2016

# Invariant Algebraic Surfaces in Three Dimensional Vector Fields

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http://hdl.handle.net/10026.1/4417

http://dx.doi.org/10.24382/3473

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# INVARIANT ALGEBRAIC SURFACES IN THREE DIMENSIONAL VECTOR FIELDS

by

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A thesis submitted to Plymouth University in partial fulfilment for the degree of

**DOCTOR OF PHILOSOPHY**

October 2015

# Abstract

This work is devoted to investigating the behaviour of invariant algebraic curves for the two dimensional Lotka-Volterra systems and examining almost a geometrical approach for finding invariant algebraic surfaces in three dimensional Lotka-Volterra systems.

We consider the two dimensional Lotka-Volterra system in the complex plane which we can simplify to the following form

$$\dot{x} = x\Big(1 - x - A(1-C)y\Big),$$
$$\dot{y} = y\Big(A - (1-B)x - Ay\Big).$$

We consider the twenty three cases of invariant algebraic curves found in Ollagnier (2001) of the above system and then we explain the geometric nature of each curve, especially at the critical points of the system above.

We also investigate the local integrability of two dimensional Lotka-Volterra systems at its critical points using the monodromy method which we extend to use the behaviour of some of the invariant algebraic curves mentioned above.

Finally, we investigate invariant algebraic surfaces in three dimensional Lotka-Volterra systems by a geometrical method related with the multiplicity of the

intersection of the algebraic surface with the axes including the lines at infinity. We will classify both linear and quadratic invariant algebraic surfaces under some assumptions and commence a study of the cubic surfaces.

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# Acknowledgements

I would first of all like to thank my God, Allah, for getting me through all the difficulties. I have experienced your guidance day by day. You are the one who let me finish my degree. Thank you Allah.

I would like to express my special appreciation and thanks to my supervisor, Dr. Colin Christopher for his invaluable support and monitoring me throughout my graduate studies program.

A special thanks to my family, words cannot express how grateful I am to my mum, your prayer for me was what sustained me thus far.

Also I would like to express my appreciation to my beloved wife Shirin who was always my support in the moments when there was no one to answer my queries. For her understanding, support and love during my years as a graduate student.

To my beloved kids Dania and Ali.

I would like to express my thanks for being such good kids always cheering me up.

Finally, I appreciate and acknowledge the Kurdistan government that provided financial support during my graduate studies program.

# Dedication

This dissertation is dedicated to:

- The memory of my Dad who passed away during my PhD study. I will never forget you Dad.
- To my Mum, you mean the world to me, but I do not tell you enough.
- To my brothers and sisters, who protect me from the things I should not do.
- To my lovely wife Shirin, who guides me in the right direction. You do not know how much I love you.
- To my kids Dania and Ali, you are the apple of my eyes. All of my troubles simple disappear when I see your smiles.

# Author's Declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award without prior agreement of the Graduate Committee.

Work submitted for this research degree at the Plymouth University has not formed part of any other degree either at Plymouth University or at another establishment.

Word count of main body of thesis: 47,000

Signed ..........................................

Date ..........................................

## Posters

"Integrability of quadratic Lotka-Volterra systems in $\mathbb{C}^2$ at the cririal points away from the origin", with Christopher C. and Z.Wang. Presented at:
*Advances in Qualitative Theory of differential Equations (Second Edition).*
20-24 April,2015. Tarragona, Spain.

## Conferences

- *The 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications*, 7-11 July, Madrid, Spain.
- *Advances in Qualitative Theory of Differential Equations (Second Edition)*, 20-24 April, 2015, Tarragona, Spain.

## Journal Publications

1. Colin Christopher and Wuria M.A. Hussein, *A geometric approach to Moulin-Ollagnier's classification of algebraic solutions of Lotka-Volterra systems. Preprint, 2015*
2. Colin Christopher, Wuria M.A.Hussein and Zhaoxia Wang, *On the integrability of Lotka-Volterra equations. Submitted, 2015*

# List of Abbreviations

$F$ ............ Field

$\mathbb{A}^n$ ........... $n$-dimensional affine space

$\mathbf{V}(f)$ .......... Variety of $f$

$\mathbf{K}$ ............. Polynomial ring

$\mathbb{C}^n$ ........... $n$-dimensional complex plane

$\mathbf{P}^n(F)$ ......... $n$-dimensional projective space over a field $F$

$\Im$ .............. Holomorphic foliation

$S(\Im)$ .......... Critical points of holomorphic foliation $\Im$

$\mathbf{P_{xy}}$ ........... Projective plane determined by the axes $x$ and $y$

$L_\infty$ ........... Line at infinity

$\boldsymbol{w}$ ............. A complex cubic root of one

$\boldsymbol{i}$ ............. $\sqrt{-1}$

# Chapter 1

# Introduction

We consider the following three dimensional Lotka-Volterra system in $\mathbb{C}$

$$\begin{aligned}
\dot{x} &= \frac{dx}{dt} = x\big(1 + a_1x + a_2y + a_3z\big), \\
\dot{y} &= \frac{dy}{dt} = y\big(b_0 + b_1x + b_2y + b_3z\big), \\
\dot{z} &= \frac{dz}{dt} = z\big(c_0 + c_1x + c_2y + c_3z\big),
\end{aligned} \tag{1.1}$$

where $a_i$, $i \in \{1, 2, 3\}$ and $b_j, c_j$, $\jmath \in \{0, 1, 2, 3\}$ are complex parameters. The name the of Lotka-Volterra system comes from the American mathematician Alfred J. Lotka, Lotka (1920) and the Italian mathematician V.Volterra, Volterra and Brelot (1931). In the application of this system, the variables $x$, $y$ and $z$ represent populations of different species: $b_0$ and $c_0$ are the growth rates among these species; and the other parameters are the interaction coefficients of the species.

Lotka-Volterra systems also have applications in physics, see for example Sprott et al. (2005). In this thesis, we are interested in investigating invariant algebraic surfaces $C_f$ with respect to the system (1.1) defined by the polynomial $f(x, y, z) = 0$. In fact, the study of such a surface $C_f$ becomes much easier by investigating the corresponding curves of $C_f$ in each face, (by the faces, we mean the $xy$, $xz$ and $yz$-planes in $\mathbb{C}^3$). For this purpose, we will first study invariant algebraic curves in two dimensional Lotka-Volterra system.

In fact, any two dimensional polynomial vector field takes the following form

$$\dot{x} = \frac{dx}{dt} = P(x, y), \qquad \dot{y} = \frac{dy}{dt} = Q(x, y), \tag{1.2}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials and not necessarily homogeneous. We assume that the polynomials $P$ and $Q$ are co-primes. If they are not co-prime, then they have a common factor say $h$ and hence $h = 0$ is a line of singularities in the case of the quadratic system. In two dimensions, we can divide out by the line of singularities, while in three dimensions, the line of singularities can not divide the system and it will appear in our study.

It is known that the system (1.2) can also be considered as a holomorphic foliation $\Im$ in the affine complex plane. We can also write the vector field in the form $\chi = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$.

An algebraic curve $C_f$, is defined by a polynomial $f(x, y) = 0$ in the polynomial ring $\mathbb{C}[x, y]$ called invariant with respect to (1.2) if $\chi(f) = fL_f$, for some polynomial $L_f$ called the cofactor of $f$. Notice that the degree of the cofactor is at most one less than the degree of $\Im$. If $f$ is irreducible, then the conditions $\chi(f) = fL_f$, for some $L_f$ is necessary and sufficient for $f = 0$ to be invariant.

In 1878, Darboux proved that invariant algebraic curves can be used for first integrals in polynomial vector fields (Darboux (1878)). By a first integral of the system (1.2), we mean a non-constant function $H$ defined on an open ball in $\mathbb{C}^2$ such that $H\big(x(t), y(t)\big)$ is constant for all values of $t$ for which $\big(x(t), y(t)\big)$ is a solution of the system (1.2). Darboux showed that a sufficient condition for a polynomial vector field of degree $n$ to possesses a first integral is that the system has at least $\frac{n(n+1)}{2} + 1$ invariant algebraic curves. In this case the first integrals can be expressed as functions of the invariant algebraic curves. Jouanolou (1979) proved that if a polynomial vector field of degree $n$ possesses at least $\frac{n(n+1)}{2} + 2$ invariant algebraic curves, then it has a rational first integral. Llibre and Pantazi, Llibre (2004) argued that, the existence of a first integral for a differential system determines the whole of the phase portrait of the system. Many authors have worked on the Darboux theory of integration for a polynomial vector field such as Schlomiuk (1993), Christopher (1994), Christopher and Llibre (1999), Christopher and Llibre (2000), Christopher and Llibre (2002), Christopher et al. (2002) and Pantazi et al. (2004).

In 2001, Ollagnier (2001) classified all invariant algebraic curves up to degree 12 for the following homogeneous quadratic Lotka-Volterra system in $\mathbb{C}^3$

$$\dot{x} = x(Cy + z), \quad \dot{y} = y(Az + x), \quad \dot{z} = z(Bx + y), \tag{1.3}$$

where $A$, $B$ and $C$ are complex parameters.

He found twenty three invariant algebraic curves including line, conic, cubic, quartic and sextic curves with an exceptional family of parameters $\Big[A = 2, \quad B = \frac{2k+1}{2k-1}, C = \frac{1}{2}\Big]$, for some integer $k$.

We are interested in studying all the twenty three cases mentioned above except the general case (Case 24), since this case has zero ratio of eigenvalues at one of the critical points of the system (1.3). We study each case individually, showing fully the behaviour of the branches of the curves at the critical points of the system (1.3). Later, Ollagnier (2004) found another invariant algebraic curve missed in the previous work. This curve has zero ratio of eigenvalues at one of its critical point, so we will not study this case either. The system (1.3) is equivalent to the normal Lotka-Voltera systems in $\mathbb{C}^2$ (Cairó et al. (2003)).

The results in Neto (1988) will also play a role in our work. We use his results to verify the fact that the sum of the ratios of eigenvalues for an invariant algebraic curve of degree $n$ is $n^2$. Authors such as Cairó and Llibre (2000) and Cairó (2000) have studied invariant algebraic surfaces in three dimensional Lotka-Volterra systems. In Cairó and Llibre (2000), the authors studied the existence of first integrals. They formed on linear and quadratic invariant surfaces for the following class of three dimensional Lotka-Volterra system

$$
\begin{aligned}
\dot{x} &= x(\lambda + Cy + z),\\
\dot{y} &= y(\mu + x + Az),\\
\dot{z} &= z(\nu + Bx + y)
\end{aligned}
$$

In Cairó (2000), the invariant algebraic surfaces for the system (1.1) up to degree two have been investigated. Twelve invariant algebraic planes have been shown with 149 invariant quadratic surfaces. However many of these cases are projectively equivalents.

We now turn to the contents of this thesis which is as follows: In Chapter 2, we review the necessary subjects related with our work. We present the notion of affine and projective algebraic sets, holomorphic foliations and the concept of invariant algebraic curves in holomorphic foliations which has a big application in this work. We also give the basic definitions, results and some proofs for the topics we considered. In Chapter 3 we study invariant algebraic hyperplanes in three and higher dimensional complex Lotka-Volterra systems. We supply the codimension associated to the different hyperplane cases. The formula is related to the non-zero terms appearing in the linear polynomial $f$, defining the hyperplane. In Chapter 4, we study the behaviour of all twenty three invariant algebraic curves found in Ollagnier (2001) mentioned above. We explain the behaviour of each curve by a graph including the ratios of eigenvalues of the corresponding system which the curve is invariant. In Chapter 5, we investigate the local integrability of two dimensional Lotka-Volterra systems at its critical points. We make use of the behaviour of some of the invariant algebraic curves which we considered in Chapter 4. In Chapter 6, we use a geometric method to classify invariant algebraic surfaces in three dimensional Lotka-Volterra systems. We use our technique to investigate many of the quadratic invariant surfaces. In the end of this chapter, we will apply our technique to reinvestigate the invariant algebraic planes that we found in Chapter 3. In Chapter 7, we commence a study of the possibilities for invariant algebraic cubic surfaces for the three dimension Lotka-Volterra systems. In this case it has been necessary to make a number of assumptions. In Chapter 8 and Chapter 9, we present respectively the main results and the personal contribution in this thesis.

# Chapter 2

# Background

# Introduction

In this chapter, we introduce and review definitions and results which are necessary for our work in this thesis. We reproof some results and showing them in different ways to seem much simpler in using for our work.

## 2.1 Affine and projective spaces

In this section, we define the notion of affine and projective algebraic sets.

### 2.1.1 Affine space and varieties

For a field, $F$, the affine $n$-dimensional space over $F$, denoted by $\mathbb{A}^n$, is defined to be

$$\mathbb{A}^n = \Big\{(a_1, \cdots, a_n)\Big|\ \ a_i \in F, \ \text{ where } \ i \in \{1, \cdots, n\}\Big\}.$$

Let $\mathbf{K}$ be the polynomial ring $F[x_1, \cdots, x_n]$. The variety, $\mathbf{V}(f)$, of a polynomial $f \in \mathbf{K}$ is the zero set of $f$, that is $\mathbf{V}(f) = \{a \in \mathbb{A}^n \mid \ f(a) = 0\}$.

For a subset $B = \{f_1, \cdots, f_r\}$ of polynomials in $\mathbf{K}$, not necessary of the same degree, the variety of $B$ is defined by

$$\begin{aligned}\mathbf{V}(B) &= \mathbf{V}(\{f_1, \cdots, f_r\})\\ &= \mathbf{V}(f_1, \cdots, f_r) = \Big\{a \in \mathbb{A}^n \mid \ f_i(a) = 0, \ \ i \in \{1, ..r\}\Big\}.\end{aligned}$$

Geometrically, $\mathbf{V}(B)$ is the solution of the system (2.1)

$$f_i(x_1, \cdots, x_n) = 0, \quad i \in \{1, \cdots, r\}. \tag{2.1}$$

A subset $X$ of $\mathbb{A}^n$ is said to be an affine algebraic set if $X = \mathbf{V}(B)$, for some $B \subseteq \mathbf{K}$. In the polynomial ring of real numbers $\mathbb{R}$, the set of all irrational numbers is not an affine algebraic set.

For an ideal $J$ in $\mathbf{K}$, the variety of $J$ is defined by $\mathbf{V}(J) = \{a \in \mathbb{A} \mid f(a) = 0, \text{ for all } f \in J\}$.

**Theorem 1** (Hilbert Basis Theorem)**.** *Any ideal $J$ in $\mathbf{K}$ is finitely generated, in other words there exists a set of polynomials $\{f_1, \cdots, f_t\}$ in $J$ such that any member of $J$ is a linear combination of $\{f_1, \cdots, f_t\}$. In this case $J$ is denoted by $J =< f_1, \cdots, f_t >$.*

For a variety, $\mathbf{V}(J)$, of an ideal $J$ in $\mathbf{K}$, the ideal, $I\big(\mathbf{V}(J)\big)$, is defined to be the set of all polynomials in $\mathbf{K}$ that vanishes on $\mathbf{V}(J)$. In other words

$$I\big(\mathbf{V}(J)\big) = \Big\{f \in \mathbf{K} \mid f(a) = 0, \text{ for all } a \in \mathbf{V}(J)\Big\}.$$

### 2.1.2 Projective spaces and projective varieties

For the $(n+1)$-dimensional vector space $F^{n+1}$, the $n$-dimensional projective space, denoted by $\mathbf{P}^n(F)$ is defined to be the set of all one-dimensional vector subspaces of $F^{n+1}$.

On the other hand, $\mathbf{P}^n(F)$ can be defined as the quotient set $F^{n+1}$- $\{0\}/\sim$, where

$\sim$ is an equivalence relation given in (2.2)

$$\sim = \Big\{(p, q) \in F^{n+1} \times F^{n+1} \mid p = \lambda q, \text{ for some } 0 \neq \lambda \in F\Big\}. \tag{2.2}$$

In other words

$$\mathbf{P^n}(F) = \Big\{(x_0 : \cdots : x_n) \mid (x_0, \cdots, x_n) \in F^{n+1}\text{-}\{0\}\Big\}.$$

We will call any point $x = (x_0 : \cdots : x_n)$ in $\mathbf{P}^n(F)$, a projective point. Geometrically, $\mathbf{P}^n(F)$ can be considered as the set of all lines passing through the origin in $F^{n+1}$.

**Theorem 2.** *The n-dimensional projective space $\mathbf{P}^n(F)$ can be covered by the charts $\{(U_i, \phi_i)\}$, where*

$$U_i = \Big\{(\cdots : x_i : \cdots) \in \mathbf{P^n}(F) \mid x_i \neq 0 \Big\},$$

*and for each $i$, $\phi_i$ is a one-to-one correspondence between $U_i$ and $F^n$.*

*Proof.* Obvious. $\square$

**Remark 1.** 1. In $\mathbf{P}^n(F)$, any projective point having a non zero coordinate $x_0$ intersects the hyperplane $\{x_0 = 1\}$ in one and only one point. Thus there is a one to one correspondence between $\{x_0 = 1\}$ and $F^n$, and hence $\mathbf{P}^n(F)$ can be identified topologically as the union of $F^n$ with the hyperplane $\{x_0 = 0\}$, which is itself is topologically equivalent to $\mathbf{P^{n-1}}(F)$.

2. A one dimensional projective space $\mathbf{P^1}(F)$ called a projective line, and it is isomorphic to the Riemann Sphere $\hat{\mathbb{C}}$.

**Definition 1.** *In $\mathbf{P}^n(F)$, the projective variety of a homogeneous polynomial $h$ in the polynomial ring $\mathbf{K}^* = F[x_0, \cdots, x_n]$ is the zero set of $h$ and is defined by*

$$\mathbf{W}(h) = \Big\{ p \in \mathbf{P^n}(F) \;\Big|\; h(p) = 0 \Big\}.$$

For a set $B = \{h_1, \cdots, h_s\}$ of homogeneous polynomials, in $\mathbf{K}^*$, not necessary of the same degree, the projective variety is given by

$$\mathbf{W}(B) = \Big\{ p \in \mathbf{P^n}(F) \;\Big|\; h_i(p) = 0, \;\; i \in \{1, \cdots, s\} \Big\}.$$

A subset $Y$ of $\mathbf{P}^n(F)$ is said to be a projective algebraic set if $Y = \mathbf{W}(B)$, for some $B \subseteq \mathbf{K}^*$.
A homogeneous ideal is the ideal, generated by a set of homogeneous polynomials in $\mathbf{K}^*$, not necessary of the same degree.

### 2.1.3 The relations between affine and projective algebraic sets

- **affine $\rightarrow$ projective**

  Let $X$ be an affine algebraic set in $\mathbb{A}^n$, so $X = \mathbf{V}(B)$, for some $B = \{f_1, \cdots, f_r\} \subseteq \mathbf{K}$. To each curve $f_i$ of degree $d_i$, we may define a homogeneous polynomial $h_i$, by adding a new variable $x_0$, such that

$$h_i(x_0, \cdots, x_n) = \sum_{j=0}^{d_i} f_j(x_1, \cdots, x_n) x_0^{d_i - j}$$

where $f_j$ is the homogeneous term of degree $j$ of $f_i$. Hence the zero set of the set of homogeneous polynomials $\{h_1, \cdots, h_r\}$ in $\mathbf{K}^*$, given a projective algebraic set in $\mathbf{P}^n(F)$. This process called the homogenisation of the affine variety to projective variety.

- **projective $\rightarrow$ affine**

  Let $Y$ be a projective algebraic set in $\mathbf{P}^n(F)$, so $Y = \mathbf{W}(B)$, for some $B = \{h_1, \cdots, h_t\}$ of homogeneous polynomial in $\mathbf{K}^*$. Fixed a chart $(U_i, \Phi_i)$ and then for each $h_j$, we may define a polynomial $f_j$ as a restriction of $h_j$ at $x_i = 1$ . Hence $f_j$ is a member of a polynomial ring $\mathbf{K_i} = F[\cdots, x_{i-1}, x_{i+1}, \cdots]$. Consequently the zero set of $\{f_1, \cdots, f_t\}$ gives an affine algebraic set. This process called the dehomogenisation of the projective variety with respect to the variable $x_i$ to the affine variety. (Cox et al. (1992)).

  Notice that, we may dehomogenise with respect to other variables, consequently, homogenisation and dehomogenisation are not isomorphic.

## 2.2 Holomorphic foliation

A holomorphic foliation, $\Im$ of the complex projective plane $\mathbf{P^2}(\mathbb{C})$, can be defined as follows

**(I)** In the affine complex plane $\mathbb{C}^2$, by either

- a polynomial differential equation system

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q(x,y), \end{aligned} \tag{2.3}$$

or

- a vector field

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \tag{2.4}$$

or

- a 1-form

$$P(x,y)dy - Q(x,y)dx = 0, \tag{2.5}$$

where, $P(x,y)$ and $Q(x,y)$ are co-prime polynomials in $\mathbf{K} = \mathbb{C}[x,y]$ not necessary of the same degree.

**(II)** In projective complex plane $\mathbf{P}^2(\mathbb{C})$, by a 1-form

$$A(X,Y,Z)dX + B(X,Y,Z)dY + C(X,Y,Z)dZ = 0, \tag{2.6}$$

where

$$A(X,Y,Z), \quad B(X,Y,Z) \quad \text{and} \quad C(X,Y,Z),$$

are homogeneous polynomials of the same degree such that

$$XA(X,Y,Z) + YB(X,Y,Z) + ZC(X,Y,Z) = 0. \tag{2.7}$$

**Theorem 3.** *Any holomorphic foliation $\Im$ in the affine complex plane $\mathbb{C}^2$ corresponds to a holomorphic foliation in the projective complex plane $\mathbf{P}^2(\mathbb{C})$ and vice versa.*

*Proof.* The proof is straightforward. $\square$

### 2.2.1 The degree of a holomorphic foliation

By a solution of the holomorphic foliation $\Im$ defined by the 1-form (2.5), we mean a pair of analytic function $(x, y) = \big(x(t), y(t)\big)$ such that

$$P\big(x(t), y(t)\big)\frac{d}{dt}y(t) = Q\big(x(t), y(t)\big)\frac{d}{dt}x(t).$$

The zero solution of (2.5) is the affine algebraic set $\mathbf{V}(P, Q)$. Points in $\mathbf{V}(P, Q)$ are called the critical points of $\Im$, denoted by $S(\Im)$. Non trivial solutions of (2.5) over the complex domain are called the leaves of $\Im$.

The solution of a holomorphic foliation in term of algebraic curve $C_f$, defined by a polynomial $f = 0$ is given by Definition 2.

**Definition 2.** *An algebraic curve $C_f$ is called a solution of a holomorphic foliation $\Im$ if $C_f - S(\Im)$ is a leaf of $\Im$.*

**Definition 3.** *Consider a holomorphic foliation $\Im$ in $\mathbf{P}^2(\mathbb{C})$ for which the projective line $L = 0$ is not a solution. A point $p \in L$ is said to be a tangency point of $\Im$ with respect to $L$ if either $p \in S(\Im)$ or the tangent space of $L$ at $p$ and the leaf of $\Im$ through $p$, are coincide. (Neto (1988))*

**Theorem 4.** *Let $\Im$ be a holomorphic foliation in $\mathbf{P}^2(\mathbb{C})$, for which a projective line $L = 0$ is not an algebraic solution. A point $p \in L$ is a tangency point of $\Im$ if* *and only if there exist* $t$*, that satisfies the equation (2.8)*

$$bP\big(\psi(t)\big) = aQ\big(\psi(t)\big), \tag{2.8}$$

*where* $\psi(t) = (x_0 - at, y_0 - bt)$ *is a parametrization of the affine version of the line* $L$*.*

*Proof.* See Neto (1988). □

**Definition 4.** *The degree of a holomorphic foliation* $\Im$*, denoted by* $|(\Im, L)|$ *is given in* (*2.9*)

$$|(\Im, L)| = \sum_{p \in L} mult_p(\Im, L), \tag{2.9}$$

*where* $mult_p(\Im, L)$*, is the multiplicity of the root* $t$ *such that* $p = \psi(t)$*.*

## 2.3 Invariant algebraic curves of holomorphic foliations in $\mathbf{P}^2(\mathbb{C})$

An algebraic curve $C_f = \big\{f \big| f(x, y) = 0\big\}$ is said to be invariant with respect to a holomorphic foliation $\Im$ if, $\chi(f) = fL_f$, for some polynomial $L_f$, called the cofactor of $f$; Moreover, $L_f$ is a polynomial of degree at most $deg(\Im) - 1$.

**Remark 2.** Since the points of $\mathbf{P}^2(\mathbb{C})$, are lines in $\mathbb{C}^3$ passing through the origin, then every line intersects the plane, $\{z = 1\}$, in a point $(x, y, 1)$, for some $x, y \in \mathbb{C}^2$, so the coordinates of any line can be given by $(x : y : 1)$ except the lines having $z = 0$ as a third coordinate. In fact, the projective plane points intersect the plane $\{z = 1\}$, can be described by a pair $(x, y)$ in the $xy$-plane. Points $p$ in the projective plane, which lie in $\{z = 0\}$ have the coordinates $p = (x : y : 0)$. Without loss of generality, if $x \neq 0$, then $\left(1 : \frac{y}{x} : 0\right)$ is another formulation of $p$. Hence by assuming $\frac{y}{x}$, as another axis, we may express $\mathbf{P}^2(\mathbb{C})$ geometrically by exactly three axes, as shown in Figure (2.1)

In Figure (2.1), any point determined by the yellow and green lines $\{x = 0\}$ and $\{y = 0\}$ respectively has a coordinate $(x : y : 1)$ or briefly $(x : y)$ and we will call it a finite point of $\mathbf{P}^2(\mathbb{C})$. The set of all finite points constitute the finite part or the affine part of $\mathbf{P}^2(\mathbb{C})$.

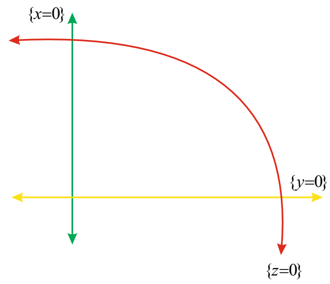



**Figure 2.1:** $\mathbf{P}^2(\mathbb{C})$.

For the red line $\{z = 0\}$. Without loss of generality, let $x \to \infty$, hence the coordinates of the points on the red line are given by $\left(1 : \frac{y}{x} : 0\right)$ or briefly $\left(0 : \frac{y}{x}\right)$, which is the projective line $\mathbf{P}^1(\mathbb{C})$. Any point located on the line $\{z = 0\}$ is called a point at infinity and the line $\{z = 0\}$ is called the line at infinity or the infinite part of $\mathbf{P}^2(\mathbb{C})$, denoted by $L_\infty$.

Recall the following results from Neto (1988)

**Theorem 5.** *Let $\Im$ be a holomorphic foliation in $\mathbf{P}^2(\mathbb{C})$, then the line at infinity is invariant with respect to $\Im$ if and only if*

$$yP_k(x, y) - xQ_k(x, y) \neq 0, \tag{2.10}$$

*where $P_k(x, y)$ and $Q_k(x, y)$ are the homogeneous terms of degree $k$ of $P(x, y)$ and*

*$Q(x,y)$ respectively and k=max {deg (P),deg (Q)}.*

*Proof.* The proof can be found in Neto (1988). □

**Remark 3.** To study the vector field at infinity, we may use the change of coordinates (2.11) as $x \to \infty$

$$Z = \frac{1}{x}, \quad Y = \frac{y}{x}, \tag{2.11}$$

or, the change of coordinates (2.12) as $y \to \infty$

$$X = \frac{x}{y}, \quad Z = \frac{1}{y}. \tag{2.12}$$

**Theorem 6.** *Let $\Im$ be a holomorphic foliation in $\mathbf{P}^2(\mathbb{C})$ defined by the 1-form given in (2.13)*

$$P(x,y)dy - Q(x,y)dx = 0, \tag{2.13}$$

*where k=max{deg (P),deg (Q)}. Then, the degree of $\Im$ is equal to $k$ if and only if $L_\infty$ is invariant; Moreover, the degree of $\Im$ is either $k$ or $k-1$.*

*Proof.* The proof can be found in Neto (1988). □

**Corollary 1.** *Let $\Im$ be a holomorphic foliation in $\mathbf{P}^2(\mathbb{C})$ defined by the 1 -form (2.13). Then the degree of $\Im$ is equal to $k$ if and only if $yP_k(x,y) - xQ_k(x,y) \neq 0$.*

*Proof.* Follows directly from Theorem 5 and Theorem 6. □

**Lemma 1.** *Any holomorphic foliation $\Im$ of degree $n$ can be written as the 1-form* *given in (2.14)*

$$(p + xg)dy - (q + yg)dx = 0, \tag{2.14}$$

*where $p$ and $q$ are polynomials of degree at most $n$ and $g$ is a homogeneous polynomial of degree $n$.*

*Proof.* Let $\Im$ defined by the following 1-form

$$P(x, y)dy - Q(x, y)dx = 0, \tag{2.15}$$

Let $d$ =max{deg $(P)$, deg $(Q)$)}, then from Theorem 6, either $d = n+1$ or $d = n$.

- Let $d = n + 1$, then there exist polynomials $p$ and $q$ of degree at most $n$ such that

  $$P(x, y) = P_{n+1}(x, y) + p,$$
  $$Q(x, y) = Q_{n+1}(x, y) + q,$$

  where $P_{n+1}$ and $Q_{n+1}$ are the homogeneous terms of $P$ and $Q$ of degree $n + 1$ respectively.
  By using Corollary 1, we have

  $$yP_{n+1}(x, y) - xQ_{n+1}(x, y) = 0, \tag{2.16}$$

  so, $y^{n+1}$ and $x^{n+1}$ are not terms in $P_{n+1}(x, y)$ and $Q_{n+1}(x, y)$ respectively.

Consequently

$$
\begin{aligned}
P_{n+1}(x, y) &= xR(x, y) \\
Q_{n+1}(x, y) &= yS(x, y),
\end{aligned}
\tag{2.17}
$$

for some homogeneous polynomials $R$ and $S$ of degree $n$.

By substituting (2.17) in (2.16), we get $xy(R - S) = 0$, hence $R = S$.

- For the case, where $d = n$, the proof is trivial since we may choose $g$ as a zero polynomial.

□

**Remark 1.** Generally there is no any relation between the degree of a holomorphic foliation and the degree of their invariant algebraic curves, for example the algebraic curve $C_f$ defined by $f = y^a - x^b$ is invariant with respect to the following holomorphic foliation

$$
axdy - bydx = 0 \tag{2.18}
$$

where $a$ and $b$ are distinct positive integers.

**Definition 5.** *Let* $\Im$ *be a holomorphic foliation of degree* $n$ *in* $\mathbf{P}^2(\mathbb{C})$ *defined by the vector field* $\chi = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$*,* $C_f$ *an invariant algebraic curve with respect to* $\Im$ *of degree* $m$ *defined by a polynomial* $f(x, y) = 0$*,* $p \in C_f$ *a critical point of* $\Im$ *and* $\phi : U \to \mathbb{C}^2$ *a local parametrization of a branch* $B$ *at the point* $p$*, where* $U$ *is an open disk in* $\mathbb{C}$ *such that* $\phi(0) = (0, 0)$*.*

*The multiplicity of* $\Im$ *at* $B$ *is defined as the order of the pullback of the vector field* $\chi$ *and denoted by* $i(\Im, B)$*.*

**Remark 4.** Let $\phi : U \to \mathbb{C}^2$ be a smooth map such that $\phi(t) = \big(\alpha_1(t), \alpha_2(t)\big)$, where $U$ is an open disk in $\mathbb{C}$. The pullback of the vector field $\chi$ in (2.4) is defined by

$$\begin{aligned} f(t) &= \frac{P\big(x, y\big)(\phi(t))}{\partial t(\alpha_1(t))} \\ &= \frac{P\big(x(t), y(t)\big)}{\partial t(\alpha_1(t))}, \end{aligned} \tag{2.19}$$

or, equivalently

$$\begin{aligned} f(t) &= \frac{Q\big(x, y\big)(\phi(t))}{\partial t(\alpha_2(t))} \\ &= \frac{Q\big(x(t), y(t)\big)}{\partial t(\alpha_2(t))}. \end{aligned}$$

**Theorem 7.** *Let $\Im$ be a holomorphic foliation of degree $n$ in $\mathbf{P}^2(\mathbb{C})$ and $C_f$ an invariant algebraic curve with respect to $\Im$ defined by an irreducible polynomial $f(x, y) = 0$, of degree $m$, then*

$$2 - 2g(C_f) = \sum_{B \in \Delta} i(\Im, B) - m(n - 1),$$

*where $\Delta$ is the set of all local branches of $\Im$ passes through the critical points of $\Im$ and $g(C_f)$ is the geometric genus of the curve $C_f$.*

*Proof.* Assume the representation (2.20) given in Lemma 1 for $\Im$

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) = p(x, y) + xh(x, y), \\ \frac{dy}{dt} &= Q(x, y) = q(x, y) + yh(x, y), \end{aligned} \tag{2.20}$$

where $p(x,y)$ and $q(x,y)$ are polynomials of degree at most $n$ and $h(x,y)$ is homogeneous of degree $n$. We may write

$$p(x,y) = p_n(x,y) + \cdots + p_0(x,y), \quad q(x,y) = q_n(x,y) + \cdots + q_0(x,y),$$

where $p_i(x,y)$ and $q_i(x,y)$ are the homogeneous terms of degree $i$ of $p(x,y)$ and $q(x,y)$ respectively. Without loss of generality, let the invariant algebraic curve $C_f$ intersect the line at infinity transversely. Assume the change of coordinates given in (2.11), hence the corresponding vector field of (2.20) is given in (2.21)

$$\tilde{\chi} = \Big( -Z^2 p\Big(\frac{1}{Z}, \frac{Y}{Z}\Big) - Zh\Big(\frac{1}{Z}, \frac{Y}{Z}\Big)\Big)\frac{\partial}{\partial Z} + \Big(Zq\Big(\frac{1}{Z}, \frac{Y}{Z}\Big) - ZYp\Big(\frac{1}{Z}, \frac{Y}{Z}\Big)\Big)\frac{\partial}{\partial Y}. \tag{2.21}$$

or, equivalently

$$\tilde{\chi} = \frac{\Big( -Z\tilde{p}(Z,Y) - \tilde{h}(Z,Y)\Big)\frac{\partial}{\partial Z} + \Big(\tilde{q}(Z,Y) - Y\tilde{p}(Z,Y)\Big)\frac{\partial}{\partial Y}}{Z^{n-1}}, \tag{2.22}$$

where

$$\tilde{p}(Z,Y) = -Z^n p\left(\frac{1}{Z}, \frac{Y}{Z}\right),\ \tilde{q}(Z,Y) = -Z^n q\left(\frac{1}{Z}, \frac{Y}{Z}\right) \text{ and } \tilde{h}(Z,Y) = -Z^n h\left(\frac{1}{Z}, \frac{Y}{Z}\right).$$

Since the line at infinity intersect the curve $C_f$ transversely, then the constant term of the part $-Z\tilde{p}(Z,Y) - \tilde{h}(Z,Y)$ is a non zero. In other words

$$-Z\tilde{p}(Z,Y) - \tilde{h}(Z,Y) = c + A(Z,Y), \quad c \neq 0.$$

Then for any local branch

$$\phi(t) = \Big(Z(t), Y(t)\Big) = (t, \alpha_1 t + \alpha_2 t^2 + \cdots),$$

passes through the origin, we get the pullback vector field given in (2.23)

$$f(t) = \frac{c + A(t, \alpha_1 t + \alpha_2 t^2 + \cdots)}{t^{n-1}}. \tag{2.23}$$

Consequently $f(t)$ is a meromorphic vector field having poles of order $n-1$. By hypothesis the invariant curve $C_f$ is of degree $m$, then we must subtract the quantity $m(n-1)$ from the Euler characteristic of the curve $C_f$. Hence by applying Poincaré-Hopf formula, we get the proof. □

### 2.3.1 Branches and index of the branches

For a curve $C_f$ defined by a polynomial $f$, the variety $\mathbf{V}(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is called the set of all singular points of $C_f$. Any non-singular point of $C_f$ is called a regular point of $C_f$.

More details about the topic below can be found in Neto (1988).

Let $\Im$ be a holomorphic foliation and $C_f$ be an algebraic solution of $\Im$ defined by the polynomial $f$. Suppose $f = f_1 \cdots f_r$ is a decomposition of $f$. For a critical point $p$ of $\Im$ belongs to the algebraic solution $C_f$, the branch $B_i$ of $f$ at $p$ is defined as follows

$$\begin{aligned} B_i = \big\{ & q \in U \,|\, f_i(q) = 0, \text{where } U \text{ is an open neighborhood of } p \text{ having} \\ & \text{the coordinates } (x, y) \text{ such that } x(p) = y(p) = 0 \big\}. \end{aligned}$$

The branch $B_i$ defined by an irreducible polynomial $f_i$ called smooth if $\frac{\partial f_i}{\partial x}\big|_p \neq 0$ or $\frac{\partial f_i}{\partial y}\big|_p \neq 0$, otherwise $B_i$ is called singular. For any regular point of $C_f$ there is a unique branch.

We will present the notion of index of the branches.

Consider the 1-form representation of $\Im$. To each branch $B$ mentioned above, there corresponds a complex number called the index of $B$ defined in terms of some residue, let $I_p(B, \Im)$ denotes the index of the branch $B$ at $p$ with respect to $\Im$. If $p$ is a regular point of $C_f$, then

$$I_p(B, \Im) = Residue\Big(\frac{Q(x,0)}{P(x,0)}, x = 0\Big). \tag{2.24}$$

If $P(x,y)dx - Q(x,y)dy = 0$ is a holomorphic foliation such that the Jacobian of $P$ and $Q$ at $(0,0)$ has eigenvalues $\mu_1 \neq 0$ and $\mu_2$, where the eigenvector relative to $\mu_1$ coincides with the tangent space of $B$, then $I_{(0,0)}(B, \Im)$ is defined to be the ratio of eigenvalues $\frac{\mu_2}{\mu_1}$. In the case where $p$ is singular and from the results in Neto (1988), we conclude that the index can be given as follows

$$I_p(B, \Im) = Residue\Big(\frac{L_f\big(x(t), y(t)\big)}{P\big(x(t), y(t)\big)}, t = 0\Big), \tag{2.25}$$

where $L_f$ is the cofactor of $f$ and $\big(x(t), y(t)\big)$ is a local parametrisation of $f$ in a small neighbourhood of $p$.

On the other hand, for the case when we have a number of smooth branches, $B_i$, $i \in \{1, ..., k\}$, of $f$ at $p$, the index can be computed from the following formula

$$\sum_{i=1}^{k-1}\Big(\sum_{j=i+1}^{k} mult_P(B_i, B_j) + r_i + r_j\Big), \tag{2.26}$$

where $mult_p(B_i, B_j)$ is the multiplicity of the intersection between the branches $B_i$ and $B_j$, while $r_i$ and $r_j$ are the ratios of eigenvalues at $p$ with respect to $B_i$ and $B_j$ respectively.

The following theorems will play a role in our work.

**Theorem 8.** *The sum of the indices of an invariant algebraic curve $C_f$ of degree $n$ with respect to a holomorphic foliation $\Im$ is equal to $n^2$.*

*Proof.* The proof can be found in Neto (1988). □

**Theorem 9.** *If the equation of a curve $C_f$ is of degree $n$, a line is either is a component of $C_f$ or has precisely $n$ points of intersection with $C_f$ (properly counted).*

*Proof.* The proof can be found in Walker (1950). □

## 2.4 Lotka-Volterra system of equations

The $n$-dimensional Lotka-Volterra systems in the complex domain is defined by

$$\dot{x}_i = \frac{dx_i}{dt} = P_i(x_1, \cdots, x_n) = x_i\Big(\lambda_i + \sum_{j=1}^{n} a_{ij}x_j\Big), \quad i \in \{1, \cdots, n\}. \tag{2.27}$$

In this section we are interesting in two dimensional Lotka-Volterra systems. Consider the following two dimensional Lotka-Volterra system

$$\begin{aligned} \dot{x} &= P(x, y) = x(1 + ax + by), \\ \dot{y} &= Q(x, y) = y(\lambda + cx + dy). \end{aligned} \tag{2.28}$$

The system (2.28) is said to be degenerate if the polynomials $P(x, y)$ and $Q(x, y)$ are co-primes. In Figure 2.2, we assume that all the parameters in (2.28) are

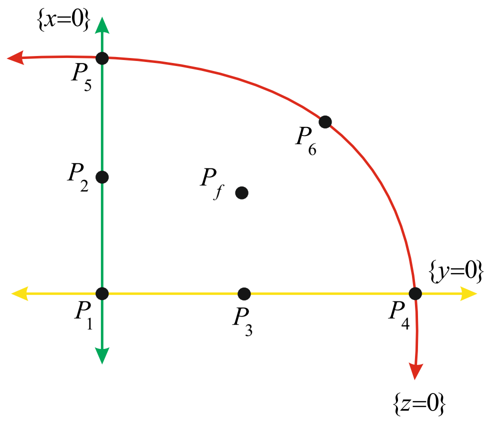



**Figure 2.2:** Critical points of (2.28).

non zero. Hence in $\mathbf{P}^2(\mathbb{C})$, the system (2.28) has exactly seven critical points as follows: $P_1 = (0, 0)$, $P_2 = (0, y_1)$, $P_3 = (x_1, 0)$ and $P_f = (x^*, y^*)$, where $x_1$, $y_1$, $x^*$ and $y^*$ are non zero numbers. All those points are located in the affine part of $\mathbf{P}^2(\mathbb{C})$ and the others are at infinity for which the critical $P_4$ located at the intersection position of the lines $\{y = 0\}$ with $\{z = 0\}$, while $P_5$ located in the intersection position of the lines $\{x = 0\}$ with $\{z = 0\}$. The other critical point at infinity has coordinates $P_6 = (0, Y_1)$ or $P_6 = (X_1, 0)$, where $X_1 \neq 0 \neq Y_1$ as $x \to \infty$ or $y \to \infty$ respectively. We call the critical points $P_1$, $P_4$ and $P_5$ the corner points, $P_2$ and $P_3$ the finite side critical points and $P_6$ the side critical point at infinity. We call the point $P_f$ the face critical point. The position of $P_f$ is various and depends on the parameters of (2.28). It is easy to verify that the lines $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$ are invariant lines with respect to the (2.28).

# Chapter 3

# The codimension of Families of Invariant Algebraic Planes and Hyperplanes

# Introduction

This chapter is divided into two sections. In the first section, we investigate invariant algebraic planes and hyperplanes in both three and four dimensional Lotka-Volterra systems. In the second section, we find a relation between the codimension of an affine variety with the number of non-zero terms appearing in an invariant hyperplane with respect to the $n$-dimensional Lotka-Volterra systems.

## 3.1 Invariant algebraic hyperplanes in Lotka-Volterra systems

In the next sections, we will investigate the codimension of invariant algebraic planes and hyperplanes in three and four dimensional Lotka-Volterra systems

### 3.1.1 Invariant algebraic planes in three dimensional Lotka-Volterra systems

Consider the following three dimensional Lotka-Volterra system

$$\dot{x_i} = P_i(x_1, x_2, x_3) = x_i\left(\lambda_i + \sum_{j=1}^{3} a_{ij}x_j\right), \quad i \in \{1, 2, 3\}. \tag{3.1}$$

Let $C_f = \left\{ f \mid f(x_1, x_2, x_3) = b_0 + \sum_{i=1}^{3} b_i x_i = 0 \right\}$, be an invariant algebraic plane with respect to the system (3.1). So we have

$$\chi(f) = f L_f, \tag{3.2}$$

where $\chi$ is the corresponding vector field of the system (3.1) and $L_f = c_0 + \sum_{i=1}^{3} c_i x_i$ is the cofactor of $f$. Clearly the space of the equation (3.2) containing the parameters $\lambda_i$ and $a_{ij}$ for $i, j \in \{1, 2, 3\}$, while the space of $C_f$ with its cofactor $L_f$ include the parameters $b_i$ and $c_i$ respectively, where $i \in \{0, 1, 2, 3\}$.

From equation (3.2), we get the following equations

$$\begin{aligned}
&a_{11} b_1 = b_1 c_1, \\
&a_{12} b_1 + a_{21} b_2 = b_1 c_2 + b_2 c_1, \\
&b_1 a_{13} + a_{31} b_3 = b_1 c_3 + b_3 c_1, \\
&\lambda_1 b_1 = b_0 c_1 + b_1 c_0, \\
&a_{22} b_2 = b_2 c_2, \\
&a_{23} b_2 + a_{32} b_3 = b_2 c_3 + b_3 c_2, \\
&\lambda_2 b_2 = b_0 c_2 + b_2 c_0, \\
&a_{33} b_3 = b_3 c_3, \\
&\lambda_3 b_3 = b_0 c_3 + b_3 c_0, \\
&b_0 c_0 = 0.
\end{aligned} \tag{3.3}$$

Now for the equations (3.3), we split into a number of cases depend on the number of non zero coefficients in $f$.

I. The case of exactly one of the coefficients is non-zero trivial, since we get either a constant or a one of the coordinate planes which already are invariants.

II. Exactly two of the coefficients are non zero.
For $b_0$, either $b_0 \neq 0$ or $b_0 = 0$. Later, we will show that both cases are equivalents under a projective transformation. Assume the case where $b_0 \neq 0$. Without loss of generality, let $b_1 \neq 0$ and $b_2 = b_3 = 0$. Hence $C_f = \{f \mid f = b_0 + b_1 x_1\}$, then from (3.3), we get the following equations

$$\begin{aligned} &c_0 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_1 = a_{11}, \\ &a_{12} = 0, \quad a_{13} = 0 \text{ and } b_0 a_{11} - b_1 \lambda_1 = 0. \end{aligned} \tag{3.4}$$

Let II be the ideal generated by (3.4), then clearly $\mathbf{V}(\mathrm{II})$ has codimension two. The invariant plane $C_f$ in this case is parallel to the $x_2x_3$-plane.

III. Exactly three of the coefficients are non zero.
Similarly as we have declared in II. above, we consider the case where $b_0 \neq 0$. Without loss of generality, let $b_1 \neq 0 \neq b_2$ and $b_3 = 0$. Hence $C_f = \{f \mid f = b_0 + b_1 x_1 + b_2 x_2\}$, then from (3.3), we get the following equations

$$\begin{aligned} &a_{13} = 0, \quad a_{23} = 0, \quad c_0 = 0, \quad c_3 = 0, \quad c_1 = a_{11}, \\ &c_2 = a_{22} \text{ and } a_{11}a_{12}\lambda_2 + \lambda_1 a_{21} a_{22} = a_{11}a_{22}(\lambda_1 + \lambda_2). \end{aligned} \tag{3.5}$$

Let III be the ideal generated by (3.5), then $\mathbf{V}$(III) has codimension three. In this case, the invariant plane $C_f$ is parallel to the $x_3$-axis.

IV. All coefficients of $f$ are non zero.

Hence from system (3.3), we get the following equations

$$\begin{aligned}
&c_0 = 0, \quad c_1 = a_{11}, \quad c_2 = a_{22}, \quad c_3 = a_{33}, \\
&a_{11}a_{12}\lambda_2 + \lambda_1 a_{21}a_{22} = a_{11}a_{22}(\lambda_1 + \lambda_2), \\
&a_{11}a_{13}\lambda_3 + \lambda_1 a_{31}a_{33} = a_{11}a_{33}(\lambda_1 + \lambda_3), \\
&a_{22}a_{23}\lambda_3 + \lambda_2 a_{32}a_{33} = a_{22}a_{33}(\lambda_2 + \lambda_3).
\end{aligned} \tag{3.6}$$

Let IV be the ideal generated by the system (3.6), then $\mathbf{V}$(IV) has codimension three. In this case, the invariant plane $C_f$ intersect each axis in a point different from the origin.

### 3.1.2 Invariant algebraic hyperplanes in four dimensional Lotka-Volterra systems

Suppose the following four dimensional Lotka-Volterra system

$$\dot{x_i} = P_i(x_1, x_2, x_3, x_4) = x_i\left(\lambda_i + \sum_{j=1}^{4} a_{ij}x_i\right), \quad i \in \{1, 2, 3, 4\}. \tag{3.7}$$

Let $C_f = \left\{f \mid f(x_1, x_2, x_3, x_4) = b_0 + \sum_{i=1}^{4} b_i x_i\right\}$ be an invariant algebraic hyperplane with respect to the system (3.7). Similarly as we have done for invariant algebraic planes and by using the corresponding vector field of (3.7) with the cofactor $L_f = c_0 + \sum_{i=1}^{4} c_i x_i$, we get the following equations

$$\begin{aligned}
&a_{11}b_1 = b_1c_1,\\
&a_{12}b_1 + a_{21}b_2 = b_1c_2 + b_2c_1,\\
&a_{13}b_1 + a_{31}b_3 = b_1c_3 + b_3c_1,\\
&a_{14}b_1 + a_{41}b_4 = b_1c_4 + b_4c_1,\\
&\lambda_1b_1 = b_0c_1 + b_1c_0,\\
&a_{22}b_2 = b_2c_2,\\
&a_{23}b_2 + a_{32}b_3 = b_2c_3 + b_3c_2,\\
&a_{24}b_2 + a_{42}b_4 = b_2c_4 + b_4c_2,\\
&\lambda_2b_2 = b_0c_2 + b_2c_0,\\
&a_{33}b_3 = b_3c_3,\\
&a_{34}b_3 + a_{43}b_4 = b_3c_4 + b_4c_3,\\
&\lambda_3b_3 = b_0c_3 + b_3c_0,\\
&a_{44}b_4 = b_4c_4,\\
&\lambda_4b_4 = b_0c_4 + b_4c_0,\\
&b_0c_0 = 0.
\end{aligned} \tag{3.8}$$

Notice that, the space of the equations (3.8) containing the parameters $\lambda_i$ and $a_{ij}$ for $i, j \in \{1, 2, 3, 4\}$, while the space of an invariant algebraic hyperplane $C_f$ with its cofactor $L_f$ include the parameters $b_i$ and $c_i$ respectively, where $i \in \{0, 1, 2, 3, 4\}$.

For the equations (3.8), we split into a number of cases depend on the number of non zero coefficients of $C_f$. Similarly as the cases of invariant algebraic planes, we assume $b_0 \neq 0$, in other words we will take one possibility for each case.

I. The case for which exactly only one of the coefficients is non-zero trivial by similar way as we have explained in the case of invariant algebraic planes.

II. Exactly two of the coefficients are non zero.

Without loss of generality, let $b_1 \neq 0$ and $b_2 = b_3 = b_4 = 0$.

Hence $C_f = \{f \mid f = b_0 + b_1 x_1\}$, then from system (3.8) we get the following equations

$$\begin{aligned} &c_0 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0, \quad c_1 = a_{11}, \\ &a_{12} = 0, \quad a_{13} = 0, \quad a_{14} = 0 \text{ and } b_0 a_{11} - b_1 \lambda_1 = 0. \end{aligned} \tag{3.9}$$

Let II be the ideal generated by (3.9), then $\mathbf{V}(\mathrm{II})$ has codimension three. The invariant hyperplane $C_f$ in this case is parallel to the 3-fold $x_2 x_3 x_4$.

III. Exactly three of the coefficients are non zero.

Without loss of generality, let $b_1 \neq 0 \neq b_2$ and $b_3 = b_4 = 0$. Hence $C_f = \{f \mid f = b_0 + b_1 x_1 + b_2 x_2\}$, then from system (3.8) we get the following equations

$$\begin{aligned} &c_0 = 0, \quad c_3 = 0, \quad c_4 = 0, \quad c_1 = a_{11}, \quad c_2 = a_{22}, \\ &a_{13} = 0, \quad a_{14} = 0, \quad a_{23} = 0, \quad a_{24} = 0, \\ &a_{11} a_{12} \lambda_2 + a_{21} a_{22} \lambda_1 = a_{11} a_{22} (\lambda_1 + \lambda_2). \end{aligned} \tag{3.10}$$

Let III be the ideal generated by (3.10), then $\mathbf{V}(\mathrm{III})$ has codimension five. The invariant hyperplane $C_f$ in this case is parallel to the $x_3 x_4$-hyperplane and not passes through the origin.

IV. Exactly four of the coefficients are non zeros.

We consider only the case where $b_i \neq 0,\ i \in \{0,1,2,3\}$ and $b_4 = 0$. Hence $C_f = \{f \mid f = b_0 + \sum_{i=1}^{3} b_i x_i\}$, then from the system (3.8) we get the following equations

$$
\begin{aligned}
& c_0 = 0, \quad c_4 = 0, \quad c_1 = a_{11}, \quad c_2 = a_{22}, \quad c_3 = a_{33}, \\
& a_{14} = 0, \quad a_{24} = 0, \quad a_{34} = 0, \\
& a_{11}a_{12}\lambda_2 + \lambda_1 a_{21}a_{22} = a_{11}\lambda_2 a_{22} + \lambda_1 a_{11}a_{22}, \\
& a_{11}a_{13}\lambda_3 + \lambda_1 a_{31}a_{33} = a_{11}\lambda_3 a_{33} + \lambda_1 a_{11}a_{33}, \\
& a_{22}a_{23}\lambda_3 + \lambda_2 a_{32}a_{33} = a_{22}\lambda_3 a_{33} + \lambda_2 a_{22}a_{33}.
\end{aligned}
\tag{3.11}
$$

Let IV be the ideal generated by (3.11), then $\mathbf{V}(\mathrm{IV})$ has codimension six. In this case, the intersection of the invariant hyperplane $C_f$ with each one of the $x_i x_4$-hyperplanes, for $i \in \{1,2,3\}$ does not contain the $x_4$-axis.

V. All the coefficients of $f$ are non zero.

Hence from system (3.8), we get the following equations

$$
\begin{aligned}
& c_0 = 0, \quad c_1 = a_{11}, \quad c_2 = a_{22}, \quad c_3 = a_{33}, \quad c_4 = a_{44}, \\
& a_{11}a_{12}\lambda_2 + \lambda_1 a_{21}a_{22} = a_{11}a_{22}(\lambda_1 + \lambda_2), \\
& a_{11}a_{13}\lambda_3 + \lambda_1 a_{31}a_{33} = a_{11}a_{33}(\lambda_1 + \lambda_3), \\
& a_{11}a_{14}b_0 + \lambda_1 b_1 a_{44} = a_{11}a_{44}(\lambda_1 + \lambda_4), \\
& a_{22}a_{23}\lambda_3 + \lambda_2 a_{32}a_{33} = a_{22}a_{33}(\lambda_2 + \lambda_3), \\
& a_{22}a_{24}b_0 + \lambda_2 b_2 a_{44} = a_{22}a_{44}(\lambda_2 + \lambda_4), \\
& a_{33}a_{34}b_0 + \lambda_3 b_3 a_{44} = a_{33}a_{44}(\lambda_3 + \lambda_4).
\end{aligned}
\tag{3.12}
$$

Let V be the ideal generated by (3.12), then $\mathbf{V}(\mathrm{V})$ has codimension six. In this case, the intersection of the invariant algebraic hyperplane $C_f$ with each axis is not the origin.

## 3.2 Codimension formula of an affine variety by an invariant hypersurfac

In this section, we investigate a formula about the codimension of affine varieties. First we need mention the concept of projective equivalence among affine varieties.

Recall the following definitions and results from Cox et al. (1992)

Consider $\mathbf{P^n}(F)$ and a non singular $(n+1)$-dimensional matrix $T$

$$T = \begin{pmatrix} c_{00} & \cdots & c_{0n} \\ . & . & . \\ c_{n0} & \cdots & c_{nn} \end{pmatrix}.$$

Since any point $p \in \mathbf{P^n}(F)$ can be written as an $(n+1) \times 1$ matrix, then $T$ can be considered as a transformation on $\mathbf{P^n}(F)$ such that for any $p = (x_0 : \cdots : x_n) \in \mathbf{P^n}(F)$

$$T(p) = \begin{pmatrix} \sum_{i=0}^{n} \frac{c_{0i}}{\mu} x_i & \cdots \sum_{i=0}^{n} \frac{c_{ni}}{\mu} x_i \end{pmatrix}.$$

$T$ is called a projective transformation on $\mathbf{P^n}(F)$ and it is exactly the same which divided by a non-zero constant $\mu$.

**Remark 5.** Any projective transformation we are interested in are exactly the same as we have explained in Chapter 2, Section 2.1.3.

From the following theorem, we show a formula related to the codimanesion of affine varieties.

**Theorem 10.** *Let the hyperplane $C_f$ defined by the polynomial*

$$f = b_0 + \sum_{i=1}^{n} b_i x_i. \tag{3.13}$$

*be invariant with respect to the $n$-dimensional non-degenrate Lotka-Volterra system (3.14)*

$$\dot{x}_i = x_i\big(\lambda_i + \sum_{j=1}^{n} a_{ij} x_j\big), \quad i \in \{1, ..., n\}, \tag{3.14}$$

*where $\lambda_i$ and $a_{ij}$ are constants in an arbitrary field $F$.*
*If $f$ has exactly $r$ non-zero terms for $1 \le r \le n+1$, then the codimension of the variety, $\mathbf{V}(J)$, where $J$ is the ideal generated by the equations introduced from the system 3.15*

$$\nu(f) = f L_f, \tag{3.15}$$

*can be given by the formula (3.16)*

$$\frac{(2n-r)(r-1)}{2}, \tag{3.16}$$

*where $\nu$ is the corresponding vector field of the system (3.14) and $L_f = c_0 + \sum_{i=1}^{n} c_i x_i$ is the cofactor of $f$.*

*Proof.* From (3.15), we get

$$b_0 c_0 = 0.$$

- $b_0 \neq 0$. So the polynomial $f$ can be rewritten as (3.17)

$$f = 1 + \sum_{i=1}^{r-1} b_i x_i. \tag{3.17}$$

Notice that in (3.17), each $b_i$ is not necessary to be the same $b_i$'s appearing in (3.13). Hence the equation (3.15) can be written as

$$\sum_{i=1}^{r-1} b_i \lambda_i x_i + \sum_{i=1}^{r-1} \sum_{j=1}^{n} b_i a_{ij} x_i x_j = \sum_{i=1}^{r-1} c_i x_i + \sum_{i=r}^{n} c_i x_i + \sum_{i=1}^{r-1} \sum_{j=1}^{n} b_i c_j x_i x_j. \tag{3.18}$$

From system (3.18), we get

$$c_i = \begin{cases} b_i \lambda_i & if \;\; i \in \{1, ..., r-1\}, \\ 0 & if \;\; i \in \{r, ..., n\}. \end{cases} \tag{3.19}$$

By substituting the values of $c_i$'s in (3.19) to the system (3.18), we get

$$\sum_{i=1}^{r-1} \sum_{j=1}^{r-1} b_i a_{ij} x_i x_j + \sum_{i=1}^{r-1} \sum_{j=r}^{n} b_i a_{ij} x_i x_j = \sum_{i=1}^{r-1} \sum_{j=1}^{r-1} b_i b_j \lambda_j x_i x_j. \tag{3.20}$$

Now, from (3.20), we have

$$a_{ij} = 0 \quad \text{for} \quad j \in \{r, ..., n\}, \tag{3.21}$$

and

$$\sum_{i=1}^{r-1}\sum_{j=1}^{r-1} b_i a_{ij} x_i x_j = \sum_{i=1}^{r-1}\sum_{j=1}^{r-1} b_i b_j \lambda_j x_i x_j. \tag{3.22}$$

Then from (3.22), we get

$$b_i = \frac{a_{ij}}{\lambda_i} \quad \text{if} \;\; i = j, \tag{3.23}$$

and

$$(b_i a_{ij} + b_j a_{ji}) = b_i b_j \lambda_j + b_j b_i \lambda_i \qquad \text{if} \quad i \neq j. \tag{3.24}$$

By substituting the values of $b_i$'s appearing in (3.23) to the system (3.24), we get

$$u_{ij} = u(a_{ij}, \lambda_i), \;\; \text{for} \;\; i \in \{1, ..., r-2\} \;\; \text{and} \;\; j \in \{i+1, ..., r-1\}, \tag{3.25}$$

where each $u(a_{ij}, \lambda_i)$ is an equation including only the parameters $a_{ij}$ and $\lambda_i$ for fixed $i$ and $j$.

Consequently, the total number of equations including the parameters $\lambda_i$ and $a_{ij}$ are given in both systems (3.21) and (3.25). For (3.21), we have exactly

$$(r-1)(n-r+1) - \text{equations}, \tag{3.26}$$

while for (3.25), we have exactly

$$(r-2) + (r-3) + ... + 1 = \frac{(r-2)(r-1)}{2} - \text{equations}. \tag{3.27}$$

So the total number of the equations is given by

$$(r-1)(n-r+1)+\frac{(r-2)(r-1)}{2}=\frac{(2n-r)(r-1)}{2}. \tag{3.28}$$

- $b_0 = 0$. Without loss of generality, the polynomial $f$ can be rewritten as in (3.29)

$$f=\sum_{i=1}^{r} b_i x_i. \tag{3.29}$$

Similarly, each $b_i$ is not necessary to be the same $b_i$'s appearing in (3.13). Consider the coordinate system $(X_1, ..., X_n)$ of $F^n$ with an arbitrary hypersurface $C_g$ defined by the polynomial $g$ having exactly $r$-non zero terms

$$g=d_0+\sum_{i=1}^{r-1} d_i X_i. \tag{3.30}$$

For

$$C_f=\Big\{x=(x_1,...,x_r)\Big|\ f(x)=\sum_{i=1}^{r} b_i x_i=0\Big\},$$

and

$$C_g=\Big\{X=(X_1,...,X_{r-1})\Big|\ g(X)=d_0+\sum_{i=1}^{r-1} d_i X_i=0\Big\}.$$

Let $\tilde{C}_f$ and $\tilde{C}_g$ be the projective versions of $C_f$ and $C_g$ in $\mathbf{P^n}(F)$ respectively

$$\tilde{C}_f=\Big\{x=(x_1,...,x_r)\Big|\ x_1+\sum_{i=2}^{r}\frac{b_i}{b_1}x_i=0\Big\}_{x_1\neq 0}\cup\mathbf{P^{n-1}}(F),$$

$$\tilde{C}_g=\Big\{X=(X_0,...,X_{r-1})\Big|\ X_0+\sum_{i=1}^{r-1}\frac{d_i}{d_0}X_i=0\Big\}_{X_0\neq 0}\cup\mathbf{P^{n-1}}(F).$$

Consider the following change of coordinates

$$\begin{aligned} x_1 &= X_0, \\ x_i &= \frac{b_1 d_{i-1}}{b_i d_0} X_{i-1}, \quad \textit{where} \;\; i \in \{2, ..., r\}. \end{aligned} \tag{3.31}$$

To show that $\tilde{C}_f$ and $\tilde{C}_g$ are projectively equivalents by the change of coordinates (3.31), let $x = (x_1, ..., x_r) \in \tilde{C}_f$, then

$$\begin{aligned} 0 = x_1 + \sum_{i=2}^{r} \frac{b_i}{b_1} x_i =& X_0 + \sum_{i=2}^{r} \frac{b_i}{b_1} x_i \\ =& X_0 + \sum_{i=2}^{r} \frac{b_i}{b_1} \Big( \frac{b_1 d_{i-1}}{b_i d_0} X_{i-1} \Big) \\ =& X_0 + \sum_{i=2}^{r} \frac{d_{i-1}}{d_0} X_{i-1} \\ =& X_0 + \sum_{i=1}^{r-1} \frac{d_i}{d_0} X_i. \end{aligned}$$

hence $X = (X_0, ..., X_{r-1}) \in \tilde{C}_g$. □

**Remark 6.** From the proof of the above theorem we can get the justification of why we considered only one possibility for each case mentioned in the first section for invariant algebraic planes and hyperplanes in three and four dimensional Lotka-Volterra systems.

# Chapter 4

# Branch Behaviour of Algebraic Curves in Two Dimensional Complex Lotka-Volterra Systems

## 4.1 Introduction

In this chapter, we investigate fully the behaviour at the branches of all invariant algebraic curves found in Ollagnier (2001).

In 2001, Moulin-Ollagnier considered the following homogeneous Lotka-Volterra system in $\mathbb{C}^3$

$$\dot{x} = x(Cy + z), \quad \dot{y} = y(Az + x), \quad \dot{z} = z(Bx + y). \tag{4.1}$$

He studied the values for the non-zero parameters $A$, $B$ and $C$ for which the above system has a homogeneous Liouvillian first integral of degree zero. For this purpose, he found twenty four irreducible invariant curves as follows: two algebraic lines, two conics, nine cubics, seven quartics, three sextics and an exceptional family of triples of parameters, $\left(A = 2, B = \frac{2l+1}{2l-1}, C = \frac{1}{2}\right)$, where $l$ is an integer. We will study all twenty three cases but not the exceptional case since the corresponding system of this case includes six critical points instead of seven and then we get zero ratio eigenvalues at one of its critical points. Also we are not interesting to study the Case 25 in Ollagnier (2004) missed in his previous work for the same reason as we have mentioned in Case 24.

We will study the branches of each invariant algebraic curve mentioned above as they have not been studied. To do that, we will use the two dimensional Lotka-Volterra system in the complex domain. Fortunately, the system (4.1) can be transform to a correspondence system in $\mathbb{C}^2$ as shown by Cairó et al. (2003). We will do something similar to what we have got in Cairó et al. (2003).

Consider the following two dimensional non-degenerate Lotka-Volterra system in $\mathbb{C}^2$

$$\begin{aligned}\dot{u} &= \frac{du}{dt} = u(1+au+bv),\\ \dot{v} &= \frac{dv}{dt} = v(\lambda+cu+dv).\end{aligned} \tag{4.2}$$

Assume $ad \neq 0$, otherwise, if $a = 0$, then the side critical point on the line $\{y = 0\}$, which has coordinates $(\xi_1, 0)$, $\xi_1 \neq 0$ coincides with the origin and then we get zero ratio of eigenvalues, similarly the side critical point on the line $\{x = 0\}$ which has the coordinates $(0, \eta_1)$, $\eta_1 \neq 0$ draws to the origin if $d = 0$. Henceforth, we consider $ad \neq 0$.

To scale the side critical points $(0, \eta_1)$ and $(\xi_1, 0)$ of the system (4.2) to the points $P_2 = (0, 1)$ and $P_3 = (1, 0)$, we may assume the change of coordinates given in (4.3)

$$u = -\frac{1}{a}x, \quad v = -\frac{\lambda}{d}y. \tag{4.3}$$

Hence, in the new coordinate system, the equation (4.2) can be rewritten as the following system

$$\begin{aligned}\dot{x} &= x\Big(1 - x - A(1-C)y\Big),\\ \dot{y} &= y\Big(A - (1-B)x - Ay\Big),\end{aligned} \tag{4.4}$$

where

$$A = \lambda, \quad B = 1 - \frac{c}{a} \ \text{ and } \ C = 1 - \frac{b}{d}.$$

We assume that the parameters $A$, $B$ and $C$ are non zero to guarantee that the system has exactly seven critical points.

**Remark 7.** Notice that we used these three parameters $A$, $B$ and $C$ to make our system much simpler than the system (4.2) by including only three parameters that correspond to the ratios of eigenvalues of the system (4.2) at its corner critical points, (see Figure 4.1) (ii). All invariant algebraic curves found in Ollagnier (2001) can be transformed to the correspondence with respect to the system (4.4). So the algebraic curves which we will study in the next section are invariant with respect to (4.4) with an exceptional of the value of $B$ become $\frac{1}{B}$.

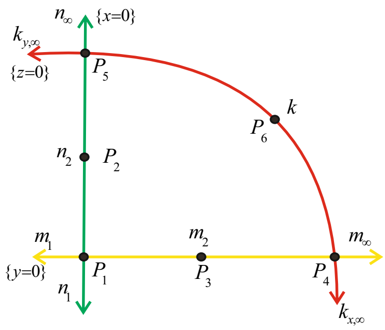
We will describe fully the behaviour of the invariant algebraic curves mentioned above.

## 4.2 Branch behaviour of invariant algebraic curves

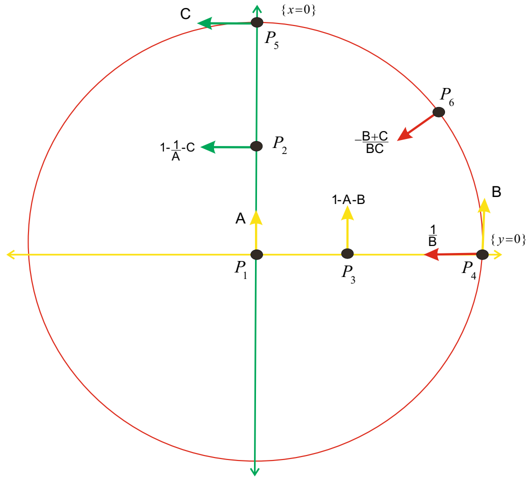
In this section, we will explain how can we find the branches of invariant algebraic curves with respect to (4.4).

First, we will use Figure 4.1 in drawing the graph of each invariant algebraic curve as follows: Figure 4.1, (i) describes the intersection multiplicity of an invariant algebraic curve $C_f$ defined by the equation $f(x, y) = 0$ with the axes and the line at infinity of $\mathbf{P}^2(\mathbb{C})$, for example, $n_2 = mult_{P_2}(\{x = 0\}, C_f)$, is the intersection multiplicity of the curve $C_f$ with $y$-axis at the point $P_2$. Notice that the sum of the number of intersection multiplicities of any curve at the points on the same axes is equal to the degree of $f$. See Theorem 9, Page 24.

In Figure 4.1, (i), we have also shown




(i) Multiplicity of $C_f$ in $\mathbf{P}^2(\mathbb{C})$.




(ii) Ratios of eigenvalues of (4.4).

**Figure 4.1:** Behaviour of an algebraic curve $C_f$ in $\mathbf{P}^2(\mathbb{C})$.

- The yellow and green lines are respectively the axes $\{y = 0\}$ and $\{x = 0\}$ of the affine part of $\mathbf{P}^2(\mathbb{C})$, while the red curve is the line at infinity.

- $P_2$ and $P_3$ are the side critical points on their corresponding lines, while $P_6$ is the side critical point at infinity. $P_4$ and $P_5$ are the critical points at infinity where $x \to \infty$ and $y \to \infty$ respectively.

- $m_1 = mult_{P_1}(\{y = 0\}, C_f)$, $\quad m_2 = mult_{P_3}(\{y = 0\}, C_f)$,
  $n_1 = mult_{P_1}(\{x = 0\}, C_f)$, $\quad n_2 = mult_{P_2}(\{x = 0\}, C_f)$,
  $m_\infty = mult_{P_4}(\{y = 0\}, C_f)$, $\quad n_\infty = mult_{P_5}(\{x = 0\}, C_f)$,
  $k_{x,\infty} = mult_{P_4}(L_\infty, C_f)$, $\quad k_{y,\infty} = mult_{P_5}(L_\infty, C_f)$,
  $k = mult_{P_6}(L_\infty, C_f)$.

In Part (ii) of Figure 4.1, we use the following conversion.

- The axes and the points are the same as we have explained in Part (i), while the red circle stand to the line at infinity and the position of the face point

$P_f$ is various and depends on the invariant algebraic curves.

- To each perpendicular arrows to the axes at the critical points, the quantity near the head of each arrow stand for the ratio of eigenvalues at the corresponding point. We apply colour formatting among the perpendicular arrows to the axes, for example at $P_4$ the quantity, $B$, for the yellow arrow is the ratio of eigenvalues at $P_4$ with respect to the line $\{x = 0\}$, while the quantity, $\frac{1}{B}$ for the red arrow is the ratio of eigenvalues at the same point with respect to the line $L_\infty$. In the applications, to each critical point, we draw only one of the arrows with the priority to the cases where the arrow is perpendicular to such an invariant algebraic curve $C_f$.
- We use blue colours for the graph of each invariant curve $C_f$ with the same arrow colour to the ratios of eigenvalues at the critical points for which $C_f$ passes through. Notice that in the case for which the branch of $C_f$ is complex, we use dotted lines points.

To find the behaviour of each invariant algebraic curve $C_f$ defined by a polynomial $f$ and sketch its graph, we apply the following steps

**Step 1** We find the associated Figure 4.1 (i) for $C_f$ by substituting the values of the parameters $A$, $B$ and $C$.

**Step 2** We will verify the type of each critical point belonging to $C_f$ by whether it is singular or regular point for the curve $C_f$.

**Step 3** We will find the ratio of eigenvalues at each critical point mentioned in Step 2 and then by using Step 1, we can get the values of the ratios of eigenvalues at the remaining critical points.

**Step 4** For any singular point, the value of the ratio of eigenvalues gives the fact: what type of singularity it is. In our work, we will find that the singularities are mainly cusps or nodes but higher order of singularities, for example tacnodes also occur.

**Step 5** For each critical point mentioned in Step 2, we find the branches of $C_f$ in a small neighbourhood containing the critical point. We apply some change of coordinates or the Puiseux's expansion method to find the branches.

**Step 6** We will find the index of the branches of $C_f$ at each critical point belongs to $C_f$. Notice that for a regular point, the index is the same as the value of the ratio of eigenvalues, while for a singular point is defined to be the residue concept on simple poles in complex analysis. (See Chapter 2, Section 2.3.1) For the quantity $m$,(index=$n$) to each arrow, the value of $m$ is the ratio of eigenvalues and $n$ is the index. The value of $n$ applies in the calculation of the sum of the ratios of eigenvalues at all critical points belong to $C_f$, while the value of $m$ contributes to the sum of the ratios of eigenvalues located on the same axis and must be equal to one. (See Theorem 9, page 24)

**Remark 8.** For each invariant algebraic curve $C_f$ with respect to (4.4)

1. We use a table to summarise the behaviour of $C_f$. The title '*Critical Points*' in the first column stands for the critical points belong to $C_f$, while the titles '*R.E.*', '*Branch*' and '*Indices*' are stand for the ratio of eigenvalues, the branches and the index of $C_f$ at each corresponding point in the first

column. Notice that the expression '*Singular*' beside each critical point in the first column indicate to a singular point of $C_f$, otherwise the point is regular.

2. In some cases, points at infinity appear in opposite position on the line at infinity, so we represent the critical point once by a dot and its correspondence by an open dot.

3. At infinity, we use the $(Z = 0, Y = Y_1)$ in the coordinate system (2.11) or $(X = X_1, Z = 0)$ in the coordinate system (2.12).

We prove the following theorems.

**Theorem 11.** *If $C_f = \{f : f(x, y) = 0\}$ is an invariant algebraic curve of degree $n$ with respect to* (4.4)*, then the number of the branches of $C_f$ passes through the critical points of the system (4.4) is given by*

$$n + 2 - 2g(C_f), \tag{4.5}$$

*where $g(C_f)$ is the geometric genus of $C_f$.*

*Proof.* From Theorem 7 of Chapter 2, we have

$$\sum_{\delta \in \Delta} i(Br_\delta, \Im) = n + 2 - 2g(C_f). \tag{4.6}$$

where $\Im$ is the holomorphic foliation defined by (4.4). Now, the corresponding vector field of the system (4.4) is given by

$$\chi = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y},$$

where $P(x,y) = x\Big(1 - x - A(1-C)y\Big)$ and $Q(x,y) = y\Big(A - (1-B)x - Ay)\Big)$. Let $\Phi(t) = \big(x(t), y(t)\big) = \big(t^k, y(t)\big)$ be a local parametrisation of an arbitrary branch $Br_\delta$ for some rational number $k$. Then the pullback $\chi^*(\Phi)$ of $\Phi$ by the vector field $\chi$ can be defined as follows

$$\begin{aligned}\chi^*(\Phi) &= \frac{P\big(x(t), y(t)\big)(\Phi(t))}{\frac{d}{dt}x(t)} \\ &= \frac{t^k - t^{2k} - A(1-C)t^k y(t)}{kt^{k-1}} \\ &= \frac{1}{k}t + \psi(t),\end{aligned} \tag{4.7}$$

where $\psi(t)$ is a polynomial in $t$ of order greater than one, hence the multiplicity $i(Br_\delta, \Im)$ of (4.7) at the branch $Br_\delta$ is equal to one. Since $\delta$ is arbitrary, then the number of the branches of the invariant algebraic curve $C_f$ at the critical points belong to $C_f$ is given by $n + 2 - 2g(C_f)$. □

**Theorem 12.** *Let $C_f$ be an invariant algebraic curve with respect to the system* (4.4) *passes through the origin, If the branch $Br$ of $C_f$ at the origin is a cusp defined as follows*

$$Br = \Big\{(x,y) \in \mathbb{C}^2 \mid y^m - kx^n = 0, \textit{ for coprime numbers } m,n \textit{ and } k \neq 0\Big\}.$$

*Then the index of $Br$ is equal to $mn$.*

*Proof.* It is easy to verify that an invariant algebraic curve $C_g$ defined by the polynomial $g = y^m - kx^n$ is invariant with respect to the system (4.4) with the cofactor $L_g = n(1-x-y)$ under the conditions, $A = \frac{n}{m}$, $B = 1 - \frac{n}{m}$ and $C = 1 - \frac{m}{n}$. For the singular point (0,0) of $C_g$, let $Br^*$ be a local branch in a small neighbourhood of $(0,0)$, then $Br^*$ can be defined as follows

$$Br^* = \big(x(t), y(t)\big) = \Big(\frac{t^m}{k^{\frac{m+1}{n}}}, \frac{t^n}{k}\Big).$$

By using the equation (2.25) in Chapter 2, we get

$$Res\left(\frac{L_g\big(x(t), y(t)\big)}{P(x(t), y(t))}, t = 0\right) = mn.$$

Since the index is an invariant quantity, then the proof is completed. □

**Proposition 1.** *The necessary and sufficient conditions for the system (4.4) having an invariant algebraic line passes through the origin transversely is that the ratio of eigenvalues must be equal to one at the origin.*

*Proof.* Suppose the ratio of eigenvalues of the system (4.4) is equal to one, then the value $A = 1$ in (4.4). It is not difficult to show that the line $f = x - \frac{C}{B}y$ is invariant with respect to (4.4) with the cofactor $L_f = 1 - x - y$.
The converse is obvious, since any algebraic line passes through the origin has multiplicity one in both axes. Consequently the ratio of eigenvalues is equal to one. □

**Remark 9.** Proposition 1 is true for any corner critical points of the system (4.4) by applying projective transformations.

We consider all invariant algebraic curves found in Ollagnier (2001) and we divide them by their degrees starting from those of degree one and so on.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2$ | $\frac{B}{B+C-BC}$ | $y = 1 - x$ | $\frac{B}{B+C-BC}$ |
| $P_3$ | $\frac{C}{B+C-BC}$ | $y = 1 - x$ | $\frac{C}{B+C-BC}$ |
| $P_6 = (Z = 0, Y = -1)$ | $-\frac{BC}{B+C-BC}$ | $Y = -1$ | $-\frac{BC}{B+C-BC}$ |

**Table 4.1:** Behaviour of $C_{f_1}$.

### 4.2.1 Invariant Lines

There are two separate cases of invariant algebraic lines with respect to (4.4). We will explain the behaviour of each case individually.

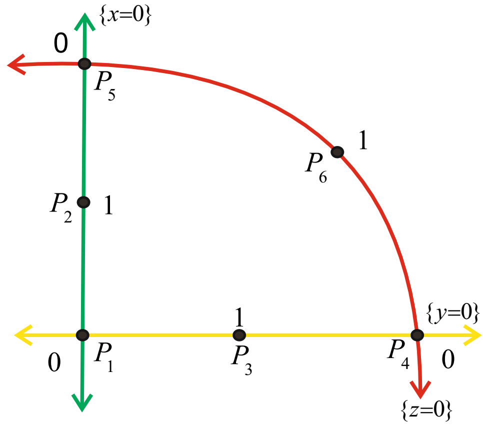
**(a)** The algebraic line, $C_{f_1}$, defined by the polynomial

$$f_1(x, y) = 1 - x - y,$$

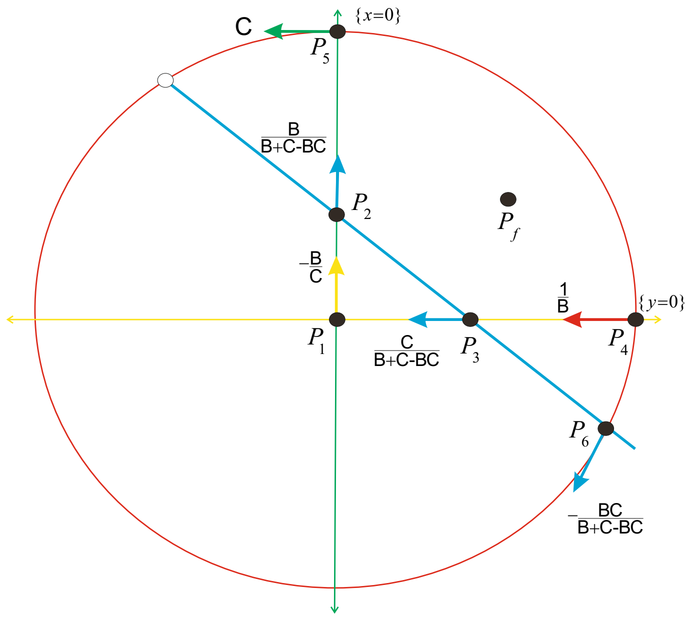
is invariant with respect to the system (4.4) under the condition $AC = -B$. Hence $C_{f_1}$ has codimension one. The invariant algebraic line $C_{f_1}$ does not pass through the face point. We will see later that $C_{f_1}$ is the only invariant algebraic curve found in Ollagnier (2001) not passing through the face point. More details about $C_{f_1}$ can be found in Table 4.1 and Figure 4.2. Notice that the sum of the indices of $C_{f_1}$ is equal to $1^2 = 1$ which satisfies Theorem 8, page 24.

**(b)** The algebraic line, $C_{f_2}$, defined by the polynomial

$$f_2(x, y) = x - \frac{C}{B}y,$$




(i) Multiplicity of $C_{f_1}$.




(ii) The graph of $C_{f_1}$.

**Figure 4.2:** Behaviour of $C_{f_1}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
| --- | --- | --- | --- |
| $P_1$ | 1 | $y = \frac{B}{C}x$ | 1 |
| $P_f = \left(-\frac{C}{B+C-BC}, -\frac{B}{B+C-BC}\right)$ | $-\frac{BC}{BC-B-C}$ | $y = \frac{B}{C}x$ | $-\frac{BC}{BC-B-C}$ |
| $P_6 = (Z = 0, Y = \frac{B}{C})$ | $\frac{BC}{BC-B-C}$ | $1 - \frac{C}{B}Y = 0$ | $\frac{BC}{BC-B-C}$ |

**Table 4.2:** Behaviour of $C_{f_2}$.

is invariant with respect to the system (4.4) under the condition $A = 1$. Hence $C_{f_2}$ has codimension one. The invariant algebraic line $C_{f_2}$ passes through the face point $P_f$. More details about $C_{f_2}$ can be found in Table 4.2 and Figure 4.3. Also notice that the sum of the indices in Table 4.2 is equal to 1.

**Remark 10.** In Ollagnier (2001), for any invariant algebraic curve of degree more than one, there is no any other invariant algebraic line passes through the corner critical points.

### 4.2.2 Invariant algebraic conics

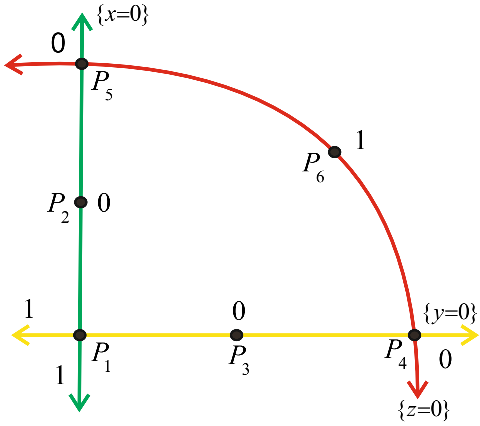
There are two separate cases of invariant algebraic conics with respect to (4.4). We will explain the properties of each one of them individually.

**(a)** The irreducible algebraic conic curve, $C_{f_3}$, defined by the polynomial

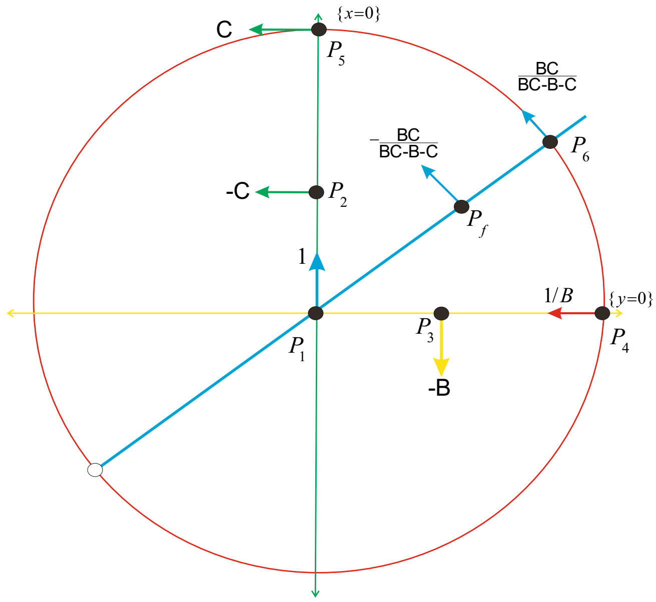
$$f_3(x, y) = (x-1)^2 + (y-1)^2 - 2xy - 1,$$

is invariant with respect to the system (4.4) under the conditions

$$A = -(B+1) \text{ and } C = -\frac{B}{B+1}.$$




(i) Multiplicity of $C_{f_2}$.




(ii) The graph of $C_{f_2}$.

**Figure 4.3:** Behaviour of $C_{f_2}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2$ | 2 | $x = \frac{1}{4}(y-1)^2 + ...$ | 2 |
| $P_3$ | 2 | $y = \frac{1}{4}(x-1)^2 + ...$ | 2 |
| $P_f = \Big( (\frac{B+1}{B})^2, (\frac{1}{B})^2 \Big)$ | -2 | $y - (\frac{1}{B})^2 = -B\Big(x - (\frac{B+1}{B})^2\Big) + ...$ | -2 |
| $P_6 = (Z = 0, Y = 1)$ | 2 | $Z = \frac{1}{4}(Y-1)^2 + ...$ | 2 |

**Table 4.3:** behaviour of $C_{f_3}$.

Hence $C_{f_3}$ has codimension two. On the other hand $C_{f_3}$ is a smooth conic since no critical point belongs to $C_{f_3}$ is a singularity. Consequently $C_{f_3}$ has exactly four branches. More details about $C_{f_3}$ can be seen in Table 4.3 and Figure 4.4. It is easy to verify that the sum of the indices in Table 4.3 is equal to $2^2 = 4$.

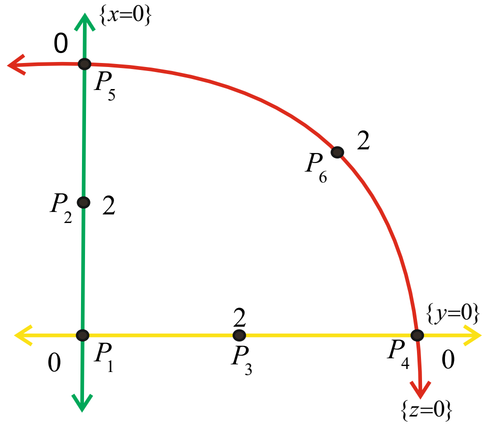
**(b)** The irreducible algebraic conic, $C_{f_4}$, defined by the polynomial

$$f_4(x, y) = x^2 + \Big(\frac{2}{B+1}\Big)^2 y(y-1) + \frac{4}{B+1}xy,$$

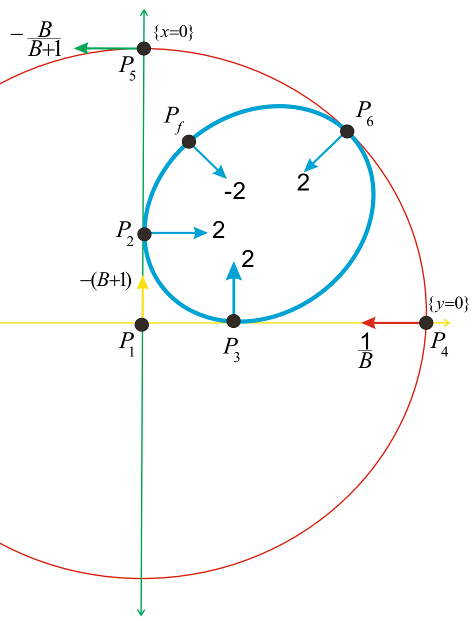
is invariant with respect to the system (4.4) under the conditions

$$A = 2 \text{ and } C = -\frac{B}{B+1}.$$

Hence $C_{f_4}$ also has codimension two. Clearly $C_{f_4}$ is a smooth conic since no critical point belongs to $C_{f_4}$ is a singularity of $C_{f_4}$. Consequently, $C_{f_4}$ has exactly four branches as $C_{f_3}$. More details about $C_{f_4}$ can be seen in Table 4.4 and Figure 4.5. It is easy to verify that the sum of the indices in Table 4.4 is equal to 4.

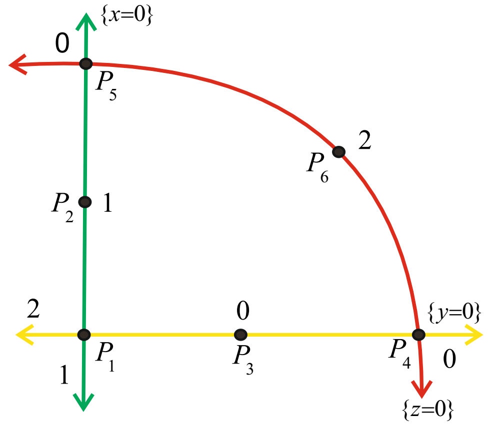



(i) Multiplicity of $C_{f_3}$.

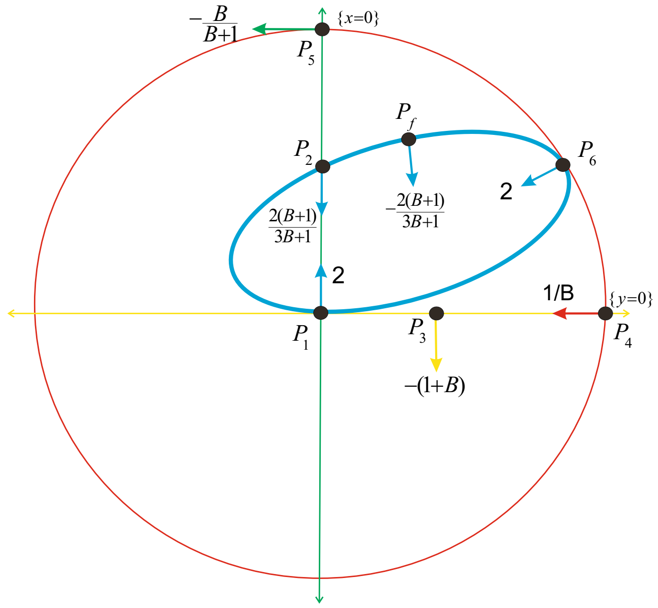



(ii) The graph of $C_{f_3}$.

**Figure 4.4:** Behaviour of $C_{f_3}$.




(i) Multiplicity of $C_{f_4}$.




(ii) The graph of $C_{f_4}$.

**Figure 4.5:** Behaviour of $C_{f_4}$.

| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_1$ | 2 | $y=\frac{1}{4}(B+1)^2x^2+...$ | 2 |
| $P_2$ | $\frac{2(B+1)}{3B+1}$ | $y-1=-(B+1)x+...$ | $\frac{2(B+1)}{3B+1}$ |
| $P_f=\Big(-\frac{3B+1}{2B^2},(\frac{B+1}{2B})^2\Big)$ | $-\frac{2(B+1)}{3B+1}$ | $y+\frac{(B+1)^2}{4B^2}=-\frac{(B+1)^2}{2(2B+1)}\Big(x+\frac{3B+1}{2B^2}\Big)+...$ | $-\frac{2(B+1)}{3B+1}$ |
| $P_6=\Big(Z=0,Y=-\frac{1}{2}B-\frac{1}{2}\Big)$ | 2 | $Z=-\frac{2}{B+1}\Big(Y+\frac{1}{2}(B+1)\Big)^2+...$ | 2 |

**Table 4.4:** behaviour of $C_{f_4}$.

### 4.2.3 Invariant algebraic cubics

There are nine separate cases of invariant algebraic cubics, $C_{f_i}, i \in \{5, ..., 13\}$ with respect to the system (4.4). We will explain each case individually.

**(a)** The irreducible cubic curve, $C_{f_5}$, defined by the polynomial

$$f_5(x,y) = (x-1)^2 + (1-y)^3 + \frac{1}{4}xy^2 - 5xy - 1,$$

is invariant with respect to (4.4) under the conditions

$$A = -\frac{3}{2}, \quad B = \frac{1}{2} \text{ and } C = -\frac{4}{3}.$$

At the affine part of $\mathbf{P}^2(\mathbb{C})$, the cubic curve $C_{f_5}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity it passes through the critical points $P_4$ and $P_6$ $\big($see Figure 4.6,(i)$\big)$. $P_f$ is the only singular point of $C_{f_5}$ among the other critical points passed through, so by assuming the change of coordinates

$$u = -1 + x - \frac{5}{2}y + \frac{1}{8}y^2, \quad v = \frac{1}{2}(y+8)\sqrt[3]{1 - \frac{1}{8}(y+8)}.$$

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2$ | 3 | $x = -\frac{4}{27}(y-1)^3 + ...$ | 3 |
| $P_3$ | 2 | $y = \frac{1}{8}(x-1)^2 + ...$ | 2 |
| $P_f(singular) = (-27, -8)$ | 3/2 | $(y+8)^2 = k(x+27)^3 + ..., k \neq 0$ | 6 |
| $P_4$ | 2 | $Z = -\frac{1}{4}Y^2 + ...$ | 2 |
| $P_6 = (Z = 0, Y = \frac{1}{4})$ | -4 | $Z = -(Y - \frac{1}{4}) + ...$ | -4 |

**Table 4.5:** Behaviour of $C_{f_5}$.

The correspondence curve of $C_{f_5}$ at $P_f$ can be given by the polynomial

$$g_5(u,v) = u^2 + v^3.$$

Hence $P_f$ is a cusp singularity of $C_{f_5}$, and then from Theorem 12, the index is equal to 6. Generally, any irreducible cubic curve with at most one singularity has a zero geometric genus. Then from Theorem 11, we conclude that $C_{f_5}$ has exactly five branches. More details about the invariant curve $C_{f_5}$ can be found in Table 4.5 and Figure 4.6. It is to easy to verify that the sum of the indices in Table 4.5 is equal to 9.

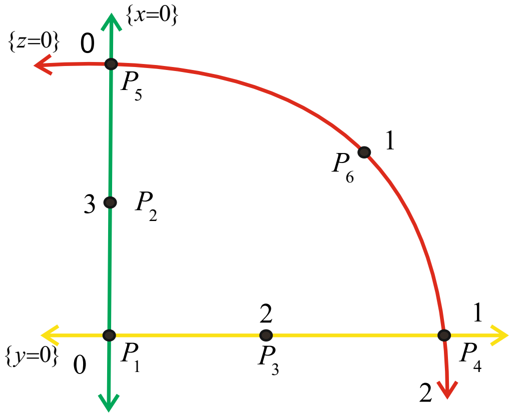
**(b)** The irreducible cubic curve, $C_{f_6}$, defined by the polynomial

$$f_6(x,y) = 27x^2(x-1) + 64y(y-1)^2 + 108x^2y + 144xy^2 - 144xy,$$

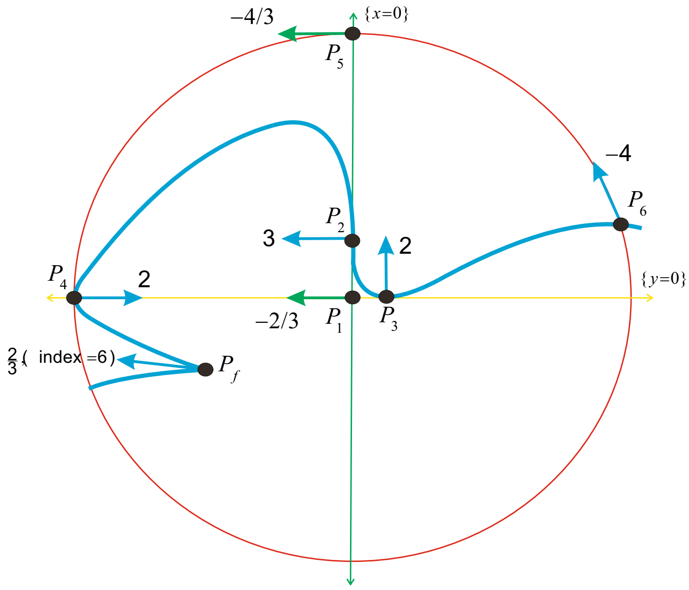
is invariant with respect to (4.4) under the conditions

$$A = 2, \quad B = \frac{1}{4} \text{ and } C = -\frac{1}{6}.$$

At the affine part of $\mathbf{P}^2(\mathbb{C})$, the cubic curve $C_{f_6}$ passes through the critical




(i) Multiplicity of $C_{f_5}$.




(ii) The graph of $C_{f_5}$.

**Figure 4.6:** Behaviour of $C_{f_5}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_1$ | 2 | $y = \frac{27}{64}x^2 + ...$ | 2 |
| $P_2(singular)$ | $\frac{2}{3}$ | $(y-1)^3 = kx^2 + ..., k \neq 0$ | 6 |
| $P_3$ | $-\frac{4}{5}$ | $y = -\frac{27}{28}(x-1) + ...,$ | $-\frac{4}{5}$ |
| $P_f = \left(-\frac{32}{3}, 5\right)$ | $-\frac{6}{5}$ | $y - 5 = -\frac{9}{16}x + \frac{32}{3} + ...$ | $-\frac{6}{5}$ |
| $P_6 = \left(Z = 0, Y = -\frac{3}{4}\right)$ | 3 | $Z = -\frac{64}{9}(Y + \frac{3}{4})^3 + ...$ | 3 |

**Table 4.6:** Behaviour of $C_{f_6}$.

points $P_1$, $P_2$, $P_3$ and $P_f$, while at infinity it passes through the critical point $P_6$ only. $\big($see Figure 4.7,(i)$\big)$. $P_2$ is the only singular point of $C_{f_6}$ among the other critical points passed through, so by assuming the change of coordinates

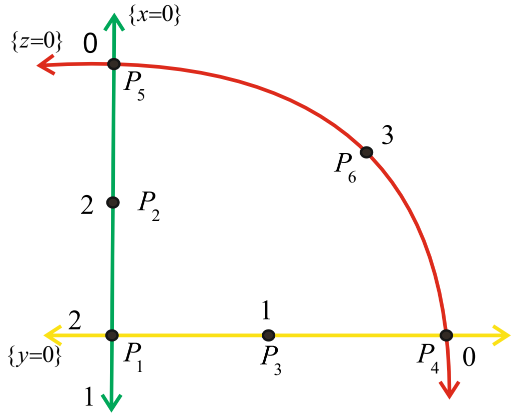
$$u = -8 + 9x + 8y, \quad v = -4 + 3x + 4y.$$

The correspondence curve of $C_{f_6}$ at $P_2$ can be given by the same polynomial $g_5(u, v)$. Hence $P_2$ is a cusp singularity of $C_{f_6}$, and then from Theorem 12, the index is equal to 6. Generally, any irreducible cubic curve with at most one singularity has a zero geometric genus. Then from Theorem 11, we conclude that $C_{f_6}$ has exactly five branches.

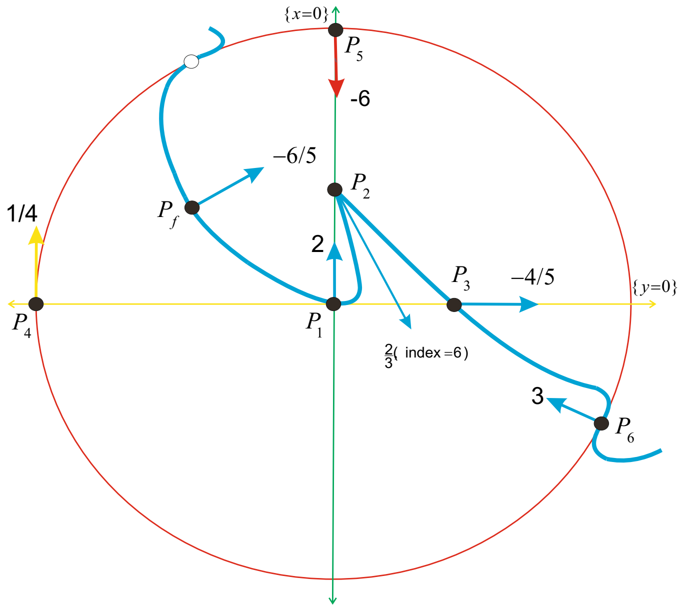
More details about the invariant curve $C_{f_6}$ can be found in Table 4.6 and Figure 4.7. It is to easy to verify that the sum of the indices in Table 4.6 is equal to 9.

**(c)** The irreducible cubic curve, $C_{f_7}$, defined by the polynomial

$$f_7(x, y) = x^2(x-1) + \frac{256}{27}y(y-1) + \frac{32}{3}xy,$$




(i) Multiplicity of $C_{f_6}$.




(ii) The graph of $C_{f_6}$.

**Figure 4.7:** Behaviour of $C_{f_6}$.

is invariant with respect to (4.4) under the conditions

$$A = 2, \quad B = -\frac{7}{8} \text{ and } C = \frac{1}{3}.$$

At the affine part of $\mathbf{P}^2(\mathbb{C})$, the cubic curve $C_{f_7}$ passes through the critical points $P_1$, $P_2$, $P_3$ and $P_f$, while at infinity it passes through the critical point $P_5$ only. $\big($see Figure 4.8,(i)$\big)$. $P_f$ is the only singular point of $C_{f_7}$ among the other critical points passed through, so by assuming the change of coordinates

$$u = -\frac{4}{3} + x, \quad v = -\frac{4}{3} + x + \frac{16}{9}\Big(y + \frac{1}{4}\Big).$$

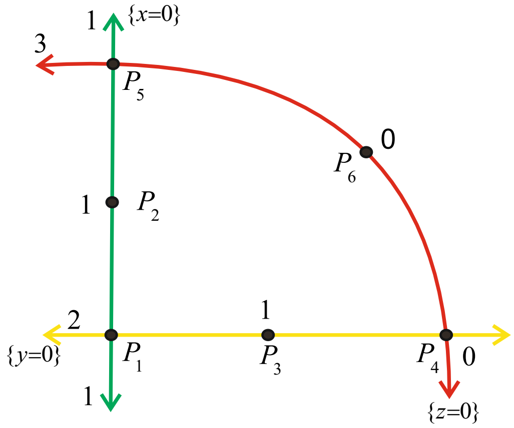
The correspondence curve of $C_{f_7}$ at $P_f$ is given by the polynomial

$$g_7(u, v) = u^3 + 3v^2.$$

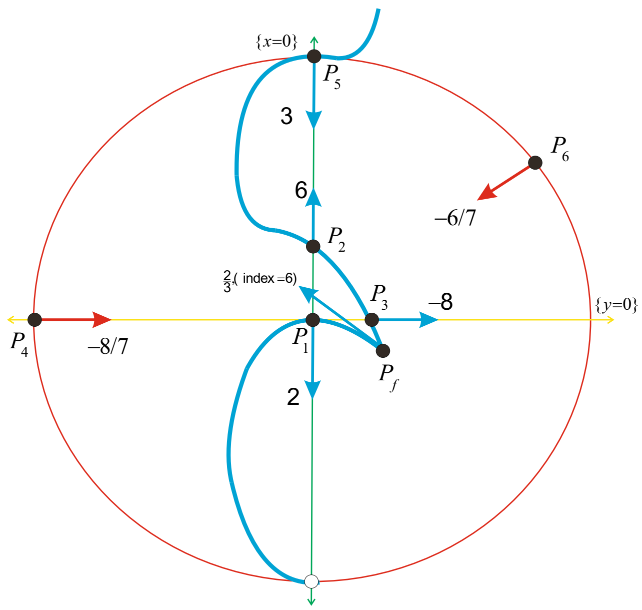
Consequently the branch of $C_{f_7}$ at $P_f$ is a cusp and then the index is equal to 6. Similarly to the other cases, $C_{f_7}$ has a zero geometric genus. Then from Theorem 11, we conclude that $C_{f_7}$ has exactly five branches.
More details about the invariant curve $C_{f_7}$ can be found in Table 4.7 and Figure 4.8. It is easy to verify that the sum of the indices in Table 4.7 is equal to 9.

**(d)** The irreducible cubic curve, $C_{f_8}$, defined by the polynomial

$$f_8(x, y) = (1 - x) + (1 - y)^3 - \frac{1}{4}x^2 y - xy^2 + 2xy - 1,$$




(i) Multiplicity of $C_{f_7}$.




(ii) The graph of $C_{f_7}$.

**Figure 4.8:** Behaviour of $C_{f_7}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_1$ | 2 | $y=-\frac{27}{256}x^2+...$ | 2 |
| $P_2$ | 6 | $x=-\frac{8}{9}(y-1)+...$ | 6 |
| $P_3$ | -8 | $y=-\frac{27}{32}(x-1)+...,$ | -8 |
| $P_f(singular)=(\frac{4}{3},-\frac{1}{4})$ | $\frac{2}{3}$ | $(y+\frac{1}{4})^2=k(x-\frac{4}{3})^3+...,k\neq 0$ | 6 |
| $P_5$ | 3 | $Z=-\frac{27}{256}X^3+...$ | 3 |

**Table 4.7:** Behaviour of $C_{f_7}$.

is invariant with respect to (4.4) under the conditions

$$A=6,\quad B=2 \text{ and } C=-\frac{2}{3}.$$

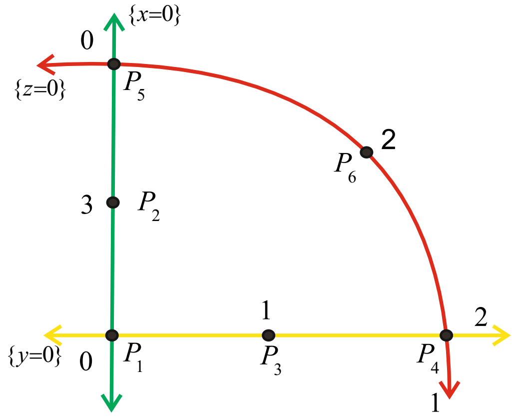
At the affine part of $\mathbf{P}^2(\mathbb{C})$, the cubic curve $C_{f_8}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity it passes through the critical points $P_4$ and $P_6$. $\big($see Figure 4.9,(i)$\big)$. $P_2$ is the only singular point of $C_{f_8}$ among the other critical points passed through, so by assuming the change of coordinates

$$u=\frac{1}{2}\Big(-\frac{1}{4}x^2-xy+2x-y^2+2y-1\Big),\quad v=\Big(\frac{1}{2}x+y-1\Big)\sqrt[3]{\frac{1}{4}\Big(\frac{1}{2}x+y-1\Big)-1}.$$

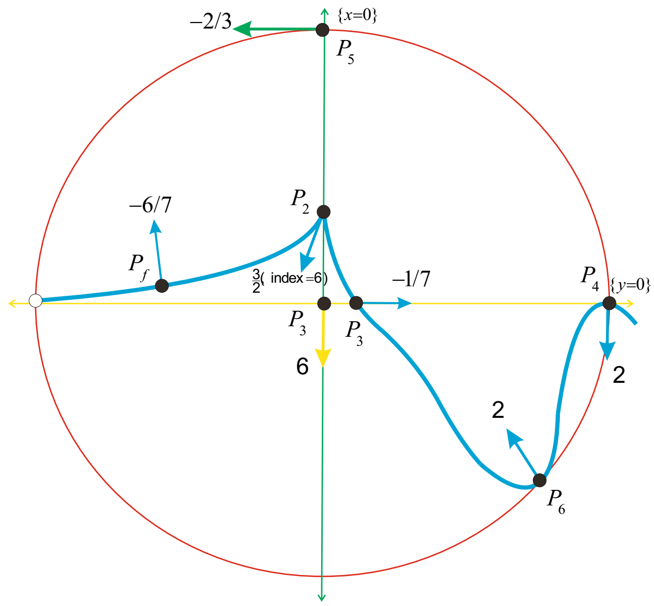
The correspondence curve of $C_{f_8}$ at $P_2$ is given by the polynomial

$$g_8(u,v)=-u^2+v^3.$$

Consequently the branch of $C_{f_8}$ at $P_2$ is a cusp and then it has index 6. Similarly to the other cases, $C_{f_8}$ has a zero geometric genus. Then from Theorem 11, we conclude that $C_{f_8}$ has exactly five branches. More details about the invariant curve $C_{f_8}$ can be found in Table 4.8 and Figure 4.9. It is to easy to verify that the sum of the indices in Table 4.8 is equal to 9.




(i) Multiplicity of $C_{f_8}$.




(ii) The graph of $C_{f_8}$.

**Figure 4.9:** Behaviour of $C_{f_8}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2(singular)$ | $\frac{2}{3}$ | $(y-1)^3 = kx^2 + ..., k \neq 0$ | 6 |
| $P_3$ | $-\frac{1}{7}$ | $y = -\frac{4}{5}(x-1) + ...$ | $-\frac{1}{7}$ |
| $P_f = \left(-\frac{27}{8}, \frac{7}{16}\right)$ | $-\frac{6}{7}$ | $y - \frac{7}{16} = \frac{1}{18}(x + \frac{27}{8}) + ...$ | $-\frac{6}{7}$ |
| $P_4$ | 2 | $Y = -4Z^2 + ...$ | 2 |
| $P_6 = \left(Z = 0, Y = -\frac{1}{2}\right)$ | 2 | $Z = 2(Y + \frac{1}{2})^2 + ...$ | 2 |

**Table 4.8:** Behaviour of $C_{f_8}$.

**(e)** The irreducible cubic curve, $C_{f_9}$, defined by the polynomial

$$f_9(x, y) = (1 - x) + (y - 1)^2 + \frac{1}{4}x^2y + xy - 1,$$

is invariant with respect to (4.4) under the conditions

$$A = -6, \quad B = 2 \text{ and } C = \frac{1}{2}.$$

At the affine part of $\mathbf{P}^2(\mathbb{C})$, the cubic curve $C_{f_9}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity it passes through the critical points $P_4$ and $P_5$. (see Figure 4.10,(i)). $P_2$ is the only singular point of $C_{f_9}$ among the other critical points passed through, so by assuming the change of coordinates

$$u = -\frac{1}{2}\sqrt[3]{1 + \frac{1}{8}x}, \quad v = \frac{1}{8}x^2 + \frac{1}{2}x + y - 1.$$

The correspondence curve of $C_{f_9}$ at $P_2$ is given by the polynomial

$$g_9(u, v) = u^3 + v^2.$$

Consequently the branch of $C_{f_9}$ at $P_2$ is a cusp and then it has index 6.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2(singular)$ | $\frac{2}{3}$ | $(y-1)^3 = kx^2 + ..., k \neq 0$ | 6 |
| $P_3$ | $\frac{1}{5}$ | $y = -\frac{4}{3}(x-1) + ...$ | $\frac{1}{5}$ |
| $P_f = (\frac{8}{3}, \frac{5}{9})$ | $-\frac{6}{5}$ | $y - \frac{5}{9} = -\frac{1}{12}(x - \frac{8}{3}) + ...$ | $-\frac{6}{5}$ |
| $P_4$ | 2 | $Y = 4Z^2 + ...$ | 2 |
| $P_5$ | 2 | $Z = -\frac{1}{4}X^2 + ...$ | 2 |

**Table 4.9:** Behaviour of $C_{f_9}$.

Similarly to the other cases, $C_{f_9}$ has a zero geometric genus. Then from Theorem 11, we conclude that $C_{f_9}$ has exactly five branches. More details about the invariant curve $C_{f_9}$ can be found in Table 4.9 and Figure 4.10. It is to easy to verify that the sum of the indices in Table 4.9 is equal to 9.

**(f)** The irreducible cubic curve, $C_{f_{10}}$, defined by the polynomial

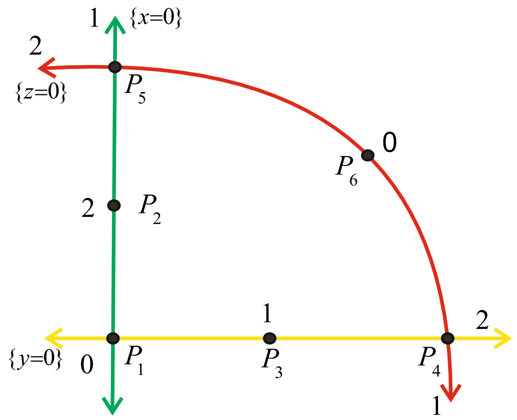
$$f_{10}(x, y) = x^2 + 8y + 8xy^2 - 12xy,$$

is invariant with respect to (4.4) under the conditions

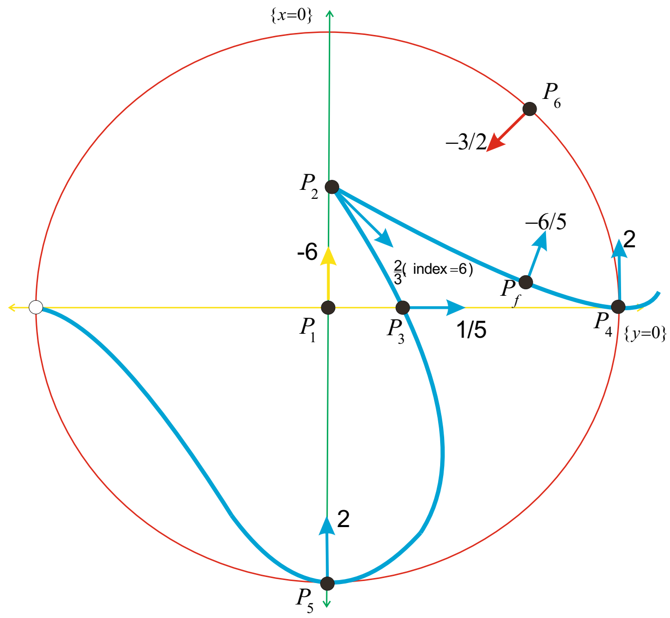
$$A = 2, \quad B = \frac{1}{2} \text{ and } C = 2.$$

At the affine part of $\mathbf{P}^2(\mathbb{C})$, the cubic curve $C_{f_{10}}$ passes through the critical points $P_1$ and $P_f$, while at infinity it passes through the critical points $P_4$ and $P_5$. $\big($see Figure 4.11,(i)$\big)$. $P_f$ is the only singular point of $C_{f_{10}}$ among the other critical points passed through, so by assuming the change of coordinates

$$u = x - 6y + 4y^2, \quad v = \sqrt{12}\big(y - \frac{1}{2}\big)\sqrt{1 + \frac{4}{3}\big(y - \frac{1}{2}\big) - \frac{4}{3}\big(y - \frac{1}{2}\big)^2}.$$




(i) Multiplicity of $C_{f_9}$.




(ii) The graph of $C_{f_9}$.

**Figure 4.10:** Behaviour of $C_{f_9}$.

| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_1$ | 2 | $y = -\frac{1}{8}x^2 + ...$ | 2 |
| $P_f(singular) = (2, \frac{1}{2})$ | $\frac{-3+i\sqrt{3}}{-3-i\sqrt{3}}$ | $(y - \frac{1}{2}) = \frac{1}{8}(1 + i\sqrt{3})(x - 2) + ...,$ or $(y - \frac{1}{2}) = \frac{1}{8}(1 - i\sqrt{3})(x - 2) + ...,$ | 3 |
| $P_4$ | 2 | $Z = -8Y^2 + ...$ | 2 |
| $P_5$ | 2 | $X = -Z^2 + ...$ | 2 |

**Table 4.10:** Behaviour of $C_{f_{10}}$.

The correspondence curve of $C_{f_{10}}$ at $P_f$ is given by the polynomial

$$g_{10}(u, v) = u^2 + v^2.$$

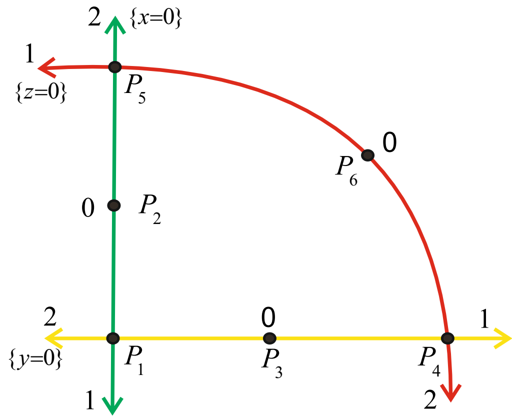
Consequently the branch of $C_{f_{10}}$ at $P_f$ is a node. Since the ratio of eigenvalues at $P_f$ is equal to $\frac{-3+i\sqrt{3}}{-3-i\sqrt{3}}$ and each branch has multiplicity one to the other one at $P_f$, then from (2.26), we can compute the index which is given by $2 + \left(\frac{-3+i\sqrt{3}}{-3-i\sqrt{3}}\right) + \left(\frac{-3-i\sqrt{3}}{-3+i\sqrt{3}}\right) = 3$. Similarly to the other cases, $C_{f_{10}}$ has a zero geometric genus. Then from Theorem 11, we conclude that $C_{f_{10}}$ has exactly five branches. More details about the invariant curve $C_{f_{10}}$ can be found in Table 4.10 and Figure 4.11. It is to easy to verify that the sum of the indices in Table 4.10 is equal to 9.

**(g)** The irreducible cubic curve, $C_{f_{11}}$, defined by the polynomial

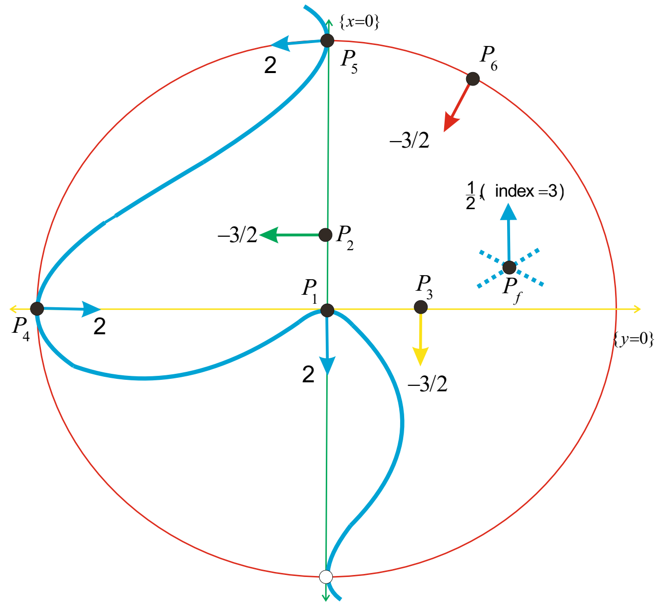
$$f_{11}(x, y) = x^2 - 9y(y - 1)^2 + 12xy,$$

is invariant with respect to (4.4) under the conditions

$$A = 2, \quad B = \frac{1}{3} \text{ and } C = -\frac{3}{2}.$$




(i) Multiplicity of $C_{f_{10}}$.




(ii) The graph of $C_{f_{10}}$.

**Figure 4.11:** Behaviour of $C_{f_{10}}$.

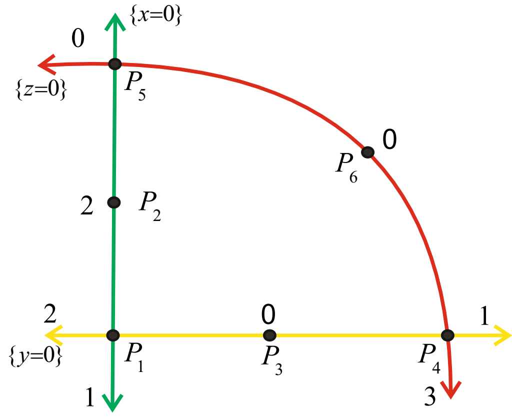
| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_1$ | 2 | $y = \frac{1}{9}x^2 + ...$ | 2 |
| $P_2$ | 2 | $x = \frac{3}{4}(y-1)^2 + ...$ | 2 |
| $P_f(singular) = (6, -1)$ | $\frac{-1+i}{-1-i}$ | $(y+1) = \frac{1}{15}(-2+i)(x-6) + ...,$ or $(y+1) = \frac{1}{15}(-2-i)(x-6) + ...,$ | 2 |
| $P_4$ | 3 | $Z = 9Y^3 + ...$ | 3 |

**Table 4.11:** Behaviour of $C_{f_{11}}$.

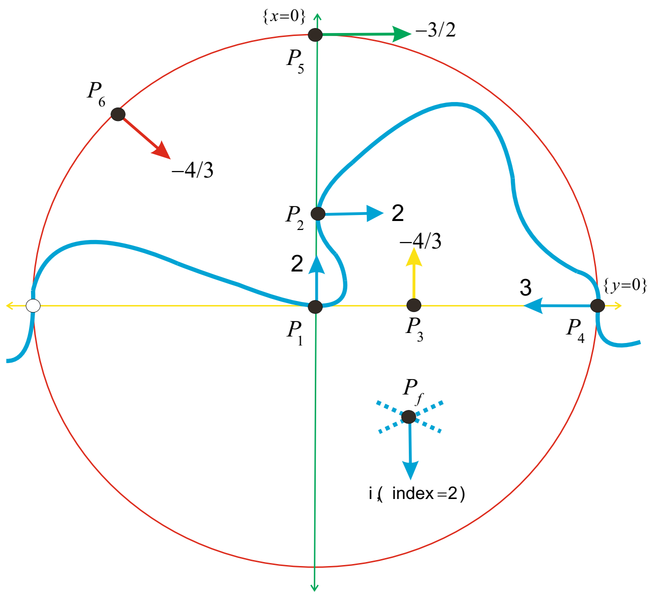
At the affine part of $\mathbf{P}^2(\mathbb{C})$, the cubic curve $C_{f_{11}}$ passes through the critical points $P_1$, $P_2$ and $P_f$, while at infinity it passes through the critical point $P_4$ only. $\big($see Figure 4.12,(i)$\big)$. $P_f$ is the only singular point of $C_{f_{11}}$ among the other critical points passed through, so by assuming the change of coordinates

$$u = x + 6y, \quad v = 3(y+1)\sqrt{-y}.$$

The correspondence curve of $C_{f_{11}}$ at $P_f$ is given by the same polynomial $g_{10}$. Consequently the branch of $C_{f_{11}}$ at $P_f$ is a node. Since the ratio of eigenvalues at $P_f$ is equal to $\frac{-1+i}{-1-i}$ and each branch has multiplicity one to the other one at $P_f$ as shown in Table 4.11, then the index is given by $2 + \left(\frac{-1+i}{-1-i}\right) + \left(\frac{-1-i}{-1+i}\right) = 2$. Similarly to the other cases, $C_{f_{11}}$ has a zero geometric genus. Then from Theorem 11, we conclude that $C_{f_{11}}$ has exactly five branches. More details about the invariant curve $C_{f_{11}}$ can be found in Table 4.11 and Figure 4.12. It is to easy to verify that the sum of the indices in Table 4.11 is equal to 9.




(i) Multiplicity of $C_{f_{11}}$.




(ii) The graph of $C_{f_{11}}$.

**Figure 4.12:** Behaviour of $C_{f_{11}}$.

**(h)** The irreducible cubic curve, $C_{f_{12}}$, defined by the polynomial

$$f_{12}(x,y) = (x-1)^2 + (1-y)^3 - 6xy - 1,$$

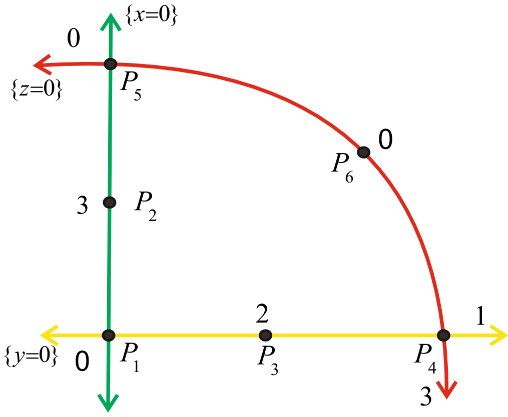
is invariant with respect to (4.4) under the conditions

$$A = -\frac{4}{3}, \quad B = \frac{1}{3} \text{ and } C = -\frac{5}{4}.$$

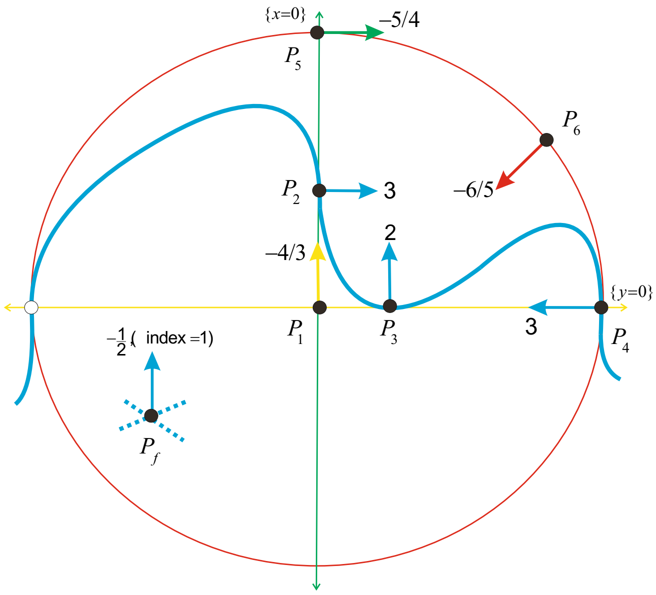
At the affine part of $\mathbf{P}^2(\mathbb{C})$, the cubic curve $C_{f_{12}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity it passes through the critical point $P_4$ only. $\big($see Figure 4.13,(i)$\big)$. $P_f$ is the only singular point of $C_{f_{12}}$ among the other critical points passed through, so by assuming the change of coordinates

$$u = -1 + x - 3y, \quad v = \sqrt{3}(y+3)\sqrt{1 - \frac{1}{3}(y+3)}.$$

The correspondence curve of $C_{f_{12}}$ at $P_f$ is given by the same polynomial $g_{10}$. Consequently the branch of $C_{f_{12}}$ at $P_f$ is a node, since the ratio of eigenvalues at $P_f$ is equal to $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$ and each branch has multiplicity one to the other at $P_f$, then the index is given by $2 + \left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right) + \left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right) = 1$. Similarly to the other cases, $C_{f_{12}}$ has a zero geometric genus. Then from Theorem 11, we conclude that $C_{f_{12}}$ has exactly five branches. More details about the invariant curve $C_{f_{12}}$ can be found in Table 4.12 and Figure 4.13. It is to easy to verify that the sum of the indices in Table 4.12 is equal to 9.




(i) Multiplicity of $C_{f_{12}}$.




(ii) The graph of $C_{f_{12}}$.

**Figure 4.13:** Behaviour of $C_{f_{12}}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2$ | 3 | $x = -\frac{1}{8}(y-1)^3 + ...$ | 3 |
| $P_3$ | 2 | $y = \frac{1}{9}(x-1)^2 + ...,$ | 2 |
| $P_f(singular) = (-8, -3)$ | $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$ | $(y+3) = \frac{1}{4}(1+\frac{1}{3}i\sqrt{3})(x+8) + ...,$ or $(y+3) = \frac{1}{4}(1-\frac{1}{3}i\sqrt{3})(x+8) + ...$ | 1 |
| $P_4$ | 3 | $Z = Y^3 + ...$ | 3 |

**Table 4.12:** Behaviour of $C_{f_{12}}$.

**(i)** The irreducible cubic curve, $C_{f_{13}}$, defined by the polynomial of complex coefficients

$$f_{13}(x,y) = (1-x)^3+(1-y)^3+\frac{3}{2}\Big((1-i\sqrt{3})xy+(1+i\sqrt{3})x^2y+(1-i\sqrt{3})xy^2\Big)-1,$$

is invariant with respect to (4.4) under the conditions

$$A = \frac{-3+i\sqrt{3}}{6}, \quad B = \frac{2}{-3+i\sqrt{3}} \quad \text{and} \quad C = \frac{-1+i\sqrt{3}}{2}.$$

At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant cubic curve $C_{f_{13}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity, the only critical point for which $C_{f_{13}}$ passes through is $P_6$, $\big($see Figure 4.14,(i)$\big)$. $P_3$ is the only singular point of $C_{f_{13}}$. By applying Puiseux's expansion, (for more about Puiseux's expansion, see Walker (1950), Chapter 4) of $C_{f_{13}}$ at $P_f$, we get two branches, one of order two and the other is of order one, (see Table 4.13), consequently $P_3$ is a node. Since the ratio of eigenvalues at one of the branch is equal to 2 and the multiplicity between them is one, then the index is given by $2 + \left(2 + \frac{1}{2}\right) = \frac{9}{2}$.

More details about the invariant curve $C_{f_{13}}$ can be found in Table 4.13 and

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2$ | 3 | $x = \frac{1}{9}i\sqrt{3}(y-1)^3 + ...$ | 3 |
| $P_3(singular)$ | 2 | $y = \frac{1}{6}(1-\frac{1}{3}i\sqrt{3})(x-1)^2 + ...$, or $y = -\frac{1}{2}(1+i\sqrt{3})(x-1) + ...$ | $2 + \frac{5}{2}$ |
| $P_f = \big(-1, -(1+i\sqrt{3})\big)$ | $-\frac{3}{2}$ | $y+(1+i\sqrt{3}) = \frac{1}{2}(3+\frac{1}{2}i\sqrt{3})(x+1)+...$ | $-\frac{3}{2}$ |
| $P_6 = \big(Z=0, Y=-\frac{1}{2}(1+i\sqrt{3})\big)$ | 3 | $Z = \frac{1}{9}i(Y-\frac{1}{2}(1-i\sqrt{3}))^3$ | 3 |

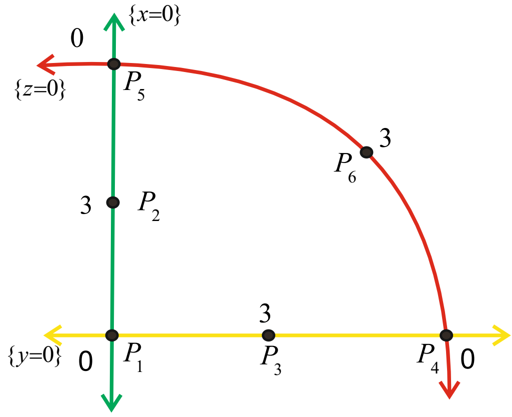
**Table 4.13:** Behaviour of $C_{f_{13}}$

Figure 4.14. It is easy to verify that the sum of the indices in Table 4.13 is equal to 9.

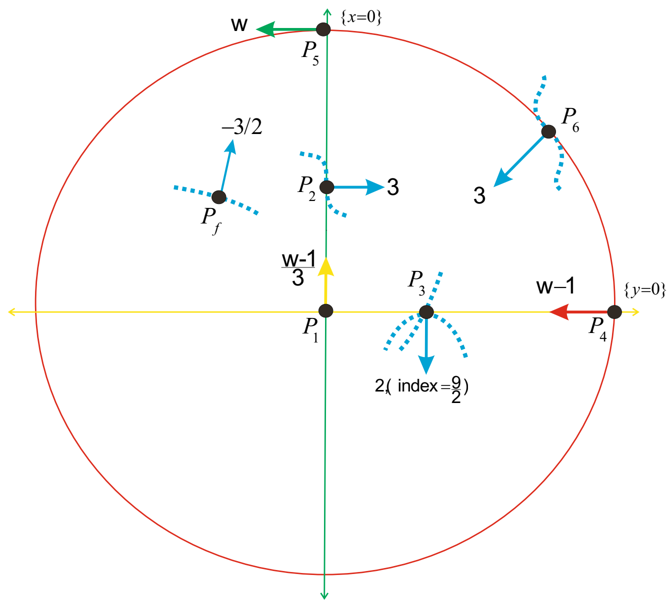
**Remark 11.** From the above results, we see that any invariant cubic curve, $C_{f_i}, i \in \{5, ..., 13\}$ has codimension three with exactly five branches.

### 4.2.4 Invariant algebraic quartics

There are seven separate cases of quartic invariant algebraic curves, $C_{f_i}, i \in \{14, ..., 20\}$ with respect to the system (4.4). We will find that to each $C_{f_i}$, exactly two of the critical points for which $C_{f_i}$ passes through are singularities for the quartic curves. Every $C_{f_i}, i \in \{14, 15, 16\}$ has a cusp branch at one of its singularity and a node at the other, while the other quartic curves $C_{f_i}, i \in \{17, 18, 19, 20\}$ have only node branches at their singularities; Moreover the quartic curve $C_{f_{20}}$ is of complex coefficients. Generally, since any irreducible invariant quartic curve having at most two singularities has a zero geometric genus. Then from Theorem 11, each invariant quartic, $C_{f_i}, i \in \{14, ..., 20\}$ has exactly six branches. We will explain and give the details of each $C_{f_i}, i \in \{14, 15, 16\}$ and $C_{f_i}, i \in \{17, 18, 19, 20\}$ individually




(i) Multiplicity of $C_{f_{13}}$.




(ii) The graph of $C_{f_{13}}$.

**Figure 4.14:** Behaviour of $C_{f_{13}}$.

I. The irreducible quartic curves, $C_{f_i}, i \in \{14, 15, 16\}$, defined by their corresponding polynomials

$$\begin{aligned}
&\text{I.1:} \;\; f_{14}(x,y) = (1-x)^3 + (y-1)^4 - \frac{1}{4}xy^3 + 5x^2y - \frac{5}{2}xy^2 + \frac{23}{4}xy - 1,\\
&\text{I.2:} \;\; f_{15}(x,y) = x^3(x-1) + \frac{1}{16}y(y-1)^3 + 2x^3y + \frac{3}{2}x^2y^2 + \frac{1}{2}xy^3 - \frac{7}{4}x^2y\\
&\qquad\qquad - xy^2 + \frac{1}{2}xy,\\
&\text{I.3:} \;\; f_{16}(x,y) = x^4 + \frac{28561}{567}y^2(1-y) + \frac{26}{21}x^3y - \frac{169}{63}x^2y - \frac{8788}{189}xy^2,
\end{aligned}$$

are respectively invariants with respect to the system (4.4) under the following correspondence conditions

$$\begin{aligned}
&\text{I.1:} \;\; A = -\frac{7}{3}, \;\; B = \frac{1}{3} \;\text{ and }\; C = -\frac{4}{7},\\
&\text{I.2:} \;\; A = 3, \;\; B = 5 \;\text{ and }\; C = -\frac{5}{6},\\
&\text{I.3:} \;\; A = 2, \;\; B = -\frac{7}{13} \;\text{ and }\; C = \frac{1}{3}.
\end{aligned}$$

We will give the details of each case one by one

I.1. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant quartic curve $C_{f_{14}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity, the critical points for which $C_{f_{14}}$ passes through are $P_4$ and $P_6$, $\big($see Figure 4.15,(i)$\big)$. $P_2$ and $P_f$ are the singular points of $C_{f_{14}}$. By applying Puiseux's expansion of $C_{f_{14}}$ at $P_3$ and $P_f$, we get a tac-node and a cusp branches respectively, $\big($see Table 4.14$\big)$. Clearly by Theorem 12, the index at $P_f$ is equal to 6. For $P_2$, since we have two branches of order two with a double multiplicity at $P_2$ and the ratio of eigenvalues is equal to 2, then by using the equation (2.26), the index is given by

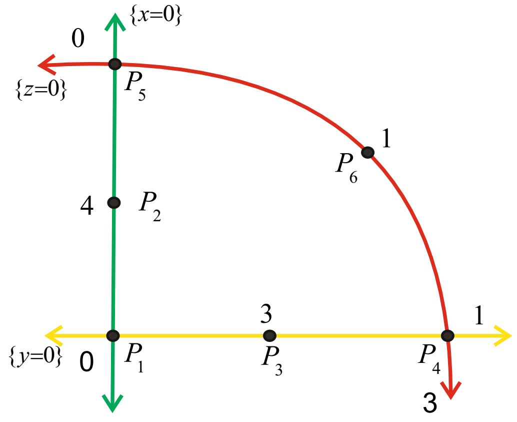
| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_2(singular)$ | 2 | $x = \frac{1}{64}(13+7i\sqrt{7})(y-1)^2 + ...$ or $x = \frac{1}{64}(13-7i\sqrt{7})(y-1)^2 + ...$ | 2(2+2) |
| $P_3$ | 3 | $y = \frac{4}{27}(x-1)^3 + ...$ | 3 |
| $P_f(singular) = (-98, -27)$ | $\frac{3}{2}$ | $(y+27)^2 = k(x+98)^3 + ..., k \neq 0$ | 6 |
| $P_4$ | 3 | $Z = -\frac{1}{4}Y^3 + ...$ | 3 |
| $P_6 = (Z=0, Y=\frac{1}{4})$ | -4 | $Z = -\frac{1}{2}(Y-\frac{1}{4}) + ...$ | -4 |

**Table 4.14:** Behaviour of $C_{f_{14}}$.

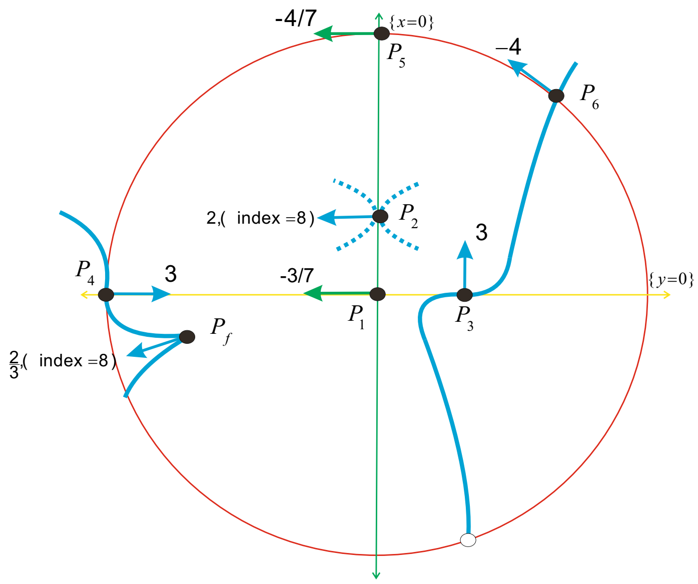
$2(2+2) = 8$. More details about the invariant quartic curve $C_{f_{14}}$ can be found in Table 4.14 and Figure 4.15. Clearly the sum of the indices in Table 4.14 is equal to $4^2 = 16$.

I.2. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant quartic curve $C_{f_{15}}$ passes through the critical points $P_1$, $P_2$, $P_3$ and $P_f$, while at infinity, the only critical point for which $C_{f_{15}}$ passes through is $P_6$, (see Figure 4.16,(i)). $P_2$ and $P_6$ are the singularities of $C_{f_{15}}$. By applying Puiseux's expansion of $C_{f_{15}}$ at $P_2$, we get a cusp branch, (see Table 4.15). Clearly the index at $P_2$ is equal to 6. For $P_6$, the branch of the projective version of $C_{f_{15}}$ at $P_6$ is of a tac-node type, since we have two branches of order two with a double multiplicity and the ratio of eigenvalues at $P_6$ is equal to 2, then the index is given by $2(2+2) = 8$. More details about the invariant quartic curve $C_{f_{15}}$ can be found in Table 4.15 and Figure 4.16 and the sum of indices is equal to 16.

I.3. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant quartic curve $C_{f_{16}}$ passes through the critical points $P_1$, $P_2$ and $P_f$, while at infinity, the critical points for which $C_{f_{16}}$ passes through are $P_5$ and $P_6$, (see Figure

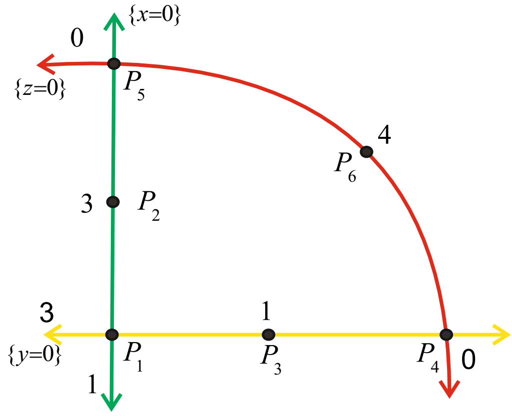



(i) Multiplicity of $C_{f_{14}}$.

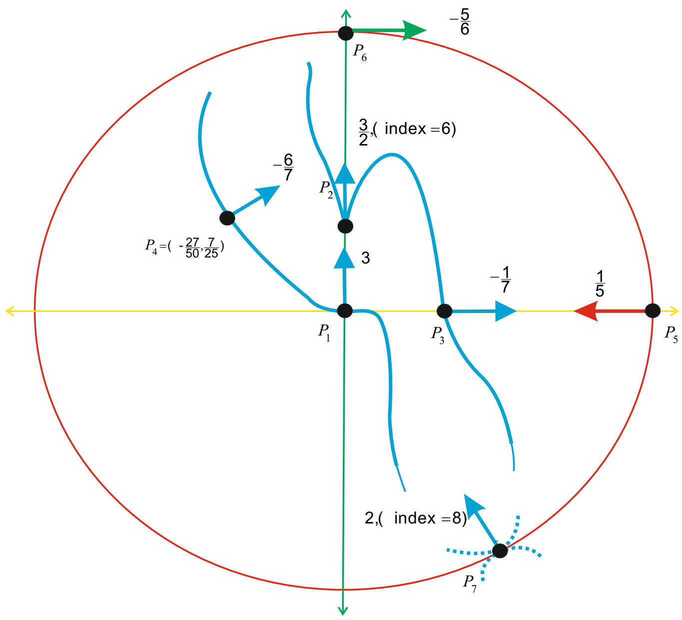



(ii) The graph of $C_{f_{14}}$.

**Figure 4.15:** Behaviour of $C_{f_{14}}$.




(i) Multiplicity of $C_{f_{15}}$.




(ii) The graph of $C_{f_{15}}$.

**Figure 4.16:** Behaviour of $C_{f_{15}}$.

| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_1$ | 3 | $y = -16x^3 + ...$ | 3 |
| $P_2(singular)$ | $\frac{2}{3}$ | $(y-1)^3 = kx^2 + ..., k \neq 0$ | 6 |
| $P_3$ | $-\frac{1}{7}$ | $y = -\frac{16}{11}(x-1) + ...$ | $-\frac{1}{7}$ |
| $P_f = \left(-\frac{27}{50}, \frac{7}{25}\right)$ | $-\frac{6}{7}$ | $y - \frac{7}{25} = -\frac{8}{9}(x + \frac{27}{50}) + ...$ | $-\frac{6}{7}$ |
| $P_6(singular) = (Z=0, Y=-2)$ | 2 | $Z = \frac{1}{4}(1+i\sqrt{5})(Y+2)^2 + ...$ or $Z = \frac{1}{4}(1-i\sqrt{5})(Y+2)^2 + ...$ | 2(2+2) |

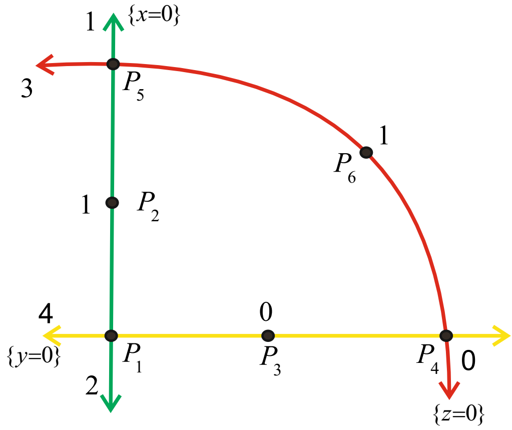
**Table 4.15:** Behaviour of $C_{f_{15}}$.

| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_1(singular)$ | 2 | $y = \frac{1}{338}(9+27i\sqrt{3})x^2 + ...$ or $y = \frac{1}{338}(9-27i\sqrt{3})x^2 + ...$ | 2(2+2) |
| $P_2$ | 6 | $y - 1 = -\frac{12}{13}x + ...$ | 6 |
| $P_f(singular) = (13, -9)$ | $\frac{2}{3}$ | $(y+9)^3 = k(x-13)^2 + ...,\ k \neq 0$ | 6 |
| $P_5$ | 3 | $Z = \frac{54}{2197}X^3 + ...$ | 3 |
| $P_6 = (X = -\frac{26}{21}, Z = 0)$ | -7 | $Z = \frac{8}{13}(X + \frac{26}{21}) + ...$ | -7 |

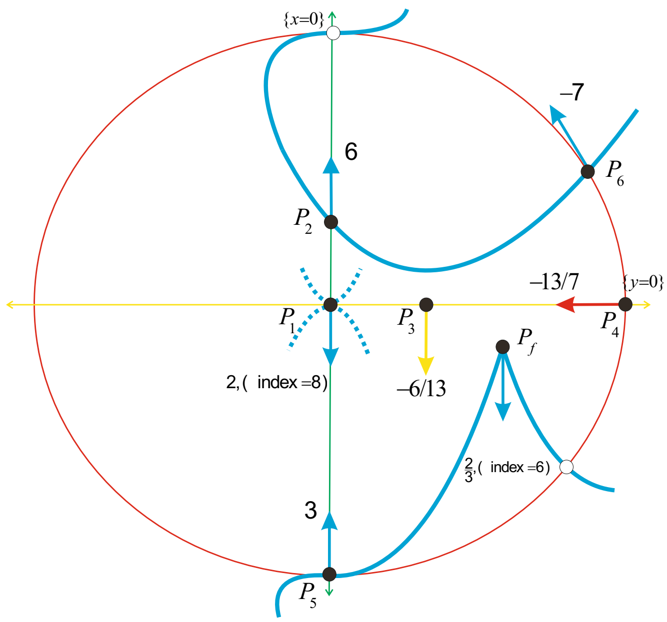
**Table 4.16:** Behaviour of $C_{f_{16}}$.

4.17,(i)). $P_1$ and $P_f$ are the singular points of $C_{f_{16}}$. By applying Puiseux's expansion of $C_{f_{16}}$ at $P_1$ and $P_f$, we get a tac-node and a cusp branches respectively, (see Table 4.16). Clearly the index at $P_f$ is equal to 6. For $P_1$, the branch is of the same what we have got in $C_{f_{14}}$ at $P_2$, hence the index is given by $2(2+2) = 8$. More details about the invariant quartic curve $C_{f_{16}}$ can be found in Table 4.16 and Figure 4.17. It is easy to verify that the sum of the indices is 16.

II. The irreducible quartic curves, $C_{f_i}, i \in \{17, 18, 19, 20\}$, defined by their




(i) Multiplicity of $C_{f_{16}}$.




(ii) The graph of $C_{f_{16}}$.

**Figure 4.17:** Behaviour of $C_{f_{16}}$.

corresponding polynomials

$$\begin{aligned}
\text{II.1: } f_{17}(x,y) =& x^2+8y(y-1)^3-4xy^2+12xy,\\
\text{II.2: } f_{18}(x,y) =& (1-x)^3+(y-1)^4+6x^2y-3xy^2+6xy-1,\\
\text{II.3: } f_{19}(x,y) =& (x-1)^2+(y-1)^4+xy^2-8xy-1 \text{ and}\\
\text{II.4: } f_{20}(x,y) =& (x-1)^4+(y-1)^4-4ix^3y-6x^2y^2+4ixy^3+\\
& 4(-1+2i)x^2y+4xy^2+4(2-i)xy-1,
\end{aligned}$$

are respectively invariants with respect to (4.4) under the following conditions

$$\begin{aligned}
&\text{II.1: } A=2,\ \ B=\frac{1}{2} \text{ and } C=-\frac{5}{2},\\
&\text{II.2: } A=-\frac{9}{4},\ \ B=\frac{1}{4} \text{ and } C=-\frac{5}{9},\\
&\text{II.3: } A=-\frac{3}{2},\ \ B=\frac{1}{2} \text{ and } C=-\frac{7}{3} \text{ and}\\
&\text{II.4: } A=\frac{i-2}{5},\ \ B=\frac{i-3}{2} \text{ and } C=i-1.
\end{aligned}$$

We will give the details of each case individually

II.1. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant quartic curve $C_{f_{17}}$ passes through the critical points $P_1$, $P_2$ and $P_f$, while at infinity, the only critical point for which $C_{f_{17}}$ passes through is $P_4$, (see Figure 4.18,(i)). $P_f$ and $P_4$ are the singularities of $C_{f_{17}}$. By applying Puiseux's expansion of $C_{f_{17}}$ at $P_f$, we get node branches, (see Table 4.17). Since the ratio of eigenvalues at $P_f$ is equal to $\frac{-3+i\sqrt{3}}{-3-i\sqrt{3}}$ and the multiplicity of both branches is equal to 1, then the index at $P_f$ is equal to $\Big(2+\big(\frac{-3+i\sqrt{3}}{-3-i\sqrt{3}}\big)+\big(\frac{-3-i\sqrt{3}}{-3+i\sqrt{3}}\big)\Big)=3$. For $P_4$, the branch of the projective version of $C_{f_{17}}$ is

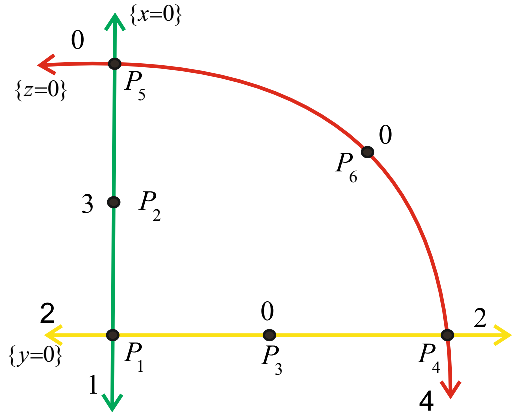
| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_1$ | 2 | $y = \frac{1}{8}x^2 + ...$ | 2 |
| $P_2$ | 3 | $x = -(y-1)^3 + ...$ | 3 |
| $P_f(singular) = (8,-1)$ | $\frac{-3+i\sqrt{3}}{-3-i\sqrt{3}}$ | $y+1 = \frac{1}{56}(-5+i\sqrt{3})(x-8) + ...,$ or $y+1 = \frac{1}{56}(-5-i\sqrt{3})(x-8) + ...,$ | 3 |
| $P_4(singular)$ | 2 | $Z = 2(1+i)Y^2 + ...$ or $Z = 2(1-i)Y^2 + ...$ | 2(2+2) |

**Table 4.17:** Behaviour of $C_{f_{17}}$.

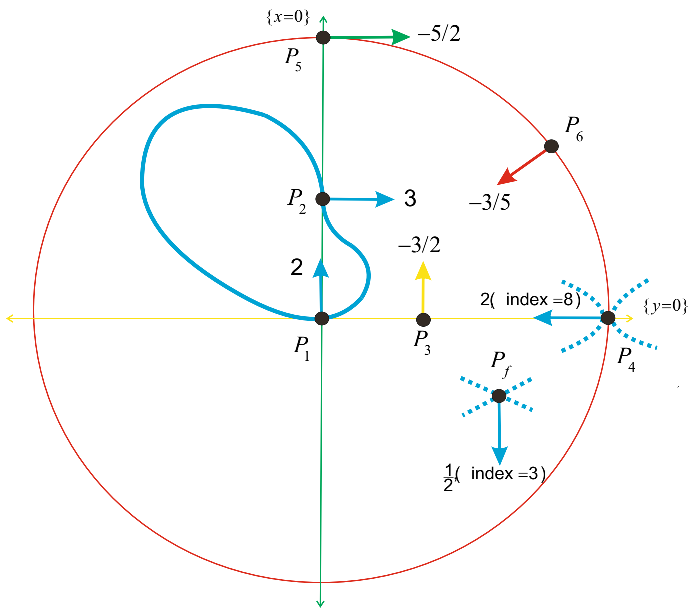
of the same what we have in $C_{f_{15}}$ at $P_6$, hence the branch has index 8. More details about the invariant quartic curve $C_{f_{17}}$ can be found in Table 4.17 and Figure 4.18. It is easy to verify that the sum of the indices is equal to 16.

II.2. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant quartic curve $C_{f_{18}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity, the only critical point for which $C_{f_{18}}$ passes through is $P_4$, $\big($see Figure 4.19,(i)$\big)$. $P_2$ and $P_f$ are the singular points of $C_{f_{18}}$. By applying Puiseux's expansion of $C_{f_{18}}$ at both singularities, we get node branches in both, $\big($see Table 4.18$\big)$. At $P_2$, we have a case similar to the quartic curve $C_{f_{14}}$, hence the index is equal to 8. For $P_f$, since we have two branches of order one and the ratio of eigenvalues is equal to $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$, then the index at $P_f$ is equal to $\Big(2 + \big(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\big) + \big(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\big)\Big) = 1$. More details about the invariant quartic curve $C_{f_{18}}$ can be found in Table 4.18 and Figure 4.19. Notice that the sum of the indices is equal to 16.

II.3. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant quartic curve $C_{f_{19}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity, the only crit-

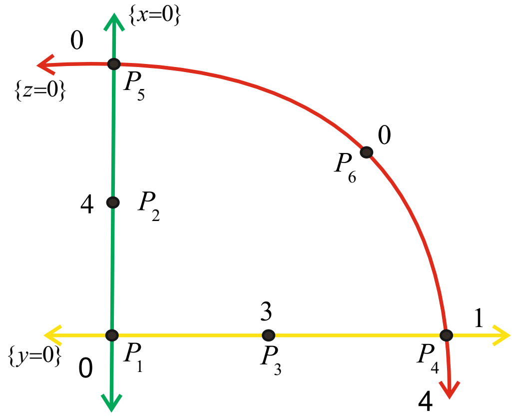



(i) Multiplicity of $C_{f_{17}}$.

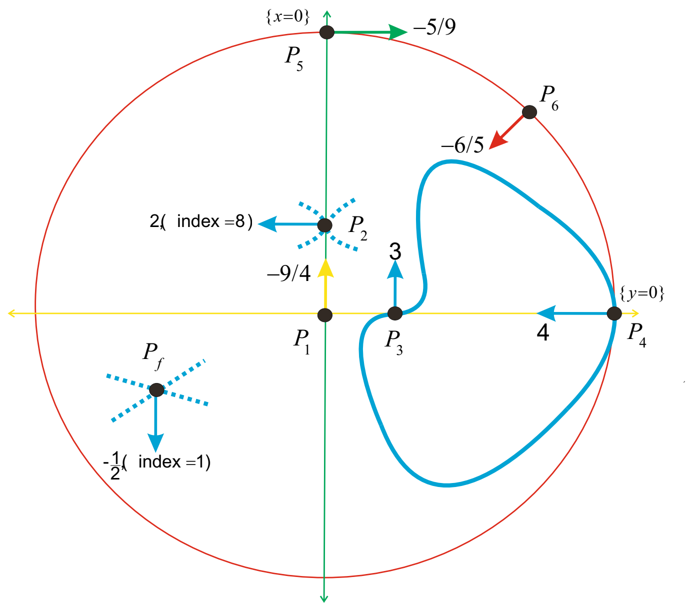



**(A)** the graph of $C_{f_{17}}$.

**Figure 4.18:** Behaviour of $C_{f_{17}}$.




(i) Multiplicity of $C_{f_{18}}$.




(ii) The graph of $C_{f_{18}}$.

**Figure 4.19:** Behaviour of $C_{f_{18}}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2(singular)$ | 2 | $x = \frac{1}{6}(1 + i\sqrt{3})(y-1)^2 + ...$ or $x = \frac{1}{6}(1 - i\sqrt{3})(y-1)^2 + ...$ | 2(2+2) |
| $P_3$ | 3 | $y = \frac{1}{8}(x-1)^3 + ...,$ | 3 |
| $P_f(singular) = (-27, -8)$ | $-\frac{1}{2}$ | $y + 8 = \frac{1}{21}(5 + i\sqrt{3})(x + 27) + ...,$ or $y + 8 = \frac{1}{21}(5 - i\sqrt{3})(x + 27) + ...,$ | 1 |
| $P_4$ | 4 | $Z = Y^4 + ...$ | 4 |

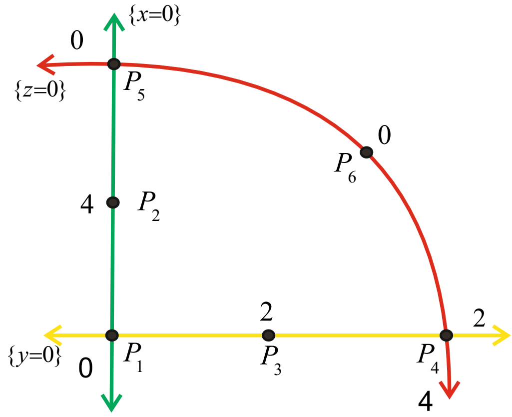
**Table 4.18:** Behaviour of $C_{f_{18}}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2$ | 4 | $x = \frac{1}{9}(y-1)^4 + ...,$ | 4 |
| $P_3$ | 2 | $y = \frac{1}{12}(x-1)^2 + ...,$ | 2 |
| $P_f(singular) = (-9, -2)$ | $\frac{1+i}{1-i}$ | $y + 2 = \frac{1}{15}(2 + i)(x + 9) + ...,$ or $y + 2 = \frac{1}{15}(2 - i)(x + 9) + ...,$ | 2 |
| $P_4(singular)$ | 2 | $Z = \frac{1}{2}(-1 + i\sqrt{3})Y^2 + ...$ or $Z = \frac{1}{2}(-1 - i\sqrt{3})Y^2 + ...$ | 2(2+2) |

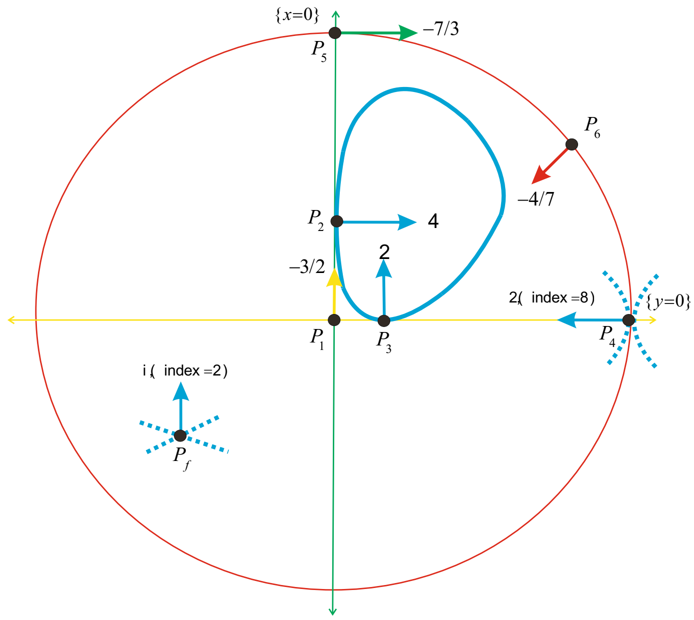
**Table 4.19:** Behaviour of $C_{f_{19}}$.

ical point for which $C_{f_{19}}$ passes through is $P_4$, $\big($see Figure 4.20,(i)$\big)$. $P_f$ and $P_4$ are the singular points of $C_{f_{19}}$. By applying Puiseux's expansion of $C_{f_{19}}$ at $P_f$, we get a node branch, $\big($see Table 4.19$\big)$. Since the ratio of eigenvalues at $P_f$ is equal to $\frac{1+i}{1-i}$ and the multiplicity of both branches at $P_f$ is equal to one, then the index is equal to $2 + \Big(\big(\frac{1+i}{1-i}\big) + \big(\frac{1-i}{1+i}\big)\Big) = 2,$. For $P_4$, the behaviour of $C_{f_{19}}$ is the same as for $C_{f_{17}}$ at $P_4$. More details about the invariant quartic curve $C_{f_{19}}$ can be found in Table 4.19 and Figure 4.20. Consequently, the sum of the indices is equal to 16.

II.4. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant quartic curve $C_{f_{20}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity, the only critical point for which $C_{f_{20}}$ passes through is $P_6$, $\big($see Figure 4.21,(i)$\big)$.




(i) Multiplicity of $C_{f_{19}}$.




(ii) The graph of $C_{f_{19}}$.

**Figure 4.20:** Behaviour of $C_{f_{19}}$.

| *Critical Points* | *R.E.* | *Branches* | *Indices* |
|---|---|---|---|
| $P_2$ | 4 | $x = -\frac{1}{8}(y-1)^4 + ...$ | 4 |
| $P_3(singular)$ | 2 | $y = \frac{1}{8}(-1+\sqrt{113-16i})(x-1)^2+...$ or $y = \frac{1}{8}(-1-\sqrt{113-16i})(x-1)^2+...$ | 2(2+2) |
| $P_f = \big(-\frac{3}{2}, -(1+\frac{4}{3}i)\big)$ | $-\frac{4}{3}$ | $y+1+\frac{4}{3}i = (\frac{9}{5}-\frac{1}{10}i)(x+\frac{2}{3}) + ...$ | $-\frac{4}{3}$ |
| $P_6$(singular)=(Z=0,Y=-$i$) | 3 | $Z = -\frac{1}{8}(Y+i)^3 + ...$ or $Z = \frac{2}{5}(1-2i)(Y+i) + ...$ | $\frac{16}{3}$ |

**Table 4.20:** Behaviour of $C_{f_{20}}$.

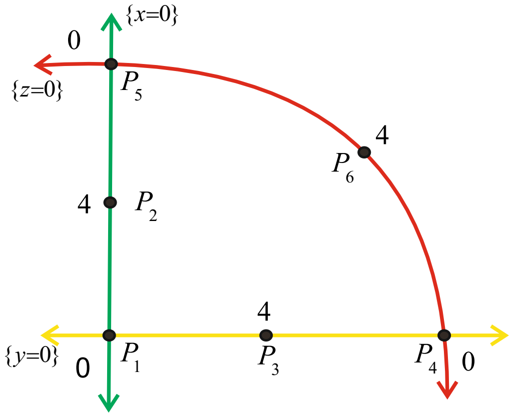
$P_3$ and $P_6$ are the singular points of $C_{f_{20}}$. By applying Puiseux's expansion of $C_{f_{20}}$ at $P_3$, we get the same node branch as we have got in the quartic curve $C_{f_{16}}$ at $P_1$, (see Table 4.20). Hence the index is equal to 8. At $P_6$, we have two branches of order three and one, since the ratios of eigenvalues in one of them is 3 and obviously with multiplicity one, then the index is equal to $2 + \big(3 + \frac{1}{3}\big) = \frac{16}{3}$. More details about the invariant quartic curve $C_{f_{20}}$ can be found in Table 4.20 and Figure 4.21. Consequently the sum of the indices is equal to 16.

**Remark 12.** From the result above, we see that, any invariant quartic curves, $C_{f_i}, i \in \{14, 15, 16, 17, 18, 19, 20\}$ has codimension three.

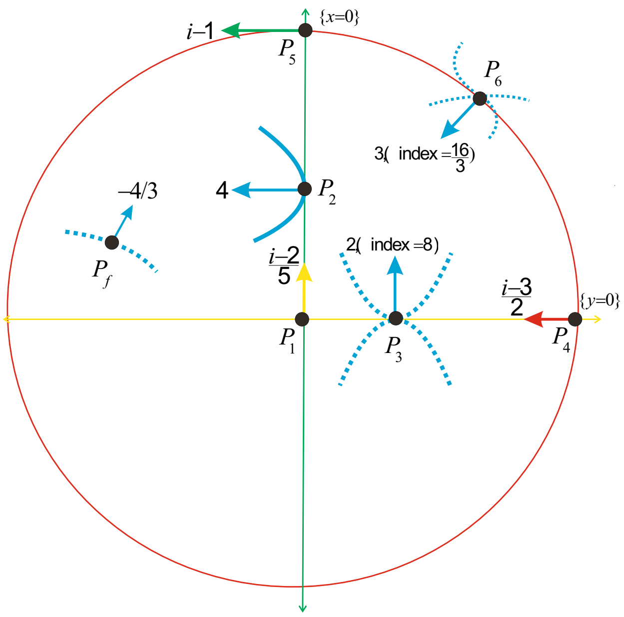
**Corollary 2.** *Any invariant quartic curve in two dimensional Lotka-Volterra system has exactly six branches.*

### 4.2.5 Invariant algebraic sextics

There are three separate irreducible invariant sextic curves $C_{f_i}, i \in \{21, 22, 23\}$, with respect to the system (4.4). We will find that to each sextic curve $C_{f_i}$,




(i) Multiplicity of $C_{f_{20}}$.




(ii) The graph of $C_{f_{20}}$.

**Figure 4.21:** Behaviour of $C_{f_{20}}$.

exactly three of the critical points for which $C_{f_i}$ passes through are singularities; Moreover the sextic curve $C_{f_{23}}$ is of complex coefficients. We will find that, each sextic curve $C_{f_i}$ has node branches of higher order at their singularities. As we will show that each sextic curve $C_{f_i}, i \in \{21, 22, 23\}$ is parametrisable, then the geometric genus of each sextic curve is equal to zero, hence each sextic curve has exactly eight branches. The irreducible sextic curves, $C_{f_i},\ i \in \{21, 22, 23\}$, defined by their corresponding polynomials

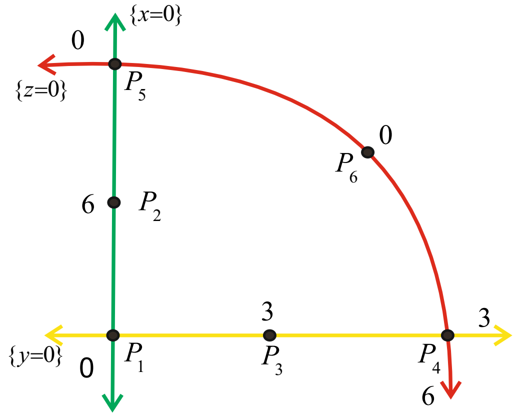
$$\begin{aligned}
\text{I: } f_{21}(x,y) =& (1-x)^3 + (y-1)^6 + \frac{1}{2}x^2y^2 + 9x^2y + 2xy^4 - 3xy^3 - 3xy^2 + 7xy \\
& - 1, \\
\text{II: } f_{22}(x,y) =& (x-1)^4 + (y-1)^6 - 8x^3y + x^2y^3 + 4x^2y^2 - 11x^2y - 4xy^4 \\
& + 16xy^3 - 24xy^2 + 16xy - 1, \\
\text{III: } f_{23}(x,y) =& (x-1)^6 + (y-1)^6 + 3(1 - i\sqrt{3})x^5y + \\
& \frac{15}{2}(-1 - i\sqrt{3})x^4y^2 + 6(-3 + 2i\sqrt{3})x^4y - 20x^3y^3 + 3(5 + 9i\sqrt{3})x^3y^2 \\
& + 6(7 - 3i\sqrt{3})x^3y + \frac{15}{2}(-1 + i\sqrt{3})x^2y^4 + 42x^2y^3 \\
& + \frac{1}{2}(15 - 63i\sqrt{3})x^2y^2 + 12(-4 + i\sqrt{3})x^2y + 3(1 + i\sqrt{3})xy^5 \\
& - 12i\sqrt{3}xy^4 + 6xy^3 + 6(-5 + 2i\sqrt{3})xy^2 + 3(9 - i\sqrt{3})xy - 1,
\end{aligned}$$

are respectively invariants with respect to the system (4.4) under the following correspondence conditions

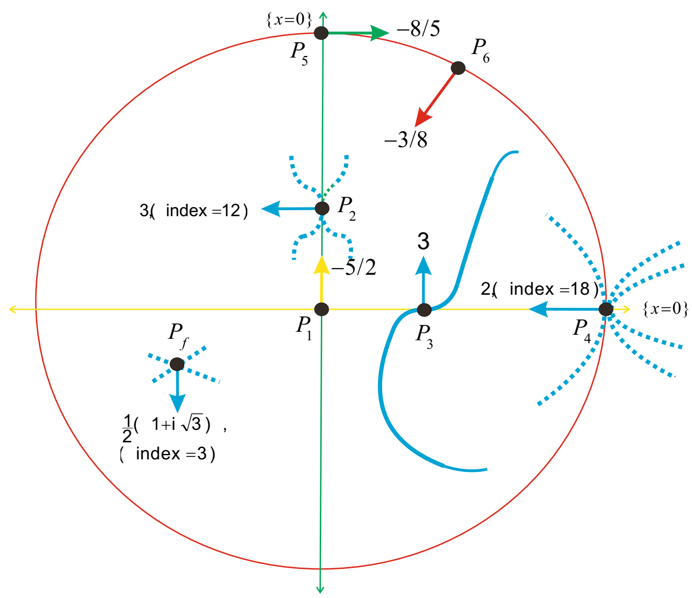
$$\begin{aligned}
&\text{I: } A = -\frac{5}{2}, \ B = \frac{1}{2} \ \text{ and } \ C = -\frac{8}{5}, \\
&\text{II: } A = -\frac{10}{3}, \ B = \frac{1}{3} \ \text{ and } \ C = -\frac{7}{10}, \\
&\text{III: } A = \frac{-5+i\sqrt{3}}{14}, \ B = \frac{6}{-9+i\sqrt{3}} \ \text{ and } \ C = \frac{-3+i\sqrt{3}}{2}.
\end{aligned}$$

I. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant sextic curve $C_{f_{21}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity, the only critical point for which $C_{f_{21}}$ passes through is $P_4$, $\big($see Figure 4.22,(i)$\big)$. $P_2$, $P_f$ and $P_4$ are the singularities of $C_{f_{21}}$. By applying Puiseux's expansion of $C_{f_{21}}$ at $P_2$ and $P_f$, we get two branches of order three at $P_2$ and two branches of order one at $P_f$, $\big($see Table 4.21$\big)$. For $P_2$, since the ratio of eigenvalues is equal to 3 and the multiplicity between both of them is 3, then the index is equal to $2(3+3)$. At $P_f$, the ratio of eigenvalues is equal to $\frac{3+i\sqrt{3}}{3-i\sqrt{3}}$ and the multiplicity is one, then the index is equal to $2 + \Big(\big(\frac{3+i\sqrt{3}}{3-i\sqrt{3}} + \frac{3-i\sqrt{3}}{3+i\sqrt{3}}\big)\Big) = 3$. At $P_4$, the projective version of $C_{f_{21}}$ has three branches of order two with multiplicity two between each other, hence the index is equal to $3(2+2+2)$. More details about the invariant sextic curve $C_{f_{21}}$ can be found in Table 4.21 and Figure 4.22. It is to verify that the sum of the indices in Table 4.21 is equal to $6^2 = 36$.

II. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant sextic curve $C_{f_{22}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity, the only




(i) Multiplicity of $C_{f_{21}}$.




(ii) The graph of $C_{f_{21}}$.

**Figure 4.22:** Behaviour of $C_{f_{21}}$.

| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_2(singular)$ | 3 | $x=\frac{1}{5}(-1+i)(y-1)^3+...$ or $x=\frac{1}{5}(-1-i)(y-1)^3+...$ | 2(3+3) |
| $P_3$ | 3 | $y=\frac{1}{10}(x-1)^3+...$ | 3 |
| $P_f(singular)=(-25,-4)$ | $\frac{3+i\sqrt{3}}{3-i\sqrt{3}}$ | $y+4=\frac{1}{65}(7+i\sqrt{3})(x+25)+...$ or $y+4=\frac{1}{65}(7-i\sqrt{3})(x+25)+...$ | 3 |
| $P_4(singular)$ | 2 | $Z=k_1Y^2+...$ or $Z=k_2Y^2+...$ or $Z=k_3Y^2+...$ | 3(2+2+2) |

**Table 4.21:** Behaviour of $C_{f_{21}}$.

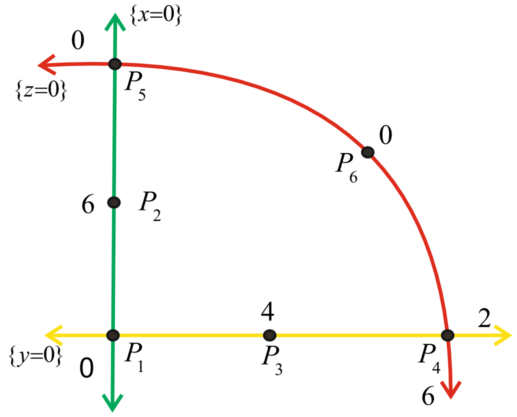
where

$$k_1=\frac{1}{6}\frac{(145+30\sqrt{6})^{\frac{2}{3}}+(145+30\sqrt{6})^{\frac{1}{3}}+25}{(145+30\sqrt{6})^{\frac{1}{3}}},$$

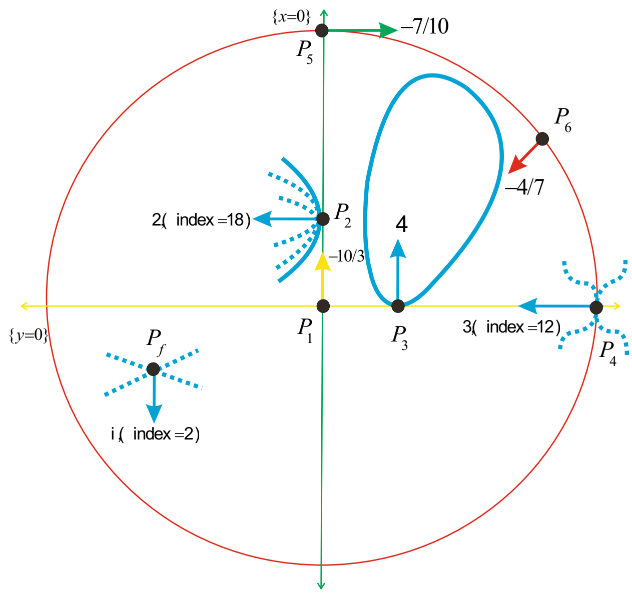
$$k_2=\frac{1}{12}\frac{i(145+30\sqrt{2}\sqrt{3})^{\frac{2}{3}}\sqrt{3}-(145+30\sqrt{2}\sqrt{3})^{\frac{2}{3}}-(25i)\sqrt{3}+2(145+30\sqrt{2}\sqrt{3})^{\frac{1}{3}}-25}{(145+30\sqrt{6})^{\frac{1}{3}}},$$

$$k_3=-\frac{1}{12}\frac{i(145+30\sqrt{2}\sqrt{3})^{\frac{2}{3}}\sqrt{3}+(145+30\sqrt{2}\sqrt{3})^{\frac{2}{3}}-(25i)\sqrt{3}-2(145+30\sqrt{2}\sqrt{3})^{\frac{1}{3}}+25}{(145+30\sqrt{6})^{\frac{1}{3}}}.$$

critical point for which $C_{f_{22}}$ passes through is $P_4$, (see Figure 4.23,(i)). $P_2$, $P_f$ and $P_4$ are the singular points of $C_{f_{22}}$. By applying Puiseux's expansion of $C_{f_{22}}$ at $P_2$ and $P_f$, we get three branches of order two at $P_2$ and two branches of order one at $P_f$, (see Table 4.22). For $P_2$, since the ratio of eigenvalues is equal to 2 and the multiplicity among pairwise branches is 2, then the index is equal to $3(2+2+2)$. At $P_f$, the ratio of eigenvalues is equal to $\frac{1+i}{1-i}$ and the multiplicity is one, then the index is equal to $2+\left(\frac{1+i}{1-i}+\frac{1+i}{1-i}\right)=2$. At $P_4$, the projective version of $C_{f_{22}}$ has two branches of order three with multiplicity three between each other, hence the index is equal to $2(3+3)$. More details about the invariant sextic curve $C_{f_{22}}$ can be found in Table 4.22 and Figure 4.23. Clearly the sum of the indices is equal to 36.




(i) Multiplicity of $C_{f_{22}}$.




(ii) The graph of $C_{f_{22}}$.

**Figure 4.23:** Behaviour of $C_{f_{22}}$.

| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_2(singular)$ | 2 | $x = \frac{1}{3}(y-1)^2 + ...$<br>or<br>$x = \frac{1}{8}(1+i\sqrt{15})(y-1)^2 + ...$<br>or<br>$x = \frac{1}{8}(1-i\sqrt{15})(y-1)^2 + ...$ | 3(2+2+2) |
| $P_3$ | 4 | $y = \frac{1}{9}(x-1)^4 + ...$ | 4 |
| $P_f(singular) = (-50, -9)$ | $\frac{1+i}{1-i}$ | $y + 9 = \frac{3}{85}(4+i)(x+50) + ...$<br>or<br>$y + 9 = \frac{3}{85}(4-i)(x+50) + ...$ | 2 |
| $P_4(singular)$ | 3 | $Z = \frac{1}{2}(-1+i\sqrt{3})Y^3 + ...$<br>or<br>$Z = \frac{1}{2}(-1-i\sqrt{3})Y^3 + ...$ | 2(3+3) |

**Table 4.22:** Behaviour of $C_{f_{22}}$.

III. At the affine part of $\mathbf{P}^2(\mathbb{C})$, the invariant sextic curve $C_{f_{23}}$ passes through the critical points $P_2$, $P_3$ and $P_f$, while at infinity, the only critical point for which $C_{f_{23}}$ passes through is $P_6$, (see Figure 4.24,(i)). $P_2$, $P_3$ and $P_6$ are the singularities of $C_{f_{23}}$. By applying Puiseux's expansion of $C_{f_{23}}$ at $P_2$ and $P_3$, we get two branches, one of order five and the other of order one at $P_2$, while we get three branches each of order two at $P_3$, (see Table 4.23). For $P_2$, since the branches intersect each other transversely and the ratio of eigenvalues with respect to one of them is equal to 5, then the index is equal to $2 + \left(5 + \frac{1}{5}\right)$. At $P_3$, the ratio of eigenvalues is equal to 2 and they have pairwise multiplicity equal to 2, then the index is equal to $3(2+2+2)$. At $P_6$, the projective version of $C_{f_{23}}$ has two branches of order three with multiplicity three between each other, hence the index is equal to $2(3+3)$. More details about the invariant sextic curve $C_{f_{23}}$ can be found in Table 4.23 and Figure 4.24. It is easy to verify that the sum of the indices is equal to 36.

| ***Critical Points*** | ***R.E.*** | ***Branches*** | ***Indices*** |
|---|---|---|---|
| $P_2(singular)$ | 5 | $x = -\frac{i\sqrt{3}}{36}(y-1)^5 + ...$ or $x = \frac{4}{9}(-4+i\sqrt{3})(y-1) + ...$ | $\frac{36}{5}$ |
| $P_3(singular)$ | 2 | $y = k_1(x-1)^2 + ...$ or $y = k_2(x-1)^2 + ...$ or $y = k_3(x-1)^2 + ...$ | 3(2+2+2) |
| $P_f = \big(-\frac{5}{9}, -\frac{1}{9}(11+5i\sqrt{3})\big)$ | $-\frac{6}{5}$ | $y + \frac{1}{9}(11+5i\sqrt{3}) = \frac{1}{7}(13-3i\sqrt{3})(x+\frac{5}{9}) + ...$ | $-\frac{6}{5}$ |
| $P_6(singular) = \big(Z=0, Y=-\frac{1}{2}(1+i\sqrt{3})\big)$ | 3 | $Z = \frac{i}{3}(2+\sqrt{3})(Y+\frac{1}{2}(1+i\sqrt{3})^3 + ...$ or $Z = \frac{i}{3}(2-\sqrt{3})(Y+\frac{1}{2}(1+i\sqrt{3})^3 + ...$ | 2(3+3) |

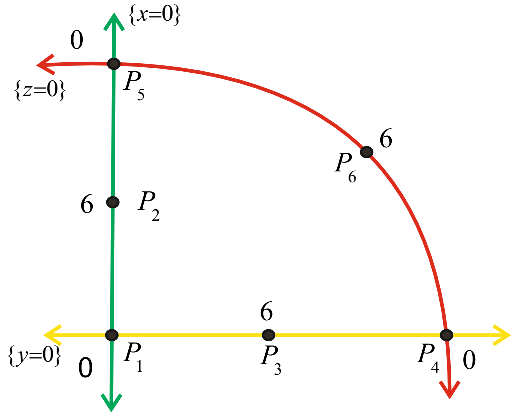
**Table 4.23:** Behaviour of $C_{f_{23}}$.

where

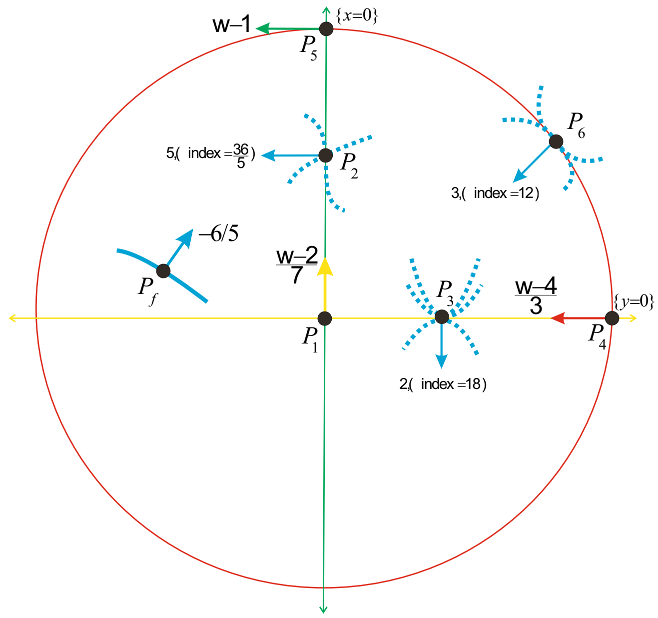
$$k_1 = -\frac{1}{16}\left(\frac{3i\sqrt{3}(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}} - 31i\sqrt{3} - 2(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})}{27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}}}, + \frac{5(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}}}{27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}}}\right),$$

$$k_2 = -\frac{1}{16}\left(\frac{i\sqrt{3}(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{2}{3}} - 3i\sqrt{3}(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}} - 23i\sqrt{3}}{(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}}}, + \frac{-(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{2}{3}} - 5(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}} + 39}{(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}}}\right),$$

$$k_3 = \frac{1}{16}\left(\frac{i\sqrt{3}(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{2}{3}} + 3i\sqrt{3}(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}} + 8i\sqrt{3}}{(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}}} + \frac{+(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{2}{3}} + 5(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}} + 54}{(27-66i\sqrt{3}+8\sqrt{54+78i\sqrt{3}})^{\frac{1}{3}}}\right).$$




(i) Multiplicity of $C_{f_{23}}$.




(ii) The graph of $C_{f_{23}}$.

**Figure 4.24:** Behaviour of $C_{f_{23}}$.

**Remark 13.** From the above results, we see that any invariant sextic curve, $C_{f_i}, i \in \{21, 22, 23\}$ has codimension three.

**Corollary 3.** *Any invariant sextic curve in two dimensional Lotka-Volterra system has exactly eight branches.*

In this chapter, we have investigated that any invariant algebraic curve has the following properties that are not mentioned in Ollagnier (2001).

**1.** Any invariant algebraic curve has zero geometric genus, and then the number of branches of each invariant algebraic curve is equal to two plus the order of the curve.

**2.** Each invariant algebraic curve except the invariant line, $C_{f_2} = 1 - x - y$ passes through the face point.

**3.** Invariant algebraic lines are of codimension one, invariant conics are of codimension two, while the others are of codimension three.

**4.** Also geometrically, we have described the behaviour of all invariant algebraic curves.

**5.** The results of this chapter can be found in Christopher and Wuria (2015), preprint.

# Chapter 5

# Integrability and Linearizibility of Two Dimensional Non-Degenrate Lotka-Volterra Systems in $\mathbf{C^2}$

We recall that the results of this chapter are obtained in collaboration with C.Christopher and Z. Wang. For more extensive presentation, see the preprint Christopher et al. (2015), already submitted for publication.

## 5.1 Introduction

In Christopher and Rousseau (2004) the authors considered various results around the integrability and linearisability of the origin for the Lotka-Volterra equations

$$\dot{x} = x(1 + ax + by), \qquad \dot{y} = y(-\lambda + cx + dy), \tag{5.1}$$

for rational values of $\lambda$. In particular, for $\lambda = p/q$ with $p + q \leq 12$ they showed that all the integrability conditions were generated by two mechanisms. First, when

$$\lambda ab + (1 - \lambda)ad - cd = 0, \tag{5.2}$$

there is an invariant line $L = 0$. Using the classical theory of Darboux, a first integral of the form $x^p y^q L^\alpha$ could be found. Furthermore, the conditions for linearisability can be given explicitly. Second, the integrability of the origin could be explained by a monodromy argument. That is, the ratios of eigenvalues of the critical points on the axes and the line at infinity implied that the monodromy map around the origin was linearisable, and hence the critical point was in fact integrable. When (5.2) does not hold, it was shown that the conditions for integrability automatically imply linearisability. Several results for more general values of $\lambda$ were given. In addition, by comparison with the results of Ollagnier (2001) on Liouvillian integrability of Lotka-Volterra systems, two exceptional cases were found when $\lambda = 8/7$ and $13/7$ which turned out to support invariant algebraic curves and were hence solvable by the Darboux method. Some further results were announced in Liu et al. (2004) and Gravel and Thibault (2002). Our aim in this work is to extend this investigation to the critical points of (5.1) which do not lie at the origin. In particular, if the critical point with ratio of eigenvalues $-p/q$ lies on one of the axes (the "side" case) we show that for $p+q \leq 17$ that the critical point is integrable if either there is an invariant line passing though the point (and the system is reducible to the conditions found above via a projective transformation), or there is a monodromy argument involving the monodromy group of the axes and the line at infinity and possibly an invariant algebraic curve passing through the critical point. If, on the other hand, the critical point does not lie on the axes (the "face" case) we show that for $p+q \leq 12$ that the critical point is integrable if either there is an invariant line passing though the point (and again the system is reducible to the conditions found in Christopher and Rousseau (2004)), or there is a monodromy argument involving an invariant algebraic curve passing through the critical point. We also return to the origin case considered in Christopher and Rousseau (2004) and extend the classification to $p+q \leq 20$. We show that no new cases appear, and furthermore, that the two exceptional cases mentioned above can be considered as arising from monodromy arguments involving the invariant curves and the axes. The work is arranged as follows. In the next section we give a brief summary of the monodromy method in the form that we use here. We will also explain how we can extend the monodromy method to some of the invariant algebraic curves found in Ollagnier (2001). The geometric classification of these curves and their singularities is part of a more extensive investigation to be published elsewhere. Here, however, we want to show that the method can still be applied in some cases where the invariant curve has singularities. Finally, in Section 5.3, we give our results. Since the computation of integrability conditions is now a well-trodden area, we do not give extensive sets of conditions, but merely indicate the classes of systems involved and some indicative examples.

## 5.2 The monodromy method

Recall that a polynomial vector field

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

gives rise to an analytic foliation (with singularities)

$$P(x, y)\, dy - Q(x, y)\, dx = 0.$$

Such a foliation extends in a natural way to $\mathbf{P}^2(\mathbb{C})$. If $P$ and $Q$ have degree $n$, then the line at infinity is invariant if $xQ_n - yP_n \neq 0$, where $P_n$ and $Q_n$ are the terms of degree $n$ in $P$ and $Q$ respectively. For the Lotka-Volterra system (5.1) we therefore have three invariant lines: the $x$ and $y$-axes and the line at infinity. Since we work over $\mathbf{P}^2(\mathbb{C})$, these lines are really copies of the Riemann Sphere. In the neighbourhood of each line we can consider the monodromy group as follows. We fix a family of transversals to the line which pass through all points on the line which are non-singular points of the vector field (call this set of points $S$).

We also fix a point $p \in S$ on the line and denote its transversal $\Sigma$. For each closed path, $\gamma$, starting at $p$ and each point, $q$, on $\Sigma$ sufficiently close to $p$ we can lift the path to a unique curve on the leaf of the foliation through $q$. On returning to $\Sigma$ this curve will intersect $\Sigma$ at a new point $q'$. If we are given a local parameter $z$ for $\Sigma$ with $z(p) = 0$, then the map from $q$ to $q'$ will define the germ of a local analytic function $M_\gamma : (\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$. This map is called the monodromy map associated to the path $\gamma$. The map $M_\gamma$ only depends on the path up to homotopy in $S$. Furthermore, a change in the transversal or its parametrisation will give a monodromy map conjugate to the original one. Finally, the monodromy map for a composition of paths is just the compositions of the monodromies ($M_{\alpha\circ\beta} = M_\alpha \circ M_\beta$). It is well known that there is a close connection between the conjugacy class of the monodromy map about a path surrounding a single critical point and the analytic classification of the critical point itself. In particular, a critical point which is a saddle is integrable (i.e. can be orbitally linearised) if and only if the monodromy map is linearisable. The monodromy method consists of the finding simple conditions which guarantee that the monodromy around a critical point is linearisable by considering the monodromy maps of the other points on the line. Since we are working on the Riemann Sphere, the monodromy $M_1$ about a critical point, $Q_1$, is just the inverse of the composition of the monodromies, $M_k$, about the other critical points, $Q_k$. Thus, if $M_2$ is linearisable and $M_k$ is the identity map for $k > 2$, then it is clear that $M_1$ is also linearisable and hence the critical point $Q_1$ must be integrable. The power of the method lies in the fact that in many cases it is easy to give conditions for a critical point to have identity or linearisable monodromy. Consider a critical point whose ratio of eigenvalues is $\lambda$. Here, we take the ratio of eigenvalues so that the eigenvalue associated to the tangential direction to the line is on the denominator. If $\lambda$ is a positive rational number which is not an integer or the reciprocal of an integer then the critical point is a linearisable node, and hence has linearisable monodromy. In the case when $\lambda$ or $1/\lambda$ is a positive integer, then the node may have at most one resonant term. If this resonant term is zero, the node must be linearisable as above. This can be established by a simple computation. However, in most cases the linearisability can be seen geometrically: if the node is resonant then there is no analytic separatrix passing through the critical point tangent to the eigenvector with smaller (in absolute terms) eigenvalue. Thus, if $\lambda$ is a positive integer greater than one, then the fact that line itself is such a separatrix shows that the node must be resonant, and the monodromy just the identity. Similarly, if the node occurs at the crossing of two invariant lines, it must also be non-resonant. What we have said about lines will also work for any smooth invariant curve of geometric genus 0 (i.e. conics). More generally, since the monodromy only "sees" the branches of the curve, we can also apply the method to geometric genus 0 curves whose singularities only have smooth branches (that is, the curve has at most ordinary multiple points). If the curve has singularities then a further investigation needs to take place. However, for our investigation we need only consider one such case: when the curve has a cusp. In this case, the associated critical point at the cusp is a node with ratio of eigenvalues 2/3. Since such a node is linearisable, we can locally find an analytic transformation bringing it to the form $\dot{x} = 2x$, $\dot{y} = 3y$ with invariant curve $y^2 = x^3$. The monodromy can be calculated directly from the parametrization of a loop around the critical point, $(x, y) = (e^{2i\theta}, e^{3i\theta})$, which shows that the monodromy is in fact the identity. Alternatively, and more geometrically, the monodromy will be preserved under blowing up. If we blow-up the cusp singularity we get a smooth branch with ratio of eigenvalues 2. We can therefore conclude that the monodromy is the identity. In what follows, we shall say that we can apply the monodromy method if all linearisability and identity monodromies are deduced in exactly the ways described above.

## Case I

Here, we have an invariant cubic curve, $C_{f_7}$, mentioned in Section 4.2.3, Case (c), Page 61. The eigenvalue ratios of the critical points at the smooth branches of the curve are 2, 3, 6, and $-8$. There is also a cusp with eigenvalue ratio $2/3$ as showed in Figure 4.8, (ii) and this has the identity monodromy as explained above. Thus, the critical point with eigenvalue ratio $-8$ must be integrable. It also follows that the critical point $P_4$ must also be integrable by considering the monodromy on the $x$-axis.

## Case II

Here, we have an invariant quartic curve, $C_{f_{16}}$, mentioned in Section 4.2.4, I.3, Page 80. The eigenvalue ratios of the critical points at the smooth branches of the curve are 2, 2, 3, 6, and $-7$. There is also a cusp with eigenvalue ratio $2/3$ as showed in Figure 4.17, (ii) and also this has the identity monodromy. Thus, the critical point with eigenvalue ratio $-7$ must be integrable. It follows that the critical point $P_4$ must also be integrable by considering the monodromy on the line at infinity. Finally, the critical point $P_3$ must also be integrable.

## Case III

Here, we have an invariant conic, $C_{f_4}$, mentioned in Section 4.2.2, (b), Page 55. The eigenvalue ratios of the critical points at the smooth branches of the curve are 2, 2, $\frac{2(B+1)}{3B+1}$, and -$\frac{2(B+1)}{3B+1}$. It is easy to verify that exactly one of $\frac{2(B+1)}{3B+1}$ or -$\frac{2(B+1)}{3B+1}$ is not the reciprocal of a positive integer, where $B \neq 1$. Hence if $\frac{2(B+1)}{3B+1}$ is positive and not the reciprocal of a positive integer, then the critical point $P_f$ has a linearisable monodromy and then the critical point $P_3$ is integrable. On the other hand if -$\frac{2(B+1)}{3B+1}$ is positive and not the reciprocal of a positive integer, then the critical point $P_3$ has a linearisable monodromy and then the critical point $P_f$ is integrable.

## Case IV

Here, we have an invariant cubic curve, $C_{f_{13}}$, mentioned in Section 4.2.3, (i), Page 76. The eigenvalue ratios of the critical points at the smooth branches of the curve are $1/2$, 2, 3, 3 and $-3/2$ (not a cusp). Thus the critical point with eigenvalue ratio $-3/2$ must be integrable.

## Case V

Here, we have an invariant quartic curve, $C_{f_{20}}$, mentioned in Section 4.2.4, II.4, Page 89. The eigenvalue ratios of the critical points at the smooth branches of the curve are $1/3$, 3, 2, 2, 4 and $-4/3$. Thus the critical point with eigenvalue ratio $-4/3$ must be integrable.

## 5.3 Results for Lotka-Volterra systems

Now, we return to consider the Lotka-Volterra equation in $\mathbf{P}^2(\mathbb{C})$. On each invariant line (including the one at infinity) we have three critical points. If one of these critical points has identity monodromy and the other monodromy is linearisable, we can conclude that the third critical point also has linearisable monodromy and is hence integrable. In more elaborate cases we might need to iterate this construction. That is, we apply the monodromy method on a line to show that a certain critical point has linearisable monodromy, and then use this knowledge to apply the monodromy method on a second line on which the critical point lies. In the "side" case mentioned below, a third iteration is sometimes needed. We now describe our results. We will split our consideration into three cases. The first considers the integrability of a saddle critical point at the origin. This is the case considered in Christopher and Rousseau (2004). For each $p, q > 0$ with $p + q \leq 20$, we take the general Lotka-Volterra system with a saddle of ratio of eigenvalues $-p/q$ and calculate the first three resonant terms of the normal form. From these calculations we obtain necessary conditions for integrability of the saddle. We then prove the sufficiency of these conditions: either by showing that (5.2) holds and hence there is a Darboux first integral, or by establishing that the critical point is integrable by a monodromy argument.

As an indication of the method, we will give examples of any case in the following theorems.

**Theorem 13.** *If a Lotka-Volterra system has an integrable saddle at the origin with ratio of eigenvalues* $-p/q$ *with* $p+q \leq 20$ *then it falls into one of the following categories*

**1.** The condition (5.2) holds and the system has a Darboux first integral.

**2.** The monodromy method can be applied using two of the invariant lines of the system.

**3.** The monodromy method can be applied using the invariant cubic given in Case $C_{f_7}$ of Chapter 4.

**4.** The monodromy method can be applied using the invariant quartic given in Case $C_{f_{16}}$ of Chapter 4.

We give an example to each case mentioned in Theorem 13.

**1.** The system

$$\dot{x} = x\Big(1 - x - \frac{11}{16}y\Big), \qquad \dot{y} = y\Big(-\frac{3}{16} - \frac{1}{8}x + \frac{3}{16}y\Big), \tag{5.3}$$

satisfies the condition (5.2), where

$$\lambda = -\frac{3}{16}, \quad a = -1, \quad b = -\frac{11}{16}, \quad c = -\frac{1}{8} \quad \text{and} \quad d = \frac{3}{16}.$$

Hence (5.3) is integrable at the origin. For more details, see Figure 5.1, (1).

**2.** Consider the system

$$\dot{x} = x\Big(1 - x + \frac{1}{2}y\Big), \qquad \dot{y} = y\Big(-\frac{3}{10} + x + \frac{3}{10}y\Big), \tag{5.4}$$

From Figure 5.1, (2), the line at infinity, $L_\infty$, has eigenvalue ratios $\frac{1}{2}$, 2 and -$\frac{3}{2}$. So, the critical point with eigenvalue ratio $\frac{1}{2}$ has linearisable monodromy, the critical point with eigenvalue ratio 2 clearly has identity monodromy and hence the critical point with eigenvalue ratio -$\frac{3}{2}$ must be integrable. With respect to the $y$-axis, $P_5$ has eigenvalue ratio -$\frac{2}{3}$, and the other critical points have eigenvalue ratios 5 and -$\frac{10}{3}$ at $P_2$ and $P_1$ respectively. Thus the critical point at the origin must also be integrable. Consequently, the system (5.4) is integrable at the origin. Hence, only two axes are required to show the integrability.

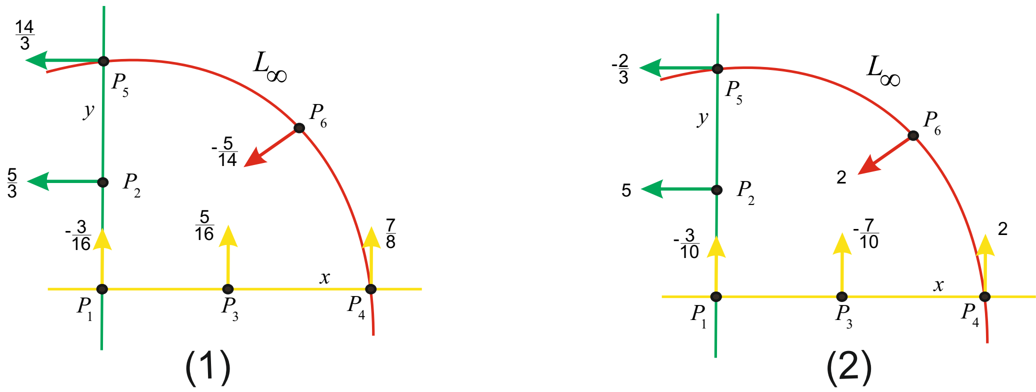



**Figure 5.1:** The Ratios of eigenvalues of the systems (5.3) and (5.4).

**3.** By using the following transformation

$$x \to x, \quad y \to L_\infty \quad \text{and} \quad L_\infty \to y, \tag{5.5}$$

we can verify that Figure 4.8, (i), Page 64 is projectively equivalent to Figure 5.2, (1).

The correspondence cubic curve of Figure 5.2, (1) is given in Figure 5.2, (2) and is defined by

$$f(x,y) = 1 - x - 14xy + \frac{112}{9}x^2y + \frac{49}{3}xy^2.$$

This cubic curve is invariant with respect to the following system

$$\dot{x} = x\Big(1 - x - \frac{7}{4}y\Big), \qquad \dot{y} = y\Big(-\frac{7}{8} + x + \frac{7}{8}y\Big). \tag{5.6}$$

Clearly, $P_3$, $P_4$, $P_5$, $P_6$ and $P_f$ are the critical points for which the invariant cubic curve passes through with eigenvalue ratios -8,2,3,6 and $\frac{3}{2}$ respectively. The critical point $P_f$ has linearisable monodromy, while $P_4$, $P_5$ and $P_6$ have identity monodromies, then the critical point $P_3$ must also be integrable. Since $P_4$ has eigenvalue ratio 2 with respect to the $x$-axis as well, then the origin must also be integrable.

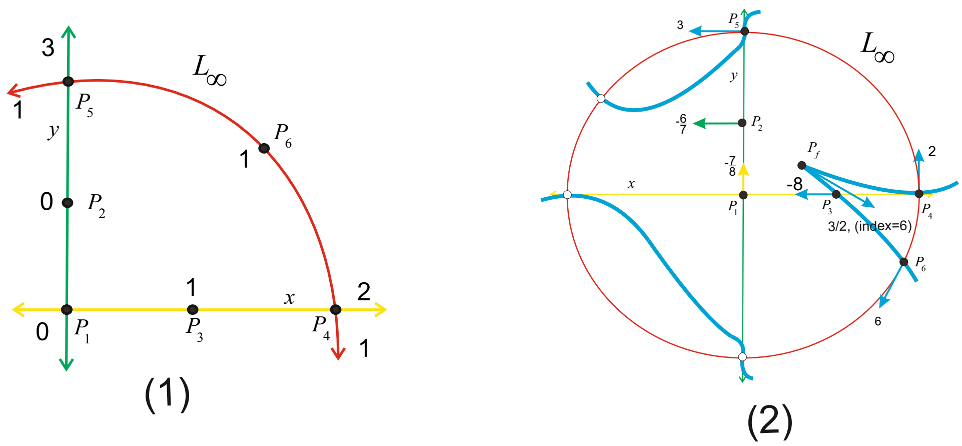



**Figure 5.2:** The Ratios of eigenvalues of the system (5.6).

**4.** Consider the system

$$\dot{x} = x\Big(1 - x - \frac{14}{13}y\Big), \qquad \dot{y} = y\Big(-\frac{7}{13} + x + \frac{7}{13}y\Big). \tag{5.7}$$

By using the transformation given in (5.5), we may verify that Figure 4.17, (i), Page 84 is projectively equivalent to Figure 5.3, (1).

The corresponding quartic curve of Figure 5.3, (1) is given in Figure 5.3, (2) and is defined by

$$f(x, y) = 1 - y + \frac{13}{6}xy - \frac{91}{3}xy^2 + \frac{637}{24}xy^3 + \frac{1183}{36}x^2y^2. \tag{5.8}$$

This quartic curve is invariant with respect to (5.7). Clearly, $P_2$, $P_4$, $P_5$, $P_6$ and $P_f$ are the critical points for which the invariant quartic curve passes through with eigenvalue ratios -7,2,2,3,6 and $\frac{3}{2}$. The critical point $P_f$ has linearisable monodromy, while $P_4$, $P_5$ and $P_6$ have identity monodromies, then the critical point $P_2$ must also be integrable. Since $P_5$ has eigenvalue ratio 3 with respect to the $y$-axis as well, then the origin must also be integrable.

**Remark 14.** The final two cases were found in Christopher and Rousseau (2004), who showed that they give Darboux first integrals. However, in the light of the other results below, it is probably better to consider them as examples of the monodromy method.

The second case is where the critical point lies on one of the invariant lines but not at the origin. We proceed as above and find the following result.

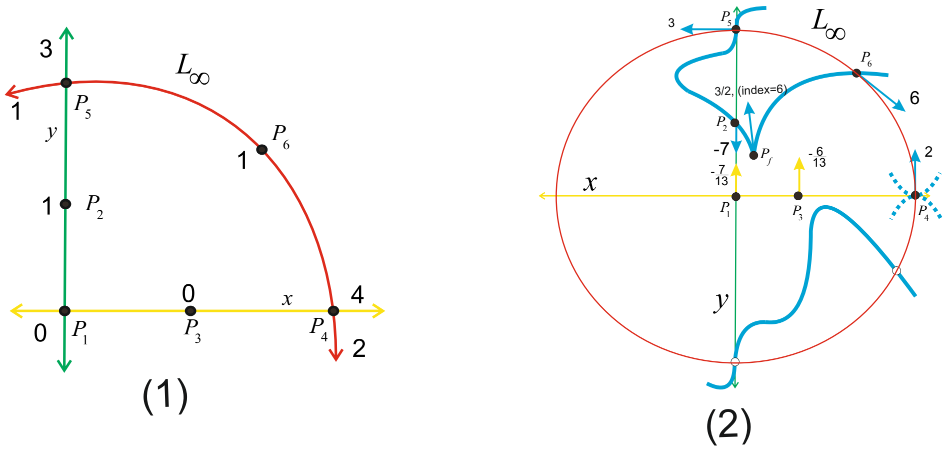



**Figure 5.3:** The Ratios of eigenvalues of the system (5.7).

**Theorem 14.** *If a Lotka-Volterra system has an integrable saddle on one of its axes but not at the origin with ratio of eigenvalues* $-p/q$ *with* $p+q \leq 17$ *then it falls into one of the following categories*

1. The condition (5.2) holds and the system has a Darboux first integral.
2. The monodromy method can be applied using one, two, or three of the invariant lines of the system.
3. The monodromy method can be applied using the invariant conic given in Case $C_{f_4}$ of Chapter 4.
4. The monodromy method can be applied using the invariant cubic given in Case $C_{f_7}$ of Chapter 4.
5. the monodromy method can be applied using the invariant quartic given in Case $C_{f_{16}}$ of Chapter 4.

We give an example to each case mentioned in Theorem 14.

**1.** Consider the system

$$\dot{x} = x\Big(1 - x - \frac{6}{5}y\Big), \qquad \dot{y} = y\Big(\frac{2}{5} - \frac{1}{5}x - \frac{2}{5}y\Big). \tag{5.9}$$

Clearly, the system satisfies the condition (5.2). Then there exist an invariant line such as one given in Section 4.2.1, (a), Page 51 passes through the critical points $P_2$, $P_3$ and $P_6$ with eigenvalue ratios 2,-5 and 4. Consequently $P_3$ must be integrable since $P_2$ and $P_6$ have identity monodromies. For more details, see Figure 5.4, (1).

**2.** Consider the following system

$$\dot{x} = x\Big(1 - x - \frac{p}{qd}y\Big), \qquad \dot{y} = \frac{p}{q}y(1 - y), \tag{5.10}$$

where $p$ and $q$ are co-prime numbers.

On the $x$-axis, the eigenvalue ratios of the critical points $P_1$, $P_3$ and $P_4$ are $\frac{p}{q}$, -$\frac{p}{q}$ and 1. Since $P_1$ has linearisable monodromy and $P_4$ has identity monodromy, then $P_3$ must also be integrable. Consequently, only the $x$-axis is required to show the integrability. For more details, see Figure 5.4, (2.1). For the case where only two axes are required to show the integrability is similar to what we have done in Theorem 13.

To see the case when three axes are required to show the integrability, we may consider the following system

$$\dot{x} = x\Big(1 - x + \frac{5}{4}y\Big), \qquad \dot{y} = y\Big(-\frac{3}{4} + x + \frac{3}{4}y\Big). \tag{5.11}$$

From $L_\infty$, $P_4$ has linearisable monodromy and $P_6$ has identity monodromy. Hence the critical point $P_5$ must be integrable. With respect to the $y$-axis, since $P_2$ has identity monodromy, then the origin with eigenvalue ratio -$\frac{4}{3}$ with respect to the $y$-axis must be integrable. From the $x$-axis, the identity monodromy of $P_4$ implies that the point $P_3$ must be integrable. Consequently all the axes are required to show the integrability at $P_3$. for more details, see Figure 5.4, (2.2).

**3.** From Figure 5.4, (3), there is a conic curve defined by

$$f(x, y) = y^2 + \frac{169}{4}x(x-1) + 13xy,$$

and is invariant with respect to the following system

$$\dot{x} = x\Big(1 - x - \frac{11}{13}y\Big), \qquad \dot{y} = y\Big(\frac{1}{2} + \frac{5}{4}x - \frac{1}{2}y\Big). \tag{5.12}$$

This conic curve passes through the critical points $P_1$, $P_3$, $P_6$ and $P_f$ with eigenvalue ratios 2,-$\frac{7}{4}$, 2 and $\frac{7}{4}$ respectively. Hence the critical point $P_3$ is integrable.

**4.** Since the ratio of eigenvalues at the side critical point $P_3$ with respect to the $x$-axis is equal to $-\frac{1}{8}$ for the Case (c) mentioned in Section 4.2.3, Page 61, then from the eigenvalue ratios of the critical points belong to the cubic curve, we see that the critical point $P_3$ must be integrable. So, in this case we have used only the cubic curve to show the integrability at $P_3$.

**5.** To show that the system (5.7) is integrable at the saddle critical point at $P_3$.

First, consider the invariant quartic curve (5.8) with respect to (5.7). The quartic curve passes through the critical poits $P_2$, $P_4$, $P_5$, $P_6$ and $P_f$ with eigenvalue ratios -7,2,2,3,6 and $\frac{3}{2}$. Hence the critical point $P_2$ is integrable. On the other hand, on the $y$-axis, the integrability of the critical point $P_2$ implies that the origin $P_1$ is integrable since $P_5$ has identity monodromy. Finally, on the $x$-axis, the identity monodromy of the critical point $P_4$ implies that $P_3$ must be integrable. For more details, see Figure 5.3, (2).

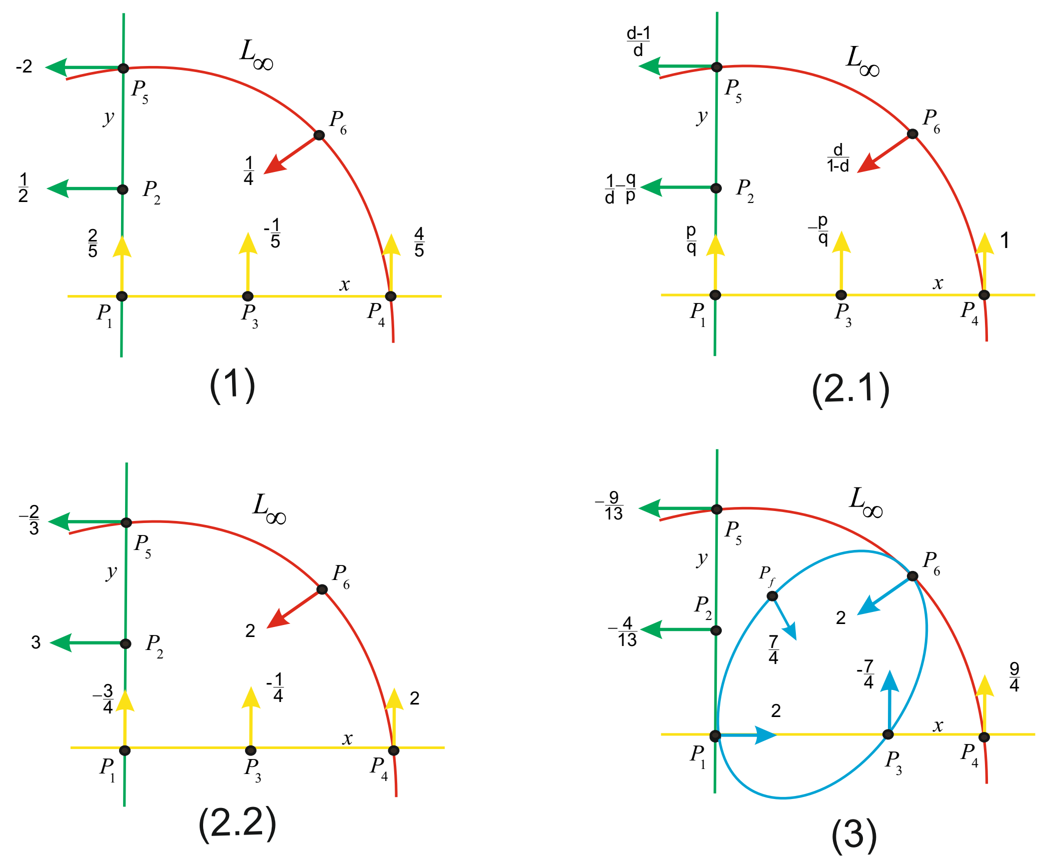



**Figure 5.4:** The Ratios of eigenvalues of the systems (5.9), (5.10), (5.11) and (5.12).

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Finally, we consider the case where the saddle is not on the axes. In this case we find the following result.

**Theorem 15.** *If a Lotka-Volterra system has an integrable saddle which does not lie on one of its axes with ratio of eigenvalues $-p/q$ with $p+q \leq 12$ then it falls into one of the following categories*

1. The monodromy method can be applied using the invariant line given in Case $C_{f_2}$ of Chapter 4.

2. The monodromy method can be applied using the invariant conic given in Case $C_{f_4}$ of Chapter 4.

3. The monodromy method can be applied using the invariant cubic given in Case $C_{f_{13}}$ of Chapter 4.

4. the monodromy method can be applied using the invariant quartic given in Case $C_{f_{20}}$ of Chapter 4.

We give an example to each case mentioned in Theorem 15.

**1.** By using Proposition 1, Page 50, there is an invariant algebraic line $f(x, y) = x - y$ with respect to the system

$$\dot{x} = x\Big(1 - x - \frac{5}{3}y\Big), \qquad \dot{y} = y\Big(1 - \frac{5}{3}x - y\Big), \tag{5.13}$$

passes through the critical points $P_f$ and $P_6$ as well. For more details, see Figure 5.5, (1). Since $P_1$ and $P_6$ have identity and linearisable monodromies, then $P_f$ must be integrable.

**2.** Consider the system

$$\dot{x} = x\Big(1 - x - \frac{13}{5}y\Big), \qquad \dot{y} = y\Big(2 - \frac{4}{7}x - 2y\Big). \tag{5.14}$$

The conic curve defined by

$$f(x,y) = x^2 + \frac{49}{25}y(y-1) + \frac{14}{5}xy,$$

is invariant with respect to the system (5.14). This conic passes through the critical points $P_1$, $P_2$, $P_f$ and $P_6$ with eigenvalue ratios 2,$\frac{5}{4}$, -$\frac{5}{4}$ and 2. Hence the face critical point $P_f$ must be integrable. For more details, see Figure 5.5, (2).

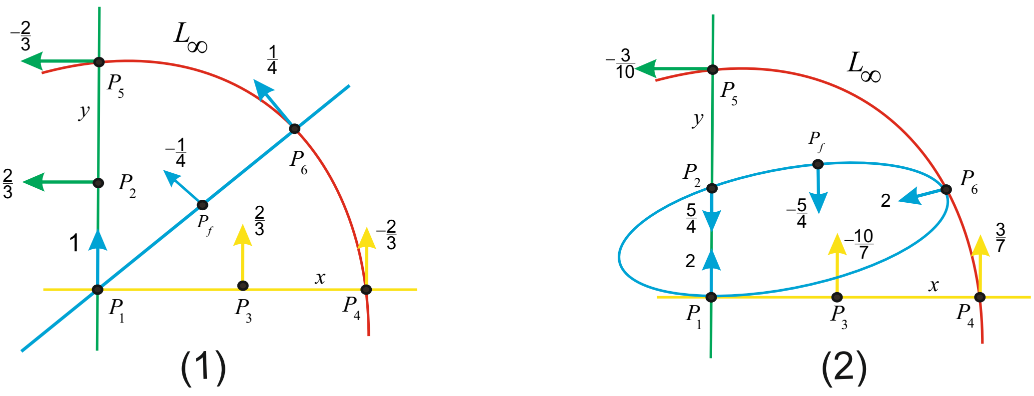



**Figure 5.5:** The Ratios of eigenvalues of the systems (5.13) and (5.14).

**3.** By using Figure 4.14 in Section 4.2.3, Page 78, we can verify that the system is integrable at the face critical point $P_f$ having -$\frac{3}{2}$ as an eigenvalue ratio since the other critical points for which the cubic curve passes through are $P_2$, $P_3$ and $P_6$ with eigenvalue ratios 3,2,$\frac{1}{2}$ and 3.

**4.** Similarly as the cubic curve explained above in Theorem 15, $P_f$ with eigenvalue ratio -$\frac{4}{3}$ is integrable since the other eigenvalue ratios for which the quartic curve passes through are 4,2,2,3,$\frac{1}{3}$. For more details, see Figure 4.21, Page 92.

We believe that these results will hold for all $p$ and $q$, but were not able to establish this yet.

# Chapter 6

# Quadratic Invariant Alebraic Surfaces in three dimensional Lotka-Volterra Systems

# Introduction

In Cairó and Llibre (2000), invariant algebraic surfaces up to degree two for the following three dimensional complex Lotka-Volterra system have been investigated

$$\begin{aligned}\dot{x} &= P(x,y,z) = x(\lambda + Cy + z),\\ \dot{y} &= Q(x,y,z) = y(\mu + x + Az),\\ \dot{z} &= R(x,y,z) = z(\nu + Bx + y),\end{aligned} \tag{6.1}$$

where, $\lambda,\ \mu,\ \nu,\ A,\ B$ and $C$ are real parameters. In Cairó (2000), linear and quadratic invariant algebraic surfaces in applying the Darboux theory of integrability in the following three dimensional Lotka-Volterra system have been studied

$$\begin{aligned}\dot{u} &= u(1 + a_1u + a_2v + a_3w),\\ \dot{v} &= v(b_0 + b_1u + b_2v + b_3w),\\ \dot{w} &= w(c_0 + c_1u + c_2v + c_3w).\end{aligned} \tag{6.2}$$

This chapter is devoted to introduce a new technique for finding invariant algebraic surfaces in three dimensional complex Lotka-Volterra systems.

This chapter is divided into three sections. In the first section, we describe the mechanism mentioned above. In the second section, we apply our technique to investigate the invariant algebraic quadratic surfaces with respect to the system (6.2) exclude those surfaces having eigenvalue ratios equal to one at the origin. In the last section, we use our mechanism to find all the invariant algebraic planes with respect to the system (6.2) found in Chapter 3.

## 6.1 A geometrical technique for finding invariant algebraic surfaces

In this section, we investigate a geometrical mechanism for finding the invariant algebraic surfaces with respect to the system (6.2). To do that, let we assume that the system (6.2) is non degenerate.

By an invariant algebraic surface with respect to the system (6.2), we mean a surface $C_f$ defined by a polynomial $f(x, y, z) = 0$ such that

$$\chi(f) = fL_f,$$

where $\chi$ is the corresponding vector field of (6.2) and $L_f$ is a polynomial of order at most one called the cofactor of $f$.

The system (6.2) has exactly fifteen critical points in $\mathbf{P}^3(\mathbb{C})$, when all the parameters in (6.2) are non-zero. Eight critical points are located in the affine part of $\mathbf{P}^3(\mathbb{C})$ and the others are at infinity. By the affine and infinite parts of $\mathbf{P}^3(\mathbb{C})$, we mean the same as we have already explained for $\mathbf{P}^2(\mathbb{C})$ in Chapter 2 and we will explain more later.

Notice that, if $a_1 = 0$, (respectively $b_0 = 0$ / $c_0 = 0$), then the critical point $\left(-\frac{1}{a_1},0,0\right)$, $\Big($respectively $\left(0,-\frac{b_0}{b_2},0\right)$ / $\left(0,-\frac{c_0}{c_3},0\right)\Big)$ of the system (6.2) moves to the origin. Henceforward, we assume that $a_1b_0c_0 \neq 0$. On the other hand, if $a_1 = b_1$ or $a_2 = b_2$, (respectively $a_1 = c_1$ or $a_3 = c_3$ / $b_2 = c_2$ or $b_3 = c_3$), then at infinity, there is no side critical point on the line $z = 0$, (respectively $y = 0$ / $x = 0$). Hence, we may assume that

$$a_1 \neq b_1, \quad a_2 \neq b_2, \quad a_1 \neq c_1, \quad a_3 \neq c_3, \quad b_2 \neq c_2, \quad b_3 \neq c_3.$$

We will explain later more about the infinite part of $\mathbf{P}^3(\mathbb{C})$.

In order to get a system much simpler than the system (6.2), we may assume the following change of coordinates

$$u = -\frac{1}{a_1}x, \quad v = -\frac{b_0}{b_2}y \text{ and } w = -\frac{c_0}{c_3}z.$$

Hence the system (6.2) can be written as

$$\begin{aligned}
\dot{x} &= P(x,y,z) = x\Big(1 - x - A(1-C)y - D(1-E)z\Big),\\
\dot{y} &= Q(x,y,z) = y\Big(A - (1-B)x - Ay - D(1-G)z\Big),\\
\dot{z} &= R(x,y,z) = z\Big(D - (1-F)x - A(1-H)y - Dz\Big),
\end{aligned} \tag{6.3}$$

where

$$\begin{aligned}
&A = b_0, \quad B = \frac{a_1 - b_1}{a_1}, \quad C = \frac{b_2 - a_2}{b_2}, \quad D = c_0,\\
&E = \frac{c_3 - a_3}{c_3}, \quad F = \frac{a_1 - c_1}{a_1}, \quad G = \frac{c_3 - b_3}{c_3} \text{ and } H = \frac{b_2 - c_2}{b_2}.
\end{aligned}$$

Clearly, all the parameters appear in (6.3) are non zero. As a result, the system (6.3) has the finite critical points $P_2 = (1,0,0)$, $P_3 = (0,1,0)$ and $P_4 = (0,0,1)$ on the $x$, $y$ and $z$-axes with $P_1 = (0,0,0)$ as well. We call $P_2$, $P_3$ and $P_4$ the side critical points on their corresponding axes.

By generalising what we have explained in Chapter 2 for $\mathbf{P^2}(\mathbb{C})$, we have considered Figure 6.1 to represent $\mathbf{P^3}(\mathbb{C})$.

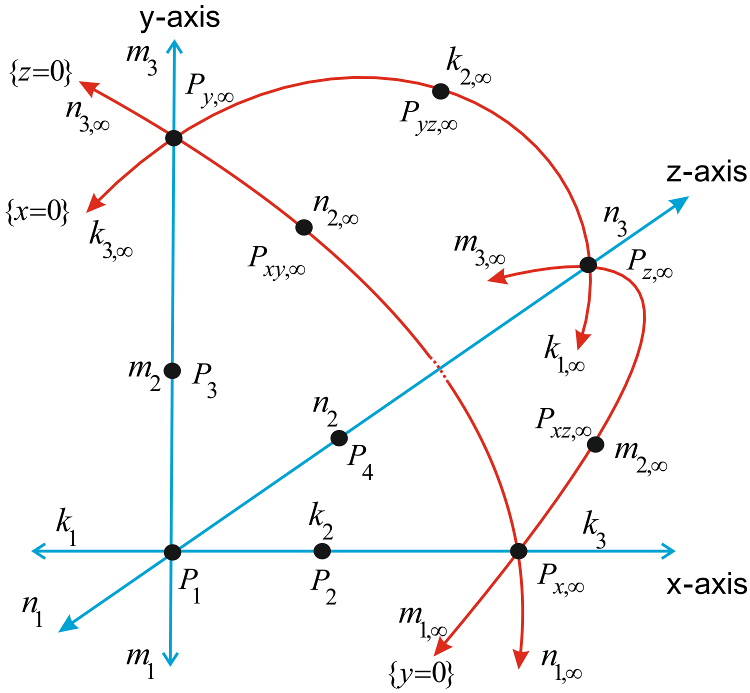



**Figure 6.1:** $\mathbf{P^3}(\mathbb{C})$.

To explain Figure 6.1, we assume an arbitrary algebraic surface $C_f$. So, in Figure 6.1

- The blue lines are the axes determine the finite part of $\mathbf{P^3}(\mathbb{C})$, while the red lines determine the infinite part of $\mathbf{P^3}(\mathbb{C})$.

- Black dots stand for the critical points located on $\mathbf{P^3}(\mathbb{C})$'s axes. $P_{x,\infty}$, $P_{y,\infty}$ and $P_{z,\infty}$ are the intersection points of $x$, $y$ and $z$-axes at infinity. $P_{xy,\infty}$, $P_{xz,\infty}$ and $P_{yz,\infty}$ are the side critical points on the axes $x = 0$, $y = 0$ and $z = 0$ respectively.

The variables $k_i$, $m_i$, $n_i$, $k_{i,\infty}$, $m_{i,\infty}$ and $n_{i,\infty}$ for $i$ in $\{1,2,3\}$ are non negative integers stand for the intersection multiplicities of the surface $C_f$ with the axes at their corresponding points such that

$$
\begin{aligned}
& k_1 = mult_{P_1}(C_f, x-axis), && k_2 = mult_{P_2}(C_f, x-axis),\\
& m_1 = mult_{P_1}(C_f, y-axis), && m_2 = mult_{P_3}(C_f, y-axis)\\
& n_1 = mult_{P_1}(C_f, z-axis) && n_2 = mult_{P_4}(C_f, z-axis)\\
& k_3 = mult_{P_{x,\infty}}(C_f, x-axis), && m_3 = mult_{P_{y,\infty}}(C_f, y-axis),\\
& n_3 = mult_{P_{z,\infty}}(C_f, z-axis), && k_{1,\infty} = mult_{P_{z,\infty}}(C_f, x=0),\\
& k_{2,\infty} = mult_{P_{yz,\infty}}(C_f, x=0), && k_{3,\infty} = mult_{P_{y,\infty}}(C_f, x=0),\\
& m_{1,\infty} = mult_{P_{x,\infty}}(C_f, y=0), && m_{2,\infty} = mult_{P_{xz,\infty}}(C_f, y=0),\\
& m_{3,\infty} = mult_{P_{z,\infty}}(C_f, y=0), && n_{1,\infty} = mult_{P_{x,\infty}}(C_f, z=0),\\
& n_{2,\infty} = mult_{P_{xy,\infty}}(C_f, z=0) \quad \text{and} && n_{3,\infty} = mult_{P_{y,\infty}}(C_f, z=0).
\end{aligned}
$$

To study the system (6.3), at infinity. We will consider the corresponding vector field $\chi$ of the differential system (6.3)

$$\chi = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}.$$

Without loss of generality, let $x \to \infty$. Hence, we may consider the following change of coordinates

$$W = \frac{1}{x}, \quad Y = \frac{y}{x} \quad \text{and} \quad Z = \frac{z}{x}. \tag{6.4}$$

Then the version of the vector field $\chi$ at infinity is given by

$$
\begin{aligned}
\tilde{\chi} =& -W^3 P\Big(\frac{1}{W},\frac{Y}{W},\frac{Z}{W}\Big)\frac{\partial}{\partial W} + W^2\Big(Q\Big(\frac{1}{W},\frac{Y}{W},\frac{Z}{W}\Big) - YP\Big(\frac{1}{W},\frac{Y}{W},\frac{Z}{W}\Big)\Big)\frac{\partial}{\partial Y} + \\
& W^2\Big(R\Big(\frac{1}{W},\frac{Y}{W},\frac{Z}{W}\Big) - ZP\Big(\frac{1}{W},\frac{Y}{W},\frac{Z}{W}\Big)\Big)\frac{\partial}{\partial Z}, \\
&= W\Big(1 - W + A(1-C)Y + D(1-E)Z\Big)\frac{\partial}{\partial W} + Y\Big(B + (A-1)W \\
&- ACY + D(G-E)Z\Big)\frac{\partial}{\partial Y} + Z\Big(F + W(D-1) + A(H_C)Y - DEZ\Big)\frac{\partial}{\partial Z}
\end{aligned}
$$

Clearly $W \to 0$ as $x \to \infty$, hence we get the coordinates of the following critical points

$P_{x,\infty} = (W = 0 : Y = 0 : Z = 0)$, $P_{xz,\infty} = \Big(W = 0 : Y = 0 : Z = \dfrac{F}{DE}\Big)$ and $P_{xy,\infty} = \Big(W = 0 : Y = \dfrac{B}{AC} : Z = 0\Big)$.

By the following change of coordinates

$$X = \frac{x}{y}, \quad W = \frac{1}{y} \quad \text{and} \quad Z = \frac{z}{y}, \tag{6.5}$$

as $y \to \infty$, or

$$X = \frac{x}{z}, \quad Y = \frac{y}{z} \quad \text{and} \quad W = \frac{1}{z}, \tag{6.6}$$

as $z \to \infty$, we can get the coordinates to the other critical points at infinity, for example the coordinate of the point $P_{yz,\infty} = \Big(X = 0 : W = 0 : Z = \frac{AH}{DG}\Big)$.

We will follow the following steps to classify invariant algebraic surfaces with respect to (6.3).

**Step 1** All possible choices for the variables mentioned in Figure 6.1 must be taken such that the sum of all three numbers located on the same axis must be equal to the degree of the algebraic surface $C_f$. For example, the sum of the numbers $m_{1,\infty}$, $m_{2,\infty}$ and $m_{3,\infty}$ on the line $y = 0$ must be equal to the degree of $C_f$.

**Step 2** We take one example as all other cases can be obtained by projective transformations.

**Step 3** Any case in Step 2 corresponds to a potential algebraic surface $C_f$ defined by a polynomial $f(x, y, z) = 0$ which will depend on a number of parameters different from what we have got in the system (6.3).

**Step 4** The values of the parameters appear in Step 3 can be computed by assuming $C_f$ as an invariant algebraic surface with respect to (6.3).

To draw and describe fully the behaviour of an invariant algebraic surface $C_f$ found from the steps above, we may use a regular tetrahedron with its sides flattened as shown in Figure 6.2.

In Figure 6.2,

- We use the equilateral triangles as follows: the blue triangle stands for the projective plane $\mathbf{P^2}(\mathbb{C})$ determined by the $x$ and $y$-axes as its affine part. Obviously, in this case, the line $z = 0$ (the line segment joined by the vertices $P_{x,\infty}$ and $P_{y,\infty}$) is the line at infinity of this projective plane. The

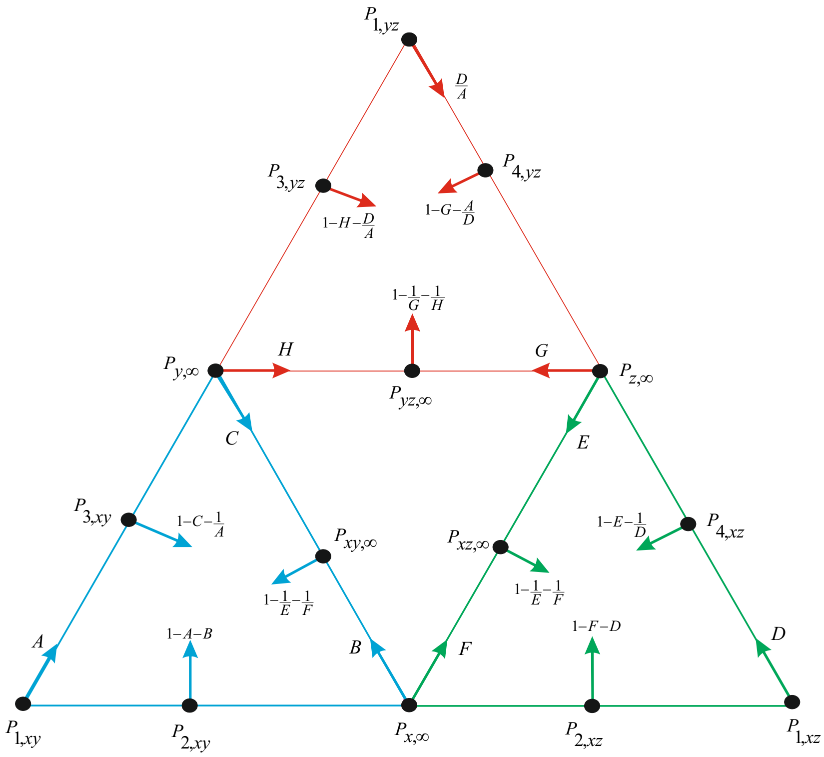



**Figure 6.2:** Ratios of eigenvalues of (6.3).

blue arrow at each point stand for the eigenvalue ratio of the system (6.4) at its corresponding point. We use the notation $\mathbf{P}^{\mathbf{2}}_{xy}(\mathbb{C})$ to represents this projective plane.

Similarly, the green triangle stands for $\mathbf{P}^{\mathbf{2}}(\mathbb{C})$ determined by the $x$ and $z$-axes, while the red triangle stands for $\mathbf{P}^{\mathbf{2}}(\mathbb{C})$ determined by the $y$ and $z$-axes. We preserve the same arrow colours for the eigenvalue ratios at their corresponding points in each part. We use the notations $\mathbf{P}^{\mathbf{2}}_{xz}(\mathbb{C})$ and $\mathbf{P}^{\mathbf{2}}_{yz}(\mathbb{C})$ for their projective planes for the green and the red triangles respectively.

- For the infinite part of $\mathbf{P}^3(\mathbb{C})$ which is itself a projective plane, we may assume the lines $x = 0$ and $y = 0$ as the affine part and then the line $z = 0$

as the line at infinity, we will use the notation $\tilde{\mathbf{P}}^{\mathbf{2}}_{xy}(\mathbb{C})$ for this case. Similarly, we use the notations $\tilde{\mathbf{P}}^{\mathbf{2}}_{xz}(\mathbb{C})$ $\big($respectively $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})\big)$ for representing the infinite part of $\mathbf{P}^3(\mathbb{C})$ where $x = 0$ and $z = 0$ (respectively $y = 0$ and $z = 0$) are the axes in the affine part.

- Each number near the head of any arrow stands for the ratio of eigenvalues at the corresponding point in the system (6.3).
- Despite the coordinates of the critical points $P_2$, $P_3$ and $P_4$ being unchanged, the notations $P_{2,xy}$ and $P_{3,xy}$, $\Big($respectively $P_{2,xz}$ and $P_{4,xz}$ / $P_{3,yz}$ and $P_{4,yz}\Big)$ are applied to their correspondence in $\mathbf{P}^{\mathbf{2}}_{xy}(\mathbb{C})$ $\Big($respectively $\mathbf{P}^{\mathbf{2}}_{xz}(\mathbb{C})$ / $\mathbf{P}^{\mathbf{2}}_{yz}(\mathbb{C})\Big)$.

To find and draw $C_f$

- We write only the corner parameters to each projective plane due to Ollagnier (2001) and what we have done in Chapter 4, since the sum of the eigenvalue ratios of the other critical points located on the same axis is equal to one. We write the parameters $A, B$ and $C$ $\big(D, E$ and $F$ respectively $G, H$ and $D/A\big)$ for $\mathbf{P}^{\mathbf{2}}_{xy}(\mathbb{C})$ $\big($ $\mathbf{P}^{\mathbf{2}}_{xz}(\mathbb{C})$ respectively $\mathbf{P}^{\mathbf{2}}_{yz}(\mathbb{C})\big)$.
- Different colours for each part of any invariant algebraic surface, $C_f$, will apply with the same arrow colours stand for the ratios of eigenvalues at the critical points for which $C_f$ passes through. We use green, red and blue colours for the parts of $C_f$ in $\mathbf{P}^{\mathbf{2}}_{xy}(\mathbb{C})$, $\mathbf{P}^{\mathbf{2}}_{xz}(\mathbb{C})$ and $\mathbf{P}^{\mathbf{2}}_{yz}(\mathbb{C})$ respectively, while we use the pink colour for the part of $C_f$ at infinity.
- In some cases, we draw and extend some parts of the surface $C_f$ or the axes by dotted curve to guarantee that there are no any intersections neither

among them nor with the axes of $\mathbf{P^3}(\mathbb{C})$.

- $P_{xy}$, $P_{xz}$ and $P_{yz}$ are the face critical points in $\mathbf{P}^{\mathbf{2}}_{xy}(\mathbb{C})$, $\mathbf{P}^{\mathbf{2}}_{xz}(\mathbb{C})$ and $\mathbf{P}^{\mathbf{2}}_{yz}(\mathbb{C})$.

- $P_{xyz}$ is the critical point of the system (6.3) in the space, while the critical point $P$ is the face point at infinity. The space point is the critical point with non-zero coordinates.

- In some cases, we use two different colour arrows at the same critical point with only one specified. In this case the other eigenvalue ratio is the same and geometrically involved.

**Remark 15.** 1. By the finite part of an invariant algebraic surface, $C_f$, we mean the behaviour of $C_f$ at the finite part of each projective plane in $\mathbf{P}^3(\mathbb{C})$, while the infinite part of $C_f$ means the behaviour of $C_f$ at infinity in $\mathbf{P^3}(\mathbb{C})$.

2. Any axis or a line with a yellow colour in applying Figure 6.2 for an invariant algebraic surface $C_f$ indicates to a line of singularities of the corresponding vector field for which the surface $C_f$ is invariant.

3. In a line of singularities, each point $Q_i$, for $i \in \{1, 2\}$ is not necessary to be one of the critical points mentioned in Figure 6.2.

**Theorem 16.** *The necessary and sufficient conditions for the system (6.3) having an invariant algebraic plane passes through the origin transversely is that* $A = D = 1$ *in the system (6.3), where* $BEH + CGF = 0$.

*Proof.* Suppose $A = D = 1$ in the system (6.3), so we have the following system

$$\begin{aligned}
\dot{x} &= x\Big(1 - x - (1-C)y - (1-E)z\Big),\\
\dot{y} &= y\Big(1 - (1-B)x - y - (1-G)z\Big),\\
\dot{z} &= z\Big(1 - (1-F)x - (1-H)y - z\Big).
\end{aligned} \tag{6.7}$$

It is easy to verify that the algebraic plane $f = x - \left(\frac{C}{B}\right)y - \left(\frac{E}{F}\right)z$ is invariant with respect to the system (6.7) with the cofactor $L_f = 1 - x - y - z$.

The converse is straightforward since any invariant algebraic plane intersect the axes at the origin transversely has a multiplicity one with each axis. □

**Remark 16.** To investigate invariant quadratic surfaces, we make some assumptions since there are many cases to be considered. One of the assumption is that we ignore the case where the algebraic surface does not intersect each axis transversely. The other assumption is that we will ignore the case having the ratios of eigenvalues equal to one at the corner critical point for the corresponding vector field for which the surface is invariant.

## 6.2 Invariant quadratic surfaces

In this section, we apply the technique of Section 6.1, to investigate all invariant algebraic quadratic surfaces with respect to the system (6.3).

By applying the first three steps in our method, we get exactly seven possible cases for invariant quadratic surfaces as can be seen in Figure 6.3.

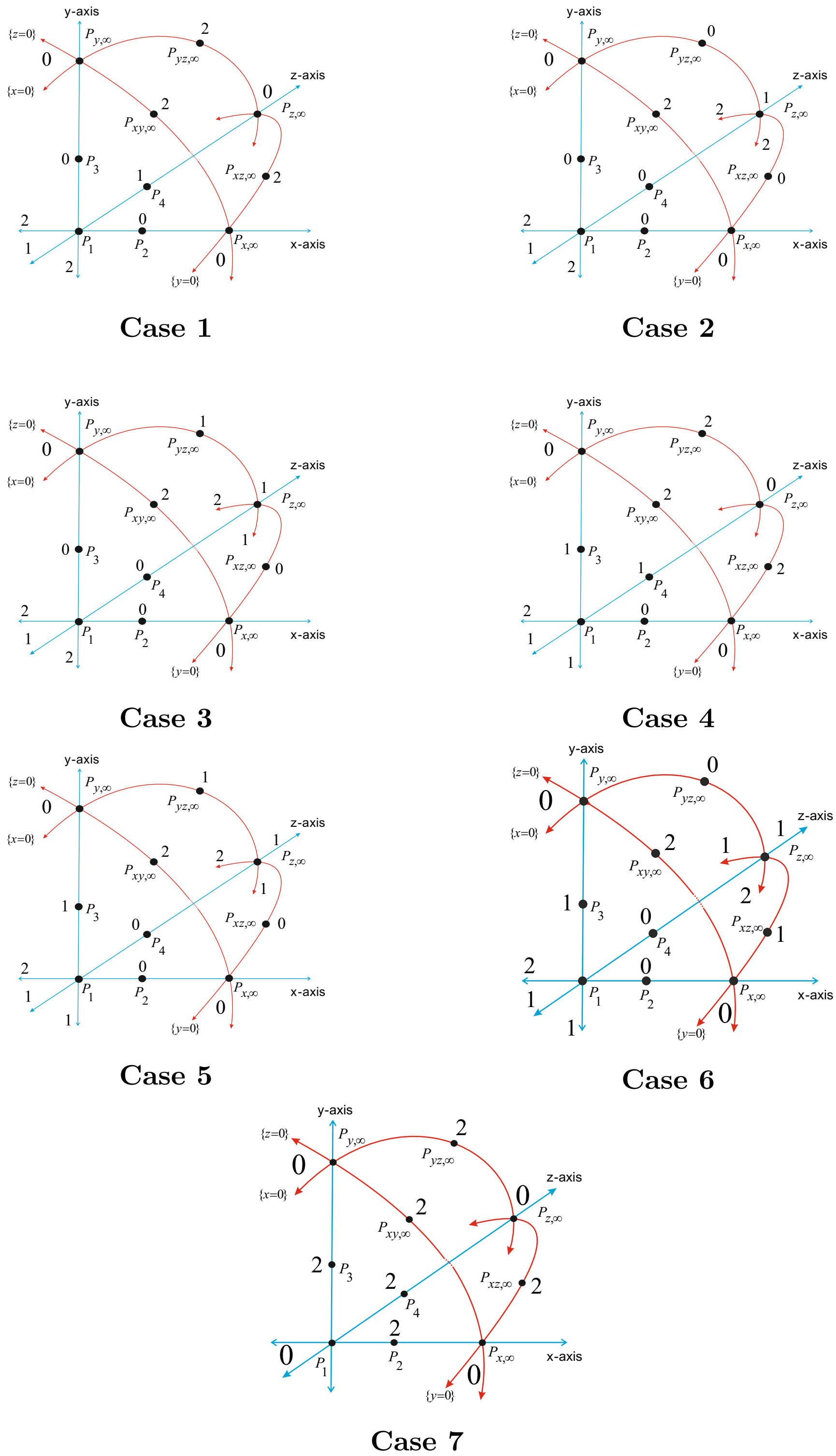



**Figure 6.3:** Multiplicity possibilities of invariant quadratic surfaces.

We split to each case in Figure (6.3) individually

**Case 1**

The general form of the quadratic algebraic surface in this case is defined by the following polynomial

$$f = x^2 + A_1 y^2 + A_2 z(z-1) + A_3 xy + A_4 xz + A_5 yz.$$

By assuming $f$ as an invariant with respect to (6.3), we get exactly three possible choices for the parameters $A_i,\ \ i \in \{1, \cdots, 5\}$.
We will explain each case individually

**I.** The quadratic surface, $C_{f_{1.1}}$, defined by the polynomial

$$f_{1.1} = x^2 + \frac{1}{(F+1)^2} y^2 + \frac{4}{(F+1)^2} z(z-1) + \frac{2}{F+1} xy + \frac{4}{F+1} xz + \frac{4}{(F+1)^2} yz, \quad F \neq -1,$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 1, \quad B = B, \quad C = -\frac{B}{F+1}, \quad D = 2, \\ &E = -\frac{F}{F+1}, \quad F = F, \quad G = 0 \ \text{ and } \ H = 0. \end{aligned} \tag{6.8}$$

Notice that, $F \neq -1$, otherwise we get $B = 0$.

$C_{f_{1.1}}$ has the following properties

- In ${\mathbf{P}_{xy}}^2(\mathbb{C})$, $f_{1.1} = \left(x + \frac{1}{F+1} y\right)^2$. So, $C_{f_{1.1}}$ is the double of a line, which is projectively equivalent to the case, $C_{f_2}$, mentioned in Chapter 4.

- In ${\mathbf{P}_{xz}}^2(\mathbb{C})$, $f_{1.1} = x^2 + \frac{4}{(F+1)^2}z(z-1) + \frac{4}{F+1}xz$. So, $C_{f_{1.1}}$ is the case, $C_{f_4}$, mentioned in Chapter 4.

- In ${\mathbf{P}_{yz}}^2(\mathbb{C})$, $f_{1.1} = \frac{1}{(F+1)^2}y^2 + \frac{4}{(F+1)^2}z(z-1) + \frac{4}{(F+1)^2}yz$. This case has not been mentioned in Chapter 4, since it is not listed in Ollagnier (2001). $C_{f_{1.1}}$ intersects, the $z$-axis, transversely at the critical points $P_{1,yz}$ and $P_{4,yz}$, while it intersects, the $y$-axis tangentially at the critical point $P_{1,yz}$. More about this part will be given later.

- $C_{f_{1.1}}$ passes through the space critical point

$$P_{xyz} = \Big(\frac{1}{B-F}, \frac{F^2+F-3BF-B}{(B-F)^2}, \frac{B^2}{(B-F)^2}\Big), \quad B \neq F.$$

- At infinity: To consider $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, we may use the change of coordinates (6.4). So the version of $f_{1.1}$ can be defined by

$$F_{1.1} = \frac{1}{(F+1)^2}\Big(1+F+Y+2Z\Big)^2.$$

Clearly, $C_{F_{1.1}}$, is the double of a line that is projectively equivalent to the case, $C_{f_1}$, in Chapter 4. The correspondence version of the vector field determined by the conditions given in (6.8) for which $F_{1.1}$ is invariant can be defined by

$$\begin{aligned} \dot{Y} &= Y\Big(B + \frac{B}{F+1}Y + \frac{2F}{F+1}Z\Big), \\ \dot{Z} &= Z\Big(F + \frac{B}{F+1}Y + \frac{2F}{F+1}Z\Big). \end{aligned} \tag{6.9}$$

$C_{F_{1.1}}$ passes through the critical points $P_{xy,\infty} = \big(W=0 : Y = -(F+1) : Z = 0\big)$ and $P_{xz,\infty} = \Big(W = 0 : Y = 0 : Z = -\frac{1}{2}(F+1)\Big)$ of (6.9), and both are singularities of $C_{F_{1.1}}$. In $\tilde{\mathbf{P}}^{\mathbf{2}}_{xz}(\mathbb{C})$, which is determined by the change of coordinates (6.5), the correspondence vector field is defined by

$$\begin{aligned}\dot{X} &= X\Big(\frac{B}{F+1} + BX + \frac{2F}{F+1}Z\Big),\\ \dot{Z} &= (F-B)XZ.\end{aligned} \tag{6.10}$$

Hence, $X = 0$, is a line of singularities of the system (6.10), and then $C_{F_{1.1}}$ intersects the line, $X = 0$, at a point $Q_1$ not necessary to be the same critical point $P_{xy,\infty}$. Consequently, in $\mathbf{P}_{yz}{}^{2}(\mathbb{C})$, $C_{f_{1.1}}$ intersects the line, $x = 0$, at the points $Q_1$ and $Q_2$. The whole behaviour of $C_{f_{1.1}}$, can be seen in Figure 6.4.

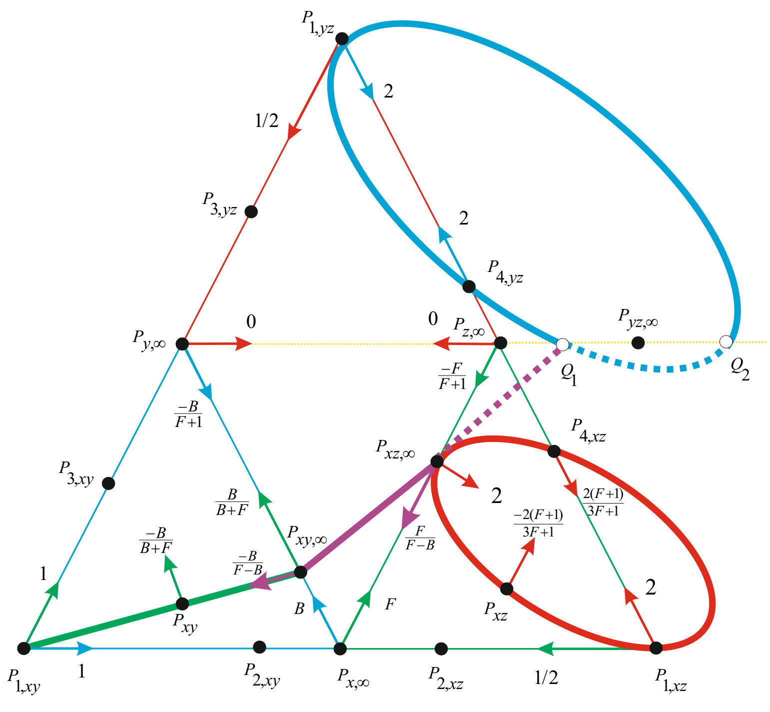



**Figure 6.4:** The whole behaviour of $C_{f_{1.1}}$.

**II.** The quadratic surface, $C_{f_{1.2}}$, defined by the polynomial

$$f_{1.2} = x^2 + \frac{C^2}{B^2}y^2 + \frac{4}{(F+1)^2}z(z-1) - \frac{2C}{B}xy + \frac{4}{F+1}xz - \frac{4C}{B(F+1)}yz, \quad B \neq 0 \text{ and } F \neq -1,$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 1, \quad B = B, \quad C = C, \quad D = 2, \quad E = -\frac{F}{F+1}, \\ &F = F, \quad G = -\frac{CF+B+C}{C(F+1)} \text{ and } H = -\frac{CF+B+C}{B}. \end{aligned} \tag{6.11}$$

$C_{f_{1.2}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$, $f_{1.2} = \left(x - \frac{C}{B}y\right)^2$. So, $C_{f_{1.2}}$ is the double of the case, $C_{f_2}$, mentioned in Chapter 4.
- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{1.2}$ is the same as $f_{1.1}$, in $\mathbf{P}_{xz}{}^2(\mathbb{C})$.
- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{1.2} = \frac{C^2}{B^2}y^2 + \frac{4}{(F+1)^2}z(z-1) - 4\frac{C}{B(F+1)}yz$. Notice that

$$\left(\frac{B}{C}\right)^2 f_{1.2} = y^2 + 4\left(\frac{B}{C(F+1)}\right)^2 z(z-1) - 4\left(\frac{B}{C(F+1)}\right)yz.$$

  Hence, $C_{f_{1.2}}$ is projectively equivalently to the case, $C_{f_4}$, mentioned in Chapter 4.
- $C_{f_{1.2}}$ does not pass through the space critical point.
- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$. So, the version of $f_{1.2}$ is defined by

$$F_{1.2} = \frac{1}{B^2(F+1)^2}\Big(B(1+F) - C(1+F)Y + 2BZ\Big)^2.$$

Clearly, $C_{F_{1.2}}$ is the double of a line which is projectively equivalent to the case, $C_{f_1}$, mentioned in Chapter 4.

The correspondence of the vector field determined by the conditions given in (6.11) for which $F_{1.2}$, is invariant can be defined by

$$\begin{aligned} \dot{Y} &= Y\Big(B - CY - 2\frac{B+C}{C(F+1)}Z\Big), \\ \dot{Z} &= Z\Big(F - \frac{BC+FC+B+C}{B}Y + \frac{2F}{F+1}Z\Big). \end{aligned} \tag{6.12}$$

$C_{F_{1.2}}$ passes through the critical points $P_{xy,\infty} = \big(W=0 : Y = \frac{B}{C} : Z = 0\big)$ and $P_{xz,\infty} = \Big(W = 0 : Y = 0 : Z = -\frac{1}{2}(F+1)\Big)$ of (6.12), and both are singularities of $C_{F_{1.2}}$. To get the other critical point for which $C_{F_{1.2}}$ passes through, we may assume, $\tilde{\mathbf{P}}^{\mathbf{2}}_{xz}(\mathbb{C})$, which can be determined by the change of coordinates (6.5). Hence, the correspondence vector field is defined by

$$\begin{aligned} \dot{X} &= X\Big(C - BX + \frac{2(B+C)}{C(F+1)}Z\Big), \\ \dot{Z} &= Z\Big(-\frac{CF+B+C}{B} + (F-B)X + \frac{2(CF+B+C)}{C(F+1)}Z\Big). \end{aligned} \tag{6.13}$$

So, the other critical point of (6.13) belongs to $C_{F_{1.2}}$ has the coordinates $P_{xz,\infty} = \big(X = 0, W = 0, Z = \frac{C(F+1)}{2B}\big)$. Also, $P_{xz,\infty}$ is a singularity of $C_{F_{1.2}}$.

The whole behaviour of $C_{f_{1.2}}$ can be seen in Figure 6.5.

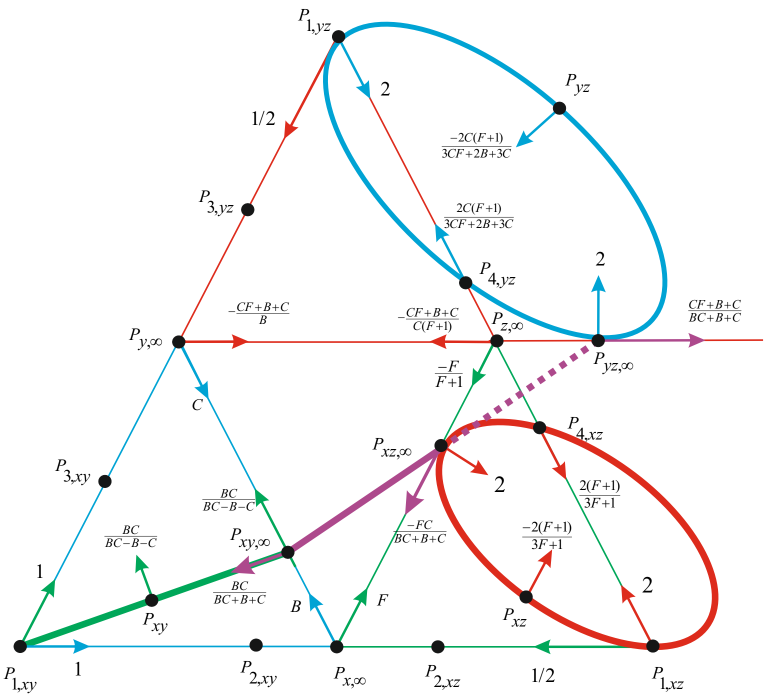



**Figure 6.5:** The whole behaviour of $C_{f_{1.2}}$.

**III.** The quadratic surface, $C_{f_{1.3}}$, defined by the polynomial

$$f_{1.3} = x^2 + \frac{1}{(B+2F+1)^2}y^2 + \frac{4}{(F+1)^2}z(z-1) - \frac{2}{B+2F+1}xy + \frac{4}{F+1}xz + \frac{4}{(B+2F+1)(F+1)}yz, \quad F \neq -1 \text{ and } B+2F \neq -1,$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 1, \quad B = B, \quad C = \frac{B}{B+2F+1}, \quad D = 2, \\ &E = -\frac{F}{F+1}, \quad F = F, \quad G = \frac{F+B}{F+1} \text{ and } H = -\frac{F+B}{B+2F+1}. \end{aligned} \tag{6.14}$$

$C_{f_{1.3}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$, $f_{1.3} = \left(x - \frac{1}{B+2F+1}y\right)^2$. So, $C_{f_{1.3}}$ is the double of a line, that is projectively equivalent to the case, $C_{f_2}$, in Chapter 4. The point $P_{xy} = \left(\frac{1}{2(F+1)}, \frac{B+2F+1}{2(F+1)}, 0\right)$ is a singularity of $C_{f_{1.3}}$. By assuming the following change of coordinates

$$\begin{aligned}
x &= \frac{1}{2(F+1)} - \frac{1}{8}u + \frac{1}{2}v + \frac{1}{2}w,\\
y &= \frac{B+2F+1}{2(F+1)} - \frac{1}{8}(B+2F+1)u + \Big(\frac{1}{2}B + F + \frac{1}{2}\Big)v - \Big(\frac{1}{2}B + F + \frac{1}{2}\Big)w,\\
z &= \frac{1}{4}(F+1)u,
\end{aligned}$$

the correspondence of $f_{1.3}$, is given by the equation of the cone $g = g(u,v,w) = uv + w^2$. Hence $C_{f_{1.3}}$ has a cone branch at the singular point $P_{xy}$.

- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{1.3}$ is the same as $f_{1.1}$.
- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{1.3} = \frac{1}{(B+2F+1)^2}y^2 + \frac{4}{(F+1)^2}z(z-1) + 4\frac{1}{(B+2F+1)(F+1)}yz$. Notice that, $(B+2F+1)^2 f_{1.3} = y^2 + 4\big(\frac{B+2F+1}{F+1}\big)^2 z(z-1) + 4\big(\frac{B+2F+1}{F+1}\big)yz$. Hence, $C_{f_{1.3}}$ is projectively equivalently to the case, $C_{f_4}$, mentioned in Chapter 4.
- $C_{f_{1.3}}$ does not pass through the space critical point.
- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, in other words, we use the change of coordinates (6.4). So the version of $f_{1.3}$ is defined by,

$$\begin{aligned}
F_{1.3} =& \Big(\frac{Y}{B+2F+1} - 1\Big)^2 + \Big(\frac{2Z}{F+1} + 1\Big)^2 + 4\frac{YZ}{(B+2F+1)(F+1)}\\
&- 1.
\end{aligned}$$

Clearly, $C_{F_{1.3}}$ is projectively equivalent to the case $C_{f_3}$, mentioned in

Chapter 4. The correspondence of the vector field determined by the conditions given in (6.14) is defined by

$$\begin{aligned} \dot{Y} &= Y\Big(B + \frac{B}{B+2F+1}Y + \frac{2(2F+B)}{F+1}Z\Big), \\ \dot{Z} &= Z\Big(F - \frac{F+2B}{B+2F+1}Y + \frac{2F}{F+1}Z\Big). \end{aligned} \tag{6.15}$$

$C_{F_{1.3}}$ passes through the critical points $P_{xy,\infty} = \big(W = 0 : Y = B + 2F + 1 : Z = 0\big)$ and $P_{xz,\infty} = \Big(W = 0 : Y = 0 : Z = -\frac{1}{2}(F+1)\Big)$ of (6.15). Similarly, as the other cases, the other critical point belongs to $C_{F_{1.3}}$ has the coordinates, $P_{yz,\infty} = \big(X = 0 : W = 0 : Z = -\frac{F+1}{2B+4F+2}\big)$. The whole behaviour of $C_{f_{1.3}}$ can be seen in Figure 6.6.

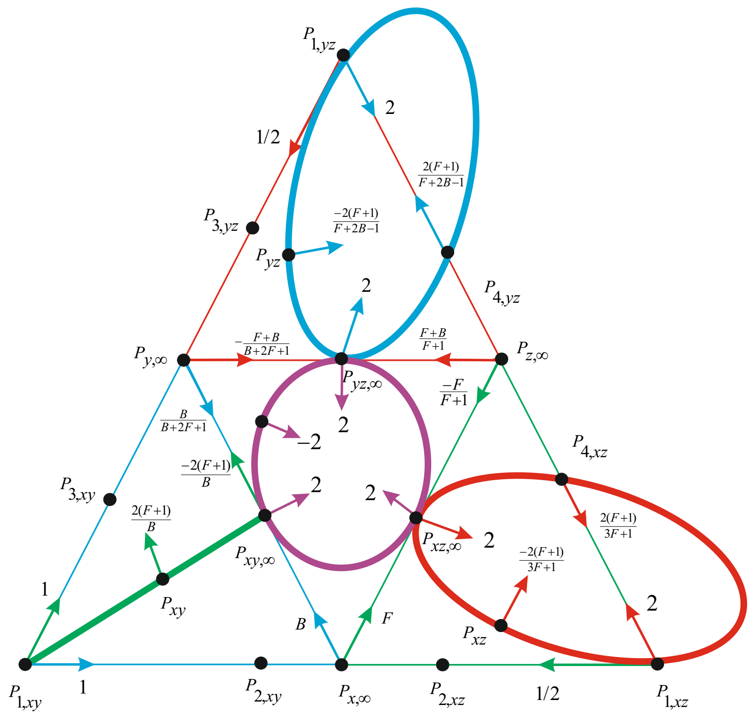



**Figure 6.6:** The whole behaviour of $C_{f_{1.3}}$.

**Case 2**

The general form of the quadratic algebraic surface in this case is defined by the following polynomial

$$f = x^2 + A_1 y^2 + A_2 z + A_3 xy.$$

By assuming $f$ as invariant with respect to (6.3), we get exactly one possible choice for the parameters $A_i, \ \ i \in \{1, 2, 3\}$.

The quadratic surface, $C_{f_2}$, defined by the polynomial

$$f_2 = x^2 + \frac{C^2}{B^2} y^2 + \alpha z - \frac{2C}{B} xy, \ \ 0 \neq \alpha \in \mathbb{C},$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 1, \quad B = B, \quad C = C, \quad D = 2, \\ &E = \frac{1}{2}, \quad F = -1, \quad G = \frac{1}{2} \ \text{ and } \ H = -1. \end{aligned} \tag{6.16}$$

$C_{f_2}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C}), \ f_2 = \left(x - \frac{C}{B} y\right)^2$. So, $C_{f_2}$, is the double of the case, $C_{f_2}$, mentioned in Chapter 4.

- In $\mathbf{P}_{xz}{}^2(\mathbb{C}), \ f_2 = x^2 + \alpha z$. This case has not been mentioned in Chapter 4, since it is not listed in Ollagnier (2001). $C_{f_2}$, intersects the $z$-axis transversely at both critical points $P_{1,xz}$ and $P_{z,\infty}$, while it intersects tangentially, the $x$-axis and the line at infinity, $y = 0$, at the critical points $P_{1,xz}$ and $P_{z,\infty}$ respectively.

- In ${\mathbf{P}_{yz}}^2(\mathbb{C})$, $f_2 = \frac{C^2}{B^2}\big(y^2 + \alpha z\big)$. So, the properties of $C_{f_2}$, is the same as we have in ${\mathbf{P}_{xz}}^2(\mathbb{C})$, by replacing the $x$-axis, by $y$-axis and the line $y = 0$, by $x = 0$.

- $C_{f_2}$ does not pass through the space critical point.

- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{xy}(\mathbb{C})$, in other words, we use the change of coordinates (6.6), so the version of $f_2$ can be defined by $F_2 = \Big(X - \frac{C}{B}Y\Big)^2$. Clearly $C_{F_2}$ is a double of a line that is projectively equivalent to the case, $C_{f_2}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.16) can be defined by

$$\begin{aligned} \dot{X} &= X\big(1 + X + (1 + C)Y\big), \\ \dot{Y} &= Y\big(1 + (1 + B)X + Y\big). \end{aligned} \tag{6.17}$$

$C_{F_2}$ passes through the critical points $P_{z,\infty} = \big(X = 0 : Y = 0 : W = 0\big)$ and $P = \big(X = -\frac{C}{BC+B+C} : Y = -\frac{B}{BC+B+C} : W = 0\big)$ of (6.17), and both are singularities of $C_{F_2}$. To find the other point for which $C_{F_2}$ passes through, we may assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$. Hence by assuming the change of coordinates (6.4), the correspondence of $f_2$ can be defined by $F_2 = \Big(\frac{C}{B}Y - 1\Big)^2$, and then the correspondence vector field for which $F_2$ is invariant can be given by

$$\begin{aligned} \dot{Y} &= Y\big(B - CY\big), \\ \dot{Z} &= Z\big( - 1 - (1 + C)Y - Z\big). \end{aligned} \tag{6.18}$$

Consequently, the critical point $P_{xy,\infty} = \Big(W = 0 : Y = \frac{B}{C} : Z = 0\Big)$ of the system (6.18) is a singularity of $C_{F_2}$. The whole behaviour of $C_{f_2}$ can be seen in Figure 6.7

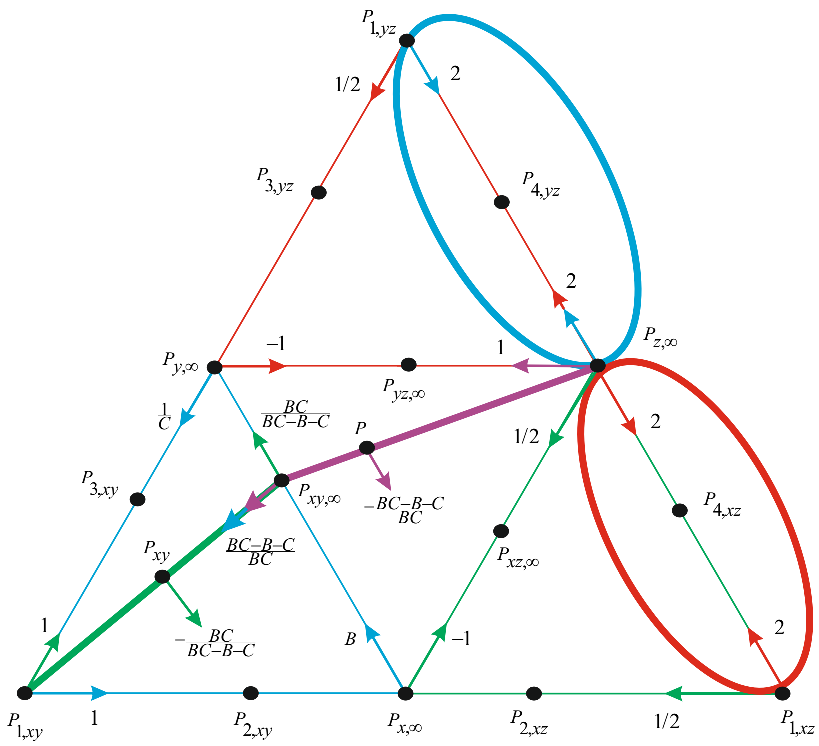



**Figure 6.7:** The whole behaviour of $f_2$.

### Case 3

The general form of the quadratic algebraic surface in this case can be defined by the following polynomial

$$f = x^2 + A_1 y^2 + A_2 z + A_3 xy + A_4 yz.$$

By assuming $f$ as invariant with respect to (6.3), we get exactly one possible choice for the parameters $A_i$, $i \in \{1, 2, 3, 4\}$.

The quadratic surface, $C_{f_3}$, defined by the polynomial

$$f_3 = x^2 + \frac{H^2}{(B-1)^2}y^2 + \frac{2H}{(B-1)^2(H+1)}z + \frac{2H}{B-1}xy - \frac{2H}{(B-1)^2}yz,$$
$$B \neq 1 \ \text{ and } \ H \neq -1,$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 1, \quad B = B, \quad C = -\frac{BH}{B-1}, \quad D = 2, \\ &E = \frac{1}{2}, \quad F = -1, \quad G = 1 \ \text{ and } \ H = H. \end{aligned} \tag{6.19}$$

$C_{f_3}$ has the following properties

- In ${\mathbf{P}_{xy}}^2(\mathbb{C})$, $f_3 = \left(x + \frac{H}{B-1}y\right)^2$. So, $C_{f_3}$ is the double of a line, that is projectively equivalent to the case, $C_{f_2}$, mentioned in Chapter 4.
  The point $P_{xy,} = \left(-\frac{H}{BH+B-H-1}, \frac{1}{H+1}, 0\right)$ is a singularity of $C_{f_3}$. By assuming the following change of coordinates

$$\begin{aligned} x &= -\frac{H}{BH+B-H-1} + \frac{H}{1-B}v + w, \\ y &= \frac{1}{H+1} + v, \\ z &= -\frac{(B-1)^2}{2H}u, \end{aligned}$$

  the correspondence of $f_3$ is given by the equation of the cone, $g = g(u, v, w) = uv + w^2$.

- In ${\mathbf{P}_{xz}}^2(\mathbb{C})$, $f_3 = x^2 + \frac{2H}{(B-1)^2(H+1)}z$. This case is similar to the case $C_{f_2}$, mentioned above in the projective plane, ${\mathbf{P}_{xz}}^2(\mathbb{C})$.

- In ${\mathbf{P}_{yz}}^2(\mathbb{C})$, $f_3 = \frac{H^2}{(B-1)^2}y^2 + \frac{2H}{(B-1)^2(H+1)}z - \frac{2H}{(B-1)^2}yz$.

This case has not been mentioned in Chapter 4, since it is not listed in Ollagnier (2001). $C_{f_3}$, intersects the $y$-axis tangentially at the critical point $P_{1,yz}$, while it intersects the $z$-axis transversely at both critical points $P_{1,yz}$ and $P_{z,\infty}$.

Also $C_{f_3}$ intersects transversely, the line at infinity, $x = 0$, at both critical points $P_{z,\infty}$ and $P_{yz,\infty}$.

- $C_{f_3}$ does not pass through the space critical point.

- At infinity:

  Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{xy}(\mathbb{C})$, so the version of $f_3$ can be defined by

$$F_3 = X^2 + \frac{H^2}{(B-1)^2}Y^2 - \frac{2H}{(B-1)^2}Y + \frac{2H}{B-1}XY.$$

Hence, $C_{F_3}$, is projectively equivalent to the case $C_{f_4}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.19) for which $F_3$ is invariant can be defined by

$$\begin{aligned}\dot{X} &= X\Big(1 + X - \frac{H(2B-1)}{B-1}Y\Big),\\ \dot{Y} &= Y\big(2 + (1+B)X - HY\big).\end{aligned} \tag{6.20}$$

$C_{F_3}$ passes through the critical points $P_{z,\infty} = \big(X = 0 : Y = 0 : W = 0\big)$, $P_{yz,\infty} = \big(X = 0 : Y = \frac{2}{H} : W = 0\big)$ and $P = \Big(X = \frac{1-3B}{2B^2} : Y = \frac{(B-1)^2}{2B^2H} : W = 0\Big)$ of (6.20). To find the other point for which $C_{F_3}$ passes through, we may use $\tilde{\mathbf{P}}^{\mathbf{2}}_{xy}(\mathbb{C})$ and similarly as we have done in the previous cases, the critical point belongs to $C_{F_3}$ has the coordinates $P_{xy,\infty} = \big(W = 0 : Y = \frac{1-B}{H} : Z = 0\big)$.

The whole behaviour of $C_{f_3}$ can be seen in Figure 6.8.

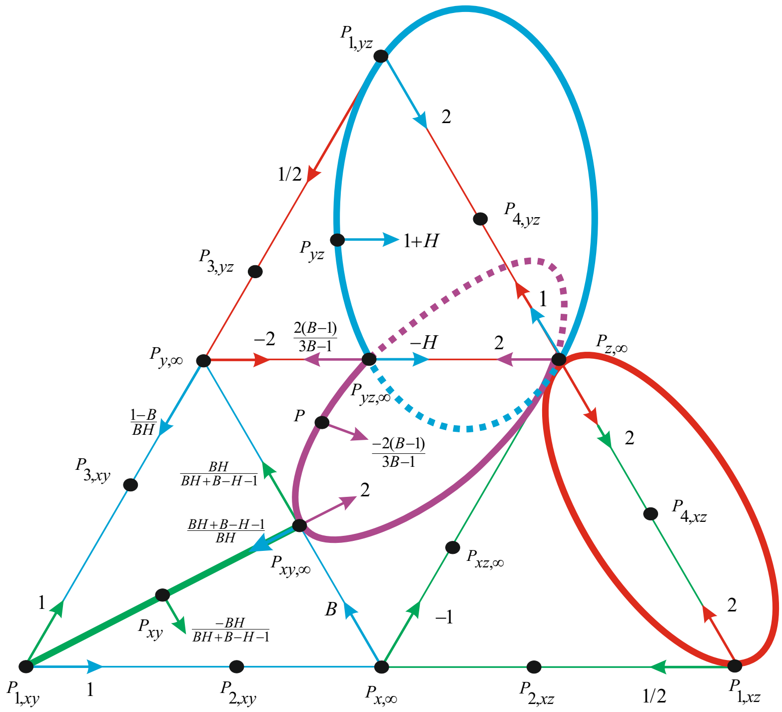



**Figure 6.8:** The whole behaviour of $f_3$.

### Case 4

The general form of the quadratic algebraic surface in this case can be defined by the following polynomial

$$f = x^2 + A_1 y(y-1) + A_2 z(z-1) + A_3 xy + A_4 xz + A_5 yz.$$

By assuming $f$ as an invariant with respect to (6.3), we get exactly three possible choices for the parameters $A_i$, $\;i \in \{1, \cdots, 5\}$.

We will explain each case individually

**I.** The quadratic surface, $C_{f_{4.1}}$, defined by the polynomial

$$f_{4.1} = x^2 + \frac{4}{(F+1)^2}y(y-1) + \frac{4}{(F+1)^2}z(z-1) - \frac{4}{F+1}xy + \frac{4}{F+1}xz + \frac{8}{(F+1)^2}yz, \quad F \neq -1,$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 2, \quad B = -(F+2), \quad C = -\frac{F+2}{F+1}, \quad D = 2, \\ &E = -\frac{F}{F+1}, \quad F = F, \quad G = -\frac{2}{F+1} \text{ and } H = \frac{2}{F+1}. \end{aligned} \tag{6.21}$$

$C_{f_{4.1}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$, $f_{4.1} = x^2 + \frac{4}{(F+1)^2}y(y-1) - \frac{4}{F+1}xy$. So, $C_{f_{4.1}}$ is projectively equivalent to the case, $C_{f_4}$, mentioned in Chapter 4.

- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{4.1} = x^2 + \frac{4}{(F+1)^2}z(z-1) + \frac{4}{F+1}xz$. So, $C_{f_{4.1}}$ is the case, $C_{f_4}$, mentioned in Chapter 4.

- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{4.1} = \frac{4}{(F+1)^2}\big(y+z\big)\big(y+z-1\big)$. So, $C_{f_{4.1}}$ is the product of two lines. The line, $-\frac{4}{(F+1)^2}\big(y+z\big) = 0$, is projectively equivalent to the cases, $C_{f_2}$, mentioned in Chapter 4, while the line, $1 - y - z = 0$, is the case, $C_{f_1}$, mentioned in Chapter 4.

- $C_{f_{4.1}}$ does not pass through the space critical point.

- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, so the version of $f_{4.1}$ can be defined by

$$F_{4.1} = \Big(\frac{2Y}{F+1} - 1\Big)^2 + \Big(\frac{2Z}{F+1} + 1\Big)^2 + 8\frac{YZ}{(F+1)^2} - 1.$$

$C_{F_{4.1}}$, is projectively equivalent to the case, $C_{f_3}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.21) can be defined by

$$\begin{aligned}\dot{Y} &= Y\Big(-(F+2) + \frac{2(F+2)}{F+1}Y + \frac{2(F-2)}{F+1}Z\Big),\\ \dot{Z} &= Z\Big(F + \frac{2(F+4)}{F+1}Y + \frac{2F}{F+1}Z\Big).\end{aligned} \tag{6.22}$$

$C_{F_{4.1}}$ passes through the critical points $P_{xy,\infty} = \Big(W = 0 : Y = \frac{1}{2}(F+1) : Z = 0\Big)$, $P_{xz,\infty} = \Big(W = 0 : Y = 0 : Z = -\frac{1}{2}(F+1)\Big)$ and $P = \Big(W = 0 : Y = \frac{1}{8}F^2(F+1) : Z = -\frac{1}{8}F^3 - \frac{5}{8}F^2 - F - \frac{1}{2}\Big)$ of (6.22). The other critical point belongs $C_{F_{4.1}}$ has the coordinates $P_{yz,\infty} = \big(X = 0 : Y = -1 : W = 0\big)$.

As a result, $C_{f_{4.1}}$ is a smooth invariant quadratic surface passing through exactly ten critical points of the system (6.3). The whole behaviour of $C_{f_{4.1}}$ can be seen in Figure 6.9.

**II.** The quadratic surface, $C_{f_{4.2}}$, defined by the polynomial

$$\begin{aligned}f_{4.2} = x^2 + \frac{4}{F^2}y(y-1) + \frac{4}{(F+1)^2}z(z-1) - \frac{4}{F}xy +\\ \frac{4}{F+1}xz + \frac{8}{F(F+1)}yz, \quad 0 \neq F \neq -1,\end{aligned}$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned}&A = 2, \quad B = -(F+1), \quad C = -\frac{F+1}{F}, \quad D = 2,\\ &E = -\frac{F}{F+1}, \quad F = F, \quad G = -\frac{1}{F+1} \text{ and } H = \frac{1}{F}.\end{aligned} \tag{6.23}$$

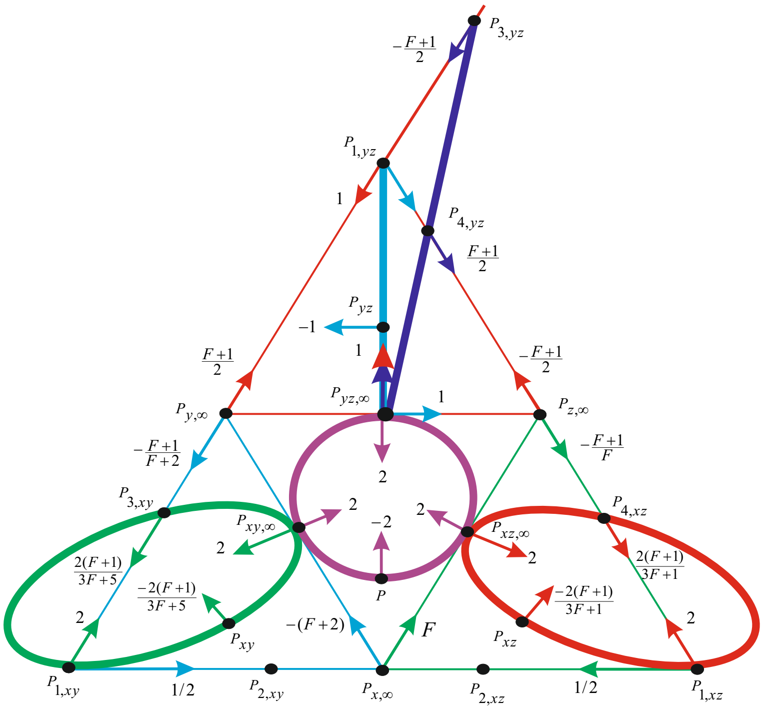



**Figure 6.9:** The whole behaviour of $f_{4.1}$.

$C_{f_{4.2}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$, $f_{4.2} = x^2 + \frac{4}{F^2}y(y-1) - \frac{4}{F}xy$. So, $C_{f_{4.2}}$, is projectively equivalent to the case, $C_{f_4}$, mentioned in Chapter 4.

- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{4.2} = x^2 + \frac{4}{(F+1)^2}z(z-1) + \frac{4}{F+1}xz$. So, $C_{f_{4.2}}$ is the case, $C_{f_4}$, mentioned in Chapter 4.

- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{4.2} = \frac{4}{F^2}y(y-1) + \frac{4}{(F+1)^2}z(z-1) + \frac{8}{F(F+1)}yz$.

  This case has not been mentioned in Chapter 4, since it is not listed in Ollagnier (2001).

  In this case, $C_{f_{4.2}}$ intersects the $x$ and $y$-axes transversely at the critical points $P_{1,yz}$, $P_{3,yz}$ and $P_{4,yz}$, while at infinity it intersects, the line, $x = 0$, tangentially at $P_{yz,\infty}$.

- $C_{f_{4.2}}$ passes through the space critical point $P_{xyz} = \Big(-\frac{1}{4F(F+1)}, -\frac{F(3F+2)}{8(F+1)}, \frac{(F+1)(3F+1)}{8F}\Big)$.

- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, so the version of $f_{4.2}$ can be defined by

$$F_{4.2} = \Big(\frac{2Y}{F} - 1\Big)^2 + \Big(\frac{2Z}{F+1} + 1\Big)^2 + 8\frac{YZ}{(F(F+1)} - 1.$$

Hence, $C_{F_{4.2}}$ is projectively equivalent to the case, $C_{f_3}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.23), can be defined by

$$\begin{aligned} \dot{Y} &= Y\Big(-(F+1) + \frac{2(F+1)}{F}Y + \frac{2(F-1)}{F+1}Z\Big), \\ \dot{Z} &= Z\Big(F + \frac{2(F+1)}{F}Y + \frac{2F}{F+1}Z\Big). \end{aligned} \tag{6.24}$$

$C_{F_{4.2}}$ passes through the critical points $P_{xy,\infty} = \Big(W = 0 : Y = \frac{1}{2}F : Z = 0\Big)$, $P_{xz,\infty} = \Big(W = 0 : Y = 0 : Z = -\frac{1}{2}(F+1)\Big)$ and $P = \Big(W = 0 : Y = \frac{1}{2}F^3 : Z = -\frac{1}{2}\big((F+1)^3\big)$ of (6.24). The other critical point for which $C_{F_{4.2}}$ passes through has the coordinates, $P_{yz,\infty} = \big(X = 0 : Y = -\frac{F}{F+1} : W = 0\big)$. As a result, $C_{f_{4.2}}$ is a smooth invariant quadratic surface passing through exactly ten critical points of the system (6.3). The whole behaviour of $C_{f_{4.2}}$ can be seen in Figure 6.10.

**III.** The quadratic surface $C_{f_{4.3}}$ defined by the polynomial

$$\begin{aligned} f_{4.3} = x^2 + \frac{4}{(B+1)^2}y(y-1) + \frac{4}{(F+1)^2}z(z-1) + \frac{4}{B+1}xy + \\ \frac{4}{F+1}xz + \frac{8}{(B+1)(F+1)}yz, \quad B \neq -1 \neq F, \end{aligned}$$

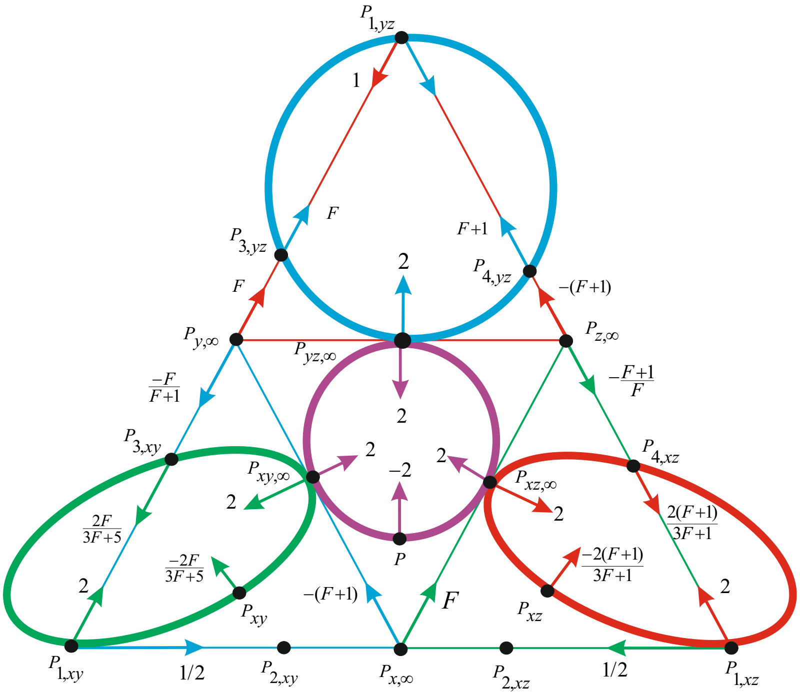



**Figure 6.10:** The whole behaviour of $f_{4.2}$.

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 2, \quad B = B, \quad C = -\frac{B}{B+1}, \quad D = 2, \\ &E = -\frac{F}{F+1}, \quad F = F, \quad G = \frac{-F+B}{F+1} \text{ and } H = \frac{F-B}{B+1}. \end{aligned} \tag{6.25}$$

$C_{f_{4.3}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$, $f_{4.3} = x^2 + \frac{4}{(B+1)^2}y(y-1) + \frac{4}{B+1}xy$. So, $C_{f_{4.3}}$ is projectively equivalent to the Case $C_{f_4}$ mentioned in Chapter 4.

- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{4.3} = f_{4.2}$.

- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{4.3} = \frac{4}{(B+1)^2}y(y-1) + \frac{4}{(F+1)^2}z(z-1) + \frac{8}{(B+1)(F+1)}yz$. It has not been mentioned in Chapter 4, due to the same reason above.

- $C_{f_{4.3}}$ does not pass through the space critical point.

- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, in other words we have used the change of coordinates (6.4), so the version of $f_{4.3}$ can be defined by

$$\begin{aligned} F_{4.3} &= \frac{1}{(F+1)^2(B+1)^2}\Big(1+B+F+BF+2Y(1+F)+2Z(1+B)\Big)^2, \\ &= \Big(1+\frac{1}{B+1}Y+\frac{1}{F+1}Z\Big)^2. \end{aligned}$$

$C_{F_{4.3}}$ is a double of a line that is projectively equivalent to the case, $C_{f_2}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.25) can be defined by

$$\begin{aligned} \dot{Y} &= BY\Big(1+\frac{1}{B+1}Y+\frac{1}{F+1}Z\Big), \\ \dot{Z} &= FZ\Big(1+\frac{1}{B+1}Y+\frac{1}{F+1}Z\Big). \end{aligned} \tag{6.26}$$

$C_{F_{4.3}}$ passes through the critical points $P_{xy,\infty} = \big(W=0 : Y = -\frac{B+1}{2} : Z=0\big)$ and $P_{xz,\infty} = \big(W=0 : Y=0 : Z = -\frac{1}{2}(F+1)\big)$. Notice that, from the definition of $F_{4.3}$, we conclude that $C_{F_{4.3}}$ is a line of singularity with respect to the system (6.26). The other critical point belongs to $C_{F_{4.3}}$ has the coordinates $P_{yz,\infty} = \Big(X=0 : W=0 : Z = -\frac{F+1}{B+1}\Big)$. The whole behaviour of $C_{f_{4.3}}$ can be seen in Figure 6.11.

**Case 5**

The general form of the quadratic algebraic surface in this case is defined by

$$f = x^2 + A_1y(y-1) + A_2z + A_3xy + A_4yz.$$

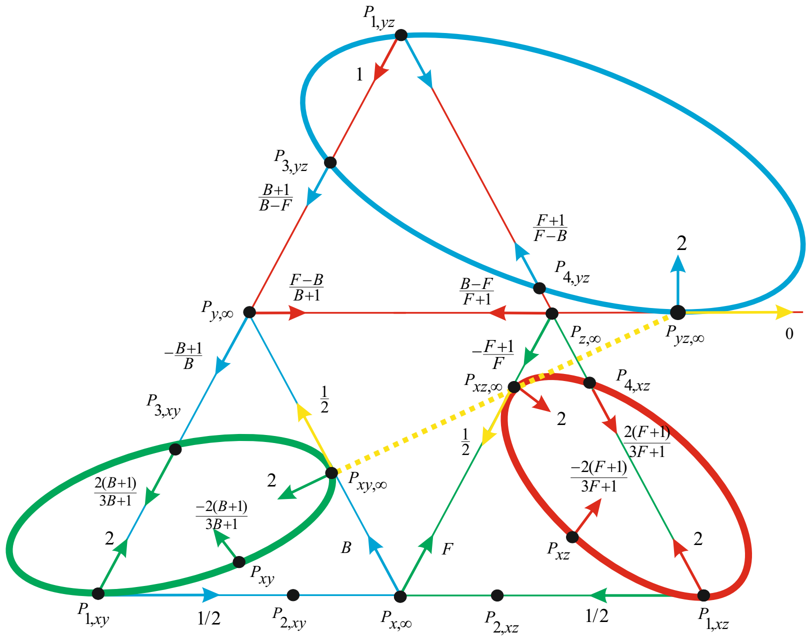



**Figure 6.11:** The whole behaviour of $f_{4.3}$.

By assuming $f$ as invariant with respect to (6.3), we get exactly one possible choice for the parameters $A_i$, $i \in \{1, 2, 3, 4\}$.

The quadratic surface $C_{f_5}$ defined by the polynomial

$$f_5 = x^2 + \frac{4}{(B+1)^2} y(y-1) + \frac{4}{B^2-1} z + \frac{4}{B+1} xy - \frac{4}{B^2-1} yz,$$
$$-1 \neq B \neq 1,$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 2, \quad B = B, \quad C = -\frac{B}{B+1}, \quad D = 2, \\ &E = \frac{1}{2}, \quad F = -1, \quad G = 1 \text{ and } H = \frac{B-1}{B+1}. \end{aligned} \tag{6.27}$$

$C_{f_5}$ has the following properties

- In ${\mathbf{P}_{xy}}^2(\mathbb{C})$, $f_5 = x^2 + \frac{4}{(B+1)^2}y(y-1) + \frac{4}{B+1}xy$. So, $C_{f_5}$ is the case, $C_{f_4}$, mentioned in Chapter 4.

- In ${\mathbf{P}_{xz}}^2(\mathbb{C})$, $f_5 = x^2 + \frac{4}{B^2-1}z$. So, $C_{f_5}$, has the same properties as the quadratic surface, $C_{f_3}$, has in the projective plane ${\mathbf{P}_{xz}}^2(\mathbb{C})$.

- In ${\mathbf{P}_{yz}}^2(\mathbb{C})$, $f_5 = \frac{4}{B+1}(y-1)\big(\frac{1}{B+1}y - \frac{1}{B-1}z\big)$. Hence, $C_{f_5}$, is the product of two lines. The line, $y-1=0$, is projectively equivalent to, $x=0$, and the other is projectively equivalent to the case, $C_{f_1}$, mentioned in Chapter 4.

- $C_{f_5}$ does not pass through the space critical point.

- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{xy}(\mathbb{C})$, so the version of $f_5$ can be defined by

$$F_5 = X^2 + \frac{4}{(B+1)^2}Y^2 - \frac{4}{B^2-1}Y + \frac{4}{B+1}XY.$$

Hence, $C_{F_5}$, is projectively equivalent to the case, $C_{f_4}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.27) for which $F_5$ is invariant can be defined by

$$\begin{aligned} \dot{X} &= X\Big(1 + X + \frac{2(1-2B)}{B+1}Y\Big), \\ \dot{Y} &= Y\Big(2 + (1+B)X + \frac{2(1-2B)}{B+1}YB\Big). \end{aligned} \tag{6.28}$$

$C_{F_5}$ passes through the critical points $P_{z,\infty} = \big(X=0 : Y=0 : W=0\big)$, $P_{yz,\infty} = \big(X=0 : Y=\frac{B+1}{B-1} : W=0\big)$ and $P = \Big(X = \frac{1-3B}{2B^2} : Y = \frac{1-B^2}{4B^2} : W=0\Big)$ of (6.28). The other point for which $C_{F_5}$ passes through has the coordinates $P_{xy,\infty} = \Big(X = \frac{-2}{B+1} : W=0 : Z=0\Big)$. The whole behaviour of

$C_{f_5}$ can be seen in Figure 6.12.

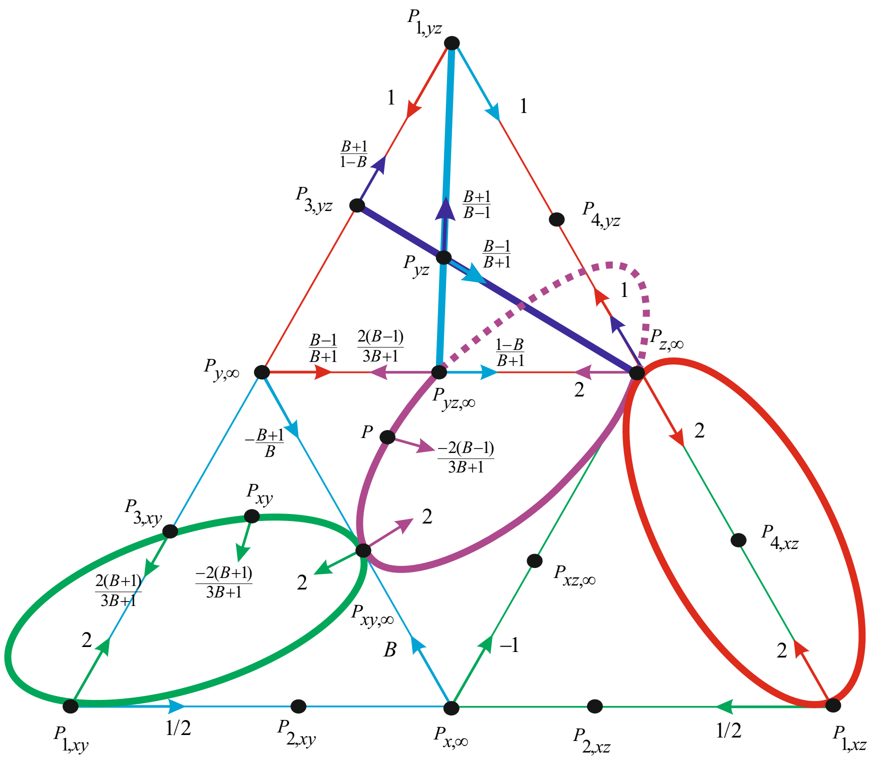



**Figure 6.12:** The whole behaviour of $f_5$.

### Case 6

The general form of the quadratic algebraic surface in this case is defined by the following polynomial

$$f = x^2 + A_1 y(y-1) + A_2 z + A_3 xy + A_4 xz.$$

By assuming $f$ as an invariant algebraic surface with respect to (6.3), we get exactly two possible choices for the parameters $A_i$, $i \in \{1, 2, 3, 4\}$. We will explain each case individually

**I.** The quadratic surface, $C_{f_{6.1}}$, defined by the polynomial

$$f_{6.1} = x^2 + \frac{4}{(B+1)^2}y(y-1) + \frac{2}{B(B+1)}z + \frac{4}{B+1}xy + \frac{2}{B+1}xz, \quad 0 \neq B \neq -1,$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 2, \quad B = B, \quad C = -\frac{B}{B+1}, \quad D = 2, \\ &E = 1, \quad F = -(1+B), \quad G = \frac{1}{2} \text{ and } H = -\frac{1}{B+1}. \end{aligned} \tag{6.29}$$

$C_{f_{6.1}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$, $f_{6.1} = x^2 + \frac{4}{(B+1)^2}y(y-1) + \frac{4}{B+1}xy$. So, $C_{f_{6.1}}$ is the case, $C_{f_4}$, mentioned in Chapter 4.

- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{6.1} = x^2 + \frac{2}{B(B+1)}z + \frac{2}{B+1}xz$. This case has not been mentioned in Chapter 4, since it is not listed in Ollagnier (2001). In this case, $C_{f_{6.1}}$, intersects the $z$-axes, transversely at both critical points $P_{1,xz}$ and $P_{z,\infty}$, while, it intersects the $x$-axis, tangentially at the critical point $P_{1,xz}$. Also it intersects the line at infinity, $y = 0$, transversely at both critical points $P_{z,\infty}$ and $P_{xz,\infty}$. Notice that, $C_{f_{6.1}}$ passes through the face critical point $P_{xz}$ as well.

- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{6.1} = \frac{4}{(B+1)^2}y(y-1) + \frac{2}{B(B+1)}z$. This case has not been mentioned in Chapter 4, since it is not listed in Ollagnier (2001). In this case, $C_{f_{6.1}}$ intersects the $y$-axis transversely at both critical points

$P_{1,yz}$ and $P_{3,yz}$, while, it intersects the $z$-axes transversely at both critical points $P_{1,yz}$ and $P_{z,\infty}$. Also, it intersects tangentially the line at infinity, $x = 0$, at the critical point $P_{z,\infty}$. Notice that, $C_{f_{6.1}}$ passes through the face critical point $P_{yz}$ as well.

- $C_{f_{6.1}}$ passes through the space critical point

$$P_{xyz} = \Big( -\frac{1}{2B}, \frac{B+1}{4B}, 1 \Big).$$

- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{xy}(\mathbb{C})$, so the version of $f_{6.1}$ can be defined by

$$F_{6.1} = X^2 + \frac{4}{(B+1)^2}Y^2 + \frac{2}{B+1}X + \frac{4}{B+1}XY.$$

Hence, $C_{F_{6.1}}$ is projectively equivalent to the case, $C_{f_4}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.29) can be defined by

$$\begin{aligned} \dot{X} &= X\Big(2 + (B+1)X - \frac{2(B-1)}{B+1}Y\Big), \\ \dot{Y} &= Y\Big(1 + (1+2B)X + \frac{2}{B+1}Y\Big). \end{aligned} \tag{6.30}$$

$C_{F_{6.1}}$ passes through the critical points $P_{z,\infty} = \Big(X = 0 : Y = 0 : W = 0\Big)$, $P_{xz,\infty} = \Big(X = -\frac{2}{B+1} : Y = 0 : W = 0\Big)$ and $P = \Big(X = -\frac{B+1}{2B^2} : Y = -\frac{(3B+1)(B+1)}{4B^2} : W = 0\Big)$ of (6.30). The other critical point for which $C_{F_{6.1}}$ passes through has the coordinates $P_{xy,\infty} = \big(X = -\frac{2}{3B+1} : W = 0 : Z = 0\big)$ in $\tilde{\mathbf{P}}^{\mathbf{2}}_{xz}(\mathbb{C})$. As a result, $C_{f_{6.1}}$ is a smooth invariant quadratic surface passing through exactly ten critical points of the system (6.3). The whole behaviour of $C_{f_{6.1}}$ can be seen in Figure 6.13.

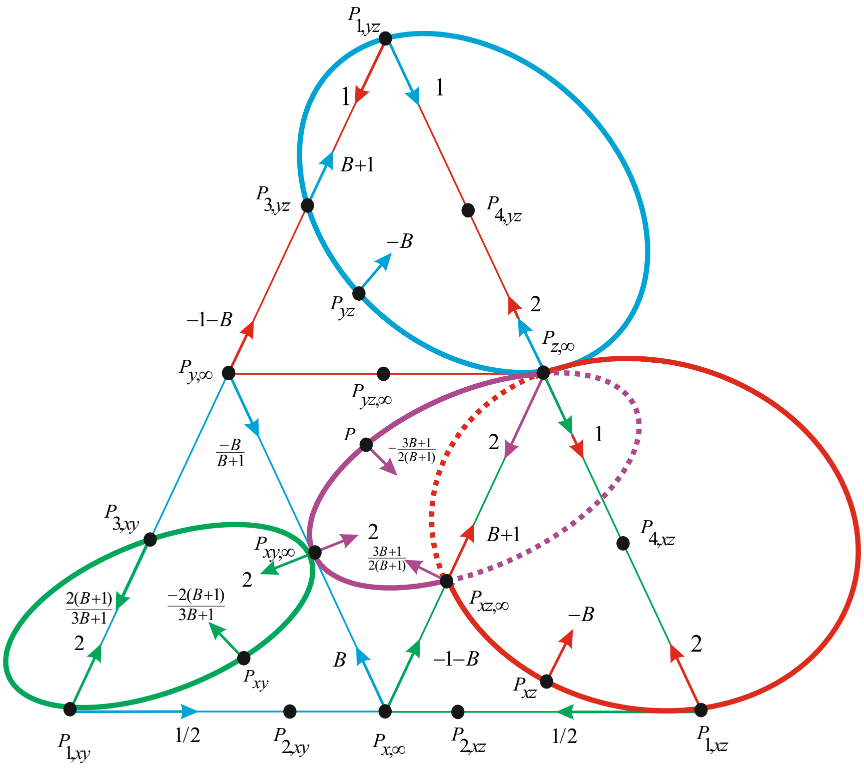



**Figure 6.13:** The whole behaviour of $f_{6.1}$.

**II.** The quadratic surface $C_{f_{6.2}}$ defined by the polynomial

$$f_{6.2} = x^2 + \frac{4}{(B+1)^2}y(y-1) + \frac{1}{(2B+1)(B+1)}z + \frac{4}{B+1}xy - \frac{2}{2B+1}xz, \quad -1 \neq B \neq \frac{-1}{2},$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = 2, \quad B = B, \quad C = -\frac{B}{B+1}, \quad D = 2, \\ &E = 1, \quad F = 2B+1, \quad G = \frac{1}{2} \text{ and } H = \frac{3B+1}{B+1}. \end{aligned} \tag{6.31}$$

$C_{f_{6.2}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$, $f_{6.2} = x^2 + \frac{4}{(B+1)^2}y(y-1) + \frac{4}{B+1}xy$. So, $C_{f_{6.2}}$ is the case, $C_{f_4}$, mentioned in Chapter 4.
- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{6.2} = x^2 + \frac{1}{(2B+1)(B+1)}z - \frac{2}{2B+1}xz$. This case has not been mentioned in Chapter 4, since it is not listed in Ollagnier (2001).
- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{6.2} = \frac{4}{(B+1)^2}y(y-1) + \frac{1}{(2B+1)(B+1)}z$.This case has not been mentioned in Chapter 4, since it is not listed in Ollagnier (2001).
- $C_{f_{6.2}}$ does not passes through the space critical point
- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{xy}(\mathbb{C})$, in other words we have used the change of coordinates (6.6), so the version of $f_{6.2}$ can be defined by

$$F_{6.2} = X^2 + \frac{4}{(B+1)^2}Y^2 - \frac{2}{2B+1}X + \frac{4}{(B+1)}XY.$$

Hence, $C_{F_{6.2}}$ is projectively equivalent to the case, $C_{f_4}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.31) can be defined by

$$\begin{aligned} \dot{X} &= X\Big(2 - (2B+1)X - \frac{2(4B+1)}{B+1}Y\Big), \\ \dot{Y} &= Y\Big(1 - (1+B)X - \frac{2(3B+1)}{B+1}Y\Big). \end{aligned} \tag{6.32}$$

$C_{F_{6.2}}$ passes through the critical points $P_{z,\infty} = \Big(X = 0 : Y = 0 : W = 0\Big)$, $P_{xz,\infty} = \Big(X = \frac{2}{2B+1} : Y = 0 : W = 0\Big)$ and $P = \Big(X = \frac{2B+1}{2B^2} : Y = -\frac{B+1}{4B^2} : W = 0\Big)$. The other critical point for which $C_{F_{6.2}}$ passes through has the coordinates $P_{xy,\infty} = \Big(W = 0 : Y = -\frac{1}{2}(B+1) : Z =$

$\left.0\right)$ in $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$. As a result, $C_{f_{6.2}}$ is a smooth invariant quadratic surface passing through exactly ten critical points of the system (6.3). The whole behaviour of $C_{f_{6.2}}$ can be seen in Figure 6.14.

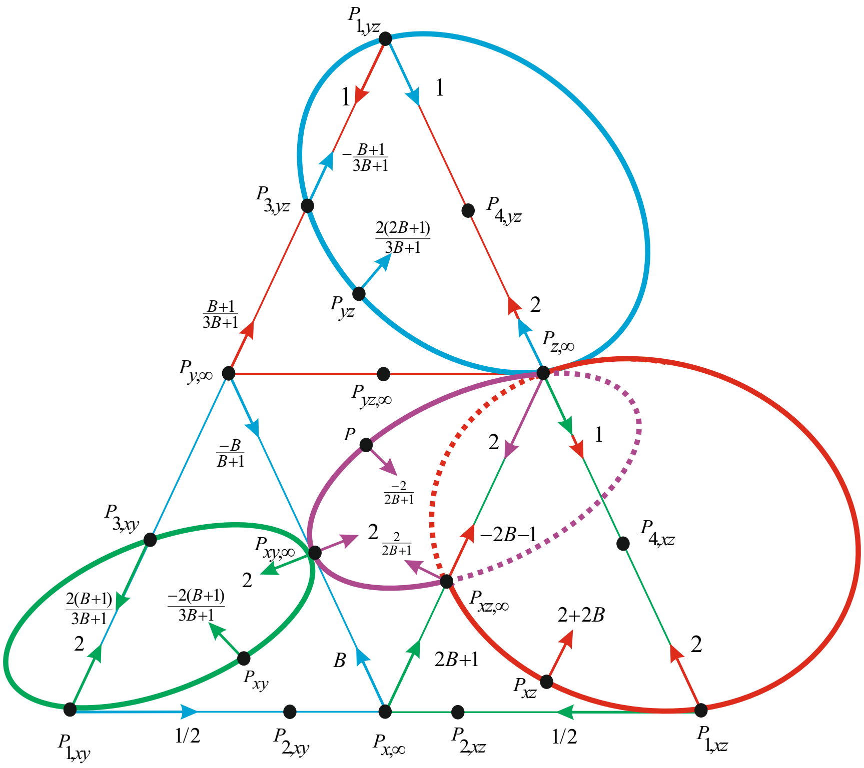



**Figure 6.14:** The whole behaviour of $f_{6.2}$.

### Case 7

The general form of the quadratic algebraic surface in this case is defined by the following polynomial

$$f = A_1x(x-1) + A_2y(y-1) + A_3z(z-1) + A_4xy + A_5xz + A_6yz + 1.$$

By assuming $f$ as an invariant algebraic surface with respect to (6.3), we get exactly two possible choices for the parameters $A_i,\ \ i \in \{1, \cdots, 6\}$.
We will explain each case individually

**I.** The quadratic surface $C_{f_{7.1}}$ defined by the polynomial

$$f_{7.1} = 1 + x^2 - 2x + y^2 - 2y + z^2 - 2z - 2xy + 2xz - 2yz,$$

is invariant with respect to (6.3) under the following conditions

$$\begin{aligned} &A = -\frac{1}{C+1}, \quad B = -\frac{C}{C+1}, \quad C = C, \quad D = -F, \\ &E = 1, \quad F = F, \quad G = -\frac{CF+F+1}{F(C+1)} \text{ and } H = -(CF+F+1). \end{aligned} \tag{6.33}$$

$C_{f_{7.1}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$, $f_{7.1} = x^2 + y^2 - 2x - 2y - 2xy + 1$. So, $C_{f_{7.1}}$, is the case, $C_{f_3}$, mentioned in Chapter 4.

- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{7.1} = (1 - x - z)^2$. So, $C_{f_{7.1}}$, is the double of the case, $C_{f_1}$, mentioned in Chapter 4.

- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{7.1} = y^2 + z^2 - 2y - 2z - 2yz$. So, $C_{f_{7.1}}$, is the case , $C_{f_3}$, mentioned in Chapter 4.

- $C_{f_{7.1}}$ passes through the space critical point

$$P_{xyz} = \Big(\frac{2F(C+1)(F+1)+1}{2(C+1)\big(F+\frac{1}{2}\big)^2}, \frac{1}{\big(2F+1\big)^2}, \frac{2CF+2C+2F+1}{2(C+1)\big(F+\frac{1}{2}\big)^2}\Big).$$

- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, so the version of $f_{7.1}$ can be defined by
$$F_{7.1} = (Y - Z - 1)^2.$$
Hence, $C_{F_{7.1}}$, is the double of a line, which is projectively equivalent to the case, $C_{f_1}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.33) can be defined by
$$\begin{aligned} \dot{Y} &= Y\Big(-\frac{C}{C+1} + \frac{C}{C+1}Y + \frac{2CF+2F+1}{C+1}Z\Big), \\ \dot{Z} &= Z\big(F + (F+1)Y + FZ\big). \end{aligned} \tag{6.34}$$
$C_{F_{7.1}}$ passes through the critical points $P_{xz,\infty} = \big(W = 0 : Y = 0 : Z = -1\big)$ and $P_{xy,\infty} = \big(W = 0 : Y = 1 : Z = 0\big)$ of (6.34) and both are singularities of $F_{7.1}$. The other critical point for which $C_{F_{7.1}}$ passes through can be given by the coordinates $P_{yz,\infty} = \big(X = 0 : W = 0 : Z = 1\big)$ in $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$ and it is a singularity of $F_{7.2}$ as well. The whole behaviour of $C_{f_{7.1}}$ can be seen in Figure 6.15.

**II.** The quadratic surface $C_{f_{7.2}}$ defined by the polynomial
$$f_{7.2} = 1 + x^2 - 2x + y^2 - 2y + z^2 - 2z - 2xy + 2xz + 2yz,$$
is invariant with respect to (6.3) under the following conditions
$$\begin{aligned} &A = -\frac{1}{C+1}, \quad B = -\frac{C}{C+1}, \quad C = C, \quad D = -1, \\ &E = -\frac{CG - C + G}{C+1}, \quad F = -\frac{CG - C + G}{C+1}, \quad G = G \end{aligned} \tag{6.35}$$
and $H = -G(C+1)$.

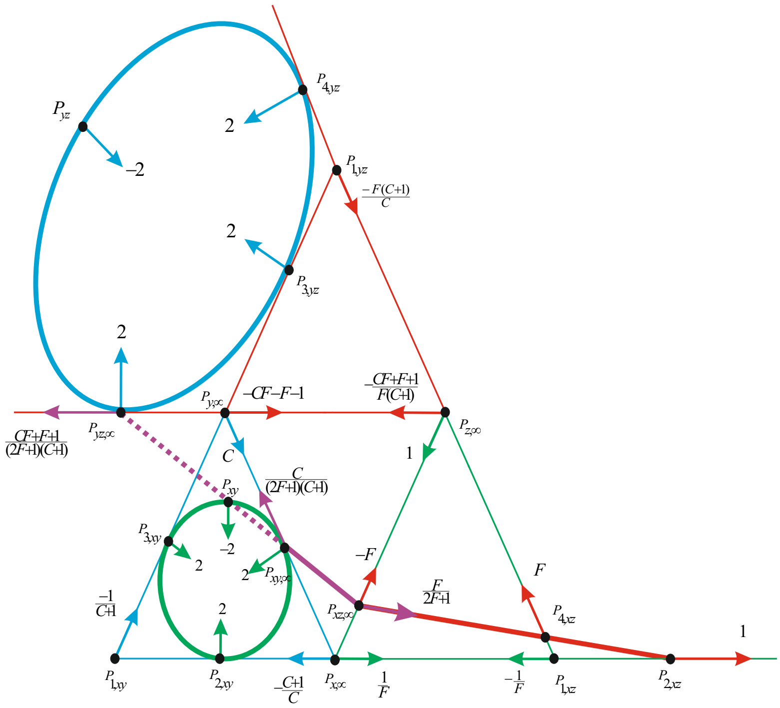



**Figure 6.15:** The whole behaviour of $f_{7.1}$.

$C_{f_{7.2}}$ has the following properties

- In $\mathbf{P}_{xy}{}^2(\mathbb{C})$. So, $C_{f_{7.2}}$, is the case, $C_{f_3}$, mentioned in Chapter 4.

- In $\mathbf{P}_{xz}{}^2(\mathbb{C})$, $f_{7.2} = (1 - x - z)^2$. So, $C_{f_{7.2}}$, is the double of the case $C_{f_1}$, mentioned in Chapter 4. The point $P_{4,xz} = (0, 0, 1)$ is a singularity of $C_{f_{7.2}}$. By assuming the following change of coordinates

$$
\begin{aligned}
x &= \frac{1}{4}\big(-u + v + 2w\big),\\
y &= \frac{1}{4}\big(-u + v - 2w\big),\\
z &= 1 + u,
\end{aligned}
$$

the correspondence of $f_{7.2}$, is given by the equation of the cone $g =$

$g(u, v, w) = uv + w^2$. Hence $C_{f_{7.2}}$ has a cone branch at $P_{4,z}$.

- In $\mathbf{P}_{yz}{}^2(\mathbb{C})$, $f_{7.2} = (1 - y - z)^2$. So, $C_{f_{7.2}}$, is the double of the case, $C_{f_1}$, mentioned in Chapter 4.

- $C_{f_{7.2}}$ passes through the space critical point.

$$P_{xyz} = \left( \frac{\left(CG - C + G\right)^2}{2\left(2C^2G^2 - C^2G + 4CG^2 + C^2 + CG + 2G^2 + 2G\right)}, \frac{\left(CG + C + G + 2\right)^2}{2\left(2C^2G^2 - C^2G + 4CG^2 + C^2 + CG + 2G^2 + 2G\right)}, \frac{C^2G - C^2 + 3CG + 2G + 2}{2\left(2C^2G^2 - C^2G + 4CG^2 + C^2 + CG + 2G^2 + 2G\right)} \right).$$

- At infinity:

  Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, so the version of $f_{7.2}$ can be defined by

$$F_{7.2} = 1 + Y^2 + Z^2 - 2Y + 2Z + 2YZ.$$

  Hence, $C_{F_{7.2}}$, is projectively equivalent to the case, $C_{f_3}$, mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.35) can be defined by

$$\begin{aligned} \dot{Y} &= Y\Big( -\frac{C}{C+1} + \frac{C}{C+1}Y + \frac{2CF + 2F + 1}{C+1}Z \Big), \\ \dot{Z} &= Z\big(F + (F+1)Y + FZ\big). \end{aligned} \tag{6.36}$$

  $C_{F_{7.2}}$ passes through the critical points $P_{xz,\infty} = \big(W = 0 : Y = 0 : Z = -1\big)$, $P_{xy,\infty} = \big(W = 0 : Y = 1 : Z = 0\big)$ and $P = \Big(W = 0 : Y = -\frac{C^2G^2 - 2C^2G + 2CG^2 + C^2 - 2CG + G^2}{G^2(C+1)^2}, -\frac{C^2}{G^2(C+1)^2}\Big)$ of (6.36). The other critical point for which $C_{F_{7.2}}$ passes through has the coordinates $P_{yz,\infty} = \big(X = 0 : W = 0 : Z = 1\big)$ in $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$. The whole behaviour of $C_{f_{7.2}}$ can be seen in Figure 6.15.

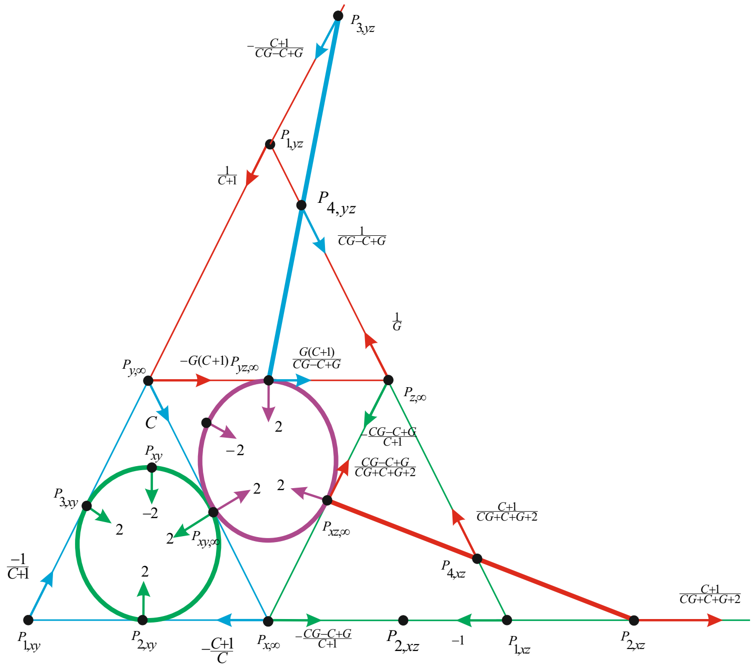
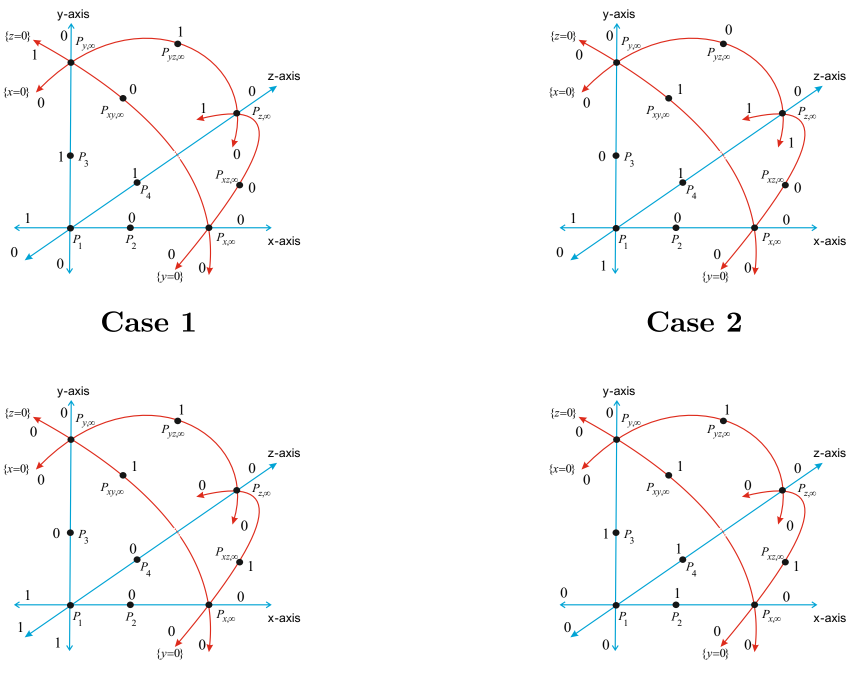



**Figure 6.16:** The whole behaviour of $f_{7.2}$.

## 6.3 Invariant algebraic planes

In this section, we will apply our mechanism to investigate the invariant algebraic planes in three dimensional complex Lotka-Volterra systems, and confirm the calculation in Chapter 3. In applying Figure 6.1 on the algebraic planes in $\mathbb{C}^3$, we get exactly four possible invariant planes with respect to the system (6.3) as can be seen in Figure 6.17. We consider each case individually




Case 1 Case 2

Case 3 Case 4

**Figure 6.17:** Multiplicity possibilities of the invariant algebraic planes.

### Case 1

This case is trivial, since it is represented the $yz$-plane which is already invariant with respect to the system (6.3), with no required conditions.

### Case 2

The general form of the algebraic plane in this case is defined by the following polynomial

$$h_2 = x + A_1 y.$$

By assuming $h_2$ as an invariant with respect to (6.3), we get $A_1 = -\frac{C}{B}$, under the following conditions

$$A = 1,\ B = B,\ C = C,\ D = D,\ E = E,\ F = F,\ G = E \text{ and } H = H. \tag{6.37}$$

$C_{h_2}$ has the following properties

- In ${\mathbf{P}_{xy}}^2(\mathbb{C})$, $h_2 = x - \frac{C}{B}y$. So, $C_{h_2}$, is the case, $C_{f_2}$, mentioned in Chapter 4.
- In ${\mathbf{P}_{xz}}^2(\mathbb{C})$, $h_2 = x$. So, $C_{h_2}$, is the trivial case.
- In ${\mathbf{P}_{yz}}^2(\mathbb{C})$, $h_2 = -\frac{C}{B}y$. So, $C_{h_2}$, is the trivial case as well.
- $C_{h_2}$ passes through the space critical point

$$P_{xyz} = \Bigg( \frac{C(D - DE - 1)}{BEH + CEF + BC - BE - BH - CE - CF}, \frac{B(D - DE - 1)}{BEH + CEF + BC - BE - BH - CE - CF}, \frac{BCD - BD - BH - CD - CF + B + C}{D(BEH + CEF + BC - BE - BH - CE - CF)} \Bigg).$$

- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, so the version of $h_2$ can be defined by $H_2 = 1 - \frac{C}{B}Y$. Hence, $C_{H_2}$, is projectively equivalent to the trivial case. The correspondence of the vector field determined by the conditions given in (6.37) for which $H_2$ is invariant can be defined by

$$\begin{aligned} \dot{Y} &= Y\big(B - CY\big), \\ \dot{Z} &= Z\big(F + (H - C)Y - DEZ\big). \end{aligned} \tag{6.38}$$

$C_{H_2}$ passes through the critical points $P_{xy,\infty} = \big(W = 0 : Y = \frac{B}{C} : Z = 0\big)$

and $P = \left(W = 0 : Y = \frac{B}{C} : Z = \frac{BH+CF-BC}{CDE}\right)$ of (6.38). The other critical point for which $C_{H_2}$ passes through has the coordinates $P_{z,\infty} = \left(X = 0 : Y = 0 : W = 0\right)$. The whole behaviour of $C_{h_2}$ can be seen in Figure 6.18.

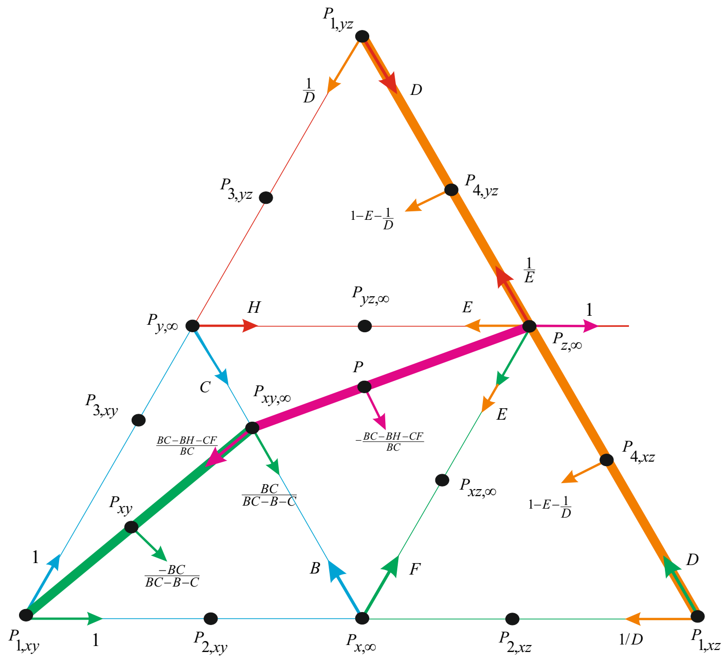



**Figure 6.18:** The whole behaviour of $h_2$.

### Case 3

The general form of the algebraic plane in this case is defined by the following polynomial

$$h_3 = x + A_1 y + A_2 z.$$

By assuming $h_3$ as an invariant with respect to (6.3), we get $A_1 = -\frac{C}{B}$ and $A_2 = -\frac{E}{F}$ where $BCEF \neq 0$, under the following conditions

$$\begin{aligned} &A = 1, \quad B = B, \quad C = C, \quad D = 1, \\ &E = E, \quad F = F, \quad G = G \text{ and } H = -\frac{CGF}{BE}. \end{aligned} \tag{6.39}$$

$C_{h_3}$ has the following properties

- In ${\mathbf{P}_{xy}}^2(\mathbb{C})$, $h_3 = x - \frac{C}{B}y$. So, $C_{h_3}$, is the case, $C_{f_2}$, mentioned in Chapter 4.
- In ${\mathbf{P}_{xz}}^2(\mathbb{C})$, $h_3 = x - \frac{E}{F}z$. So, $C_{h_3}$ is the case, $C_{f_2}$, mentioned in Chapter 4.
- In ${\mathbf{P}_{yz}}^2(\mathbb{C})$, $h_3 = -\frac{B}{C}y - \frac{E}{F}z$. So, $C_{h_3}$ is projectively equivalent to the case, $C_{f_2}$, mentioned in Chapter 4.
- $C_{h_3}$ does not passes through the space critical point.
- At infinity: Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, so the version of $h_3$ is defined by $H_3 = 1 - \frac{C}{B}Y - \frac{E}{F}Z$. Hence, $C_{H_3}$, is projectively equivalent to the case, $C_{f_1}$ mentioned in Chapter 4. The correspondence of the vector field determined by the conditions given in (6.39) for which $H_3$ is invariant can be defined by

$$\begin{aligned} \dot{Y} &= Y\big(B - CY + (G - E)Z\big), \\ \dot{Z} &= Z\Big(F - \frac{C(BE + FG)}{BE}Y - EZ\Big). \end{aligned} \tag{6.40}$$

  $C_{H_3}$ passes through the critical points $P_{xy,\infty} = \big(W = 0 : Y = \frac{B}{C} : Z = 0\big)$ and $P_{xz,\infty} = \big(W = 0 : Y = 0 : Z = \frac{F}{E}\big)$ of (6.40). The other critical point for which $C_{H_3}$ passes through has the coordinates $P_{yz,\infty} = \big(X = 0 : Y = -\frac{BE}{CF} : W = 0\big)$. The whole behaviour of $C_{h_3}$ can be seen in Figure 6.19.

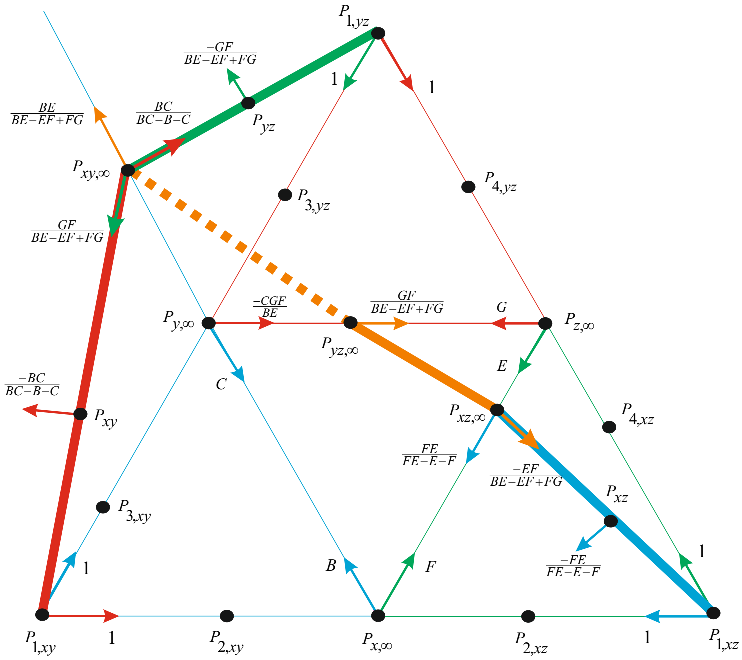



**Figure 6.19:** The whole behaviour of $h_3$.

## Case 4

The general form of the algebraic plane in this case is defined by the following polynomial

$$h_4 = 1 + A_1x + A_2y + A_3z.$$

By assuming $h_4$ as an invariant with respect to (6.3), we get $A_i = -1$ for all $i \in \{1,2,3\}$, under the following conditions

$$\begin{aligned} &A = -\frac{B}{C}, \quad B = B, \quad C = C, \quad D = -\frac{F}{E}, \\ &E = E, \quad F = F, \quad G = G \quad \text{and} \quad H = -\frac{CGF}{BE}. \end{aligned} \tag{6.41}$$

$C_{h_4}$ has the following properties

- In ${\mathbf{P}_{xy}}^2(\mathbb{C})$, $h_4 = 1 - x - y$. So, $C_{h_4}$ is the case, $C_{f_1}$, mentioned in Chapter 4.

- In ${\mathbf{P}_{xz}}^2(\mathbb{C})$, $h_4 = 1 - x - z$. So, $C_{h_4}$ is the case, $C_{f_1}$, mentioned in Chapter 4.

- In ${\mathbf{P}_{yz}}^2(\mathbb{C})$, $h_4 = 1 - y - z$. So, $C_{h_4}$ is the case, $C_{f_1}$, mentioned in Chapter 4.

- $C_{h_4}$ passes through the space critical point

$$P_{xyz} = \Bigg( \frac{CFG + BE - CF}{C(BE - EF + FG)}, \frac{E + F - EF}{(BE - EF + FG)}, \frac{E(BC - B - C)}{C(BE - EF + FG)} \Bigg).$$

- At infinity:
  Assume $\tilde{\mathbf{P}}^{\mathbf{2}}_{yz}(\mathbb{C})$, so the version of $h_4$ can be defined by $H_4 = -1 - Y - Z$. Hence, $C_{H_4}$ is projectively equivalent to the case, $C_{f_1}$ mentioned in Chapter 4.
  The correspondence of the vector field determined by the conditions given in (6.41) for which $H_4$ is invariant can be defined by

$$\begin{aligned} \dot{Y} &= Y\Big(B + BY + \frac{F(E-G)}{E}Z\Big), \\ \dot{Z} &= Z\Big(F + \frac{BE + FG}{E}Y + FZ\Big). \end{aligned} \tag{6.42}$$

$C_{H_4}$ passes through the critical points $P_{xy,\infty} = \big(W = 0 : Y = -1 : Z = 0\big)$ and $P_{xz,\infty} = \big(W = 0 : Y = 0 : Z = -1\big)$ of (6.42). The other critical point for which $C_{H_4}$ passes through has the coordinates $P_{yz,\infty} = \big(X = 0 : Y = -1 : W = 0\big)$.

The whole behaviour of $C_{h_4}$ can be seen in Figure 6.20

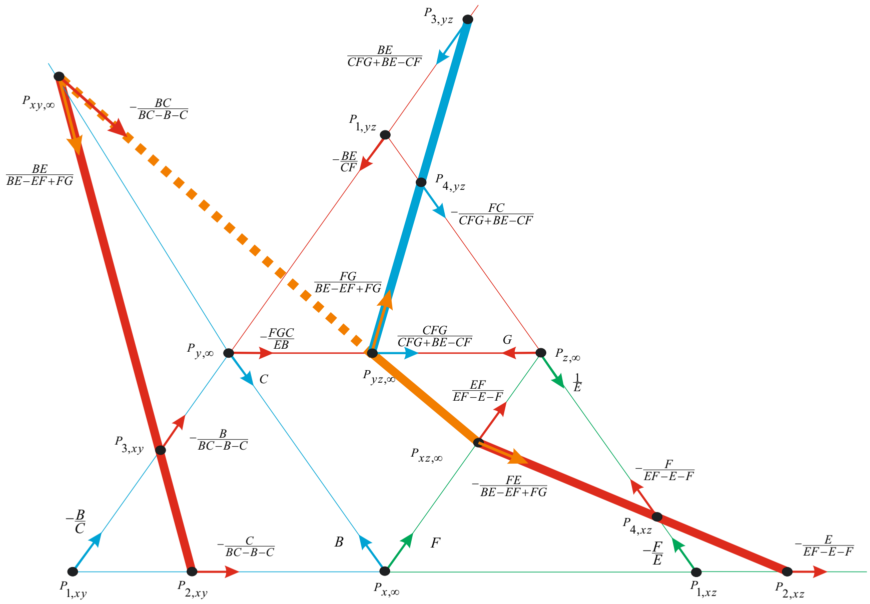



**Figure 6.20:** The whole behaviour of $h_4$.

# Chapter 7

# Invariant Cubic Surfaces in Three Dimensional Lotka-Volterra Systems

In this chapter, we make a start at applying the only technique of counting the multiplicity possibilities given in Chapter 6, Figure 6.1 for the cubic surfaces. By applying the technique of Chapter 6 together with Remark 16 for investigating invariant algebraic cubic surfaces with respect to (6.3), we get exactly 184 non projectively equivalent cases as shown in Figure 7.1. We will explain how can we get those figures.

We assume, Figure 6.1 and an arbitrary algebraic cubic surface $C_f$ defined by a polynomial $f(x, y, z) = 0$. Without loss of generality, we may fix the critical point $P_1$ and start with the $x$-axis. So, for $k_1 = mult_{P_1}(C_f, x - axis)$, we have four possible choices for the value of $k_1$, hence $k_1 \in \{3, 2, 1, 0\}$.

1. Let

$$k_1 = 3. \tag{7.1}$$

Then by applying Theorem 9, Page 24, we get

$$k_2 = k_3 = 0. \tag{7.2}$$

Consequently from $k_3 = 0$, we get

$$m_{1,\infty} = n_{1,\infty} = 0. \tag{7.3}$$

On the other hand, for $m_1 = mult_{P_1}(C_f, y - axis)$, we have $m_1 \in \{3, 2, 1, 0\}$.

1.1. Let

$$m_1 = 3. \tag{7.4}$$

Similarly, we get

$$m_2 = m_3 = k_{3,\infty} = n_{3,\infty} = 0, \tag{7.5}$$

and hence by applying Theorem 9, Page 24 on the line at infinity, $z = 0$, we get

$$n_{2,\infty} = 3. \tag{7.6}$$

For $n_1 = mult_{P_1}(C_f, z - axis)$, we have, $n_1 \in \{2, 1, 0\}$. Notice that $n_1 \neq 3$, otherwise we get the ratios of eigenvalues 1:1:1 at $P_1$ which contradicts to what we have considered in Remark 16.

i. Let

$$n_1 = 2. \tag{7.7}$$

Hence, for $n_2 = mult_{P_4}(C_f, z - axis)$, we have, $n_2 \in \{1, 0\}$

- Let

$$n_2 = 1, \tag{7.8}$$

then we get

$$n_3 = 0, \tag{7.9}$$

and hence

$$m_{3,\infty} = k_{1,\infty} = 0. \tag{7.10}$$

Consequently, on the line at infinities, $x = 0$ and $y = 0$, we get

$$k_{2,\infty} = m_{2,\infty} = 3. \tag{7.11}$$

Hence from the equations $(7.j)$, $j \in \{1, ..., 11\}$, we get Case 1.

- Let $n_2 = 0$, then we get $n_3 = 1$. Consequently $k_{1,\infty} \neq 1$ or $m_{3,\infty} \neq 1$ due to Remark 16. Without loss of generality, let $m_{3,\infty} \in \{3, 2\}$ and $k_{1,\infty} \in \{3, 2, 1\}$.

  - Let
  $$m_{3,\infty} = 3 \text{ and } k_{1,\infty} = 3. \tag{7.12}$$
  Hence, the equation (7.11) can be rewritten as
  $$k_{2,\infty} = m_{2,\infty} = 0. \tag{7.13}$$
  Consequently, from the equations $(7,j)$, $j \in \{1, 2, ..., 7, 12, 13\}$, we get Case 2.

  - Let
  $$m_{3,\infty} = 3 \text{ and } k_{1,\infty} = 2. \tag{7.14}$$
  Hence, the equation (7.13) can be rewritten as
  $$k_{2,\infty} = 1 \text{ and } m_{2,\infty} = 0. \tag{7.15}$$
  Consequently, from the equations $(7,j)$, $j \in \{1, 2, ..., 7, 14, 15\}$, we get Case 3.

  - Let
  $$m_{3,\infty} = 3 \text{ and } k_{1,\infty} = 1. \tag{7.16}$$

Hence, the equation (7.15) can be rewritten as

$$k_{2,\infty} = 2 \text{ and } m_{2,\infty} = 0. \tag{7.17}$$

Consequently, from the equations $(7{,}j)$, $j \in \{1, 2, ..., 7, 16, 17\}$, we get Case 4.

− Let

$$m_{3,\infty} = 2 \text{ and } k_{1,\infty} = 3 \tag{7.18}$$

This case is projectively equivalent to Case 3, by the following transformation

$$x \to y, \quad y \to x \quad \text{and} \quad z = z.$$

− Let

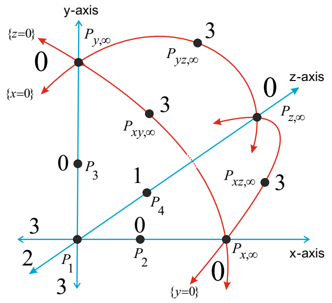
$$m_{3,\infty} = 2 \text{ and } k_{1,\infty} = 2. \tag{7.19}$$

Hence, the equation (7.17) can be rewritten as

$$k_{2,\infty} = 1 \text{ and } m_{2,\infty} = 1. \tag{7.20}$$

Consequently, from the equations $(7{,}j)$, $j \in \{1, 2, ..., 7, 19, 20\}$, we get Case 5.

So, by continuing this process for all other possible choices, we get exactly 184 possible cases as given in Figure 7.1.

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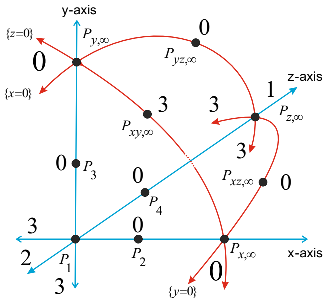

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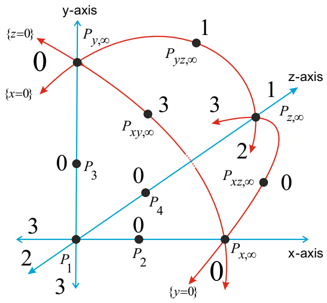

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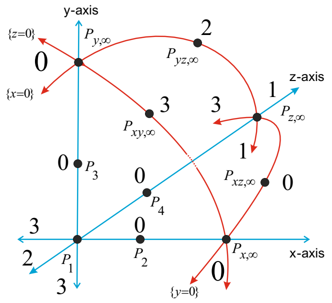

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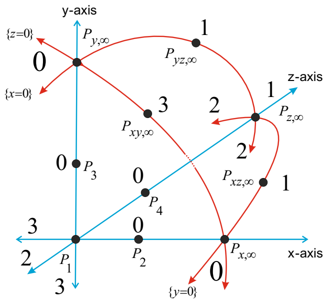

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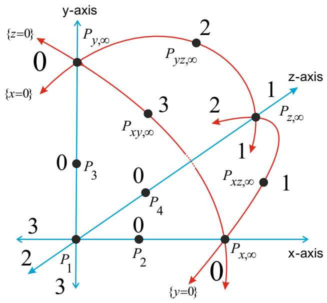

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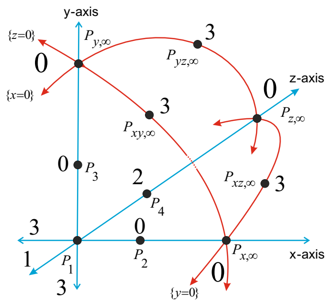

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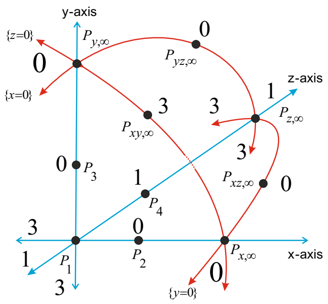

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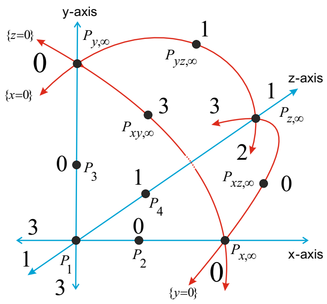

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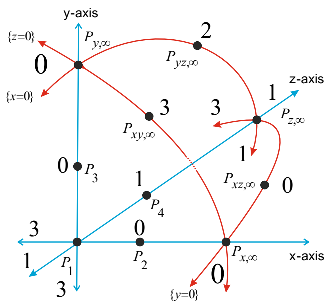

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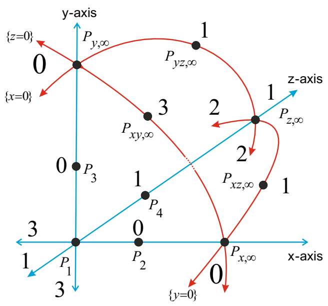

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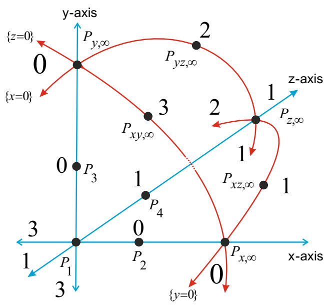

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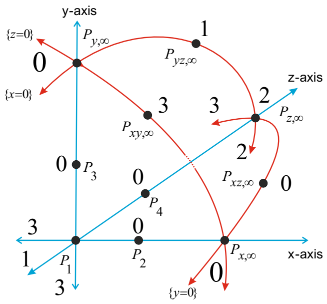


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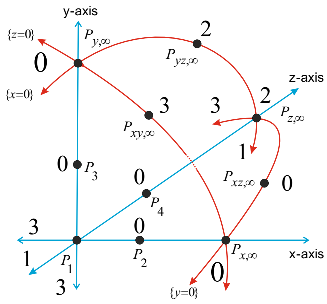


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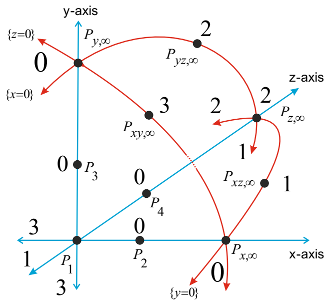


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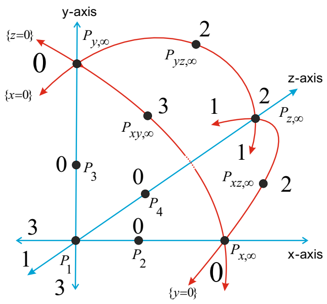


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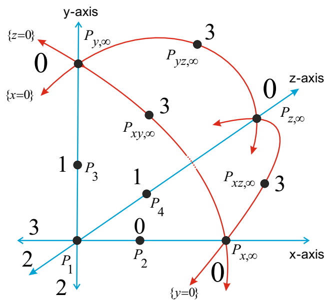


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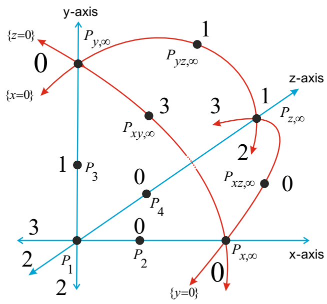


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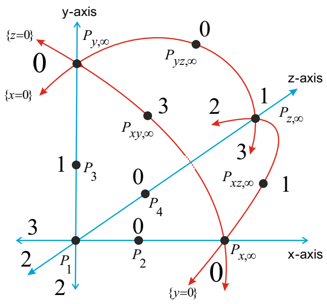


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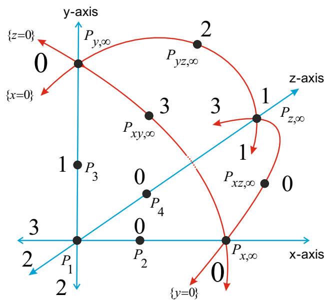


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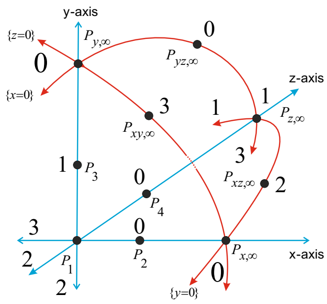


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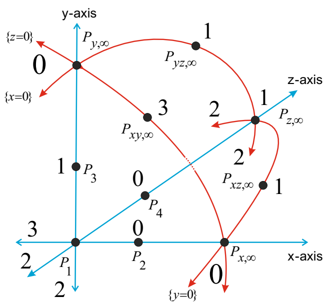


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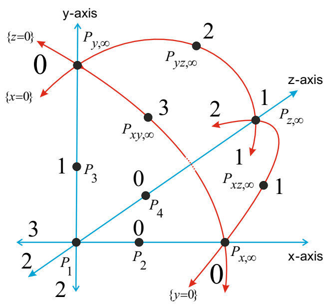


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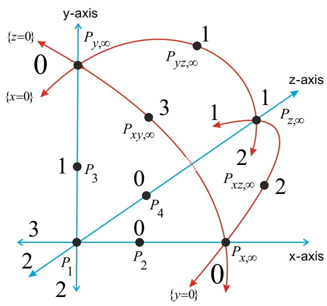


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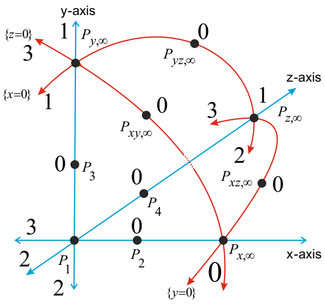


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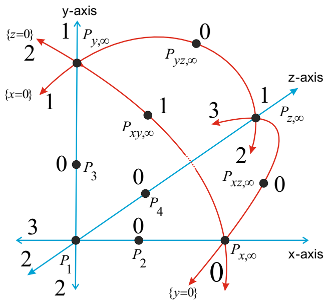


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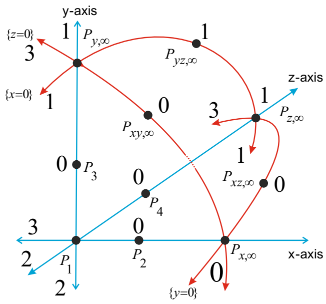


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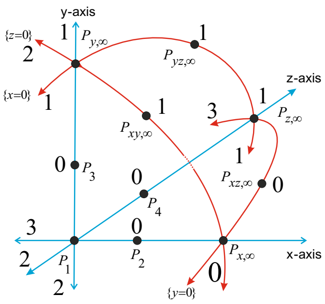


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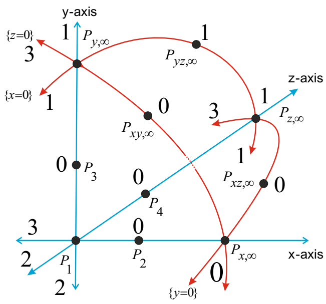


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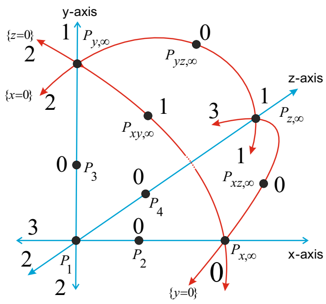


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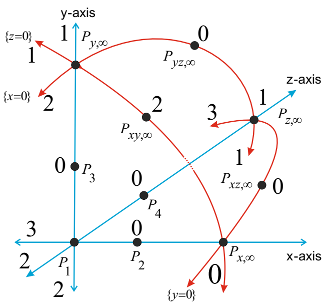


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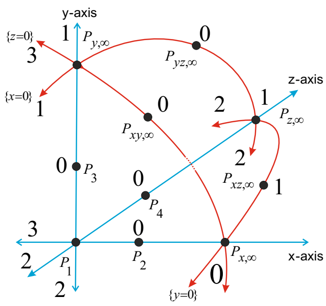


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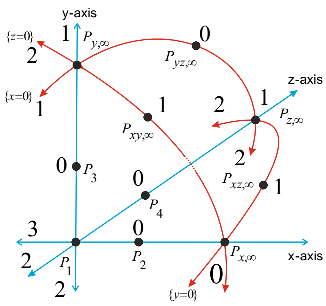


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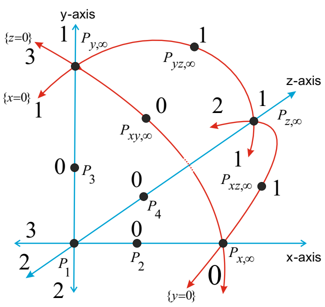


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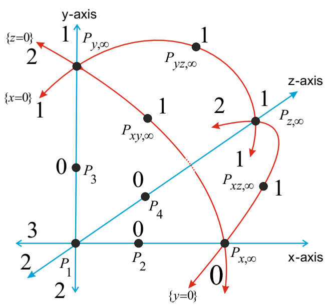


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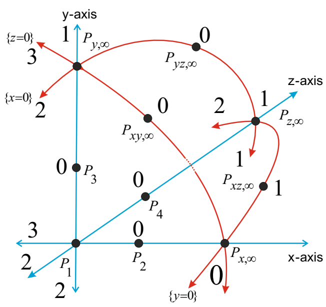


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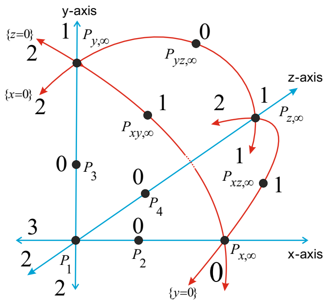


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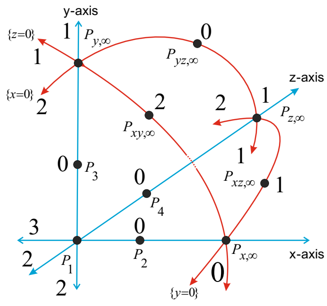


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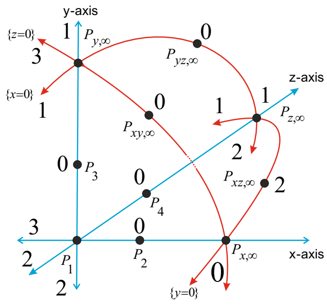


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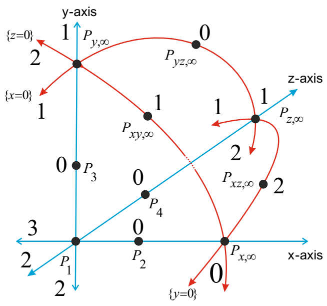


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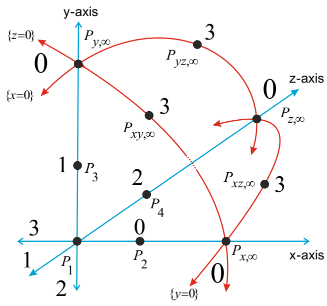


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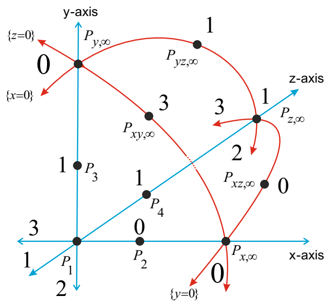


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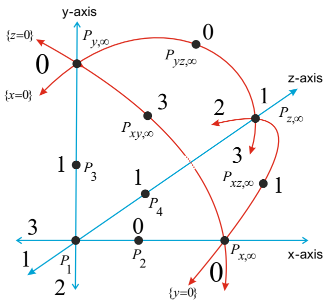


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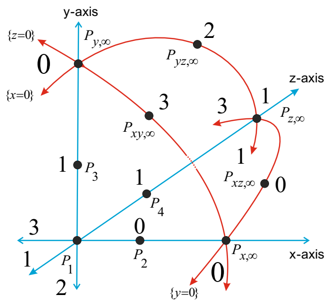


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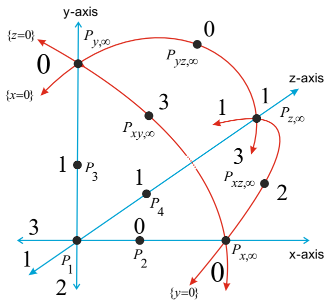


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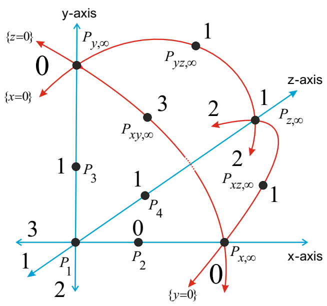


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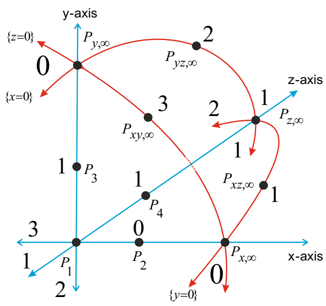


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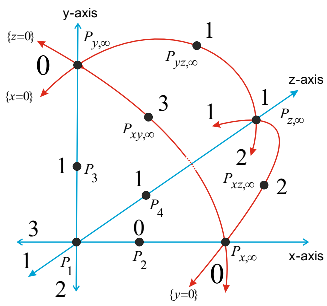


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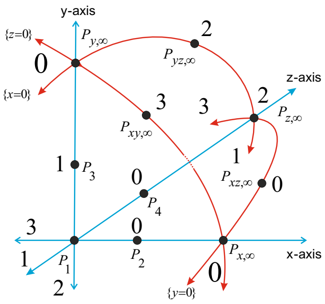


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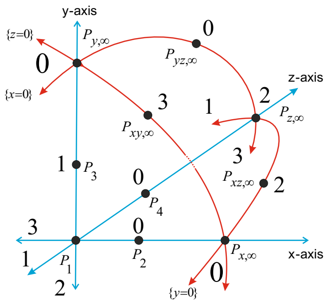


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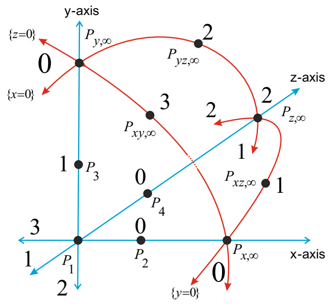


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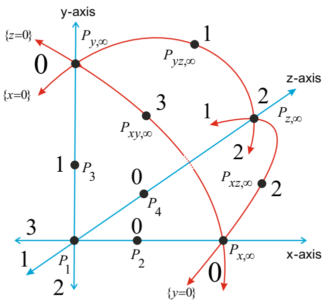


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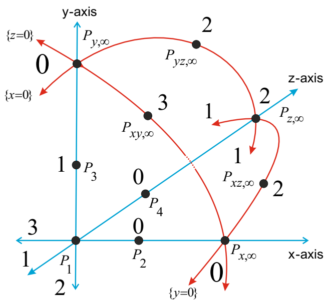


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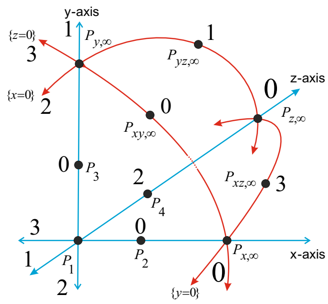


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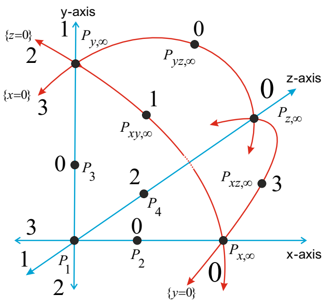


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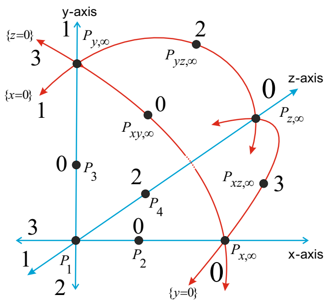


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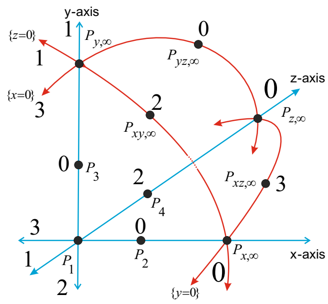


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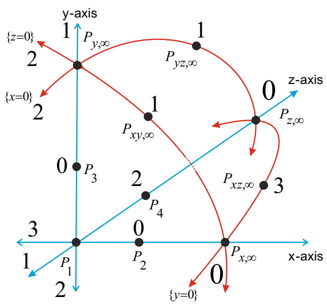


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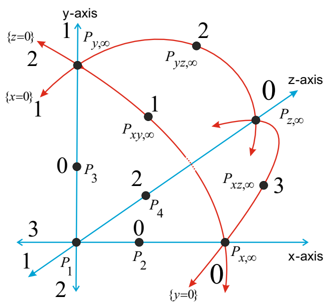


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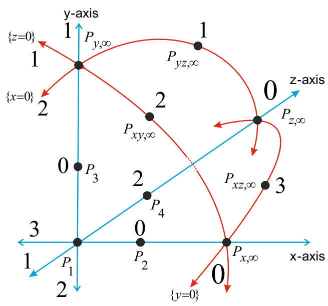


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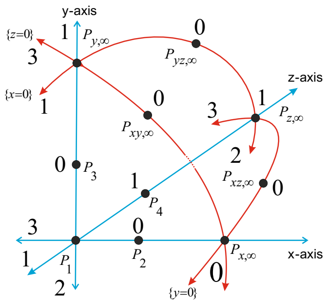


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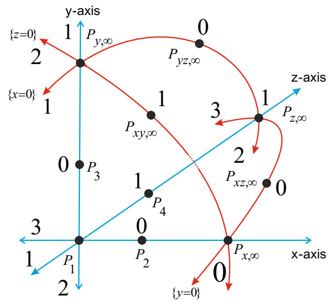

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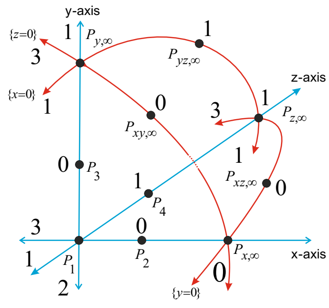

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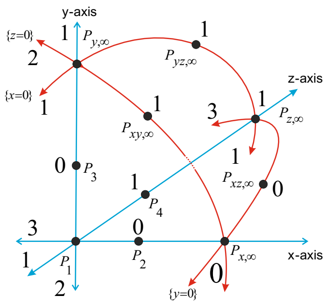

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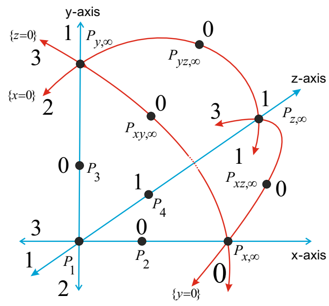

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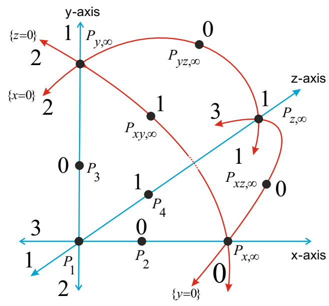

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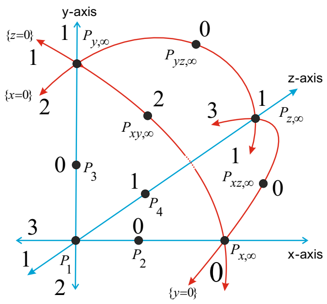

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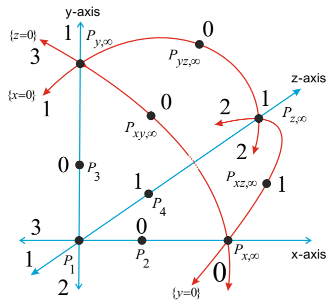

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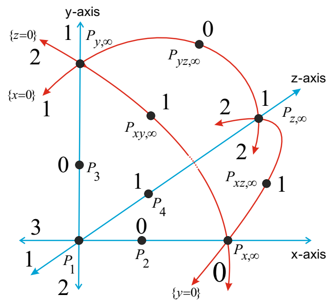

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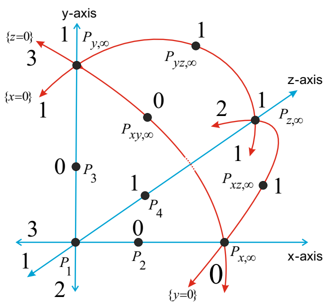

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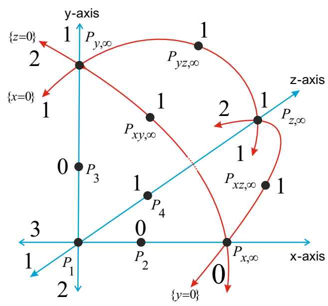

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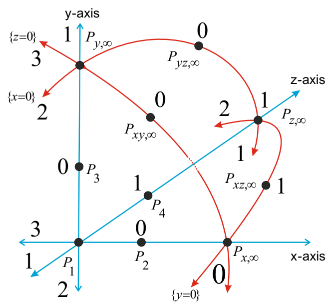

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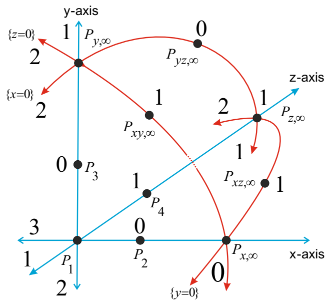


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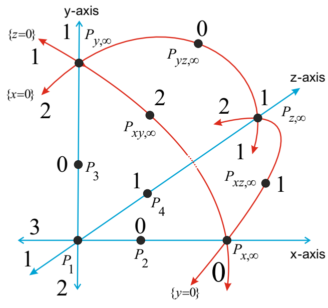

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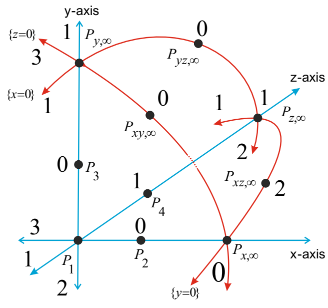

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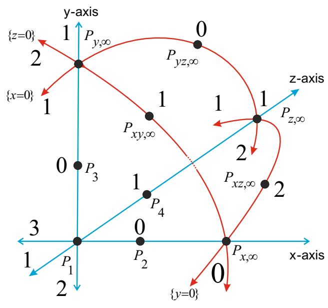

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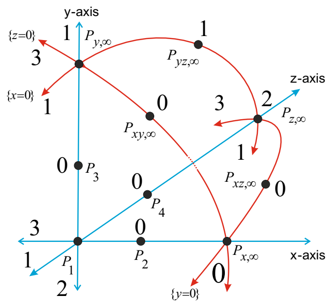

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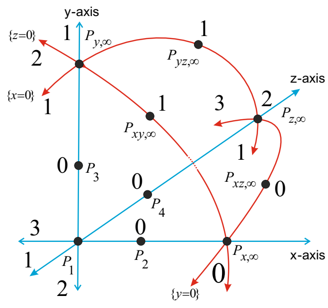

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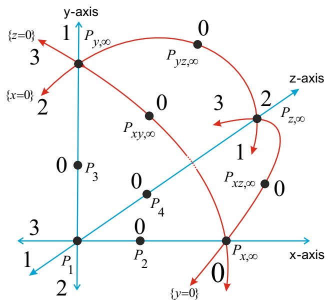

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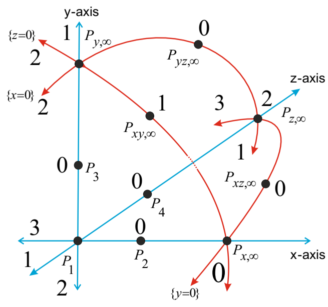

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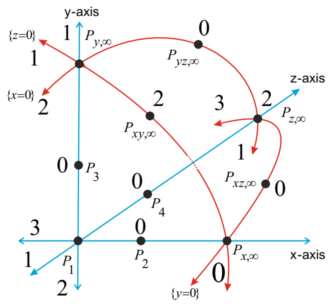

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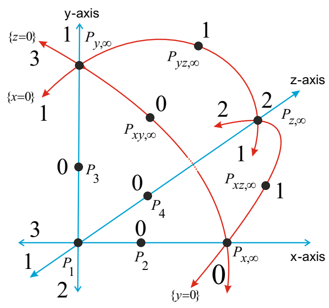

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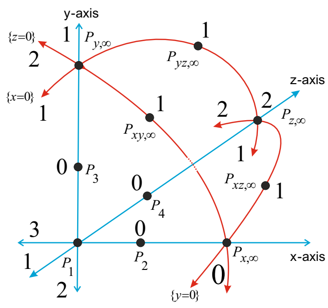

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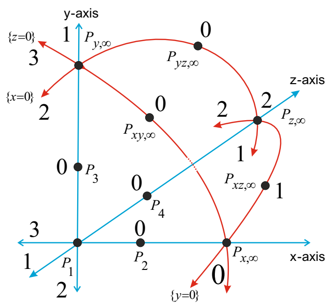

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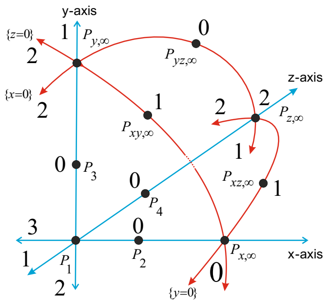


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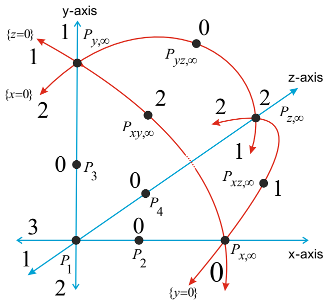


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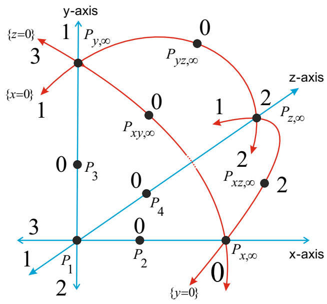


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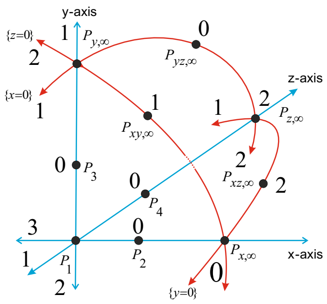


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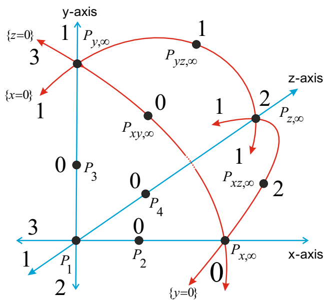


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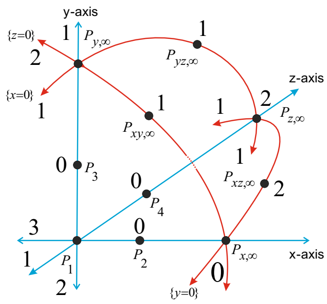


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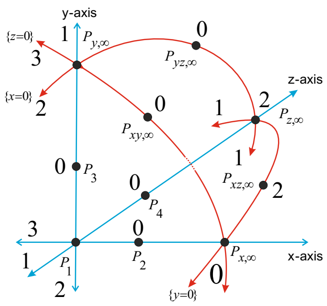


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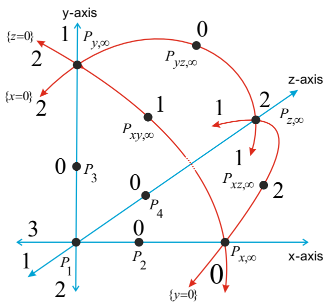


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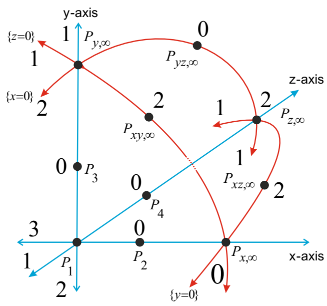


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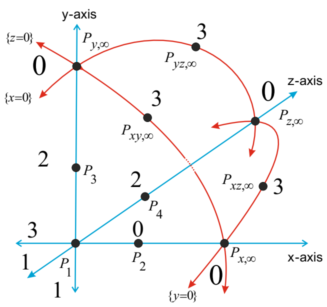


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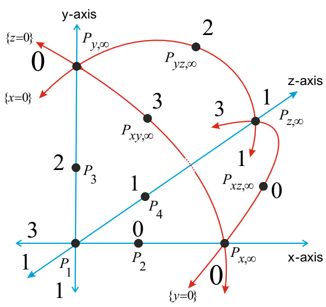


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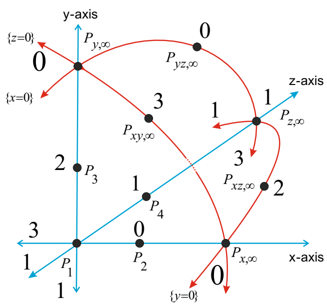


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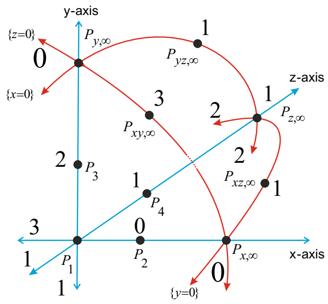


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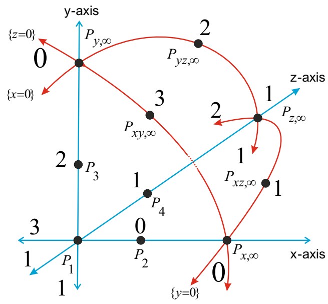


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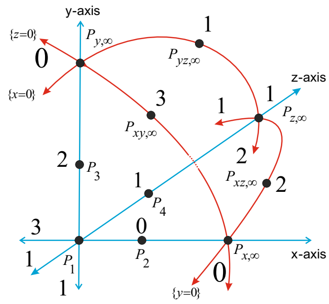


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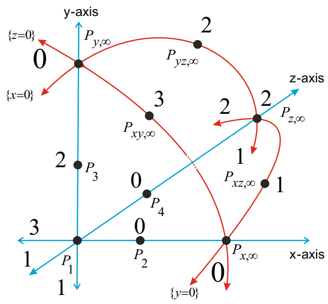


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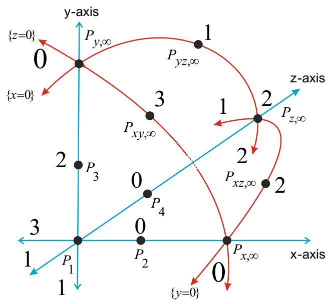


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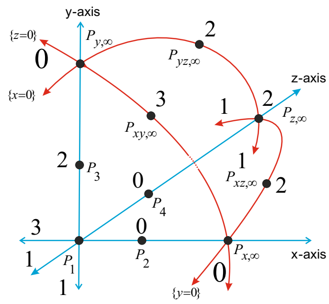


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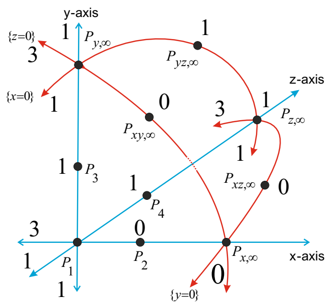


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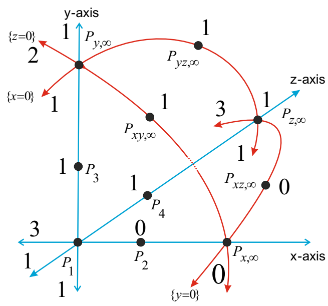


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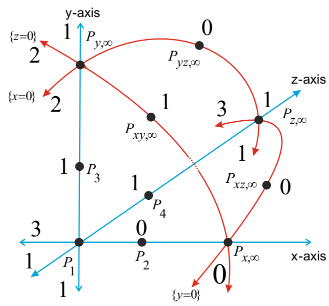


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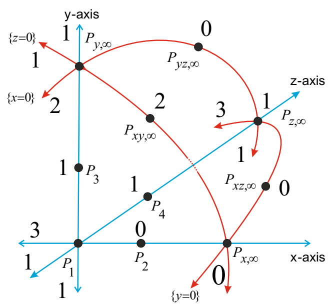


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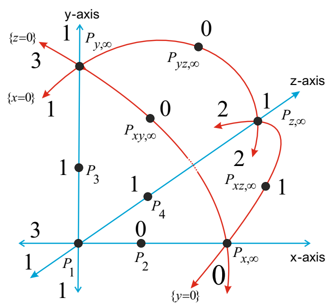


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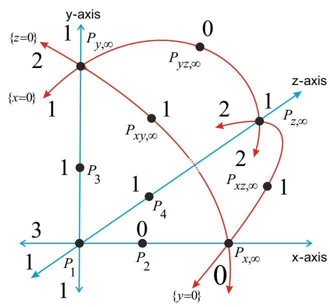


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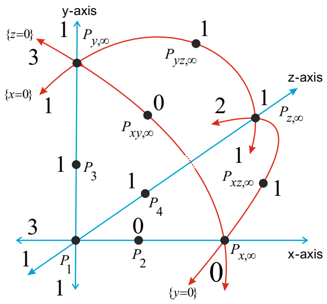


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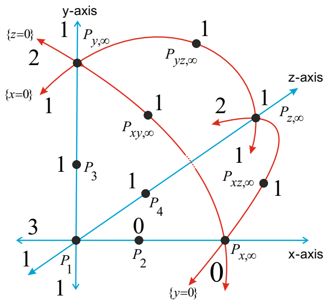


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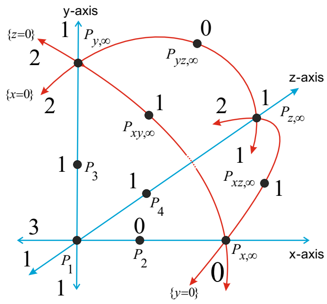


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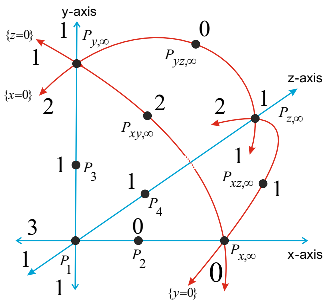


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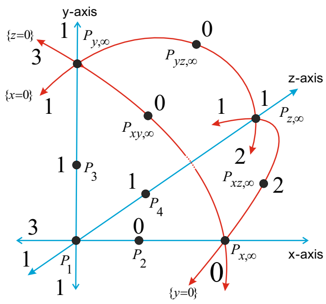


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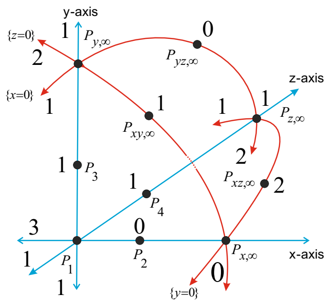


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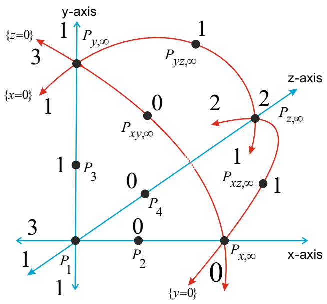


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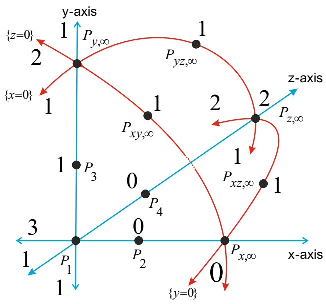


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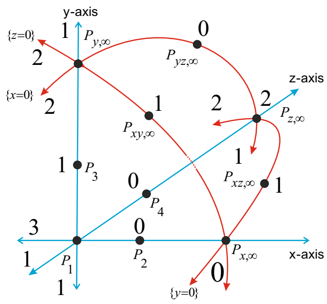


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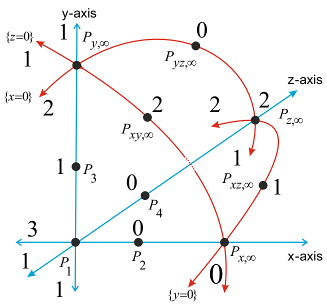


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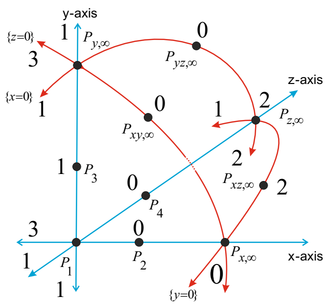


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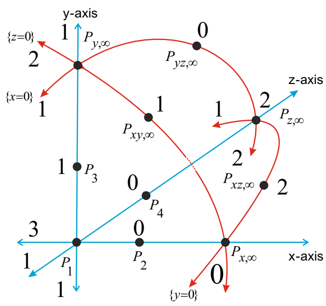


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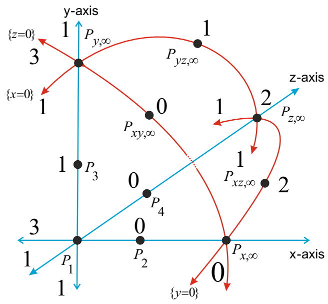


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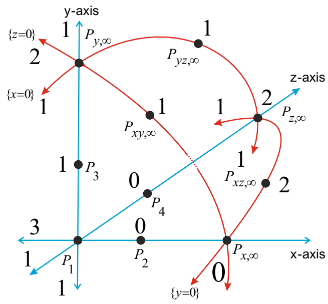


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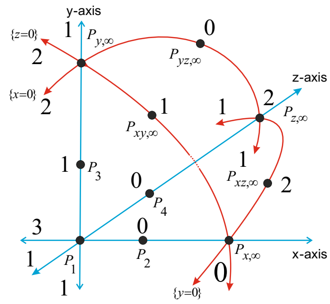


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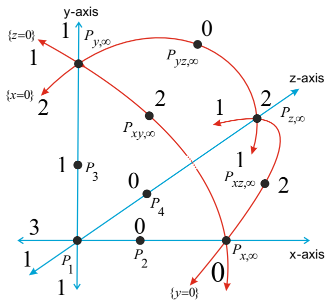


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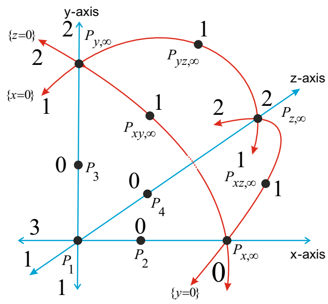


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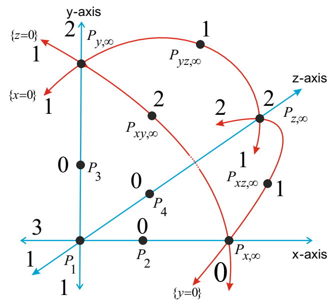


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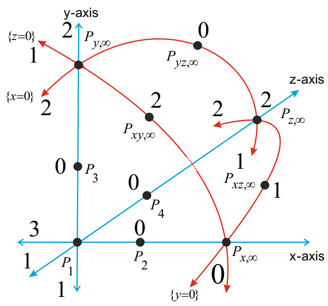


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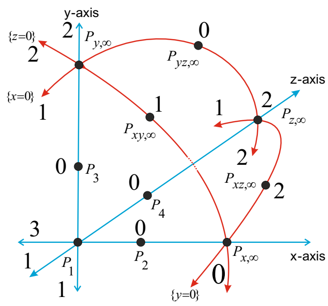


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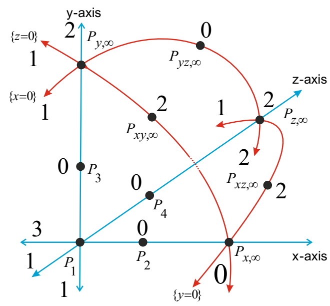


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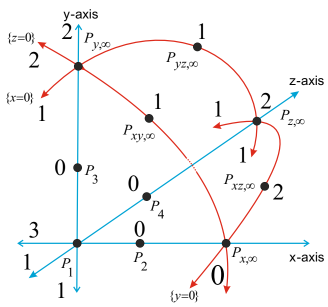


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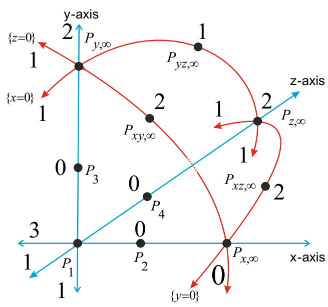


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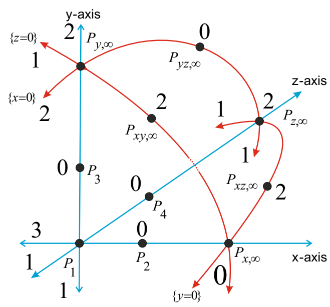


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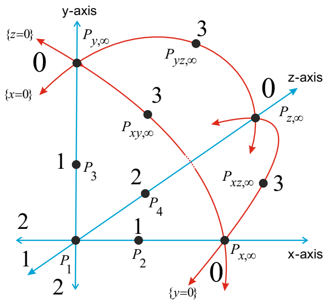
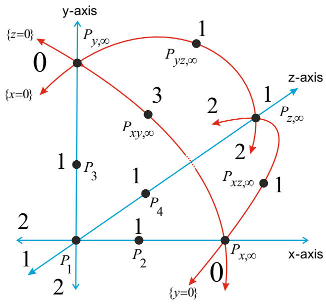



**Case 133**



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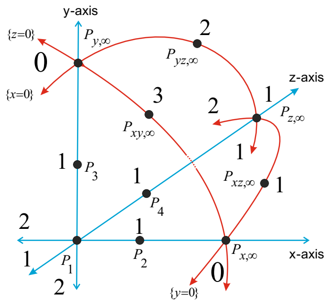

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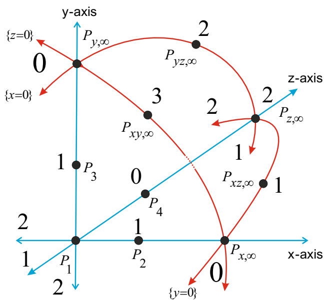

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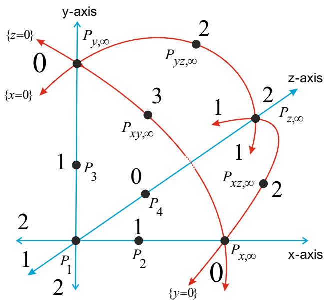

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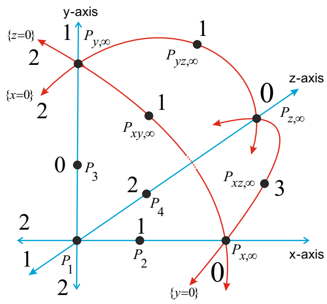

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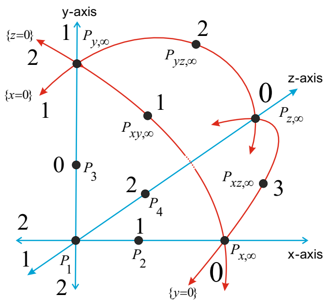

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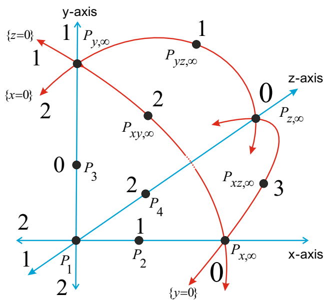

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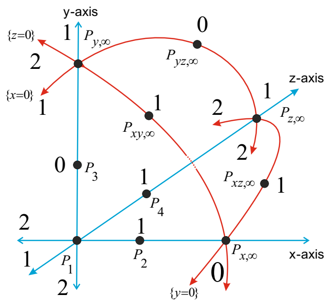

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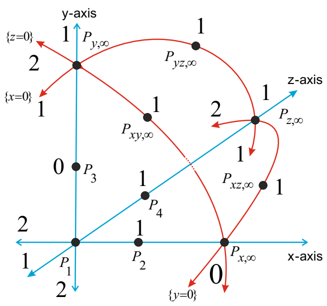

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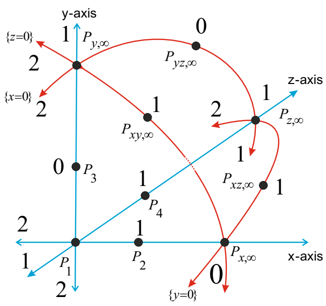

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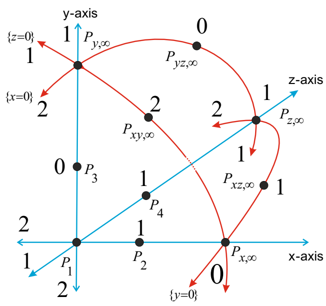

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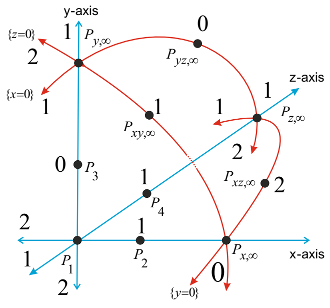


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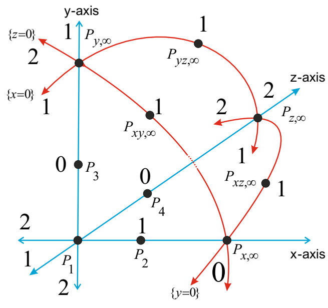

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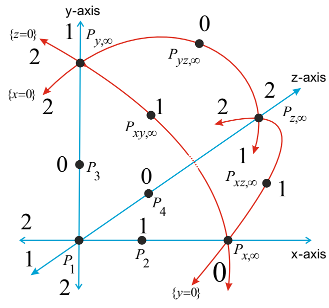

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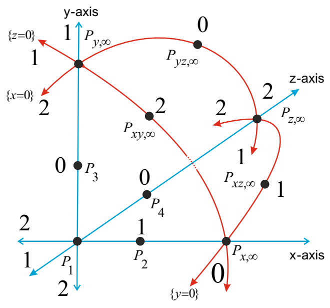

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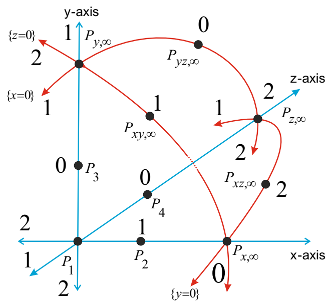

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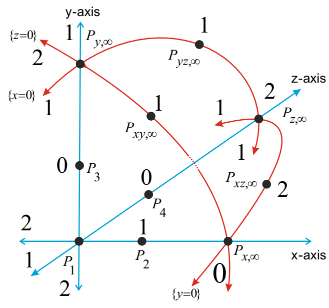

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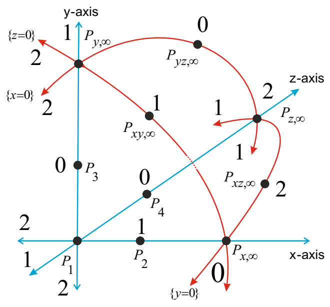

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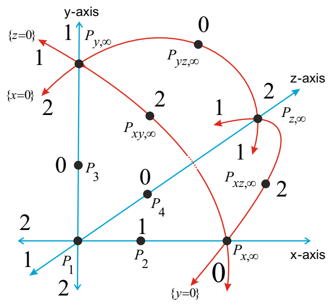

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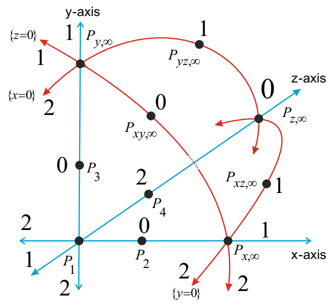

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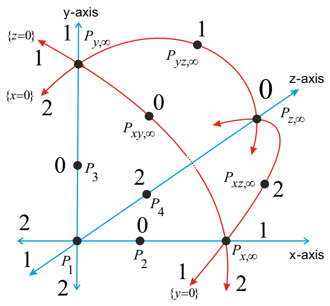

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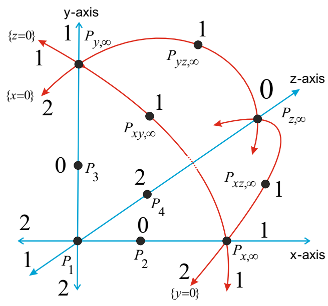

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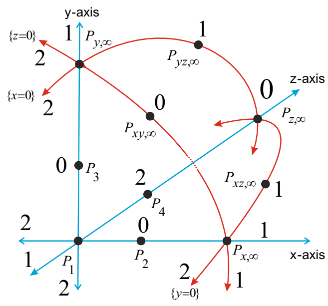

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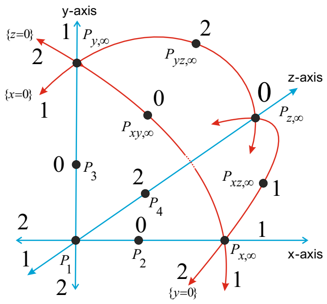

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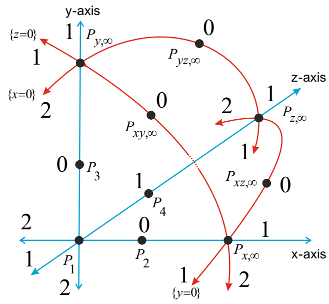

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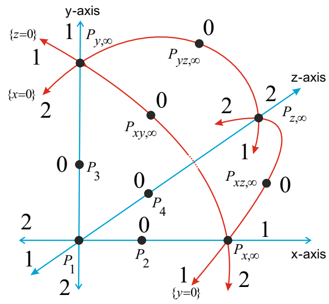

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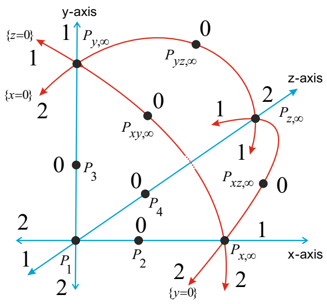

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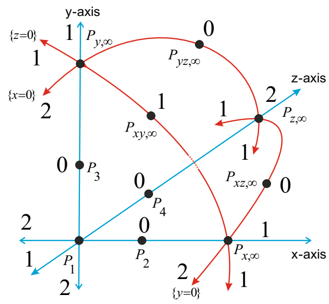

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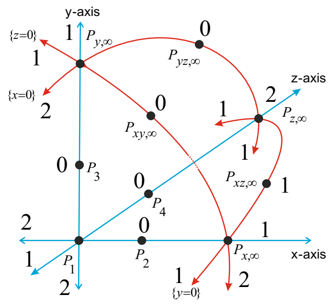

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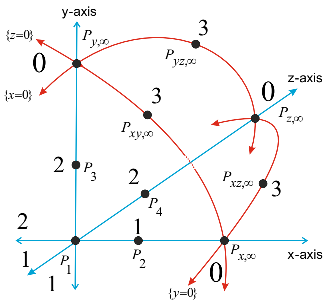

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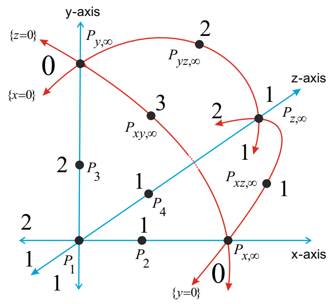

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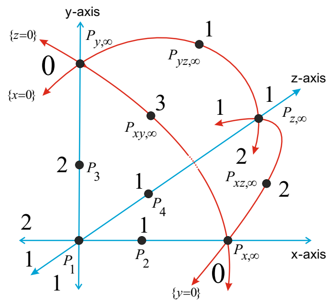

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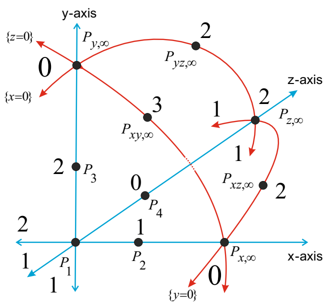

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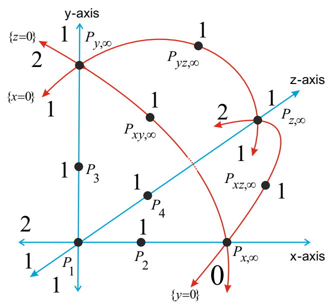

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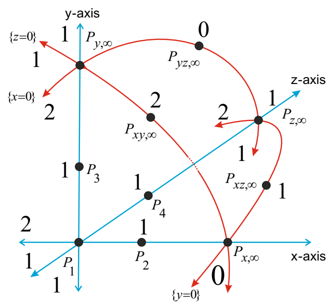

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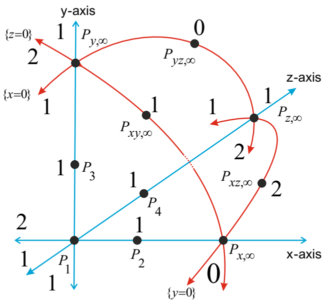

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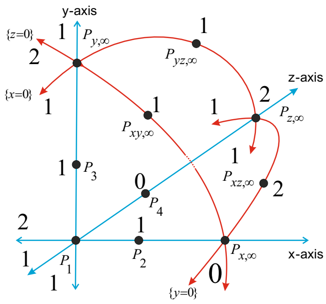

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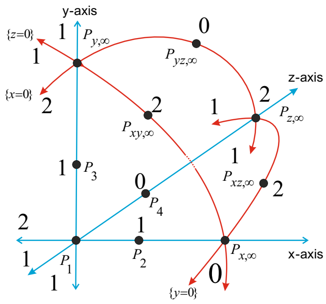

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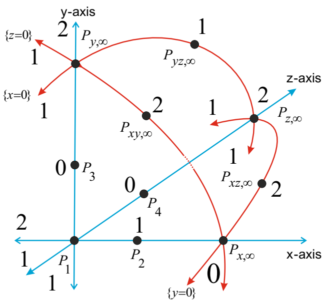

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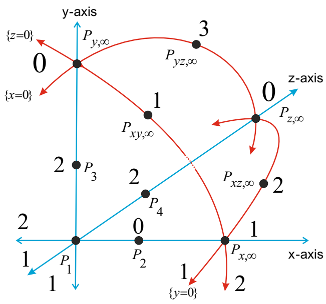

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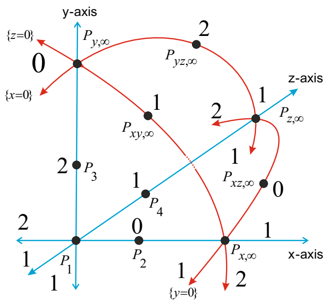

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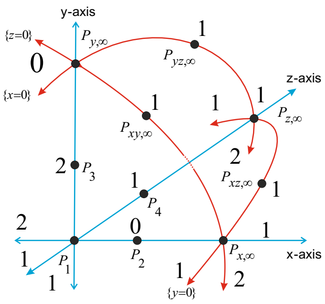

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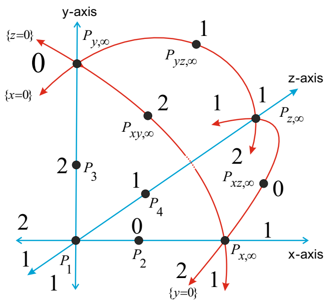

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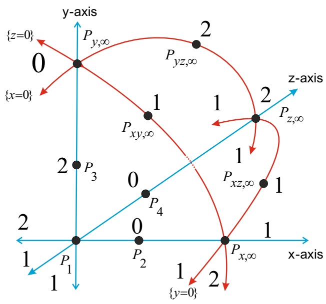

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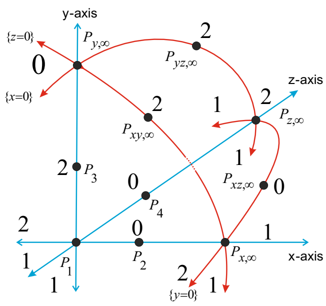

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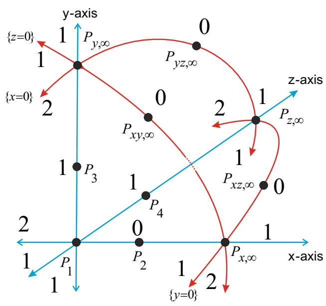

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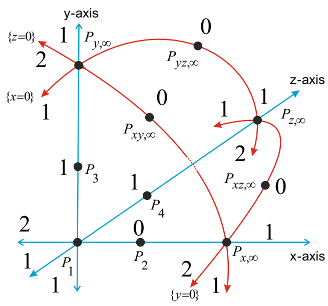

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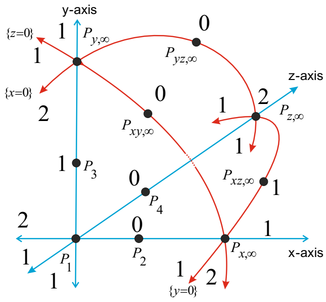


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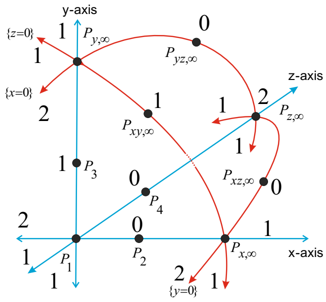


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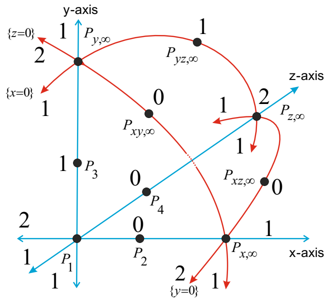


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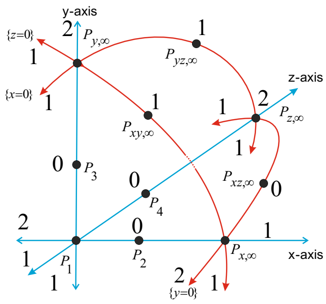


**Case 184**

**Figure 7.1:** Multiplicity possibilities of cubic surfaces.

For many of the cases above, there are no corresponding invariant cubic surfaces with these properties with respect to (6.3). For example, in the case when we have multiplicities 3 and 2 at the origin in the $xy$-plane, there is no invariant cubic curve in this plane, therefore 3:2:2 and 3:2:1 are cases that can not occur. To show why, without loss of generality, let the algebraic surface $C_f$ has multiplicity 3 and 2 with respect to the $x$-axis and the $y$-axis respectively. So the multiplicity possibilities of the corresponding curve of $C_f$ in the $xy$-plane face can be given in Figure (7.2). We will explain, why no invariant algebraic curve exists to each case in Figure (7.2).

Case 1, is not possible, since we have a cusp and therefore at most another two smooth branches at $P_5$ and $P_f$. In other words, we have at most three branches which contradicts Theorem 11, Page 48 for invariant singular cubic curves. Similarly, it is easy to verify that we get at most four branches in the other cases which again contradicts Theorem 11, Page 48.

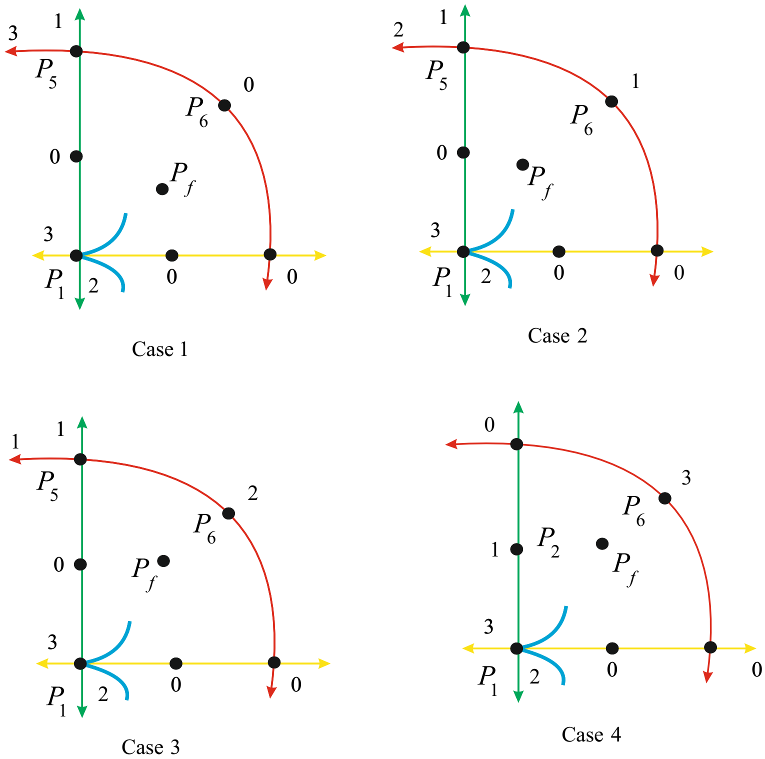


**Figure 7.2:** Multiplicity possibilities of $C_f$ in the $xy$-plane.

# Chapter 8

# Summary of Results

In this chapter, we give a summary of what we have achieved in this thesis.

**In Chapter 3:**

1. We have classified all non-projective invariant algebraic hyperplanes in three and four dimensional Lotka-Volterra systems.

2. We have given a formula between the number of non zero terms appearing in an invariant algebraic hyperplane and the codimension of a variety of the $n$-dimensional Lotka-Volterra systems with that type of hyperplane.

**In Chapter 4:**

We have investigated the behaviour of invariant algebraic curves in two dimensional Lotka-Volterra systems found by Moulin-Ollagnier (2001). In particular, we formed the following results.

- For an invariant algebraic curve $C_f$ defined by a polynomial $f(x, y) = 0$ of degree $n$ with respect to the two dimensional Lotka-Volterra systems in $\mathbf{P}^2(\mathbb{C})$, we have shown that the number of branches of $C_f$ is given by the formula: $n + 2 - 2g(C_f)$, where $g(C_f)$ is the geometric genus of $C_f$.

- Each invariant algebraic curve $C_f$ mentioned above has a zero geometric genus. Consequently each $C_f$ has exactly $n+2$ branches, for a polynomial of degree $n$.

- The type of singularities of each singular invariant algebraic curve are cusps, simple nodes and higher order singularities such as tac-nodes.

**In Chapter 5:**

We have studied the integrability and the linearisability of the critical points of the two dimensional Lotka-Volterra systems. We summarise our results below. Notice that, in each case, the quantity $-\frac{p}{q}$ is the ratio of eigenvalues at the critical point examined for integrability

1. We have extended the investigation for the origin (corner case) for $p+q \leq 20$. No new cases have been used other then the cubic and the quartic invariant curves which were known in Christopher and Rousseau (2004) but we have shown the integrability by monodromy.

2. We have investigated the integrability and the linearisability for the side critical point where $p+q \leq 17$. The monodromy argument used a conic invariant curve from Chapter 4 together with the same cubic and quartic curves mentioned in the corner case.

3. Also, we have studied the integrability conditions for the face point, where $p+q \leq 12$. The same conic invariant curve mentioned above and new cubic and quartic invariant curves other than the one's mentioned above have been applied.

**In Chapter 6:**

We have investigated invariant algebraic surfaces in three dimensional Lotka-Volterra systems. A summary of the results of this chapter are shown below

1. We have used a geometric approach technique to classify smooth and singular invariant algebraic quadratic surfaces under some assumptions.

2. We have shown that each smooth invariant quadratic surface passes through exactly ten critical points of the three dimensional Lotka-Volterra systems.

3. We have reconfirmed the results of invariant algebraic planes in three di-

mensional Lotka-Volterra systems mentioned in Chapter 3.

**In Chapter 7:**

We have applied the same technique mentioned in Chapter 6 for enumerating invariant cubic surfaces in $\mathbf{P}^3(\mathbb{C})$.

The basic results of this chapter are shown below

1. Generally, there are exactly 184 non-projectively cases as candidate for invariant cubic surfaces. In fact, the cases that succeed in being invariants divide into sub-cases such that their polynomials have the same terms with different coefficients.

2. We have shown that, in particular: the case where we have 3:2:2 or 3:2:1 as the ratios of eigenvalues at one of the corner critical point is not possible to has invariant algebraic cubic surface with respect to the three dimensional Lotka-Volterra systems.

# Chapter 9

# My Personal Contribution

The main contribution of this thesis has been explained below

1. We have found that there are exactly four non projectively equivalent classes of invariant algebraic plane in three and four dimensional Lotka-Volterra systems.

2. In $n$-dimensional Lotka-Volterra systems, for an invariant algebraic hyperplane $C_f$ defined by a polynomial $f(x_1, \cdots, x_n) = 0$, if $f$ has exactly $k$ non-zero terms, then the codimension of the variety of the system mentioned above is given by the formula

$$\frac{1}{2}\big(2n-k\big)\big(k-1\big).$$

3. We have investigated an algorithm to show the behaviour of each invariant algebraic curve in two dimensional Lotka-Volterra systems.

4. During our study of invariant algebraic curves in two dimensional Lotka-Volterra systems, we have found the number of branches for each invariant curve and the index of the cusp branches.

5. We have investigated an algorithm to classify invariant algebraic surfaces for any degree in three dimensional Lotka-Volterra systems. We have used a geometric approach in our algorithm. Also we have applied a two dimensional shape to describe the behaviour of each invariant algebraic surface in each face of $\mathbf{P}^3(\mathbb{C})$. We have applied this shape for describing the invariant algebraic planes and quadratic surfaces.

6. We have applied the above algorithm in three dimensional Lotka-Volterra

systems to classify all non-projectively equivalent cubic surfaces that are candidate to be invariant.

Finally, we present the following suggestions for future work

- For each invariant algebraic curve in Chapter 4: to consider the phase portrait of the corresponding vector field.
- To classify all invariant algebraic cubic surfaces.
- To apply the algorithms mentioned in Chapter 4 and Chapter 6 on more general vector fields.

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