\right\rangle=\frac{\rho^{2} \pi n(8 n+3) a^{\frac{5 n+3}{n}}}{4(2 k)^{3 / n}(3 n+1)^{2}(5 n+3)(2 n+1)}\left\langle\frac{|P|^{3 / n^{-1}}}{P} \frac{d^{2}}{d t^{2}}\left(|P|^{1 / n}\right)\right\rangle
$$

In the experiment carried out by Chakrabati and Davies the flow rate was known and the pressure gradient was measured. Hence, we require to make $P(t)$ the subject of equation (6.75). This was achieved using a Taylor's series expansion as described below.

Equation (6.75) can be written as

$$
\begin{align*}
Q(t)= & k_{0} P|P|^{1 / n-1}+k_{1} P^{|P|^{1 / n-1} \frac{d}{d t}\left(P|P|^{\frac{1}{n}-1}\right)} \\
& +k_{2} P^{2}\left[-(n-1) \frac{|P|^{\frac{1}{n}-1}}{P}\left(\frac{d}{d t}\left(P|P|^{\frac{1}{n}-1}\right)\right)^{2}\right. \\
& \left.+2|P|^{\frac{1}{n}-1} \frac{d}{d t}\left(|P|^{\frac{1}{n}-1} \frac{d}{d t}\left(P|P|^{\frac{1}{n}-1}\right)\right)\right] \tag{6.86}
\end{align*}
$$

where

$$
\begin{align*}
& k_{0}=\frac{\pi n a^{\frac{3 n+1}{n}}}{(2 k)^{1 / n}(3 n+1)}  \tag{6.87}\\
& k_{1}=\frac{-\pi n a^{\frac{4 n+2}{n}}}{(2 k)^{2 / n}(3 n+1)(2 n+1)}  \tag{6.88}\\
& k_{2}=\frac{\pi n(8 n+3) a^{\frac{5 n+3}{n}}}{2(2 k)^{3 / n}(3 n+1)^{2}(5 n+3)(2 n+1)} \tag{6.89}
\end{align*}
$$

we allow $P(t)$ to be given by

$$
\begin{equation*}
P(t)=P_{0}(t)+\rho P_{1}(t)+\rho^{2} P_{2}(t) \tag{6.90}
\end{equation*}
$$

where $P_{0}, P_{1}$ and $P_{2}$ are functions of $t$ which must be determined. From equation (6.81) it is evident that we need to evaluate the term $P|P|^{-1+1 / n}$ to order $\rho^{2}$ and the term $|P|^{-1+1 / n}$ to order $\rho$. From equation (6.90) we have

$$
\begin{equation*}
P\left|P^{\frac{1}{n}-1}=\left(P_{0}+P P_{1}+P^{2} P_{2}\right)\right| P_{0}+P P_{1}+\left.P^{2} P_{2}\right|^{\frac{1}{n}-1} \tag{6.91}
\end{equation*}
$$

Using a Taylor's series expansion and ignoring terms of order $p^{3}$ and higher we have

$$
\begin{align*}
& P|P|^{\frac{1}{n}-1}=P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}+P_{n} P_{1}\left|P_{0}\right|^{\frac{1}{n}-1} \\
& \quad+P^{2}\left(\frac{1}{n} P_{2}\left|P_{0}\right|^{\frac{1}{n}-1}+\frac{1-n}{2 n^{2}} \frac{P_{1}^{2}}{P_{0}}\left|P_{0}\right|^{\frac{1}{n}-1}\right) \tag{6.92}
\end{align*}
$$

Similarly the term $|\mathrm{P}|^{-1+\frac{1}{n}}$, is given, to order $\rho$, by

$$
\begin{equation*}
|P|^{\frac{1}{n}-1}=\left|P_{0}\right|^{\frac{1}{n}-1}+P \frac{1-n}{n} \frac{P_{1}}{P_{0}}\left|P_{0}\right|^{\frac{1}{n}-1} \tag{6.93}
\end{equation*}
$$

Substituting equations (6.90), (6.92) and (6.93) into equation (6.86) we obtain

$$
\begin{align*}
Q(t)= & k_{0}\left[P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}+P_{n} P_{1}\left|P_{0}\right|^{\frac{1}{n}-1}\right. \\
& \left.+P^{2}\left(\frac{1}{n} P_{2}\left|P_{0}\right|^{\frac{1}{n}-1}+\frac{1-n}{2 n^{2}} P_{1}^{2}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right] \\
& +k_{1} p\left[\left|P_{0}\right|^{\frac{1}{n}-1}+\rho \frac{1-n}{n} P_{1}\left|P_{0}\right|^{\frac{1}{n}-1}\right] \\
& \cdot\left[\frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}+\frac{\rho}{n} P_{1}\left|P_{0}\right|^{\frac{1}{n}-1}\right]\right. \\
& +k_{2} P^{2}\left[-\frac{n-1}{P_{0}}\left|P_{0}\right|^{\frac{1}{n}-1}\left(\frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right)^{2}\right. \\
& \left.+2\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right)\right] \quad(6 \tag{6.94}
\end{align*}
$$

By grouping terms we obtain

$$
Q(t)=k_{0} P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}+\quad \text { (CONT.) }
$$

$$
\begin{align*}
& P\left[\frac{k_{0}}{n} P_{1}\left|P_{0}\right|^{\frac{1}{n}-1}+k_{1}\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right] \\
+ & P^{2}\left[\frac{k_{0}}{n} P_{2}\left|P_{0}\right|^{\frac{1}{n}-1}+\frac{k_{0}(1-n) P_{1}^{2}}{2 n^{2} P_{0}}\left|P_{0}\right|^{1 / n-1}\right. \\
+ & k_{1}\left(\frac{(1-n)}{n} \frac{P_{1}}{P_{0}}\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right. \\
+ & \frac{k_{1}}{n}\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(P_{1}\left|P_{0}\right|^{\frac{1}{n}-1}\right) \\
& -k_{2}(n-1) \frac{\left|P_{0}\right|^{\frac{1}{n}-1}}{P_{0}}\left(\frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right)^{2} \\
& \left.+2 k_{2}\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right)\right] \tag{6.95}
\end{align*}
$$

But $Q(t)$ is a fixed quantity and independent of the fluid inertia. Therefore, the coefficients of $p$ and $p^{2}$ must vanish in equation (6.95) giving

$$
\begin{equation*}
Q(t)=k_{0} P_{0}\left|P_{0}\right|^{\frac{1}{n}-1} \tag{6.96}
\end{equation*}
$$

which, on rearrangement gives

$$
\begin{equation*}
P_{0}=\frac{1}{k_{0}^{n}} Q|Q|^{n-1} \tag{6.97}
\end{equation*}
$$

Substituting equation (6.87) into equation (6.97) gives

$$
P_{0}=\frac{2 K(3 n+1)^{n}}{\pi^{n} n^{n} a^{3 n+1}} Q|Q|^{n-1} \quad \text { (6.98) }
$$

By comparing terms of order $p^{2}$ in equation (6.95) we obtain

$$
\begin{equation*}
\frac{k_{0}}{n} P_{1}\left|P_{0}\right|^{\frac{1}{n}-1}+k_{1}\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)=0 \tag{6.99}
\end{equation*}
$$

which, on rearrangement, gives

$$
\begin{equation*}
P_{1}=-n \frac{k_{1}}{k_{0}} \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right) \tag{6.100}
\end{equation*}
$$

Using the expression for $k_{0}$ and $k_{1}$ given by equations (6.87) and (6.88) we have

$$
\begin{equation*}
P_{1}=\frac{3 n+1}{\pi a^{2}(2 n+1)} \frac{d}{d t} Q(t) \tag{6.101}
\end{equation*}
$$

By comparing terms of order $\rho^{2}$ in equation (6.95) and dividing through by $\left|\mathrm{PO}_{\mathrm{O}}\right|-1+1 / \mathrm{n}$ we obtain

$$
\begin{align*}
& k_{0}\left[\frac{P_{2}}{n}+\frac{(1-n)}{2 n^{2}} \frac{P_{1}^{2}}{P_{0}}\right]+k_{1} \frac{(1-n)}{n} \frac{P_{1}}{P_{0}} \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right) \\
& +\frac{k_{1}}{n} \frac{d}{d t}\left(P_{1}\left|P_{0}\right|^{\frac{1}{n}-1}\right)-k_{2} \frac{(n-1)}{P_{0}}\left(\frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right)^{2} \\
& +2 k_{2} \frac{d}{d t}\left(\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right)=0 \tag{6.102}
\end{align*}
$$

Substituting $P_{1}$ from equation (6.100) into this equation and grouping terms gives

$$
\begin{align*}
P_{2}= & \frac{n}{k_{0}}\left(k_{1}^{2}-2 k_{0} k_{2}\right)\left[\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d^{2}}{d t^{2}}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right. \\
& +\frac{1-n}{2 P_{0}}\left(\frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right)^{2} \\
& \left.+\frac{d}{d t}\left(\left|P_{0}\right|^{\frac{1}{n}-1}\right) \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right] \tag{6.103}
\end{align*}
$$

Substituting the expressions for $k_{0}, k_{1}$ and $k_{2}$ from equations (6.87), ( 6.88 ) and ( 6.89 ) gives

$$
\begin{align*}
P_{2}= & \frac{-n^{3} a^{2+\frac{2}{n}}}{(2 k)^{2 / n}(3 n+1)(2 n+1)(5 n+3)}\left[\left|P_{0}\right|^{\frac{1}{n}-1} \frac{d^{2}}{d t^{2}}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n-1}}\right)\right. \\
& +\frac{(1-n)}{2 P_{0}}\left(\frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{-\frac{1}{n}-1}\right)^{2}\right. \\
& \left.+\frac{d}{d t}\left(\left|P_{0}\right|^{\frac{1}{n}-1}\right) \frac{d}{d t}\left(P_{0}\left|P_{0}\right|^{\frac{1}{n}-1}\right)\right] \tag{6.104}
\end{align*}
$$

Substituting equation (6.97) into equation (6.104) gives

$$
\begin{gathered}
P_{2}=\frac{-n^{n+1}(3 n+1)^{1-n} \pi^{n-2} a^{3 n-3}}{2 k(2 n+1)^{2}(5 n+3) Q|Q|^{n-1}}\left[Q Q^{\prime \prime}+\frac{1-n}{2} Q^{\prime 2}\right. \\
\left.+Q|Q|^{n-1} \frac{d}{d t}\left(|Q|^{1-n}\right) Q^{\prime}\right] \quad \text { (6.105) }
\end{gathered}
$$

Equation (6.105) simplifies to give

$$
P_{2}=\frac{-n^{n+1}(3 n+1)^{1-n} \pi^{n-2} a^{3 n-3}}{2 k(2 n+1)^{2}(5 n+3)}\left[\frac{Q Q^{11}+\frac{3}{2}}{Q \mid(1-n) Q^{12}} \underset{\left(\left.6\right|^{n-1}\right.}{(6.106)}\right.
$$

Substituting the expression for $P_{0}, P_{1}$ and $P_{2}$ given in equations (6.97), (6.100) and (6.106) into equation (6.90) we obtain

$$
\begin{aligned}
& P(t)=\frac{2 k(3 n+1)^{n}}{\pi^{n} n^{n} a^{3 n+1}} Q|Q|^{n-1}+\frac{\rho(3 n+1)}{\pi a^{2}(2 n+1)} Q^{\prime} \\
& -\rho^{2} \frac{n^{n+1}(3 n+1)^{1-n} \pi^{n-2} a^{3 n-3}}{2 k(2 n+1)^{2}(5 n+3)}\left[\frac{Q Q^{\prime \prime}+\frac{3}{2}(1-n) Q^{12}}{Q|Q|^{n-1}}\right] \text { (6.107) }
\end{aligned}
$$

and $P(t)$ is now expressed in terms of $Q(t)$. Substituting equation (6.38) into equation (6.107) and using equation (6.41) it is evident that a singularity occurs in $P(t)$ when $Q(t)=0$ and $\epsilon \geqslant 1$. The mean theoretical pressure gradient, $P(t)$ is, using equation (6.107) given by

$$
\begin{aligned}
\bar{P}(t) & \left.=\left.\frac{2 K(3 n+1)^{n}}{\pi^{n} n^{n} a^{3 n+1}}\langle Q| Q\right|^{n-1}\right\rangle \\
& -\frac{\rho^{2} n^{n+1}(3 n+1)^{1-n} \pi^{n-2} a^{3 n-3}}{2 K(2 n+1)^{2}(5 n+3)}\left\langle\frac{Q Q^{\prime \prime}+\frac{3}{2}(1-n) Q^{12}}{Q|Q|^{n-1}}\right\rangle
\end{aligned}
$$

or, using equation (6.90)

$$
\begin{equation*}
\bar{P}(t)=\left\langle P_{0}\right\rangle+P^{2}\left\langle P_{2}\right\rangle \tag{6.109}
\end{equation*}
$$

6.3 .3

Non-Dimensional Analysis

Let

$$
\begin{align*}
\rho^{2} P_{2} & =\bar{P}_{s} P_{2}^{*}  \tag{6.110}\\
T & =\frac{\omega t}{2 \pi} \tag{6.111}
\end{align*}
$$

where $\bar{P}_{S}$ is the mean steady pressure gradient given by equation (6.12).

Using equations (6.106) and (6.38)

$$
\begin{align*}
& \rho^{2} P_{2}= \frac{n^{n+1} p^{2}(3 n+1)^{1-n} \pi^{n-2} a^{3 n-3} \pi^{2} a^{4} \bar{u}^{2} \omega^{2}}{2 k(2 n+1)(5 n+3) \pi^{n} a^{2 n} \bar{u}^{n} Q^{*}\left|Q^{*}\right|^{n-1} 4 \pi^{2}} \\
& \times\left[\begin{array}{c}
3 \\
2
\end{array}(n-1)\left(Q^{*^{\prime}}\right)^{2}-Q^{*} Q^{* 4}\right] \quad(6 \cdot 112)  \tag{6.112}\\
&= \frac{p^{2} n^{n+1}(3 n+1)^{1-n} a^{n+1} \omega^{2}}{8 k(2 n+1)^{2}(5 n+3) \pi^{2} \bar{u}^{n-2}}\left[\frac{\frac{3}{2}(n-1)\left(Q^{* 1}\right)^{2}-Q^{*} Q^{* 11}}{Q^{*}\left|Q^{*}\right|^{n-1}}\right] \\
& \quad(6 \cdot 113)
\end{align*}
$$

Dividing equation (6.113) through by $P_{S}$, using equations (6.20), (6.22), (6.24) and (6.110) and simplifying we obtain

$$
P_{2}^{*}=\frac{F^{2}(3 n+1) n}{256(2 n+1)^{2}(5 n+3)}\left[\frac{3 / 2(n-1)\left(Q^{*^{\prime}}\right)^{2}-Q^{*} Q^{* 11}}{Q^{*}\left|Q^{*}\right|^{n-1}}\right]
$$

(6.114)

Using equation (6.98) we obtain

$$
\begin{equation*}
P_{0}=2 k\left(\frac{3 n+1}{n}\right)^{n}{\frac{\bar{u}^{n+1}}{}}^{n} Q^{*}\left|Q^{*}\right|^{n-1} \tag{6.115}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
P_{0}^{*}=\frac{P_{0}}{\bar{P}_{s}}=Q^{*}\left|Q^{*}\right|^{n-1} \tag{6.116}
\end{equation*}
$$

Finally the normalised pressure gradient is given by

$$
\begin{equation*}
P^{*}=\frac{P}{\bar{P}_{s}} \tag{6.117}
\end{equation*}
$$

From equation (6.109) we have

$$
\begin{equation*}
\bar{P}^{*}=\left\langle P_{0}^{*}\right\rangle+\left\langle P_{2}^{*}\right\rangle \tag{6.118}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{P}^{*}=\left.\left\langle Q^{*}\right| Q^{*}\right|^{n-1}+\frac{F^{2}(3 n+1) n}{256(2 n+1)^{2}(5 n+3)}\left\langle\frac{\frac{3}{2}(n-1)\left(Q^{*}\right)^{2}-Q^{*} Q^{* \prime \prime}}{Q^{*}\left|Q^{*}\right|^{n-1}}\right\rangle \tag{6.119}
\end{equation*}
$$

Therefore, the normalised mean pressure gradient is given by

$$
\begin{equation*}
\overline{P^{*}}=\int_{0}^{1} Q^{*}\left|Q^{*}\right|^{n-1} d T+\frac{F_{n}^{2}(3 n+1)}{256(2 n+1)^{2}(5 n+3)} \int_{0}^{3 / 2(n-1)\left(Q^{* 1}\right)^{2}-Q^{*} Q^{* 1}} \frac{Q^{*}\left|Q^{*}\right|^{n-1}}{1} d T \tag{6.120}
\end{equation*}
$$

Since $Q^{*}=1+\epsilon \operatorname{Cos} 2 \pi T$ then it appears that a singularity occurs in the second integrand when $\epsilon \geqslant 1$. However, this is not the case since rewriting this expression we have

$$
\begin{equation*}
\int_{0}^{1} \frac{\frac{3}{2}(n-1)\left(Q^{* \prime}\right)^{2}}{Q^{*}\left|Q^{*}\right|^{n-1}} d T=\frac{3}{2}(n-1) \int_{0}^{1}\left(\frac{Q^{*^{\prime}}}{Q^{*}\left|Q^{*}\right|^{n-1}}\right) Q^{*^{\prime}} d T \tag{6.121}
\end{equation*}
$$

$$
\begin{array}{r}
=\frac{3}{2}(n-1)\left[Q^{*^{\prime}}\left|Q^{*}\right|^{1-n}\right]_{0}^{1}-\frac{3}{2}(n-1) \int_{0}^{1} \frac{\left|Q^{*}\right|^{1-n}}{1-n} Q^{* \prime \prime} d T  \tag{6.122}\\
=\frac{3}{2} \int_{0}^{1}\left|Q^{*}\right|^{1-n} Q^{*^{n}} d T
\end{array}
$$

The second integral in (6.120) becomes

$$
\begin{aligned}
& \int_{0}^{1}\left(\frac{\left.3(n-1)\left(Q^{*}\right)^{2}\right)^{2}}{2 Q^{*}\left|Q^{*}\right| n^{\left(n^{*}\right.}}-\frac{Q^{*}}{\left(Q^{*} \mid n^{-1}\right)}\right) d T \\
& =\int_{0}^{1}\left(\frac{3}{2}\left|Q^{*}\right|^{\mid-1} Q^{* *}-Q^{*}\left|Q^{*}\right|^{1-n}\right) d T(6.124) \\
& =\frac{1}{2} \int_{0}^{1}\left|Q^{*}\right|^{1-n} Q^{*} d T \\
& \text { (6.125) }
\end{aligned}
$$

and the normalised mean pressure gradient is given by

$$
\bar{P}^{*}=\int_{0}^{1} Q^{*}\left|Q^{*}\right|^{n-1} d T+\frac{F^{2}(3 n+1) n}{512(2 n+1)^{2}(5 n+3)} \int_{0}^{1}\left|Q^{*}\right|^{1-n} Q^{* 1} d T
$$

The results from equation (6.126) are plotted in Figs. 13-16.
6.4 Results and Discussion

As expected there is total agreement in the Newtonian case, between the data derived numerically and that obtained using the perturbation theory. There is also good agreement between the two sets of data when the fluid inertia is zero (ie. when $F=0$ ). This is evident from Figs. 13-16 and from Tables 4-7 and Table 12.

The numerical variation of the normalised mean pressure gradient against $F$ for $\epsilon=10.0$ and $\epsilon=5.0$ is small. Therefore for this reason it is difficult to know whether there is good agreement between the perturbation analysis and the numerical data. However, for the case when $\epsilon=1.0$ or 0.5 the variation of the mean pressure gradient with $F$ is more pronounced and the disagreement between the two relevant curves is more noticeable. Since the range of validity of the perturbation theory and of the numerical procedure is uncertain then this disagreement might be removed by using a modified perturbation theory which takes into account higher order terms and a numerical procedure which is free of instabilities. It should be noted that, since terms of third order and higher were ignored in the perturbation theory then the variation of mean pressure gradient against $F$ is quadratic. The modified perturbation theory mentioned might be obtained by including terms of third (and possibly fourth) order and ignoring all higher order terms.

The downwards direction of variation of the mean pressure gradient with F, predicted by the perturbation theory, is consistent with the corresponding numerically derived variation of these quantities. Also, trends similar to those evident in the complete, numerically derived curves, have been observed from the results of the experimental work carried out by Chakrabati and Davies already mentioned in the introduction to this Chapter. However, Davies 47 (private communication 1986) has recently shown that similar trends occur when elasticity is included in the theory.

Figures $17-20$ show the relationship between phase angle and $F$ for various $\in$ and $N$. The variation of the phase angle with $F$ shows the same trend with $N$ for all $\epsilon$, being more pronounced for larger $\epsilon$. As $F$ gets large then, in all cases, the phase angle tends to a $90^{\circ}$ lag.

Figures 21-24 show the relationship between the fundamental pressure amplitude and $F$ for various values of $\epsilon$ and $N$. The variation of the pressure amplitude with $F$ shows the same trend with $N$ for all $\in$. For large $F$ the pressure amplitude, for each value of $\epsilon$, appears to be independent of $N$ and proportional to a power of F。

Finally, Fig. 25 shows the relationship between the normalised mean pressure gradient and $\in$ for various values of $F$ when $N=0.5$. In general the flow enhancement increases with increasing and decreasing fluid inertia. Interestingly, however, when $\epsilon=1$ then the flow enhancement is practically independent of the fluid inertia.

The work presented in this chapter has shown the effect of a realistic viscosity function and the effect of fluid inertia on the flow enhancement and pressure gradient reduction predictions. It is hoped that this work will be extended to include the effect of elasticity. However, it is important to choose a model which gives good predictions for steady shear behaviour and oscillatory shear flow behaviour.

In the oil drilling industry, large positive displacement mechanical pumps are used to circulate the drilling fluid. These pumps use an oscillatory mechanism which results in a pulsatile pressure gradient imposed on the fluid. The properties of the fluid will obviously be affected by the superposition of an oscillation on the steady flow. This type of flow behaviour also occurs in the removal of rock cuttings from the recovered oil drilling fluid. Due to the cost of this fluidit is necessary to separate the drill cuttings from it enabling re-circulation to take place. This is achieved by passing the contaminated oil drilling fluid over a vibrating table. The work presented in this Cnapter will obviously be relevant in describing the pulsatile flow behaviour of fluids used in the oil drilling industry.

APPENDIX 1

THE GOVERNING EQUATION FOR A SPHERE
FALLING SLOWLY THROUGH A NEWTONIAN FLUID

The constituitive equations for a Newtonian fluid are, using spherical polar coordinates and taking the symmetry of Fig. 3 into account:

$$
\begin{align*}
& \tau_{r r}=-2 \eta \frac{\partial V_{r}}{\partial r}  \tag{A1.1}\\
& \tau_{\theta \theta}=-2 \eta\left(\frac{1}{r} \frac{\partial V_{\theta}}{\partial \theta}+\frac{V_{r}}{r}\right)  \tag{A1.2}\\
& \tau_{\phi \phi}=-2 \eta\left(\frac{V_{r}}{r}+V_{\theta} \frac{\cot \theta}{r}\right)  \tag{A1.3}\\
& \tau_{r \theta}=-\eta\left(r \frac{\partial}{\partial r}\left(\frac{V_{\theta}}{r}\right)+\frac{1}{r} \frac{\partial V_{r}}{\partial \theta}\right)
\end{align*} \quad \text { (Al.2) }
$$

The equations of motion, ignoring the inertial terms are:

$$
\begin{align*}
-\frac{\partial p}{\partial r}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \tau_{r r}\right) & +\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\tau_{r \theta} \sin \theta\right) \\
& -\frac{\left(\tau_{\theta \theta}+\tau_{\phi \phi}\right)}{r} \tag{A1.5}
\end{align*}
$$

and

$$
\begin{align*}
-\frac{1}{r} \frac{\partial p}{\partial \theta}=\frac{1}{r^{3}} \frac{\partial}{\partial r}\left(r^{3} \tau_{r \theta}\right) & +\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\tau_{\theta \theta} \sin \theta\right) \\
& -\frac{\tau \phi \phi \cot \theta}{r} \tag{A1.6}
\end{align*}
$$

The equation of continuity is

$$
\begin{equation*}
\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} v_{r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(v_{\theta} \sin \theta\right)=0 \tag{A1.7}
\end{equation*}
$$

From the Stoke's streamline formulation

$$
\begin{equation*}
V_{r}=-\frac{1}{r^{2} \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad V_{\theta}=\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \tag{A1.8}
\end{equation*}
$$

From the equation of continuity

$$
\begin{equation*}
\frac{\partial V_{r}}{\partial r}+\frac{2}{r} V_{r}+\frac{1}{r} \frac{\partial V_{\theta}}{\partial \theta}+\frac{\cot \theta}{r} V_{\theta}=0 \tag{A1.9}
\end{equation*}
$$

Using

$$
\begin{gather*}
\tau_{\theta \theta}+\tau_{\phi \phi}=-2 \eta\left[\frac{V_{r}}{r}+\frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta}+\frac{v_{r}}{r}+v_{\theta} \frac{\cot \theta}{r}\right]  \tag{Al.10}\\
=2 \eta \frac{\partial V_{r}}{\partial r} \tag{AI.II}
\end{gather*}
$$

the equations of motion become

$$
\begin{gather*}
\frac{1}{\eta} \frac{\partial p}{\partial r}=\frac{2}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial V_{r}}{\partial r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left[\operatorname { s i n } \theta \left(r \frac{\partial}{\partial r}\left(\frac{V_{\theta}}{r}\right)\right.\right. \\
\left.\left.+\frac{1}{r} \frac{\partial V_{r}}{\partial \theta}\right)\right]+\frac{2}{r} \frac{\partial V_{r}}{\partial r} \tag{Al.12}
\end{gather*}
$$

ie.

$$
\begin{equation*}
\frac{1}{\eta} \frac{\partial p}{\partial r}=\frac{2}{r^{3}} \frac{\partial}{\partial r}\left(r^{3} \frac{\partial V_{r}}{\partial r}\right)+\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left[\sin \theta\left(r \frac{\partial}{\partial r}\left(\frac{V_{\theta}}{r}\right)+\frac{1}{r} \frac{\partial V_{r}}{\partial \theta}\right)\right] \tag{A1.13}
\end{equation*}
$$

and

$$
\begin{align*}
\frac{1}{\eta} \frac{\partial p}{\partial \theta}= & \frac{1}{r^{2}} \frac{\partial}{\partial r}\left[r^{3}\left(r \frac{\partial}{\partial r}\left(\frac{V_{\theta}}{r}\right)+\frac{1}{r} \frac{\partial V r}{\partial \theta}\right)\right] \\
& +\frac{2}{\sin \theta} \frac{\partial}{\partial \theta}\left[\sin \theta\left(\frac{1}{r} \frac{\partial V_{\theta}}{\partial \theta}+\frac{V_{r}}{r}\right)\right]-2 \cot \theta\left[\frac{V_{r}}{r}+V_{\theta} \frac{\operatorname{Ct} \theta}{r}\right] \tag{A1.14}
\end{align*}
$$

Written in terms of $\psi$ the equations of motion become

$$
\begin{align*}
\frac{1}{\eta} \frac{\partial p}{\partial r}= & -\frac{2}{r^{3} \sin \theta} \frac{\partial}{\partial r}\left(r^{3} \frac{\partial}{\partial r}\left(\frac{1}{r^{2}} \frac{\partial \psi}{\partial \theta}\right)\right) \\
& +\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(r \frac{\partial}{\partial r}\left(\frac{1}{r^{2}} \frac{\partial \psi}{\partial r}\right)\right) \\
& -\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\frac{\sin \theta}{r} \frac{\partial}{\partial \theta}\left(\frac{1}{r^{2} \sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right)(A 1.15) \\
= & \frac{2}{r^{3} \sin \theta} \frac{\partial}{\partial r}\left(r^{3} \frac{\partial}{\partial r}\left(\frac{1}{r^{2}} \frac{\partial \psi}{\partial \theta}\right)\right)+\frac{1}{\sin \theta} \frac{\partial^{2}}{\partial \theta \partial r}\left(\frac{1}{r^{2}} \frac{\partial \psi}{\partial r}\right) \\
& -\frac{1}{r^{4} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right) \quad \text { (A1.16) } \tag{A1.16}
\end{align*}
$$

and

$$
\begin{align*}
\frac{1}{\eta} \frac{\partial p}{\partial \theta}= & \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{4} \frac{\partial}{\partial r}\left(\frac{1}{r^{2} \sin \theta} \frac{\partial \psi}{\partial r}\right)\right) \\
& -\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial}{\partial \theta}\left(\frac{1}{r^{2} \sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right) \\
& +\frac{2}{\sin \theta} \frac{\partial}{\partial \theta}\left(\frac{\sin \theta}{r} \frac{\partial}{\partial \theta}\left(\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}\right)\right) \\
& +\frac{2 \cot \theta}{r^{3} \sin \theta} \frac{\partial \psi}{\partial \theta}-\frac{2}{\sin \theta} \frac{\partial}{\partial \theta}\left(\frac{1}{r^{3}} \frac{\partial \psi}{\partial \theta}\right) \\
& -\frac{2 \cot }{r^{2} \theta} \cdot \frac{\partial \psi}{\partial r}  \tag{A1.17}\\
= & \frac{1}{r^{2} \sin \theta} \frac{\partial}{\partial r}\left(r^{4} \frac{\partial}{\partial r}\left(\frac{1}{r^{2}} \frac{\partial \psi}{\partial r}\right)\right)-\frac{1}{r^{2}} \frac{\partial^{2}}{\partial r \partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right) \\
& +\frac{2}{r^{2} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial r}\right)\right) \\
& +\frac{2 \cot \theta}{r^{3} \sin \theta} \frac{\partial \psi}{\partial \theta}-\frac{2}{r^{3} \sin \theta} \frac{\partial^{2} \psi}{\partial \theta^{2}}-\frac{2 \cot \theta}{r^{2} \sin \theta} \frac{\partial \psi}{\partial r} \tag{A1.18}
\end{align*}
$$

i.e.

$$
\begin{aligned}
\frac{1}{\eta} \frac{\partial p}{\partial \theta}= & \frac{1}{r^{2} \sin \theta}\left(r^{2} \frac{\partial^{3} \psi}{\partial r^{3}}-2 \frac{\partial \psi}{\partial r}\right) \\
& -\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(\frac{1}{\sin \theta} \frac{\partial^{2} \psi}{\partial \theta^{2}}-\frac{\cot \theta}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right) \\
& +\frac{2}{r^{2} \sin \theta}\left(\frac{\partial^{3} \psi}{\partial \theta^{2} \partial r}-\cot \theta \frac{\partial^{2} \psi}{\partial r \partial \theta}+\frac{1}{\sin ^{2} \theta} \frac{\partial \psi}{\partial r}\right)
\end{aligned}
$$

$$
\begin{equation*}
+\frac{2 \cot \theta}{r^{3} \sin \theta} \frac{\partial \psi}{\partial \theta}-\frac{2}{r^{3} \sin \theta} \frac{\partial^{2} \psi}{\partial \theta^{2}}-\frac{2 \cot ^{2} \theta}{r^{2} \sin \theta} \frac{\partial \psi}{\partial r} \tag{A1.1.9}
\end{equation*}
$$

$$
=\frac{1}{\sin \theta} \frac{\partial^{3} \psi}{\partial r^{3}}+\frac{2}{r^{2}} \frac{\partial \psi}{\partial r}\left(\frac{-1}{\sin \theta}+\frac{1}{\sin ^{3} \theta}-\frac{\cot ^{2} \theta}{\sin \theta}\right)
$$

$$
-\frac{1}{r^{2} \sin \theta} \frac{\partial^{3} \psi}{\partial r \partial \theta^{2}}+\frac{1}{r^{2}} \frac{\cot \theta}{\sin \theta} \frac{\partial^{2} \psi}{\partial r \partial \theta}+\frac{2}{r^{2} \sin \theta} \frac{\partial^{3} \psi}{\partial \theta^{2} \partial r}
$$

$$
-\frac{2 \cot \theta}{r^{2} \sin \theta} \frac{\partial^{2} \psi}{\partial r \partial \theta}+\frac{2 \cot \theta}{r^{3} \sin \theta} \frac{\partial \psi}{\partial \theta}-\frac{2}{r^{3} \sin \theta} \frac{\partial^{2} \psi}{\partial \theta^{2}}
$$

(A1.20)
i.e.

$$
\begin{align*}
\frac{1}{\eta} \frac{\partial p}{\partial \theta}= & \frac{1}{\sin \theta} \frac{\partial^{3} \psi}{\partial r^{3}}+\left(\frac{1}{r^{2} \sin \theta} \frac{\partial^{3} \psi}{\partial \theta^{2} \partial r}-\frac{\cot \theta}{r^{2} \sin \theta} \frac{\partial^{2} \psi}{\partial r \partial \theta}\right) \\
& -\frac{2}{r^{3} \sin \theta} \frac{\partial^{2} \psi}{\partial \theta^{2}}+\frac{2 \cot \theta}{r^{3} \sin \theta} \frac{\partial \psi}{\partial \theta} \quad \text { (A1.21) }  \tag{A1.21}\\
= & \frac{1}{\sin \theta} \frac{\partial^{3} \psi}{\partial r^{3}}+\frac{1}{r^{2}} \frac{\partial^{2}}{\partial r \partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)-\frac{2}{r^{3}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)(A 1.22)
\end{align*}
$$

Therefore

$$
\frac{1}{\eta} \frac{\partial p}{\partial \theta}=\frac{1}{\sin \theta} \frac{\partial^{3} \psi}{\partial r^{3}}+\frac{\partial}{\partial r}\left(\frac{1}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right)
$$

(A1.23)

Now

$$
\begin{aligned}
\frac{1}{\eta} \frac{\partial p}{\partial r}= & \frac{-2}{r^{3} \sin \theta} \frac{\partial}{\partial r}\left(r \frac{\partial^{2} \psi}{\partial r \partial \theta}-2 \frac{\partial \psi}{\partial \theta}\right) \\
& +\frac{1}{\sin \theta}\left(-\frac{2}{r^{3}} \frac{\partial^{2} \psi}{\partial r \partial \theta}+\frac{1}{r^{2}} \frac{\partial^{3} \psi}{\partial \theta \partial r^{2}}\right) \\
& -\frac{1}{r^{4} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right)(A 1.24) \\
= & \frac{-2}{r^{2} \sin \theta} \frac{\partial^{3} \psi}{\partial r^{2} \partial \theta}+\frac{1}{r^{2} \sin \theta} \frac{\partial^{3} \psi}{\partial \theta \partial r^{2}} \\
& -\frac{1}{r^{4} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right)(A 1.25)
\end{aligned}
$$

Therefore

$$
\begin{equation*}
\frac{1}{\eta} \frac{\partial p}{\partial r}=-\frac{1}{r^{2} \sin \theta} \frac{\partial^{3} \psi}{\partial r^{2} \partial \theta}-\frac{1}{r^{4} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right) \tag{A1.26}
\end{equation*}
$$

Eliminating $p$ from equations (A1.23) and (A1.26) by setting $\frac{\partial^{2} p}{\partial r \partial \theta}=\frac{\partial^{2} p}{\partial \theta \partial r}$

$$
\begin{aligned}
& \frac{1}{\sin \theta} \frac{\partial^{4} \psi}{\partial r^{4}}+\frac{\partial^{2}}{\partial r^{2}}\left(\frac{1}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right) \\
= & \frac{1}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial^{3} \psi}{\partial r^{2} \partial \theta}\right)-\frac{1}{r^{4}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right)\right)
\end{aligned}
$$

ie.

$$
\begin{align*}
0=\frac{\partial^{4} \psi}{\partial r^{4}} & +\sin \theta \frac{\partial^{2}}{\partial r^{2}}\left(\frac{1}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right)  \tag{A1.27}\\
& +\frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial^{3} \psi}{\partial r^{2} \partial \theta}\right) \\
& +\frac{\sin \theta}{r^{4}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right)\right) \tag{Al.28}
\end{align*}
$$

Therefore
$\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\right)\right)\left(\frac{\partial^{2} \psi}{\partial r^{2}}+\frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta}\right)\right)=0$
i.e.

$$
\begin{equation*}
E^{4} \psi=0 \tag{A1.29}
\end{equation*}
$$

where

$$
\begin{equation*}
E^{2}=\frac{\partial^{2}}{\partial r^{2}}+\frac{\sin \theta}{r^{2}} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\right) \tag{A1.30}
\end{equation*}
$$

Equation (A1.31) is the governing equation for the Newtonian fluid.

## APPENDIX 2

## DERIVING THE BOUNDARY CONDITIONS FOR $\psi_{\text {AN WHEN THE }}$ OUTER SPHERE HAS A FINITE RADIUS

The boundary conditions derived here first are true for both Newtonian and power law fluids. The stream function, $\psi$, refers therefore to either fluid.

Let the inner sphere have a radius a and the outer sphere a radius b. Using equations (4.3) and (4.4) and the no slip condition on the inner sphere, we obtain

$$
\begin{equation*}
\left.\frac{\partial \psi}{\partial r}\right|_{r=a}=\left.\frac{\partial \psi}{\partial \theta}\right|_{r=a}=0 \tag{A2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\psi\right|_{r=a}=\text { Const } \tag{A2.2}
\end{equation*}
$$

The constant is arbitrary and without loss of generality is chosen to be zero. Therefore

$$
\begin{equation*}
\left.\psi\right|_{r=a}=0 \tag{A2.3}
\end{equation*}
$$

From Fig. 3 and the no slip condition on the outer sphere, we obtain
and

$$
\begin{equation*}
\left.V_{\theta}\right|_{r=b}=-U \sin \theta \tag{A2.4}
\end{equation*}
$$

$$
\begin{equation*}
\left.V_{r}\right|_{r=b}=U \cos \theta \tag{A2.5}
\end{equation*}
$$

From equations (4.3), (4.4), (A2.4) and (A2.5), we obtain
and

$$
\begin{equation*}
\left.\frac{\partial \psi}{\partial \theta}\right|_{r=b}=-U b^{2} \sin \theta \cos \theta \tag{A2.6}
\end{equation*}
$$

$$
\begin{equation*}
\left.\frac{\partial \psi}{\partial r}\right|_{r=b}=-u b \sin ^{2} \theta \tag{A2.7}
\end{equation*}
$$

Equation (A2.6) can be rewritten as

$$
\begin{equation*}
\left.\frac{\partial \psi}{\partial \theta}\right|_{r=b}=-\frac{U b^{2}}{2} \frac{\partial}{\partial \theta} \sin ^{2} \theta \tag{A2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\psi\right|_{r=b}=-\frac{U b^{2}}{2} \sin ^{2} \theta+\text { const } \tag{A2.9}
\end{equation*}
$$

From equation (4.4), we obtain

$$
\begin{equation*}
\left.\frac{\partial \psi}{\partial r}\right|_{\theta=0}=\left.\frac{\partial \psi}{\partial r}\right|_{r=\pi}=0 \tag{A2.10}
\end{equation*}
$$

From equations (A2.3) and (A2.10)

$$
\begin{equation*}
\left.\psi\right|_{\theta=0}=\left.\psi\right|_{\theta=\pi}=0 \tag{A2.11}
\end{equation*}
$$

Fran equations (A2.9) and (A2.11) we he

$$
\begin{equation*}
\left.\psi\right|_{r=b}=-\frac{1}{2} U b^{2} \sin ^{2} \theta \tag{A2.12}
\end{equation*}
$$

Let $\psi$ now refer to the stream function for the power law fluid. Let the Newtonian stream function be $\psi_{N}$. Thus

$$
\begin{align*}
& \left.\psi_{N}\right|_{r=a}=0 \\
& \left.\psi_{N}\right|_{r=b}=-\frac{1}{2} U b^{2} \sin ^{2} \theta \\
& \left.\frac{\partial \psi_{N}}{\partial r}\right|_{r=a}=0  \tag{A2.13}\\
& \left.\frac{\partial \psi_{N}}{\partial r}\right|_{r=b}=-U b \sin ^{2} \theta \\
& \left.\frac{\partial \psi_{N}}{\partial r}\right|_{\theta=0}=\left.\frac{\partial \psi_{N}}{\partial r}\right|_{\theta=\pi}=0 \\
& -A 8=
\end{align*}
$$

Using equation (4.26) we obtain

$$
\begin{align*}
& \left.\psi_{N N}\right|_{r=a}=\left.\psi_{N N}\right|_{r=b}=0 \\
& \left.\frac{\partial \psi}{\partial r} N N\right|_{r=a}=\left.\frac{\partial \psi}{\partial r} N N\right|_{r=b}=0 \tag{A2.14}
\end{align*}
$$

and

$$
\left.\frac{\partial \psi}{\partial \theta} N\right|_{\theta=0}=\left.\frac{\partial \psi}{\partial \theta} A N\right|_{\theta=\pi}=0
$$

the required boundary conditions for $\psi_{\text {NS }}$ when the outer sphere has a finite radius.

DERIVING THE NEWTONIAN STREAM FUNCTION
WHEN THE OUTER SPHERE HAS A FINITE OR AN INFINITE RADIUS

The Newtonian stream function, $\psi_{N}$, was obtained by solving equation (A1.31) under the boundary conditions for $\psi_{N}$ derived in Appendix 2. A variable separable solution of the form

$$
\psi_{N}=F(r) \sin ^{2} \theta \quad(A 3.1)
$$

was assumed. Substituting equation (A3.1) into equation (A1.29) we obtain

$$
\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r^{2}}\right)\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r^{2}}\right) F(r)=0 \quad(A 3 \cdot 2)
$$

Let

$$
\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r^{2}}\right) F(r)=F^{*}(r)
$$

Then equation (A3.2) may be rewritten as

$$
\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r^{2}}\right) F^{*}(r)=0 \quad(A 3 \cdot 4)
$$

By inspection, a solution is given by

$$
\begin{equation*}
F^{x}(r)=\frac{a}{r}+C r^{2} \tag{A3.5}
\end{equation*}
$$

where a and c are arbitrary constants. From equations (A3.3) and (A3.5) we have

$$
\left(\frac{d^{2}}{d r^{2}}-\frac{2}{r_{2}}\right) f(r)=\frac{a}{r}+C r^{2} \quad(13,6)
$$

By inspection, a particular integral is given by

$$
\begin{equation*}
F_{p}(r)=b^{*} r+d^{*} r^{4} \tag{A3.7}
\end{equation*}
$$

and the complementary function is given by

$$
\begin{equation*}
F_{c}(r)=\frac{a}{r}^{*}+c^{*} r^{2} \tag{A3.8}
\end{equation*}
$$

where $a^{*}, b^{*}, c^{*}$ and $d^{*}$ are arbitrary constants. Let

$$
\begin{aligned}
& a^{*}=-\frac{U}{2} a^{3} A \\
& b^{*}=-\frac{U}{2} a B \\
& c^{*}=-\frac{U}{2} C
\end{aligned}
$$

$$
d^{*}=-\frac{U}{2} \frac{D}{a^{2}}
$$

Then

$$
\psi=\left(F_{p}(r)+F_{c}(r)\right) \sin ^{2} \theta
$$

(A3.10)
ie.

$$
\psi=-\frac{1}{2} U a^{2}\left(A \frac{a}{r}+B \frac{r}{a}+C \frac{r^{2}}{a^{2}}+D \frac{r^{4}}{a^{4}}\right) \sin ^{2} \theta
$$

(A3.11)
The constants $A, B, C$ and $D$ are now derived. From equations (A2.1), (A2.3), (A2.7), (A2.12) and (A3.11) we obtain

$$
\begin{gather*}
A+B+C+D=0  \tag{3}\\
-A+B+2 C+4 D=0  \tag{A3.13}\\
A+B d^{2}+C d^{3}+D d^{5}=d^{3}
\end{gather*}
$$

$$
(A 3.14)
$$

and

$$
\begin{equation*}
-A+B d^{2}+2 C d^{3}+4 D d^{5}=2 d^{3} \tag{A3.15}
\end{equation*}
$$

Eliminating $A$, we have

$$
2 B+3 C+5 D=0
$$

from (A3.12) + (A3.13)

$$
\begin{equation*}
2 B+3 C d+5 D d^{3}=3 d \tag{A3.17}
\end{equation*}
$$

from (A3.14) + (A3.15)

$$
3 C(d-1)+5 D\left(d^{3}-1\right)=3 d \quad(A 3.18)
$$

from (A3.17) - (A3.16)

$$
\begin{equation*}
\left(d^{2}-1\right) B+\left(d^{3}-1\right) C+\left(d^{5}-1\right) D=d^{3} \tag{A3.19}
\end{equation*}
$$

from (A3.12) - (A3.14) and

$$
\begin{equation*}
\left(d^{2}-1\right) B+\left(d^{3}-1\right) 2 C+4\left(d^{5}-1\right) D=2 d^{3} \tag{A3.20}
\end{equation*}
$$

from (A3.13) - (A3.15).
Subtracting equation (A3.19) from (A3.20) we obtain

$$
\begin{equation*}
\left(d^{3}-1\right) C+3\left(d^{5}-1\right) D=d^{3} \tag{A3.21}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
c=\frac{d^{3}-3\left(d^{5}-1\right) D}{d^{3}-1} \tag{A3.22}
\end{equation*}
$$

Substituting equation (A3.22) into equation (A3.18) we obtan

$$
\begin{equation*}
3\left[\frac{d^{3}-3\left(d^{5}-1\right) D}{1+d+d^{2}}\right]+5 D\left(d^{3}-1\right)=3 d \tag{A3.23}
\end{equation*}
$$

But

$$
\begin{equation*}
d^{3}-1=(d-1)\left(d^{2}+d+1\right) \tag{A3.24}
\end{equation*}
$$

and

$$
\begin{equation*}
d^{5}-1=(d-1)\left(d^{4}+d^{3}+d^{2}+1\right) \tag{A3.25}
\end{equation*}
$$

Transposing equation (A3.23) we obtain

$$
\begin{gathered}
D\left[d+6 d^{2}+d^{3}-4 d^{4}-4\right]=\frac{-3 d(1+d)}{(d-1)}(A 3.26) \\
-A 12-
\end{gathered}
$$

which may be written as

$$
D(d-1)^{2}\left(4 d^{2}+7 d+4\right)=\frac{-3 d(1+d)}{d-1}(A 3.27)
$$

and

$$
D=\frac{-3 d(1+d)}{(d-1)^{3}\left(4 d^{2}+7 d+4\right)} \quad(A 3.28)
$$

By back substitution into equations (A3.18), (A3.16) and (A3.12) we obtain

$$
\begin{aligned}
& C=\frac{d\left(9+9 d+4 d^{2}+4 d^{3}+4 d^{4}\right)}{(d-1)^{3}\left(4+7 d+4 d^{2}\right)}(A 3.29) \\
& B=\frac{-6 d(1+}{(d-1)^{3}\left(4+7 d+4 d^{2}\right)} \quad(A 3.30)
\end{aligned}
$$

and

$$
A=\frac{2 d^{3}\left(1+d+d^{2}\right)}{(d-1)^{3}\left(4+7 d+4 d^{2}\right)} \quad(A 3.31)
$$

and the expression for $\psi_{N}$ is known.
For an unbounded Newtonian fluid $d \rightarrow \infty$ and we obtain

$$
\begin{align*}
& A=\frac{1}{2}  \tag{A3.32}\\
& B=-\frac{3}{2}  \tag{A3.33}\\
& C=1  \tag{A3.34}\\
& D=0
\end{align*}
$$

(A3.35)

EVALUATING THE INTEGRAL $I_{2}$

The integral, $I_{2}$, is given by

$$
I_{2}=\int_{0}^{\pi} \sin ^{3} \theta \log \sin ^{2} \theta d \theta \quad \text { (A 4.1) }
$$

Now

$$
d \cos \theta=-\sin \theta d \theta
$$

(A4.2)
Therefore

$$
I_{2}=\int_{\theta=\pi}^{0}\left(1-\cos ^{2} \theta\right) \log \left(1-\cos ^{2} \theta\right) d \cos \theta(A 4 \cdot 3)
$$

Let

$$
\cos \theta=X \quad(A 4.4)
$$

Then

$$
I_{2}=\int_{-1}^{1}\left(1-x^{2}\right) \log \left(1-x^{2}\right) d x \quad(A 4.5)
$$

Integrating by parts

$$
I_{2}=\left[\left(x-\frac{x^{3}}{3}\right) \log \left(1-x^{2}\right)\right]_{-1}^{1}+I \quad(A 4.6)
$$

where

$$
\begin{aligned}
I & =\int_{-1}^{1}\left(x-\frac{x^{3}}{3}\right) \frac{2 x}{1-x^{2}} d x \quad(A 4.7) \\
& =\frac{4}{3} \int_{0}^{1} \frac{x^{2}\left(3-x^{2}\right)}{1-x^{2}} d x \quad \text { (A4.8) }
\end{aligned}
$$

By repeatedly expressing the numerator of the integrand in terms of the denominator we find that

$$
I=-\frac{20}{9}+\frac{4}{3}\left[\log 2-\left.\log (1-x)\right|_{x=1}\right] \quad(A 4.9)
$$

Now

$$
\begin{array}{r}
{\left[\left(x-\frac{x^{3}}{3}\right) \log \left(1-x^{2}\right)\right]_{-1}^{1}=\left.\frac{2}{3} \log \left(1-x^{2}\right)\right|_{x=1}+\left.\frac{2}{3} \log \left(1-x^{2}\right)\right|_{x=-1}} \\
\text { (A4.10) } \\
=\frac{2}{3} \log (1+x)_{x=1}+\frac{2}{3} \log (1-x)_{x=1}+\frac{2}{3} \log (1+x)_{x=-1} \\
+
\end{array} \begin{array}{r}
\frac{2}{3} \log (1-x)_{x=-1} \quad \text { (A4.11) } \\
=\frac{4}{3} \log 2+\frac{4}{3} \log (1-x)_{x=1} \quad \text { (A4.12) } \tag{A4.12}
\end{array}
$$

Therefore

$$
\begin{equation*}
I_{2}=-\frac{20}{9}+\frac{8}{3} \log 2 \tag{A4.13}
\end{equation*}
$$

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## NOMENCLATURE



| $r$ | : | radial distance. residual, A $\hat{u}-A u$. |
| :---: | :---: | :---: |
| $u$ | : | the solution to a differential equation. |
| $\hat{u}$ | : | a trial function, an approximation to $u$. |
| $\hat{u}(\mathrm{e})$ | : | local approximation to $u$ in element $e$. |
| $\hat{u}_{i}(e)$ | : | local approximation to $u$ at the ith local node of element $e$. |
| (u, v) | : | local coordinates. |
| $\overline{\mathrm{u}}$ | : | mean fluid velocity. |
| v | : | fluid velocity vector. |
| $v_{i, k}$ | : | covariant derivative of $\mathrm{v}_{1}$. |
| w | : | $z$ component of fluid velocity vector. |
| $w_{0}, w_{1}, w_{2}$ | : | coefficients of $p^{0}, \rho^{1}$ and $\rho^{2}$ in the perturbation expansion of $w$ about $p=0$. |
| $w_{i}$ | : | weighting function. |
| A | $:$ $:$ $:$ | material constant for the Robertson-Stiff fluid. <br> a differential operator. <br> coefficient in the equation for $\psi N$. |
| $A^{\prime}$ | : | coefficient in the equation for $\psi_{\mathrm{NN}}$. |
| A* | : | coefficient in the final equation for $\psi \mathrm{NN}$. |


| $\mathrm{A}_{1}$ |  | a function of $\dot{\gamma}_{i k}, r$ and $\theta$. |
| :---: | :---: | :---: |
| $A(t)$ |  | a function arising from integration with respect to $r$. |
| B |  | coefficient in the equation for $\Psi_{N}$. a material constant for the Robertson-Stiff fluid. |
| $B^{\prime}$ | : | coefficient in the equation for $\psi_{\text {NN }}$. |
| B* | : | coefficient in the final equation for $\Psi_{\text {NN }}$. |
| $\mathrm{B}_{1}$ |  | a function of $\dot{\gamma} i k, r$ and $\theta$. |
| $B(t)$ |  | a function arising from integration with respect to $r$. |
| C | : | a material constant for the Robertson-Stiff fluid. <br> a material constant for the Casson fluid. coefficient in the equation for $\psi_{N} \cdot$ |
| $C_{i k}$ | : | the Cauchy deformation tensor. |
| C ${ }^{\text {l }}$ | : | coefficient in the equation for $\psi \mathrm{NN}^{\text {• }}$ |
| $\mathrm{C}_{1}$ |  | a function of $\dot{\gamma} i k, r$ and $\theta$. |
| C* |  | an expression used in the analytical expression for $F(\Psi N)$. |
| $C(R)$ | : | the correction term applied to V(R)ORIG* |
| D | : | coefficient in the equation for $\psi_{N}$. |
| D' | : | coefficient in the equation for $\psi_{\mathrm{NN}}$. |



| $L \psi_{N N}$ |  | load vector for $\psi_{\mathrm{NN}}$. |
| :---: | :---: | :---: |
| $\underline{L} \omega_{N N}$ | : | load vector for $\omega_{\text {NN }}$. |
| $N(\tau)$ | : | relaxation spectrum over $\tau$ 。 |
| $P_{1}, \ldots \ldots, P_{5}$ | : | expressions used in the analyical expression for $F\left(\psi_{N}\right)$. |
| $P(T)$ | : | pressure gradient. |
| P* (T) | : | the $T$ dependent factor of $P(T)$ |
| ) |  |  |
| $\mathrm{P}_{0}, \mathrm{P}_{1}, \mathrm{P}_{2}$ | : | coefficients of $\rho^{0}, \rho^{1}, \rho^{2}$ in the perturbation expansion of P about $\rho$ equals zero. |
| $\bar{Q}$ | : | the mean flow rate of the fluid. |
| Q* | : | the flow rate non-dimensionalised with respect to the mean flow rate. |
| Q NUM | : | numerical value of $Q^{*}$ before correction. |
| $Q_{0}, Q_{1}, Q_{2}$ | : | coefficients of $\rho^{0}, \rho^{1}$ and $\rho^{2}$ in the perturbation expansion of $Q$ about $p$ equals zero. |
| R | : | non-dimensional variable. |
| Re | : | Reynold's number. |
| S | : | stiffness matrix. |
| $s(e)$ | : | element stiffness matrix. |
| $S_{i j}(\mathrm{e})$ | : | i, jth member of $S^{(e)}$. |


| $S_{1}, \ldots, S_{6}$ |  | expressions used in the analytical expression for $F\left(\psi_{N}\right)$. |
| :---: | :---: | :---: |
| T | : | non-dimensional variable. |
| $T_{i j}, T^{i j}, T_{i}{ }^{j}$ | : | covariant, contravariant and mixed tensors. |
| U | : | instantaneous velocity of the outer sphere. |
| $\left(U^{(e)}, V^{(e)}\right)$ | : | global coordinates of a point in element (e). |
| $\left(U_{i}(\mathrm{e}), \mathrm{V}_{\mathrm{i}}(\mathrm{e})\right.$ ) | : | global coordinates of the ith local node of elemet (e). |
| V | : | non-dimensional variable. |
| $V(R)$ ORIG | : | numerical value of $V(R)$ before correction. |
| $V(R)_{\text {NEW }}$ | : | numerical value of $V(R)$ after correction. |
| $\mathrm{V}_{\mathrm{rNN}}$ | : | the non-Newtonian radial velocity component. |
| $\mathrm{V}_{\mathrm{r}}, \mathrm{V}_{\theta}, \mathrm{V}_{\phi}$ | : | components of the fluid velocity vector in spherial polar coordinates. |
| $\mathrm{V}_{\mathrm{r}}, \mathrm{V}_{\theta}, \mathrm{V}_{\mathrm{z}}$ | : | components of the fluid velocity vector in cylindrical polar coordinates. |
| $V_{0}, V_{1}, V_{2}$ | : | the coefficients of $p^{0}, p^{1}$ and $p^{2}$ in the perturbation expansion for $V$ about $\rho$ equals zero. |
| X | : | variable of transformation. |
| Z | : | variable of transformation. |


$\lambda_{1}, \lambda_{2}:$ material constants for the Oldroyd four constant fluid.

: coloumn vector of $\Psi_{\text {NN }}$ values at node points of finite element mesh.
: the frequency of the imposed pulsatile pressure gradient (rads $\mathrm{s}^{-1}$ ).
: equal to $E^{2} \psi$.
$w_{N}$
$w_{\mathrm{NN}}$
: equal to $E^{2} \psi_{N N}$.
${\underset{\sim}{W N}}^{\text {N }}$
$\Gamma$
$: \quad$ the boundary of $\Omega$
$\Omega$
: a physical region in the ( $r, \theta$ ) plane.

D/Dt : material derivative.
$\vartheta / \vartheta_{t} \quad: \quad$ a derivative independent of absolute motion in space.
$-\partial p / \partial z \quad: \quad$ pressure gradient in tube.
$\partial \psi_{\mathrm{NN}} / \partial \mathrm{n} \quad: \quad$ normal derivative of $\psi_{\mathrm{NN}} \cdot$
$\partial \omega_{\mathrm{NN}} / \partial_{\mathrm{n}} \quad: \quad$ normal derivative of $\boldsymbol{\omega}_{\mathrm{NN}}$.

I, II, III : the invariants of the rate of strain tensor.

〈. : average value taken over one cycle.
$\left\langle r, w_{i}\right\rangle \quad: \quad$ the inner product of $r$ with $w_{i}$.

## FIGURE 1

# THIS FIGURE SHOWS THE RELATIONSHIP BETWEEN SHEAR STRESS AND SHEAR RATE FOR A COMMERCIAL DRILLING FLUID. 

THE SAME RELATIONSHIP IS ALSO SHOWN FOR
(i) THE BINGHAM FLUID

AND (ii) THE POWER LAW FLUID
CONSTRUCTED TO REPRESENT THE DRILLING FLUID.


## FIGURES 2(a) - 2(d)

FIG. 2(a): THE LOCAL COORDINATE SYSTEM FOR A TYPICAL FINITE ELEMENT.

FIG. 2(b): THE LOCAL COORDINATES OF THE VERTICES OF A TYPICAL FINITE ELEMENT AND OF ONE OF ITS INTERIOR POINTS.

FIG. 2(c): A TYPICAL FINITE ELEMENT REFERRED TO A RECTILINEAR COORDINATE SYSTEM WITH EQUAL SCALES.

FIG. 2(d): A SIMPLE TWO ELEMENT MESH.


## FIGURE 3

THIS FIGURE SHOWS A SCHEMATIC DIAGRAM FOR THE FLOW SITUATION WHEN THE OUTER SPHERE HAS A FINITE RADIUS.

$V_{r}=U \cos \theta$
$V_{\theta}=-$ USIN $\theta$

## FIGURE 4

THIS FIGURE SHOWS THE ORIENTATION OF THE COMPONENTS OF THE TOTAL STRESS TENSOR AT THE SURFACE OF THE INNER SPHERE.


WHERE $\pi=p \delta+\tau$

## FIGURE 5

THIS FIGURE SHOWS THE DOMAIN IN THE ( $r, \theta$ ) PLANE, OVER WHICH THE FINITE ELEMENT MESH IS CONSTRUCTED, WHEN THE OUTER SPHERE HAS AN 'INFINITE' RADIUS.


FIGURE 6

THIS FIGURE SHOWS A FINITE ELEMENT MESH IN THE ( $\mathrm{r}, \theta$ ) DOMAIN WHEN THE OUTER SPHERE HAS AN 'INFINITE' RADIUS. THE BROKEN LINE SHOWS THE POSITION OF THE MATCHING SURFACE THE RADIUS OF WHICH IS 10 OR 20 TIMES THAT OF THE INNER SPHERE.
FINITE ELEMENT MESH
ANAIYTICAL SOLUTION KNOWN FOR LARGE RADIUS
OMEGA AND PSI MATCHED AT SUFFICIENTLY LARGE RADIUS


## FIGURE 7

THIS FIGURE SHOWS THE ITERATIVE PROCEDURE USED TO CALCULATE AND PRINT $\mathrm{F}^{\prime}(\mathrm{n})$ WHEN THE FLUID IS SLIGHTLY POWER LAW AND THE OUTER SPHERE HAS AN 'INFINITE' RADIUS.


FIGURE 8

THIS FIGURE SHOWS THE RELATIONSHIP BETWEEN $F(n, d)$ AND n FOR A SLIGHTLY POWER LAW FLUID WHEN d TAKES THE VALUES 2, 10, 20, 40 AND 100.

TWO SETS OF RESULTS ARE PRESENTED:
(i) THE FINITE ELEMENT RESULTS.
(ii) THE RESULTS OF LOCKYER, DAVIES AND JONES OBTAINED USING FINITE DIFFERENCE METHODS.


## FIGURE 9

THIS FIGURE SHOWS THE RELATIONSHIP BETWEEN F'(n) AND THE NUMBER OF RADIAL ELEMENTS USED IN THE FINITE ELEMENT MESH. THE MATCHING SURFACE USED HAD A RADIUS OF EITHER 10 OR 20 TIMES THAT OF THE INNER SPHERE.


## FIGURE 10

THIS FIGURE SHOWS THE RELATIONSHIP BETWEEN $F(n)$ AND $n$ FOR VARIOUS NUMBERS OF RADIAL ELEMENTS USED IN THE FINITE ELEMENT MESH. THE MATCHING SURFACE USED HAD A RADIUS OF EITHER 10 OR 20 TIMES THAT OF THE INNER SPHERE.


## FIGURE 11

THIS FIGURE SHOWS A COMPARISON OF THE RESULTS OBTAINED FOR THE DRAG CORRECTION FACTOR, F(n), WITH THE RESULTS OF OTHER WORKERS.

## CURVE

1,3 J.C. SLATTERY48
2 Y. TOMITA ${ }^{49}$
4,6 M.L. WASSERMAN, J.C. SLATTERY25
4,5 Y.I. CHO AND J.P. HARTNETT ${ }^{28}$
7 A. FARAROURI, R.C. KINTNER50
8 M.A. LOCKYER, J.M. DAVIES AND T.E.R. JONES 37
9 G.U. DAZHI AND R.J. TANNER 31
10 M.J. CROCHET, A.R. DAVIES AND K. WALTERS 30
11 FINITE ELEMENT RESULT (OUTER SPHERE 'INFINITE')
12 FINITE ELEMENT RESULT ( $\mathrm{d}=100$ )

THIS FIGURE SHOWS A SCHEMATIC DIAGRAM OF THE APPARATUS OF CHAKRABATI AND DAVIES 45 IN WHICH THE MEAN FLOW RATE AND THE PULSATILE FLOW RATE ARE CONTROLLED AND THE PRESSURE GRADIENT IS MEASURED.


THESE FIGURES SHOW THE VARIATION OF THE NORMALISED MEAN PRESSURE GRADIENT WITH F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. THE VALUE OF EPSILON IS $0.5,1,5$ OR 10. THE CONTINUOUS LINES REFER TO NUMERICAL DATA, THE BROKEN LINES TO. RESULTS OBTAINED FROM PERTUBATION THEORY AND THE DOTTED LINES TO THE RESULTS WHEN FLUID INERTIA IS ZERO.





## FIGURES 17 - 20

THESE FIGURES SHOW THE VARIATION OF THE PHASE ANGLE WITH F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. THE VALUE OF EPSILON IS $0.5,1,5$ OR 10.




THESE FIGURES SHOW THE VARIATION OF THE FUNDAMENTAL PRESSURE AMPLITUDE WITH F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. THE VALUE FOR EPSILON IS 0.5, 1, 5 OR 10. THE SLOPE OF THE GRAPH TENDS TO 1 FOR HIGH VALUES OF F FOR ALL EPSILON.




TVINヨWVONn」


THIS FIGURE SHOWS THE VARIATION OF THE NORMALISED MEAN PRESSURE GRADIENT WITH EPSILON WHEN $N=0.5$ AND F IN THE RANGE 5 TO 100.


## TABLES 1-3

TABLE 1 TABULATION OF $F^{\prime}(n, d)$ AGAINST d (REFER FIGURE 8).

TABLE 2 TABULATION OF $F^{\prime}(n)$ AGAINST NUMBER OF RADIAL ELEMENTS IN THE FINITE ELEMENT MESH. THE RADIUS OF THE MATCHING SURFACE IS 10 TIMES THAT OF THE FIXED SPHERE.

TABLE 3 AS FOR TABLE 2, EXCEPT THAT THE RADIUS OF THE MATCHING SURFACE IS 20 TIMES THAT OF THE FIXED SPHERE. FOR TABLES 2 AND 3 REFER FIGURE 9.

## TABLE 1

(MESH PARAMETERS 100*20)

| VALUE $O F d$ |  | $F^{\prime}(n, d)$ |
| :---: | :---: | :---: |
| 2 |  | 1.76936 |
| 10 |  | -0.39778 |
| 20 |  | -0.74215 |
| 40 |  | -0.96722 |
| 60 |  | -1.06053 |
| 100 |  | -1.14473 |

TABLE 2
(NATAH RADIUS OF 10 )
MESH PARAMETERS
F'(n)
$10 * 20 \quad-0.71128$
20*20 -1.09128
40*20
-1.33409
50*20
$-1.3876$
70*20
$-1.45035$
100*20
$-1.49911$

TABLE 3
(:AATAI PADIUS CF 20)

| MESH PARAMETERS |  | $F^{\prime}(\mathrm{n})$ |
| :---: | :---: | :---: |
| $10 * 20$ | -0.58647 |  |
| $20 * 20$ | -0.97847 |  |
| $40 * 20$ | -1.24522 |  |
| $50 * 20$ | -1.30509 |  |
| $70 * 20$ | -1.37626 |  |
| $100 * 20$ | -1.43143 |  |
| $140 * 20$ |  | -1.46893 |

## TABLES 4-7

TABULATIONS OF THE NUMERICAL VALUE OF THE NORMALISED MEAN PRESSURE GRADIENT AGAINST F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. FOR TABLES 4-7 THE VALUE OF EPSILON IS 0.5, 1, 5 AND 10 RESPECTIVELY. REFER FIGURES 13-16.

## NORMALISED MEAN PRESSURE GRADIENTS

## TABLE 4 (EPSILON=0.5)

| F | 0.5 | 1 | 5 | 10 | 50 | 100. |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.9 | 0.99401 | 0.99396 | 0.99269 | 0.99017 | 0.98076 | 0.98166 |
| 0.7 | 0.98567 | 0.98553 | 0.98178 | 0.97472 | 0.93755 | 0.94064 |
| 0.5 | $\mathbf{6} .98220$ | 0.98197 | 0.97611 | 0.96557 | 0.87644 | 0.89554 |
| 0.3 | $*$ | 0.98514 | 0.97632 | 0.96442 | 0.77136 | 0.80799 |

TABLE 5 (EPSILON=1.0)

| F | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 1.00000 | 1.00001 | 1.0000 | 1.00000 | 1.00000 | $1.0 n 000$ |
| 0.9 | 0.97072 | 0.97069 | 0.96321 | 0.95611 | 0.94944 | 0.95459 |
| 0.7 | 0.92544 | 0.92462 | 0.89857 | 0.87487 | 0.85709 | 0.87561 |
| 0.5 | $*$ | 0.89767 | 0.84746 | 0.80095 | 0.78187 | 0.81415 |
| 0.3 | $*$ | $*$ | 0.81482 | 0.73250 | 0.74078 | 0.77536 |

TABLE 6 (EPSILON=5.0)

| $F$ | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  | 1.00000 | 1.00000 | 0.99999 | 0.99999 | 1.00000 |
| 1.0 | 1.0 .99998 |  |  |  |  |  |
| 0.9 | 0.82467 | 0.82426 | 0.81839 | 0.81667 | 0.84228 | 0.86467 |
| 0.7 | 0.55725 | 0.55552 | 0.53988 | 0.53994 | 0.60622 | 0.66333 |
| 0.5 | 0.37237 | 0.36810 | 0.34611 | 0.35069 | 0.43482 | 0.49442 |
| 0.3 | 0.24461 | 0.23556 | 0.21211 | 0.22543 | 0.28344 | 0.31057 |

TABLE 7 (EPSILON=10.0)

| F | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | 1.00000 | 1.00000 | 0.99999 | 1.00000 | 0.99999 | 0.99998 |
| 1.0 | 0.76938 | 0.76895 | 0.76322 | 0.76210 | 0.79214 | 0.82056 |
| 0.9 | 0.45218 | 0.45010 | 0.43745 | 0.44083 | 0.51236 | 0.57189 |
| 0.7 | 0.26191 | 0.25671 | 0.24481 | 0.25444 | 0.33128 | 0.37780 |
| 0.5 | 0.13725 | 0.13544 | 0.14699 | 0.18570 | $0.2 n 046$ |  |

## TABLES 8-11

TABULATIONS OF THE THEORETICAL MEAN PRESSURE GRADIENTS AGAINST F FOR $N$ IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. FOR TABLES 8-11 THE VALUES OF EPSILON ARE 0.5, 1, 5 AND 10 RESPECTIVELY. THE MEAN PRESSURE GRADIENTS HAVE BEEN DERIVED USING THE PERTURBATION ANALYSIS. REFER FIGURES 13-16.

NORMALISED MEAN PRESSURE GRADIENTS DERIVED USING THE PERTURBATION ANALYSIS

TABLE 8 ( BPS ILON $=0.5$ )

| F | 1 | 3 | 6 |  | 8 | 10 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| N | 3 |  | 15 |  |  |  |
| 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 0.9 | 0.9941 | 0.9936 | 0.9921 | 0.9904 | 0.9884 | 0.9811 |
| 0.7 | 0.9860 | 0.9846 | 0.9799 | 0.9750 | 0.9687 | 0.9469 |
| 0.5 | 0.9830 | 0.9808 | 0.9732 | 0.9654 | 0.9553 | 0.9201 |
| 0.3 | 0.9855 | 0.9828 | 0.9736 | 0.9642 | 0.9520 | 0.9097 |


| F | TABLE 9 (EPS ILON $=1.0$ ) |  |  |  | 10 | 15 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 3 | 6 | 8 |  |  |
| N |  |  |  |  |  |  |
| 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 0.9 | 0.9709 | 0.9678 | 0.9574 | 0.9466 | 0.9328 | 0.8846 |
| 0.7 | 0.9252 | 0.9178 | 0.8927 | 0.8666 | 0.8331 | 0.7169 |
| 0.5 | 0.8990 | 0.8885 | 0.8530 | 0.8162 | 0.7688 | 0.6045 |
| 0.3 | 0.8994 | 0.8878 | 0.8485 | 0.8078 | 0.7554 | 0.5735 |

TABLE 10 (EPSILON=5.0)

| F | 1 | 3 | 6 | 8 | 10 | 15 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N 101010 |  |  |  |  |  |  |
| 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 0.9 | 0.8244 | 0.8206 | 0.8078 | 0.7944 | 0.7773 | 0.7178 |
| 0.7 | 0.5564 | 0.5419 | 0.4931 | 0.4425 | 0.3775 | 0.1515 |
| 0.5 | 0.3707 | 0.3410 | 0.2408 | 0.1368 | 0.0032 | 0.1515 |
| 0.3 | 0.2429 | 0.1968 | 0.0412 |  | . |  |

TABLE 11 (EPSILON=10.0)

| F | 1 | 2 | 3 | 4 | 6 | 8 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| N |  |  |  |  |  |  |
| 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 0.9 | 0.7688 | 0.7672 | 0.7647 | 0.7611 | 0.7508 | 0.7364 |
| 0.7 | 0.4495 | 0.4428 | 0.4316 | 0.4160 | 0.3713 | 0.3087 |
| 0.5 | 0.2559 | 0.2400 | 0.2137 | 0.1767 | 0.0723 |  |
| 0.3 | 0.1372 | 0.1090 | 0.0619 |  |  |  |

## TABLE 12

TABULATION OF THE THEORETICAL NORMALISED MEAN PRESSURE GRADIENTS (WHEN FLUID INERTIA IS IGNORED) WHEN N IS IN THE RANGE 0.3 TO 1.0 AND EPSILON HAS THE VALUES 0.5, 1 , $5,10,50$ AND 100.

TABLE 12

|  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
| EPS | 0.5 | 1 | 5 | 10 |
| $N$ |  |  |  |  |
| 0.3 | 0.9858 | 0.9002 | 0.2537 | 0.1536 |
| 0.4 | 0.9839 | 0.8963 | 0.3090 | 0.2020 |
| 0.5 | 0.9833 | 0.9002 | 0.3768 | 0.2652 |
| 0.6 | 0.9841 | 0.9104 | 0.4593 | 0.3472 |
| 0.7 | 0.9862 | 0.9261 | 0.5592 | 0.4537 |
| 0.8 | 0.9895 | 0.9465 | 0.6798 | 0.5915 |
| 0.9 | 0.9942 | 0.9713 | 0.8251 | 0.7698 |
| 1.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

## TABLES 13-16

TABULATION OF THE FUNDAMENTAL PHASE ANGLES FOR THE NORMALISED PRESSURE GRADIENTS AGAINST F WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. FOR TABLES 13-16 THE VALUES OF EPSILON ARE $0.5,1,5$, AND 10 RESPECTIVELY. REFER FIGURES 17-20.

## TABLE 13 (EPSILON=0.5)

| F | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 14.555 | 27.256 | 64.964 | 73.240 | 82.686 | 84.817 |
| 0.9 | 15.888 | 29.432 | 66.578 | 74.235 | 83.108 | 85.146 |
| 0.7 | 19.471 | 34.959 | 70.034 | 76.430 | 84.023 | 85.878 |
| 0.5 | 25.244 | 42.893 | 73.984 | 76.729 | 84.980 | 86.731 |
| 0.3 |  | 55.070 | 78.798 | 82.207 | 85.663 | 87.517 |

TABLE 14 (EPSILON=1.0)

| F | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 14.554 | 27.255 | 64.973 | 73.253 | 82.686 | 84.817 |
| 0.9 | 15.577 | 28.987 | 66.233 | 74.033 | 83.120 | 85.223 |
| 0.7 | 18.169 | 33.056 | 68.784 | 75.654 | 84.069 | 86.093 |
| 0.5 | $*$ | 38.514 | 71.547 | 77.486 | 85.175 | 87.037 |
| 0.3 | $*$ | $*$ | 75.018 | 79.840 | 86.527 | 88.043 |

TABLE 15 (EPSILON=5.0)

| F | 0.5 | 1 | 5 | 10 | 50 | ino |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 14.555 | 27.256 | 64.964 | 73.240 | 82.686 | 84.817 |
| 0.9 | 16.553 | 30.492 | 67.320 | 74.877 | 83.611 | 85.607 |
| 0.7 | 21.126 | 37.361 | 71.581 | 77.976 | 85.422 | 87.140 |
| 0.5 | 26.331 | 44.374 | 75.470 | 80.901 | 87.088 | 88.377 |
| 0.3 | 31.750 | 51.139 | 79.126 | 83.574 | 88.358 | 89.150 |

TABLE 16 (EPSILON=10.0)

| F | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 14.555 | 27.256 | 64.964 | -73.253 | 82.686 | 84.812 |
| 0.9 | 17.615 | 32.138 | 68.244 | 75.435 | 83.839 | 85.776 |
| 0.7 | 25.230 | 42.788 | 73.797 | 79.351 | 85.994 | 87.553 |
| 0.5 | 34.613 | 53.408 | 78.478 | 82.758 | 87.803 | 88.812 |
| 0.3 | 44.879 | 63.042 | 82.427 | 85.592 | 88.953 | 89.466 |

## TABLES 17-20

TABULATION OF THE FUNDAMENTAL AMPLITUDES OF THE NORMALISED PRESSURE GRADIENTS WITH N IN THE RANGE 0.3 TO 1.0 FOR GIVEN EPSILON. FOR TABLES 17-20 THE VALUES OF EPSILON ARE 0.5, 1, 5 AND 10 RESPECTIVELY. REFER FIGURES 21-24.

FUNDARENTAL AMPLITUDES FOR THE NORMALISED PRESSURE GRADIENTS.

TABLE 17 (EPSILON=0.5)

| F | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 0.51774 | 0.56745 | 1.38809 | 2.50434 | 10.81144 | 20.82173 |
| 0.9 | 0.47073 | 0.52375 | 1.35566 | 2.46539 | 10.74772 | 20.71905 |
| 0.7 | 0.37723 | 0.43819 | 1.28780 | 2.38013 | 10.61056 | 20.50891 |
| 0.5 | 0.28454 | 0.35592 | 1.21336 | 2.28110 | 10.45428 | 20.27707 |
| 0.3 | $*$ | 0.27920 | 1.12640 | 2.15994 | 10.28515 | 20.01692 |

## TABLE 18 (EPSILON=1.0)

| F | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 1.03547 | 1.13491 | 2.77587 | 5.00853 | 21.62289 | 41.64345 |
| 0.9 | 0.95753 | 1.06244 | 2.72651 | 4.94801 | 21.42151 | 41.29931 |
| 0.7 | 0.80463 | 0.92106 | 2.62549 | 4.82373 | 21.02242 | 40.66699 |
| 0.5 | $*$ | 0.77856 | 2.51421 | 4.69301 | 20.63118 | 40.13605 |
| 0.3 | $*$ | $*$ | 2.38546 | 4.55598 | 20.24588 | 39.72621 |

## TABLE 19 (EPSILON=5.0)

| $F$ | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 5.17735 | 5.67447 | 13.8809 | 25.0434 | 108.1144 | 208.2172 |
| 0.9 | 4.52314 | 5.07056 | 13.3857 | 24.3117 | 106.0478 | 205.0170 |
| 0.7 | 3.49215 | 4.14046 | 12.4938 | 22.9658 | 102.4726 | 200.1439 |
| 0.5 | 2.74466 | 3.48359 | 11.7022 | 21.7795 | 99.9814 | 197.6141 |
| 0.3 | 2.19497 | 3.00483 | 11.0069 | 20.7643 | 98.7472 | 196.6952 |

TABLE 20 (EPSILON $=10.0$ )

| F | 0.5 | 1 | 5 | 10 | 50 | 100 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  |  |  |  |  |  |
| 1.0 | 10.3547 | 11.3489 | 27.7619 | 50.0831 | 216.2289 | 416.4166 |
| 0.9 | 8.51607 | 9.66984 | 26.4788 | 48.2292 | 211.2660 | 408.8970 |
| 0.7 | 5.90482 | 7.38512 | 24.2991 | 44.9993 | 203.2164 | 398.4077 |
| 0.5 | 4.29174 | 6.04360 | 22.5082 | 42.3934 | 198.5377 | 394.1187 |
| 0.3 | 3.30455 | 5.22265 | 21.07021 | 40.5053 | 196.8544 | 392.9862 |

## TABLE 21

TABULATION OF NORMALISED MEAN PRESSURE GRADIENT WITH EPSILON WITH F IN THE RANGE 5 TO 100 AND N EQUAL TO 0.5. REFER FIGURE 25.

## NORMALISED MEAN PRESSURE GRADIENTS

## TABLE $21(\mathrm{~N}=0.5)$

| EPS 0.1 | 0.5 | 1 | 2 | 3 | 4 | 5 | 7 | 10 |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| F |  |  |  |  |  |  |  |  |  |
| 5 | .99808 | .97611 | .84746 | .55845 | .45035 | .38794 | .34611 | .29200 | .24481 |
| 10 | .99776 | .96557 | .80095 | .54965 | .44856 | .38994 | .35069 | .29946 | .25444 |
| 50 | .99515 | .87644 | .78187 | .61896 | .53056 | .47442 | .43482 | .38116 | .33128 |
| 100 | .99237 | .89554 | .81415 | .67739 | .59396 | .53662 | .49442 | .43435 | .37780 |

## PROGRAM 1

THIS PROGRAM GENERATES THE NODAL POINTS AND THE GEOMETRICAL DATA FOR THE FINITE ELEMENT MESH. THE FILE FOLLOWING THE PROGRAM REPRESENTS A TYPICAL OUTPUT FILE CONTAINING THIS DATA.

## C

```
C
```

            INTEGER 1,J,TOTNOD, V1,N,N,NX,NY,ISTAR,IFINI,
    ```
            INTEGER 1,J,TOTNOD, V1,N,N,NX,NY,ISTAR,IFINI,
            W1,W2,W3,W4,W5,NELX,DOFNOD, DIMEN, ELTYP,NEL,NODEL
            W1,W2,W3,W4,W5,NELX,DOFNOD, DIMEN, ELTYP,NEL,NODEL
        DOUBLE PRECISION L,V2,V3,D,NXD,NYD
        DOUBLE PRECISION L,V2,V3,D,NXD,NYD
        DIMENSION V1(4000),V2(4000),V3(4000).
        DIMENSION V1(4000),V2(4000),V3(4000).
        * W1(4000),W2(6400),W3(6400),W4(6400),WS(6400)
        * W1(4000),W2(6400),W3(6400),W4(6400),WS(6400)
        COMON/A/W1,W2
        COMON/A/W1,W2
        COMMON/B/W3,W4
        COMMON/B/W3,W4
        COMNON/C/WS
        COMNON/C/WS
        COMTDN/D/V1
        COMTDN/D/V1
        COMMON/E/V2
        COMMON/E/V2
        COMNON/F/V3
        COMNON/F/V3
        DATA NX/101/,NY/21/,TOTNOD/2121/,DIMEN/2/,
        DATA NX/101/,NY/21/,TOTNOD/2121/,DIMEN/2/,
        * NEL/2000/,DOFNOD/1/,ELTYP/1/,NODEL/4/,NOUT/6/
        * NEL/2000/,DOFNOD/1/,ELTYP/1/,NODEL/4/,NOUT/6/
        DO 10 1=1,TOTNOD
        DO 10 1=1,TOTNOD
        V1(1)=1
        V1(1)=1
10 CONTINUE
10 CONTINUE
    N=1
    N=1
        L=0.000
        L=0.000
        DO 20 J=1,NX
        DO 20 J=1,NX
        ISTAR=N
        ISTAR=N
        IFINI=N+NY-1
        IFINI=N+NY-1
        DO 30 I=ISTAR,IFINI
        DO 30 I=ISTAR,IFINI
        V2(1)=L
        V2(1)=L
30 CONTINUE
30 CONTINUE
        N=N+NY
        N=N+NY
        NXD=NX*1.0D0
        NXD=NX*1.0D0
L=L+0.6931472/(NXD-1)
L=L+0.6931472/(NXD-1)
C L=L+2.9957323/(NXD-1)
C L=L+2.9957323/(NXD-1)
C L=L+3.4011974/(NXD-1)
C L=L+3.4011974/(NXD-1)
L=L+3.6888795/(NXD-1)
L=L+3.6888795/(NXD-1)
    L=L+2.30258/(NXD-1)
    L=L+2.30258/(NXD-1)
    L=L+4.09434/(NXD-1)
    L=L+4.09434/(NXD-1)
20 CONTINUE
20 CONTINUE
    N=1
    N=1
    DO 40 I=1,NX
    DO 40 I=1,NX
    \therefore=0
    \therefore=0
    JSTAR=N
    JSTAR=N
    JFINI=N+NY-1
    JFINI=N+NY-1
    DO 50 J=JSTAR,JFINI
    DO 50 J=JSTAR,JFINI
    NYD=NY*1.0DO
    NYD=NY*1.0DO
    D=-2*Mi/(NYD-1)+1
    D=-2*Mi/(NYD-1)+1
    V3(J)=D
    V3(J)=D
    M=N+1
    M=N+1
5 0 ~ C O N T I N U E ~
5 0 ~ C O N T I N U E ~
    N=N+NY
    N=N+NY
40 CONTINUE
40 CONTINUE
    WRITE(NOUT, 8010)TOTNOD, DIMEN
```

    WRITE(NOUT, 8010)TOTNOD, DIMEN
    ```

DO \(60 \quad 1=1\), TOTNOD
WRITE(NOUT, 8020)V1 (1),V2(1),V3(1)
60 CONTINUE
C
DO \(70 \quad 1=1\), NEL
\(W 1(1)=1\)
70 CONTINUE
C
NELX \(=N X-1\)
\(M=1\)
\(N=0\)
DO \(80 \quad 1=1\), NELX
J STAR \(=N+1\)
JFINI \(=N+N Y-1\)
DO 90 J=J STAR, JFINI
W2(J) \(=\mathrm{in}\)
\(M=M+1\)
\(N=N+1\)
90 CONTINUE
\(M=M+1\)
80 CONTINUE
C
DO \(100 \quad 1=1\),NEL
W3 (I) \(=W 2(1)+1\)
\(W 4(1)=W 3(1)+N Y\)
\(W 5(1)=W 2(1)+N Y\)
100 CONT INUE
C
WRITE(NOUT, 8010)ELTYP,NEL, NODEL
DO \(110 \quad 1=1\),NEL
WRITE(NOUT, 8010)W1 (1), W2 (1), W3 (1), W4 (1),W5 (1)
110 CONT INUE
C
VRITE(NOUT, 8010) DOFNOD
C
STOP
8010 FORAMAT (1615)
8020 FOPMAT (15,6D12.5)
END
the data file for
A finite element niesh

\section*{total number of nodes in fe mesh dinens ionality of problem 162}

NODAL GEOMETRY
GLOBAL NODE NUABER AND
COORDINATES OF NODE
\(10.00000 \mathrm{D}+000.10000 \mathrm{D}+01\)
\(20.00000 \mathrm{D}+000.33333 \mathrm{D}+00\)
\(30.00000 \mathrm{D}+00-0.33333 \mathrm{D}+00\)
\(40.00000 \mathrm{D}+00-0.10000 \mathrm{D}+01\)
\(50.76753 \mathrm{D}+000.10000 \mathrm{D}+01\)
\(60.76753 \mathrm{D}+000.33333 \mathrm{D}+00\)
\(70.76753 \mathrm{D}+00-0.33333 \mathrm{D}+00\)
\(80.76753 \mathrm{D}+00-0.10000 \mathrm{D}+01\)
\(90.15351 \mathrm{D}+010.10000 \mathrm{D}+01\)
\(100.15351 \mathrm{D}+010.33333 \mathrm{D}+00\)
\(110.15351 \mathrm{D}+01-0.33333 \mathrm{D}+00\)
\(120.15351 \mathrm{D}+01-0.10000 \mathrm{D}+01\)
\(130.23026 \mathrm{D}+010.10000 \mathrm{D}+01\)
\(140.23026 \mathrm{D}+010.33333 \mathrm{D}+00\)
\(150.23026 \mathrm{D}+01-0.33333 \mathrm{D}+00\)
\(160.23026 \mathrm{D}+01-0.10000 \mathrm{D}+01\)
element geomietry
ELEMENT TYPE,NU:BER OF ELEMENTS and number of nodes per element 194
number of element and tie global NUNEERING OF IT'S VERTEX NODES
\begin{tabular}{rrrrr}
1 & 1 & 2 & 6 & 5 \\
2 & 2 & 3 & 7 & 6 \\
3 & 3 & 4 & 8 & 7 \\
4 & 5 & 6 & 10 & 9 \\
5 & 6 & 7 & 11 & 10 \\
0 & 7 & 8 & 12 & 11 \\
7 & 9 & 10 & 14 & 13 \\
8 & 10 & 11 & 15 & 14 \\
9 & 11 & 12 & 16 & 15
\end{tabular}

DEGREES OF
FREEDOM PER NODE

\section*{PROGRAM 2}

THIS PROGRAM CALCULATES THE DRAG CORRECTION FACTOR FOR A SPHERE FALLING THROUGH AN INFINITE EXPANSE OF SLIGHTLY POWER LAW FLUID.

\title{
THIS PROGRAM CALCULATES THE DRAG CORRECTION FACTOR FOR A SPHERE FALLING SLONLY THROUGH AN INFINITE EXPANSE OF PONER LAW FLUID．
}

INTEGER＊4 BDCND，BTYPES，DIF，DIMEN，DOFEL，DIAG，ILO，ISTIF，
＊DOFNOD，ELNUM，ELTOP，ELTYP，HBAND，I，IABSS，CI，
＊ICOORD，ILOAD，ICNT，LL，IPFINI，SP，TCI，
＊IDTPD，IELK，IELTOP，IFUN，IGDER，IPTR，IPSTAR，
＊IGEOM，IJAC，IJACIN，ILDER，INF，IP，
＊IQLAD，ISTEER，ITEST，IWGHT，REG，NN，KKI，KSP，SN，
＊J，JABSS，JCOORD，LAST，IRN，ILOCAL，II，POD，KOUNT，
＊JDTPD，JELK，JELTOP，J GDER，J GEOM，J JAC，
＊JJACIN，JLDER，JNF，JP，K，NELE，ILV，ILEC，
＊NF，NIN，NODEL，NODNUM，NOUT，NQP，
＊STEER，TOTDOF，TOTELS，TOTNOD，CCC，COUNT1，COUNT2，COUNT3，
＊BCDI，BCD2，IMID，JMID，
＊IF，JF，COUNT，ILN，IBEV，IRIH，
＊ICP，ICCP，INTRA，INTRB
DOUBLE PRECISION ABSS，DENOMA，SUM，STIF，RRQ，SINARG，VN1，N，
＊COORD，DET，DTPD，ELK，ETA，FUN，GDER，DENOIB，TY，
＊GDERT，GEOA，JAC，JACIN，LDER，P，PD，QUOT，DIV，
＊SCALE，wGit，X，XI，XX，Y，Yy，load，lec，lv，lo，
＊VGA，H1 ，H2，MID，F，R，RQ，TQ，MGDER \((2,4), I W, S L O P E, A R G L O G\),
＊FMD，PI，LN，BEV，RH，U，CMVAA，SLOP，FN，RSTAR，GSTAR，PMMARG，
＊OMNAB，S（5），GU，TTT，RRR，G1，G2，G3，G4，G5，G6，G7，G8，IPOINT，
＊S1，S2，S3，S4，S5，S6，A1，B1，P1，P2，P3，P4，P5，P6，BETAP，JTC4，
＊C1，D1，COEFF，XE \((3000,4,4)\) ，YE \((4,4)\), ALPHA，
＊ \(\operatorname{DGU}(40), \operatorname{DANEW}(40), \operatorname{DRGU}(40), \operatorname{DRAPIEW}(40)\), LRMULT1 ，LMULT2 LOGICAL FIRST
DIMENS ION ABSS \((2,9)\), ILOCAL（9），
＊ \(\operatorname{DTPD}(4,4)\) ．
＊ \(\operatorname{ELK}(4,4), \operatorname{FUN}(4), \operatorname{CDER}(2,4), \operatorname{CDERT}(4,2), \operatorname{CEOin}(4,2)\) ，
＊ \(\operatorname{JAC}(2,2), \operatorname{JACIN}(2,2), \operatorname{LDER}(2,4), P(2,2), \operatorname{PD}(2,4)\) ，
＊ \(\operatorname{STEER}(4), \operatorname{VGHT}(9), \operatorname{MID}(2,2), F(4,2), R(2), \operatorname{LN}(4), \operatorname{LV}(4)\),
＊ \(\operatorname{FADD}(4,4), \operatorname{BEV}(4), \operatorname{LEC}(4), \operatorname{LO}(6561)\)
C PROBELM SIZE DEPENDENT ARRAYS
DIMENS ICN LOAD（6561），GU（6561），COORD（6561，2），
＊ \(\operatorname{ELTOP}(6561,6), \operatorname{NF}(6561,1), \operatorname{COEFF}(3000,9)\) ．
＊ \(\operatorname{ANEW}(6561), \mathrm{RH}(6561), \operatorname{RSTAR}(6561), \operatorname{GSTAR}(6561)\),
＊ \(\operatorname{onvaA}(81), \operatorname{arvab}(81), \operatorname{INTRA}(81)\),
＊ \(\operatorname{INTRB}(81)\) ，
＊ \(\operatorname{IW}(81), \operatorname{BCD1}(5,81), \operatorname{BCD}(5,81)\) ，
＊ \(\operatorname{BDCND}(2,2,81), \operatorname{REG}(6562), \operatorname{IRN}(65620), \operatorname{DIAG}(6561)\) ，
＊STIF（65620）
COMDN／A／ELTOP
COMMON／P／DIAG
COMON／C／COORD
COMMDN／D／RH，ATJEW，NF
－OMON／E／ABSS，BCD1，ECD2，BOCND，BEV，DTPD，
＊ELK，F，FMD，FLN，CDER，
＊GDERT，GEOM，GU，INTRA，INTRE，IW
COMMON／F／JAC，JACIN，LCER，LN，LOAD，MGDER，MID，
＊OMNAA，OMVAB，P，PD，R，S，STEER．
* W1, W2, WGHT,REG, IRN COWMON/G/STIF,COEFF COMMON/H/GSTAR, RSTAR COMADON/I/XE COMMON/J / YE DATA IABSS /2/.ISTIF/65620/,
* IDTPD /4/, IELK /4/,
* IFUN /4/, IGDER /2/, IGEOM /4/, IJAC /2/,
* \(1 \mathrm{JACIN} / 2 /\). ILDER /2/, IP /2/,ISTEER /4/,
* IWGHT /9/, JABSS /9/,JCOORD /2/, ILV/4/. ILEC/4/,
* JDTPD /4/, JELK /4/, J GDER /4/, J GEOM /2/,
* JJAC /2/, JJACIN /2/, JLDER /4/, JNF /1/, JP /2/,
* SCALE /1.0D+25/, ILOAD/6561/,
* IMID /2/,JMID/2/,IF/4/,JF/2/, ILO/6561/

\section*{PROBELA SIZE DEPENDENT DATA STATEMENTS} DATA ICOORD/6561/,IELTOP/6561/,INF/6561/,
* JELTOP/6/, LIULT1/0.5/, LMULT2/0.5/,
* |RRH/6561/, IBEV/4/,ILN/4/,
* RR/10.0/, BETA/-1.5/, COUNT/21/,
* ALPHA/0.0/ DATA NIN /5/, NOUT /6/
C \(\quad \mathrm{WGA}(X X, Y Y)=1.0^{*}((\operatorname{DS} \operatorname{IN}(Y Y) * * 2) / 34.0) *(90 * X X * * 2-372 / X X)\) WGA \((X X, Y Y)=-0.5^{*}\left(-2 * B B / X X+10 * D D^{*} X X * * 2\right) * D S I N(Y Y) * * 2\)
\(\mathrm{HI}(\mathrm{XX}, \mathrm{YY})=0.0000 \mathrm{DO}\)
C \(\quad \mathrm{H} 2(\mathrm{XX}, \mathrm{YY})=-(\mathrm{DSIN}(\mathrm{MY}) * * 2 * 2.0)\) \(\mathrm{H} 2(\mathrm{XX}, \mathrm{YY})=-\left(\mathrm{DS} \operatorname{IN}(\mathrm{YY})^{* * 2 * 0.5 * R R^{* *}} 2\right)\)
C \(\quad \mathrm{H} 3(\mathrm{XX}, \mathrm{YY})=-0.5^{*}\left(\mathrm{BETA} * R R+R R^{* *} 2\right) * \operatorname{DS} \operatorname{IN}(\mathrm{YY}) * * 2\) H3 (XX, YY) \(=(3 *(A L P H A / R R+R R * E E T A+2 * R R * A L O G(R R)) / 4.0) * D S I N(Y Y) * * 2\)
C \(\quad \mathrm{H} 3(\mathrm{XX}, \mathrm{YY})=-\left(\mathrm{DS} I \mathrm{~N}(\mathrm{YY}) * * 2 * \mathrm{RR}^{* *} 2\right) / 2.0\)
C \(\quad \mathrm{H} 3(X X, Y Y)=0.0000\)
C SET ITEST FOR FULL CHECKING ITEST = 0
PI=4.0DO*DATAN(1.0D0)
    \(F N=24 * R R^{*}\left(1+R R+R R * * 2+R R^{* *} 3+R R * * 4\right) /\)
* ((RR-1)**3* (4+7*RR+4*RR**2)*1.0)
    ARGLOG \(=2 * A A+2 * C C+12 * D D\)
    DIV=4* (5*AA+2*CC)/3.0
    SLOPE \(=2 *(0.75 *(2 *\) DLOG(ARGLOG) \(/ 3.0-5 / 9.0)\)
* \(\quad+(4 * A A+2 * C C+4 * D D) / D I V)\)
    \(A A=0.5\)
    \(R B=-1.5\)
    \(C C=1.0\)
    \(D D=0.0\)
    ARGLOG \(=2 * A A+2 * C C+12 * D D\)
    DIV=4*(5*AA+2*CC)/3.0
    SLOPE \(=2 *(0.75 *(2 * D L O G(A R G L O G) / 3.0-5 / 9.0)\)
    * \(\left.\quad+\left(4^{*} A A+2^{*} C C+4^{*} D D\right) / D I V\right)\)
```

C (1) NODAL GEOMETRY
C
OPEN(FILE='LLDATA',UNIT=7,STATUS='UKNOON')
READ(NIN,8010) TOTNOD,D MEN
WR ITE (7,8010) TOTNOD
DO 1010 I=1,TOTNOD
READ(NIN, 8020) NODNUM, (COORD(NODNLM, ) , J=1,D IMEN)
1010 CONTINUE
C
C (2) ELEMENT TOPOLOGY
C
READ (NIN,8010) ELTYP, TOTELS, NODEL
WRITE(7,8010)TOTELS
DO 1020 I=1,TOTELS
READ (NIN, 8010) ELNUM, (ELTOP(ELNCM,j+2),j=1,NODEL)
ELTOP(ELNLM,1) = ELTYP
ELTOP(ELNUM,2) = NODEL
1020 CONTINUE
C
READ (NIN, 8010) DOFNOD
BTYPES=1
DO 2324 I=1,COUNT
J=1+3
BCD1 (1, ) =1
BCD2(1,1)=1+TOTNOD-COUNT
2324 CONTINUE
DO 1075 J=1,COLNT
|]=1+3
BDCND}(1,1,j1)=\operatorname{BCD1}(1,11
BDCND}(2,2,1))=\operatorname{BCD}2(1,1]
1075 CONTINLE
CCC=COLNT+3
K=COUNT+3
DO 2071 l=1,COUNT
INTRA(1)=ECD1 (1,1+3) +COUNT
INTRB(1)=BCD2(1,1+3)-COUNT
2071 CONTINUE
WRITE(1,8010) (INTPA(1),I=1,COUNT)
WRITE(1,8010) (INTRE(1),1=1,COUNT)
TOTDOF = 0
DO 1050 I=1,TOTIND
DO 1040 J=1,DOFNOD
TOTDOF = TOTUOF + 1
NF}(1,1)= TOTDO
1040 CONTINUE
1050 CONTINUE
C
C ASSEMBLY OF TIEE MODIFIED STIFFNESS MATRIX
C: (ONLY NON-ZERO ELENENTS ARE INCLLOED)
C
|PTR=1
CI=1
WIRITE(1,8023)
J JF=1
ICOUN1=COUNT-1
1 TOTC2=TOTELS-COUNT+2
ITIC=TOTNOD-COUNT

```
```

    NNN=0
    DO 23 1=1,TOTNOD
    TCl=Cl-1
    IF(TCI.GT.0.00001)GO TO 231
    Cl=C1+50
    WRITE(1,8010)I
    231 IPSTAR=|PTR
REG(I)=IPTR
|RN(|PTR)=1
|PTR=1PTR+1
IF(I.LE.COUNT)GO TO 6765
IF(I.GT.ITNC)GO TO 7765
1TT=1-1-(COUNT+NNN*COUNT)
IF(ITT.NE.O)GO TO }876
JJF=1 +NNN* (COUNT-1)
j JL=j JF+2*(COUNT-1)-1
NMN=NNN+1
8765 DO 24 J=JJF.JJL
CALL DIRECT(J,ELTOP, IELTOP,JELTOP,NF, INF,JNF,
DCFNOD, STEER, ISTEEP, ITEST)
DO 25 K=1,4
IF(STEER(K).EQ.I)GO TO 15
25 CONTINUE
GO TO 24
15 CONTINUE
DO 30 K=1,4
IPF|N|=|PTR-1
DO 26 L=IPSTAR, IPFIN|
IF(IRN(L).EQ.STEER(K))CO TO 30
26 CONTINUE
I RN(IPTR)=STEER(K)
IPTR=IPTR+1
30 CONTINUE
24 COFTTINLE
GO TO 7799
7 7 6 5 ~ D O ~ 2 2 4 ~ J = 1 T O T C 2 , T O T E L S ~
CALL DIRECT (J,ELTOP, IELTOP,JELTOP,NF, INF,JNF,
* DOFNOD, STEER, I STEER, ITEST)
DO 225 K=1,4
IF(STEER(K).EQ. 1)CO TO 215
225 COMTTINUE
GO TO 224
215 CONTINLE
DO 230K=1,4
|PF IN| = IPTK-1
DO 226 L=IPSTAR,IPFINI
IF(IRN(L).EQ.STEER(K))GO TO 230
226 CONTINUE
I PNN(IPTR) =STEER(K)
| PTR= IPTR+1
230 CONTINUE
224 CONTINUE
GO TO }779
6765 DO 1224 J=1, I COUN1
CALL DIRECT(J,ELTOP, IELTOP ,JELTOP ,NF, INF , JNF,
DOFNOD,STEER, I STEER, ITEST)
DO 325 K=1,4

```

IF (STEER (K).EQ.I)GO TO 315
325 CONTINUE
GO TO 1224
315 CONTINUE
DO \(330 \mathrm{~K}=1,4\)
\(|\mathrm{PF}| \mathrm{NI}=\mathrm{IPTR}-1\)
DO 326 L=IPSTAR, IPFINI
IF (IRN(L).EQ.STEER(K))GO TO 330
326 CONTINUE
IRN: (IPTR) =STE: \(:\) (K. (K)
\(1 P T R=1 P T R+1\)
330 CONTINUE
1224 CONTINUE
7799 LAST=IPSTAR+8
23 CONTINUE
REG (TOTNOD +1 ) \(=1\) PTR
1 TOT \(1=\) TOTNOD +1
DO \(474 \quad 1=1\), 1 TOT1
WRITE \((7,8010)\) REG(1)
474 CONTINUE
\(\mathrm{Cl}=1\)
\(\operatorname{VRITE}(1,8037)\)
DO \(36 \quad 1=1\), TOTNOD
\(T C I=C 1-1\)
IF(TCI.GT.0.00001) 00 TO 361
\(\mathrm{Cl}=\mathrm{Cl}+400\)
WRITE \((1,8010) \mathrm{I}\)
\(361 \quad \mid P 1=\operatorname{REG}(1)\)
1 P2=REG( \(1+1\) )-1
\(K=0\)
DO \(31 J=I P 1, I P 2\)
\(K=K+1\)
1 LOCAL \((K)=1\) RN (J)
31 CONTINUE
\(K K 1=K-1\)
DO \(32 \mathrm{SP}=1\),KK1
\(\mathrm{R}:=0\)
\(K S P=k-S P\)
DO \(33 \mathrm{NN}=1, \mathrm{KSP}\)
IF(ILOCAL(NN).LE.ILCCAL(NN+1)) CO TO 33
\(\therefore=1\) LOCAL (NJ)
I LOCAL (NN) = I LOCAL \(\left(\mathrm{Ni}^{+}+1\right)\)
\(1 \operatorname{LOCAL}(N N+1)=X\)
\(\mathrm{N}=1\)
33 CONTINUE
IF (M.EQ.O) GO TO 34
32 CONTINUE
\(34 K=0\)
DO \(35 \mathrm{~J}=\mathrm{IP} 1, \mid \mathrm{P} 2\)
\(K=K+1\)
\(\operatorname{IDN}(1)=1 \operatorname{LOCAL}(K)\)
35 CONTINUE
36 CONTINLE
1PEND=J-1
\(\mid\) PEND \(|=|\) PEND +1
IPENDT = 1 PEND+TOTNOD
WRITE \((7,8010)\) IPEND
```

    WRITE(7,8010) IPEND1
    VIRITE(7,8010)IPENDT
    CALCULATION OF SEMI-BANDWIDTH
    FIRST = .TRUE.
    DIF=COUNT+1
    |BAND = DIF + 1
    WRITE (1,8038)
    WRITE(1,8010) HRAND
    DOFEL=NODEL*DOFNOD
    CALL VECNUL(STIF,ISTIF,ISTIF,ITEST)
    CALL MATNUL(MID,IMID,JMID,DIMEN,DIMEN,ITEST)
    CALL MATNUL (P, IP,JP,DINEN,DIMEN, ITEST)
    CALL MATNUL(F,IF,JF,DOFEL,DIMEN,ITEST)
    CALL QQUA4(VGIIT, IWGHT, ABSS, IABSS,JABSS, NQP, ITEST)
    Cl=1
    WRITE(1, 8043)
    DO 1100 NELE=1,TOTELS
    TCI=CI-NELE
    IF(TCI.GT.0.00001)CO TO 1101
    Cl=Cl+200
    WRITE(1,8010)NELE
    1101 KCOLNT=0
CALL DIRECT(NELE,ELTOP, IELTOP,J ELTOP,NF, INF, INF,
* DOFNOD,STEER,ISTEER, ITEST)
CALL ELGEOM(NELE, ELTOP, IELTOP, JELTOP, COORD, ICOORD,
* JCOORD, GEOri, I CEOM, ; GEOM, DIMEN, ITEST)
CALL MATNI:LELK, IELK, JELK, DCFEL, LCFEL, ITEST)
CALL VECY!UL(EEV,IREV,DOFEL,ITEST)
DO 1090 IQUAD=1,NQP
XI = ABSS (1, IQUAD)
ETA = ABSS (2,1OLHD)
CALL ?UA:A(FLN, IFUN, LDER, ILDER, JLCER, XI, ETA, ITEST)
CALL M!ATMUL(LDER, ILDER, JLDER, GEOM, I CEOM, J GEOM, JAC,
* ||AC, JJAC, DIMEN, NODEL, DMMEN, ITEST)
CALL MATINV(|AC, |JAC, |JAC, JACIN, |JACIN, |JACIN, DIMEN,
* DET, ITEST)
CALL MATMUL(JACIN, IJACIN, JJACIN, LDER, ILUER, JLDER, GDER,
* IGDER, J GDER, DINEN, DIMEN, NODEL, ITEST)
CALL SCAPRD(FLN, IFUN,GEOM(1,1), IGEO:I,NODEL, RQ, ITEST)
CALL SCAPRD(FUN, IFUN, GEOA(1, 2), I GEOM,NODEL,TQ, ITEST)
DO 1212 I=1,4
DO 1313 J=1,4
DTPD(1, ) =-CDER(1,1)*GDER(1,1)-
*GDER(1, ) *FLN(1)-
* (1-TQ**2)*GDER(2,1)*GDER (2,j) +
*2*TQ*GDER(2,j)*FUN(1)
1313 CONTINLE
1212 CONTINUE
QLOT = DET*VGGIT(IQUAD)
DO 1080 I=1,DOFEL
DO 1070 J=1,DOFEL
DTPD(1,1) = DTPD(1,1)*QUOT
1070 CONTINUE
1080 CONTIN:UE
CALL MATADD(ELK, IELK, JELK, DTPD, IDTPD, JDTPD, DCFEL,
DOFEL; ITEST)
1090 CONTINUE

```
```

    DO 51 H=1,TOTNOD
    IF(KCOLNT.EQ.NODEL)GO TO 1100
    DO 52 I=1,4
    IF(H.EQ.STEER(1))CO TO 56
    52 CONTINUE
    GO TO 51
    56 KCOUNT=KCOUNT+1
    I P1 =REG(H)
    I P2=REG(H+1)-1
    DO 53 K=1,4
    DO 54 L=IP1,IP2
    IF(IRN(L).EQ.STEER(K))STIF(L)=STIF (L)+ELK(1,R)
    54 CONTINUE
    53 CONTINUE
    51 CONTINUE
    1100 CONTINUE
DO 774 I=1,1PEND
WRITE(7,8011)STIF(I)
774 CONTINUE
DO 779 I=1,IPEND
WRITE(7,8010)IRN(1)
779 CONTINLE
DO 58 I=1,TOTNOD
|P1=REG(1)
IP2=REG(1+1)-1
58 CONTINUE
DO 59 1=1,TOTMOD
IP1=REG(1)
|P2=REG(1+1)-1
KOUNT=1
DO 60 L=IP1,IP2
IF(IRN(L).LT.I)CO TO 61
IF(IRN(L).EQ.I)GO TO 62
61 KOUNT=KOUUNT+1
6 0 CONTINUE
6 2 DIAG(1) そ゚OLNT
59 CONTINUE
ICP = TOTNOD-2*COLNT+1
ICCP = TOTNOD-COUNT
[C) 1115 1=1,2
DO 1120 J=1,COUNT
K=RDCND (1,1,j+3)
STIF(POD)=STIF(POD)*SCALE
POD=REG(K ) +DIAG(K)-1
1120 CONTINUE
1115 CONTINUE
DO 473 1=1,TOTNOD
WRITE(7,8010)DIAG(1)
473 CONTINUE
IPA=COLNT+1
DO 1253 K=IPA, ICP,COUNT
POD=REG(K)+D|AG(K)-1
STIF(POD)=STIF (POD)*SCALE
1253 CONTINUE
1PB=2*COUNT
DO 1252K=IPB, ICCP, COLNNT
POD=REG(K)+D|AG(K)-1

```
```

    STIF(POD)=STIF(POD)*SCALE
    1252 CONTINUE
        DO 68 I=1,TOTNOD
        IP1=REG(1)
        1P2=REG(1+1)-1
    6 8 \text { CONTINUE}
    ```
```

ASSEMBLY OF THE LOAD VECTOR
FOR RHS $=F(R, T H E T A)$
CALL VEGNUL(LO, ILO, ILO, ITEST)
DO 2100 NELE=1,TOTELS
CALL ELGEOM(NELE, ELTOP, IELTOP, JELTOP, COORD, ICOORD,
JCOORD, GEOM, I GEOB, J GEOM, DIMEN, I TEST )
CALL VECNU(LV,ILV,DOFEL,ITEST)
CALL DIRECT (NELE, ELTOP, IELTOP, JELTOP,NF, INF, JNF, DOFNOO, STEER, ISTEER, ITEST)
DO 2091 QUAD=1,NQP
$X I=\operatorname{ABSS}(1,1 Q \operatorname{AD})$
ETA=ABSS (2, IQLAD)
CALL QUAM4 (FUN, IFUN, LDER, ILDER, I LDER, XI, ETA, ITEST)
CALL MATMUL(LDER, ILDER, JLDER, GEOM, ICEOM, J GEO: $:$, JAC,

* IJAC, JJAC, DIMEN,NODEL, D IMEN, ITEST)
$D E T=J A C(1,1) * J A C(2,2)-J A C(1,2) * J A C(2,1)$
CALL SCAPRD(FUN, IFUN, GEOH( 1,1 ), IGEON, NODEL, HQ, ITEST)
CALL SCAPRD(FUN, IFUN, $\operatorname{GEGM}(1,2), 1$ GEOM, NOCEL, TQ, ITEST $)$
QUOT=DET*WGHT (IQUAD)
RRR=DEXP (RQ)
IF(TQ.EQ.0) CO TO 45
$T Y=\operatorname{DSQRT}\left((1-T Q * * 2) / T Q^{* *} 2\right)$
TTT=DATAN(TY)
GO TO 46
$45 \mathrm{TTT}=\mathrm{P} 1 / 2.0$
$46 \mathrm{Gl}=2 * \operatorname{DCOS}(\mathrm{TTT}) *\left(-3 * A A / R R R^{*} * 4-\mathrm{BD} / R R R^{*} * 2+2 * D D * R R R\right)$
$\mathrm{G} 2=-\mathrm{G} 1 / 2$
G3 $=$ G2
GA=-3*DS IN(TTT)*(AA/RRR**4+DD*RRR)
G5 $=2 * \operatorname{DCOS}(T T T) *(12 * A A / R R R * * 5+2 * B E / R R R * * 3+2 * D D)$
G6 $=-2 * \operatorname{DSIN}(T T T) *\left(-3 * A A / R R R * * 4-\right.$ EB/RRR**2 $\left.+2 * D D^{*} R R R\right)$
G7 $=-3 * \operatorname{DS} \operatorname{IN}(\mathrm{TTT}) *(-4 * A A / R R R * * 5+D D)$
$G 8=-3 * D C O S$ (TTT)* (AA/RRR**4+DD*RRR)
S1 $=2 * \operatorname{DCOS}(T T T) *(-60 * A A / R P R * * 6-6 * R E / R P R * * 4)$
S2=-2*DS IN(TTT)*(12*AA/RRR**S +2*EB/RRR**3+2*DU)
S3 $=-\mathrm{G} 1$
S4 $=-3^{*} \operatorname{DS} \operatorname{IN}(T T T) *\left(20 * A A / R R R^{* *} 6\right)$
$\mathrm{S5}=-3 * \operatorname{DCOS}(\mathrm{TTT}) *\left(-4 * A A / R R R^{* * 5}+D D\right)$
$\mathrm{S} 6=-\mathrm{G4}$
$A 1=-D S I N(T T T) *(10 * D D * R R R+C B / R R R * * 2)$
E1=2*(5*DD-BB/RRR**3)*DCOS(TTT)
P1=3*G1**2/2.0+2*G4**2
P2 $2=3 * \mathrm{G} 1 * \mathrm{G} 5+4^{*} \mathrm{G} 4 * \mathrm{G} 7$
P3=3*G1*G6+4*G4* CS
P4=3*G1*S1 3 * G ** $2+4 * \mathrm{G} 4 * \mathrm{~S} 4+4 * \mathrm{G} 7 * * 2$
P5 =3*G1*S2+3* G6* G5 +4*G4*S5 +4* C8* G7

```

```

$\mathrm{C} 1=\mathrm{G} 1 *(\mathrm{P} 1 * \mathrm{P} 5-\mathrm{P} 2 * \mathrm{P} 3) /(\mathrm{P} 1 * * 2)+\mathrm{G} 6 * \mathrm{P} 2 / \mathrm{P} 1$
$\mathrm{C} 1=\mathrm{C} 1+\left(\mathrm{P} 6 / \mathrm{P} 1-(\mathrm{P} 3 / \mathrm{P} 1)^{* *} 2\right) * \mathrm{G} 4 / R R R+\mathrm{CB} * \mathrm{P} 3 /\left(R R R^{*} \mathrm{P} 1\right)$

```
```

    D1=(G4+RRR*G7)*P2/P1+RRR*G4*(P1*P4-P 2**2)/(P1**2)
    D1=D1-G5*P3/(2*P1)-(G1/2.0)*(P1*P5-P 3*P2)/(P1**2)
    COEFF(NELE, IQUAD) =DSIN(TTT)/2.0* (C1 -D1+P 3*B1/P1
    *
                                    -P2*A1/P1)
    VN1=N-1
    DO 7086 K=1,DOFEL
    RPQ=2.0*RQ
    AE (NELE, IQUAD,K)=FUN(K)*QUOT*DEXP (RPQ)
    YE(IQUAD,K)=FUN(K)
    LEC(K)=F UN(K)*COEFF (NELE , IQUAD) *Q LOT*DEXP (RPX)
    7086 CONTINUE
    CALL VECADD(LV, ILV,LEC, ILEC, DOFEL, ITEST)
    2091 CONTINUE
    DO 2222 L=1,DOFEL
    SN=STEER(L)
    LO(SN)=LO(SN)+LV(L)
    2222 CONTINUE
    2100 CONTINLE
    DO 776 I=1,TOTNOD
    WRITE(7,8011)LO(1)
    76 CONTINUE
            DO 512 l=1,4
            DO 513 J=1,4
            WRITE(7,8011)VE(1,J)
    513 CONTINUE
    512 CONT INUE
    DO 514 1=1,TOTELS
    DO 515 J=1,4
    DO 516K=1,4
    WRITE(7, 8011)XE(I,J,K)
    516 CONTINUL
    515 CONTINUE
    514 CONTINUE
    C
COLNT1=COLNT+1
COUNT2=2*COUNT+1
COLNT3 = TOTNOD-COLNT
DENDMA=COORD(COUNT1,1)
DENOMJ = DEXP (COORD (COUNT2,1) )-UEXP (COORD (1, 1))
DEMOMB=DEXP(COORD(TOTNOD,1))-DEXP(COORL(COUNT3,1))
C
C CALCULATION OF THE NEWTONIAN PSI FIELO
C
DO 1255 1=1,TOTNOD
IF(COORD(1,2).EQ.0)GO TO 14
TY=DSQRT((1-COORD(1, 2)**2)/COORD(1,2)**2)
S INARG=DATAN(TY)
CO TO 17
14 SINARG=P 1/2.0
17 U=DSIN(S INARG)
XY=DEXP(COORD(1,1))
ANEW(1)=-0.5* (AA/XY+BB*XY+
CC*XY**2+DD*XY**4)*U**2
RH(1)=ANEW(1)
RSTAR(1)=ANEW/ 1)
1255 CONTINUE
2057 FORNAT(51H

```
```

C
C CALCULATION OF THE NEWTONIAN OANEGA FIELD
C
DO 719 I=1,TOTNOD
IF (COORD(1,2).EQ.0) GO TO }
TY=DSQRT((1-COORD(1,2)**2)/COORU(1,2)**2)
S INARG=DATAN(TY)
GO TO 10
9 S INARG=P 1/2.0
10 GU(1)=WGG(DEXP (COORD (1,1)), SIMARG)
GSTAR(1)=GU(1)
719 CONTINUE
C
IPO INT=0.0000DO
J TC=0
ICNT=0
WRITE (1,8050)
WRITE(NOUT,8050)
WR ITE(1,787)
WRITE(1,8036)DENOMA, DENON:B
787 FORNAT(37H INTO ITERATIVE LOOP //)
CLOSE(7)
IDGU=COLNT- ICOUN1 /2.0
C
C MAIN ITERATIVE LOOP
C ...................
C CALCULATION OF TIE BOLNDARY OMEGA VALUES
C
725 DO 2073 1=1,COUNT
JC1=1+1*COUNT
OMVAA(1)=2*RH(JC1 )/DENOMA/DENOMAA
C OMVAA(1)=-1.5*DSIN(S INARG)**2
C SINARG=DACOS(COORD(INTRB(1),2))
C OANAB(I)=0.1*BETA*DS IN(S INARG)**2
IF(COORD(INTRB(1),2) .EQ.0)CO TO 47
TY=DSQRT((1-COORD(INTRR (1),2)**2)/COORD(INTRB(1),2)**2)
S INARG=DATAN (TY)
CO TO 48
47 SINARG=P 1/2.0
48 AMVAB(1)=(-3*(BETA-1+2*ALOG(RR))*DSIN(S INAFG)**2)/(2.0*RR)
C OKNAB(1)=2*PH(INTRB(1))/DENNONB/DENOMB
C * (RR-DENOKR )**2*DS IN(S INARG)**2/DENO*i!
| DENOME
DRGU(1)=(-3)*(3-BETA-2*AL()G(RR))*DSIN(S INARG)**2/(2.0*RR**2)
DRANNEW(1)=(3/4.0)*(-ALPHA/RR**2+BETA+2+2*ALOG(RR))*
DS IN(S INARG)**2
2073 CONTINUE
C
CALL VECNUL (LOAD, I LOAD, I LOAI), ITEST)
DO 2074 I=1,COUNT
J=1+3
LOAD(ECD1 (1, ) ) = OMNAA(1)
LOAD(BCD2 (1, ) ) = MavAB ( 1)
2074 CONTINUE
LL=TOTNOD-COLNT+1
DO 1111 I=1,TONNOD
RH(1)=LO(1)

```

1111 CONTINUE
C
C APPLICATION OF DIRICHLET GOUNDARY
C CONDITIONS FOR OMEGA
c
DO 4115 I=1,2
DO \(4120 \mathrm{~J}=1\), COUNT
\(K=\operatorname{BDCND}(1,1,1+3)\)
\(\mathrm{X}=\operatorname{COORD}(\mathrm{K}, 1)\)
GO TO(4125,4135), 1
\(4125 \operatorname{POD}=\operatorname{REG}(\mathrm{K})+\mathrm{DIAG}(K)-1\)
\(R H(K)=S T I F(P O D) * \operatorname{LOAD}(K)\)
GO TO 4120
\(4135 \operatorname{POD}=\operatorname{REG}(K)+\operatorname{DIAG}(K)-1\)
\(R H(K)=S T I F(P O D) * \operatorname{LOAD}(K)\)
4120 CONTINUE
4115 CONTINUE
ICCP = COUNT+COLNT* (COUNT-1)
DO \(4253: K=1\), ICP , COLNT
\(R H(K)=0.0000 D 0\)
4253 CONTINUE
DO \(4252 \mathrm{~K}=\) COUNT, ICCP, COUNT
\(R H(K)=0.000000\)
4252 CONTINUE
C
C GAUSS-SEIDEL ITERATICN FOR QUEGA
C
DO \(755 K=1,1\)
DO \(727 \mathrm{l}=1\), TOTNOD
\(S U N=0\)
\(1 P 1=R E G(1)\)
1 P \(2=\operatorname{REG}(1+1)-1\)
CNT \(=0\)
DO \(729 L=I P 1, I P 2\)
IF (IRN(L).LT.I)CO TO 728
IF (IRN(L).EQ. I)GO TO 729
SUN=SUM+STIF(L)*GU(IRN(L))
GO TO 729
728 CNT=CNT+1
SUA: \(=S U A+S T I F(L) * G U(I R N(L))\)
729 CONTINUE
\(|P 3=| P 1+C N T\)
\(G U(1)=(R H(1)-S U M) / S T I F(I P 3)\)
727 CONTINUE
755 CONTINLE
8011 FOPMAT(D21.14)
C
C ASSEMPLY OF LOAD VECTOR FOR RHS=OMEGA
C
CALL VECNU (RH, I RH, I RH, I TEST)
DOFEL=NODEL*DOFNOD
CALL QQUA4 (WGHT, IWGHIT, ABSS, IABSS,IABSS, MDP, ITEST) DO 6100 NELE \(=1\), TOTELS
CALL VECNUL (BEV, IBEV, DOFEL, ITEST)
CALL DIRECT (NELE, ELTOP, IELTOP, JELTOP, NF, IPF, JNF,
* DOFNOD, STEER, ISTEER, ITEST)

DO 6091 IQUAD=1, MQP
```

    S(1)=0.00
    DO 6086 I=2,5
    S(1)=S(1-1)+GU(STEER(1-1))*VE(IQUAD, 1-1)
    6 0 8 6 ~ C O N T I N U E ~
    DO 6085 K=1,DOFEL
    LN(K)=XE (NELE, IQUAD,K)*S(5)
    6085 CONTINUE
    CALL VECADD(BEV,IDEV,LN, ILN,DOFEL, ITEST)
    6 0 9 1 ~ C O N T I N U E ~
        CALL ASRHS (RH, IRH, BEV, IBEV,STEER, ISTEER,
        *NODEL, I TEST)
    6100 CONTINUE
    C
C APPLICATION OF BOUNDARY
C CONDITIONS FOR PSI
C
DO 6115 I=1,2
DO 6120 j=1,COLNT
K = BDCND (1,1,1+3)
X = COORD(K,1)
IF(COORD(K, 2).EQ.0)GO TO 3
TY=DSQRT ((1-COORD(K, 2)**2)/COORD (K,2)**2)
Y=DATAN(TY)
GC TO 2
3 Y=P I/2.0
2 CO TO(6125,6135),1
6125 POD=REG(K)+DIAG(K)-1
RH(K)=STIF (POD)*H1 (X,Y)
GO TO 6120
6135 POD=REG(K)+DIAG(K)-1
RH(K)=STIF(POD)*|{3(X,Y)
6120 CONTINUE
6115 CONTINLE
DO 6253 K=1,ICP , COUNT
RH(K)=0.000000
6253 CONTINUE
DO 6252:R=COUNT, ICCP , COUNT
RH(K)=0.0000DO
6252 CONTINUE
C
C GAUSS-SEIDEL ITERATION FOR PSI
C
DO 766K=1,1
DO 827 l=1,TOTNOD
SUA:=0.0
| P1=REG(1)
IP2=REG(I+1)-1
CNT=0
DO 829 L=IP1,IP2
IF(IRN(L).LT.I)CO TO 828
IF(IRN(L).EQ.I)CO TO }82
SLM:=SUM+STIF(L)*ANEW(IRN(L))
GO TO }82
828 CNT=CNTT+1
SUM=SUNI+STIF(L)*ANEW(IRN(L))
829 CONTIINUE
IP3= 1P1 +CNT

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```

            ANEW(1)=(RH(I)-SUN)/STIF(IP 3)
    827 CONTINUE
    766 CONTINUE
    C
| TCC=| TC-IPOINT
IF(JTCC.GT.0.001)GO TO 433
| TC=J TC+20
J TC4 =MOD( ) TC ,40)
IF(JTC4.LT.00001.0)GO TO 434
IF(JTC4.GT.0.001)GO TO 435
434 OPEN(FILE='GUANEW2', UNITT=8,STATUS='UNKNOWN')
DO 756 1=1,TOTNOD
WRITE(8,8011) GU(1)
756 CONTINUE
DO 747 l=1,TOTNOD
WRITE(8,8011)ANEW(1)
747 CONTINUE
CLOSE(8)
GO TO 433
435 OPEN(FILE='GUANEW', UNIT=9,STATUS='UNKNONN')
DO 977 1=1,TOTNOD
WRITE(9,8011) GU(1)
977 CONTINUE
DO 978 I=1,TOTNOD
WRITE(9,8011)ANEW(1)
978 CONTINUE
CLOSE(9)
433 OPEN(FILE='F)TC',UNIT=10,STATUS='UNNOVN')
WRITE(10,8010)]TC
CLOSE(10)
DO 777 l=1,TOTNOD
RH(1)=ANEW(1)
777 CONTINUE
IF(IPOINT.GT.O.5)GO TO }777
WRITE(1,8041)
WRITE(NOUT,8041)
C
C CALCULATION OF THE INTEGRAL '|'
C
7777 DO 1017 I=1,COUNT
IVI(1)=(GU(INTRA(1))-GU(1))/DENOASA
WW(1)=(IW(I)-2*GU(1))
1017 CONTINLE
IW(2)=4.0*IW(2)
ICT1 =COUNT-1
ICT2=COLNT-2
IW(3)=2.0*IW(3)
DO 1018 I=4,1CT1,2
IW(1)=IW(1)*4.0+IW(1-2)
1018 CONTINUE
DO 1019 I=5, ICT2,2
IW(1)=IW(1)*2.0+1W(1-2)
1019 CONTINUE
TOTAL=IW(1)+IW(ICT2)+IW(ICT1)+IW(COLNT)
DELTA=2.0/((COUNT-1)*1.0)
AREA=DELTA* TOTAL/3.0
C

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```

    RMARG=20.000000
    RM=DMOD(IPOINT, R\UARG)
    SLOP=SLOPE+AREA/DIV+ALOG(2.0)
    | TER= I POINT +1
    IPO INT= IPOINTT+1
    ALPHA=0.0
    OPEN(FILE='FBETA',UNIT=11,STATUS='UKKNONN')
    WRITE(11,8011)BETA
    CLOSE(11)
    C
C CALCULATION OF BETA
C
DO 479 1=1,COUNT
1 TC3 =COUST3 + 1
DGU(1)=(GU(1TC3)-GU(INTRB(1)))/DENOMD
DANEN(1)=(ANEVi(1TC3)-ANEW(INTRB(1)))/DENNMB
479 CONTINUE
BETA=LMULT1* ((4/3.0) * DANEV(IDGU)-3-2*LOG(RR))
BETA=RETA+LA:ULT2*(3-2*LOG(RR)+(2/3.0)*RR**2*DGU(IDGU))
C
IF(ABS(RM).GT.0.005)GO TO 39
BETAP=BETA+0.1972246
WRITE(1, 8039) SLOP , AREA, ITER
WRITE(1,2057)
C
C COAPARING THE NUAERICAL DERIVATIVES FOR
C OMEGA AND PSI WITH THE CORPESPONDING
C TIIEORETICAL VALLES
C
WRITE(1,8039)DGU(IOGU),DANEW(IDGU)
WRITE(1,8039)DRGU(IDGU),DRANEW(IDGU)
C
WRITE(NOUT, 8039)SLOP,AREA, ITER
39 IF(IPOINT.LT.20000)CO TO }72
STOP
8050 FOPAIAT (//41H MATCH RADIUS 020 MESH 140\times20 //)
8010 FORPMAT (1615)
8023 FORMAT (//14H INTO LOOP 23 )
8037 FOR::AT(14H INTO LOOP 36)
8041 FOFMAT (//42H1 SLOPE INTGR //)
8043 FOPAAT (17H STIFFNESS VIATRIX)
8038 FORNAAT (7H HIBAND )
8036 FORMAT (1H , 110F10.5)
8039 FORMAT (2F20.5,18)
8020 FOFN:AT (15, 6F12.5)
END
SUBROUTINE ASRHS(RHS, IRHS, VALUE, IVALLE, STEER, ISTEER,
* DOFEL, ITEST)
C-m-m,
* K, L, STEER
DOURLE PRECISION RHYS, SFNAN:E, VALLE
DIMENSION RIIS(IEHS), STEER(ISTEER), VALUE(IVALUE)
DATA SRNAHEE / 8H ASPUIS I
IF (ITEST.EQ.-1) GO TO 1010
IERROS: = 0
IF (ISTEER.LT.CNFEL) IERROI: = 3

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IF (IVALUE.LT.DOFEL) IEFROR = 2
IF (DOFEL.LE.O) IERSOR = 1
ITEST = ERFNGES(ITEST, IERKOR,SRNAAME )
1010
IF (ITEST.NE.O) RETURN
DO 1030 K=1,DOFEL
IF (STEER(K).EQ.0) GO TO 1030
L = STEER(K)
IF (ITEST.EQ.-1) GO TO 1020
IERROR = 0
IF (L.GT.IRHS ) IERFOR = 4
|TEST = ERRNES(ITEST, IERROR,SRNNAME)
IF (ITEST.NE.O) RETURN
1020 RHS(L) = RIIS(L) + VALLE(K)
1030 CONTINUE
RETURN
END

```

\section*{PROGRAM 3}

THIS PROGRAM CONTINUES THE RUN IN THE EVENT OF DOWN-TIME DURING THE RUNNING OF PROGRAM 2.

> THIS PRIOGRAM CONTINUES THE RUN IN THE EVENT OF DONN-TIME DURING THE RUNNING OF THE PREVIOUS PROGRAM

INTEGER BDCND，BTYPES，DIMEN，DOFEL，DIAG，
＊DOFNOD，ELNLM，ELTOP，ELTYP，I，IABSS，
＊ILOAD，ICNT，LL，IELTOP，INF，JELTOP，
＊IQUAD，ISTEER，ITEST，IWGHT，REG，J，JABSS，IRN，POD，
＊NF，NIN，NODEL，NODNUM，NOUT，NQP，JNF，K，NELE，
＊STEER，TOTDOF，TOTELS，TOTNOD，CCC，COUNT1，COUNT2，
＊COUNT3，COLNT，ILN，IBEV，IRH，BCDI，BCD2，
＊ICP，ICCP，INTRA，INTRB
DOUBLE PRECISION ABSS，DENOMA，SLIM，STIF，S INARG，，MULT1，MULT2，
＊COORD，DTPD，EUK，FUN，GDER，DENOMB，TY，BETAP．
＊GDERT，GEOi，JAC，JACIN，LDER，P，PD，DIV，
＊SCALE，WGHT，X，XX，Y，YY，LOAD，LO，GU，IPOINT，S（5），
＊WGA，H1，H2，MID，F，R，MGDER \((2,4), I W, S L O P E, A R G L O G\),
＊FMD，PI，LN，BEV，RH，CMIVAA，SLOP ，FN，RSTAR，GSTAR，KAARG，
＊COEFF，XE \((3000,4,4), Y E(4,4)\)
DIMENSION ABSS \((2,9)\) ，
＊ \(\operatorname{DTPD}(4,4)\) ，
＊ \(\operatorname{ELK}(4,4), \operatorname{FLN}(4), \operatorname{GDER}(2,4), \operatorname{GDERT}(4,2), \operatorname{GEO}(4,2)\) ，
＊JAC \((2,2), \operatorname{JACIN}(2,2), \operatorname{LDER}(2,4), \operatorname{P}(2,2), \operatorname{PD}(2,4)\) ，
＊ \(\operatorname{STEER}(4), \operatorname{WGHT}(9), \operatorname{MID}(2,2), F(4,2), R(2), \operatorname{LN}(4)\) ，
＊\(\quad \operatorname{FMD}(4,4), \operatorname{BEV}(4), \operatorname{LO}(6561)\)

DIMENS ION LOAD（6561），GU（6561）， \(\operatorname{COORD}(6561,2)\),
＊ \(\operatorname{ELTOP}(6561,6), N F(6561,1), \operatorname{COEFF}(3000,9)\) ．
＊ \(\operatorname{ANEW}(6561), \operatorname{RH}(6561), \operatorname{RSTAR}(6561), \operatorname{GSTAR}(6561)\) ，
＊\(\quad \operatorname{amNAA}(81), \operatorname{OMAB}(81), \operatorname{INTRA}(81), \operatorname{INTRB}(81)\) ，
＊\(\quad \operatorname{IW}(81), \operatorname{BCD}(5,81), \operatorname{BCD}(5,81)\) ，
＊\(\quad \operatorname{BDCND}(2,2,81), \operatorname{REG}(6562), \operatorname{IRN}(65620), \operatorname{DIAG}(6561)\) ，
＊STIF（65620）
COMDN／A／ELTOP
COMHON／B／DIAG
COMTMON／C／COORD
CO：iON／D／RH，ANEW，NF
COW：WN／E／ABSS，BCD1，BCD2 ，BDCND，BEV，DTPD，
＊ELK，F，FMD，FUN，GDER，
＊GDERT，GEOM，GU，INTRA，INTRB，IW
COA：ION／F／JAC，JACIN，LDER，LN，LOAD，NGGER，MID，
＊OíVAA，OMVAB，P，PD，R，S，STEER，
＊W1，W2，VIGHT，REG，IRN
COMDN／G／STIF，COEFF
COA：BON／H／GSTAR，RSTAR
COMADON／I／XE
COALHON／J／YE
C
DATA IABSS／2／，1STEER／4／，
＊IWGHT／9／，JABSS／9／．JNF／1／．
＊SCALE／1．0D＋25／，ILOAD／6561／
        DATA IELTOP/6561/,INF/6561/.jELTOP/6/,
    * |R11/6561/,|BEV/4|,ILN/4|.
    * RR/10.0/, BETA/-1.3/,COLNT/21/,
    * multi/0.5/, mult2/0.51
C
    DATA NIN /5/, NOUT /6/
C
C \(\quad\) WGA \((X X, Y Y)=1.0^{*}\left(\left(D S I N(Y Y)^{* * 2}\right) / 34.0\right)^{*}\left(90^{*} X X * * 2-372 / X X\right)\)
    WGA \((X X, Y Y)=-0.5^{*}\left(-2 * B B / X X+10^{*} D D^{*} X X^{* *} 2\right) * \operatorname{DS} \operatorname{IN}(Y Y) * * 2\)
    \(111(X X, Y Y)=0.0000 D 0\)
C \(\quad \mathrm{H} 2(X X, Y Y)=-(\mathrm{DSIN}(\mathrm{YY}) * * 2 * 2.0)\)
    \(H 2(X X, Y Y)=-\left(D S I N(Y Y) * * 2 * 0.5 * R R^{* *} 2\right)\)
C \(\quad \mathrm{H} 3(X X, Y Y)=-0.5^{*}\left(B E T A * R R+R R^{* *} 2\right) * D S I N(Y Y) * * 2\)
    \(H 3(X X, Y Y)=\left(3 *\left(R R * B E T A+2 * R R^{*} A L O G(R R)\right) / 4.0\right) * D S I N(Y Y) * * 2\)
C \(\quad \mathrm{H} 3(X X, Y Y)=-\left(D S I N(Y Y) * * 2 * R R^{* *} 2\right) / 2.0\)
C \(\quad \mathrm{H} 3(X X, Y Y)=0.0000\)
C
C
C
    1 TEST \(=0\)
    PI=4.0DO*DATAN(1.0DO)
C \(\quad A A=0.5\)
C \(\quad B B=-1.5\)
C \(\quad C C=1.0\)
C \(\quad \mathrm{DD}=0.0\)
9088 FORMAT (10F10.5)
C
    READING DATAF ILES CREATED BY
C:
    OPEN(FILE = 'ABCD', UNIT \(=12\), STATUS = \({ }^{\prime}\) LNKNOMN')
    READ (12,9088)AA
    \(\operatorname{READ}(12,9088) \mathrm{BB}\)
    READ \((12,9088)\) CC
    READ \((12,9088)\) DD
    CLOSE(12)
    FN=24*RR* (1+RR+RR**2+RR**3+RR**4)/
    * ((RR-1)**3*(4+7*RR+4*RR**2)*1.0)
    APGLOG=2*AA+2*CC+12*DD
C
    DIV=4*(5*AA+2*CC)/3.0
    SLCPE \(=2 *\) ( \(0.75^{*}(2 *\) DLOG(ARGLOG) \(/ 3.0-5 / 9.0)\)
    * \(\left.\quad+\left(4 * A A+2 * C C+4^{*} D D\right) / D I V\right)\)
C
C
C
C
C
    WRITE \((1,8036)\) DIV, SLOPE, ARGLOG
                                    INPUT OF NODAL GEOMETRY

```

    1 TOT 1 = TOTMOD+1
    DO 4551 l=1,ITOT1
    READ(7,8010)REG(1)
    4551 CONTINUE
    READ(7,8010)IPEND
    READ(7;8010) IPEND1
    READ(7, 8010) IPENDT
    DO 4552 l=1,1PEND
    READ(7, 8011)STIF(1)
    4552 CONTINUE
    DO 7559 1=1,1PEND
    READ(7, 8010)IRN(1)
    7559 CONTINUE
    DO 4553 l=1, TOTNOD
    READ(7, 8010)DIAG(1)
    4553 CONTINUE
    DO 4554 i=1, TOTNOD
    READ(7,8011)LO(1)
    4554 CONTINUE
            DO 754 I=1,4
            DO 753 J=1,4
            READ(7,8011) YE (1,J)
    753 CONTINLE
    754 CONTINUE
    C
DO 889 I=1,TOTELS
DO 888 J=1,4
DO 887K=1,4
READ(7,8011)XE(1,J,K)
87 CONTINUE
8 8 8 CONTINUE
89 CONTINUE
J TC4 =MOD (J TC , 40)
IF(JTC4.GT.0.01)GO TO 678
OPEN(F ILE='GUANEW2', UNIT=9,STATUS='OLD')
DO 4591 l=1, TOTNOD
READ(9,8011)GU(1)
4591 CONTINUE
DO 4592 I=1,TOTNOD
READ(9,8011)ANEW(1)
4592 CONTINUE
CLOSE(8)
GO TO 611
678 OPEN(FILE='GUANEW', UNIT=8,STATUS='OLD')
DO 6543 i=1,TOTNOD
READ(8,8011)GU(1)
6543 CONTINLE
DO 6542 I=1,TOTNOD
READ(8,8011)ANEW(1)
6542 CONTINUE
CLOSE(9)
611 CONTINUE
CLOSE(7)
READ(NIN,8010) TOTNOD,DIMEN
DO 1010 I=1,TOTNOD
READ(NIN, 8020) NODNUM,(COORD(NODNLM,J),J=1,DIMEN)
1010 CONTINLE

```

READ (NIN, 8010) ELTYP, TOTELS; NODEL
DO 1020 I=1, TOTELS
READ (NIN, 8010) ELNLM, (ELTOP(ELNUM, \(1+2\) ), \(1=1\), NODEL)
ELTOP (ELNUM,1) = ELTYP
ELTOP (ELNUM,2) \(=\) NODEL
1020 CONTINLE
READ (NIN, 8010) DOFNOD
BTYPES=1
DO \(2324 \quad l=1\),COLNT
\(\mathrm{J}=1+3\)
\(\operatorname{BCD1}(1,1)=1\)
\(\operatorname{BCD2}(1,1)=1+\) TOTNOD-COUNT
2324 CONTINUE
DO \(1075 \mathrm{~J}=1\),COUNT
\(\mathrm{JJ}=\mathrm{j}+3\)
\(\operatorname{BDCND}(1,1, j \jmath)=\operatorname{BCD}(1, j 1)\)
\(\operatorname{BDCND}(2,2,1 \mathrm{j})=\operatorname{BCD} 2(1,11)\)
1075 CONTINUE
CCC \(=\) COLNT +3
\(K=C O U N T+3\)
DO \(2071 \mathrm{I}=1\),COLNT
\(\operatorname{INTRA}(1)=\operatorname{BCD1}(1,1+3)+C O U N T\)
\(\operatorname{INTRB}(1)=\operatorname{BCD} 2(1,1+3)-C O U N T\)
2071 CONTINUE
WRITE \((1,8010)\) (INTRA \((1), 1=1, C O U N T)\)
WRITE \((1,8010)\) (INTRB(1), \(1=1\), COLNT)
TOTDOF \(=0\)
DO \(1050 \quad 1=1\), TOTNOD
DO \(1040 \mathrm{j}=1\),DOFNOD
TOTDOF \(=\) TOTDOF +1
\(\mathrm{NF}(1,1)=\) TOTDOF
1040 CONTINUE
1050 CONTINUE
c
ICP = TOTNOD-2 *COUNT+1
ICCP=TOTNOD-COUNT
C
DO 1115 1=1,2
DO \(1120 \mathrm{~J}=1\), COUNT
\(K=\operatorname{BCCND}(1,1, j+3)\)
\(P O D=\operatorname{REG}(K)+D I A G(K)-1\)
\(\operatorname{STIF}(P O D)=S T I F(P O D) * S C A L E\)
1120 CONTINUE
1115 CONTINUE
C
IPA \(=\) COUNT +1
DO \(1253 \mathrm{~K}=\) IPA, ICP, COUNT
\(P O D=\operatorname{REG}(K)+D \operatorname{IAG}(K)-1\)
\(\operatorname{STIF}(P O D)=S T I F(P O D) * S C A L E\)
1253 CONTINUE
C
IPB=2*COUNT
DO \(1252 K=1 P B, I C C P, C O U N T\)
\(\mathrm{POD}=\operatorname{REG}(\mathrm{K})+\mathrm{DIAG}(\mathrm{K})-1\)
STIF(POD)=STIF(POD)*SCALE
1252 CONTINUE
COUNT \(1=\) COLNT +1
```

    COLNT2=2*COLNT+1
    COLNT3 = TOTNOD-COLNT
    DENOMMA=COORD(COUNT1,1)
    DENOX:. = DEXP (COORD(COUNT2,1))-DEXP (COORD(1,1))
    DENOR:IE = DEXP (COORD(TOTNOD, 1)) -DEXP (COORD(COUNT3,1))
    DO 1255I=1,TOTNOD
    RH(1)=ANEW(1)
    1255 CONTINUE
    |PO INT=0.0000DO
    ICNT=0
    WRITE(1,8050)
    WRITE(NOUT,8050)
    WRITE(1,787)
    787 FOR:'AT(37H
    J TC=0
    725 DO 2073 1=1,COUNT
    JCl=1+1*COLNT
    OMNAA(1)=2*RH(JC1)/DENOMA/DEROAMA
    C ONNAA(1)=-1.5*DSIN(SINARG)**2
C SINARG=DACOS (COORD(INTRB (1),2))
C }\quad\operatorname{AIVAB}(1)=0.1*BETA*DSIN(SINARG)**
IF (COORD(INTRB(1),2).EQ.0)GO TO 47
TY=DSQRT((1-COORD(INTRB(1),2)**2)/COORD(INTRR (1),2)**2)
SINARG=DATAN(TY)
GO TO 48
47 S INARG=P //2.0
4 8 \operatorname { A i r N A B ~ ( 1 ) = ( - 3 * ( B E T A - 1 + 2 * A L O G ( R R ) ) * D S I N ( S I N A R G ) * * 2 ) / ( 2 . 0 * R R ) }
C 48 OMNAB(I)=2*RH(INTRB(1))/DENOMB/DENANB
C * (RR-DENOMB )*2*DSIN(SINARG)**2/DENON:B
C * /DEINOMB
2073 CONTINUE
CALL VECNUI(LOAD, I LOAD, I LOAD, ITEST)
DO 2074 l=1,COUNT
J=1+3
LOAD(BCD1 (1,j))=OMNAA(1)
LOAD(BCD2(1, ) ))=O:NAB(1)
2074 CONTINUE
LL=TOTROD-COUNT+1
DO 1111 I=1,TOMMOD
RH(1)=I_C(1)
1111 COINTINUE
DO 4115 1=1,2
CO 4120 J=1, COUNT
K}=\operatorname{BLCND}(1,1,j+3
x = COORD (K,1)
GO TO(4125,4135),1
4125 POD=REG(K)+DIAG(K)-1
FHH(K)=STIF(POD)*LOAD(K)
GO TO 4120
4135 POD=PEG(K)+DIAG(K)-1
RH(K)=STIF (POD)*LOAD(K)
4120 CONTINLE
4115 CONTINLE
ICCP =COLNTT+COLNT * (COLNT-1)
DO 4253 K=1, ICP ,COUNT
RH(K)=0.000000
4253 CONTINUE.

```
```

            DO 4252 K=COUNT, ICCP,COUNT
            RH(K)=0.0000DO
    4252 CONTINUE
    DO 755 K=1.1
    DO 727 l=1,TOTNOD
    S LMM=0
    |P1=REG(1)
    1P2=REG(1+1)-1
    CNT=0
    DO 729 L=IP1,IP2
    IF(IRN(L).LT.I)GO TO 728
    IF(IRN(L).EQ.I)GO TO 729
    SUM=SURI+STIF(L)*GU(IRN(L))
    GO TO 729
    728CNT=CNT+1
    SUA=SUN+STIF(L)*GU(IRN(L))
    79 CONTINUE
            IP3= IP1+CNT
            GU(1)=(RH(1)-SLM)/STIF(IP3)
        727 CONTINUE
        755 CONTINUE
    8011 FORIAT(D21.14)
    C
C
CALL VECNUL(RH, IRH, IRH, ITEST)
DOFEL=NODEL*DOFNOD
CALL QQUA4(WGHT, IWGHT, ABSS, IABSS,JABSS, NQP, ITEST)
DO 6100 NELE=1,TOTELS
CALL VECNUL(BEV,IBEV,DOFEL,ITEST)
CALL DIRECT (NELE,ELTOP,IELTOP,JELTOP,NF,IIF,JNF,
* DOFNOD,STEER, ISTEER, ITEST)
DO 6091 IQUAD=1,NQP
S(1)=0.00
DO 6086 I=2,5
S(1)=S(1-1)+GU(STEER(1-1))* YE(IQLAD, I-1)
6 0 8 6 CONTINUE
DO 6085 K=1,DOFEL
LN(K)=XE (NELE, IQUAD,K) *S(5)
6 0 8 5 CONTINUE
CALL VECADD(BEV,IBEV,LN,ILN,DOFEL,ITEST)
6 0 9 1 ~ C O N T I N U E ~
C
C : ASSEMBLY OF STIFFNESS :MATRIX
C
CALL ASRHIS (RH, IRH, BEV,IDEV,STEER, ISTEER,
*NODEL,ITEST)
6 1 0 0 CONTINUE
DO 6115 l=1,2
DO 6120 J=1,COUNT
K = BDCND (1,1,1+3)
X = COORD (K,1)
IF(COORD(K,2).EQ.0)GO TO 3
TY=DSQRT((1-COORD(K,2) **2)/COORD(K,2)**2)
Y=DATAN(TY)
GO TO 2
3 Y=P 1/2.0
2 GO TO(6125,6135),1

```
```

6125 POD=REG(K)+DIAG(K)-1
RII(K)=STIF(POD)*H1 (X,Y)
GO TO 6120
6135 POD=REG(K)+DIAG(K)-1
RH(K)=STIF(POD)*H3(X,Y)
6120 CONTINUE
6115 CONTINUE
DO 6253 K=1,ICP ,COUNT
\#H(K)=0.0000D0
6253 CONTINUE
DO 6252 K=COUNT,ICCP,COUNT
RH(K)=0.0000D0
6252 CONTINUE
DO 766 K=1,1
DO 827 l=1,TOTNOD
SLA:=0.0
I P1 =REG( 1)
IP2=REG(1+1)-1
CNT=0
DO }829\mathrm{ L=|P1,IP2
IF(IRN(L).LT.I)GO TO 828
IF(IRN(L).EQ.I)GO TO }82
SUAi=SUAN+STIF(L)*ANEV(IRN(L))
GO TO 829
828 CNT=CNT+1
SLM=SUM+STIF(L)*ANEW(IRN(L))
829 CONTINUE
IP}= |P} +CNT
ANEW(1)=(RH(1)-SLN)/STIF(IP3)
827 CONTINLE
7 6 6 CONTINLE
JTCC=J TC-IPOINT
IF(JTCC.GT.0.001)GO TO 433
JTC=| TC+ 20
| TC4 =\#OD() TC,40)
IF(J TC4.LT.0.001)GO TO 434
IF(JTC4.GT.0.001)GO TO 435
434 OPEN(FILE='GLANEW2',UNIT=9,STATUS='U*UNONN')
DO 756 1=1,TOTNOD
WRITE(9,8011)GU(1)
756 CONTINUE
DO 747 l=1,TOTNOD
\RRITE(9,8011)ANEW(1)
747 CONTINUE
CLOSE(9)
GO TO 433
435 OPEN(FILE='GUANEW', UNIT=8,STATUS='UKNOUN')
DO 977 1=1,TOTNOD
WRITE(8,8011)GU(1)
977 CONT INUE
DO 978 I=1,TOTNOD
WRITE(8,8011)ANEW(1)
978 CONTINUE
CLOSE(8)
433 OPEN(FILE= 'F) TC',UNIT=10,STATUS='UNKNOIN')
WRITE(10,8010)JTC
CLOSE(10)

```
```

C
C
C
DO 777 1=1,TOTNOD
RH(1)=ANEW(1)
77 CONTINUE
IF(IPOINT.GT.0.5)GO TO 7777
WRITE(1,8041)
WRITE(NOUT, 8041)
7777 DO 1017 I=1,COUNT
IW(1)=(GU(INTRA(1))-GU(1))/DENOMA
W(1)=(IW(1)-2*GU(1))
1017 CONTINUE
IW(2)=4.0*IW(2)
ICTI=COUNT-1
ICT2=COUNT-2
WW(3)=2.0* NW(3)
DO 1018 I=4,1CT1,2
IW(1)=IW(1)*4.0+IW(1-2)
1018 CONTINUE
DO 1019 I=5, ICT2,2
IW(1)=IW(1)*2.0+IW(1-2)
1019 CONTINUE
C
TOTAL=IW(1)+IW(ICT2)+IW(ICT1)+IW(COUNT)
DELTA=2.0/((COUNT-1)*1.0)
AREA=DELTA*TOTAL/3.0
RMMARG=20.0000DO
RIF=DMIOD(I POINT, RNARG)
C IF(ABS(RM).GT.0.01)GO TO 39
C
C
SLOP=SLOPE+AREA/DIV+ALOG(2.0)
ITER=IPOINT+1
IPO INT= IPO|NT+1
C
C Calculation of beta
C
DO 479 I=1,COUNT
1 TC3=COUNT3+1
DGU(1)=(GU(1TC3)-GU(INTRD(1)))/DENNA:B
DANEW(1)=(ANEW(1TC3)-ANEW(INTRB(1)))/DENCM1B
479 CONTINUE
BETA=L:MULT1*((4/3.0) *DANEW(IDGU)-3-2*LOG(RR))
BETA=BETA+LMULT2*(3-2*LOG(RR)+(2/3.0)*RR**2*DGU(IDGU))
C
C BETA=AREA/6.0+2.0945349
IF(ABS(RM).GT.0.005)GO TO 39
BETAP = BETA +0.1972246
WRITE (1, 8039)SLOP,AREA, ITER
WRITE(1,2057)
C
C COMPARING THE NLMERICAL DERIVATIVES FOR
C OMEGA AND PSI WITH THE CORRESPONDING
C THEORETICAL VALUES
C
VRITE(1, 8039)DGU(IDGU),DANEW(IDGU)

```
```

        WRITE(1,8039)DRCU(IDGU),DRANEW(IDGU)
    C
WRITE(NOUT,8039)SLOP,AREA, ITER
39 IF(IPOINT.LT.20000)CO TO }72
STOP
8050 FORNLAT ///41H NMATCH RADIUS 020 MESH 140\times20 //)
8010 FORNAT(1615)
8023 FORWAT (//14H INTO LOOP 23 )
8037 FORMAT (14IH INTO LOOP 36 )
8041 FOR:IAT (//42H SLOPE INTGR //)
8043 FORMAT (17H STIFFNESS MATRIX.)
8038 FORMAT (7H HBAND )
8036 FORMAT(1H, 110F10.5)
8039 FOR:AAT (2F20.5,18)
8020 FORNAT (15, 6F12.5)
END
SUBROUTINE ASRHS(RHIS, IRHS, VALLE, IVALUE, STEER, ISTEER,
* DOFEL, ITEST)
C-MTFGER DOFEL, ERRIIES, IENROR, IRHS, ISTEER, ITEST, IVALUE,
* K, L, STEER
DOUBLE PRECISION RHS, SRNAMAE, VALLE
DIMENSION RHS(IRHS), STEER(ISTEER), VALUE(IVALUE)
DATA SRNAME /8H ASRHS /
IF (ITEST.EQ.-1) GO TO 1010
IERROR = 0
IF (ISTEER.LT.DOFEL) IERROR = 3
IF (IVALUE.LT.DOFEL) IERROR = 2
IF (DOFEL.LE.O) IERROR = 1
ITEST = ERFMES (ITEST,IERROR,S FNMAME )
1010 IF (ITEST.NE.0) RETURN
DO 1030 K=1,DOFEL
IF (STEER(K).EQ.O) GO TO 1030
L = STEER(K)
IF (ITEST.EQ.-1) GO TO 1020
IERROR = 0
IF (L.GT.IRHS) IERROR = 4
ITEST = ERRMES(ITEST, IERROR,SPTJAME )
IF (ITEST.NE.O) RETURN
1020 RHS(L) = RHS(L) + VALUE(K)
1030 CONTINUE
RETURN
END

```

\section*{PROGRAM 4}

THIS PROGRAM SOLVES THE DIFFERENTIAL EQUATION FOR PULSATILE FLOW. THE NORMALISED PRESSURE GRADIENTS ARE CALCULATED. THE NORMALISED MEAN PRESSSURE GRADIENTS AND THE FUNDAMENTAL NORMALISED PRESSURE GRADIENT AMPLITUDES AND PHASE ANGLES ARE OBTAINED.
            INTEGER*4 1,J,NC,COUNT,L1,L2,L3,L4,
    * NR,NUT,J \(2, N C C, N R 1, C T R, C T R 2, N V A L\),
    * NUT1,NUTM1, IX,JX,KX,NAX,
    - MEM1, NEM2,N(M,FLAG
    REAL*8 V(151,2),R(151),T(2),PI,QNUM,TOTAL,N,EPS,F,
    * DELT, DELR,W(2),MULTAE,NN,NNN, DRNNN,DSGN3,DSGN4,
    * INT(160), PSTAR(40010), GRAD,DRN,DSGNS,DSON6,
    * AiPl, GRR1, GR1, ARG, DIFFP (40010) ,DSGN7, DSGN8,
    * MEAN,DSGN1,DSGN2,PHASE,
    * \(\operatorname{AN}(20), \operatorname{MF}(20), \operatorname{MEPS}(20)\),
    * ALPHA, AJ, BIJ, CIJ, EPSS, EPSTAR,
    * \(\quad A 1, A 2, A 3, A 4, A 5, B 1, B 2, B 3, B 4, B 5\)
    COMMON /A/INTO
    COMMDN/B/V
    COMAON/C/PSTAR
    COMMON/E/T
    COW:ON/D/INT
    COMBON/E/DIFFP
C
    DATA MN(1)/0.3D0/, MEPS (1)/0.1D0/, \(\operatorname{MF}(1) / 0.500 /\),
    * \(\operatorname{MN}(2) / 0.500 /, \operatorname{MEPS}(2) / 0.500 /, \mathrm{MF}(2) / 1.0001\),
    * MN(3)/0.7D0 , \(\operatorname{MEEPS}(3) / 1.0 \mathrm{DO} / \mathrm{MF}(3) / 2.0 \mathrm{D} 0 /\),
    * NN(4)/0.9D0/,NEPS (4)/5.0D0/, NF (4)/3.000/,
    * \(\operatorname{AN}(5) / 1.000 / \operatorname{MEPS}(5) / 10.000 /, \mathrm{MF}(5) / 5.0001\),
    * \(\operatorname{MiF}(6) / 7.000 /\).
    * \(\operatorname{MiF}(7) / 10.0001\).
    * \(\operatorname{imF}(8) / 15.0001\).
    * MF(9)/20.000/,
    * \(\quad M F(10) / 30.0001\),
    * \(\operatorname{MiF}(11) / 50.0001\),
    * AF(12)/60.000/,
    * NF(13)/70.0D0/.
    * \(\operatorname{arf}(14) / 80.0001\),
    * \(\operatorname{MiF}(15) / 100.000 /\),
    * NC/3/,NUT/1000/,NR/20/,NVAL/1/, IMAX/10000/
2333 FOPFAT (//7X, 'NUMBER')
3333 FOPSAT (41H OF CYCLES STEP/CICLE RADIAL DIV.)
4444 FORNAT(1112,1115,1113)
S999 FORMAT ( 3014 FOURIER CCEFFICIENTS:A'S)
7777 FOMAAT \(30 \mathrm{H} \quad\) FOURIER COEFFICIENTS: E'S)
4445 FORIAT(5X,'NO. OF SIGN OSCILLATIONS IN PRESSLRE VALUES',113)
WRITE \((6,2333)\)
    WRITE \((6,3333)\)
    WRITE \((6,4444) N C, N U T, N R\)
1111 FOPMAT(1F13.3,1F14.3,1F14.3)
2222 FOPMAT (41H VISCOSITY EPSILON F )
C
C

VARIABLE LIST
NCC \(=\mathrm{NC}+1\)
\(\mathrm{NUT} 1=\mathrm{NUT}+1\)
NUTM1 \(=\) NUT- 1
```

DELR=1.0/(DBLE(FLOAT(NR)))
NR1 =NR+1
DELT=1/(DBLE(FLOAT(NUT)))
NRRM1 =NR-1
COUNT=0
PI=4.000*DATAN(1.0DO)
C
C INITIAL CONDITIONS
DO 1 I=1,NR
V(1,1)=0.000
1 CONTINUE
V(NR1,1)=0.0DO
C
R(1)=0.0DO
DO 6 I=1,NR
R(1+1)=R(1)+DELR
6 CONTINUE
C
T(1)=0.0D0
DO 1812 IX=4,4
N=NN(1X)
ALPHA=((3.0DO*N+1.0DO)/N)**N
NN=N-1
NNN=N+1
DRNNN=(1.0DO/DELR)**NNN
DO 1944 JX=2,2
EPS=MEPS (JX)
DO 1957KX=5,5
F=MF (KX)
MULTAE=2.ODO*ALPHA*EPS*P | F / 8.0DO
WRITE(6,2222)
WRITE(6,1111)N,EPS,F
WRITE(6,9999)
WRITE(6,7777)
WRITE(6,4321)
WRITE(6,2057)
2057 FORHAAT (5X,43H
WRITE(1,1111)N,EPS,F
C:IMP LEMENT ING FINITE DIFFERENCE SCHEME
C======================================
CTR=0
7CTR2=0
iNU:i=0
: :- 1=1
FLAG=0
DO 2 J=1,NUT
T(2)=T(1)+DELT
T(2)=2.0CO*PI*T(2)
T(1)=2.0DO*PI*T(1)
| 2=j+1
V(NR1,2) =0.000
W(1)=V(2,1)-V(1,1)
GRR1 = (4.0DO*V(NR,1)-V(NRM1,1))/(2.0DO*DELR)
DSGN1=DABS (GRR1)**N
; DSGN2=DABS (W(1))**N
V(1, 2) =MULTAE* (-1.0D0) *DSIN(T(1))+2.000*DSIGN(DSCN1,GRR1)
+4.0DO*DSIGN(DSGN2,W(1))*(1.0D0/DELR)**NNN

```
```

    V(1,2)=V(1,2)*DELT*8.0iO/(F*ALPHIA )+V(1,1)
    DO 3 I=2,NR
    K2=1+1
    AJ =MULTAE* (-1.0DO)*DSIN(T(1))+2.000*DSIGN(DSGN1,GRR1)
    DRN=DELR**N
    DIRN=1.ODO/DRN
    DSGN3=DABS (V(K2,1)-V(1,1))**N
    DSGN4 =V (K2,1)-V(1,1)
    E|J=(R(1)+DELR/2.0DO)*DSIGN(DSGN3,DSGN4)
    DSGN5 = V(1,1)-V (1-1,1)
    DSGN6=DABS (DSGN5) **N
    CIJ=(R(1)-DELR/2.0D0)*DSIGN(DSGN6,DSGN5)
    V(1, 2)=Al + (1.ODO/R(1))*DRNNN*(B|J-C1J)
    V(1,2)=V(1,2)*DELT*8.0DO/(F*ALPHA)+V(1,1)
    3 CONTINUE
C
CTR=CTR+1
IF(CTR.NE.NVAL)GO TO 17
CTR2=CTR2+1
CTR=0
C CALCULATING THE NON-DIMENSIONALISED FLON RATE
DO 12 LI=1,NR1
INT(L1) =V (L1, 2)*R(L1)*2.0DO
12 CONTINUE
INT (2) = 4.0DO*INT (2)
INT(3) =2.0DO* INT(3)
DO 13 L2=4,NR,2
INT(L2) = INT(L2) *4.0DO+1NT(L2-2)
13 CONTINUE
DO 14 L3=5,NRM1,2
INT(L3) = INT(L3) *2.0DO + INT(L3 -2)
14 CONTINLE
TOTAL=INT(1) +INT(NR) +INT(NRM1) + INT (NR1)
QNUA=DELR*TOTAL/3.0DO
MODIFYING THE NON-DIMENSIONALISED VELOCITIES
C MODIFYING THE NON-DIMENSIONALISED VELOCI'
DO 15 L4=1,NR1
EPSS=1.0DO+EPS*DCOS(T(2))
EPSTAR=3.0DO* (EPSS-QNUM)
V(L4, 2) =V (L4, 2)+EPSTAR* (1.0DO-R(L4))
15 CONTINUE
17 CONTINUE
C
C
C
CTR=CTR +1
IF (CTR.NE.NVAL) GO TO 17
CTR2 $=$ CTR2 +1
CTR=0
C CALCULATING THE NON-DIMENSIONALISED FLOW RATE
DO $12 \mathrm{Li}=1$, NR1
INT(L1) $=V(L 1,2) * R(L 1) * 2.0 D 0$
12 CONTINUE
$\operatorname{INT}(2)=4.00^{*} \operatorname{INT}(2)$
$\operatorname{INT}(3)=2.00^{*} \operatorname{INT}(3)$
DO 13 L2=4,NR,2
INT(L2) $=$ INT (L2) *4.0D0+1NT(L2-2)
13 CONTINUE
DO 14 L3 $=5$, NRM1, 2
INT(L3) $=1 \mathrm{NT}(\mathrm{L} 3) * 2.0 \mathrm{DO}+1 \mathrm{NT}(\mathrm{L} 3-2)$
14 CONTINLE
TOTAL = INT (1) + INT (NR) + INT (NRM1) + INT (NR1)
QNUA=DELR*TOTAL/3.0D0
C MODIFYING THE NON-DIMENSIONALISED VELOCI
C USING THE NON-DIMENSIONALISED FLOW RATE.
DO 15 L4 = 1 , NR1
EPSS $=1.0 D 0+E P S * \operatorname{COS}(T(2))$
EPSTAR $=3.00^{*}$ (EPSS-QNUM1)
$V(L 4,2)=V(L 4,2)+\operatorname{EPSTAR}^{*}(1.0 D 0-R(L 4))$
17 CONTINUE
CALCULATING THE NORMAIISED PRESSURE GRADIENTS.
GRAD $=(4.0 D 0 * V(N R, 2)-V(N R M 1,2)) /(2.0 D 0 * D E L R)$
DSGN7=DABS (GRAD)**N
GRAD=DS IGN(DSGN7, GRAD)
GR1 $=(4.000 * V(N R, 1)-V(N R M 1,1)) /(2.0 D 0 * D E L R)$
DSGN8 $=\operatorname{DARS}(G R 1) * * N$
IF(J.GT.1)GO TO 77
$\operatorname{PSTAR}(1)=\operatorname{SSIGN}(D S G N 7, G R 1)$
$\operatorname{PSTAR}(1)=P \operatorname{STAR}(1) / A L P H A$
$\operatorname{DIFFP}(1)=0.000$
77 CONTINUE
$\operatorname{PSTAR}(/ 2)=(F / 16.000) *(-2.000 * P 1 * E P S * \operatorname{DSIN}(T(2)))$

* . $\because$. +GRAD/ALPHA

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```

            DIFFP(J 2) =PSTAR(J 2)-PSTAR(J 2-1)
            T(1)=T(2)/(2.0DO*P1)
            DO 44 KK=1,NR1
            V(KK, 1) =V (KK, 2)
        44 CONTINUE
            MEM12 = - 1
            IF(DIFFP(1 2).GT.0) AIEM2=1
            IF (MEM2 .EQ. MEM1)GO TO 231
            MEM1 =MEM2
            FLAG=FLAG+1
            IF(FLAG.LE.1)GO TO 2.
            NUN=NUM+1
    C WRITE(6,11111)NUM
IF(NUM.GT.MAX)GO TO 8
231 FLAG=0
2 CONTINUE
WRITE(1,11111)NUS
C FOURIER ANALYSIN OF
C NORMAIISED PRESSURE GRADIENTS.
CALL FANAL(PSTAR,DELT,PI,NUT,NUT1,NUTM11,
* MEAN,A1,A2,A3,A4,A5,B1, B2,B3,
* B4,B5,A贝P L,ARG)
1235 FORMAT(1F14.5,2F10.5,1 110)
1234 FORSMAT(//1F14.5,11F10.5)
1236 FORMNAT(1F14.5,11F10.5)
WRITE (6,1234)A1,A2,A3,A4,A5
WRITE(6,1236)B1, B2, B3,B4,B5
PHASE=DATAN(-B1/A1)
PHASE=PHASE* 180/P 1
COUNT=COUNT+1
4 3 2 1 ~ F O R Y M T ~ ( 3 4 H ~ M E A N ~ A M P L ~ C O U N T )
WRITE (6,1235)MEAN, AMIPL, PHASE,COUNT
WRITE(1, 1235)PHASE
WRITE(1,1235)MEAN,AMPL,COLNT
WRITE (1,1235)COUNT, MEAN
CONTINUE
C
IF(COLNT.EQ.NC)GO TO 8
T(1)=0.0DO
GO TO 7
11111 FORSAAT(12110)
8 CONTINUE
COUNT=0
T(1)=0.0DO
DO 112 l=1,NR1
V(1,1)=0.0D0
112 CONTINUE
1957 CONTINUE
1944 CONTINUE
1812 CONTINUE
WRITE (6, 2057)
WRITE(6,4445)NUM
WRITE (6;2057)
STOP
END
C
SLEROUTINE FANAL(PSTAR,DELT, PI,NUT.

```
```

            * NUT1,NUTM1,I.LCAN,A1,A2,A3,A4,A5, B1,B2,B3,B4,
            * B5,AMPPL,ARG)
            INTEGER NUT1 ,NUT,NUTM11
            DOUBLE PRECISIONS PI,MEAN,A1,A2,A3,A4,A5,
            * B1,B2,B3,B4,B5,ANiPL,ARG,
            * PSTAR(40010).
            * X,DELT,
                            * TOTALC,TOTALS,TOTC(5),TOTS(5),
    * TOTALO,CF(5),SF(5),CE(5).
* SE(5),CO(5),SO(5),CL(5),SL(5)
C
DO 1 1=1,5
CF(1)=2.000*PSTAR(1)
SF(1)=0.000
CE(1)=2.0DO*PSTAR(2)*DCOS(DELT*2.0DO*1*P I)
SE(1)=2.000*PSTAR(2)*DSIN(CELT*2.0DO*1*P 1)
CO(I)=2.0DO*PSTAR(3)*DCOS(DELT*4.0DO*I*P I)
SO(1)=2.0DO*PSTAR(3)*DSIN(DELT*4.0DO*1*PI)
CL(1)=2.0DO*PSTAR(NUT1)*DCOS (DELT*NUT*1*2.0DO*PI)
SL(1)=2.0DO*PSTAR(NUT1)*DS IN(DELT*NUT*1*2.0DO*PI)
1 CONTINUE
C WRITE (1, 1000)(\operatorname{CF}(1),1=1,5)
C WRITE(1,1000)(CE (1), I=1,5)
C VIRITE(1,1000)(SE(1), l=1,5)
C URITE(1,1000)(CO(1), I=1,5)
C WRITE(1,1000)(SO(1),I=1,5)
C WRITE(1,1000)(CL(1), I=1,5)
C WRITE(1,1000)(SL(1),1=1,5)
1000 FORMAT (1 OF10.4)
DO 90 l=4,NUT,2
X=(1-1)*DELT*2.0DO*P1
DO 2 J=1,5
CE(J)=CE(J)+2.0DO*PSTAR(1)*DCOS (J *X)
SE(J)=SE(J)+2.0DO*PSTAR(I)*DSIN(J*X)
2 CONTINLE
VRITE(1,1000) (CE (K),K=1,5)
C WRITE(1,1000)(SE(K),K=1,5)
90 CONTINUE
DO 91 I=5,NUTM1,2
X=(1-1)*DELT*2.0DO*P I
DO 3 J=1,5
CO(J)=CO(1)+PSTAR(1)*2.000*COS(1*X)
SO(J)=SO(J)+2.0DO*PSTAR(1)*\operatorname{DSIN(J*X)}
3 CONTINUE
WRITE (1, 1000)(CO(K),K=1,5)
C WRITE(1,1000)(SO(K),K=1,5)
91 CONTINUE
DO 4 l=1,5
CE(1)=CE (1)*4.0DO
SE(1)=SE(1)*4.0DO
CO(1)=CO(1)*2.000
SO(1)=SO(1)*2.000
4 CONTINUE
TOTALC=CF(1)+CE(1)+CO(1)+CL(1)
TOTALS =SF(1)+SE(1)+SO(1)+SL(1)
DO 6 1=2,5

```
```

$\operatorname{TOTC}(1)=\operatorname{CF}(1)+\operatorname{CE}(1)+\operatorname{CO}(1)+C L$ ( 1$)$
$\operatorname{TOTS}(1)=S F(1)+S E(1)+S O(1)+S L(1)$

```

6 CONTINLE
AI =TOTALC*DELT/3.0DO
\(B 1=\) TOTALS * DELT \(/ 3.000\)
\(A 2=\operatorname{TOTC}(2) * D E L T / 3.000\)
\(\mathrm{B} 2=\operatorname{TOTS}(2) * D E L T / 3.0 D 0\)
\(A 3=\operatorname{TOTC}(3) * D E L T / 3.000\)
\(B 3=\operatorname{TOTS}(3) * D E L T / 3.0 D 0\)
A4 \(=\operatorname{TOTC}(4) * D E L T / 3.0 D 0\)
B4 \(=\operatorname{TOTS}(4)\) *DELT/3.0D0
\(\mathrm{A} 5=\operatorname{TOTC}(5) * D E L T / 3.0 D 0\)
\(B 5=\operatorname{TOTS}(5) * D E L T / 3.0 D 0\)
\(A R G=A 1 * A 1+B 1 * B 1\)
\(A N P L=D S Q R T(A R G)\)
\(\operatorname{PSTAR}(2)=4.0 \mathrm{DO} * \operatorname{PSTAR}(2)\)
\(\operatorname{PSTAR}(3)=2.0 D 0 * \operatorname{PSTAR}(3)\)
DO \(53 \mathrm{~N} 2=4\), NUT, 2
\(\operatorname{PSTAR}(N 2)=P S T A R(N 2) * 4.000+P S T A R(N 2-2)\)
53 CONTINUE
DO 54 N3=5, NUTM1, 2
\(\operatorname{PSTAR}(N 3)=\operatorname{PTAR}(N 3) * 2.0 D 0+P \operatorname{STAR}(N 3-2)\)
54 CONTINUE
TOTALO \(=\operatorname{PSTAR}(1)+P \operatorname{STAR}\) (NUT) \(+P \operatorname{STAR}\) (NUTR:i1) +PSTAR(NUT1)
MEAN=DELT*TOTALO/3.ODO
RETURN
END

\section*{PROGRAM 5}

THIS PROGRAM CALCULATES THE NORMALISED MEAN PRESSURE GRADIENT FROM THE POWER LAW MODEL PERTURBATION ANALYSIS.

10 REM PROGRAM TO CAICULATE NORMALISED MEAN PRESSURE GRADIENT 20 REM FROM THE PONER LAW MODEL PERURBATION ANALYSIS
30 :
40 DIM \(\mathrm{Y}(100), \mathrm{Z}(100)\)
\(50 \mathrm{~N}=.9\)
\(60 \mathrm{E}=10\)
\(70 \mathrm{ND}=100\)
\(80 \mathrm{~F}(1)=1: \mathrm{F}(2)=2: F(3)=3: F(4)=4: F(5)=6: F(6)=8\)
\(90 \mathrm{Pl}=3.1415927 \mathrm{E}\)
\(100 \mathrm{P} 2=2\) *P1
110 FOR I=0 TO ND
\(120 \mathrm{U}=\mathrm{P} 2 * \mathrm{I} / \mathrm{ND}\)
\(130 \mathrm{Q}=1+\mathrm{E}^{\star} \cos (\mathrm{U})\)
\(140 \mathrm{Y}(\mathrm{I})=(\operatorname{ABS}(\mathrm{Q}))^{\wedge} \mathrm{N} * \operatorname{SGN}(\mathrm{Q})\)
\(150 \mathrm{Z}(\mathrm{I})=-\mathrm{E} * \mathrm{P} 2 * \mathrm{P} 2 * \operatorname{Cos}(\mathrm{U}) *(\operatorname{ABS}(\mathrm{Q}))^{\wedge}(1-\mathrm{N})\)
160 NEXT I
170 GOSUB 340
180 Q0=RESULT
190 gosub 400
200 Ql=RESULT
\(210 \mathrm{Kl}=\mathrm{N} *(3 * \mathrm{~N}+1) * \mathrm{Q} 1 /\left(512 \star(2 * \mathrm{~N}+1)^{\wedge} 2 \star(5 * \mathrm{~N}+3)\right)\)
220 FOR \(\mathrm{I}=1\) TO 6
\(230 \mathrm{Q}=\mathrm{Q} 0+\mathrm{F}(\mathrm{I}) * \mathrm{~F}(\mathrm{I}) * \mathrm{Kl}\)
240 PRINT Q;" ";
250 NEXT
260 PRINT
\(270 \mathrm{~F}(1)=10: F(2)=15: F(3)=20: F(4)=25: F(5)=30: F(6)=35\)
280 FOR I=1 TO 6
\(290 \mathrm{Q}=\mathrm{Q} 0+\mathrm{F}(\mathrm{I}) * \mathrm{~F}(\mathrm{I}) * \mathrm{Kl}\)
300 PRINT Q;" ";
310 NEXT I
320 PRINT
330 END
\(340 \mathrm{Sl}=0\) : \(\mathrm{S} 2=0\)
350 FOR \(\mathrm{I}=1\) TO ND-1 STEP 2:Sl=Sl+Y(I):NEXT
360 FOR I=2 TO ND-2 STEP 2:S2=S2+Y(I) :NEXT
370 RESULT \(=(\mathrm{Y}(0)+4 * \mathrm{~S} 1+2 * \mathrm{~S} 2+\mathrm{Y}(\mathrm{ND})) /(3 * \mathrm{ND})\)
380 REIURN
390 :
\(400 \mathrm{Sl}=0: \mathrm{S} 2=0\)
410 FOR I=1 TO ND-1 STEP 2:Sl=Sl+Z(I):NEXT
420 FOR I \(=2\) TO ND-2 STEP 2:S2=S2+Z(I):NEXT
430 RESULT \(=(Z(0)+4 * S 1+2 * S 2+Z(N D)) /(3 * N D)\)
440 RETURN

\section*{PROGRAM 6}

THIS PROGRAM CALCULATES THE THEORETICAL NORMALISED MEAN PRESSURE GRADIENTS WHEN FLUID INERTIA IS IGNORED.

C THIS PROGRAM CALCULATES THE THEORETICAL NORMALISED MEAN PRESSURE GRADIENTS. FLUID INERTIA HAS BEEN IGNORED.

INTEGER NR,NRP 1 ,NIRM1
DOUBLE PRECISION DELR, INT(200),TOTAL,PR(10), * X(200),ARG,N(10),EPS(10)

DATA EPS(1)/.1/, EPS(2)/0.5/, EPS(3)/1.0/, * \(\operatorname{EPS}(4) / 5.0 /, \operatorname{EPS}(5) / 10.0 /, \operatorname{EPS}(6) / 50.0 /, \operatorname{EPS}(7) / 100.0 /\), * \(\quad N(1) / .3 /, N(2) / .4 /, N(3) / .5 /, N(4) / .6 /\), * \(N(5) / .7 D 0 /, N(6) / .8 D 0 / . N(7) / .9 D 0 /, N(8) / 1.0 D 0 /\), * NR/100/,NOUT/61
    600 FORALT 4 X, 'EPS', \(5 \mathrm{X}, 1 \mathrm{~F} 3.1,8 \mathrm{X}, 111,9 \mathrm{X}, 111,8 \mathrm{X}, 112,8 \mathrm{X}\),
        * 112,8X,113)
    800 FORAAT \(\left(5 X, N^{\prime} N^{\prime}\right)\)
        WRITE(NOUT, 600)EPS(2), (IDINT(EPS(1)), \(1=3,7)\)
        WRITE(NOUT, 800)
        DO \(5 \mathrm{~J}=1,8\)
        DO \(6 K=1,7\)
C
    PI=4.0DO*DATAN(1.0DO)
    NRM1 \(=\) NR- 1
    NRP \(1=N R+1\)
    DELR=2*PI/100.0D0
    DO 1 I=1,NRP 1
        \(X(1)=(1-1) *\) DELR
    1 CONTINUE
        DO \(21=1\), NRP 1
        ARG=x(1)
        IF (DABS (1+EPS(K)*DSIN(ARG)).LT.0.000001)GO TO 7
        INT( 1 ) \(=(1+\operatorname{EPS}(K) * \operatorname{DS} \operatorname{IN}(\operatorname{ARG})) * \operatorname{DABS}(1+\operatorname{EPS}(K) * \operatorname{DS} \operatorname{IN}(A R G))\)
    * **(N(J)-1)
        GO TO 2
        \(7 \operatorname{INT}(1)=0.0\)
        2 CONTINUE
        \(\operatorname{INT}(2)=4.0 \mathrm{DO} * \operatorname{INT}(2)\)
        \(\operatorname{INT}(3)=2.0 \mathrm{DO} * \operatorname{INT}(3)\)
        DO 3 I=4,NR,2
        \(\operatorname{INT}(1)=\operatorname{INT}(1) * 4.0 \mathrm{DO}+\operatorname{INT}(1-2)\)
    3 CONTINLE
        DO 4 I=5,NRM1,2
        \(\operatorname{INT}(1)=\operatorname{INT}(1) * 2.0 \mathrm{DO}+\operatorname{INT}(1-2)\)
    4 CONTINUE
        TOTAL=INT(1)+INT(NR)+INT(NRMI) + INT(NRPI)
        \(P R(K)=D E L R * T O T A L / 3.0 D O\)
        \(P R(K)=P R(K) /(2.000 * P I)\)
    6 CONTINUE
        WRITE(NOUT,500)N(J), (PR(KK), KK=2,7)
500 FORMAT (/1F7.1.7F10.4)
    5 CONTINUE
        STOP```

