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The Geometrical Thought of Isaac Newton: An Examination of the Meaning of Geometry between the 16th and 18th Centuries

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The Geometrical Thought of Isaac Newton

**An Examination of the Meaning of Geometry between the 16th and
18th Centuries**

by

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requirements for the degree of

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Abstract

Our thesis explores aspects of the geometrical work and thought of Isaac Newton in order to better understand and re-evaluate his approach to geometry, and specifically his synthetic methods and the organic description of plane curves.

In pursuing this research we study Newton's geometrical work in the context of the changing view of geometry between the late 16th and early 18th centuries, a period defined by the responses of the early modern geometers to a new Latin edition of Pappus' *Collectio*. By identifying some of the major challenges facing geometers of this period as they attempted to define and practice geometry we are able to contrast Newton's own approach to geometry.

The themes emerging from the geometrical thought of early modern geometers provide the mathematical context from which to understand, interpret and re-evaluate the approach taken by Newton. In particular we focus on Newton's profound rejection of the new algebraic Cartesian methods and geometrical philosophies, and the opportunity to focus more clearly on some of his most astonishing geometrical contributions.

Our research highlights Newton's geometrical work and examines specific examples of his synthetic methods. In particular we draw attention to the significance of Newton's organic construction and the limitations of Whiteside's observations on this subject. We propose that Newton's organic rulers were genuinely original. We disagree with Whiteside that they were inspired by van Schooten, except in the loosest sense. Further, we argue that Newton's study of singular points by their resolution was new, and that it has been misunderstood by Whiteside in his interpretation of the transformation effected by the rulers. We instead emphasise that it was the standard quadratic transformation.

Overall we wish to make better known the importance of geometry in Newton's

scientific thought, as well as highlighting the mathematical and historical importance of his organic description of curves as an example of his synthetic approach to geometry. This adds to contemporary discourse surrounding Newton's geometry, and specifically provides a foundation for further research into the implications of Newton's geometrical methods for his successors.

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Author's declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award.

Relevant scientific seminars and conferences were regularly attended at which work was often presented. One paper has been accepted for publication in a refereed journal.

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Chapter 1

Introduction

1.1 Newton's Geometry

The work of Isaac Newton has been widely studied and yet as a geometer he remains relatively unknown. For most people Newton is known for his gravitational theory. In science and academia, much attention has been paid to other areas of his work, for example, in optics and the calculus. In contrast, we seek to explore aspects of Newton's geometrical work that remain overshadowed and, we argue, undervalued.

For Newton, geometrical thought was not confined to pure geometry, it extended to other aspects of his scientific work, especially in physics, and many examples can be found in the *Principia*. That is certainly not to say that Newton did not use other methods. Far from it, he was a skilled algebraist. However, Newton's use of geometric methods was conscious and selective, especially in his published works.

In order to gain a thorough understanding of the development of Newton's geometry we look at his first readings of geometrical texts from his time at Trinity College, Cambridge. We also consider Newton's close study of the methods of the ancients. We follow the progression of Newton's growing anti-Cartesianism through examples of his contributions to geometry.

The main part of our study is set between the mid-17th and mid-18th centuries, where we focus on the geometrical work and thought of Isaac Newton, and his influence upon those who succeeded him. Newton is known to have adopted anti-Cartesian views, which he expressed through his geometrical work. Whilst these views have been fairly well documented (see, for example, [Guicciardini, 2009, part II]), they only evolved sometime

after 1670, strengthening with Newton's admiration for the ancient Greek geometers. Newton's close study of the ancient geometry, especially through his reading of Pappus' *Collectio*¹, was particularly important in reinforcing his anti-Cartesian ideals. It was Newton's defence of a more classical style of geometry against the new Cartesian geometry that led to one of his most astonishing, and yet under appreciated, geometrical discoveries: the organic description of curves.

We contrast Newton's work with that of René Descartes. The differences in geometrical thought between Descartes and Newton are perhaps nowhere more evident than in their approaches to the four-line locus problem of Pappus. Newton had used his solution to this problem specifically to attack Descartes' methods. Newton's resolution of the problem, which would eventually be published in the *Principia*, leads naturally to his remarkable organic description of curves. Newton's organic rulers allowed him to perform what are now referred to as Cremona transformations to resolve singularities of plane algebraic curves. Newton is generally thought to have been inspired by van Schooten², who also used quite subtle ruler constructions in his *De organica conicarum sectionum in plano descriptione, tractatus* (1646)³ [van Schooten, 1646]. We question this, and point to the uniqueness of Newton's rulers compared with the mechanical instruments of both van Schooten and Descartes.

Finally, we look at how Newton's anti-Cartesian and classical geometric thought influenced his publication decisions. This affected how his mathematics was received by his contemporaries and successors, and the influence he had over the following decades.

We wish to make better known Newton's geometrical thought and work, and to identify the significance of his connection with the geometry of the ancients coupled with a detachment from the foundational aspects of mathematics. In order to understand Newton's view of geometry, we first need to understand the questions and challenges facing geometers of this and the immediately preceding period.

¹Commandino's edition [Pappus of Alexandria, 1588].

²For example, by Whiteside in [MP, 1967–1981, 2, p.9, note 22], Guicciardini in [Guicciardini, 2009, p.97], and Shkolenok in [Shkolenok, 1972, pp.24–25].

³Reprinted in Book 4 of [van Schooten, 1657].

1.2 Context

In the late 16th century a renewed enthusiasm for geometry was emerging in Europe. The Renaissance had inspired an interest in many aspects of classical civilisation, including the restoration and translation into Latin of ancient texts. This, along with the invention of the printing press, meant that works including those by some of the most famous Greek geometers⁴ were much more widely accessible.

In particular, Federico Commandino's edition of Pappus' *Collectio* (1588) [Pappus of Alexandria, 1588] attracted much attention. Pappus was a fourth century commentator on Greek mathematics, and his eight volume work included extracts from and commentaries on the work of the ancient geometers and famous problems of the time. It covered a wide range of geometrical topics, from the three *classical problems* (books 3 and 4) to problems in mechanics (book 8), as well as numerical and arithmetical methods (books 1 and 2). Pappus also treated the foundations of geometrical problem solving, recounting details of the ancient methods of analysis and synthesis (book 7). Commandino's new edition of this work was to play a most significant role in the understanding of Greek geometry in the 16th and 17th centuries. The two aspects of geometrical problem solving, analysis and synthesis, became a focus for geometers during this period.

Pappus' work inspired geometers to question the foundations of geometry as well as its practice. This drove them to attempt to define what geometry was for, what it was capable of, and what were its limits. Through this period of questioning and close examination, geometry underwent dramatic changes, in particular the introduction by François Viète of algebraic means as a method of analysis, and the exploration of various construction methods. But Pappus' words were subject to interpretation, and geometers of the period were not unanimous in their approaches. The challenge was to allow the development of geometry using modern techniques, but to retain the perceived logic, certainty, and foundational strength of the classical subject. Here we also highlight the geometrical thought of Johannes Kepler. In stark contrast to the approach of Viète, Kepler retained a strict Euclidean view of geometry, deeply influenced by his religious and cosmological beliefs.

⁴For example, [Euclid, 1533], [Archimedes, 1544], [Apollonius of Perga, 1566].

In the first decades of the 17th century the French philosopher and mathematician Descartes played a major role in the algebraisation of geometry. By exploring his discourse and contemporary commentaries on his work, we will observe that Descartes' achievements were made at the cost of his rejection of the methods of the ancients. In reviewing this and similar aspects of his work we suggest that Descartes saw geometry as a subject to be utilised in the study of more practical mathematics, the “mechanical arts”. However, in order to do so, geometry needed to be “tidied” and “finished-off” so that it might be put to its proper use. We follow the development of Descartes' geometrical thought, culminating in his *Géométrie* (1637). Descartes used this opportunity to address the questions surrounding geometry of the previous generation, testing and exemplifying his methods, both analytical and constructive, through Pappus' problem.

Descartes' *Géométrie*⁵ is crucial in that it represents a significant part of the mathematical context from which Newton's work emerges. As we shall see, the young Newton, who is reported to have struggled at first with Euclid's *Elements*⁶ was receptive to the new analytical geometry of Descartes. However, Newton's view of the Cartesian methods did not remain favourable for long. In his classical approach to geometrical thinking, and his challenge to presumptions and restrictions on what geometry was, Newton was explicitly and forcefully to reject the Cartesian methods.

Undoubtedly, Newton's reluctance to publish caused him many problems later in his career. He was seen as proud and difficult, and it led to accusations of plagiarism and other disputes with fellow mathematicians. The most famous of these was the calculus priority dispute with Leibniz. These events marked the beginning of a divide between British and Continental approaches to geometry in the early 18th century. There is a view that there was a steep decline in the study of geometry in Britain after Newton. However, we will highlight examples of Newton's direct impact through the work of his contemporaries and successors.

⁵The *Géométrie*, originally written in French, was translated into Latin and edited with a great number of additions by van Schooten in 1649 [Descartes, 1649]. A second edition was published in 1659–1661, and a third in 1683. This greatly improved the circulation of the *Géométrie* and meant it was accessible to scholars such as Newton (likely through the first or second edition).

⁶See, for example, De Moivre's account in *Memorandums relating to Sr Isaac Newton given me by Mr Abraham Demoivre in Novr 1727* [Ms. 1075.7, 1r–1v, Joseph Halle Schaffner Collection, University of Chicago Library, Chicago, Illinois, USA].

1.3 Themes and questions

In our exploration of the meaning of geometry during this period we find several recurring themes, which we use to draw distinctions between our various geometers.

Geometry and Mechanics. A distinction that has been made not just throughout the period of our study, but also in ancient times, is that between *geometry* and *mechanics*, or *geometrical* and *mechanical*. We find that this distinction has been used in different ways by our various protagonists and has often been used to separate geometrical from non-geometrical.

In his discussion of Descartes' relationship with the geometry of the ancients, Molland points out the Greek distinction between geometry and mechanics, which can be seen as a distinction between geometrical and instrumental; between pure geometry and practical geometry [Molland, 1976, p.24]. We tend to think of pure geometry in this sense as relating to *ideal* objects under the influence of Platonism. This distinction, if made by the ancients, did not deter an extensive exploration into many different types of instrumental construction, particularly when the ruler and compasses were insufficient. Many of these methods would later be rejected *because* they were judged mechanical by modern standards.

As we shall see, Kepler, for example, criticised what he viewed as mechanical procedures in Pappus' *Collectio* [Kepler, 1997, p.87]. The editors of his *Hamonices mundi* indicate that Kepler adopted a Hellenistic definition of mechanical, where it was considered to be any kind of procedure of minor adjustment and readjustment, and that it had a lower intellectual status than construction by ruler and compasses [Kepler, 1997, p.87, note 259]. Kepler's criticism of the Hellenistic notion of "mechanical" is an example of a preference for strict Euclidean certainty over approximate algorithmic methods, at least where *knowledge* was concerned. Kepler himself, of course, was famous for his dogged and brilliant use of approximate algorithmic methods in his exploration of the orbit of Mars.

Descartes, on the other hand criticised what he saw as the ancient distinction between mechanical and geometrical. He argued that he did not understand why some instruments were considered to be mechanical, and therefore given a lower status, whilst ruler and

compasses which were instruments themselves were judged more favourably. Further, he claimed that mechanics required more accuracy than geometry [Descartes, 1954, pp.40–43]. It has been suggested that Descartes misunderstood this distinction and that this allowed him to give his own methods greater precedence⁷. All of this resulted in a change of meaning of the word “mechanical” for Descartes. Rather than resulting from description by motion or instrument, instead it would pertain to mechanics.

Finally we note that in his preface to the *Principia* Newton distinguishes between “rational” and “practical” mechanics, defining the relationship between rational mechanics and geometry, in particular, he claims that geometry is founded upon mechanics, and that the “description of straight lines and circles, which is the foundation of geometry, appertains to mechanics” [Newton, 1999, p.381]⁸. Newton eventually came to completely reject Descartes’ distinction between “geometrical” and “mechanical” curves. In particular, he objected to the idea that geometry could somehow be restricted to curves that could be constructed by ruler and compasses or, in Descartes’ case, generalised compasses [Domski, 2002, pp.1120–1123].

Instruments. The ancients were inspired by the classical problems to explore a wide variety of instruments and construction methods, both before and after Euclid’s *Elements*, which had an implied restriction to straightedge and compasses. As we noted in our discussion on *mechanical*, a distinction was drawn between Platonic idealised curves and those described by instrument. In the *Collectio* Pappus listed a number of different methods without judgment, focusing instead on the classification of geometrical curves.

We note that in the 16th century, Viète had shown that all problems of third and fourth degree could be reduced to the two classical problems of cube duplication and angle trisection which could be constructed by neusis (as shown by Pappus). Viète opted to accept neusis as a postulate, eliminating the need to justify particular construction methods.

As we shall see, Kepler believed that only strictly Euclidean means of construction were geometrical, but as noted above, he accepted numerical approximations in other areas of his work.

⁷See, for example, [Domski, 2002, 1117], [Israel, 1997, pp.28–29], and [Molland, 1976, p.36].

⁸See also [Guicciardini, 2004b].

In chapter 3, we see that Descartes defined geometrical curves as those curves which could be drawn by *single continuous motion*. This restricted his geometry to algebraic curves, but at the same time made the subject of geometry broader than either Viète or Kepler had allowed. Descartes' restriction to singular continuous motion implied that transcendental curves were non-geometric. Outside of the scheme of geometry, Descartes allowed various other methods of curve tracing, such as the use of strings.

Having examined the work of his predecessors we observe how Newton developed his organic rulers which, ironically, fulfilled Descartes criteria of single continuous motion. These powerful rulers allowed him to go much further in the field of curve description than anyone before him. They allowed him to think of curve description as a transformation. We will observe how this enabled him to generate a general conic, and to study and resolve singularities.

The “domain” of geometry. We notice, particularly in the earlier part of our study, that the subject of geometry has been viewed as something that can be restricted and bounded. In our next chapter we see that whilst Pappus placed a strict classification on geometric problems, this did not necessarily restrict geometry. However, it did place a perceived priority on the first two classes, plane and solid, perhaps because the third class, linear, was not well defined. This was certainly true for Viète who believed that geometry contained only the plane and solid problems. With such a firm restriction placed upon geometry, it was possible to imagine that one could “finish” geometry in the sense that all “geometrical” problems could be reduced to a standard construction. By accepting neusis as a postulate, Viète did not need to justify other means of construction, or how such constructions might be performed.

Descartes viewed geometry as subservient to practical sciences, but was very much dissatisfied with what his predecessors had achieved. He saw geometry as restricted to only algebraic curves. Whilst his geometry was not limited to problems of a particular degree, he still placed a restriction on geometry. By developing algorithmic methods, he saw that his procedures could be applied to problems of any degree. This very much

meant reducing problems to standardised constructions. The idea was that geometry would provide unchallengeable truth and knowledge that could be applied to any science.

Far apart from his predecessors Newton did not focus anywhere near as much on the foundational aspects of geometry. In that sense he was open minded about geometry, and any limitations that might be applied to it. In our study we will note that Newton frequently used projective transforms in his work with ease, although no formal subject existed yet.

Hidden analysis. Classically, the analysis phase of geometrical problem solving was something to be concealed, or hidden from view. This particular topic demonstrates the great differences between Descartes and Newton. In chapter 3 we note that Descartes saw the ancients as purposefully and deviously concealing their analyses (so that “simple” methods would not be exposed, and they would look clever).

In contrast, Newton saw an elegance and certainty in the geometrical methods of the ancients. We highlight the significance of the ancient methods of analysis, as described in Pappus’ *Collectio*, and as understood by Newton. In particular, we study the role of porisms, often dismissed as akin to locus problems, which remained a large part of the British study of geometry in the 18th century. In spite of the significance of porisms in the ancient method of analysis, this aspect of the subject has received relatively little attention.

New Geometries. It is worth noting that throughout our period of interest there was only one type of geometry under consideration. Saccheri’s 1733 work on what became non-Euclidean geometry was not well known until Beltrami drew attention to it in the mid-19th century, so the question of other geometries simply did not arise.

In the first half of 17th century Desargues had made early contributions to the theory of perspective and to what would eventually become projective geometry, it was largely ignored until the 19th century. Whilst there was no formal subject of projective geometry, Newton used projective methods with ease. For example, in his Waste Book Newton considered a problem in the work of van Schooten which he identified “may be solved more easily by supposing the Ellipsis to be a circle first & then Figure reducing it to the

desired circle”⁹. We note also that in whilst he did not have the projective plane, both Newton’s organic construction and his enumeration of cubic curves made use of projective transformations.

1.4 Literature

In chapters 2 and 3, we have made particular use of Bos’ *Redefining Geometrical Exactness* [Bos, 2001], which spans the period immediately after the publication of the Commandino’s Latin edition of the *Collectio* in 1588 up to the death of Descartes in 1650, and provides us with a good starting point to give an overview of geometry during this period. Bos focuses on geometrical procedures and the idea of geometrical acceptability. These points of interest inspired major questions for geometers during this period, as we have described above. The second part of Bos’ research concentrates on the foundational aspects of the Descartes’ *Géométrie*, a work that was the culmination of many years of development. Bos gives an excellent survey of this period, and in particular, has been able to go into much more detail about the range of approaches to geometry in this period. We instead focus on two key examples, Viète and Kepler, in order to give starkly contrasting views on the reception of the *Collectio* in this period. This is our main resource for this period, supported by modern English translations and commentaries on key texts.

It has been particularly helpful to be able to consult modern editions of books 4 [Pappus of Alexandria, 2010] and 7 [Pappus of Alexandria, 1986] of Pappus’ *Collectio* with commentary. We give additional information from these works, noting that our principal study, Newton, obtained much of his information on the ancient methods from, in particular, book 7. We are aware however that both of these works are based on original Greek sources, so they do not have the additional information and interpretation provided by Commandino [Pappus of Alexandria, 1588]. The three main texts of Viète that we discuss are available in English translation as [Viète, 2006]. We are also fortunate to have an English translation of Kepler’s main work, supported by extensive footnotes and commentary, in the form of [Kepler, 1997].

The second half of [Bos, 2001] guides us through the geometry and geometrical devel-

⁹Newton’s Waste Book, f.96v, [MS Add. 4004:50v–198v, Cambridge University Library, Cambridge].

opment of Descartes, as does [Sasaki, 2003] from Descartes' early contact with Beeckman, through his development of the *Regulae*, the *Géométre*, and beyond. Whilst [Sasaki, 2003] is the less well received of these two, its first half does provide us with a model for the development of Descartes' geometrical thought. This time we are able to go into more detail than in chapter 2. Again, we reinforce these works with key translations, for example, the Dover edition of Descartes' *Géométrie* [Descartes, 1954], which has become the standard text for study. The *Regulae* is also available in English translation in part in [Descartes, 1997] with some of Descartes' other philosophical texts, and in [Descartes, 1998]. The latter of these also contains a substantial introduction in which the editor describes historical points of interest, such as the dating of the *Regulae*, and its relationship to other Cartesian texts such as the *Discours de la méthode*.

In addition to the main Cartesian texts, there are many respectable articles worthy of note, and we have used these to present additional, and sometimes conflicting, commentary. In particular, we wish to point out Boyer's *Descartes and the Geometrization of Algebra* [Boyer, 1959]. Boyer presents a view which, at the time, was somewhat contradictory to that of the first half of the 20th century. However this view has evolved and become more accepted in recent works.

Our work could not have been undertaken without the invaluable resource of Whiteside's *Mathematical Papers of Isaac Newton* [MP, 1967–1981]. It is the study of these 8 volumes, along with Whiteside's expert commentary, that has given us a much greater insight into Newton's mathematical development over a number of decades. It is also through working in depth with Whiteside's notes that we have been able to challenge certain perceived views of Newton's work, and to propose alternatives. In addition, a vast amount of Newton's papers and correspondence is also now available digitally through the Newton Project [newtonproject.sussex.ac.uk]. Images of original papers held by Cambridge University are also now easily available through the Cambridge Digital Library [cudl.lib.cam.ac.uk/collections/newton].

The main authority on the geometrical work of Newton, and the scholar to whom we are closest, is undoubtedly Niccolò Guicciardini. This is especially reflected in his

book *Isaac Newton on mathematical certainty and method* (2009) [Guicciardini, 2009]. Guicciardini has identified a gap in the knowledge of what he terms *Newton's philosophy of mathematics* compared with, say, Descartes or Leibniz [Guicciardini, 2009, pp.xiii–xiv]. We suggest that there is a particular gap in Newton's *geometrical thought*, and that this underpins the *certainty* to which Guicciardini refers. We are interested in exploring the development of Newton's geometrical thought and how it differed dramatically from his immediate predecessors, especially Descartes.

Guicciardini's body of work extends to an impressive list of well considered papers, especially in the areas of the calculus and the *Principia*. We refer to many of these works, particularly in support of our discussion on the reception of Newton's work in the 18th century whilst maintaining an emphasis on geometry.

1.5 Structure of the thesis

Each of our four principal chapters explores one or more of the key protagonists of geometry between the late 16th and early 18th centuries. In order to examine and understand their respective approaches to geometry, we explore the geometrical work of each of our main protagonists in a historical progression, starting with the response of the early modern geometers to the 1588 publication of Pappus' *Collectio*. During this period we identify several themes and questions, outlined above, which are repeated and amplified in the work of Descartes.

We explore specific aspects and examples of their respective approaches in order to provide a context from which to observe the evolution of geometrical thought during this period, and to give context to the approach offered by Newton. In response to Descartes' prescribed approach, we see Newton contest the Cartesian methods through his own approach to geometry. Finally, having examined his challenges to Descartes, we go on to examine briefly aspects of Newton's geometrical influence after his death. In examining Newton's work in this way, and by understanding the surrounding context, we are able to explore key thematic issues concerning what it meant to do geometry during this period.

In our next chapter we identify the fundamental questions facing geometers of the 16th century, arising from their interest in classical geometry. It is the response to such questions, in particular the algebraisation of geometry, that had a profound impact on the practice of geometry and its evolution over the following century. We present two highly contrasting approaches to respond to these questions.

In chapter three we follow the route of the algebraic approach to geometry further into the 17th century. Here we focus on Descartes who is recognised to have made major contributions to this aspect of geometry. We maintain a focus on Descartes' response to the questions we have identified throughout the development of his mathematical ideas which were eventually expounded in his most influential mathematical text, the *Géométrie*. Although the *Géométrie* received at first a luke warm reception, its translation into Latin by van Schooten in 1646 meant it was much more accessible in the second half of the 17th century, and its popularity spread.

The new edition of the *Géométrie* was to have a most profound impact in the early education of Isaac Newton, the main focus for our discussion, and the subject of chapter four. We discuss Newton as a geometer from the development of his early education where he received the Cartesian ideas with enthusiasm. In the 1670s there was an abrupt change in Newton's geometrical thought. This was combined with a reverence for the geometrical practice of the ancients which Newton did not find to be compatible with the Cartesian approach. Over the decades that followed Newton's anti-Cartesian thought strengthened, and this is reflected in many of his manuscripts as well as affecting his approach to publication. We discover this through examples of his geometrical work.

In the following chapter we give a discussion of how Newton's geometrical thought affected his publication strategy, and in turn the impact of this on how his work was received by his contemporaries and immediate successors. Our discussion in this chapter highlights some of the types of geometrical research being undertaken in Britain in the first half of the 18th century. It is at this point, due in no small part to the publication of the calculus, that there is a distinctive divide in the approaches to geometry in Britain and the Continent. With a completely new perspective on geometry and the development of

new geometries on the horizon this makes for an appropriate place to end our discussion on the questions facing geometers between the 16th and 18th centuries.

Finally, we summarise our discussion, and reflect upon how not just the responses to questions of geometrical foundations, but the questions themselves have changed over this period.

It is important to state clearly how our premise is situated within wider commentaries and questions surrounding Newton's work. First, we note that we do not seek to provide a complete survey of all of Newton's geometry. We have explicitly selected a number of key examples of his work that provide a mechanism from which to understand his approach in contrast to his early modern predecessors. Similarly, we have by no means sought to provide a complete survey of all the geometers of the early modern period. Once again we have chosen eminent mathematicians from the period, whose respective work exemplifies key aspects of the changes in thought and approach to geometry that occurred as a result of the publication of a Latin edition of Pappus' *Collectio*.

Our study of these protagonists also highlights some of the issues that connect many of the great minds of the period, for example claims of plagiarism and priority disputes. However, we do not seek to comment in depth on the various claims and counter-claims that affected Newton's life. Instead we seek to understand the complex period in the history of mathematics in which he worked, and the effects this may have had on his work as a geometer. Finally, we need to make it clear that we do not seek to claim that Newton was solely a geometer, or that this was the only way that he thought. Instead we highlight key examples drawn from an overlooked aspect of Newton's work that define how he approached geometry in a way that contrasted greatly to Descartes.

In exploring Newton's work in contrast to that of Descartes, we identify the significance of his connection with the geometry of the ancients coupled with a detachment from the foundational aspects of mathematics. It was not just that Newton did geometry, he thought in a geometrical way, and it provided him with a standard of certainty that he could not

obtain from the new algebraic methods.

We propose that Newton's organic rulers were genuinely original. We disagree with Whiteside that they were inspired by van Schooten, except in the loosest sense. Further, we argue that Newton's study of singular points by their resolution was new, and that it has been misunderstood by Whiteside in his interpretation of the transformation effected by the rulers. We instead emphasise that it was the standard quadratic transformation.

Overall we wish to make better known the importance of geometry in Newton's scientific thought, as well as highlighting the mathematical and historical importance of his organic description of curves as an example of his synthetic approach to geometry.

Chapter 2

Early modern geometry: two points of view

In this chapter we will discuss the attitudes to and exploration of geometry in what is known as the *early modern* period. Broadly speaking, we will take this to mean 16th and early 17th century Europe—a time when geometry was flourishing, and mathematicians had begun a complete reappraisal of what geometry was, and just as importantly, how it should be treated. The geometry of this period is pre-Cartesian, and a clear link can be drawn between the classical geometry of ancient Greece and the geometry of this time.

The early modern period saw the restoration and translation into Latin of many ancient texts. During this time Latin was the most important language of culture in Europe, especially amongst scholars. Such texts included the advanced mathematical treatises Archimedes' *On the Sphere and Cylinder* and Apollonius' *Conics*. Whilst this meant that the classical texts were both preserved and made more accessible for close study, it also left them open to interpretation and criticism by translators and commentators. As we discuss below, such commentary was not taken lightly and had a profound effect on the interpretation of classical geometry in the early modern period. In particular, we refer to the publication of Federico Commandino's 1588 edition of Pappus's *Collectio* [Pappus of Alexandria, 1588], and how this brought a classical understanding of geometry and geometrical constructions into the early modern period.

Bos is noted for his appraisal of *geometrical exactness* during the early modern period. He comments that the revival of classical mathematical texts during the sixteenth century

prompted several writers to comment on the geometrical status of constructions, usually in connection with discussion on the three classical problems. For example, in his *De quadratura circuli libri duo* [Buteo, 1559], Johannes Buteo rejected ten different attempts to resolve the classical problem. Further, Bos states that, at least in his opinion, there are no explicit criteria for “accepting or rejecting geometrical procedures in the relevant literature before c.1590” [Bos, 2001, pp.23–24].

However, as we shall see, the discussion of acceptable geometrical procedures is a subject that pervades the exploration of geometry throughout our work. Here we introduce the three classical problems as a vehicle to analyse the exploration of this question in the early modern period.

In the sixteenth century there existed a clear interest in the three classical problems, but in particular, the construction of two mean proportionals¹. Bos gives two possible reasons for this: firstly, the availability of Eutocius’ commentary on Archimedes’ *Sphere and Cylinder*, which included 12 different constructions of the problem in [Valla, 1501] and [Werner, 1522], whereas constructions of the angle trisection only became known later. Secondly, there were various discoveries that many problems not solvable by straight lines and circles could be reduced to the problem of two mean proportionals, whereas the reduction of problems to the trisection of an angle was possible in fewer cases [Bos, 2001, p.27].

In his commentary, Pappus did not discuss the geometric acceptability of the various constructions, considering only how easy it might be physically to perform them. Bos notes that “[t]he general attitude of sixteenth-century writers to the problem seems to have been the same. Most of them stated explicitly that no truly geometrical solution had yet been found. The only explicit argument against the available constructions was that they involved the use of instruments and were therefore “mechanical” ”[Bos, 2001, p.34].

Proclus² identified, in particular, the special status given to straight lines and circles, but gave no indication that geometry should be restricted to special curves or methods

¹Thought to have been a reduction of the original problem of doubling the cube and attributed to Hipocrates of Chios [Knorr, 1986, pp.23–24].

²Proclus was a 5th century Greek philosopher and Platonist.

of construction. It was only after the 1588 publication of Commandino's translation of Pappus' *Collectio*, in which Pappus commented extensively on the appropriate construction methods, that the early modern commentators began to concentrate their efforts on demarcating the boundaries of geometry and determining the geometric acceptability of various curves and constructions. In the first half of the century Proclus' commentary on Euclid's *Elements* [Euclid, 1533] had become available in print.

The use of various construction methods and their acceptability will play a major role in our understanding of what it meant to do geometry both here, in the transition from the classical methods to the early modern period, and into the seventeenth century. We will see later that Viète favoured the neusis method of construction, and gave it the status of a postulate in order that problems of this type could be solved in a way considered geometrically acceptable. In contrast, Descartes favoured particular mechanical methods, which we discuss more in the next chapter, but in particular, he devised an instrument which could find not just one or two, but any number of mean proportionals.

In exploring these examples, we are interested in the various creative and often ingenious methods of constructing problems for which ruler and compasses were not sufficient. Whilst not a new challenge³, this in itself led to questions of acceptability in geometry during this period.

In this chapter we outline the main features of Pappus' *Collectio*, focusing on Books 4 and 7. In our discussion of Book 4 we consider Pappus' famous passage on the demarcation of "classes" in geometry, and construction within those classes. We also include some examples of ancient construction of geometrical problems, and consider how Pappus' commentary on those supported his strict ideas on the classification of problems.

A main theme for this study is the analysis of geometrical problems, that is, the methods by which solutions were found. In particular, we consider the concealment of that process in classical texts, and its subsequent discovery and interpretation between the 16th and 18th centuries. Book 7 of the *Collectio*, above any other text, provided our main protagonists

³The ancient geometers were well versed in tackling such problems. See, for example, the numerous expositions in [Thomas, 2006].

with a guide to the ancient analysis of geometrical problems. Here, Pappus collected together key ideas from Euclid, Apollonius, Aristaeus, and Eratosthenes. We discuss the content of this work as a reference point for our later chapters.

Having set out the main ideas and questions which faced the early modern geometers, we next consider two very different approaches. Firstly, we consider the development of the ‘new analysis’ by François Viète towards the end of the 16th century. Here we observe the historical significance of Viète’s *Isagoge* (1591), which laid out an entirely new method for resolving geometrical problems as well as the introduction of algebraic methods. We consider the problems faced by Viète in setting out his new method and how he addressed these. In addition, Viète presented a somewhat new concept to the geometrical community, namely the idea that geometry was somehow bounded and could be *solved* in its entirety.

In our next section we look at the strict Euclidean demarcation of geometry adopted by Johannes Kepler in the early 17th century, who addressed the question of what it meant for a geometrical object to be truly “known”. We compare Kepler’s foundational way of thinking about geometry with his use of it in his other endeavours. For example, in his calculation of orbits he was happy to accept algorithmic approximations, which stood in contrast to his strict geometrical ideals.

Finally we compare the relative successes and influences of the approaches to geometry we have discussed, as we move into the 17th century and towards our next topic: the geometrical work of Descartes.

2.1 Pappus’ *Collectio* in the 16th century

Very little is known about the life of Pappus himself. He is thought to have lived and worked in the first half of the 4th century in Alexandria, and that he was an extraordinary geometer. Pappus’ major contribution to mathematics was his *Synagoge* or *Mathematical Collection*, thought to have been written around 340. Sefrin-Weis, who bases her summary on the commentaries of [Ziegler, 1949] and [Pappus of Alexandria, 1986], adds that Pappus was born into a “pagan cultural elite” with a strong tradition in mathematics, and likely had “Neoplatonic leanings”. She comments also that Pappus lived during a time of transition in religion and culture, one which was not accepting of Platonic traditions, and she suggests

that this may help to explain why Pappus had wanted to look backward and to make the classics more accessible to future generations [Pappus of Alexandria, 2010, pp.xv–xvi].

The *Collectio* was a work of eight volumes describing various elements of mathematics and mathematical problems including areas in applied geometry such as geography and astronomy. It also included geometry and mathematical puzzles as well as an extensive body of work on the three *classical problems*⁴. Pappus included many constructions from ancient geometry adding his own commentary and insights.

By the 16th century some of Pappus' *Collectio* in the original Greek had been lost, in particular Book 1 and most of Book 2, but much of it had been translated into Latin by Federico Commandino. Commandino was an Italian medical advisor to the Duke of Urbino, and spent his life translating mathematical classics from Greek into Latin. He translated almost all of books 3 to 8 of the *Collectio*, which was to be finished and published posthumously by his student Guidobaldo de Monte in 1588.

Commandino's translations and commentaries brought previously inaccessible classical Greek works into the Latin speaking world. In particular, his translation of the *Collectio* was to have a profound effect and motivate a reappraisal of the ideas and philosophies of geometry. Commandino's edition had a significant influence on the early modern geometers. There were two reissues, in 1589 and 1602.

It is the influence of this publication on the early modern geometers in which we are interested. This renewed interrogation of geometry was something that had not happened in such a way for many hundreds of years. The early modern geometers were inspired to question their own ideas on what exactly geometry was, what it meant to do geometry, and to solve geometrical problems. In particular, it prompted the early modern mathematicians to question two aspects of geometry, firstly, the classification of geometrical problems, and secondly, the acceptability of the construction methods themselves.

⁴For a detailed description of the precise nature of each volume see [Pappus of Alexandria, 2010, pp.xvi–xxi].

2.1.1 The classification of geometrical problems

We will focus now on Pappus' classification of geometrical problems, and his description of how each type should be treated. In his discussions on the classical problems in Books 3 and 4, Pappus made strong statements on the classification of problems in order to explain the approaches of the early geometers and the reasons why they did not succeed in their construction of these problems. In Book 3 Pappus considered the problem of finding two mean proportionals. He stated that the geometers were prevented from constructing two mean proportionals in a geometrical way because it was not easy to draw conics in a plane. However, he observed that they had fared better when they devised special instruments for the task.

As there is this difference between problems, the ancient geometers did not construct the aforementioned problem of the two straight lines, which is solid by nature, following geometrical reasoning, because it was not easy to draw the sections of the cone in a plane, but by using instruments they brought it to a manual construction and fit preparation, as is seen in the *Mesolabe* of Eratosthenes, and the *Mechanics* of Philo and Hero. [Pappus of Alexandria, 1965, pp.54–57] in [Molland, 1976, p.28]

Molland suggests that this passage indicates that “Pappus regarded instrumental solutions as being something of a concession to human weakness, or at least to human practical needs. Instrumental constructions were not properly geometrical, but they could indicate how a solution was physically to be performed. The imagination of idealised instruments can give constructions as exact as those of pure geometry, but they did not fit into the canons of Greek geometry, and were strictly regarded as part of mechanics” [Molland, 1976, p.28]. This observation and distinction between geometric and mechanical methodologies is of keen interest to us here. It highlights a divergence in geometrical thinking and exploration during the early modern period, particularly when it came to distinguishing legitimate curves and means of construction in geometry. We see this distinction being made especially in the works of Descartes and Newton, which we will discuss in the next two chapters.

As we noted above, Pappus had preferred mechanical instruments, giving examples from Eratosthenes, Nicomedes, and Hero. Similarly, in Book 4 Pappus chose as his

example the problem of trisecting an angle, where we observe that he valued the use of neusis⁵. He stated that the ancient geometers had succeeded by using neusis in their solutions [Bos, 2001, p.48].

In addition to his critique of construction methods, Pappus is also noted for his strict classification of geometrical problems. In the key passage given by Pappus in Book 3, geometrical problems belonged to one of three kinds: *plane*, *solid* or *linear*. The first two definitions are relatively clear. A plane problem could be constructed by straight lines and circles alone. A solid problem would require conic sections, as generated from solid cones or cylinders, for its construction. The third definition, linear, is somewhat more complicated since it apparently must include all other constructions. In Commandino's translation these curves are described as "having an inconstant and changeable origin" [Bos, 2001, p.38]. Pappus gives the examples of spirals, the quadratrix, conchoids, and cissoids. It is not clear from Pappus' words which curves might be more or less acceptable in geometry. He does however suggest the use of intersection of curves in the solution of geometrical problems. He uses the Euclidean terms of circle and straight line, rather than ruler and compasses, and thinks of conic sections as being formed on the surface of a solid cone. It is also worth noting that Pappus may not have viewed the *linear* class as any less geometrical since he includes a full section treating it before his final section on solid problems.

It is the second passage in Book 4 which is perhaps most striking in the context of Pappus' formal classification. Having set out the classes of problems, Pappus gives stern advice that the means of construction should remain within the class of the problem.

Among geometers it is in a way considered to be a considerable sin when somebody finds a plane problem by conics or [linear]⁶ curves and when, to put it briefly, the solution of the problem is of an inappropriate kind. [Bos, 2001, p.49]⁷

⁵A description of neusis is given in Appendix B.

⁶Bos uses the word "line-like" to mean curves of degree higher than 2. We will follow [Pappus of Alexandria, 2010] and use "linear".

⁷Bos translates into English Commandino's Latin edition. Compare this with a translation directly from the Greek: "Somehow, however, an error of the following sort seems to be not a small one for geometers, namely when a plane problem is found by means of conics or of linear devices by someone, and summarily, whenever it is solved from a non-kindred kind" [Pappus of Alexandria, 2010, p.145].

The intention of this classification was much more than just a way of identifying and grouping together geometrical problems, it was an assertion that a geometrical problem should only be constructed by means appropriate to it (that is, of the same *kind*), and never by means of a more complicated nature.

Sefrin-Weis notes that Pappus' comments seem only to represent his own opinion, and not a general consensus of classical mathematics. In his work on locus problems Apollonius sometimes classified problems into two classes according to the methods needed for their construction. However, he did not so in such a rigorous and general way as to fulfil Pappus' philosophies on the subject [Pappus of Alexandria, 2010, p.145; p.145 note 6]. Jones says that "Pappus is our only explicit authority on this mathematical pigeon-holing, and he says nothing about how it developed and when. However, it is difficult not to see Apollonius' two books on *Neuses* as inspired by the constraints of method imposed on the geometer [...] The only conceivable use for such a work would be as a reference useful for identifying *plane* problems" [Pappus of Alexandria, 1986, p.530].

Here we note the significance and implications of Pappus' work in the early modern period as a primary commentary on ancient geometry. His work claims classical methods and structures that began to define and demarcate geometry. And yet, as contemporary historians have pointed out, these claims are based on the assumptions of a historian writing several hundred years after the original texts and may have no firm basis. As we shall see in this chapter and throughout, the issue of assumptions and bias drawn from interpretations of classical methods defined many of the attempts during the early modern period to determine the nature of geometry.

Bos suggests that Pappus' words could have been read by the early modern geometers in an even stricter sense, where "the only legitimately geometrical constructions were those that employed the intersection of straight lines and curves—thus excluding the use of instruments or shifting rulers" [Bos, 2001, p.50]. In the *Collectio* Pappus included few examples of linear problems, and the only curves he used in those cases were the spiral and quadratrix, although he had reservations about these since they were generated by motion.

With reference to the quadratrix, he also expressed the strong objections of Sporus⁸.

In spite of his clear directions, Pappus explored a variety of curves and a range of methods of construction in the *Collectio*. This left it open to interpretation and exploration by the early modern geometers and those who wanted to understand the methods of the ancients. The *Collectio* led to more interest in not just the classification of problems but also the curves by which they would be constructed, and further the means by which those curves would be traced.

Analysis was the means by which a geometrical problem could be understood, classified, and in turn solved. For the following generations of geometers, this was a key part of the process in understanding the ancient methods of geometrical problem solving. Pappus helped to uncover these methods in Book 7 of the *Collectio*, and this is what we discuss next.

2.1.2 Geometrical analysis

Two kinds of analysis could be identified in the early modern period: classical and algebraic. We will look at the origins of algebraic analysis in the work of Viète in the next section. First, let us see how the *Collectio* helped to bridge the gap between classical analysis and the early modern period.

Analysis was the method by which solutions to problems were found, and classically it was often concealed from the final presentation of a solution, known as the *synthesis*. Analysis could also help to understand what *kind* a problem was. According to Pappus' classification problems should not be solved using any means outside of the class of that problem. Of course in one sense this meant not using curves of lesser degree, since the problem simply will not be solvable at all. For example, in his original passage on classification in Book 4, Pappus did in fact criticise the classical mathematicians for attempting to solve the problem of trisecting an angle by circle and straight line alone.

⁸See for example in [Molland, 1976, pp.26–27]: “Pappus expounded with approval Sporus’s objections [to the quadratrix], but he himself had another difficulty with the curve, namely the extent to which its genesis was mechanical. [...] It seems clear that Pappus regarded the spiral and the cylindrical helix as having a firmer claim to the status of being geometrical than the quadratrix, which could however receive authentication by being derived from them.”

Now, since a difference of such a sort belongs to problems, the earlier geometers were not able to find the above mentioned problem on the angle, given that it is by nature solid, and they sought it by plane devices. For the conic sections were not yet common knowledge for them, and on account of this they got into difficulties. [Pappus of Alexandria, 2010, p.146]

He goes on to note that they later redeem themselves by using more appropriate methods, that is, neusis by means of conic sections⁹.

Pappus gave several methods of analysis in the early parts of Book 4, starting with the plane problems in which he first assumed the problem to be solved and then showed that the resolution is independent from the initial assumptions made. He also showed how to work with the *givens* or *data* of a problem to determine those properties necessary for construction and the criteria for solvability [Pappus of Alexandria, 2010, pp.83–103].

But how does Pappus' analysis help us to understand to which *class* a problem belongs? In the passage on appropriate use of geometry Pappus criticised Archimedes for his use of a solid neusis in *Spiral Lines* where it may have been possible to construct the particular problem by plane means [Pappus of Alexandria, 2010, pp.145–146]. According to Sefrin-Weis the comment refers to proposition 18 of *Spiral Lines* which says that *the circumference of a circle circumscribed around a spiral of first rotation is equal in length to the subtangent for a tangent in the endpoint of the first rotation*. In the proof Archimedes uses solid neuses from propositions 7 and 8 of *Spiral Lines*. Pappus shows that the neuses are solid in his propositions 42–44¹⁰ [Pappus of Alexandria, 2010, pp.163–165]. His argument is that Archimedes could have proved the proposition using a plane construction. As Sefrin-Weis points out, it is not clear whether Pappus meant by plane neusis or by other plane means of construction [Pappus of Alexandria, 2010, p.146, note 1].

Pappus had intended to show, by analysis, that the neuses are solid, and to provide a methodological approach to solving other solid problems [Pappus of Alexandria, 2010, p.163]. In propositions 42 (Figure 2.1) and 43 Pappus shows only that under certain

⁹Pappus described this construction in propositions 31–34 of Book 4 [Pappus of Alexandria, 2010, pp.146–155].

¹⁰Sefrin-Weis notes that whilst the propositions do not refer directly to propositions 7 and 8 of *Spiral Lines*, but rather to proposition 9, the argument may be applied in an analogous way [Pappus of Alexandria, 2010, p.303]. Sefrin-Weis also gives here an account of the likelihood that Pappus' propositions 42–44 are targeted at Archimedes' solid neuses in *Spiral Lines*, propositions 5–9.

conditions a certain point must lie on the hyperbola or parabola, respectively. Here he uses only the *apagoge*¹¹ phase of analysis, that is, starting with assuming the proposition to be true and then by a series of deductions reaching a situation known to be true.

Proposition 2.1.1 (*Pappus, Book 4, prop. 42*) *Let a straight line AB be given in position, and from a given point C let a certain line CD be drawn forward so as to fall onto it in D , and let DE be drawn at right angles to AB , and let the ratio of CD to DE be given. I claim that E lies on a uniquely determined hyperbola.*

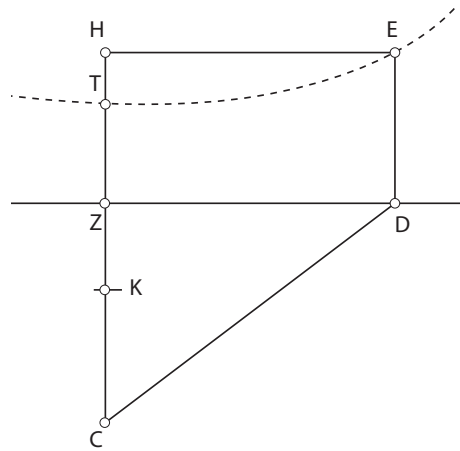


Figure 2.1: Hyperbola for the Archimedean neusis

Through C , draw the parallel CZ to the line drawn at right angles to AB . Then Z is given¹². Draw the parallel EH to AB as well, and let the ratio of CZ to both ZT and ZK be the same as the ratio of CD to DE . Then both T and K are given. Now, since the square over CZ is to the square over ZT as the square over CD is to the square over DE , the ratio of the remaining square over ZD (of the square over EH), to the remaining rectangle between KH and HT is therefore given, also. And K and T are given. Therefore, E lies on the hyperbola passing through T and E . \square

In proposition 44 Pappus' analysis leads him to the conditions of propositions 42 and 43, and thereby showing that the point lies on a hyperbola and on a parabola, and hence, in

¹¹See [Pappus of Alexandria, 2010, pp.xxii–xxiii].

¹²Since Z is the point of intersection with AB . This follows from Euclid's *Data*, propositions 25 and 28.

general, is constructible by solid loci. Pappus' analysis does lead him to conics so he takes this as having shown that the neusis is solid, but what he fails to show is that there are no cases that could be constructed by plane methods. What he does not attempt to do is the stage of analysis that allows him to find any special cases of the problem. Such conditions have been reconstructed by, for example, Tannery who, with a complete analysis and the intent of actually constructing the neusis, leads to several cases both plane and solid (and even unsolvable) [Heiberg and Zeuthen, 1912, pp.307–308]. There is also no synthesis or proof. It may be that Pappus had intended to include these, or maybe they have been lost.

What Pappus has shown is that the problem is *at most* solid. Sefrin-Weis notes that it is in fact very difficult to achieve what Pappus set out to do by a geometrical analysis since the focus is on specific problems only. Therefore, it is virtually impossible to guarantee that all the available information has been used that may “lead to a specifying condition, pushing the level of the problem down” [Pappus of Alexandria, 2010, p.304]. It should also be noticed that Pappus does not attempt to demarcate by analysis between linear and solid problems. In light of the observations made above we now move on to discuss Pappus' discourse on the domain of analysis.

The domain of analysis

Pappus reconstructed and commented on several ancient sources related to the subject of analysis, which he presented in Book 7 of the *Collectio*. This volume was particularly influential during the 17th and 18th centuries as geometers tried to gain a better understanding of the ancient methods of analysis. Of particular interest to us, of course, is its use by Newton (see chapter 4). According to Jones, “Pappus is our only substantial source of knowledge of the *Domain of Analysis*” [Pappus of Alexandria, 1986, 1, p.70]. Bos also notes that whilst the early modern interpretations of Pappus' words were many and varied, the general view was that “classical analysis was a procedure for finding constructions of problems or proofs of theorems in which the concept of “given” played a central role” [Bos, 2001, p.96].

Book 7 of Pappus' *Collectio* contains a collection of lemmas of the *Domain of Analysis*. Its preface is addressed to a pupil, Hermodorus, and explains geometrical analysis and synthesis. The domain of analysis contains several ‘books’ which give the instructions for

the analysis of theorems and problems. Pappus gives synopses of the ‘books’ and then gives selected lemmas of analysis [Pappus of Alexandria, 1986, 1, p.8]. In his commentary on Book 7, Jones discusses the transmission of the *Collectio* in the first few centuries after Pappus’ writing of it. In a 5th century commentary on Euclid’s *Data* which, according to Jones, may refer to Book 7, Marinus¹³ talks about the relationship between *data* or *givens* and the domain of analysis:

Now that the concept of *given* has been defined more broadly and with a view to immediate application, the next point would be to reveal how the application of it is useful. This is in fact one of the things that have their goal in something else; for the knowledge of it is absolutely necessary for what is called the *Domain of Analysis*. What power the *Domain of Analysis* has in the mathematical sciences and those that are closely related to it, optics and music theory, has been precisely stated elsewhere, and that analysis is the way to discover proof, and how it helps us in finding the proof of similar things, and that it is a greater thing to acquire the power of analysis than to have proofs of many particular things.[Euclid, 1896, 6, pp.252–254] in [Pappus of Alexandria, 1986, 1, p.21]

Jones describes Book 7 as a “companion” to the several treatises in the domain of analysis to which it acts as a preface or guide. Together with the various geometrical treatises the geometer is equipped with a “special resource” enabling him to solve geometrical problems by the argument of analysis [Pappus of Alexandria, 1986, 1, p.66].

Pappus draws the distinction between the analysis of theorems (where the validity of the assertion is determined) and the analysis of problems (requiring the construction of a described object from given *data*) (See [Pappus of Alexandria, 1986, 1, pp.82–84]).

In the case of the problematic kind, we assume the proposition as something we know, then, proceeding through its consequences, as if true, to something established, if the established thing is possible and obtainable, which is what mathematicians call ‘given’, the required thing will also be possible, and again the proof will be the reverse of the analysis; [...] Diorism is the preliminary distinction of when, how, and in how many ways the problem will be possible. [Pappus of Alexandria, 1986, 1, p.84]

¹³Marinus of Neapolis was a 5th century teacher and commentator of mathematics, and a pupil of Proclus.

Jones notes also the lack of “theorematical” analyses in ancient texts, which he describes as a “comparatively naïve technique”, and that it “guarantees neither correctness of the proposition nor the possibility of obtaining a valid proof by inverting the steps of the argument”. On the other hand, analyses of problems are common in Greek treatises. Jones suggests the possible reasons for this. Firstly, they formed an “expandable repertory of operations that were reversible as steps in geometrical construction”. Secondly, “an analysis could yield information about the conditions of possibility and number of solutions of a problem” [Pappus of Alexandria, 1986, 1, p.67]. It was the concealment of the analysis which proved problematic for the early modern geometers who had tried to understand how the ancients had found their solutions to geometrical problems. This was also a source of criticism for Descartes who believed that the ancients had purposefully hidden their methods [Descartes, 1997, p.15].

Bos also notes that despite many examples of solutions of solid problems by reduction to a standard problem in the early modern texts, there are few examples of formal analyses, and that “[i]t appears that the construction of solid problems by the intersection of conics, although recognized as a method of classical standing, was very little practised before Fermat and Descartes” [Bos, 2001, p.110]. As such geometrical analysis was left open to a broad and diverse range of interpretation and practice. This led most notably to a conflict between the advances of modern algebraic methods and the classical ideas of developing an argument where the concept of a “given” was central.

The first two ‘books’ in the *Domain of Analysis* were Euclid’s *Data* and *Porisms*. The *Data* contained basic definitions and theorems required for the analysis of problems. The *Porisms* elaborated on the themes of the *Data* but was more complex. Both of these works were thought to be crucial in understanding the ancient methods of analysis by Newton, who paid particular attention to the *Collectio*, Book 7. The notion of a *porism* is a complicated one and struggled to be understood well into the 18th century. We introduce the idea, as presented by Pappus in Book 7, below but it is also one that we will return to in our following chapters.

The *Domain of Analysis* also contains Apollonius' *Conics*, which is somewhat separate in character to the rest of the collected works, but builds essential theory for what is to follow. The remaining books were collections of either problems¹⁴ or locus theorems¹⁵. This would account for the interest in and emphasis which was placed upon the resolution of locus problems in the 16th to 18th centuries.

We select a particular locus problem which was given by Pappus in Book 7 and which attracted a significant amount of attention from both Descartes and Newton. We will see that the problem played an important role in the elucidation of *proper geometry* for both of them. In particular, Newton actually used his own solution of it to attack the Cartesian methods (see section 4.4).

The four-line locus problem

This problem has come to be known as Pappus' problem and is often attributed to him due to its appearance in the *Collectio*, but it is thought to have been introduced by Euclid and studied by Apollonius¹⁶. Pappus also gave a more general problem in the *Collectio*. The classic case, however, is the *four-line locus* (Figure 2.2):

Given four lines and four corresponding angles, find the locus of a point such that the angled distances d_i from the point to each line maintain the constant ratio $d_1 d_2 : d_3 d_4$.

The three-line problem occurs when two of these four given lines are coincident. In the general case of many lines, the angled distances must maintain the constant ratio $d_1 \dots d_k : d_{k+1} \dots d_{2k}$ for $2k$ lines, or $d_1 \dots d_{k+1} : \alpha d_{k+2} \dots d_{2k+1}$ for $2k+1$ lines.

In his introduction to their works Pappus gave some information about the characters of Euclid and Apollonius, and this is also where he talks about the three- and four-line locus problem (and its unsolved generalisation). He also remarked on the incompetence of

¹⁴Five books by Apollonius: *Cutting off of a ratio*, *Cutting off of an area*, *Determinate section*, *Neuses*, and *Tangencies*.

¹⁵Four locus books: Apollonius' *Plane loci*, Aristaeus' *Solid loci*, Euclid's *Loci on surfaces*, and Eratosthenes' *Loci with respect to means* in his *On means*.

¹⁶In his preface to *Conics*, Book 1, Apollonius commented that "we knew that the three-line and four-line locus had not been constructed by Euclid, but only a chance part of it and that not very happily. For it was not possible for this construction to be completed without the additional things found by us" [Apollonius of Perga, 2000, p.1].

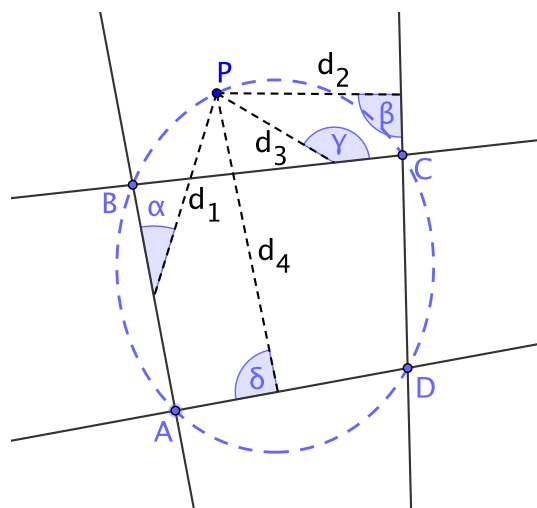


Figure 2.2: Pappus' problem

his contemporaries [Pappus of Alexandria, 1986, 1, p.70]. Pappus took his information on the problem from Apollonius' *Conics*, stating that

the synthesis of the locus on three and four lines was not made by Euclid, but merely a fragment of it, nor this felicitously. For one cannot complete the synthesis without the things mentioned above [in *Conics*].

Thus Apollonius. The locus on three and four lines that he says, in his account of the third book, was not completed by Euclid, neither he nor anyone else would have been capable of; no, he could not have added the slightest thing to what was written by Euclid, using only the conics that had been proved up to Euclid's time, as he himself confesses when he says that it is impossible to complete it without what he was forced to write first. [Pappus of Alexandria, 1986, 1, p.118]

Pappus gives the problem of three lines stating that the locus will be solid, and similarly for four lines. He then goes on to say that it has been proved that for two lines the locus will be plane, but for more than four lines "the point will touch loci that are as yet unknown, but just called *curves*, and whose origins and properties are not yet known" [Pappus of Alexandria, 1986, 1, p.120]. Pappus notes that for five or even six lines, where the ratio is defined between two solid parallelepipeds, the locus will be some *curve*. However, if more than six lines are given he says that "one can no longer say *the ratio is given of the something contained by four to that by the rest*, since there is nothing contained by more than three dimensions". Pappus is critical of his "immediate predecessors" who

have “allowed themselves to admit meaning to such things” [Pappus of Alexandria, 1986, 1, p.122]¹⁷.

As we mentioned above, the problem became a focus of study in the early modern period¹⁸ (probably inspired by Pappus), and especially for Descartes and Newton whose solutions we discuss in the following chapters (see sections 3.2.2 and 4.4). For now, we will introduce the idea of porisms, which may have been used in the solution of such locus problems. Here we give the ideas as presented by Pappus in Book 7 of the *Collectio*, which was the main source on the subject of porisms in the 17th and 18th centuries.

Porisms

In Book 7 of the *Collectio* the only source of information on the ancient analytical technique is Euclid’s *Porisms*. Pappus introduces Euclid’s three books of *Porisms* as “for many people a very clever collection for the analysis of more weighty problems”. He describes the nature of porisms:

All of them are in form neither theorems nor problems, but of a type occupying a sort of mean between them, so that their propositions can assume the form of theorems or problems, and it is for this reason that among the many geometers some have assumed them to be of the class of theorems, others of problems, looking only at the form of the proposition. [Pappus of Alexandria, 1986, 1, p.94]

Pappus remarks that the ancients had best defined the distinction between theorems, problems and porisms. He says

[A] theorem is what is offered for proof of what is offered, a problem what is proposed for construction of what is offered, a porism what is offered for the finding of what is offered. [Pappus of Alexandria, 1986, 1, p.94]

He is critical of his contemporaries for not fully understanding the depth of porisms and thereby altering the definition to “what is short of a hypothesis of being a theorem or

¹⁷Pappus is clearly perturbed by this. He says further: “They who look at these things are hardly exalted, as were the ancients and all who wrote the finer things. When I see everyone occupied with the rudiments of mathematics and of the material for inquiries that nature sets before us, I am ashamed; I for one have proved things that are much more valuable and offer much application” [Pappus of Alexandria, 1986, 1, p.122].

¹⁸See also [Whiteside, 1961, sections 6 and 7].

locus”. We will later see that in the 17th century struggle to understand porisms, and even sometimes in the modern literature, porisms had come to be thought of as akin to locus problems. It seems that the meaning, whilst difficult to ascertain completely, is much more nuanced than this, but that there is a close relationship with loci. Pappus continues

The form of this class of porisms is the loci, and these abound in the *Domain of Analysis*. This kind, separated from the porisms, has been accumulated and named and handed down because of its being more diffusible than the other forms. [Pappus of Alexandria, 1986, 1, p.96]

He also goes on to say that porisms have “terse propositions because of their complexity, and many things are customarily left to be understood, with the result that many of the geometers comprehend them in part, but are ignorant of the more essential of things signified”. He tells us that Euclid himself had given few examples, just one or two for each ‘kind’, except in the case of “that more abundant kind of loci” [Pappus of Alexandria, 1986, 1, p.96]. Pappus is referring here to the first ten propositions of the first of Euclid’s books, which he then seeks to summarise in one proposition followed by a summary of the three books [Pappus of Alexandria, 1986, 1, pp.98–104].

We give now two examples. The first is the so-called *hyptios porism*, which is Pappus’ supposed general statement for Euclid’s first ten porisms [Pappus of Alexandria, 1986, 1, p.98]. In his commentary on the work, Jones suggests that “hyptios” and “paryptios” are conventional terms for a quadrilateral with, respectively, no parallel sides or two parallel sides. He notes also the rediscovery of the porism by Simson in the 18th century, which was anticipated by Newton [Pappus of Alexandria, 1986, 2, p.549; p.549 note 3]. We discuss the work on porisms carried out by Newton later (see section 4.3.3).

Example If in a *hyptios* or *paryptios* three points on one line, or both the points on a parallel line are given, while each of the rest except one touches a line given in position, then that one too will touch a line given in position.

It is helpful to restate the porism in modern terminology. We here follow Jones’ description for convenience and clarity [Pappus of Alexandria, 1986, 2, p.549] (Figure 2.3).

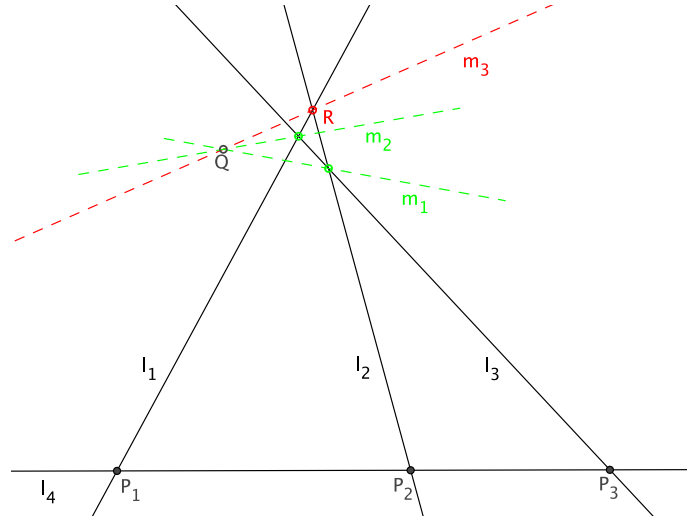


Figure 2.3: Hyptios porism

If the intersections P_1, P_2, P_3 of three variable straight lines l_1, l_2, l_3 with a straight line l_4 are given, while the intersection of l_2 and l_3 lies on a given straight line m_1 and the intersection of l_1 and l_3 lies on a given straight line m_2 , then it is possible to construct a straight line m_3 on which the intersection of l_1 and l_2 lies.

The point of the porism is to find a geometrical object (in this case the line m_3) which, although not given from the outset, is implied as given. That is, the straight lines m_1 and m_2 are given in position and we may therefore also take the constructible straight line m_3 as given. Since m_1 and m_2 intersect in a point, say Q , and l_1 and l_2 intersect in a point, say R , then by Euclid's first postulate we may *draw a straight line from any point (Q) to any point (R)*. To consider this another way, we have four lines intersecting in six points. As the lines vary about their intersections, we wish to know what the locus of point R will be. It is not clear how Pappus knew that the locus would be straight line, but he did also identify that the same would be true for any number of lines, no more than two of which intersecting in the same point [Pappus of Alexandria, 1986, 1, p.98].

Jones discusses the projective character of the porism, where m_1 and m_2 are parallel, and several sub cases in ancient works [Pappus of Alexandria, 1986, 2, pp.556–560], but we reserve such comments for the work carried out by Newton.

Our second example is Euclid's *first porism* [Pappus of Alexandria, 1986, 1, p.100].

Example If lines from two given points inflect on a line given in position, and one line cuts off an abscissa from a line given in position up to a point given on it, the other line too will cut it off from another line given in position an abscissa having a given ratio to the first.

Again, we give Jones' restatement in modern terminology for clarity [Pappus of Alexandria, 1986, 2, p.549] (Figure 2.4).

Given two straight lines h and e , point H on h , and two points A, B on neither line, and a ratio $\alpha : \beta$, it is possible to construct a line m and a point M on m such that if variable lines a through A , and b through B intersect each other in a point E that lies on e , and the intersection of a with h is G , and the intersection of b with m is N , then the ratio of intervals $HG : MN$ equals $\alpha : \beta$.

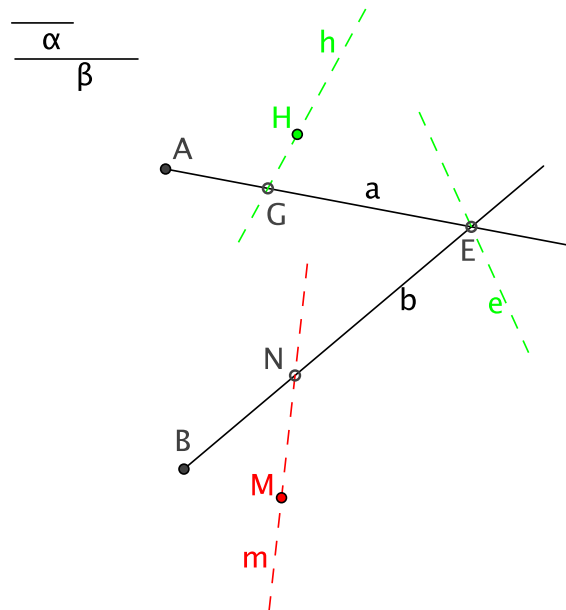


Figure 2.4: First porism

Much as with the hyptios porism above, this one gives a certain configuration from which it is possible to construct other geometrical objects which may then be taken as given. The porism may be generalised such that the lines A and B are constrained to intersect on a conic instead of another straight line, and this case would be helpful in the resolution of the four-line locus problem. Jones suggests an analysis of the four-line locus problem

due to Aristaeus making use of the porism in this way [Pappus of Alexandria, 1986, 2, pp.587–591]¹⁹.

Here we note the particular importance of porisms in Newton’s 1690s reconstructions of classical analysis based on his readings of the *Collectio*, which we discuss in chapter 4.

We move next to our first of two main early modern protagonists in this chapter. Whilst Viète was certainly not the only early modern mathematician to have been inspired by the new edition of Pappus’ *Collectio* we focus on his role in the introduction of algebraic methods in geometric analysis. Examples of work in this area and at this time, especially in the analysis of solid problems, can be found in [Bos, 2001, pp.95–117]. However, Viète was more focused on the *foundations* of geometrical analysis.

2.2 Viète and the *new analysis*

One of the biggest influences in the use of algebra in analysis was Viète. As Bos notes “[f]rom 1591 onward Viète consciously and explicitly advocated the use of algebra as an alternative method of analysis, applicable in geometry as well as in arithmetic” [Bos, 2001, p.97]. Viète was one of the first mathematicians to undertake this new expression of geometrical methods by algebraic means. Furthermore, Viète, in his *In artem analyticam isagoge* (1591), had expressed the bold view that every problem in geometry could be solved in this manner.

François Viète was born in Fontenay-le-Comte, France in 1540. He was educated locally before moving to Poitiers to attend the university there, where he graduated with a law degree in 1560. Viète remained in the legal profession for just four years before becoming the private teacher of the daughter of Antoinette d’Aubeterre. In 1566 Viète moved with the family to La Rochelle, before leaving for Paris in 1570. During this time, Viète had worked independently on mathematics and astronomy with his first publication *Universalium inspectionum ad canonem mathematicum liber singularis*, a work on trigonometry using decimal notation, appearing in 1571.

¹⁹For an example of how the porism fits into an analysis of a geometrical problem see [Pappus of Alexandria, 1986, 2, pp.587–591].

In 1573 Charles IX appointed Viète to a position in government in Rennes. During the religious unrest in France over the next two decades, Viète was banished for five years to Beauvoir-sur-Mer where he was able to devote himself to his work in mathematics. In 1589 he was able to return to a position in parliament under Henry IV, for whom he decoded messages being sent to the King's enemy Philip II of Spain.

In addition to his private work on mathematics, Viète also gave public lectures at Tours on topics such as the recent work on the three classical problems. He had shown that claims that the problems could be constructed using just ruler and compasses were false [Busard, 1981, 14, pp.23–24].

Viète completed many works on algebra, but the most important was his *In artem analyticam isagoge* (1591), in which he promoted the use of algebra as the correct method of analysis in problems of both geometry and arithmetic. We discuss this work in more detail next, and look at the development of his new method of analysis.

2.2.1 *In artem analyticam isagoge* (1591)

The *Isagoge* is one of the earliest works on symbolic algebra. In it Viète employed letters for both known (represented by consonants) and unknown (represented by vowels) quantities. Viète had certainly been inspired by Diophantus' *Arithmetica*, and possibly also Book 7 of Pappus' *Collectio*. Whilst there are explicit references to Diophantus' *Arithmetica* by Viète [Viète, 2006, p.19; p.27], the supposed influence of Pappus is less clear. On the ancient definitions of *analysis*, which Viète gives in the first paragraph of his work, he cites Plato and Theon [Viète, 2006, p.11]. Heath points out that the terms were interpolated in Book 13 of Euclid's *Elements* before Theon's time, and that they have also been attributed to Theaetetus, Eudoxus, and Heron [Euclid, 1956, 3, p.442]. Pappus also defines the same terms, which Heath translated in [Heath, 1921, p.102]. Viète states that the ancients had two kinds of analysis: *zetetics* and *poristics* [Viète, 2006, p.11]. The terms ($\xi\eta\tau\eta\tau\iota\kappa\omicron\nu$ and $\pi\omicron\rho\iota\sigma\tau\iota\kappa\omicron\nu$) are borrowed from Pappus [Pappus of Alexandria, 1986, p.83], but the meanings he assigns to them are different [Viète, 2006, p.11, note 2; note 3]. We discuss Viète's use of the terms, as well as his third added term, *exegetics*, below.

Viète used the strength of the Pappusian analysis, the combining of *data* or *givens*, with the procedural methods of Diophantus' *Arithmetica*. Viète's new analysis was the method by which a sought magnitude could be produced from an equation, which he would show how to form.

In his *Introduction to the analytic art* of 1591 Viète set out to introduce a complete analytical process in order to solve all problems, not just geometrical but also arithmetical, by way of an algebraic manipulation. His method comprised three parts which took the original problem through an algebraic form to a solution. He labelled each of these three parts with the existing Greek terms: *zetetics*, *poristics* and *exegetics* (for geometrical magnitudes) or *rhetics* (for numerical magnitudes). Viète then explained that it was by *zetetics* that one sets up an equation or proportion between the known and unknown terms, *poristics* determines the truth of a theorem by way of the equation or proportion, and by *exegetics* the value of the unknown term is identified, in what he called the "science of correct discovery in mathematics" [Viète, 2006, pp.11–12].

The first stage, *zetetics*, translates a problem, regardless of its type, into algebraic equations. Viète discusses first the "Rules of Zetetics", explaining how a problem is to be turned into an equation including labelling unknowns and comparing terms. Understanding the meaning of the next phase, *poristics*, is more difficult and, as Bos points out "[t]he relevant sections in the *Isagoge* admit various translation and interpretations" [Bos, 2001, p.147]. He notes the various translations given by Witmer of Viète's definition of *poristics*, which "differ considerably" [Bos, 2001, p.147, note 8]. If we are to take *poristics* to mean the porismatic phase of analysis, which it seems reasonable to do given the nature of its exposition, it is not surprising that it was open to a variety of different interpretations. In the 16th and 17th centuries, it was generally agreed that porisms were vital to understanding the ancient analysis, but there was no real clarity as to what this meant. We will encounter this again in our following chapters as we explore the complexity of porisms, and analysis more generally, as understood by Descartes, Newton, and others. The final phase, *exegetics*, was the derivation of a solution from the equations and information obtained in the previous stages.

What sets Viète's attempt to introduce an algebraic method of analysis apart from others is that the algebra, which he used purely as a tool, was not thought of in terms of numbers. Viète spoke of magnitudes "in species", and of his new algebra "regarding species", deriving the term "specious logistics" [Viète, 2006, p.13; p.17]. It was about abstract magnitudes, which he denoted with letters of the alphabet, set apart from the problem in hand. Of course, this abstraction would introduce a new problem that had to be dealt with: how to contend with the manipulation of magnitudes with no context. For example, in geometrical terms one way to consider a length multiplied by another length is to produce a rectangular area, but why not produce another length? Viète required a definitive reinterpretation of the algebraic operations. Bos comments that in making his decisions, Viète chose to be inspired by geometry [Bos, 2001, p.148]. That is, he had chosen to maintain a change of dimension as in geometric operations. However, rather than be limited by the dimensions of space Viète conceived of a scale of higher dimensional magnitudes, for which he used the term "grade" or "degree". To this Viète added his fundamental rule: the "law of homogeneity", which said that only magnitudes of the same degree could be compared, added, or subtracted, and that multiplication would provide the link between magnitudes of varying degree [Viète, 2006, p.15].

As well as his crucial law of homogeneity Viète formulated the operations of addition, subtraction, multiplication, division, root extraction, and the formation of ratios. Note that Viète did not require a unit element with respect to multiplication as Descartes later would (see section 3.3.1). Viète also took the classical approach of understanding ratios as relations between magnitudes rather than as forming a new magnitude. He used geometric terms to describe dimensions such as "planum", "solidum" [Viète, 2006, pp.16–17], and in doing so his equations resembled full sentences rather than the contracted symbolic equations that Descartes would later use.

Viète ended his work with the statement [Viète, 2006, p.32]:

Finally the analytic art, endowed with its three forms of zetetics, poristics and exegetics, claims for itself the greatest problem of all, which is

To solve every problem.

From this very ambitious claim Viète proceeds to tell us how he will achieve this. His method is to introduce *neusis* as a postulate so that it may be taken without proof. He also says that it may be used to solve many of the problems we have already encountered which are not solvable by ruler and compasses, as we will discuss below.

In a second treatise, *Effectuum Geometricarum Canonica Recensio* (1592), Viète treated the exegetics of geometrical problems which could be resolved by Euclidean means, that is, by circles and straight lines. This meant the geometrical construction of square roots of quadratic equations. Viète's execution of these geometrical procedures was not new or novel but, as Bos comments, it added to the completeness of Viète's programme [Bos, 2001, p.152].

***Supplementum Geometriae* (1593)**

A year later, in the *Supplementum Geometriae*, Viète explored the exegetics of geometrical problems whose analysis led to third- and fourth-degree equations. He proved that all such problems could be reduced to the classical problems of angle trisection or the finding of two mean proportionals, which he chose to resolve by *neusis*. At the time, construction of these problems by *neusis* was already known from Pappus [Pappus of Alexandria, 2010, pp.243–245].

In the work Viète ventured into the construction methods beyond ruler and compasses. For example, he used *neusis* to construct a regular heptagon [Viète, 2006, p.413] and explored curves such as the *conchoid* [Viète, 2006, p.388]. But most significantly of all was his proof that all problems whose resolution resulted in an equation of at most degree 4 (the *solid* class of Pappus' classification) could be reduced to the construction of either the angle trisection or the finding of two mean proportionals. This crucial result showed just how important the *neusis* construction could be to non-plane geometry since Pappus had already demonstrated in the *Collectio* how both of these problems could be constructed by *neusis*, and is the justification for Viète's opening sentence in the *Supplementum*.

He says [Viète, 2006, p.388]:

In order to make up for a deficiency in geometry, let it be agreed that

*One can draw a straight line from any point to any two given straight lines,
the intercept between these being any possible predefined distance.*

In his earlier work the *Isagoge* (1591) Viète stated that there is “a deficiency of geometry in the case of cubic and biquadratic equations” [Viète, 2006, p.32] showing that he had already considered how he might deal with problems that could not be resolved by straight lines and circles. In fact, at this time he makes a very similar statement concerning neusis.

In order to supply quasi-geometrically a deficiency of geometry in the case of cubic and biquadratic equations, [the analytic art] assumes that

[It is possible] to draw, from any given point, a straight line intercepting any two given straight lines, the segment included between the two straight lines being prescribed beforehand. [Viète, 2006, p.32]

Viète’s solution to this defect in geometry was to introduce a new postulate in addition to Euclid’s original five. What he suggested was that the ancient neusis construction be allowed as an axiom without proof. If accepted, the postulate allowed problems resulting in third- and fourth-degree equations by the preceding zetetics to be resolved in a way that could be considered geometric.

Although the idea of accepting a new postulate seemed progressive Viète claimed that it was actually what the ancient geometers would have done. He had also previously stated in the *Isagoge* that neusis was the *preferred* method of construction, rather than say direct construction by conic sections. In a few further comments he said that Nicomedes’ conchoids were likely to have been devised in order to perform the neusis construction and that Archimedes would have accepted the postulate without question [Viète, 2006, pp.388–389]²⁰.

Viète published a second mathematical work that same year addressing various methods of construction. In his *Book VIII of various replies on mathematical matters* (1593)

²⁰See also [Bos, 2001, p.169].

[Viète, 1646], Viète commented on the suitability of methods such as the use of the quadratrix and the spiral, stating that shifting rulers were *mechanical, but not geometrical* [Viète, 1646, p.359] in [Bos, 2001, p.176].

On the use of spirals for problems of squaring the circle Viète made this interesting remark:

Although the spirals are not described in the way of true knowledge, and neither are their tangents found in that way, still we can reason truly about questions of how large the angles are in the case of tangents, how large the lines are that are subtended by these angles, and thus art helps mechanics and mechanics helps art. This I wanted to show in this chapter, as well as a good method to square the circle as near to the true value as one wishes; it is a not too difficult method and I don't think that a more general and artful method can be proposed.[Viète, 1646, p.393] in [Bos, 2001, p.177]

Once again we see a distinction being drawn between the mechanical and geometrical. Viète seems to be making a compromise between what can realistically, or physically, be constructed and an ideal geometrical solution. In this sense he retains the classical character of wanting to produce something which is exact and ideal, but acknowledges that where this is not possible (or at least was not known at that time) we may still gain knowledge from an *accurate* description. As we will see in the next section, Kepler, whilst taking a completely different approach to geometry, would reach similar conclusions on the acquisition of knowledge about certain geometrical objects which could not be *geometrically* constructed.

Not only did Viète's treatises aim to teach the new analysis, but he had also set down a firm means of resolution of solid problems. However, due to Viète's insistence on the use of language rather than arithmetic symbols, of which he used very few, his algebra lacked the clarity of his successors. It is possible that this restricted Viète in using his new methods to their full potential. For example, he did not use it for the independent study of curves as Fermat and Descartes would later do, choosing instead to focus on curves and construction methods separately from equations.

2.3 Kepler

We now introduce Kepler, whose work provides a counterpoint to the new analytical methods of Viète. Kepler had similarly been influenced by Pappus, but took an entirely different view of what it meant to do geometry. The differences between Kepler and Viète exemplify the diverse spectrum of approaches that emerged through the exploration of geometry in the early modern period.

Whilst Kepler is primarily remembered for his work in planetary motion, he also contributed to optics, the understanding of logarithms, and geometrical topics including regular polyhedra and solids of revolution. Like his contemporaries, Kepler had strong opinions on geometrical practice. He was fiercely critical of certain developments in constructive methods and had a particular distaste for the recent use of algebra as a geometric method of analysis. We present Kepler here as a sharply contrasting figure in comparison with Viète. Although he rarely targeted Viète by name²¹, Kepler was clear in his criticisms of the new algebraic methods as a means of analysis for geometric problems.

Johannes Kepler was born in Weil der Stadt, Württemberg, Holy Roman Empire (now Germany) in 1576. He was the son of a soldier who died at war when Kepler was young. He grew up with his mother in his grandfather's inn, and was educated locally. From there, Kepler went to the University of Tübingen with the initial intention of being ordained.

At the University, Kepler attended courses on arithmetic, geometry, astronomy and music as was usual in the 16th century. The astronomy Kepler was taught was geocentric—where it was still believed that the planets revolved around the Earth in uniform circular orbits. However, in his earliest published work, *Mysterium cosmographicum* (1596), Kepler proposed to consider the actual paths of the planets²². Kepler also proposed that the distances between the planets could be understood in terms of the relationships between the five Platonic solids enclosed in a sphere, and that geometry reflected God's plan of

²¹Kepler made just one reference to Viète in the *Harmonices Mundi* criticising him, among others, of using methods of curve construction that were *ungeometric* [Kepler, 1997, p.87].

²²A second edition of the work was published in 1621. Kepler had changed his mind about a number of ideas in the first edition and added many notes to the second [Field, 1988, p.34].

the universe [Field, 1988, p.78]. Kepler was persuaded by his astronomy teacher, Michael Mästlin, to give up on ordination and instead take up a teaching position in mathematics in Graz, Austria.

Kepler sent a copy of the *Mysterium cosmographicum* to one of the foremost observational astronomers of the time, Tycho Brahe in Prague, who at the time was in need of an assistant. He was offered, and accepted, the post. When Tycho died in 1601 Kepler succeeded him as Imperial Mathematician. The next few years were extremely productive for Kepler, and he was able to publish many more works on his astronomical findings. However, in 1611 Kepler's oldest son died, followed by his wife. On the abdication of the current Emperor Rudolf, who was succeeded by his brother Matthias, Protestants were no longer accepted in Prague, and so he was forced to move with his remaining children to Linz, Austria.

Kepler was finally able to work on his *Harmonices mundi* (1619), a work which developed his earlier *Mysterium cosmographicum*. Both of these works contain his famous polyhedral model. We will focus on the geometrical content of the *Harmonices mundi* below.

Kepler remained a deeply religious man throughout his life. All his writings refer to God, and he saw it as his duty to understand the universe that God had created. In particular, he held the Platonic belief that the universe had been made according to a mathematical plan, and that mathematics was the method by which truths could be found [Gingerich, 1981, 7, pp.289–312].

2.3.1 *Harmonices Mundi* (1619)

Kepler's *Harmonices mundi* [Kepler, 1619] developed his earlier work on cosmology, the *Mysterium cosmographicum*. Kepler maintained strong opinions on the correct geometrical practices, especially on methods of construction and the use of algebra, and he expounded these forcefully in his writings.

His mathematical thought was closely related to his religious and philosophical beliefs, unified by the principle of *harmony*, and expressed through geometry. The elements of

harmony were the five Platonic solids and the regular polygons that make up their faces. To Kepler the ratios of the sides of the polygons to the diameters of their circumscribed circles were also of utmost importance, as we explain below.

Kepler took a strict Platonic approach to his mathematical thought. His polyhedral model for the radii of the orbits of the planets, for example, is based on Plato's "formal cause" structure of the universe, which describes the form taken by matter into a recognisable object. In addition, an "efficient cause" describes the need to find the underlying reason for the structure. For Kepler, this was God who "accomplished his work according to the model of the five regular polyhedra". Even God and creation could be represented geometrically. The Trinity, for example, "taking the centre for the Father, the spherical surface for the Son, and the intermediate space, which is mathematically expressed in the regularity of the relationship between the point and the surface, for the Holy Spirit" [Di Liscia, 2011].

Kepler explained the concept of *harmonious* and *knowable* as follows. Given a circle, if a regular polygon may be inscribed in that circle by strictly Euclidean means, then the ratio between the side of the polygon and the diameter of the circle was said to be *harmonious* and *knowable*.

Definition VII In geometrical matters, to know is to measure by a known measure, which known measure in our present concern, the inscription of Figures in a circle, is the diameter of the circle. [Kepler, 1997, p.18]

If the polygon is not constructible by Euclidean means (for example, the regular heptagon), then the ratio was said to be *unharmonious* and *not knowable*. Kepler saw harmonious ratios occurring in nature by God's will, therefore unharmonious ratios (and construction not by circles and straight lines) were not acceptable.

Kepler argued that the original intentions of Euclid, and as explained by Proclus, had been misunderstood by Peter Ramus in the 16th century. In particular, Kepler said that the tenth book of *Elements*, on commensurable and incommensurable magnitudes (a principle crucial to *harmony* and *knowability*), was "condemned to the atrocious sentence of not being read, though if it were read and understood it could lay bare the secrets of philosophy" [Kepler, 1997, p.10].

Kepler also defined a relationship between the concepts of *knowable* and *constructible*, which he gave in Book 1 of *Harmonices Mundi*. Bos gives this summary:

A magnitude was *measurable* if its ratio to the basic measure was rational. Lines that could be measured by a basic measure (in the case of regular polygons, the diameter of the given circle) and areas that could be measured by the square of the basic measure were *knowable*. [Bos, 2001, p.184]

On the other hand, the ratio of a knowable magnitude to its basic measure could be irrational provided such a magnitude could be constructed by Euclidean means. Kepler turned to Euclid's *Elements*, Book 10 for guidance in this respect. This ruled out for Kepler three of the main ways of constructing curves: intersection of solids, tracing by motion, and pointwise construction, since none of these means could produce *knowable figures*.

In the *Harmonices Mundi* Kepler criticised Pappus' classification of geometrical problems since he believed the only truly *geometrical* class was the plane class [Kepler, 1997, p.86]. Kepler gave a critical analysis of methods of angle trisection. In particular, he looked at a construction given by Pappus in the *Collectio* using neusis by the intersection of conic sections which he referred to as "mechanical"²³.

He noted that Pappus had labelled the problem *solid* because he required the use of a cone (to construct a hyperbola), but that in actual fact the hyperbola is constructed by a pointwise method, and it may as well therefore be classified as linear.

Pappus makes this problem a Solid one because he used a Cone [...] the problem seems equally to be classifiable as Linear. For such a line is generated by Geometrical motion, and a continuous change in distances, that is, it is represented by a collection of points, of indeterminate number; and this is no less true [of this curve] than of the Quadratrix and the Spiral, the lines which he [Pappus] uses in Proposition 35 to carry out the Trisection and General division [of an angle]. This is Pappus' mechanical procedure. [Kepler, 1997, pp.86–87]

This reaffirmed for Kepler the futility of Pappus' classification and the *ungeometric* status of any curves other than the circle and straight line. Here we once again see a discussion of a strict definition of geometry based on a reading of Pappus' classical text.

²³See [Kepler, 1997, p.87, note 259] on Kepler's use of the word "mechanical".

However, due in no small part to Kepler's religious context, he produced very contrasting results to Viète.

As well as the regular polygons which could be obtained by plane methods, the *knowable figures*, Kepler also confronted the status of the *unknowable* figures at the end of *Harmonices mundi*, Book 1, the regular heptagon most notably of all [Kepler, 1997, pp.60–79]. Whilst the heptagon could not be constructed by circles and straight lines, Kepler did not question the possibility of its existence.

Kepler sought to construct a side of the regular heptagon by reducing it to a cubic equation. This was the approach taken by both Kepler and Cardano. However, this did not solve the problem for Kepler because he had no construction for the roots of this equation²⁴. For Cardano, having the equation was the same as having the solution. Kepler was critical of Cardano's method, stating that "Cardano [...] boasted, falsely, that he had found the side of the heptagon" [Kepler, 1997, p.62]. In this earlier period (before Viète's *new analysis*) algebra was almost always a practical matter, used by instrument makers. There was no expectation of the Platonic certainty sought by Kepler: the constructible numbers could be specified precisely in a finite way, while the roots of a cubic could not be²⁵ [Field, 1994, p.230]. However, it is not clear how Kepler knew that cube roots were not constructible.

Kepler was critical of the new algebraic methods, which he saw as undermining the classical authority of Euclid and Proclus. He referred to the *analytic art* as "cossic" methods [Kepler, 1997, p.66]. As noted by Charrak "cossics, for Kepler, demonstrate that numbers lose their basis in being when their use is not strictly subordinate to the condition of measurement" [Charrak, 2004, p.367].

Kepler did however embrace these methods for their application to numerical problems, such as finding approximations and for calculating trigonometric tables. Whilst he praised the use of equations in this respect Kepler referred to them as "semimechanical" [Kepler, 1997, p.84]. In a letter to Vincenzo Bianchi (17 February, 1619) he wrote "don't

²⁴ Although he was aware of methods for obtaining approximations to the roots. In fact he was a master of approximate calculations, as his work on Mars demonstrates [Kepler, 1609].

²⁵ This is exactly analogous to Fowler's argument that Euclidean-constructible numbers are those having eventually periodic anthyphairctic sequences, or in other words those having a finite description [Fowler, 1999, p.30].

sentence me completely to the treadmill of mathematical calculations. Leave me time for philosophical speculations, my sole delight!” [Gingerich, 1993, p.396].

We introduced Kepler in order to provide a counterpoint to the new analytical methods of Viète. We saw that he was critical of Pappus’ classification, in particular, believing that the only truly geometrical class was the plane class. This immediately restricted his view of geometrical constructions to those that could be effected by ruler and compasses. In doing so he limited what it meant to do geometry, more so than Viète.

In the *Harmonices Mundi* Kepler did not criticise Viète’s new analysis explicitly. In fact, he made just one reference to Viète, criticising his use of ungeometric methods of constructing curves [Kepler, 1997, p.87]. Whereas in his objections to constructive methods beyond straight lines and circles he cited the authority of Euclid and Proclus, he was more definite about his reasons for objecting to the use of algebraic methods in geometry. Kepler gave an extensive argument on the inability to obtain knowledge from non-plane problems by either geometric or algebraic methods, referring to equations for the side of the regular heptagon given by the Swiss mathematician Jost Bürgi. Bos tells us that he “gave five separate arguments in support of his view that algebra, in particular equations, did not provide the means to solve non-plane problems in geometry” [Bos, 2001, p.190].

2.4 Into the 17th century

With a renewed focus on uncovering the analysis of the ancients and on developing an effective new method, Viète’s *new analysis* received much attention. In the next chapter we will look at the methods developed by Descartes to tackle these questions. As we shall discuss, whilst claiming to have no knowledge of Viète’s work, Descartes went on to develop his own algebraic methods and to find different ways to resolve some of the difficulties that both he and Viète faced.

Viète had given a complete method in his new analysis, if a little clumsy due to the need for better developed symbolic algebra. This may partly explain why he and his first followers did not utilise it in the independent study of curves, as Fermat and Descartes would some forty years later. Descartes had been inspired to do so through his study

of locus problems and a need to classify loci by equations. Fermat, on the other hand, considered the construction of solid problems by the intersection of conics. Bos proposes that “[h]ad Viète opted, in the classical manner, for construction by intersection of curves, he might well have been led to the relation between curves and equations. Thus we may consider Viète’s adoption of the neusis as postulate to supplement geometry and his interest in special problems and standard forms of equations in one unknown, as reasons for a delay in the development of analytic geometry ”[Bos, 2001, p.154].

Kepler’s strategy was far from successful. In the 17th century there remained a strong desire to tackle higher-order problems and to consider variations on geometrical construction. Bos suggests that

Kepler’s arguments invite a comparison with Viète’s ideas on construction and the use of algebra. Indeed Viète had dealt with the same issues and arrived at a contrary conclusion, namely, that algebra was an appropriate means for use in geometry. [Bos, 2001, pp.192–193]

As we noted, Kepler made very little in the way of direct reference to Viète (only one passing comment in the *Harmonicis mundi*), and in that case with reference to constructive methods rather than algebraic analysis. It is not certain how much he knew of Viète’s work in that area, but we can suppose that he would not have taken kindly to these new methods. Bos comments further that

The basis for Kepler’s restrictive interpretation of geometrical exactness was philosophical; he needed the classification of ratios that resulted from the strict adherence to Euclidean constructions; any extension would explode his theory of harmony.[Bos, 2001, p.194]

Other approaches

An intermediary method of algebraic analysis was developed after Viète but before Descartes by van Ceulen. Van Ceulen was a German mathematician born in 1540. He had a limited education due to the poor fortunes of his family, and was not able to attend university. He had not learned Latin or Greek which made for a difficult mathematical education in the 16th century, and he had to rely on friends to make translations of the

most important works. In spite of the difficult start, van Ceulen held many posts teaching mathematics, and spent most of his life fascinated with the Archimedean approximation of π as well as working on his own approximations [Struik, 1981, 3, p.181].

Van Ceulen's algebraic approach to geometry was published in a work whose title translates as *Arithmetical and geometrical elements, with their uses in solving various geometrical problems, partly by the tracing of lines only, partly by irrational numbers, sine tables and algebra* [van Ceulen, 1615]²⁶, which was edited and translated by Snellius after van Ceulen's death.

The work explored the correspondence between the arithmetic of irrational numbers of the form $a + \sqrt{b}$ and the geometry of line segments [van Ceulen, 1615, p.115]. His method was to introduce a unit length in order to compare geometric lengths with numbers. However, he was still faced with the same problem as his predecessor Viète, of how to translate algebraic operations into meaningful geometric ones. He was able to show that any length $a + \sqrt{b}$ could be constructed by straight lines and circles. In contrast to Viète before him, the use of a unit segment was crucial to van Ceulen's method of multiplication. Later, Descartes also made use of a unit line segment allowing him to avoid the need to introduce many abstract dimensions as in Viète's method [Bos, 2001, p.155].

Van Ceulen used his method on various problems including those resulting in quadratic irrationals from geometrical configurations. Snellius gave that portion of van Ceulen's work the title *De dedomenoon geometricorum per numeros solutione*²⁷ to which Kepler had objected strongly. Van Ceulen went so far as to work out geometrical constructions corresponding to quadratic algebraic equations, but did not go on to equations of higher degree, and so was not required to construct with means beyond straight lines and circles [Bos, 2001, p.157].

The examples introduced in this chapter provide a clear foundation for thinking about the questions and challenges that faced geometers of the early modern period. As we have seen, much of the exploration that took place was a reflection upon Pappus' commentary of the ancient methods of analysis and synthesis.

²⁶The translation is from [Bos, 2001, p.155].

²⁷*On the numerical solution of geometrical data* [van Ceulen, 1615, pp.137–183]

By redefining algebraic operations in order to apply them to geometrical processes, Viète's method had retained a connection to the classical approach. There was also a considerable effort made towards finding a geometrically acceptable means of construction, although not everyone agreed on what those means should be. In contrast, Kepler sought to retain a much closer attachment to a Platonic view of geometry. However, in spite of their differences Kepler and Viète both demonstrated a desire to formally encompass geometry.

Even with the introduction of an algebraic method there was still a firm emphasis on construction and on looking back to reflect on what the ancients had done. However, now there was potentially a new method for the classification of geometric problems. Whilst considerable developments had been made the questions and preoccupations of the early modern period persisted, but the focus and resolutions would evolve over the next century. As we shall see in the following chapters, such attempts to define geometry in this way were taken even further by Descartes, before being radically challenged by Newton.

Chapter 3

The Geometry of René Descartes

As we discovered in the last chapter mathematicians of the early modern period relied heavily on the new translations of Pappus' *Collectio* as their guide to ancient geometry and, especially, to rediscovering the art of geometrical analysis. Descartes was no exception to this and, just as those before him, he believed the ancients had cleverly concealed their methods of discovery. For example, in his *Regulae ad directionem ingenii* (1628), Descartes commented

Indeed I seem to recognise certain traces of this true mathematics in Pappus and Diophantus, who though not belonging to the earliest age, yet lived many centuries before our own times. But my opinion is that these writers then with a sort of low cunning, deplorable indeed, suppressed this knowledge. [Descartes, 1997, p.15]¹

He goes on to speculate on the reason for this concealment, suggesting that the ancients may have feared a loss of admiration for their work if the ease and simplicity of their methods were to be discovered. We will continue to discuss Descartes' relationship with the ancient methods below, in particular, his interpretation of the 'correct' resolution of geometrical problems.

In the last chapter we discussed the *new analysis* of Viète, identifying two key points. Firstly, the introduction of algebraic symbolism and methods into the analysis of geometric problems, and secondly Viète's view that, with the aid of his new analysis, every problem in geometry could be solved. We discussed the focus on the classification of geometric problems, and the idea that geometry could be somehow bounded with the inclusion of

¹For an alternative translation see [Lenoir, 1979, p.365].

certain objects and methods, whilst others were to be excluded. In this chapter we continue to observe and comment upon the development of such ideas by Descartes in the 17th century in order that we might understand better what the practice of geometry meant at this time.

3.0.1 Early life of Descartes

Descartes was born in La Haye, France in 1596. His family were considered as *noblesse de robe*². Descartes was educated at a Jesuit school, the *Collège Henri IV*, in La Flèche. The school was opened in 1604 during the Catholic Reformation. There was a particular emphasis on mathematics during this time, and Descartes is likely to have studied classical Euclidean geometry as well as the works of the 16th century mathematician Christopher Clavius. Descartes later spoke highly of the education he received there. In the *Discourse de la méthode* (1637) he said of La Flèche “there must be learned men if they existed anywhere on earth” [AT, 1964–1974, 6, p.5]. Descartes reflected on his first impressions of mathematics:

Above all I delighted in mathematics, because of the certainty and self-evidence of its reasonings. I did not yet notice its real use; and since I thought it was of service only in the mechanical arts, I was surprised that nothing more exalted had been built upon such firm and solid foundations. [AT, 1964–1974, 6, p.7]

The “mechanical arts” meant practical or applied mathematics, which may have included geography, mechanics, or military architecture. The mathematical portion of the curriculum at La Flèche for young scholars of philosophy contained little in the way of elementary geometry, arithmetic, and astronomy in favour of the mechanical arts³. The strong emphasis on the mechanical arts would have a profound effect on Descartes’ mathematical

²“Nobles of the Robe” were French aristocrats who owed their rank to judicial or administrative positions. Such positions and status were also often inherited. In his own *Cogitationes privatae* (c.1619), Descartes says that he was expected to “mount the stage of the theatre or the world” [AT, 1964–1974, 10, p.213].

³See also [Schuster, 2013, p.33]: “Their attention to mixed and practical mathematics spoke well for the Jesuits’ awareness of the needs and changing aspirations of their clientele. The educated gentleman-officer was increasingly expected to command a knowledge of practical mathematical arts. The shift in emphasis in the training of the secular elite in the late sixteenth century is indicative of a temporary lowering of caste barriers to the acceptance of mechanical arts, including practical mathematics, as elements of higher culture”.

reasoning and development over the coming years, and is reflected especially in the *Regulae*, which we discuss below (section 3.2.1). In his reflection on his early education given above, Descartes suggests that in later life he no longer viewed mathematics as subservient to the mechanical arts. However, we see below that, in the *Regulae* at least, Descartes maintained an emphasis on the use of mathematical reasoning in practical matters.

After a brief time in Paris, Descartes went on to university in Poitiers, graduating with a law degree in 1616. He then went to military school in the Dutch city of Breda. It was there that Descartes met and studied under Isaac Beeckman, with whom he would continue to correspond on his developing mathematical ideas [Crombie, 1981, 4, pp.51–65].

Descartes was attracted to the “firm and solid foundations” of mathematics⁴. He also said that crucial to the formation of his *méthode* were the disciplines of logic, analysis, and algebra, although he felt he had to “seek some other method comprising the advantages of these three subjects but free from their defects” [AT, 1964–1974, 6, p.7; p.18] in [Sasaki, 2003, pp.14–15]. These passages are from Descartes’ *Discours de la méthode* (1637), where he reflects on his initial learning of these three subjects. Descartes discusses his identification of the “defects” in them:

But in examining them I observed in respect to logic that the syllogisms and the greater part of the other teaching served better in explaining to others those things that one knows [...] than in learning what is new. [...] And as to the analysis of the ancients and the algebra of the moderns [...] the former is always so restricted to the consideration of symbols that it cannot exercise the understanding without greatly fatiguing the imagination; and the latter one is so subjected to certain rules and formulas that the result is the construction of an art which is confused and obscure, and which embarrasses the mind [...]. [AT, 1964–1974, 6, pp.17–18] in [Descartes, 1997, pp.81–82]

It is clear from his comments above that Descartes was dissatisfied with both the ancient methods of analysis as well as the *new analysis* of the moderns. It was his intention to completely revise the analysis of geometrical problems. He did so in the appended

⁴That is, its “certainty and self-evidence of its reasonings” [AT, 1964–1974, 6, p.7] in [Sasaki, 2003, p.14]. We note here Newton’s attraction to geometry for very similar reasons (see, for example, [Guicciardini, 2009]), as we discover in the following chapter.

Géométrie, but it was not without many decades of work and thought that Descartes developed his finished *méthode*.

Descartes' formal and conventional education had quite an effect on his approach to mathematics and geometry, as would his lasting relationship with Beeckman. In the next chapter we will see that Newton's education was quite different and this likely inspired many of the differences between his and Descartes' views of geometry.

In the emergence of his work from formal and strict teachings we can begin to perceive a connection between his early influences and his lifelong need to complete geometry and to find a formal resolution in his method.

3.0.2 Early influences on Descartes

The influences on Descartes are widely acknowledged in mathematical history as being hard to define with confidence. As we shall see, it is not clear if Descartes' mathematical work is genuinely original, but he rarely acknowledged the works of his contemporaries. In response to this, we address here two questions which, although unlikely to be resolved in this short space, are central to understanding the development of Descartes' mathematical thought, and have prompted many studies and partial conclusions from Cartesian scholars. Firstly, we ask by whom Descartes may have been influenced and, in particular, what knowledge of Viète's work he may have had. Secondly, we look at the sequence of events which led to the development of Descartes' algebraic-geometric methods that would eventually culminate in the *Géométrie* of 1637.

In his early education, Descartes would almost certainly have studied the works of Christopher Clavius owing to his Jesuit education at La Flèche. Descartes referred to Clavius' *Euclidis Elementorum* in a letter to Mersenne (13 November, 1629) discussing the quadratrix [AT, 1964–1974, 1, pp.70–71]. In an exchange with the English mathematician John Pell in about 1646, Descartes is said to have claimed he “had no other instructor for Algebra than ye reading of Clavy Algebra above 30 yeares agoe” [Hervey, 1952, p.78].

Clavius, who was a contemporary of Viète, worked on many historical restorations, including that of Pappus' *Collectio*, especially Book 7. In the preface to his own *Euclidis Elementorum* (1574), an extensively annotated edition of Euclid's *Elements*, Clavius praised

Commandino's translation of the *Collectio* and referred to it throughout [Clavius, 1612, e.g. 1, p.10]. Clavius wrote a number of treatises on both mathematics and astronomy. His five volume *Opera mathematica* was published in 1611–1612. Bos notes that “Clavius’ texts, whilst not innovative, were widely used throughout the 17th century”. Clavius was especially concerned with the acceptability of geometrical constructions and “he was the first to take up a theme that was to become crucial later on, namely, the legitimacy of various methods of generating curves, in particular tracing by motion and pointwise construction” [Bos, 2001, p.160]⁵. This was probably Descartes’ first experience of the type of questions that were facing late 16th and early 17th century geometers, namely, the legitimacy of construction methods and various curves, and the appropriate uses of geometry.

Sasaki describes how, in contrast to his contemporary Viète, Clavius was interested mainly in the synthesis of geometrical problems rather than the analytic method of discovery, so thinks it is unlikely that he inspired Descartes’ algebraic method [Sasaki, 2003, p.63]. If Clavius’ main concern was geometrical synthesis, where did Descartes learn about analytic methods? In the passage from Descartes’ *Discours*, which we quoted above (section 3.0.1), Descartes comments merely on the “analysis of the ancients” and the “algebra of the moderns”. How could he have made such statements without knowing about Pappus and Viète? We know he had read Pappus at some point since Descartes explicitly refers to him in the *Géométrie* (section 3.3). Given Clavius’ influence it is likely he knew about Pappus from early on. But what was meant by the “algebra of the moderns”? At first it seems unlikely that Descartes would have been unaware of the recent developments in algebraic analysis. However Mahoney sheds some light on the dissemination of Viète’s work in Paris, and on the all important role of Beeckman:

For all Isaac Beeckman’s mathematical erudition, he does not seem to have known about Viète. Beeckman’s most famous protege, Descartes, claimed he first read Viète’s work only after the appearance of his own *Géométrie*, and the recorded genesis of the latter treatise in the *Rules for the Direction of the Mind* (1628) acts to support that claim. Mersenne and the Parisian

⁵On the mathematics Descartes learned from Clavius’ works at La Flèche see also [Milhaud, 1921, p.235].

circle of mathematicians about him knew about Viète but did not follow his mathematical lead. Viète's far superior algebraic works had been in print for more than a decade when Clavius published his *Algebra*, so different in tone and style, in retrospect so backward. [Mahoney, 1973, pp.27–28]

So perhaps Descartes was genuinely unaware of Viète's work, at least up until 1628, if he had not learnt of its existence from Beeckman. Whilst Descartes claims further never to have read Viète's work until after he established his own methods in the *Géométrie* (1637)⁶, this was in light of an accusation of plagiarism⁷. Both Mahoney [1994, p.28, note 6] and Sasaki [2003, p.244] refer to a letter from Descartes to Mersenne (20 February, 1639):

I have no acquaintance with the geometer about whom you wrote to me and I am surprised with what he says, that we have studied Viète together in Paris; for this is a book of which I do not remember to have seen even the cover while I was in France. [AT, 1964–1974, 2, p.524] in [Sasaki, 2003, p.244]

In his review of [Sasaki, 2003], Serfati also disagrees that Descartes borrowed anything from Clavius, and that the main influence on his mathematical thought would have been his readings of Pappus, Cardano, and Viète. In a letter to Mersenne (end December, 1637), Descartes declared that he had “begun where Viète had left off” [AT, 1964–1974, 1, p.479] in [Serfati, 2005, p.658]. Serfati's central question is: under what conditions did Descartes become acquainted with Viète's work? He suggests that the question “should be analyzed with respect to the origins in Descartes (through Viète) of mathematical symbolic writing, especially in rule 16 of the *Regulae*” [Serfati, 2005, p.658]. That is, lower case a, b, c, \dots for known quantities, upper case A, B, C, \dots for unknown quantities⁸.

The relationship between Descartes' and Viète's respective algebraic engagement with geometry is a subject of great complexity, and is likely to remain a topic of debate for contemporary scholars for some time to come. Whilst a precise answer to this question is not central to our work, it is valuable in terms of contextualising Descartes' geometric

⁶This is a work which Michel Chasles famously declared “*prolem sine matre creatam*” (a child without a mother) [Chasles, 1837, p.94]. Michel Serfati regards the search for such direct influences in the *Géométrie* as “usually a fruitless task” [Serfati, 2005, p.658].

⁷We acknowledge this in section 3.4.

⁸See also [Rodis-Lewis, 1998, p.64]

thought. Here we provide a brief assessment of contemporary thought on the subject. Serfati comments that a few scholars, including himself, have made some headway with the problem, for example, his own [Serfati, 1998]. We add to this [Macbeth, 2004] who says that Descartes' method is the *Géométrie* "can seem to be essentially that of Viète" but identifies that there are "differences in aim and orientation" which "do not penetrate the use of symbolic language itself". Macbeth goes on to say that "Descartes' understanding of his symbolic language is very different from Viète's understanding of the *logistique speciosa* (specious logistics) [...] Viète abstracts from the particular subject matter, either that of arithmetic or that of geometry. An expression such as $a^2 + bc$ of Descartes' symbolism is not an empty formalism interpretable either arithmetically or geometrically; it is itself fully meaningful, a representation of an arbitrary line segment" [Macbeth, 2004, p.93]. In the relatively recent [Stedall, 2011, p.29], Stedall comments that the question remains "tantalisingly unanswered" and prefers not to draw any firm conclusions.

Let us turn now to the second of our questions: when did Descartes develop his mathematical ideas as presented in the *Géométrie*? Once again, this question is a subject of contemporary discussion and debate. Here we note Lenoir's comments on the lack of firm evidence on the formation of Descartes' mathematical thought. He suggests that we might turn to Beeckman's *Journal*⁹ for answers, and that it may have been under Beeckman's influence that Descartes studied Pappus¹⁰ [Lenoir, 1979, p.363]. Again, we do not seek to find a clear resolution to this, but to provide a brief overview of current discussion on this topic.

We present two points of view put forth in recent works by Cartesian scholars. First let us briefly state the sequence of events of Descartes' contact with Beeckman with respect to the development of his "new science". In a letter to Beeckman (26 March, 1619) Descartes announced that he wanted to launch a "completely new science (*scientia penitus nova*) by which all questions in general may be solved that can be proposed about any

⁹Isaac Beeckman kept an extensive journal of his studies, but it was not widely publicised until Cornelis de Waard, a historian of science, rediscovered it and edited it for publication as *Journal tenu par Isaac Beeckman de 1604 à 1634* between 1939 and 1953 [Beeckman, 1953].

¹⁰Although, as we noted above, Descartes is likely to have known of the work of Pappus through his study of Clavius.

kind of quantity, continuous as well as discrete. But each according to its own nature” [Rabouin, 2010, p.434]¹¹.

In 1628 Descartes visited Beeckman in Dordrecht. Descartes described to Beeckman a sample of a “general algebra”, which he had been developing over that past decade, commenting that he “had made as much progress as was possible for a human mind”. Descartes said that he would send Beeckman a treatise on algebra, which he claimed could bring geometry to perfection. Beeckman documented this work under the title *Algebrae Des Cartes specimen quondam* [Rabouin, 2010, p.429].

In a more substantial document sent to Beeckman a few months after his visit, Descartes is supposed to have described his “most outstanding discovery”: the “construction” of the third and fourth degree equation by the intersection of a parabola and a circle. Rabouin’s criticism is that this is inconsistent with any suggestion that the classification of curves was at the core of Descartes’ programme since 1619. He believes that this is “a technique much more sophisticated than anything that could be expected from the “sample” presented in 1628”. He goes on to tell us that “[t]here is no question of studying curves through algebraic techniques in the documents produced in 1628–1629, and the program presented to Beeckman is not that of a new classification of curves” [Rabouin, 2010, p.430].

Rabouin notes further that in his text of 1628–9, Descartes used the method of indeterminate coefficients, which “was known and used by cossist algebraists, but in an arithmetical context. By using it in a geometrical context in which curves are represented and manipulated through their equations, Descartes would have made his first step in what would be the core of the *Géométrie*’s new technique” [Rabouin, 2010, p.456]¹².

Rabouin identifies a gap in the work presented to Beeckman in 1628 and takes issue with any reconstruction of Descartes’ 1628 work which suggests he was in fact studying curves through their equations at that time, giving the following reasons. Firstly, he says, concerning the circle, Descartes never uses its equation, even in later works, relying instead on the Pythagorean theorem. Secondly, concerning the parabola, Descartes had already

¹¹The quote from Descartes is given in full in [Rabouin, 2010, p.434] and [Bos, 2001, p.232].

¹²See also [Bos, 2001, p.258] who also suggests that “it may be that Descartes arrived at the general construction [of third- and fourth- degree equations by parabola and circle] by the method of indeterminate coefficients.”

sent to Beeckman other texts in 1628 concerning conic sections, which he had studied for his work on optics. And finally, that the main document concerns ovals, which Descartes had studied in an algebraic way in a fragment preserved in the *Excerpta mathematica* [AT, 1964–1974, 10, pp. 310–324]. He comments that “[i]n this context, it is very striking that the documents preserved by Beeckman present a purely geometrical analysis with no use of algebraic techniques at all” [Rabouin, 2010, p.457].

He concludes that it may be that what Beeckman had collated were fragments of separate projects and that “[t]his would confirm Bos’s judgement that the Pappus problem, studied at the end of 1631 on Golius’ suggestion, was for Descartes the *crucial catalyst*” [Rabouin, 2010, p.457]. Rabouin’s point is that the documents held by Beeckman do not serve as evidence for a unified view of “algebraic geometry” by Descartes as later presented in the *Géométrie*. He suspects, in agreement with Bos [Bos, 2001, p.283], that it was only after Descartes had been presented with the Pappus problem that he went back to his earlier work of 1619 (comparing the classification of geometrical problems with arithmetic ones), and combined this with the *Regulae* (treating problems as equations). It was therefore after 1632, and the combination of these earlier works that provided Descartes with a firm basis for an analytical geometry as presented in 1637.

Alternatively, Sasaki believes that Descartes’ central ideas might have been developed before 1623. This assumption is based on a letter from Descartes to Mersenne (31 March, 1638): “You know that more than 15 years have already passed since I declared I would disregard geometry and never to dwell on solving any problem unless it is at the request of some friend” [AT, 1964–1974, 2, p.95] in [Sasaki, 2003, p.3]. Sasaki does not believe there is any evidence to contradict this statement, and that whilst it is not clear when exactly he composed his *Algebra* (1628), “we may safely consider him to have possessed its central idea before about 1623” [Sasaki, 2003, p.3].

As we mentioned above, we do not seek to propose a resolution of this issue. To do so would be facile. However, we err on the side of Rabouin who presents an objective study of the mathematical development, and the order in which these events are likely to have occurred, and takes into account the fact that we are already receiving second

hand information from Beeckman. In that light, his suggestion is a reasonable one. We might also cast some doubt on the reliability of Descartes' account to Mersenne given the plagiarism charge.

Whilst these issues continue to be a subject of ongoing discussion, it has been valuable here to present current thinking on the complex context from which Descartes' seminal work on geometry emerged. In the next two sections we look more closely at Descartes' mathematical writings between 1619–1628, and then his early work on Pappus' problem in 1631–1631, another crucial step in the formation of the ideas which would later be found in his most significant geometrical work, the *Géométrie* (1637).

3.1 First mathematical writings: 1619–1628

In order to understand the early development of Descartes' geometrical methods, here we will examine his first exploration of a unified approach to geometry, the *Cogitationes Privatae*. This work was undertaken after his first meeting with Isaac Beeckman and before the *Regulae* of 1628.

The *Cogitationes privatae* is thought to have been written between 1619 and 1621, and is a documentation of Descartes' work for that period.

In particular, he gave more details of his plan for “an entirely new science”, which he had expressed earlier to Beeckman. This manuscript was copied by Leibniz in 1676, and was later published by Foucher de Careil, a French nobleman, under the title *Cartesii Cogitationes privatae* (1859). However, Careil's edition contained many errors, and Leibniz's was lost, but the work was later restored by Gustav Enestöm, Henri Vogt, and Henri Adam to be included in [AT, 1964–1974, volume 10]¹³.

The *Cogitationes privatae* contains both mathematical and non-mathematical parts. Near the beginning of the work Descartes expressed a plan for a complete mathematical treatise, *Thesaurus mathematicus*:

This work lays down the true means of solving all the difficulties in the science of mathematics, and demonstrates that the human intellect can achieve nothing further on these questions. The work is aimed at certain people who promise to show us miraculous discoveries in all the sciences, its purpose

¹³See [Bos, 2001, p.237, note 17] and [Sasaki, 2003, p.109].

being to chide them for their sluggishness and to expose the emptiness of their boasts. [AT, 1964–1974, 10, p.214] in [Sasaki, 2003, p.110]

Sasaki reminds us also of Descartes' use of such phrases as “almost nothing will remain to be found in geometry” in his letter to Beeckman (26 March, 1619) [Sasaki, 2003, p.111]. Much later, in the *Géométrie*, Descartes retained this ambitious aim to resolve *all* of geometry. The passage from Descartes very much sets out the potential for a boundary around the topic of geometry, which was also a focus for his predecessors. We note also Descartes' pointed language showing his dissatisfaction with recent developments in approaches to mathematics and their haste in applying these to other scientific endeavours. This again leaves us with the impression that it was Descartes' desire to somehow *complete* geometry, before it might be put to other uses. This was very much in line with Descartes' philosophy of building knowledge from the ground up.

3.1.1 Mesolabe and other instruments

In the *Cogitationes privatae* Descartes became focused on finding practical solutions to geometrical problems, including the invention of various moving instruments, which would become Descartes' preferred method of curve construction. In contrast to Viète, who had somewhat avoided the subject, the refinement of such instruments was to become fundamental to Descartes' attempt to define geometry. His use of instruments would later reinforce his strict ideas on geometry, in particular its restriction to algebraic curves.

In the letter to Beeckman Descartes referred to a curve tracing instrument which he believed to be crucial to his new science saying that he had found four demonstrations with the help of the instrument. First, the division of an angle, and secondly the solution of three types of cubic equation¹⁴ [Sasaki, 2003, p.113]. In the *Cogitationes privatae* Descartes made several further references to such instruments which he called “new compasses”. The new compasses could be used for finding angle divisions and mean proportionals, and for the solution of certain cubic equations, so it is likely that these are what Descartes had in mind when he wrote to Beeckman.

Descartes described three instruments consisting of a series of hinged rods, one for the

¹⁴(1) $\pm a \pm bx = x^3$, (2) $\pm a \pm bx^2 = x^3$, (3) $\pm a \pm bx \pm cx^2 = x^3$.

trisection of an angle and two for solving certain cubic equations. This is the trisecting instrument (Figure 3.1) [AT, 1964–1974, 10, p.240] in [Bos, 2001, pp.237–238]:

Instrument 3.1.1 *Four rulers OA , OB , OC , and OD , are connected in the point O , around which each can turn. Four equal rods EI , FJ , GI , HJ , with length a , can turn around the points E, F, G, H , which are on the four arms at distance a from O . The rods are pairwise joined in hinges at I and J ; the hinges themselves can move freely along OB and OC . It is easily seen that by this arrangement the two arms OA and OD can form any angle within a large range and that the three inner angles AOB , BOC , and COD will always be equal; hence the instrument can serve to trisect any angle.*

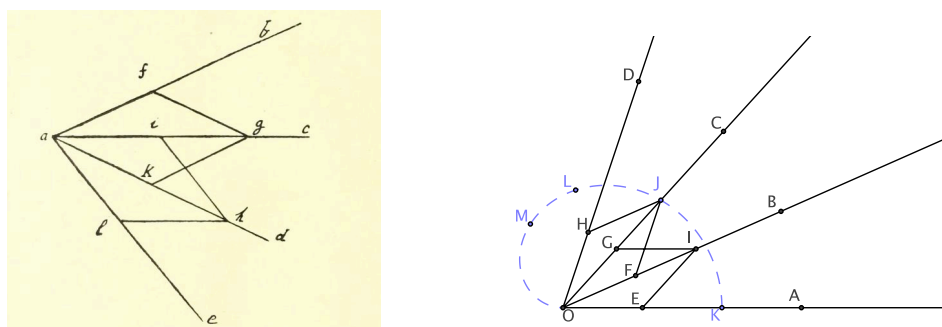


Figure 3.1: Descartes' "new compasses" [AT, 1901, 10, p.240] (left); with curve $KJLM$ (right)

In his example, Descartes' instrument is used to divide an angle into three parts, but he envisaged that with the addition of more arms the device could be used for dividing angles into any number of equal parts. The instrument was not intended to directly divide the angle. It was to be used as a curve tracing instrument to supply a curve ($KJLM$) which could then be used to trisect an angle¹⁵.

Descartes also described a method for the resolution of certain cubic equations using a second type of instrument. As Bos notes, Descartes did not directly describe the instrument itself, but it later appeared in the *Géométrie* [Descartes, 1954, p.46] and is consistent with the text in the *Cogitationes privatae* [AT, 1964–1974, 10, pp.238–239]. Descartes does, however, refer to the "curve of the mesolabum compass"¹⁶ and it has come to be

¹⁵The procedure can be found in [AT, 1964–1974, 10, pp.240–241] and [Bos, 2001, p.239].

¹⁶"*linea circini mesolabi*" [AT, 1964–1974, 10, pp.238–239].

known as the mesolabum. The idea behind the instrument is that if given an angle AOB , the consecutive perpendiculars along the arms OA and OB intercept the opposite arms respectively in geometric progression. That is, given the figure below (Figure 3.2)

$$e : x = x : y = y : z = z : u = u : v = v : w = w : \dots$$

If e is taken to be the unit measure, then the progression becomes $1, x, x^2, x^3, \dots$ and the progression can be continued indefinitely. Equations in x can be interpreted as relations between the line segments. The instrument is given in the *Géométrie* as follows [Descartes, 1954, p.44]:

Instrument 3.1.2 *Consider the lines AB , AD , AF , and so forth, which we may suppose to be described by means of the instrument YZ . This instrument consists of several rulers hinged together in such a way that YZ being placed along the line AN the angle XYZ can be increased or decreased in size, and when its sides are together the points B , C , D , E , F , G , H , all coincide with A ; but as the size of the angle is increased, the ruler BC , fastened at right angles to XY at the point B , pushes toward Z the ruler CD which slides along YZ always at right angles. In like manner, CD pushes DE which slides along YX always parallel to BC ; DE pushes EF ; EF pushes FG ; FG pushes GH , and so on. Thus we may imagine an infinity of rulers, each pushing another, half of them making equal angles with YX and the rest with YZ .*

Now as the angle XYZ is increased the point B describes the curve AB , which is a circle; while the intersections of the other rulers, namely the points, D , F , H describe other curves AD , AF , AH , of which the latter are more complex than the first, and this more complex than the circle.

Again, the instrument is to be used to draw the curves which may then be used to determine the solution of the cubic or to find mean proportionals, including two mean proportionals. Taking into account both of these instruments, Descartes now had a way to “geometrically” resolve the two classical problems of angle trisection and the finding of two mean proportionals. It is not clear whether Descartes was aware of Viète’s work at

1628 Beeckman finally collated Descartes' ideas of the previous nine years in his *Journal* under the title *Algebrae Des Cartes specimen quoddam* [Beeckman, 1953, 3, p.95].

He, I say, came to Dordrecht to call on me on October 8, 1628, after he had gone first to Middleburg from Holland in order to look for me there. He told me that insofar as arithmetic and geometry were concerned, he had nothing more to discover; that is, in these branches during the past nine years he had made as much progress as was possible for the human mind. He gave me perspicacious specimens of this and promised to send me his *Algebra* a little later from Paris, which he said, was finished and by which he not only arrived at a perfect knowledge of geometry but also claimed to embrace all human knowledge. [AT, 1964–1974, 10, p.331] in [Sasaki, 2003, p.159]

In our discussion of the early development of Descartes' mathematical ideas, Sasaki [2003] drew our attention to a letter from Descartes to Beeckman (31 March, 1638) in which Descartes claimed that “more than 15 years have passed since I declared I would disregard geometry” [AT, 1964–1974, 2, p.95].

As we have discussed, the evolution of Descartes' mathematical ideas is quite complicated. It is not known for certain when exactly he developed the geometrical canon that he laid down in the *Géométrie* of 1637. It may be that Descartes' developed his ideas before 1628, or that it was his first work on Pappus' problem between 1631–1632 that solidified for Descartes the concept of unifying algebraic and geometric methods. In his early text, *Cogitationes privatae* (1619–1621), Descartes had already started to build upon his idea of “an entirely new science”, which he had expressed to Beeckman. His plan, he declared, was to complete mathematics. In the work, Descartes had resolved to find practical solutions to solving geometrical problems, focusing on the resolution of the classical problems of angle trisection and the finding of two mean proportionals. However, as we saw in the previous section, Descartes had already gone much further than this by proposing instruments that could be applied to more general versions of these problems.

Before we move on to discuss the ideas presented by Descartes in the *Regulae ad directionem in genii* (1628)—the subject of our next section—let us first make a few comments on the general style of Descartes' mathematical writing. Molland perhaps sums

it up best when he says, “Descartes’s mathematical laziness is notorious [. . .] He had the type of mind that was happy in producing bold general conceptions, but became bored when it was a question of working out the detail, although he was quite capable of doing this”. He gives as an example Descartes’ construction of ovals, which he had applied to his work in optics, and says that “when treating these his criteria for geometrical construction are more lax than his norm” [Molland, 1976, p.40].

Further, at the end of his discussion of Pappus’ problem, Descartes excused himself for not treating more cases and also says that he has explained how any number of points may be found. He continues

But the fact that this method of tracing a curve by determining a number of its points taken at random applies only to curves that can be generated by a regular and continuous motion does not justify its exclusion from geometry. [Descartes, 1954, p.91]

Molland believes that Descartes is justifying pointwise descriptions of curves:

Here Descartes’s indolence seems to have led him to the brink of admitting definition by equation, but from what follows it is clear that he regarded this mode of description as subsidiary to genesis by determined motions [. . .] Thus some point-wise descriptions are allowed, but with an inferior status. Descartes also feels it necessary to make a similar concession for certain constructions making use of strings. [Molland, 1976, p.40].

However, we believe that Descartes is saying, not that the curve is determined in a pointwise manner, but that it would only be possible to find points in the way he has described if the curve can also be described by his geometrical criteria of “regular and continuous motion”. We agree that Descartes is more lackadaisical in his approach to the use of strings, about which he says “[n]or should we reject the method in which a string or loop of thread is used”, in that he feels he must justify this with hindsight having used it in the *Dioptrique*. With this in mind, it may well have been Descartes’ intention, if not wholly successfully, to apply his strict rules of geometry across other areas of his work.

Returning to the point of Descartes’ “mathematical laziness”, in the *Géométrie* Descartes closed with the words “I hope that posterity will judge me kindly, not only as to the things

which I have explained, but also as to those which I have intentionally omitted so as to leave others the pleasure of discovery” [Descartes, 1954, p.240]. In spite of his wish to set out a clear canon of geometrical method, it appears that he found the process somewhat tiresome. By his own admission in the preceding paragraph [Descartes, 1954, p.240], what Descartes has “intentionally omitted” is rather a lot. This seems all the more ironic in light of Descartes’ criticisms of the ancients’ failure to make their methods explicit.

3.2.1 *Regulae ad directionem ingenii* (1628)

The *Regulae ad directionem ingenii* is, broadly speaking, a set of rules regarding the discovery of knowledge and the correct method for scientific and philosophical thought.

At the end of rule 12, Descartes explained how he had originally intended for the *Regulae* to consist of three parts: on the theory of knowledge (rules 1–12), on mathematics (rules 13–21), and on natural philosophy. The second part remained unfinished, and the third left completely unwritten. The work was eventually published posthumously in its unfinished state. A Dutch translation appeared in 1684 [Descartes, 1684], and a Latin one in 1701 [Descartes, 1701].

Much of Descartes’ inspiration for the *Regulae* came from geometry and arithmetic. He believed these to be the basis for solving all problems of science. The deductive nature of geometry appealed to Descartes. In his commentary on rule 2, Descartes says

[O]f all the sciences known as yet, arithmetic and geometry alone are free from any taint of falsity or uncertainty. We must note then that there are two ways by which we arrive at the knowledge of facts: by experience and by deduction. We must observe further that while our inferences from experience are frequently fallacious, deduction, or the pure illation of one thing from another, though it may be passed over, if it is not seen through, cannot be erroneous when performed by an understanding that is in the least degree rational. [AT, 1964–1974, 10, p.365] in [Descartes, 1997, p.6]

He continued “in our search for the direct road towards truth we should busy ourselves with no object about which we cannot attain a certitude equal to that of the demonstrations of arithmetic and geometry” [Descartes, 1997, p.7].

Descartes was also critical of his predecessors who had approached geometry in a way

that meant their analyses, and hence their methods of solution, were hidden. In spite of this Descartes admits that “we have sufficient evidence that the ancient geometers made use of a certain analysis which they extended to the resolution of all problems, though they grudged the secret to posterity” [Descartes, 1997, p.13]. Descartes was intent on making his deductive methods explicit, where he believed the ancient geometers to have failed. In rule 4 he expressed his dissatisfaction with the “authors” of arithmetic and geometry [Descartes, 1997, p.14]. His aim for the *Regulae* was to develop a method of discovering the truth and knowledge of all things. He found the certainty he required in arithmetic and geometry. In the second part, Descartes concentrated on applying his method to mathematics with a consistent certainty.

Descartes stated his aim for the *Regulae* in his commentary to rule 4: *There is need of a method for finding out the truth*. The “need” arose from Descartes’ dissatisfaction with his predecessors. In his closing paragraph to this rule Descartes says

[I] have resolved that in my investigation into truth I shall follow obstinately such an order as will require me first to start with what is simplest and easiest, and never permit me to proceed farther until in the first sphere there seems to be nothing further to be done. This is why up to the present time to the best of my ability I have made a study of this universal mathematics; consequently I believe that when I go on to deal in their turn with more profound sciences, as I hope to do soon, my efforts will not be premature. But before I make this transition I shall try to bring together and arrange in a orderly manner the facts which in my previous studies I have noted as being more worthy of attention. Thus I hope both that at a future date, when through advancing years my memory is enfeebled, I shall, if need be, conveniently be able to recall them by looking in this little book, and that having now disburdened my memory of them I may be free to concentrate my mind on my future studies. [AT, 1964–1974, 10, p.379] in [Descartes, 1997, pp.16–17]

Firstly, we notice Descartes’ firm statement of his reluctance to build upon the work of others. He wished to discover for himself the *truths* of mathematics, starting with the most basic of principles. He later made further statements to this effect in the *Géométrie*. Secondly, we interpret his remark “never [...] to proceed farther until in the first sphere there seems to be nothing further to be done” as Descartes’ desire to resolve *all* of geometry

within a defined boundary, and as symbolic of the block he experienced. He could not move on until the subject of geometry had been resolved and overcome. Details such as these are important in trying to understand how Descartes thought of geometry. In contrast, and as we shall see in the next chapter, Newton did not see geometry in this way, but instead saw its greater potential. Rather than seeking to overcome geometry, he would utilise it throughout his career in a continual and open process.

We note also Descartes' choice of terminology *mathesis universalis*; a universal mathematics that could be applied to the study of any logical discipline. In fact the whole concept of *mathesis universalis* is contained entirely in rule 4 of the *Regulae*. Descartes says that

[I]t gradually came to light that all those matters only were referred to mathematics in which order and measurement are investigated, and that it makes no difference whether it be in numbers, figures, stars, sounds or any other object that the question or measurement arises. I saw consequently that there must be some general science to explain that element as a whole which gives rise to the problems about order and measurement, restricted as these are to no special subject matter. This I perceived was called *universal mathematics* [...] [AT, 1964–1974, 10, p.378] in [Descartes, 1997, p.16]

Descartes was blocked by a need to find such a method as could be applied to other sciences. He did not feel able to move on until the matter had been resolved.

In his early education at La Flèche, Descartes had been influenced by an emphasis on the “mechanical arts”. This influence is evident in the *Regulae*. Descartes made it apparent that he had little interest in the study of mathematics for its own sake. Still in rule 4 he says

I should not think much of these rules, if they had no utility save for the solution of the empty problems with which logicians or geometers have been wont to beguile their leisure; my only achievement thus would have seemed to be an ability to argue about trifles more subtly than others. Further, though much mention is here made of numbers and figures, because no other sciences furnish us with illustrations of such self-evidence and certainty, the reader who follows my drift with sufficient attention will easily see that nothing is less in my mind than ordinary mathematics, and that I am expounding quite another science, of which these illustrations are rather the outer husk than the constituents. Such a science should contain the primary rudiments of human

reasons and its province ought to extend to the eliciting of true results in every subject. [AT, 1964–1974, 10, p.374] in [Descartes, 1997, p.13]

Descartes saw, what we can assume to be, the classical problems which preoccupied geometers as “empty problems”. He saw a higher, more worthy, purpose for mathematics, and yet remained unsatisfied with the state in which he had received it. Further, it was not simply the application of geometry to “applied mathematics”, but a method of reasoning. Descartes saw a “self-evidence and certainty” in both arithmetic and geometry. We find later that Newton would find a similar certainty in geometry, and yet his motivations were quite different from those of Descartes.

The relationship between deductive reasoning and the procedures of mechanical arts is explored by Israel who describes a parallelism in the *Regulae*, which he suggests translates to geometrical constructions [Israel, 1997, p.18]. He says that this acts as a basis for demarcation between admissible and inadmissible curves, which allowed a reclassification of curves that coincided with the modern classification of algebraic and transcendental curves.

The relationship between mechanical arts and geometry can be first identified in rule 8 (If in the matters to be examined we come to a step in the series of which our understanding is not sufficiently well able to have an intuitive cognition, we must stop short there. We must make no attempt to examine what follows; thus we shall spare ourselves superfluous labour) where Descartes says

This method of ours resembles indeed those devices employed by the mechanical crafts, which do not need the aid of anything outside of them, but themselves supply the directions for making their own instruments. [AT, 1964–1974, 10, p.397] in [Descartes, 1997, p.29]

This fits nicely with what we have seen of Descartes’ constructive instruments in the *Cogitationes privatae*. In the procedures of his solutions the instruments are almost suggested by the translated movement of lines and curves¹⁷.

¹⁷Israel proposes that Descartes’ sliding rulers instrument (instrument 3.1.2) may have been inspired by textile looms—an application of the “mechanical arts”—the functioning of which is based on a concatenation of coordinated movements according to a well-defined rule. He notes that “This concatenation is determined

Descartes' aim for the first part of the *Regulae* was to develop a method of discovering the truth and knowledge of all things. He found the certainty he required in arithmetic and geometry. In the second part of the work, Descartes focused his method on mathematics, which meant finding a method of analysis suitable for solving problems. In choosing algebra as a definitive method of analysis, Descartes needed to find a way to adapt its rules to be applied to geometry, or to change geometry to fit it. He partially addressed this in the second part of the *Regulae* where his language remains very broad and generalised.

Bos explains that “a large part of the [*Regulae*] may be characterised as Descartes' endeavour philosophically to understand the application of algebraic methods in solving problems about magnitudes in general” [Bos, 2001, p.263]. Let us now look at Descartes' emerging procedure of analysis in the *Regulae*.

In the beginning (rules 13 and 14), Descartes tells us that our first task is to truly understand the problem, which meant abstraction from “every superfluous construction” (rule 13), and then to translate the “perfectly understood” problem into algebraic form (rule 14). Descartes says that “all these previously known entities [(extension, shape, motion . . .)] are recognised by means of the same idea in different subjects”, maintaining a uniformity and abstraction across the sciences, and that knowledge is acquired through the comparison of such entities [Descartes, 1998, p.179].

In his commentary on rule 14 Descartes states that

In order that we may also use the imagination as an aid, however, one also has to note that, whenever something unknown is deduced from something else already known, not on that account is some new kind of entity discovered; rather, this entire knowledge is merely extended to the point where we may perceive that the thing sought participates, in this or that manner, in the nature of things that are given in the proposition. [Descartes, 1998, p.177]

He gives an example of restating a sentence in a number of ways. This bears a striking resemblance to the Newtonian understanding of a porism, from which information that is already implicit may be deduced from a problem. This is reinforced by Descartes' sugges-

by precise number relations and consequently based on the theory of proportions. All crucial conceptional knots of the Cartesian method (continuous and uninterrupted movement, theory of proportions) can be found in these examples of mechanical arts” [Israel, 1997, p.27].

tion that we may establish proportions between what is known and what is required. In any case, Descartes' principal aim was to "reduce the proportions, however complicated, to the point where what is unknown may be found equal to something known" [Descartes, 1998, p.189] and to determine a method by which all magnitudes may be compared.

In the following two rules (15 and 16), Descartes informs us about the representation of problems, saying that it may help us to draw figures representing magnitudes. A similar notion is represented in Beeckman's *Journaal* [Beeckman, 1953, 3, p.96]. The figures are then replaced by "concise symbols" since it will "be impossible for the memory to be misled [or to be distracted] while it is involved in deducing other matters" [Descartes, 1998, p.197].

Descartes then attends to the resolution of a problem (rules 17 and 18). Firstly, the "indirect resolution of questions". That is, resolution by regarding unknown terms as if they were known, and in a series of deductive steps, determining what is known from what is unknown. Whilst in rule 18 Descartes' details his geometric algebra as set out in the previous rules. Descartes here says that only four rules are required (addition, subtraction, multiplication, division). He also highlights the importance of the "unit", which is necessary "[i]n order that these two operations [multiplication and division] may be clearly explained" [Descartes, 1998, p.207].

Descartes outlines the next three rules, but then breaks off completely, perhaps because of his "mathematical laziness", or perhaps because he could not see a way to completely unify his theory. Bos suggests that he may have been blocked by such a strict analogy between geometrical construction and philosophical problem solving, and that this signifies the beginning of a divide between "Descartes the mathematician and Descartes the philosopher" [Bos, 2001, p.270].

In the introduction to [Descartes, 1998, pp.1–62], Heffernan discusses possible reasons for the break, and the dating of the *Regulae*, and surveys the work of others in this area, stating that there have been two "contrary patterns of thought on this issue". He cites, for example, the account of [Millet, 1867] who considers the *Regulae* as a complete and independent text, and who concludes that it was written in its entirety between 1628–

1629. On the other hand, a more modern account by Weber, given in [Weber, 1964a] and [Weber, 1964b], suggests the text to have been written continuously, but intermittently, between 1619–1628. This led Weber to consider the text as being more fragmented. In spite of this discrepancy, the *Regulae* is considered to be a key text in illuminating the early work and development of Descartes [Descartes, 1998, p.6]. We recall Rabouin’s [Rabouin, 2010] convincing comments that Descartes likely revisited this and his earlier work after addressing the Pappus problem in 1632, the culmination of which was set down in the *Géométrie*.

Descartes’ aims for the *Regulae* were not dissimilar from those of Viète in his *new analysis*. As we stated above, Descartes appeared to want to find a way of resolving all problems, not just within the bounds of geometry, but in all areas of science more generally. However, Viète had stayed more closely within mathematics and made a careful decision to choose algebra as his method of resolution. Descartes had identified the similarities between the scientific subjects, and from there he chose mathematics, and more specifically geometry and arithmetic, as his universal language.

Israel suggests that

The close link between the general methodical principles uncovered in the *Regulae* and their application in *La Géométrie* is more than evident. Actually, we could say almost that the whole procedure to “develop” the unknown quantity into equations described in *La Géométrie* is already contained in the *Regulae*. [Israel, 1997, p.26]

Shortly after the *Regulae*, Descartes focused his attention on an ancient problem that would help him to find a way to unify his geometrical ideals and methods. His work on the Pappus problem, in various incarnations, would later make up a large portion of the *Géométrie* and allow him to *test* his new methods.

3.2.2 Early work on Pappus’ problem (1631–1632)

Molland considers the possible connections which may be drawn between Descartes’ geometric methods and those of the ancients, in particular, the notion of finding a locus [Molland, 1976, p.32]. Naturally, the most famous ancient locus problem tackled by Descartes was the problem of Pappus. Molland says of the problem:

In some earlier writings we see him making use of compound ratios in moving towards his developed doctrine of multiplication of lines, and these may have played an explicit role in his original solution of the problem of Pappus. This problem was proposed to Descartes by Golius in 1631, and Descartes's occupation with it seems to have played a very important role in the development of his mature system of geometry. His original solution is lost, but the problem forms a central theme in his *Géométrie*, and an examination of it can give us much insight into Descartes's geometrical procedures. [Molland, 1976, p.38]

It was known that the solution to the three- or four-line case was a conic, that is, it was classified as a *solid* problem. More lines would generally result in a locus of the *linear* kind, but up to the time Descartes had studied the problem very little was known about these cases. We recall the problem from the previous chapter.

Given four lines and four corresponding angles, find the locus of a point such that the angled distances d_i from the point to each line maintain the constant ratio $d_1 d_2 : d_3 d_4$.

The three-line problem occurs when two of these four given lines are coincident. In the general case of many lines, the angled distances must maintain the constant ratio $d_1 \dots d_k : d_{k+1} \dots d_{2k}$ for $2k$ lines, or $d_1 \dots d_{k+1} : \alpha d_{k+2} \dots d_{2k+1}$ for $2k + 1$ lines.

Descartes found his breakthrough in the ancient Pappus problem. In the *Géométrie* he used Pappus' problem as a vehicle for demonstrating and testing his new methods, but Descartes' interest in the problem came earlier. In fact, his attention was not drawn to the problem by studying Pappus' *Collection*, but at the suggestion of Dutch mathematician Jacob Golius in 1631, who prompted Descartes to use his new method to solve the problem. The following year Descartes replied to Golius with his solution. Although the manuscript is now lost Bos writes:

The letters of 1632, together with the passages on Pappus' problem in the *Geometry*, strongly suggest that Golius' challenge gave him the ideas by which he could overcome the obstacles blocking his progress at the time he left the *Rules* unfinished. [Bos, 2001, p.271]

The first part of Descartes' method was to derive an algebraic representation. As he had detailed in rules 13–21 of the *Regulae*, the first step in the analysis of a problem was

to find its equation. Descartes used one of the given lines as an axis for his coordinate system, and its intersection with one of the other lines to be taken as the origin. He denoted these lines x and y . He was then able to express the other angled distances in terms of x and y , and combine these expressions in order to derive an equation in x and y . Descartes referred to these equations in his letter to Golius [AT, 1964–1974, 1, p.234]. Bos suggests that Descartes had already identified a relation between the number of lines given in the problem and the degree of the resulting equation, which is that if the number of lines is increased by two, then the degree is increased by one [Bos, 2001, p.274].

From the forms of the second degree equations Descartes had found in his study of the four-line problem he was able to identify if the curve would be a parabola, hyperbola, ellipse or circle. However, he still needed a way to *geometrically* construct the curve given by the equation. In this instance he appealed to Apollonius for the geometric construction of the conic sections. This was inconsistent with his idea that all Pappus curves could be traced by singular motion (for example, by his instruments given above), and this seems curious given his previous explicit comments about his reluctance to build upon the work of others. In fact, nowhere did Descartes explicitly present a method for describing *all* Pappus curves by motion. He also needed to find a way of constructing curves of degree greater than two, such as those resulting from the case with five lines which Descartes also tackled.

Descartes' next move was to look at the particular five-line case where four of the lines are parallel and one line is perpendicular to these four. In this case the ratio which must be maintained is $\alpha d_3 d_5 : d_1 d_2 d_4$, where d_k are the perpendicular distances from the point on the locus to each of the respective given lines. Bos surmises that this is, in fact, the case which inspired Descartes' statement about the tracing of all Pappus curves by singular motion. Bos outlines Descartes' solution as follows (Figure 3.3) [Bos, 2001, pp.275–277].

Let P be a point on the locus and consider the line OPQ with Q on L_3 . Let R be the intersection of L_3 and the line through P parallel to L_5 . If we call $QR = z$, we have $d_5 : d_1 = z : d_3$. The condition of the problem implies $\alpha d_3 d_5 = d_1 d_2 d_4$, which may be

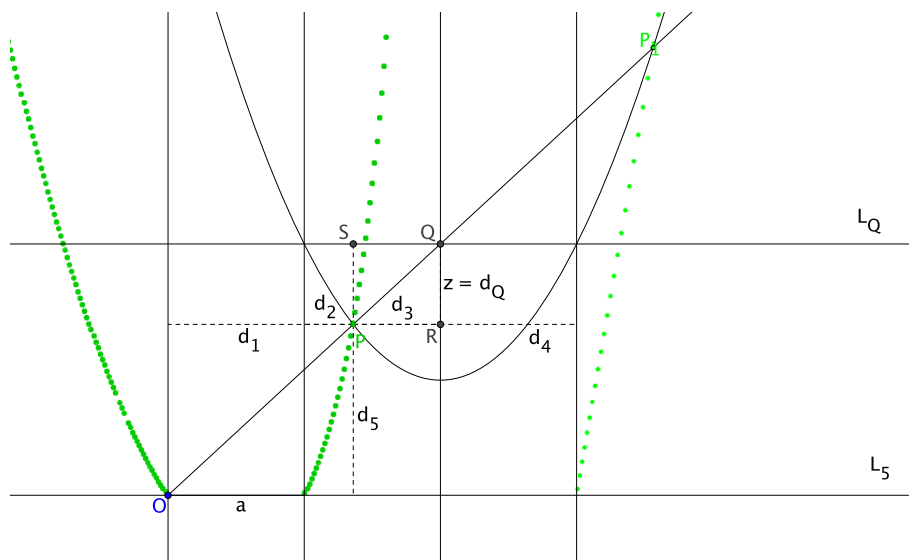


Figure 3.3: Descartes' Solution to a five-line Pappus problem

rewritten as $d_5 : d_1 = d_2 d_4 : \alpha d_3$. Combining the two equations gives $\alpha z = d_2 d_4$, where $z = d_Q$ is equal to the distance PS of P to a horizontal line L_Q through Q , to give $\alpha d_Q = d_2 d_4$. The point P therefore lies on a three-line Pappus locus, specifically a parabola with axis along L_3 .

If OQ is thought of as a moving ruler rotating around O and forcing Q along L_3 , the line L_Q moves up or down. Since L_2 and L_4 are vertical we may conceive the system of three lines L_Q , L_2 and L_4 as moving up and down with Q . Hence as the system of three lines moves, so too does the locus together with L_Q . Meanwhile the point P on the five-line Pappus curve is at the intersection of the ruler and the parabola. The construction is essentially given by the method of analysis, but Descartes did not detail it fully until the *Géométrie*, which we discuss below.

In the letter to Golius, Descartes claimed that all Pappus loci could be traced by “one single continuous motion completely determined by a number of simple relations” and that these curves were acceptable in geometry. On the other hand, curves such as the spiral and quadratrix, which could not be traced in this way, were to be excluded from geometry [AT, 1964–1974, 1, p.233]. Further, Pappus curves could be classified according to the degree of the curve which, as Descartes had identified, depended on the number of given lines in the problem. Higher order curves could be generated by the sliding of lower order ones in conjunction with the intersection of a rotating straight line. This may

have reinforced for Descartes the idea that geometrically constructed curves corresponded directly to the loci of Pappus problems.

So far in this chapter we have explored the progression of Descartes' ideas and attempts to provide his resolution to the subject of geometry. In the next section we look at the *Géométrie* and the crucial role the Pappus problem played within it. The *Géométrie* was a culmination of Descartes' attempts to set out a concise geometrical method that could be universally applied within his strictly defined boundaries of geometry. He addressed several crucial questions, which both he and his immediate predecessors had faced. For example, he established the application of algebra to geometry as a means of analysis, and he determined which curves were geometrically acceptable, and appropriate methods of construction.

3.3 The *Géométrie*

The *Géométrie* (1637) was initially published as one of three appendices¹⁸ to Descartes' *Discours de la méthode* [Descartes, 1637]—a method for obtaining truths in the sciences. In this main treatise, Descartes' showed no partiality to either geometry or algebra. In fact, he was critical of both arts. As in his earlier works, Descartes sought to take the best elements from the mathematical sciences. In particular, he used line segments to represent all magnitudes, regardless of their origin, but used symbolic representation to manipulate such objects. Boyer comments that “Descartes in a sense was returning in thought to the ancient geometrical algebra, while at the same time he encouraged the development of symbolic forms of expression”. In doing so, “he would borrow the best of both geometric analysis and algebra, correcting the faults of each” [Boyer, 1959, pp.390–391].

The *Géométrie* was by far the most significant undertaking of Descartes in the field of geometry. It unified his early work in terms of finding a universal method of resolution for geometric problems. It established a clear link between algebra and geometry, and allowed Descartes to demonstrate his ideas through the example of Pappus' problem.

In the *Géométrie*, Descartes commented that we should not take for granted the teach-

¹⁸The other two being *Dioptriques* (written in 1635) and *Météores* (1636). The complete work was initially published anonymously [Descartes, 1997, p.xiii].

ings of others, but that our knowledge should be built upon truths we have come to discover for ourselves. This was a long held belief of Descartes' and very much echoes his statements from the *Regulae*. Descartes wrote in the *Regulae* as well as the *Meditations* (1641) [Descartes, 1641] that "from the earliest period of his youth he had acquired false opinions from others, mostly teachers" and that his "analytic method was the tool with which he proposed to dispel his skeptical doubt and provide a secure foundation for knowledge" [Lenoir, 1979, p.371].

In the first book of the *Géométrie*, Descartes demonstrated that the roots of algebraical equations of first and second degree involving a single variable can be constructed geometrically using ruler and compasses, the basis for which had been established in Books 2, 5, and 6 of Euclid's *Elements*. Similar ideas had been presented previously by Viète, but it was Descartes' intention to devise geometric techniques for finding the roots of equations in two unknowns.

Lenoir suggests that the issue for Descartes here was *extending* the class of objects that could be included in geometry, rather than to identify the domains of algebra and geometry [Lenoir, 1979, p.358]. Here we observe that whilst both Descartes and Viète had tried to define the field of geometry, Descartes had also succeeded in expanding that field. As a consequence of developing such geometric techniques, for example, by implementing instruments such as his mesolabe and the notion of single continuous motion, Descartes simultaneously identified a new boundary for geometry, namely, its restriction to algebraic curves.

As we saw previously, construction had always played an important role for Descartes. Without these precise geometric methods, algebraic equations were essentially meaningless. Here we note that for Descartes the *method* of construction remained particularly important, just as it had been in both ancient and early modern times.

Descartes had already devised some instruments, such as his mesolabum and his sliding curve construction, which satisfied his criteria for constructing *geometric curves*. He restated these methods in the *Géométrie*. Further, whilst Descartes was often critical of his predecessors, both ancient and early modern, he wrote these familiar words:

[I]t is then as great a mistake to try to construct it [a solid problem] by using only circles and straight lines as it is to use the conic sections to construct a problem requiring only circles; for any evidence of ignorance may be termed a mistake. [Descartes, 1954, p.180]

It was as much an error for Descartes, as it had been for Pappus, to try to construct a geometric problem with means outside of its class. However, Descartes went further than either Pappus or Viète had been able to with both his classification of problems, and in finding a method of analysis that could determine to which class a problem belonged.

In his essay on the structure of the *Géométrie*, Bos suggests a twofold approach to understanding its content. Firstly, the *technical* aspect, that is, Descartes' approach to developing an algebraic analysis. And secondly, Descartes' *methodological* approach to finding appropriate means of construction, especially when ruler and compasses were insufficient [Bos, 1993, p.43]. We heed this advice as we consider the *Géométrie* in the context of our own research questions.

Briefly, the *Géométrie* is divided into three parts, or Books. The first of these concerns the analysis of plane problems. Here Descartes explains how the arithmetic operations are related to the operations of geometry, and he tackles the principle of homogeneity. The method of construction is essentially Euclidean. Descartes also introduces the Pappus problem which features prominently throughout the work.

In the second book, Descartes focuses on the classification of curves, and which curves should be accepted into geometry. He again uses Pappus' problem to demonstrate his ideas and concentrates on the resolution of loci problems. Descartes determines that geometrical problems should be reduced to a set of standard constructions. He introduces means for describing curves of higher order, including his mesolabum instrument. Descartes also addresses methods which he finds geometrically unacceptable, but may be utilised in other areas, such as the study of optics.

In the final book, Descartes concentrates on the construction of problems which he classified as *solid* or *supersolid*, that is, problems which result in algebraic equations of degree higher than four. He also describes a means of determining which curves are *simpler* than others, thus establishing a hierarchy of curves.

3.3.1 Algebraic operations and homogeneity

Descartes' first task in the *Géométrie* was to explain how to apply algebraic operations (addition, subtraction, multiplication, division, and root extraction) to line segments. Addition and subtraction were the intuitive joining and removing of line segments. Classically, the multiplication and division of line segments a, b , would form a rectangle of sides a, b , and a ratio $a : b$, respectively. Early modern interpretations generally followed this order of dimensional arithmetic. Viète had chosen to generalise this interpretation to higher abstract dimensions, avoiding the need for the introduction of a unit. Descartes opted to define a unit line segment, which may be chosen arbitrarily, in order to “relate it as closely as possible to numbers” [Descartes, 1954, p.2].

Having identified an arbitrary unit segment, e , Descartes showed how to regard the product of two lengths as a length, thus making all algebraically combined segments lengths. His method was as follows. Given two line segments, a and b , we are required to find a fourth line segment, c , such that c “shall be to one of the given lines as the other is to unity” [Descartes, 1954, p.2]. That is, to find c such that

$$c : a = b : e.$$

Division is then to find c such that

$$c : a = e : b.$$

The extraction of roots is to find “one, two, or several mean proportionals between unity and some other line”, which correspond to finding the square, cube, ... roots of the given line [Descartes, 1954, p.5]. For example, the square root of the line segment a is given by x where

$$e : x = x : a.$$

Descartes gave geometrical constructions of these operations for the “sake of greater clearness”, but reserved a more detailed explanation of the extraction of the cube and

higher order roots for later. Any number of mean proportionals may be geometrically constructed with Descartes' mesolabum instrument, for instance, but he does not introduce this until book 2. At this point, he insists that "it is not necessary to draw the lines on paper, but it is sufficient to designate each by a single letter" [Descartes, 1954, p.5].

Descartes gave examples of a notation much more advanced than Viète's. His method of applying algebraic operations to geometric magnitudes meant that he could successfully avoid the concept of dimension and avoid the need for a *law of homogeneity* as Viète had required. It would be possible to envisage geometrical problems and curves of arbitrarily high degree.

However, and as Bos points out, establishing the relationship between curves and equations becomes somewhat of a side issue in the *Géométrie*. Descartes retains a focus on demarcating geometrical correctness. For Descartes, geometry was concerned only with those curves which could be represented by algebraic equations [Bos, 1993, pp.37–38].

3.3.2 Acceptable curves and their generation

In book 2 of the *Géométrie* Descartes tackled the question of which curves were permissible in geometry and how they should be constructed. Descartes sought to be able to reduce any geometrical problem to a standard construction. This was achieved by the determination and reduction of its equation, and then by constructing the solution using methods which Descartes saw as geometrically appropriate.

An algebraic equation alone was not sufficient for a curve to be admitted into geometry. A geometric construction was also required, and furthermore Descartes imposed the condition that curves should be described by "continuous motion" [Descartes, 1954, p.152]. Descartes had achieved this with the instruments we have described above.

Other mathematicians, such as Fermat and Roberval, had ventured into the description of curves of degree higher than four. However, Descartes was the first to achieve a construction which satisfied his criterion of singular continuous motion, although his method is restricted to particular curves. As Whiteside has identified, starting with a straight line a single species of conic (an hyperbola) is produced, and from a parabola a single cubic (the Cartesian trident) [MP, 1967–1981, 2, p.9, note 22]. A method for

describing a general conic would not be found until Newton, who achieved this with his organic rulers.

These curves coincided precisely with those having algebraic equations, and whilst Descartes claimed that if a curve was algebraic then it could be constructed in a “geometrical” way, this remained unproven [Domski, 2010, p.69; p.70]. The general problem of constructing algebraic curves by linkages was solved by Kempe in the 19th century [Kempe, 1876].

It is important to note that Descartes’ interpretation of this demarcation was not based on algebraic arguments. His criterion for acceptability was to be based on the manner by which the curve was traced.

Molland contrasts what he calls the “modes of specification”, noting that for both Descartes and the ancients, a distinction is made between these modes. For example, in the generation of curves, Descartes distinguishes between “specification by property”, that is, the equation by which a curve is given, and “specification by genesis”, or the construction method of a curve [Molland, 1976, p.22]. He comments further that the distinction between geometry and mechanics made by the Greek geometers, and especially the distinction between geometrical and instrumental, may have been “blurred or misinterpreted” by Descartes [Molland, 1976, p.33]. In his discussion of Descartes’ misinterpretation of an ancient distinction between geometrical and mechanical, Molland notes that this in fact allows for an easier, more natural introduction by Descartes of his own foundations for geometry based on certain articulated instruments [Molland, 1976, p.36]. He further identifies that “Descartes regarded the distinguishing characteristic of geometry as opposed to mechanics as being that the geometrical was “precise and exact” [Molland, 1976, p.37].

Commenting in Book 2 on the classification of curves given by Pappus, and which Descartes saw as insufficient, he says:

I am surprised, however, that they did not go further, and distinguish between different degrees of these more complex curves, nor do I see why they called the latter mechanical, rather than geometrical.

If we say that they are called mechanical because some sort of instrument has to be used to describe them, then we must, to be consistent, reject circles

and straight lines, since these cannot be described on paper without the use of compasses and a ruler, which may also be termed instruments. It is not because the other instruments being more complicated than the ruler and compasses, are therefore less accurate, for if this was so they would have to be excluded from mechanics, in which the accuracy of construction is even more important than in geometry. [Descartes, 1954, p.40–43]

First we note the repeated use of the word “mechanical”. In the previous chapter we identified its use by both Viète and Kepler. However, Descartes’ view is quite different. He says that he does not understand the rejection of any curve on the grounds that it needs to be constructed by an instrument since he sees rulers and compasses as mechanical instruments. In a sense, Descartes saw his instruments as being generalised compasses.

Descartes thought of the subject of mechanics as requiring more precision than geometry, which suggests that his rejection of certain methods not arising from a “single continuous motion” was not on the basis of accuracy. In a sense, this could be considered quite a Platonic way of thinking about geometry. Descartes was not so concerned with the representation of a figure, but the ideal figure.

For Descartes the classification of constructions would adhere to rules based on the type of motion required to trace the curve. For example, he objected to the quadratrix on the grounds that it required both circular and linear motions which could not be strictly co-ordinated by one motion, because this would amount to a rectification of the circumference of a circle, which he believed “cannot be discovered by human minds” [Descartes, 1954, p.91].

We described above Descartes’ mesolabum, which he had devised early in his career and included in the *Cogitationes privatae*. He included it in an early part of Book 2 of the *Géométrie* [Descartes, 1954, p.44–47]. Having described the instrument, Descartes makes a number of statements about the degree of the curve that may be constructed which he says may go on to “infinity” and that “these statements are easily proved by actual calculation.” He continues

I see no reason why the description of the first [curve *AD*] cannot be conceived as clearly and distinctly as that of the circle, or at least as that of

the conic sections; or why that of the second $[AF]$, third $[AH]$, or any other that can thus be described, cannot be as clearly conceived of as the first: and therefore I see no reason why they should not be used in the same way in the solution of geometric problems. [Descartes, 1954, p.47]

Aside from the mesolabum, Descartes described just one other “instrument” in the *Géométrie*, possibly inspired by his work on the Pappus problem. The instrument is described as follows (Figure 3.4) [Descartes, 1954, p.51]:

Instrument 3.3.1 *Suppose the curve EC to be described by the intersection of the ruler GL and the rectilinear plane figure $CNKL$, whose side KN is produced indefinitely in the direction of C , and which, being moved in the same plane is such a way that its side KL always coincides with some part of the line BA (produced in both directions), imparts to the ruler GL a rotary motion about G (the ruler being hinged to the figure $CNKL$ at L).*

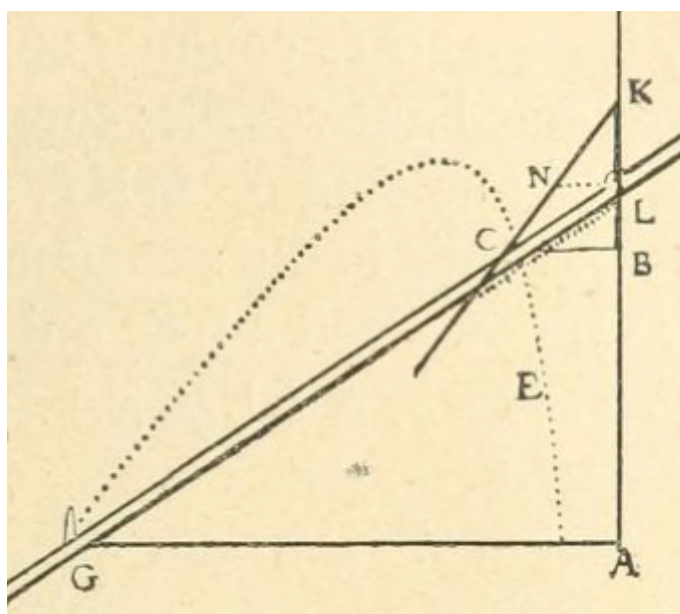


Figure 3.4: Descartes' moving ruler [AT, 1901, 6, p.393]

The procedure—which is somewhat implausible as an instrument—works by way of a rotating ruler and moving curve. A ruler rotates about a fixed point, the motion of which is determined by its intersection with a curve which may slide up and down a plane on a vertical axis. In the first instance Descartes describes a rectilinear plane figure $CNKL$, which may move up and down such that “ KL always coincides with some part of the line

BA (produced in both directions)” [Descartes, 1954, p.51]. Following this Descartes shows (by analytical methods) how to determine to which “class” the produced curve belongs.

Descartes goes on to explain that we may replace the rectilinear figure *CNK* with some other plane curve of the “first class” (that is, some other conic section including circles). Its intersection with the ruler *GL* will then describe a curve of the “second class”. He says, for example, “if *CNK* be a circle having its centre at *L*, we shall describe the first conchoid of the ancients, while if we use a parabola having *KB* as axis we shall describe the curve which [...] is the first and simplest of the curves required in the problem of Pappus [...] when five lines are given in position”. Descartes continues to say that the “first class” curve may be replaced with a “second class” curve to produce a “third class curve”, or a third class to produce a fourth class, “and so on to infinity. These statements are easily proved by actual calculation” [Descartes, 1954, pp.55–56]. (We describe the mapping effected by Descartes’ procedure in Appendix C.)

Descartes’ criterion of single continuous motion also meant the exclusion of transcendental curves, which Descartes classified as “mechanical”. In the *Géométrie*, transcendental curves, such as the quadratrix and the spiral, were to be excluded from geometry since “they must be conceived of as described by two separate movements whose relation does not admit of exact determination” [Descartes, 1954, p.44].

Lenoir suggests that Descartes’ motivation for such an exclusion is revealed by his consideration of a problem, proposed to him by Florimond De Beaune (via Mersenne), involving the construction of a logarithmic curve [Lenoir, 1979, p.360]. De Beaune had circulated a number of such problems around the Parisian mathematical circles. The particular problem which Descartes sought to resolve required the determination of a curve from a given property of its tangents. The problem is this:

Problem 3.3.2 *Being given any straight line b whatsoever, draw two infinite straight lines BK and LQ which cut one another at a point A forming an angle of 45° . It is required to construct the curve AXO such that, if from any point X taken at random on that curve, the tangent XG be drawn, as well as the ordinate XY with respect to the axis, the ratio between XY and GY will be constantly equal to the ratio of the given line b to the segment of the ordinate XI .*

That is, it was required to find a construction for the locus of points satisfying

$$XY : GY = b : XI.$$

The solution led to a transcendental curve (given by the equation $x = y + b + bCe^{-y/b}$, where C is an arbitrary constant) which could not be produced by Descartes' algebraic methods detailed in the *Géométrie*. Descartes gave two solutions, neither of which he found satisfactory. His first method was to generate a curve by a pointwise approximation, each point being the intersection of two infinitely close tangents.

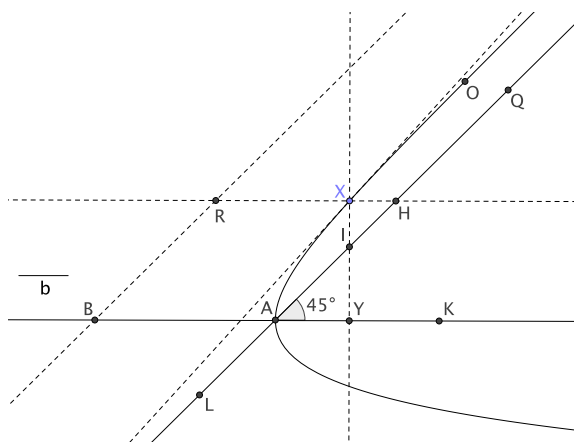


Figure 3.5: Descartes' solution of Debeaune's problem

Descartes second approach was to devise a mechanical method for giving a precise construction of the curve. His method was to suppose the intersection of two lines AH and AB , and moving BR and RH , respectively (Figure 3.5). The problem, however, was to determine the speed of the two motions [Lenoir, 1979, p.363]. If one is considered to have constant velocity, then the other would have velocity inversely proportional to $1 - s$, where s is the distance traversed by the first motion [Bos, 2001, p.421]. Descartes therefore concluded that

I suspect that these two movements are incommensurable to such an extent that it will never be possible for one to regulate the other exactly, and thus this curve is one of those which I excluded from my *Geometry* as being mechanical; hence I am not surprised that I have not been able to solve the

problem in any other way than I have given here, for it is not a geometrical line. [AT, 1964–1974, 2, p.517] in [Lenoir, 1979, pp.362–363]¹⁹

Whilst Descartes had quite sufficient means to generate a series approximation, the curve could not be constructed in a manner that he could consider to be sufficiently geometric. His only conclusion could be that the problem was not geometrical. By the time Descartes addressed De Beaune's problem he had firmly established, both in his own mind and in the *Géométrie* the connection between algebraic and geometric curves. It may not have satisfied him to have his methods reaffirmed by DeBaune's problem, for over the next few years Descartes dedicated much of his attention to such problems. Bos suggests that the appearance of these problems so soon after the publication of the *Géométrie* “was almost symbolic” for it “foreshadowed the turn mathematicians were soon to make toward problems that fell outside the domain and the power of Descartes' new methods” [Bos, 2001, p.421]. We note also that Descartes' “method of normals” [Maronne, 2010, pp.461–463], included at the end of Book 2 of the *Géométrie*, does not work for transcendental curves.

Having determined which curves were allowable in geometry, Descartes defined a hierarchical structure of simplicity. To put it plainly, if the degree of the defining equation was lower, then the curve was thought to be simpler.

While it is true that every curve which can be described by a continuous motion should be recognised in geometry, this does not mean that we should use at random the first one that we meet in the construction of a given problem. We should always choose with care the simplest curve that can be used in the solution of a problem, but it should be noted that the simplest means not merely the one most easily described, nor the one that leads to the easiest demonstration or construction of the problem, but rather the one of the simplest class that can be used to determine the required quantity.[Descartes, 1954, pp.154–155]

Descartes exemplified this by showing how to find two mean proportionals using his mesolabum and cutting the generated curve with a circle. He stated that the procedure may be generalised to any number of mean proportionals by replacing the circle with an

¹⁹See also [Bos, 2001, p.421]; [Whiteside, 1961, pp.368–370]; [Scriba, 1961].

appropriate *class* of curve. Descartes reiterated that it would be a “geometric error” to use a curve of higher degree where a lower class curve will suffice. These are precisely the errors which Pappus spoke of, but Descartes aimed to help us avoid them. We observe that Descartes’ solution for avoiding a “geometric error” was by algebraic means [Descartes, 1954, p.156].

Descartes dedicated several pages to identifying the number and types of roots of an equation before describing his method for avoiding a “geometric error”, showing, for example, how to reduce a cubic when the problem is plane [Descartes, 1954, p.175]. As we identify in the next chapter, Newton would later criticise Descartes in his algebraic classification of simplicity on the basis that a curve with a somewhat complicated equation may be generated by some simple motion.

3.3.3 Supersolid problems and the role of Pappus’ problem

In his examination of Pappus’ classification of geometrical problems Descartes found the distinctions between plane, solid, and linear to be insufficient. He saw that through his own definition of “geometrical” problems, he could include a whole sub-set of problems that would have been classed as “linear” by Pappus. These were what he called “super-solid”, that is, problems whose equations were of degree greater than four.

Both of the instruments included by Descartes in the *Géométrie* could, in theory, produce curves of arbitrary degree. His turning ruler and sliding curve procedure—virtually implicit in Descartes’ resolution of Pappus’ problem—could, in particular be used to describe curves of any order.

For example, in his resolution of the five-line Pappus problem, which we described above, and which Descartes also tackles in the *Géométrie*, Descartes describes a cubic curve with equation $xy = ax^3 + bx^2 + cx + d$ (known today as the Cartesian parabola). The curve is formed by the intersection of a parabola sliding along a vertical axis and its intersection with a ruler rotating about a fixed point.

Descartes’ idea was that the new curve (the Cartesian parabola) could then be taken as the starting curve to produce a curve of degree four, and so on. This was precisely the same principle as Descartes had developed in his early work on Pappus’ problem in 1631, but at that time he had not given the full details.

The Pappus problem played a crucial role in the *Géométrie*. Descartes used it as an example both to test his methods and to demonstrate his superiority over the ancients. He was critical of his predecessors for what he viewed as a lack of methodology, and suggested that they would have been more successful if they had managed to reduce their problems to standard constructions, as he had.

These same roots can be found by many other methods, I have given these very simple ones to show that it is possible to construct all the problems of ordinary geometry by doing no more than the little covered in the four figures that I have explained. This is one thing which I believe the ancient mathematicians did not observe, for otherwise they would not have put so much labour into writing so many books in which the very sequence of the propositions shows that they did not have a sure method of finding it at all, but rather gathered together all those propositions on which they happened by accident.

This is also evident from what Pappus has done in the beginning of his seventh book, where, after devoting considerable space to an enumeration of the books on geometry written by his predecessors, [Pappus] finally refers to a question which he says that neither Euclid nor Apollonius nor any one else had been able to solve completely. [Descartes, 1954, p.17]

Descartes explained that it would be possible to extend the Pappus problem to any number of lines. Classically, the problem had been limited to six lines so that the ratio of quantities to be compared could be envisaged as volumes of solid figures.

Descartes recalls Pappus' comments on the absurdity of considering more than three dimensions:

But if there be more than six lines, we cannot say whether a ratio of something contained by four lines is given to that which is contained by the rest, since there is no figure of more than three dimensions.²⁰

To which he responds:

[I]f there be seven [lines] that the product obtained by multiplying four of them together shall bear a given ratio to the product of the other three [...] Thus the question admits of extension to any number of lines. [Descartes, 1954, p.22]

²⁰From Pappus' *Collectio*, Book 2 as quoted by Descartes in [Descartes, 1954, p.21], and translated by the editors in note 33.

As we discussed earlier, Descartes was able to do this because he had essentially discarded the idea that the product of line segments results in a change of dimension. Bos argues that the study of Pappus' problem convinced Descartes more than anything else of the power of algebraic methods [Bos, 2001, ch 19; ch 23]. Indeed, Descartes claimed that every algebraic curve is the solution of a Pappus problem of n lines. As we shall see in the next chapter, Newton later showed this to be conclusively false. In doing so he greatly undermined a foundational principle of Descartes' view of geometry, and revealed the limitations of Descartes' attempt to strictly define geometry.

3.4 Reception of Descartes' *Géométrie*

The initial reaction to the *Géométrie* was relatively muted. Its circulation was small at first, with just a few amateur Dutch mathematicians reading the treatise.

One of the first commentators on the *Géométrie* was the French mathematician Florimond de Beaune. De Beaune's annotations were first published in van Schooten's 1649 translation of the work [Descartes, 1649], in which much attention was paid to the dependence of products of line segments on the unit. De Beaune gave examples showing that Descartes' interpretation of the operations were compatible with the results of a dimensional interpretation, and concluded that it would be best if the unit were left undetermined and calculations performed with homogeneous formulas unless a unit measure was explicitly given [Bos, 2001, pp.300–301]. Descartes received a copy of de Beaune's notes, and he expressed his appreciation to de Beaune in a letter [AT, 1964–1974, 2, pp.45–50].

Van Schooten, who had assisted in the publication of the first edition, translated the French text into Latin, and added extensive commentary and explanatory notes. Van Schooten's edition helped considerably with the spread of Descartes' ideas on geometry.

However, Descartes' ideas were not uniformly well received. First, Fermat noticed similarities between his own method of extreme values and Descartes' tangent method. This was communicated via Mersenne [AT, 1964–1974, 1, pp.481–486; pp.486–496; pp.499–504]. Next came attacks from Roberval and Beaugrand who accused Descartes of plagiarism (of Viète and Harriot) and algebraic incompetence [AT, 1964–1974, 2, p.82; pp.103–115; pp.457–461; pp.508–509]. However, Bos comments that these accusations

were due to “elementary misunderstandings and misinterpretations” [Bos, 2001, p.417].

Bos also notes an overall “lack of interest in Descartes’ program for geometry” [Bos, 2001, p.417]. We maintain that it had been Descartes’ intention to *resolve* geometry, which he saw as extending to include only algebraic curves, by reducing it to a procedural method and standard set of constructions. However, the criticisms Descartes’ *Géométrie* received were limited to individual elements of his theory.

On the charge of plagiarism mentioned above, Descartes denied any influence from the work of Viète. He made curious and contradictory claims that he did not know of Viète’s work in the area of algebra applied to geometry. On the other hand, Descartes also remarked that Viète’s notations were confusing and used unnecessary geometric justifications.

We identified in section 3.0.2 that it remains unclear how much of Viète’s work was known to Descartes before he composed the *Géométrie*. Whilst we did not seek to offer a conclusion there, we point out some of the key differences between their work. We also noted that Viète had avoided the issue of construction methods, whereas Descartes had emphasised the use of instruments. In doing so he had been able to generalise two of the classical problems. Lenoir had argued that Descartes’ intentions were focused on the extension of the geometrical class of problems rather than on defining a domain of geometry, as Viète had done. We suggested further that whilst this may have been Descartes’ intention, the two aspects were interdependent (section 3.3). Finally, Descartes had chosen to adopt a unit, which allowed him to avoid the issue of dimension altogether.

Lenoir further points out that “in contrast to Viète and Fermat, Descartes claimed not to concern himself with the method of the ancients” [Lenoir, 1979, p.367]. In fact, we propose that it had been Descartes’ intention to transcend the ancients, by refusing to engage with the geometrical foundations that had been established by Pappus. As we noted this was in spite of occasional lapses, such as his use of Apollonius’ *Conics* in his early work on Pappus’ problem where he apparently deemed it acceptable (section 3.2.2).

Here we also note that Rabouin compares Descartes’ work with that of the late-16th century Flemish mathematician Simon Stevin, and claims that the development of Descartes’ ideas is very similar to that of Stevin. He points to Stevin’s *Arithmetique* (1585), where

Stevin “insisted on the fact that one could interpret algebraic powers in terms of geometric magnitudes in a continuous proportion and therefore proposed a very simple geometric schematism, in which there would be no need to escape from the three dimensions of everyday space” [Rabouin, 2010, p.438].

The points outlined above represent a rapid development of geometrical thought at this key period in mathematical history due to a reflection on the geometry of the ancients. Our examination of these developments provides a context from which to compare Newton’s exploration of and approach to geometry, in particular, his response to the Cartesian methods.

In the modern commentary on Descartes’ geometrical thought, there seems to have been a shift in opinion over the last thirty or forty years. Initially it was accepted that Descartes had arithmetised geometry (e.g. by Coolidge [Coolidge, 1936, p.242]), but in 1959 Boyer gave a counter argument suggesting that Descartes’ influence should be termed “geometrisation of algebra” [Boyer, 1959]. Boyer suggests that whilst it is in some part true that Descartes had arithmetised geometry, it is not the whole story. He proposes that the issue is much more complex, and claims that “Descartes had no intention of arithmetising geometry”, and that we might just as well interpret the intention of the *Géométrie* as “the translation of algebraic operations into geometry” [Boyer, 1959, p.390].

In support of his argument, Boyer comments that Descartes found Viète’s work obscure, and criticised it for “marking too great a separation of algebra from geometry”. Descartes had intended to show that both algebra and geometry were a description of magnitude. Boyer claims that in this way Descartes was “returning in thought to the ancient geometrical algebra, while at the same time he encouraged the development of symbolic forms of expression” [Boyer, 1959, p.391].

Molland agrees that it is not so straight forward to characterise the mathematical work of Descartes. He states that the view of Descartes as “inventor of analytical geometry” is “not satisfactory”.

Molland briefly outlines the situation at present which accounts for “strong ancient roots [in Descartes’] analytic geometry”. He argues that, as Boyer has stated, the achievement

may be more appropriately labelled the “geometrization of algebra” [Molland, 1976, p.22]. The view is also supported by Lenoir [Lenoir, 1979, p.356]. We also compare this with Guicciardini who suggests that Descartes was somewhat “forced” into making assertions about algebra and geometry that may not have been in line with his original intentions [Guicciardini, 2009, pp.64–65].

Finally, based on our reading of Descartes’ work and of secondary commentaries, here we conclude that Descartes approached geometry as a means to an end. It was something to be completed and “tidied” in order to be put to use in other sciences and the “mechanical arts”. In a letter to Mersenne (27 July, 1638) Descartes says:

I have resolved to quit only abstract geometry [...] in order to have all the more leisure to cultivate another sort of geometry, which proposes as its questions the explanation of all the phenomena of nature. [AT, 1964–1974, 2, p.268] in [Rodis-Lewis, 1998, p.125]

Chapter 4

The Geometry and Geometrical Thought of Isaac Newton

4.1 Introduction

In this chapter we will discuss the geometrical work of Isaac Newton in order to better understand and re-evaluate his approach to geometry. Here we build upon our examination of Descartes in the previous chapter, in particular observing the contrasting methods and approaches of Newton. We will first outline his early geometrical development and the works that influenced him. We also consider his own geometrical discoveries, and the way in which he presented them. For example, it is well known that his classification of cubic curves used ideas which were to be taken up in the 19th century by the creators of projective geometry. And, thanks to Arnol'd, Newton's lemma on the areas of oval figures is now much better appreciated [Arnol'd, 1990]. We will introduce these examples, along with his less well known but extraordinary work on the *organic construction*, which allowed him to perform what are now referred to as Cremona transformations to resolve singularities of plane algebraic curves.

Through examples such as these we will demonstrate our premise that for Newton geometry was not simply a branch of mathematics. Newton saw geometry as a way of doing mathematics, and he defended it fiercely, especially against the new Cartesian methods. In this chapter we will explore why Newton was so sceptical of what most mathematicians regarded as a powerful new development. This will lead us to consider Newton's methods of curve construction, his affinity with the ancient mathematicians, and his wish to uncover the mysterious analysis supposedly underlying their work.

As we discovered in the previous chapters, these were all important topics in early modern geometry, and we provide fundamental examples of Newton's challenge to the Cartesian methods that dominated this period¹. Newton also explored questions such as which problems were to be regarded as geometric and which methods might be allowable in their solution. We will see how he first learnt from and then later challenged and contested Descartes' *Géométrie* which was largely responsible for the introduction of algebraic methods and criteria. We shall observe how this debate continued and amplified the demarcation disputes which arose, originally, from the ancient focus on allowable rules of construction.

4.1.1 Historical context

Under the influence of the Renaissance, a European cultural period spanning the previous three hundred years, an interest in many aspects of classical life had evolved, from art and architecture to ancient scientific texts. As we saw in the previous chapters many of these ancient texts, including those in subjects such as natural philosophy and mathematics, were rediscovered and translated into Latin. Thanks to these restorations and translations, often with extensive commentaries, and helped also by the invention of the printing press, such texts became more easily accessible and were disseminated widely throughout Europe. For us, the most significant of these is Commandino's edition of Pappus' *Collectio*, which influenced the work of the early modern geometers (see chapter 2), and was also studied in depth by Newton. We will look at the importance of this particular text in Newton's endeavours to understand the methods of the ancients (section 4.3.2).

In the previous chapters we witnessed a dramatic shift in attitudes towards geometry. At first, an interest in the life and works of the ancients resulted in a renewed enthusiasm for geometry. We saw a focus on the classification of geometry, and attempts to look at geometry in a more complete way, which led Viète, and later Descartes, to develop algebraic methods of analysis, moving them away from the ancient style of geometry. In addition, the 17th century saw the emergence of projective methods, initially through the

¹According to David Gregory, Newton referred to people using Cartesian methods as the "bunglers of mathematics"! This dates to May 1708 [MS 346:96, Christ Church College, Oxford], first published in [Hiscock, 1937, p.42]. See also [MP, 1967–1981, 5, p.xii, note 6] and [Westfall, 1980, p.380].

study of perspective, and then in independent studies by Desargues [Desargues, 1639] and Kepler [Kepler, 1619] who had developed the idea of points and lines “at infinity”. Meanwhile, English and Scottish mathematicians such as Gregory, Halley, and Maclaurin continued to pursue a study of the ancients, distinguishing them from the continental geometers². These examples provide a representation of the geometrical exploration that surrounded Newton’s life and work.

4.1.2 Early life of Newton

Isaac Newton was born into a family of farmers in Woolsthorpe, Lincolnshire. His father, an uneducated but monied man, died before Newton’s birth in 1642. Two years later Newton’s mother Hannah married Barnabas Smith, a church minister of a nearby village, at which time the young Isaac was sent to live with his grandmother, Margery Ayscough.

After the death of Smith in 1653, Newton returned to live with his family, but soon left to attend the Free Grammar school in Grantham, some five miles away, where he stayed with a local family. However, Newton did not thrive at school, having been described as ‘idle’, and his mother requested his return in order to manage the family farm, an occupation in which he showed very little interest. This experience can be perceived as contrasting significantly to the education that we observed in the early life of Descartes. Later, and thanks to Isaac’s uncle, William Ayscough, he was allowed to return to his school in 1660, this time lodging with the school’s Headmaster, Henry Stokes. Stokes must have identified some academic potential in Newton as he later persuaded Isaac’s mother to allow him to attend university [Westfall, 1980, p.55].

In 1661 Newton entered Trinity College, Cambridge, where he followed a traditional curriculum. During his first years there Newton kept brief notes of his studies, mainly of Aristotelian texts as was usual for Cambridge undergraduates at the time. In his final undergraduate year Newton began to keep records of his own independent studies in a notebook which he entitled *Quæstiones quædam Philosophicæ*³. Newton noted down ideas and questions related to his studies of natural philosophy, including Descartes’ *Prin-*

²See [Guicciardini, 1998, p.337] and [Guicciardini, 2003, pp.425–6].

³*Certain Philosophical Questions*, [MS Add. 3996:87r–135r, Cambridge University Library, Cambridge]. See also [McGuire and Tamny, 1983].

*cipia Philosophiæ*⁴ and various works of Boyle and Hobbes [McGuire and Tamny, 1983, p.24]. By the summer of 1664 Newton had begun to compose his own mathematical essays.[MP, 1967–1981, 1, pp.4–5].

Following a slow start to his education, Newton developed an interest in many aspects of both scientific and philosophical thought including physics, optics, alchemy, and theology. He studied what was known in the 17th century as *natural philosophy*, that is, a philosophical study of nature and the physical universe, which was the main study of ‘scientists’ before the development of modern science (the word “scientist” not coming into use until the mid-19th century). Branches of natural philosophy included physics, astronomy, and mechanics. For the purposes of this study we are, of course, mainly interested in Newton’s geometrical pursuits, whether they be pure and foundational or applied to other aspects of his work.

4.2 Early influences on Newton

Next we will survey the works that Newton studied during his education and the earliest part of his career. Newton’s mathematical learning was mainly independent of the prescribed education he received at Cambridge, and fortunately for us he made careful notes on the texts he consulted⁵. We are especially interested in Newton’s close reading of the works of Viète and Descartes and his reaction to the *new analysis*. In his early annotations, Newton rarely commented on the propositions, theorems, and constructions which he copied from these various texts. Whiteside takes this to mean that Newton was reproducing items that he found to be of particular interest, and we see no reason to doubt this [MP, 1967–1981, 1, p.11]. Whiteside’s examination will help us to build a picture of Newton’s early geometrical interests, and later, to highlight his changing views on geometry. For example, one of Newton’s early studies was Descartes’ *Géométrie*⁶, but the Cartesian methods were later fiercely rejected by him.

⁴[Descartes, 1644]. Originally written in Latin, a French edition was published in 1647, with a further Latin edition appearing in 1656.

⁵See, in particular, [MP, 1967–1981, 1, pp.25–142].

⁶Through Schooten’s second Latin edition (1659–1661).

We know very little of Newton's school curriculum. Some believe that Newton may have first been introduced to Euclid's *Elements* by Henry Stokes, the Headmaster of the Grantham Free Grammar school, which Newton attended from 1655 to 1660. The bylaws of the school as described in *A Concise Description of the Endowed Grammar Schools of England and Wales* state that the boys be "taught and instructed [. . .] in Greek, Latin, English, Writing, Mathematics and Arithmetic" [Carlisle, 1881, 1, p.810]⁷. Others suggest that there is evidence that Newton did not encounter Euclid before 1663⁸.

Following the completion of his school education, Newton entered Trinity College, Cambridge in July 1661 where his studies were dominated by the philosophy of Aristotle. However, in his third year, he found himself able to study the philosophy of Descartes, Gassendi, Hobbes, and Boyle, as well as the mechanics of the Copernican astronomy of Galileo and Kepler's Optics. Newton there began a notebook entitled *Quaestiones Quaedam Philosophicae*⁹ in which he recorded his thoughts.

De Moivre¹⁰ reports that Newton's interest in mathematics, and especially geometry, started in late 1663 when Newton could not understand the mathematics in an astrology book, and upon trying to learn trigonometry, he found he lacked the necessary skills in geometry, a problem which he intended to remedy by picking up a copy of Euclid's *Elements*. At first, finding the text self-evident, Newton's respect for the ancient geometry was not forthcoming, but in the same memorandum, De Moivre recounts how Newton

⁷Carlisle's record explicitly describes the teaching of "Euclid" and "Geometry" in other schools, but makes no remark to this effect at Grantham. However, a handwritten book dated 1654, and probably by Stokes, has since been discovered in Grantham Museum. The book contains the extraction of square and cube roots, geometry, and trigonometrical functions and may have been used by Stokes to teach the boys some mathematics.

⁸See the discussion in [MP, 1967–1981, 1, pp.6–7].

⁹*Certain Philosophical Questions*, [MS Add. 3996:88r–135r, Cambridge University Library, Cambridge]. He headed the book with the phrase *Amicus Plato amicus Aristoteles magis amica veritas*, "Plato is my friend, Aristotle is my friend, but my best friend is truth".

¹⁰De Moivre arrived in England, from France, in 1688 to escape religious persecution. In 1692 he became acquainted with Halley who was then assistant secretary to the Royal Society. Soon after, De Moivre also became close friends with Newton, probably through Halley, and was elected as a fellow of the Royal Society. In 1710 De Moivre was appointed by the Royal Society to review the calculus priority dispute between Newton and Leibniz. After Newton's death in 1727, his nephew by marriage John Conduitt wrote a biography of Newton, obtaining information from many of Newton's close friends. De Moivre had recounted many details of events that occurred well before he and Newton met. Bellhouse suggests that "[t]he information must have accumulated in De Moivre's memory from the several evenings over the years they had spent together at Slaughter's Coffeehouse"[Bellhouse, 2011, p.99]. See also *Memorandums relating to Sir Isaac Newton*, [Joseph Halle Schaffner Collection Ms. 1075.7, Joseph Halle Schaffner Collection, University of Chicago Library, Chicago, USA].

persisted with it and “began to change his mind when he read that *parallelograms upon the same base and between the same parallels are equal*, and that other proposition, that *in a right angled triangle the square of the hypotenuse is equal to the squares of the two other sides*”¹¹. Whilst De Moivre claims Newton to have then read Euclid a further time, John Conduitt gives a slightly different account with no mention of the ‘parallelogram’ anecdote or further readings of Euclid¹². It seems unlikely that Newton had paid such attention to Euclid as De Moivre suggests, or that if he had, he had certainly not appreciated it as he would much later. Newton’s unfamiliarity with Euclid is consistent with this oft-quoted remark made by Henry Pemberton¹³:

Of their taste, and form of demonstration Sir Isaac always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret for his mistake at the beginning of his mathematical studies, in applying himself to the works of Des Cartes and other algebraic writers, before he had considered the elements of Euclide with that attention, which so excellent a writer deserves.[Westfall, 1980, p.378]

and with Conduitt’s account of the occasion (April 1664) of Newton’s examination by Barrow in which he was asked of his knowledge of Euclid:

When he stood to be scholar of the house his tutour sent him to D^r Barrow then Mathematical professor to be examined, the D^r examined him in Euclid w^{ch} S^r I. had neglected & knew little or nothing of, & never asked him about Descartes’s Geometry w^{ch} he was master of S^r I. was too modest to mention it himself & D^r Barrow could not imagine that any one could have read that book without first being the master of Euclid, so that D^r Barrow conceived then but an indifferent opinion of him but however he was made scholar of the house.¹⁴

In contrast to Descartes’ Jesuit education which strongly focused on practical mathematics, Newton found that there was no formal mathematical instruction to be had at

¹¹ *Memorandums relating to Sir Isaac Newton*, [Joseph Halle Schaffner Collection Ms. 1075.7, 1r, Joseph Halle Schaffner Collection, University of Chicago Library, Chicago, Illinois, USA].

¹² *Draft account of Newton’s life at Cambridge*, [Keynes Ms. 130.04, King’s College, Cambridge].

¹³ A friend of Newton and editor of the third edition of *Principia Mathematica*.

¹⁴ *Anecdotes about Newton from various sources*, 31 August 1726, [Keynes Ms. 130.10, 2v, King’s College, Cambridge] in [Westfall, 1980, p.102]. Note that in [Keynes Ms. 130.04, 4r, King’s College, Cambridge] Conduitt gives a slightly altered account claiming that the event resulted in Newton being made scholar of the house in the following year.

Cambridge. Instead he was forced to pursue his new found interest in geometry in yet more independent consultation of texts such as Oughtred's *Clavis Mathematica* (1631), Schooten's recent two-volume *Geometria a Renato Des Cartes* (1659–1661), which contained appendices by three of Schooten's students, Jan de Witt, Johan Hudde, and Hendrick van Heuraet, as well as the *new analysis* in Schooten's collected works of Viète, *Viète Opera mathematica* (1646). In a memoir of July 1699 Newton recalls his reading material for the years 1663–1665:

July 4th 1699. By consulting an accept of my expenses at Cambridge in the years 1663 & 1664 I find that in y^e year 1664 a little before Christmas I being then senior Sophister, I bought Schooten's Miscellanies & Cartes's Geometry (having read this Geometry and Oughtreds Clavis above half a year before) & borrowed Wallis's works and by consequence made these Annotations out of Schooten & Wallis in winter between the years 1664 & 1665. At w^{ch} time I found the method of Infinite series. And in the summer 1665 being forced from Cambridge by the Plague I computed y^e area of y^e Hyperbola at Boothby in Lincolnshire to two & fifty figures by the same method. [MP, 1967–1981, 1, pp.7–8].

Aside from these geometrical texts, Newton also studied Wallis's *Arithmetica infinitorum* (1656), from which his first original text may have been derived. For example, on reading Wallis's method for finding a quadrature of the parabola and hyperbola which used indivisibles, Newton made comments on Wallis's treatment of series, devising proofs of his own, writing for example “Thus Wallis doth it, but it may be done thus [...]”¹⁵.

Perhaps Newton's greatest direct influence in matters of mathematics was Isaac Barrow, elected to the Lucasian chair in 1663. However, even he did not recognise Newton's mathematical genius immediately. Newton's most significant period of scientific work was to come in the following eighteen months between 1665–1666 when the University closed due to the Plague. On returning to Lincolnshire for this relatively short period, Newton made advances in several areas of science, including mathematics, optics, physics, and astronomy, and laid the foundations for his method of fluxions.

¹⁵*Mathematical notebook*, [MS Add. 4000, 17r, Cambridge University Library, Cambridge].

Barrow had strong views of his own on geometry and algebra. In his inaugural lecture series as Lucasian Chair in 1664–1665, and which Newton likely attended, he saw cause to give a survey of mathematics, and in doing so, condemned both John Wallis and Thomas Hobbes. He dismissed algebra as “only a Part or Species of Logic . . . an Instrument subservient to the Mathematics”¹⁶. Guicciardini comments how Barrow had been the “deepest influence” on Newton in his battle against the algebraists [Guicciardini, 2006, p.1730]. Feingold explains how after an initially warm reception of the Cartesian teachings, the Cambridge Platonists became wary of Descartes’ philosophies, theology, and metaphysics [Feingold, 1990, pp.25–27]. He notes further that Barrow, in particular, went from “enthusiastic, if discerning, advocate of Cartesianism while a student to opponent after the Restoration” [Feingold, 1990, p.28].

In 1662 Barrow became professor of geometry at Gresham College, and was elected as the first ever holder of the Lucasian Chair at Cambridge in 1663. Barrow published two mathematical works during his six years as Lucasian professor, one on geometry and one on optics. His lectures of 1664–1666 were published in 1683 as *Lectiones Mathematicae*. His 1667 lecture suggested the analysis by which Archimedes was led to his main results and was also published in 1683. In 1669 Barrow published his *Lectiones Opticae et Geometricae*, the portion being on Optics edited by Newton. In 1675 Barrow also published editions of various ancient Greek texts including Apollonius’ *On Conic Sections*, Books 1–4, and also of the extant works of Archimedes, which played a significant role in the development of Newton’s geometrical thought in the 1680s and 1690s.

At the end of the decade, Barrow was to take it upon himself to communicate Newton’s work to the mathematical community, sending Newton’s *De Analysi per æquationes numero terminorum infinitas* to John Collins. It was also in this year, 1669, that Newton was to begin the next significant phase of his career. Barrow resigned from his position as Lucasian Chair, recommending Newton as his successor.

¹⁶Isaac Barrow, *Lectiones Mathematicae* (1683), translated and reproduced by John Kirkby in [Kirkby, 1734], and quoted in [Willmoth, 1993, p.9]. See also [Mahoney, 1990, pp.200–201].

4.2.1 Newton's first commentaries on Viète, Schooten, and Descartes

Let us now look more closely at Newton's early annotations made during his final year at Cambridge. Newton's *Mathematical notebook*¹⁷ contains a number of geometrical propositions copied from the works of Viète, Schooten, Descartes, and Huygens, which Whiteside has carefully and systematically organised in [MP, 1967–1981, 1, pp.25–142]. In the introduction to the annotations he says, “[i]nvariably we are led back to examine those papers and they must remain our fundamental source of knowledge of Newton's mathematical development [...] They give us a vivid picture of Newton's likes and dislikes, of what he thought significant and what he passed lightly over in his reading” [MP, 1967–1981, 1, pp.10–11]. We use this as our guide from which to explore Newton's early view of geometry and its connections to our wider premise, namely, a re-evaluation of Newton's innovative approach to geometry.

Newton's actual annotations of the works are generally limited. He had a tendency to take factual copies, perhaps indicating which passages interested him most. Whiteside notes that even in the case of Descartes' *Géométrie*, “whose decisive influence on Newton is clear in his early research papers, explicit notes are few and those almost wholly refer to the appendices added by Schooten in his Latin editions” [MP, 1967–1981, 1, p.11]. He also identifies a bias towards analytical mathematics¹⁸ which is reflected in Newton's own early researches. We note also the absence of many whose works, Whiteside suggests, may also have been read in part by Newton: Napier, Briggs and Harriot; Desargues, Pascal and Fermat; Stevin, Girard and Kepler¹⁹. Perhaps most surprisingly of all is the absence of any Greek geometer except Euclid, and even in that case he is represented only by Barrow's translations of his *Elements* and *Data*.

The first entries in Newton's notebook of late 1664 are some simple Euclidean theorems taken from Oughtred's *Clavis Mathematica* [MP, 1967–1981, 1, p.25, note 3]. Following these he dedicates several pages to the description of solid loci giving various mechanical

¹⁷c.1664–1665, [MS Add. 4000, Cambridge University Library, Cambridge].

¹⁸Above all in Descartes and Wallis.

¹⁹Whiteside details why he thinks these may have influenced Newton in his notes on the relevant pages. See [MP, 1967–1981, 1, pp.13–14, notes 32–35].

descriptions, many from Schooten's *Exercitationes*, Book 4, and from de Witt's *Elementa Curvarum*, but also the description of a conic taken as a section from a cone (for example, [MP, 1967–1981, 1, p.31, e]), from Schooten's *Geometria*. In his notes on how to describe hyperbolae Newton even includes, as Whiteside supposes, an original construction and “probably his first reference to interpolating a continuous curve between given points on it by drawing the tangents at these points [...] or by freely joining up the constructed points with circle arcs or arbitrary portions of smooth curves drawn in with a *steady hand*” [MP, 1967–1981, 1, p.39, note 48]. It is interesting to see Newton collecting together such a range of methods of curve description, without any notable judgement. This following a period of intense reflection by his predecessors on the correct use of geometry, as identified in our previous chapters. This is a distinction between Newton and his predecessors which we will continue to see as his mathematical confidence develops. That is, a significantly reduced emphasis on the classification and demarcation of geometry along with the setting of theoretical foundations compared with Newton's increasingly pioneering nature.

In addition to his detailed study of Book 4, Newton also made many notes on the contents of Book 5 of Schooten's *Exercitationes* and the appended *De ratiocinis in ludo aleae* by Huygens²⁰, showing Newton's interest in many areas of mathematics aside from geometry. In the annotations, we also find several methods on the numerical extraction of roots, Newton having taken his cue from both Viète's *De Numerosa Potestatum Ad Exegesisin Resolutione* (1600) and Oughtred's *Clavis*²¹. The final section of Newton's annotations from Viète and Oughtred is perhaps of most interest to us in the context of this thesis, that is, Newton's notes on Viète's geometrical propositions taken from various works including Schooten's *Vieta*, which includes work taken from Viète's *Supplementum Geometriæ*²². Once again, almost all of the propositions are listed without additional comment from Newton. However, he makes one curious side note regarding three propositions taken from Viète's *Pseudo-Mesolabum*²³. Newton wrote in the margin: “prop 12. 13 & I think 11 are trew onely mechanically” [MP, 1967–1981, 1, p.77].

²⁰See [MP, 1967–1981, 1, pp.47–62]. Book 5 is titled *Sectiones triginta miscellaneas* and develops techniques for counting problems, for example, methods for finding ‘amicable numbers’. Huygens was among Schooten's students. His paper contained important work on probability.

²¹[MP, 1967–1981, 1, pp.63–71]. See also [MP, 1967–1981, 1, p.63, note 1].

²²See [MP, 1967–1981, 1, pp.72–88].

²³Again, likely from Schooten's *Vieta*. See [MP, 1967–1981, 1, pp.76–77].

Whether he has a strong opinion about “geometrical” and “mechanical” at this point or not, he is at least aware of the type of comments that would have been made by Viète, Schooten and Descartes. Newton leaves very few clues at this stage as to his thoughts on the matter. His ideas on geometry become much clearer towards the end of the decade, for example, in his *Errores Cartesij Geometriae* (late 1670s) [MP, 1967–1981, 4, pp.336–345]. Galuzzi comments on “another lemma” used by Newton when reading Jan de Witt’s *Elementa Curvarum Linearum* where “conics are considered in a way very different from that of Apollonius” which shows the influential nature the new Latin texts had on Newton’s early education and thought, compared with that of the ancient texts, which he, in fact, had little contact [Galuzzi, 2010, p.546]. Guicciardini also comments on Newton’s preference for mechanical description of conic sections rather than by slicing a cone with a plane because it gave a “continuity of tracing” [Guicciardini, 2009, p.73].

4.2.2 The influence of Descartes’ *Géométrie*

We have seen varied accounts of Newton’s first studies of Euclid, however his initial encounter with Descartes’ *Géométrie* has been reported on more consistently and seems to have been more successful²⁴. For example, in Newton’s own reflection of 4 July 1699 he recalls having bought “Cartes’s Geometry (having read this [...] above half a year before)” [MP, 1967–1981, 1, pp.7–8]. Both de Moivre and Conduitt, for example, report of his tackling the *Géométrie* a few pages at a time in 1663.

According to Conduitt’s account Newton had marked in his copy of the *Géométrie* notes such as “Error—Error non est Geom.” but Whiteside reports that a thorough study has cast some doubt on the origin of the comments, and that any such comments, if indeed made by Newton, may be attributed to “an excess of undergraduate callowness and incomprehension”. Furthermore, and as Whiteside identifies, the story conflicts with “the unchallengeable manuscript evidence of Descartes’ overwhelming influence on Newton’s early mathematical development” [MP, 1967–1981, 1, pp.17–18, note 11]. Nonetheless, in the same note, Whiteside confirms that Newton read and annotated the second Latin edition of the *Géométrie* (Leiden, 1659–1661), edited by Schooten, and with appendices

²⁴In that he was more persistent in his initial study of the text.

contributed by Schooten's students Hudde, Heuraet, and de Witt²⁵.

During Newton's first thorough studies of the *Géométrie* in 1664 he had concentrated on formulating general methods for determining diameters and asymptotes of a curve given by its Cartesian equation [MP, 1967–1981, 1, pp.155–212]. Over the next few months he developed his techniques of co-ordinate transformation before setting aside the work to concentrate on calculus, optics and astronomy. As Whiteside points out, Newton had reached a barrier in this area of research where very little was known about particular curves and even less about their basic properties. Descartes' had made a considerable effort to advance the knowledge of conics in the second book of his *Géométrie*, but the realm of cubics and higher degree curves remained very much untouched. Newton's original attempt at a classification of the cubics by way of simplifying a general equation was largely unsuccessful. Returning to the subject some three years later he found a way to reduce the general cubic equation in two variables by a transformation of co-ordinates [MP, 1967–1981, 2, pp.3–5]. We will give a more complete discussion of Newton's classification below (section 4.5).

Domski remarks that Newton's initial reaction to Descartes and the new analysis was more positive than his first experiences of Euclid, and that he even identified greater potential in the algebraic methods [Domski, 2010, p.70]. It is important to note that Descartes' *Géométrie* was largely responsible for Newton's own early interest in mathematics, and geometry in particular. It was not until the 1670s that he focused his attention on ancient geometrical methods and became dismissive of Cartesian geometry.

We shall see that in the mid 1670s Newton became sceptical of the algebraic methods, and the idea of an algebraic analysis no less so. We have seen the importance placed on the analytical stage of problem solving throughout the history discussed so far. By continuing to focus on this we are able to draw a clear distinction between the methods of Descartes and Newton.

²⁵Whiteside adds that whilst this is certainly the copy Newton had in 1664, the copy he purchased in the following winter may have been the 1649 first edition, which was found in the Portsmouth Collection in the University Library, Cambridge, but contains no annotations by Newton himself [MP, 1967–1981, 1, p.20; 1, p.21, note 7].

As we have seen, the second stage of problem solving, *synthesis*, in which a demonstration of the construction or solution is given, was a crucial requirement before a geometrical problem could be considered solved. Indeed, following the ancient geometers, early modern mathematicians usually removed all traces of the underlying analysis, leaving only the geometrical construction.

Of course, in many cases this geometrical construction was simply the reverse of the analysis, and Descartes tried to maintain this link between analysis and synthesis even when the analysis, in his case, was entirely algebraic. Newton argued, however, that this link was broken:

Through algebra you easily arrive at equations, but always to pass therefrom to the elegant constructions and demonstrations which usually result by means of the method of porisms is not so easy, nor is one's ingenuity and power of invention so greatly exercised and refined in this analysis.²⁶

There are two points being made here that are fundamental to Newton's rejection of the Cartesian methods. One is that the constructions arising from Cartesian analysis were anything but elegant, and that one should instead use the method of porisms, about which we will say more in a moment. The other is that the Cartesian procedures are algorithmic, and allow no room for the imagination.

4.3 Newton's changing view of Cartesian geometry

There are two distinctive periods in Newton's mathematical thinking. His early reading of Viète's new analysis and Descartes' *Géométrie* is reflected in an analytical period of Newton's mathematics up to around 1670. However, he had independently chosen to study the philosophies of Aristotle and Plato during his undergraduate studies, and had started writing on his philosophical thoughts. From the 1670s onward Newton focused his studies on the ancients, which reflects a transition to a more synthetic and geometric period in his thinking [Guicciardini, 2004c, p.455]. This change in approach was an abrupt one and coincided with a distinct anti-Cartesian feeling, which Newton expressed freely throughout his work, and for many years to come.

²⁶This dates from the 1690s. See [Guicciardini, 2009, p.102] and [MP, 1967–1981, 7, p.261].

In spite of the methods in Descartes' *Géométrie* becoming increasingly accepted by Newton's contemporaries, he believed that there not only could but should be a geometrical analysis. Early in his studies he mastered the new algebraic methods, and only later turned his attention to classical geometry, reading the works of Euclid and Apollonius, and Commandino's Latin translation of Pappus' *Collectio* of 1588. According to his friend Henry Pemberton, editor of the third edition of *Principia Mathematica*, Newton had a high regard for the classical geometers:

Of their taste, and form of demonstration Sir Isaac always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret for his mistake at the beginning of his mathematical studies, in applying himself to the works of Des Cartes and other algebraic writers, before he had considered the elements of Euclide with that attention, which so excellent a writer deserves. [Westfall, 1980, p.378]

It was from Pappus' work that Newton learned of what he believed to be the ancient method of analysis: the porisms. Guicciardini explores the possibility that Newton may have been trying to somehow recreate Euclid's work on porisms in order to identify ancient geometrical analysis [Guicciardini, 2009, p.82]. As we noted in chapter 2, agreement on precisely what the classical geometers meant by a porism is still elusive. However, as the early modern geometers understood it, the porisms required the construction of a locus satisfying set conditions, such as the ancient problem that came to be known as Pappus' Problem. We discuss Newton's interpretation below (4.4).

In the late 1670s Newton began his work on loci problems including the Pappus problem and he continued his work on the organic construction of curves. It is at this point that we start to see his anti-Cartesian feelings developing, especially in his *Errores Cartesij Geometriae* [MP, 1967–1981, 4, pp.336–345].

In his commentary of the *Mathematical Papers*, Whiteside notes that some historians “have been tempted to suppose (on little real evidence) that [Newton] had from the first a deep-felt sympathy with the rigour and elegance of Greek geometry”²⁷ [MP, 1967–1981,

²⁷Whiteside cites for example, [Huxley, 1959].

4, p.217]. He presumes such interpretations to have been “unduly influenced by what they take to be the Grecian façade of [the] *Principia*” and the oft-quoted remark of Pemberton²⁸, but strongly disagrees that Newton had always had such a close appreciation of the ancients. More recent commentaries²⁹ present a more balanced view and, along with a study of Newton’s own work and comments have allowed us to identify a distinct and abrupt divide between Newton’s favour of Cartesian and ancient geometries.

Up until his appointment as Lucasian chair in 1669, of the ancient geometers Newton had studied only Euclid’s *Elements* in any depth. Whiteside states that there is no evidence to suggest Newton had more than “the bare working knowledge of Apollonius and Archimedes he needed to get by in his own researches and professorial lectures” [MP, 1967–1981, 4, p.218]. Further, when Newton did come to study the ancients, whilst he had an adequate knowledge of Greek he preferred to make use of the Latin editions which were becoming more widely available³⁰. It was not until the late 1670s that Newton began to study in close detail the seventh and eighth books of the *Collectio* in Commandino’s Latin translation of the work³¹.

Whiteside suggests two “stimuli” for Newton’s newly found enthusiasm for the work of Pappus. Firstly, Newton had recently reacquainted himself with the *Géométrie*, where Descartes quotes directly from Commandino’s edition of Pappus’ *Collectio* with regard to the three- and four-line locus problem. The problem drew a great deal of attention from Newton, his study of which re-enforced his scepticism of the Cartesian methods. Whiteside concludes that even if Newton’s reading of the quoted passages in the *Géométrie* did not lead him directly to the original, it would not have taken him long to make the connection during his study of the seventh and eighth books of Pappus’ *Collectio* [MP, 1967–1981, 4, p.222].

Whatever his reasons for turning to Pappus, according to Whiteside Newton “became

²⁸That is, “Of their taste, and form of demonstration Sir Isaac always professed himself a great admirer”. Whiteside notes also that the immediately following phrase “I have heard him even censure himself for not following them yet more closely than he did” is less often quoted [Pemberton, 1728, Preface] in [MP, 1967–1981, 4, p.217, note 2].

²⁹See, for example, [Guicciardini, 2009].

³⁰Such as Barrow’s editions of *Elements* and later, Apollonius’s *Conics*, Book I–IV. For information on Newton’s library copies of these see [MP, 1967–1981, 1, p.11, note 26].

³¹[Pappus of Alexandria, 1588]. See also [MP, 1967–1981, 4, p.218, note 6].

deeply interested in the account given by Pappus in the preface to his seventh book of the lost works of Euclid, Aristaeus and Apollonius (notably, Euclid's three books of *Porisms* and Apollonius' two of *Plane Loci*, three on *Vergings*, two on *Determinate Sections* and two of *Contacts*), which, together with Euclid's extant *Data* and the full eight books of Apollonius' *Conics*, dealt with the 'analysed locus' "[MP, 1967–1981, 4, p.223].

Upon his study of the sketchily reconstructed works, Newton listed several simple loci and eventually a wider range of problems in his Waste Book³², focusing more closely on the geometrical analyses rather than their syntheses. However, as Whiteside identifies, Newton had not yet adopted a typically classical style, often making use of limiting arguments, and he claims that Newton's work at this point is incomparable with "contemporary restorers of the ancients' analysis [such as] Fermat" [MP, 1967–1981, 4, p.224]. In spite of their differences at this stage, Whiteside suggests that Fermat's reconstructions of Apollonius' *Plane Loci* and Euclid's *Porisms*³³, published posthumously in 1679, may too have served as inspiration for Newton's thorough researches into the ancient analysis. Whilst Newton never mentions Fermat's collected works, Whiteside points to "strong circumstantial evidence that he did, in fact, study these Apollonian and Euclidean reconstructions with some care"³⁴. In particular, Newton's synthetic solution to the three- and four-line locus problem can be strongly identified with propositions from Apollonius' *Conics*, book 3.

4.3.1 Before 1670

As we shall see, Newton developed strong opinions against many aspects of Cartesian thought, not in the least Descartes' ideas on geometry. Newton seems to have developed these views rapidly beginning around 1670, and we explore what led to this dramatic change in perspective.

Some time between his graduation from Trinity in 1665 and the end of the decade Newton accomplished some of his most exceptional work. In particular, in the period

³²Under the title *Quæstionum solutio Geometrica*, [MS Add. 4004:95v–96v, Cambridge University Library, Cambridge] in [MP, 1967–1981, 4, pp.230–269].

³³In [de Fermat, 1679, pp.28–43; pp.116–119].

³⁴See the relevant notes in [MP, 1967–1981, 4, p.225, note 26]. Westfall also suggests that Newton may have been influenced by these reconstructions of Fermat's as well as La Hire's treatise on solid loci, *Nouveaux élémens des sections coniques* (1679) [Westfall, 1980, p.378].

between 1665–1667 when Newton returned to Woolsthorpe to avoid the outbreak of plague in Cambridge, he established the fundamental principles of the calculus³⁵, and identified the heterogeneity of white light through its separation by refraction³⁶. It was also towards the end of this period that Newton turned his attention to the subject of gravitation.

Having developed the basic rules of finding fluxions whilst away from Cambridge in 1666, Newton's *De Analysi*³⁷ (1669) was a significant milestone on the road to calculus. It was followed in 1671 by *De methodis serierum et fluxionum* (On the methods of series and fluxions) [MP, 1967–1981, 3, pp.32–328]. Using his method of fluxions Newton produced simple analytical procedures that unified many separate techniques previously developed to solve apparently unrelated problems such as finding areas, tangents, the lengths of curves, and the maxima and minima of functions.

Arthur³⁸ comments, however, that Newton remained unsatisfied with the foundations of his methods [Arthur, 2008, p.14]. Shortly after completing the *De methodis*, Newton developed a wholly synthetic approach “based on the genesis of surfaces by their motion and flow” which he presented in *An addendum on the theory of geometrical fluxions* (1671) [MP, 1967–1981, 3, pp.328–353].

Throughout the 1670s Newton also began to focus more closely on Descartes' *Géométrie* and the Pappus problem, writing his criticisms of Descartes in *Errores Cartesij Geometria* (c.1678) [MP, 1967–1981, 4, pp.336–345]. It was during this same period that he reworked a solution to the Pappus problem³⁹, explicitly expressing his distaste for the Cartesian analytic solution and coming up with an entirely synthetic one based on Apollonian propositions.

Westfall suggests that “Nearly all of Newton's burst of mathematical activity in the period 1669–71 can be traced to external stimuli, to Barrow (armed initially with Mercator's work) and to Collins” but that “[h]is own interests had moved on” [Westfall, 1980, p.232]. By 1675 Collins reported in a letter to James Gregory (19 October, 1675) that Newton

³⁵October 1666, [MS Add. 3958.3:48v–63v, Cambridge University Library].

³⁶Ideas which he addressed to the Royal Society on 19 February 1671, [Newton, 1671].

³⁷*De Analysi per æquationes numero terminorum infinitas* (On analysis by infinite series). Communicated by Barrow to Wallis in a letter (31 July, 1669), [MS/81/4, Royal Society Library, London].

³⁸See also [Guicciardini, 2003, p.315].

³⁹A problem to which Descartes had dedicated much of the *Géométrie*.

was “intent upon Chemicall Studies and practises, and both he and Dr Barrow &c [were] beginning to thinke mathcall Speculations to grow at least nice and dry, if not somewhat barren . . .” [Turnbull et al., 1977, 1, p.356] in [Westfall, 1980, p.232].

4.3.2 After 1670

Newton’s rejection of Cartesian methods was deeply intertwined with his close study of the ancient texts. From his writings we will see that the more Newton studied the ancients, the more his anti-Cartesian feelings were reinforced, feelings which developed especially during the 1670s, and remained throughout his career. When Newton came to refocus his attention on classical geometry, one of his main points of study was Pappus’ *Collectio*⁴⁰, a text which he considered in detail and returned to many times, allowing it to guide him through the geometry of the ancients.

Throughout the 1670s Newton worked on several researches, which reaffirmed his rejection of Cartesian analysis. In particular, Newton developed a synthetic solution to the Pappus problem, which is closely related also to his method of organic description of curves. It was also around this time that Newton began a formal study of the cubic curves, although he had made some preliminary researches in the previous decade, for what would later become the *Enumeratio*. However, his most remarkable geometric achievement in this respect would not come until he took up the subject once more in the 1690s after a renewed focus on classical geometry. We will discuss both of these aspects of Newton’s work in the following sections (4.5 and 4.4).

Newton’s anti-Cartesianism grew with his study of and admiration for the ancients. Domski suggests that what Descartes had identified as the *inability* of the ancients to combine arithmetic methods with geometry, Newton interpreted as a conscious and philosophically motivated distinction [Domski, 2010, p.71].

Guicciardini proposes that Newton’s rejection was of everything Cartesian, not just geometry and states that “Cartesian mathematics had to be refused just as much as Cartesian

⁴⁰See, for example, [Guicciardini, 2003, p.425]: “[i]n his numerous manuscripts related to the method of analysis and synthesis, Newton often quoted from the introduction to the seventh book of Pappus’ *Collectio*.”

philosophy” [Guicciardini, 1998, pp.315–6]. He notes also that Descartes presented his resolution of problems in “aggressively anti-classicist terms” [Guicciardini, 2006, p.1727]. Both of these interpretations would suggest that Newton’s anti-Cartesianism was rooted not just in his own admiration for the ancients, but also in Descartes’ rejection of them. In his unpublished *Geometria curvilinear* (c.1680), Newton criticised the “men of recent times” who “eager to add to the discoveries of the ancients, have united the arithmetic of variables with geometry” [MP, 1967–1981, 4, p.421]. Even in his *Lectures on Algebra*⁴¹ Newton freely expressed what he saw as the virtues of geometry compared with algebra:

“[. . .] for anyone who examines the constructions of problems by straight line and circle devised by the first geometers will readily perceive that geometry was contrived as a means of escaping the tediousness of calculation by the ready drawing of lines.”[MP, 1967–1981, 5, p.429]

Guicciardini also comments that whilst Newton “developed a more and more acute hostility towards modern analytics” it would be “excessive to say that Newton abandoned completely the “new analysis” that he had developed in his *anni mirabiles*” [Guicciardini, 2003, p.415]. Guicciardini highlights Newton’s mathematical achievements in algebra which are exemplified in the *Arithmetica universalis* (1707). As we noted above the *Arithmetica* was based on Newton’s lectures on algebra, supposed to have been given between 1673–1683⁴². However, even here, Newton took several opportunities to express his view of the superior nature of geometry compared with algebra, and to reinforce his esteem for the ancients [MP, 1967–1981, 5, pp.423–427].

In particular, Newton made it quite clear that he rejected the idea that a curve could be defined by its equation, and that certain curves might be admitted into geometry according to the degree or simplicity of their equations. He says, for example, that “the equation to a parabola is simpler than that to a circle, and yet because of its simpler construction the circle is given prior admission” [MP, 1967–1981, 5, p.425]. On the inclusion of particular curves he says “[e]ither, then, we are, with the Ancients, to exclude from geometry all

⁴¹Newton’s *Lectures on Algebra, 1673–1683* [MP, 1967–1981, 5, pp.54–517]. What would eventually be, in revised form, Newton’s *Arithmetica universalis* (1707). See also [MP, 1967–1981, 5, p.54, note 1] for Whiteside’s account of the transmission of the papers.

⁴²There is no record of the lectures ever having been presented [MP, 1967–1981, 5, p.54, note 1]

lines except the straight line and maybe the conics, or we are to admit them all according to the simplicity of their description” [MP, 1967–1981, 5, p.425].

Pappus’ remark on the classification of problems (and on the use of means appropriate to each class) depended upon having a clear criterion for the simplicity of a construction. Here Descartes adopted an unequivocally algebraic view: simplicity was defined by the degree of the equation. In the *Lectures on Algebra* Newton says:

In contemplating curves and deriving their properties I commend their distinction into classes in line with the dimensions of the equations by which they are defined. Yet it is not its equation but its description which produces a geometrical curve. [MP, 1967–1981, 5, p.425]

Guicciardini argues that Newton was in a weak position when he criticised this criterion, because Newton’s arguments were based on aesthetic judgements, while Descartes’ criterion, whether rightly or wrongly, was at least precise [Guicciardini, 2009, p.104]. We note, however, that it explains the apparent contradiction between Newton’s use of the degree of a curve and his criticism of Descartes.

Guicciardini suggests that the sentiments expressed by Newton here are continued in the first lines of the preface to the *Principia*, namely that “geometrical objects should be conceived of as generated mechanically, that geometry is subsumed under mechanics” [Guicciardini, 2003, p.416]. He continues

Newton affirms that geometry is founded upon mechanical practice and that it is part of universal mechanics. He also denies that exactness appertains exclusively to geometry: quite the contrary, geometry receives its exactness from mechanical practice. [...] Newton was convinced that studying geometrical magnitudes in terms of their mechanical construction opened access to a much more general approach than Descartes’.[Guicciardini, 2003, p.417]

In the *Principia* Newton distinguishes between “rational” and “practical” mechanics, defining the relationship between rational mechanics and geometry, in particular, he claims that geometry is founded upon mechanics, and that the “description of straight lines and circles, which is the foundation of geometry, appertains to mechanics” [Newton, 1999, p.381]⁴³. Newton eventually came to completely reject Descartes’ distinction between

⁴³See also [Guicciardini, 2004b].

“geometrical” and “mechanical” curves. In particular, he objected to the idea that geometry could somehow be restricted to curves that could be constructed by ruler and compasses or, in Descartes’ case, generalised compasses [Donski, 2002, pp.1120–1123].

This meant that Newton’s view of geometry was already much broader than that of either Viète or Descartes. Whilst Descartes had extended the margins of geometry by admitting all algebraical curves, Newton wanted to allow all mechanical curves, such as the spiral or cycloid. However, Newton did not accept all methods of constructing curves. For example, he disapproved of pointwise constructions because one has to complete the curve by “a chance drawing of the hand” [MP, 1967–1981, 7, p.385]. It was Newton’s aim to admit mechanical curves, but he needed a way to define these. Guicciardini writes that “any curve generated by continuous motion is, in Newton’s terminology, a “fluent quantity” and, as such, a legitimate object of geometry” [Guicciardini, 2003, p.417].

It is ironic that Newton’s organic construction satisfied Descartes’ criteria for allowable constructions, given that Newton so explicitly distanced himself from Descartes’ own construction methods. The underlying difference was that (in modern terminology) to Descartes only algebraic curves were geometrical, the others being “mechanical”, while to Newton *all* mechanical curves were geometrical⁴⁴.

For Newton, the relationship between geometry and mechanics was an essential one. Newton’s view that geometry receives its exactness from mechanical practice is, however, a step away from Platonic idealism. The idea of slicing a cone with a plane, for example, is an intuitive Platonic definition of a conic section, but Newton preferred a method which described the actual movement and tracing of the curve⁴⁵. He very much considered curves as having a parameter, and often used language which expressed this such as “mobile” or “motion” (e.g., [MP, 1967–1981, 4, p.301; p.467; 7, p.122]).

Of course, before one reaches the stage of construction, one has to perform an analysis of the problem, and here the distinction between Newton and Descartes is even clearer. For Newton, the link between analysis and construction was extremely important:

⁴⁴On this point see also [Guicciardini, 2009, p.104].

⁴⁵Newton’s rejection of “solid” constructions involving intersections of planes and cones is reminiscent of Kepler’s [Bos, 2001, p.188].

Whence it comes that a resolution which proceeds by means of appropriate porisms is more suited to composing demonstrations than is common algebra. [MP, 1967–1981, 7, p.261]

But it was not merely a question of adopting a method which would lead to clear and elegant constructions. Newton also felt that mechanical (that is, geometrical) constructions had another crucial feature:

[I]n definitions [of curves] it is allowable to posit the reason for a mechanical genesis, in that the species of magnitude is best understood from the reason for its genesis. [MP, 1967–1981, 7, p.291]

4.3.3 The analysis of the ancients and the role of porisms

Newton's view of geometry in the 1690s

Above we saw that Collins identified something of a break in Newton mathematical studies around 1675 (section 4.3.1). However, this is not entirely representative, for in the early 1680s Newton attempted to write a treatise of geometry in which he would express all of his mathematics in the style of the ancients. However, the *Geometria Curvilinear* [MP, 1967–1981, 4, pp.407–518] was never published, and in its most finished form contains much of Newton's own original work including the classification of the cubics and the method of fluxions, rather than being an explicit expression of ancient analysis and synthesis [Galuzzi, 2010, p.548]. By 1684 the work was abandoned after Halley had encouraged Newton to consider some problems on planetary motion, which would eventually form results in the *Principia* [Guicciardini, 2009, p.217–219].

Nonetheless, the 1690s saw the height of Newton's interest in ancient geometry. This hiatus in his mathematical work emphasises a distinction we drew in chapter three. Unlike Descartes, Newton certainly did not view geometry as something to be resolved and completed. Whilst his view of geometry changed dramatically over this period, he returned to it again and again. In the 1690s, with a renewed emphasis on classical geometry, Newton even returned to work on many topics which he had studied many years previously. For example, he redrafted his enumeration of the cubics, and returned to his work on loci problems.

His *Geometria libri duo* (c.1693) [MP, 1967–1981, 7, pp.402–561] reflects the close attention Newton paid in particular to Book 7 of Pappus’ *Collectio*. He took this work as his main cue for reconstructing the ancient methods of analysis. Whereas Newton’s *Geometria curvilinear* of the previous decade was perhaps intended to be more foundational (even if it did not emerge as being so), here Newton focuses on the resolution of problems and the construction of curves.

Against the Cartesian definition of *simplicity*—whereby a curve was considered to be more simple if the degree of its equation was lower—Newton asserted that curves in geometry may be described by “any manual operation which shall seem simplest” [MP, 1967–1981, 7, pp.302–303]. He continues:

But the geometer does not order that operation in composition—he merely hints it as a possibility, or proposes it hypothetically and as a species of theorem, or deduces it from the assumption of what is required, or finally assumes it granted in the circumstances of the problem. [MP, 1967–1981, 7, pp.302–303]

Whereas Descartes had required “singular continuous motion”, Newton did not require an instrument to be defined at all, and so there is no restriction put upon the curves which may be included in geometry. The irony is that whilst Newton did, we believe, actually produce many physical instruments, Descartes’ procedures were sometimes idealised. For example, in the case of his instrument (instrument 3.3.1), one is required to imagine the translation of a curve along an axis. It is interesting that when Newton did invent instruments he did not do so in order to define geometry, and yet they often fulfilled Descartes’ criteria more effectively than the ones Descartes himself had described.

The role of porisms

Deeply entwined in Newton’s rejection of the Cartesian methods was his growing admiration for the ancients. As a part of this, Newton sought to understand the ancient methods of geometrical analysis. On his rereading of Pappus’ *Collectio* in its Latin translation, Newton found the method which Pappus had described as “for those who want to acquire a power in geometry that is capable of solving problems set to them” [Pappus of Alexandria, 1986,

1, p.82]. In an earlier paper entitled *Inventio porismatum* (The finding of porisms), Newton comments that “[t]he finding of *porisms* was a key element in the ancients’ technique of analysis. Since this art lies hidden to geometers of our time, it is agreeable to set out some points regarding it” [MP, 1967–1981, 7 p.231].

In chapter 2 we gave a brief description of the role of porisms in Pappus’ *domain of analysis*. Here we take a particular interest in this aspect of analysis because it seems to have been crucial in the discovery of solutions for certain problems. Pappus had described Euclid’s *Porisms* as “a very clever collection for the analysis of more weighty problems” [Pappus of Alexandria, 1986, 1, p.94]. And yet, their exact nature has remained somewhat mysterious. A particular criticism of the ancients (by Descartes, for example) was that they did not make clear their methods of discovery. Porisms were a vital link somewhere between what was known and what was sought. We here identify their importance to Newton who devoted several draft manuscripts to understanding them, and setting out examples he had learned from the *Collectio*. Finally, when the next generation of geometers turned their own focus towards reconstructing and understanding the ancient methods, many of them dedicated their time to porisms (we will say more on this in the next chapter).

Porisms are often acknowledged but dismissed as being akin to locus problems. Guicciardini, for example, comments that Newton was following an early modern understanding of porism as a locus problem, and that he saw the Pappus’ problem as a porism [Guicciardini, 2009, p.83]. If the porisms were to be thought of as closely linked to locus problems then Newton had already developed many strategies for resolving these in the 1670s [MP, 1967–1981, 4, pp.274–335].

Newton’s interpretation may have been quite different from what the ancient geometers had in mind, but we will try to grasp what Newton, his contemporaries and successors understood by porism. Newton relied heavily on Pappus for his comprehension of an ancient analysis⁴⁶, and for examples taken from ancient texts⁴⁷. Newton suggested that the porisms assisted the discovery of analysis, especially for the more difficult problems, and

⁴⁶Book 7 of the *Collectio* contained a partial reconstruction of Euclid’s *Porisms*.

⁴⁷For instance, Newton refers to examples of the use of porisms in Archimedes’ *Of the sphere and cylinder* [MP, 1967–1981, 7, p.261].

that they were just one of a number of tools for doing so.

[N]o one ought to marvel that the ancients, notwithstanding their cultivation of analysis, had recourse to a variety of aids and put out various books regarding the way to diminish the difficulty of resolution: Euclid's three books of *Porisms*, for instance, which Pappus calls "a most skilful work, one extremely useful in the resolution of the obscurer problems". [MP, 1967–1981, 7, p.257]

It is not easy to find a clear definition of *porism* in Newton's writings. He hints at a hierarchy of problem solving, and places porisms somewhere between a problem and theorem, but not quite either. A porism, according to Newton, is

a proposition whereby out of the circumstances of a problem we gather some given thing of use to its resolution. It takes on the form either of a theorem or of a problem at pleasure, and is in consequence reckoned to be a sort of mean between each. [MP, 1967–1981, 7, p.231]

Newton describes the gathering of givens, or *data*. Via a set of theorems we deduce relationships between the data, usually in the form of a ratio or proportion.

The given things, however, which are to be thus gathered are direct and inverse proportions and other relationships of unknown quantities; likewise the species of figures and the lines in which unknown points are located, and which in consequence are usually said to be the loci of the points; and also lengths, angles and points which either regard the determination of loci or otherwise contribute to the resolution of a problem. But proportions hold first place and are tracked down by means of the following theorems [...]" [MP, 1967–1981, 7, p.231]

In the previous chapter we saw that Descartes had expressed a similar notion in the *Regulae*. Descartes had consistently placed an emphasis on proportions and implicit "knowledge". However, in his discussion he of course made no connection with porisms or any classical methods.

In modern commentary porisms are often thought of as the finding of loci. This does seem to have been a large part of what was understood by "porisms", and indeed, Pappus criticised more recent geometers of altering the definition to essentially being just short

of a locus theorem [Pappus of Alexandria, 1986, 1, p.96]. The following passage from Newton helps us to better understand the emphasis on loci.

That technique consists in the ready cognisance of what is given from what is had or known or can be derived therefrom by porisms, and above all of loci. Loci, however, are none other than lines in which required points are located and by whose description problems are constructed. Two loci of this sort are to be ascertained in every problem and described so that a required point may be found at their intersection. Porisms are propositions by whose aid some mean between what is to be resolved and what to be composed is gathered from the premisses. What is gathered is either a given locus or another given thing which regards the finding and determination of a locus, and not infrequently is the bare truth, that is, a theorem. [MP, 1967–1981, 7, p.259]

In fact, Newton claims in the *Geometria* that

[G]eometry in its entirety is nothing else than the finding of points by the intersections of loci. [MP, 1967–1981, 7, p.217]

Newton stated that the resolution of problems usually involves the description of two loci so that “a required point may be found at their intersection. Porisms are an aid to “gathering from the premisses”, which may well be a given locus (or something else) in order to find a required “locus” (some curve). Newton came to think of the porisms as a tool in the route of analysis used by the ancients. He says that it is analysis which enables the geometer to “pass therefrom to the elegant constructions and demonstrations which usually result by means of the method of porisms” (and which cannot be achieved so easily through the method of algebra) [MP, 1967–1981, 7, p.261]. Perhaps more helpful is when Newton tells us what porisms are not, rather than what they are. For example, he says:

They are, of course, propositions about discerning givens and gathering them. As such, they are not merely theorems because they tell how to discover givens, nor are they merely problems because they do not present constructions for ascertaining givens other than in demonstrations; rather, they share the quality of both species and are in consequence easily changed into either—into problems by adjoining constructions, and into theorems by omitting the demonstration of givens and only positing their relationship. Porisms assumed such a form, however, in that they were designed for discovering quantities, that is, gathering givens. [MP, 1967–1981, 7, p.263]

The role of a porism, then, is to help us make sense of what we have been given, in order that we might make use of those relationships to connect what we know (and what is known implicitly) to what we are required to find.

In chapter 2 we outlined two examples of porism given by Pappus in the *Collectio*: the *hyptios* porism and Euclid's *first* porism. Newton also listed these, amongst others, in the *Geometria*. In a draft of the *Geometria* [MP, 1967–1981, 7, pp.248–285], Newton says, in his interpretation of Pappus, that porisms “were designed for discovering quantities, that is, gathering givens”. He concluded also that Euclid's *Data* “are nothing else than porisms” and that the three books of *Porisms* are therefore “a continuation of the data” [MP, 1967–1981, 7, p.263]. Newton then reconstructs twelve porisms, excessively contracted by Commandino in his edition of the *Collectio* [MP, 1967–1981, 7, pp.262–267; p.263 note 46]. Following the text of the *Collectio*, Newton writes “Porisms of this sort, indeed, were had at the beginning of the first book⁴⁸, and Pappus embraces all these in the following general proposition”:

Example 1 (*Hyptios porism*) *If from three given points A, B, C lying in a straight line there are drawn three straight lines mutually cutting one another in the points D, E, and F, and any two of the intersections D and F should trace the straight lines DG, FG given in position, then the third one E also will trace a straight line given in position. The same proposition is valid when the points A, B, C are not in a straight line provided that the points G, B, and C be set in a straight line.*

Newton here uses the *hyptios* porism as a summary of the previous twelve porisms that he has identified [MP, 1967–1981, 7, p.268–269]. In the example given in chapter 2 we explained that we could think of the porism as describing a line between two points, given by the intersections of two pairs of lines. Here we see that Newton is thinking of the porism as the variation of the lines rotating about A, B, and C, with their intersections E, D, and F tracing loci. We then see that if the intersections D and F are made to trace a straight lines, the locus of E will also be a straight line. In this sense, we see the resemblance to Newton's organic rulers, which we discuss below.

⁴⁸That is, Euclid's *Porisms*.



It is clear that Newton saw the porisms as an essential component of uncovering the analysis of the ancients. Furthermore, Guicciardini suggests that “Newton also speculated on the possibility that by porisms the ancients might have meant theorems related to the projective properties of conic sections” and that Newton believed the ancients would have been able to discover theorems and solve problems on conic sections by examining properties invariant under central projection. He points out further that Newton interpreted porisms in projective terms [Guicciardini, 2009, pp.82–84]⁴⁹. Indeed, this is not hard to imagine, and Whiteside gives a similar interpretation. To see this, let us look at another example.

⁴⁹See also [Guicciardini, 2006, p.1730].

Example 2 (Main porism) Porism 6. *From the given points A, B let there concur at the given straight line CG two mobile straight lines ACD, BCE , meeting, when further produced, the straight lines FD, FE , given in position, the former, AC , intersecting FD at D and the latter, BC, FE at E : let there then be required the simplest relationship of the lines FD, FE to each other.*⁵⁰

He then gives three cases, each intended to demonstrate a different theorem from his list. Newton's approach is to establish the correspondence between the points D and E , and to use his previously determined theorems to deduce certain relationships. For example, his first case is this:

Seeing that the points D and E mutually determine one another simply, let point D pass away to infinity and let FE 's position be such as to let point E pass simultaneously to infinity. Now let point E fall at some given point F ⁵¹ and H be the point at which D then falls, and there will then be FE to HD in a given ratio by Theorem 1. [MP, 1967–1981, 7, p.243]

Newton identifies that the points D and E are in 1,1-correspondence⁵². He then supposes that when D passes to infinity, the line FE may be adjusted such that the point E will also pass to infinity. Newton then establishes a second condition. He allows the point E to meet F , and lets H be the point at which D falls. In other words, the vanishing of FE implies that HD will vanish simultaneously. These conditions are enough for Newton to be able to apply his Theorem 1, which states that if two quantities are in 1,1-correspondence, and they both simultaneously vanish and simultaneously become infinite, then they will be in a given ratio⁵³. Applying this to the porism, Newton determines that FE and HD will be in a given ratio, as required.

⁵⁰A slightly different formulation is given in [Guicciardini, 2009, p.87], where the lines FD and FE are not given in position.

⁵¹As Whiteside identifies, Newton chooses F to be the meeting of the lines on which D and E lie, but any fixed point taken on the latter line would do [MP, 1967–1981, 7, p.242, note 36]

⁵²This is what is meant by “determine one another simply”. If two points are in 2,2-correspondence, Newton says they “determine each other doubly”. See, for example, [MP, 1967–1981, 7, p.233].

⁵³*Theorem 1.* If two indeterminate quantities, or their uniform powers, mutually determine each other simply, and these either simultaneously vanish and simultaneously become infinite, or simultaneously vanish and once return to the same ratio, or finally if they twice return to the same ratio, then they will be in a given ratio. [MP, 1967–1981, 7, p.233]. We explain this in modern terms with the aid of [MP, 1967–1981, 7, p.232, notes 8–10]. In an earlier work on solid loci, Newton had identified the general 1,1-correspondence between x and y :⁵⁴

$$axy + bx + cy + d = 0,$$

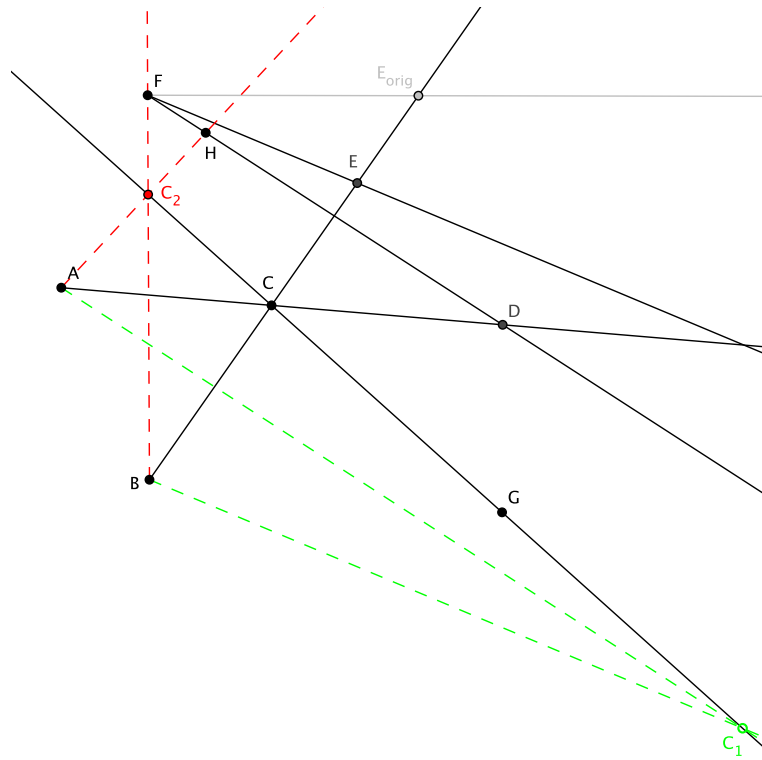


Figure 4.2: Main porism - first case

Newton next considers a second similar case, except that points D and E do not pass simultaneously to infinity. He lets K be the point at which E falls when D “passes to infinity”, and L be “the point at which D falls when E passes to infinity”. By a second theorem [MP, 1967–1981, 7, p.233], Newton determines that the product of KE and LD will be a given constant.

Perhaps the most significant part comes next when Newton determines the given relationships if the given straight line CG were to be replaced by a circle or conic. This indicates firstly the resemblance to the organic description of conics as identified by Guicciardini [Guicciardini, 2009, p.84], and secondly Newton’s ease of use of projective concepts, that is, generalising between circles and other conics, and his implicit use of the properties of cross-ratios of line-pencils.

where a, b, c, d are constants. By the condition $x \rightarrow 0 \Leftrightarrow y \rightarrow 0$, $d = 0$, and by the second condition $x \rightarrow \infty \Leftrightarrow y \rightarrow \infty$, $a = 0$, giving $bx + cy = 0$, and therefore

$$\frac{x}{y} = -\frac{c}{b} = \text{const.}$$

We see in the next chapter that an interest in porisms was maintained in the 18th century by many of the Newtonian geometers, at first due to an interest in the methods of the ancients, but with many novel projective results being discovered. The projective properties of porisms were also identified by Chasles in the 19th century [Heath, 1981, p.436].

4.4 Pappus's problem and the organic description of curves

The contrast between Newton and Descartes is perhaps nowhere more evident than in their approaches to the *Pappus problem*. We have already seen a description of the problem in our chapter on Descartes. We will now revisit our discussion of the problem, considering Newton's solution and exploration. We compare the methods of Descartes and Newton, focusing on Newton's synthetic approach as opposed to Descartes' algebraic approach.

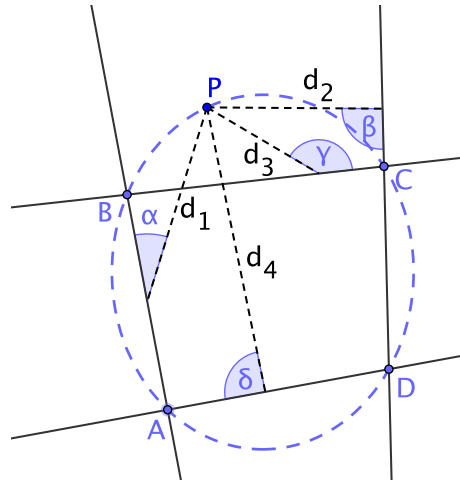


Figure 4.3: The four-line locus problem

Descartes dedicated much time to the problem, constructing solutions in the case of five lines⁵⁵. In his extensive study of the problem in the *Géométrie*, Descartes introduces a coordinate system along two of the lines, and points on the locus are described by coordinates in that system. He was able to reduce the four-line problem to a single quadratic equation in two variables. Bos argues that the study of Pappus' problem convinced Descartes more than anything else of the power of algebraic methods [Bos, 2001, ch.19, ch.23].

⁵⁵The general solution to this is the Cartesian parabola. See [Bos, 2001, §§19.2, 19.3]

Indeed, Descartes claimed that every algebraic curve⁵⁶ is the solution of a Pappus problem of n lines, which Newton showed to be false. He says:

In the complete equation defining curves of sextic order there are 28 terms, 27 of which contain given quantities capable of being varied arbitrarily. But in Pappus' twelve-line problem twenty-two quantities of this sort determine the position of eleven lines with regard to the twelfth and a further one [is needed] to determine the ratio of the product of six lines to the product of the other six. The character of curves of this order is consequently broader than those which may wholly be defined by means of Pappus' problem. [MP, 1967–1981, 4, pp.343–345]

Newton considered the case $n = 12$. He noted that 6th degree curves have 27 parameters, whilst the corresponding Pappus problem would involve 11 or 12 lines. But the 12 line problem requires that

$$d_1 d_2 d_3 d_4 d_5 d_6 = k d_7 d_8 d_9 d_{10} d_{11} d_{12},$$

which has 22 parameters in determining the position of 11 lines with respect to the 12th, and the factor k , making 23 parameters. So, there must exist algebraic curves that are not solutions of Pappus problems.

He developed a completely synthetic solution, in his manuscript *Solutio problematis veterum de loco solido* [MP, 1967–1981, 4, p.282], a version of the first section of which was later included in the first edition of the *Principia* [Newton, 1999, 1, §V, lemmas 17–22]. Newton needed these results in this part of the *Principia* in order to find orbits of comets, but in the 1690s he considered removing them from the second edition and publishing them separately. Sections IV and V are also discussed in [Milne, 1927].

Guicciardini describes how Newton's solution is in two steps [Guicciardini, 2009, pp.90–93]. Firstly, from propositions 16–23 of Apollonius' *Conics*, Book 3, he shows that (in the words of Lemma 17)

If four straight lines PQ, PR, PS, PT are drawn at given angles from any point P of a given conic to the four infinitely produced sides AB, CD, AC, DB of some quadrilateral $ABCD$ inscribed in the conic, one line being drawn to each side, the rectangle $PQ \cdot PR$ of the lines drawn to two opposite sides will

⁵⁶He thought of these as the geometrical curves.

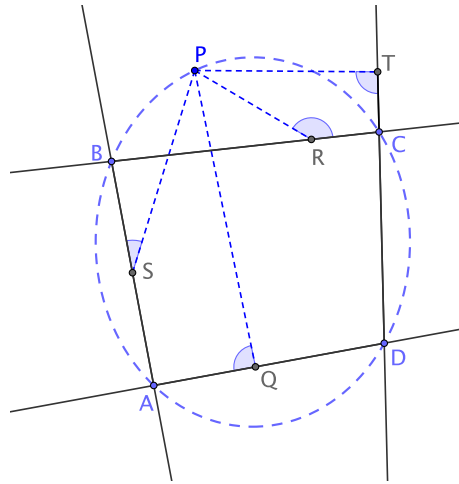


Figure 4.4: Lemma 17

be in a given ratio to the rectangle $PS \cdot PT$ of the lines drawn to the other two opposite sides.⁵⁷

The converse is Lemma 18: if the ratio is constant then P will be on a conic. Then Lemma 19 shows how to *construct* the point P on the curve.

Newton’s second step is to show how the locus which solves the problem—a conic through five given points—can be constructed. Commenting that this was essentially given by Pappus, Newton then introduces the startling *organic construction*. We will discuss this in much more detail later, but the essence is this. Newton chose two fixed points B and C called poles, and around each pole he allowed to rotate a pair of rulers, each pair at a fixed angle (the two angles not having to be equal). See figure 4.5. In each pair he designated one ruler the directing “leg” and the other the describing “leg”.

There is a third special point: when the directing legs are chosen to coincide then the point of intersection of the describing legs is denoted A .

In general, of course, the directing legs do not coincide, and as their point M of intersection moves, it determines the movement of the point D of intersection of the describing legs. Newton showed that if M is constrained to move along a straight line then D describes a conic through A , B , and C , and conversely that any such conic arises in this way.

⁵⁷[Whiteside, 1961] observes that this is equivalent to Desargues’ *Conic Involution Theorem*, and also notes that the condition amounts to the constancy of a cross-ratio.

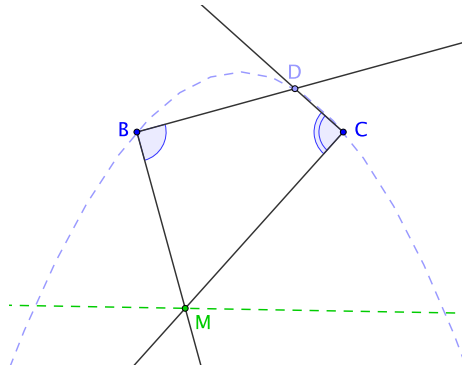


Figure 4.5: The organic construction

This result appears in the *Principia* as Lemma 21 of Book 1 Section V:

If two movable and infinite straight lines BM and CM , drawn through given points B and C as poles, describe by their meeting-point M a third straight line MN given in position, and if two other infinite straight lines BD and CD are drawn, making given angles MBD and MCD with the first two lines at those given points B and C ; then I say that the point D , where these two lines BD and CD meet, will describe a conic passing through points B and C . And conversely, if the point D , where the straight lines BD and CD meet, describes a conic passing through the given points B, C, A , and the angle DBM is always equal to the given angle ABC , and the angle DCM is always equal to the given angle ACB ; then point M will lie in a straight line given in position.

Newton's proofs of both the result and its converse are elegant and clear⁵⁸. They follow from the anharmonic property of conics (his Lemma 20) and the fact that two conics do not intersect in more than four points (his Lemma 20, Corollary 3). Guicciardini argues that this sequence of ideas came from an extension of the "main porism" of Pappus to the case of conics, and Newton had indeed been determined to restore this ancient method [Guicciardini, 2009, pp.86–87]. This reflects our previous identification of Newton's grasp of projective properties in his work on the porisms, and their similarity to the organic description of curves.

Newton's description of conics was in a fairly strong sense what we would now refer to as the projective description. In Proposition 22 he shows how to construct the conic

⁵⁸The point A is crucial to the construction, and it may be helpful to the reader to note that in his thesis [Whiteside, 1961] did not appear to grasp its importance and drew the conclusion that the proof of the converse was flawed. He corrected this misunderstanding in [MP, 1967–1981, 4, p.298]

through five given points. In fact he gives two constructions. Whiteside and others interpret the first as evidence that Newton had at least an intuitive if not explicit grasp of Steiner's Theorem⁵⁹. The second uses the organic construction, but this should not be taken as indicating any reserve about this construction on Newton's part, as he also published it in the *Enumeratio* (1704) and the *Arithmetica Universalis* (1707).

However, in the *Principia* Newton's solution of the classical Pappus problem appears as a corollary to Lemma 19, after which he cannot resist the following comments:

And thus there is exhibited in this corollary not an [analytical] computation but a geometrical synthesis, such as the ancients required, of the classical problem of four lines, which was begun by Euclid and carried on by Apollonius.

4.4.1 The organic construction of higher degree curves

We have seen Newton's use of the organic construction of a conic in his solution of the Pappus problem, and indeed Whiteside does note that the organic construction can, in fact, be derived almost as a corollary of Newton's work on that problem [Whiteside, 1961, p.277]. Furthermore, we have pointed out that the construction can also be derived from Newton's work on the porisms. But Newton knew that these rotating rulers could do much more: he thought of them as giving a transformation of the plane.

It was therefore natural for him to think of the construction in Lemma 21 as a transformation taking the straight line (on which the directing legs intersect) to the conic (on which the describing legs intersect). In an early draft manuscript of about 1667 he says (see figure 4.6):

And accordingly as the situation or nature of the line PQ varies from one place to another, so will a correspondingly varying line DE be described. Precisely, if PQ is a straight line, DE will be a conic passing through A and B ; if PQ is a conic through A and B , then DE will be either a straight line or a conic (also passing through A and B). If PQ is a conic passing through A but not B and the legs of one rule lie in a straight line [...], DE will be a curve of the third degree [...]⁶⁰. [MP, 1967–1981, 2, p.106; p.135]

⁵⁹Steiner's Theorem (1833) states that if p and p' are pencils of lines through vertices A and B respectively, and if there is a correspondence between the lines of p and p' having the property that the cross-ratio of any four lines in p is equal to the cross-ratio of the corresponding four lines in p' , then the locus of the point of intersection of corresponding lines is a conic through A and B .

⁶⁰The general problem of constructing algebraic curves by linkages was solved in [Kempe, 1876].

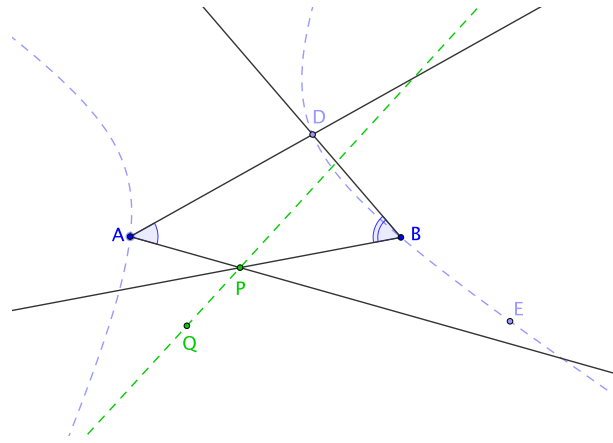


Figure 4.6: The organic construction

In the *Enumeratio* Newton described how to find a 7-point cubic (Figure 4.7)⁶¹. In this extract, note that “curves of second kind” are cubics.

All curves of second kind having a double point are determined from seven of their points given, one of which is that double point, and can be described through these same points in this way. In the curve to be described let there be given any seven points A, B, C, D, E, F, G , of which A is the double point. Join the point A and any two other of the points, say B and C , and rotate both the angle \widehat{CAB} of the triangle ABC round its vertex A and either one, \widehat{ABC} , of the remaining angles round its vertex, B . And when the meeting point C of the legs AC, BC is successively applied to the four remaining points D, E, F, G , let the meet of the remaining legs AB and BA fall at the four points P, Q, R, S . Through those four points and the fifth one A describe a conic, and then so rotate the before-mentioned angles $\widehat{CAB}, \widehat{CBA}$ that the meet of the legs AB, BA traverses that conic, and the meet of the remaining legs AC, BC will by the second Theorem describe the curve proposed. [MP, 1967–1981, 7, p.639]

Even in his earlier manuscript (1667), Newton studied various types of singular point, and indeed he went so far as to devise a little pictorial representation of them. He also gave a long list of examples, up to and including quintics and sextics. Finally, we note that just after the construction of the 7-point cubic he considers the case in which the double point A is at infinity, as he often did elsewhere, thus in effect working in the projective plane.

⁶¹Unfortunately Newton’s original accompanying figures for the *Enumeratio* have not been preserved [MP, 1967–1981, 7, p.588, note 2].

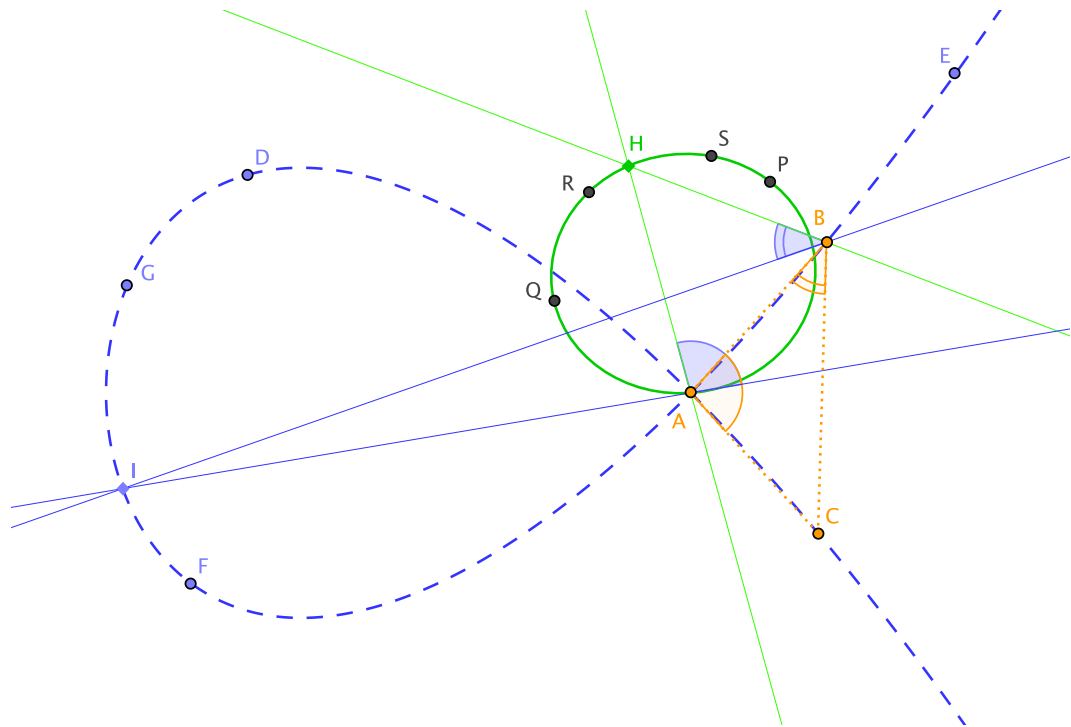


Figure 4.7: Organic construction of a cubic

As identified by Tweedie⁶² in his commentary on Colin Maclaurin’s *Geometria organica* [Tweedie, 1916, pp.94–95], the transformations effected by the organic construction are in fact birational maps from the projective plane to itself, now known as Cremona transformations⁶³. (A short technical account of this is given in [Bloye and Huggett, 2011, pp.25–26]. See Appendix D.)

Of course one wonders how Newton could possibly have discovered such extraordinary results. It seems clear at least that Newton actually made a set of organic rulers. For example, in the 1667 manuscript referred to above Newton uses terms such as “manufactured”, “steel nail”, and “threaded to take a nut” [MP, 1967–1981, 2, p.119]. We also note Newton’s choice of language in his letter to Collins explaining his constructing instrument:

And so I dispose them that they may turne freely about their poles A & B without varying the angles they are thus set at. [MP, 1967–1981, 2, p.156]

⁶²This has also been identified (independently) by Shkolenok ([Shkolenok, 1972, pp.25–30]) to whom contemporary commentators such as Guicciardini usually refer.

⁶³These were first studied by Luigi Cremona and published in [Cremona, 1862]

4.4.2 Newton's inspiration for the rulers

In the *Mathematical papers*, Whiteside suggests that “[t]he source of Newton’s inspiration should be evident to anyone who compares Schooten’s figure on p.347 of his *Exercitationes* and that of Newton’s in his letter to Collins of 20 August 1672” [MP, 1967–1981, 2, p.9, note 22]⁶⁴, that is van Schooten’s *Exercitationum mathematicarum libere quinque* (1657) [van Schooten, 1657]. Whilst we do not seek to disagree that Newton is likely to have drawn inspiration for experimenting with mechanical rulers from this and other works, we argue that this is only true in the loosest sense.

Whiteside claims that, in the fourth book of the *Exercitationes*⁶⁵, it had been van Schooten’s aim to “construct any given conic [...] by a uniform method and in one continuous, uninterrupted motion” [MP, 1967–1981, 2, p.8]. Van Schooten had been highly successful at creating elegant constructions of the individual conic sections under various conditions, but was unable to develop a more general method. Newton succeeded in this respect with his *organic* construction. Further, he was also able to generate cubics, quartics, and curves of higher degree.

In his letter to Collins, Newton describes the method of constructing a conic through five given points (Figure 4.8). The straight line (directrix) is not given here: the conic is constructed using the five points and the initial set-up of the rulers. Compare this with the instrument referred to by Whiteside, in van Schooten’s chapter called *The method of describing hyperbolae in the plane, given foci and vertices*⁶⁶ (Figure 4.9). Schooten is here describing how to produce a straight line, tangent to the hyperbola in any given point. In this particular work, Schooten develops procedures for describing specific conic sections under a variety of conditions, such as given foci, vertices, or asymptotes, but does not achieve a method for describing a general conic as Newton does.

As we noted above (section 4.2.1), it is well documented that Newton had indeed studied

⁶⁴Reproduced in [MP, 1967–1981, 2, pp.156–159]. The picture appears on p.157, but we will reproduce the original (Figure 4.8) [MS Add. 3977.10, 1v, Cambridge University Library, Cambridge].

⁶⁵*De organica conicarum sectionum in plano descriptione*, [van Schooten, 1657, pp.293–368]. The fourth book was essentially a reproduction of Schooten’s earlier *Organica conicarum sectionum* (1646) [van Schooten, 1646], which describes methods for mechanically tracing conic sections given various properties.

⁶⁶*De modo describendi hyperbolas in plano, dates focus & vertice* [van Schooten, 1657, pp.344–353].

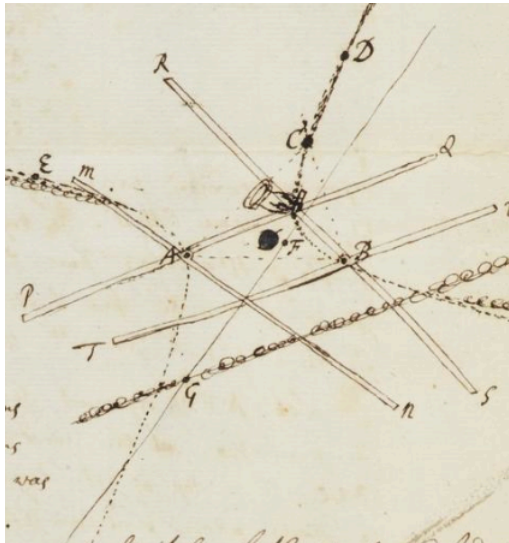


Figure 4.8: Newton to Collins, 20 August, 1672

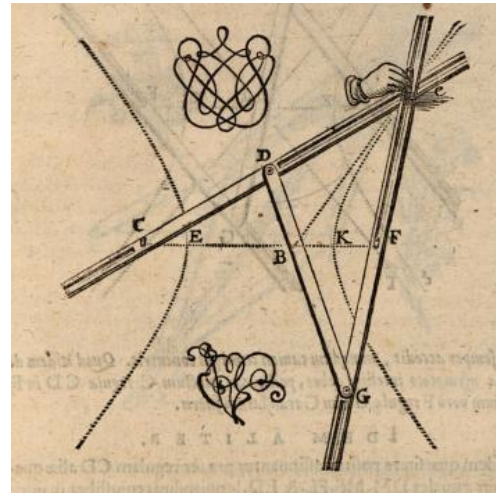


Figure 4.9: Schooten, p.347 *Exercitationes*

closely Schooten's *Exercitationes*. In his early annotations he made several notes on this work and directly copies many different constructions of conic sections [MP, 1967–1981, 1, pp.29–45], but there are some immediate differences between these two constructions. We refer to Figure 4.9.

Theorem 1 *As CD rotates about C , point e describes a hyperbola.*

Proof Note first that $CD = GF$ and $CF = DG$. Therefore triangles CDF and GFD are congruent.

Hence angle eCF is equal to angle eGD , and so triangles eCF and eGD are similar. But $CF = DG$, and therefore triangles eCF and eGD are congruent.

Hence $eC - eF = GF$, which is constant. So e moves in such a way that the difference between its distances from two fixed points is constant.

Therefore e describes a hyperbola. □

This proof is entirely Euclidean, except for the last step. Presumably, van Schooten got this step from Apollonius.

Ingenious though these rulers are, they are quite different from Newton's organic rulers, because they can only draw this hyperbola. They are specifically, and very cleverly,

designed for this one construction. They are not presented as giving a mapping from the plane to itself, as Newton does with his organic rulers. Here it remains unclear why Whiteside picks out this set of rulers as the crucial influence on Newton.

However, Whiteside does point out that of the three mathematicians Schooten identifies as having ventured into the description of curves of higher degree [Descartes, Fermat, and Roberval]⁶⁷, only Descartes had managed to develop a mechanical construction for curves *uno ductu continuo*⁶⁸. He refers to an instrument in Book 2 of the *Géométrie*, which we described in the previous chapter (instrument 3.3.1), but notes that “[t]his, however, is a poor construction in comparison with Newton’s ‘organic’ one, for from a straight line it constructs a single conic species (the hyperbola) and from a parabola a single cubic (the Cartesian trident)” [MP, 1967–1981, 2, p.9, note 22].

We consider algebraically the curves produced by Descartes’ two main instruments in Appendix C. The mesolabum (instrument 3.1.2) is an ingenious device for constructing any number of mean proportionals, but as an instrument for tracing curves it is limited: it only traces a specific family of curves, and only in the positive quadrant. We note that the mesolabum could be made into a practicable instrument. In contrast, the turning ruler and sliding curve procedure (instrument 3.3.1) is not really an instrument at all, even though it does “trace” arbitrarily complicated curves. Neither of these instruments resolves singularities of curves, and neither of them was used, in effect, in the projective plane. These two features are in sharp distinction with Newton’s organic rulers.

4.4.3 Whiteside’s interpretation of the rulers

In his introduction to Newton’s early manuscript *Researches in the organic construction of curves* (c.1667) [MP, 1967–1981, 2, pp.106–151], Whiteside gave an interpretation of the transformation effected by Newton’s rulers in which the degree of the curve does not change under the transformation, and nor does the number of singularities. He writes:

In outline, two fixed angles rotating each round a fixed pole in their plane determine a one-to-one continuous correspondence between the two meets of their arms (or ‘legs’ as Newton chooses to call them). If, consequently, one

⁶⁷See [van Schooten, 1657, 4, p.302] in [MP, 1967–1981, 2, p.9, note 22].

⁶⁸Single continuous motion.

meet is set on a known ‘describing’ curve (Newton’s ‘directrix’) the second will trace a ‘described’ curve (Newton’s ‘describend’) of equal algebraic degree and equivalent singularities. The apparent power of the construction to raise the dimension of the describing curve in drawing the corresponding described one is explained by the former’s (and indeed the latter’s) ability on occasion to conceal in itself the polar⁶⁹ line taken once or several times. [MP, 1967–1981, 2, p.8]

He later gives a more detailed account of what he means, starting as follows.

In effect, the apparatus of the organic construction sets up a one-to-one continuous correspondence between the points (f) and (c) of the describing and described curves which preserves topological configurations, tangency properties and, in general, the algebraic degree of the curves thus transmuted [...] [MP, 1967–1981, 2, p.107, note 8]

In this manuscript Newton opens by listing many cases of construction given various types of special points. As the title of Newton’s manuscript indicates, his purpose was to construct curves, and in particular to construct conics. This clearly requires a change in the degree of the curve: Newton wanted to construct conics from a straight line, as he does in his “Problems” ([MP, 1967–1981, 2, p.119]) when he constructs a conic through five given points, a conic through four given points at one of which it is tangent to a given line, and so on.

Whiteside aptly describes Newton’s subsequent use of the organic rulers to generate cubics, quartics, and higher degree curves with prescribed singularities as “a dazzling virtuoso display of talent” [MP, 1967–1981, 2, p.8]. Clearly, in these constructions of one curve from another, the number of singularities cannot be expected to remain the same. Indeed, from the modern point of view the whole point of the “blowing up” procedure is to resolve a singularity of a curve. Newton’s approach was rather the other way around: he wanted to *construct* these singularities, and then study them.

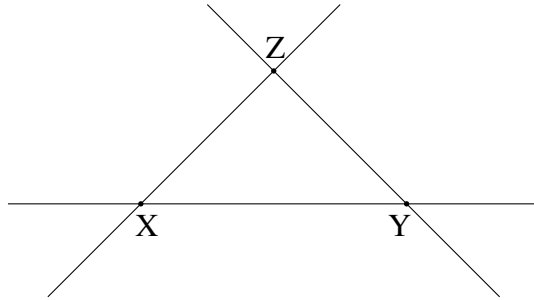
In order to analyse Whiteside’s interpretation we have to give a brief modern description of the transformation effected by Newton’s rulers.

⁶⁹Note that the “polar” line refers to the straight line between the two poles about which the rulers rotate, and not “polar” in the projective sense.

We have seen that Newton's organic rulers perform the standard quadratic transformation

$$\begin{aligned}\phi : \mathbb{P}^2 &\rightarrow \mathbb{P}^2 \\ (x : y : z) &\mapsto (yz : zx : xy)\end{aligned}\tag{4.1}$$

where $(x : y : z)$ are the usual homogeneous coordinates. We denote by X , Y , and Z the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$ respectively, and visualise them as follows:



The mapping ϕ is well-defined everywhere in \mathbb{P}^2 except at X , Y , and Z , and it is self-inverse everywhere in \mathbb{P}^2 except on the lines XY , YZ , and ZX (which, for future reference, have equations $z = 0$, $x = 0$, and $y = 0$ respectively).

The whole line XY is mapped to the single point Z by ϕ , and similarly for the other two lines.

These properties follow immediately from (4.1), or from setting up Newton's organic rulers at what he called the poles X and Y , with one pair of rulers lying along XY and XZ , and the other along YX and YZ .

It is crucial to note that although ϕ is not defined as a function at X , Y , or Z , it has extremely important properties at these points. Consider X , for example. A straight line l through X has the form

$$by + cz = 0$$

and its image $\phi(l)$ is

$$bzx + cxy = 0$$

which is the straight line

$$bz + cy = 0$$

cutting $x = 0$ at $(0 : b : c)$. Now consider a point p moving along l towards X . While p is not at X , ϕ is self-inverse and maps the line $x = 0$ to X . So p has image $\phi(p)$ moving along $\phi(l)$ towards its intersection with $x = 0$.

This is still true even when l is a curve through X : one just considers its tangent at X . Hence ϕ maps lines l through X to lines $\phi(l)$ cutting $x = 0$ in such a way that the “gradient” $(b : c)$ of l at X is recorded in the position in which $\phi(l)$ cuts $x = 0$.

A mapping which replaces a point by a line in this way, is said to “blow up” that point. This procedure is a well-known and key ingredient of modern algebraic geometry, used for example in the resolution of singularities, to which we now turn.

Let C be a conic. In general, it will intersect each of the three lines XY , YZ , and ZX twice. Suppose that these intersections avoid X , Y , and Z . Then, $\phi(C)$ will pass through each of X , Y , and Z twice, giving $\phi(C)$ double points at X , Y , and Z . If C itself passes through any of X , Y , or Z then the corresponding double points disappear.

Our conic has equation

$$ax^2 + bxy + cy^2 + dyz + ez^2 + fzx = 0, \quad (4.2)$$

and $\phi(C)$ is

$$a(yz)^2 + byz^2x + c(zx)^2 + dzx^2y + e(xy)^2 + fxy^2z = 0, \quad (4.3)$$

which is a quartic having double points at X , Y , and Z .

Now suppose that C passes through X . Then $a = 0$, and $\phi(C)$ becomes

$$byz^2 + cz^2x + dzxy + exy^2 + fy^2z = 0, \quad (4.4)$$

which is a cubic having a double point at X .

Next, suppose that C passes through both X and Y . Then we also have $c = 0$, and $\phi(C)$ becomes

$$bz^2 + dzx + exy + fyz = 0, \quad (4.5)$$

which is a conic passing through X and Y .

Finally, let C pass through X , Y , and Z . Then we also have $e = 0$, and $\phi(C)$ becomes

$$bz + dx + fy = 0, \quad (4.6)$$

which is a straight line. □

Each of these cases is dealt with by Newton, but always of course geometrically, in terms of his organic rulers, instead of algebraically, in homogeneous coordinates. Having demonstrated above the transformation effected by Newton's rulers we are now able to discuss Newton's work, together with Whiteside's interpretation. Recall that Whiteside had suggested that the degree of the curve could be preserved by the transformation [MP, 1967–1981, 2, p.8].

Newton did refer to the third base point (which we have denoted Z in our description), but not on an equal footing with the two poles (X and Y above) of his organic rulers [MP, 1967–1981, 2, p.139].

Whiteside goes on to explain that if the describing curve passes through X and Y (to use our notation) then the described curve will have the same degree and singularities. In the case of conics, this is our equation (4.5). Next, if the describing curve only passes through one of X and Y , say X , Whiteside suggests including the line XY as part of the describing curve. This makes the describing curve of one higher degree, with a double point at X . When the describing curve is a conic, the described curve will have our equation (4.4). Finally, if the describing curve passes through neither of X and Y , Whiteside suggests including the line XY twice as part of the describing curve. This increases the degree of the describing curve by two, and doubles it at X and Y . When the describing curve is a conic, the described curve will have our equation (4.3). (Whiteside reiterates this procedure, briefly, in [MP, 1967–1981, 2, p.139, note 18].)

We do not fully understand this view of the effect of Newton's organic rulers, for the following two reasons. Firstly, nothing in Newton's work suggests that he thought of it this way. On the contrary, as we noted above, Newton *wanted* the degree of the curve to change, and rather than trying to explain away the singularities which appeared, he made a very detailed study of them. Secondly, when one considers all three base points, and the three lines joining them, Whiteside's procedure does not always work. It does in the case where the original describing conic C only passes through one of X and Y : then both $C \cup XY$ and $\phi(C \cup XY)$ are of degree three, each having two double points, those of $\phi(C \cup XY)$ being at X and Z . But in the case where the original describing conic C passes through neither of X and Y things go wrong. Granted, both $C \cup 2XY$ and $\phi(C \cup 2XY)$ are of degree four, but now $C \cup 2XY$ has two triple points (where C intersects $2XY$) and a double line, while $\phi(C \cup 2XY)$ has double points at X and Y and a quadruple point at Z .

We conclude that if one wishes to understand the mathematics behind Newton's organic construction it is better to regard it as the standard quadratic transformation from \mathbb{P}^2 to \mathbb{P}^2 , as we have done above.

4.5 The enumeration of the cubics

The three non-degenerate forms of curves of degree two, the conics, were known to the ancient geometers, especially Apollonius, who had explored them in great depth, including their projective properties. Descartes made significant progress in the area of plane conics by translating the curves into analytical terms. Very little was known about curves of higher degree by either the Greeks, or in the seventeenth century. A very small number of individual curves of higher degree were known, such as the conchoid, for their use in the solution of other problems such as finding mean proportionals. As Whiteside points out, “[w]hen in the late 1630s Descartes described his trident and folium to his contemporaries he literally tripled the number of known cubics, but over the next quarter century only two more curves of third degree were discussed in print, the cubical and semicubical parabolas, first described by John Wallis and William Neil, respectively” [MP, 1967–1981, 2, p.4]. Descartes had made some progress towards reducing the general equation of the cubic into simplified forms, but no one before Newton had been able to manage a sub-classification

of the curves as had been achieved with the conic sections. However, it was not a simple task for Newton, and the work occupied him periodically over four decades.

Newton started work on the content for the *Enumeratio* in the late 1670s, but it was not published for some thirty years. Whilst it was not made public for many years, the *Enumeratio* contained one of Newton's most monumental pieces of work in geometry. The significance of Newton's enumeration of cubic curves was reflected in the many reproductions and commentaries, first as an appendix to Newton's *Opticks* (1704), and later his *Analysis per Quantitatum* (1711). Just a few years later a further edition with commentary by James Stirling appeared (1717) [Stirling, 1717], and eventually, an English translation with commentary by C.R.M. Talbot [Talbot, 2007]⁷⁰.

The existence of early manuscripts from the 1670s was not known until substantial extracts were published by Rouse Ball in 1890 [Rouse Ball, 1890]. Whiteside notes that it is not known what prompted Newton to return to the topic as too little is known about his life at this time [MP, 1967–1981, 4, p.354, note 1]. Guicciardini suggests that it was “Newton's quest for the ancient, non-algebraical, porismatic analysis led him to develop an interest in projective geometry” and that “[h]e convinced himself that the ancients had used projective properties of conic sections in order to achieve their results. Moving along these lines he classified cubics into five projective classes.”[Guicciardini, 2006, p.1730].

Newton revealed the potential complexities of these curves, including the various types of singularities, which as Guicciardini puts it “reinforced his conviction that Descartes' criteria of simplicity were foreign to geometry” [Guicciardini, 2009, p.112]⁷¹.

However, the early development of Newton's classification of the cubics is particularly interesting since he had not given up algebraic and algorithmic techniques entirely. This was a source of conflict for Newton between his private work and public presence, and may partly explain his reluctance to publish in this as well as other areas.

It is clear that at times Newton obtained results via algorithmic techniques, in spite

⁷⁰See also [Struik, 1969, pp.168–180].

⁷¹See also [Guicciardini, 2006, p.1730, note 13]: “From his work on cubics ([MP, 1967–1981, 2, pp.137–161]) Newton derived two lessons. First, Descartes' classification of curves by degree is an algebraic criterion which has little to do with simplicity. Indeed, cubics have rather complex shapes compared to mechanical (transcendental) curves such as the Archimedean spiral. Second, it is by making recourse to projective classification that one achieves order and generality.”

of his observed preference for geometrical methods. This use of symbolical methods generated what appears to be somewhat of a disparity in his work. It is intriguing that his publication strategy reflects his unease with this issue, limiting his algebraic and series methods to “scribal publication”. As Guicciardini notes, Newton explained to David Gregory in 1694 that: “Our specious algebra is fit enough to find out, but entirely unfit to consign to writing and commit to posterity” [Guicciardini, 2012, p.6].

Let us now look more closely at the development of Newton’s work in this area. Firstly as an example of Newton’s significant contribution to the advance of an important topic in geometry, but also to assist us in our understanding of the evolution of his geometrical ideas between the 1670s and 1700s.

Development of the manuscripts: 1664–1695

In the autumn of 1664 Newton began a few preliminary researches into analytical geometry, co-ordinate systems, transforms of axes, and a few attempts at curve tracing [MP, 1967–1981, 1, pp.155–212]. Whiteside reports how in 1664 Newton had “come face to face with the difficulties of attempting any synthetic classification and to realise the perils which lay waiting when a given cubic was to be traced from even a much simplified defining equation” [MP, 1967–1981, 2, pp.4–5].

On returning to the study three years later, Newton made improvements to his first investigations and began laying the foundations for his analysis of the cubics. Newton attempted to reduce the general cubic equation to four cases by a transformation of co-ordinates. Whilst the attempt was largely successful, much of the work has been lost. Whiteside had identified that in the first papers to have been preserved, Newton is already “at work strengthening and systematising the first gains of battle” [MP, 1967–1981, 2, pp.4–5]. This is his first substantial manuscript on the subject, *Enumeratio Curvarum Trium Dimensionum* (c.1667)⁷² [MP, 1967–1981, 2, pp.10–89]. Of the existing papers,

⁷²See Whiteside’s comment on the dating: “We have been unable to glean any information relating to these researches from extant sources, printed or manuscript. Newton, it would seem, left the world wholly in ignorance of them till he chose to reformulate them in the middle 1690s as his *Enumeratio Curvarum Tertij Ordinis*, there suppressing all but an abrupt sketch of his analysis. We base our conjectured dating on our estimation of Newton’s handwriting and the relative immaturity of the technical content of the present papers.

Whiteside helpfully divides them into three parts:

In the first part⁷³, Newton describes a simple geometric method by which a curve may be described. This is a typical Newonian method, based on motion, in which he considers a point on the end of a mobile line segment to trace the curve.

Starting with a fixed line AB which he calls a “base”, Newton says a “describer” BC may be inclined to the base at any angle ABC . As BC moves along AB a given point C on BC traces the curve ε (Figure 4.10).



Figure 4.10: Projection of cubics [Add. 3961.1:2r, Cambridge University Library, Cambridge]

From this he determines that it is “evident” that the same curve may be expressed in an “infinity” of ways [MP, 1967–1981, 2, p.11]. Newton expresses the relationship between the “rulers” by an equation in terms of x and y . Taking $AB = x$ and $BC = y$,⁷⁴ Newton gives an equation for the general cubic

$$ay^3 + bxy^2 + cx^2y + dx^3 + ey^2 + fxy + gx^2 + hy + kx + l = 0.$$

The next step was to “seek a relationship between $[AB]$ and $[BC]$ such that I may have in one equation all possible ways of expressing the nature of the curve by a relationship between $[AB]$ and $[BC]$ ” [MP, 1967–1981, 2, p.13]. Newton examined a number of cases dependent on the angle ABC . However, he was dissatisfied with the unwieldy nature of

These present researches are, of course, a great improvement on his first investigations into cubics and higher order algebraic curves by analytical techniques in the autumn of 1664” [MP, 1967–1981, 2, pp.10–11, note 1].

⁷³[Add. 3961.1:2r–3r, Cambridge University Library, Cambridge]

⁷⁴We have changed Newton’s notation: $AB = z$ and $BC = v$.

the equation, saying that it would “grow too big” [MP, 1967–1981, p.15]. He proposed an alternative where the relationship between x (AB) and y (BC) is invariant under a change of angle ABC , resulting in a simple cubic in one variable of which he determined the root by making a series of substitutions.

In the second⁷⁵ of the three parts, Newton considered the enumeration of component species of the first, most general canonical case of the cubic curve [MP, 1967–1981, 2, p.18, note 1]. In a redraft under the same title he supposed the relationship between $AB = x$ and $BC = y$ to be

$$bxy^2 = dx^3 + gx^2 + hy + kx + l.$$

He then gave this in the form of the root $y = \dots$. By examining the signs of the constant Newton was then able to determine what form the curve would take, and considered individual species. He also described how to find particular points on the curve [MP, 1967–1981, 2, p.23].

The final section combines the work of two parts of the manuscript⁷⁶. This was Newton’s first attempt at giving an exhaustive list of cubic curves by analysis of the four standard forms, where he found 58 individual species. Newton returned to the equation for a general cubic which he determined may be reduced to one of nine cases⁷⁷. [MP, 1967–1981, 2, p.39]

Around ten years later, in the late 1670s, Newton took up the project again [MP, 1967–1981, 4, pp.354–401]. His work from this period would form the basis for the published *Enumeratio* (1704). Newton once again revised what he had done previously, classifying the cubics into “genera” and “species” based on their equations.

Newton returned to his reduction of the general cubic equation into the four standard forms:

⁷⁵[Add. 3961.1:10r–13r, Cambridge University Library, Cambridge]

⁷⁶[Add. 3961.1:6r–9r; 14r–16r, Cambridge University Library, Cambridge]

⁷⁷“The first, general canonical form is divided into six cubic ‘cases’, the first of which is subclassified into three component ‘species’ and the rest into two each, while the remaining three standard forms are each further ‘cases’ containing a unique species. Each species, finally, is further subclassified into ‘forms’, each of which may take on one or more of the three ‘grades’ according as the three real diameters are or are not copunctual or as the cubic has a centre” [MP, 1967–1981, 2, pp.36–37, note 1].

$$Axy^2 + By = Cx^3 + Dx^2 + Ex + F$$

$$xy = Ax^3 + Bx^2 + Cx + D$$

$$y^2 = Ax^3 + Bx^2 + Cx + D$$

$$y = Ax^3 + Bx^2 + Cx + D$$

His first attempt was to change the origin and axes. According to Rouse Ball, who studied Newton's manuscripts from the 1670s and the 1690s, "he has left two independent proofs of this method" [Rouse Ball, 1890, p.107]. In the *Enumeratio* Newton's process depended on finding the diameter conjugate to an asymptote. In his first method, Newton gives the most general cubic equation of a cubic curve in (z, v) . He takes a new origin (r, s) and axes making angles α and β with the axis of z and obtained relations for z and v in terms of r and s , respectively. In his description, Rouse Ball gives trigonometric functions, but notes that Newton did not use these, instead making use of similar triangles [Rouse Ball, 1890, p.109]. Following a series of substitutions Newton reached an equation of 84 terms! Taking roots he was able to reduce the equation to its four forms. Rouse Ball comments that "[t]he algebraic analysis above described involves considerable labour, but the details of the work are not given in the manuscript. Newton's second proof by transformation of axes is much more ingenious" [Rouse Ball, 1890, p.109]⁷⁸.

In 1695, possibly urged by David Gregory to compile such a book⁷⁹, Newton began compressing his earlier researches into a short text [MP, 1967–1981, 7, pp.579–655]. He gave brief properties of curves and an enumeration of species. This was essentially the treatise that was published as an appendix to Newton's *Opticks* in 1704.

⁷⁸The general method is described by Rouse Ball in [Rouse Ball, 1890, pp.109–110]. He gives the various cases which determine the four forms in [Rouse Ball, 1890, pp.110–112].

⁷⁹For example, Whiteside says "[w]hether or not David Gregory had (as we have suggested in our preceding introduction) already urged him to compile such a book *de Curvis secundi generis* during his visit to Cambridge in May of the previous year, it is certain that Newton began, in or soon after the early summer of 1695, to condense the multi-layered bulk of his earlier researches into the theory and construction of cubic curves into a short text which should briefly enunciate their main properties, but above all give an exact enumeration of their component species according to the scheme of such division which he had laid down a quarter of a century before." [MP, 1967–1981, 7, p.579, note 1]

In the *Mathematical Papers* early drafts are omitted, but Whiteside identifies that the main difference is that in these Newton combined particular species to give a total of 69, compared with 72 in the final tract [MP, 1967–1981, 7, pp.579–580, note 2]. The manuscript, both here, and in its final published form, is in seven sections: (1) On the orders of lines, measured by the degrees of their respective equations; (2) properties of cubics analogous to conics; (3) the equation of a cubic can always be written as one of the four canonical forms; (4) the nature of curves indicated by these equations, curves are divided by species considering the maximum and minimum values of x ; (5) curves by shadows: all cubics may be obtained from one of the five divergent parabolas; (6) (organic) description of curves; (7) graphical solution of problems by the use of curves.

The genesis of curves by shadows

Perhaps the most astonishing of Newton's additions to the enumeration of cubics in 1695 was the discovery that every cubic can be generated by centrally projecting one of the five species of divergent parabola (encompassed by the equation $y^2 = Ax^3 + Bx^2 + Cx + D$). Whilst in the 1695 draft Newton also enumerated the curves by this method, the only part that made it into the 1704 publication was his statement that this could be done. Newton began with the following evocative phrase.

If onto an infinite plane lit by a point-source of light there should be projected the shadows of figures ... [MP, 1967–1981, 7, p.635]

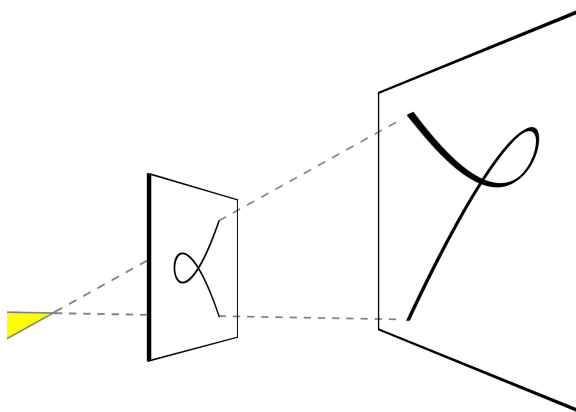


Figure 4.11: Projection of cubics

In the published work Newton gave no proof of the result, and no examples of enumeration effected by this method. In fact, it was unproved until 1731, and was first demonstrated independently by François Nicole⁸⁰ and Alexis Clairaut⁸¹.

Newton appears to have first worked on the idea of enumerating the cubic curves by projection in his *Geometria libri duo* (c.1693) [MP, 1967–1981, 7, pp.402–561]. He systematically works through each of the five divergent parabolas, characterising the species of cubic that may be obtained by projecting each one. By this method Newton obtained all 78 species of cubic, but this full enumeration did not make it into the *Enumeratio*. As Guicciardini points out, Newton had in fact identified the remaining six, but they did not materialise in his paper [Guicciardini, 2009, p.111, note 8]. The reason for this omission is not known, but Whiteside has made some headway with the problem, carefully documenting where the species go astray and noting by when and by whom they were first publicly identified. Examination of these notes reveals certain patterns in the missing species, for example, the species occur in pairs of variations either with a node or with an oval. With further study it may be possible to make additions to what little is known already of the missing species. See in particular, [MP, 1967–1981, 7, p.426, note 54; p.431, note 65].

It seems extremely plausible that Newton's intuition was supported by his use of an actual projection from a point source of light, but Guicciardini notes that there have been differing views on this question [Guicciardini, 2009, pp.123–129]. In his assessment of the manuscripts from the 1670s and 1690s, Rouse Ball argues that the result was obtained using projective transformations:

I have little doubt that Newton had arrived at this remarkable result, which proved a puzzle to most of his contemporaries, by the method of projection indicated in the *Principia*, Bk. I., sect. 5, lemma XXII. (See also propositions XXV., XXVI., XXVII.)

It is not difficult to prove the property by analytical geometry (see, for example, Salmon's *Higher Plane Curves*, second edition, Arts. 195, 196); but to Newton's contemporaries, who relied largely on pure geometry, it was by

⁸⁰In a paper read to the Académie des Sciences, Paris (1731) [Nicole, 1733].

⁸¹In a paper read to the Académie des Sciences, Paris (1731) [Clairaut, 1733].

no means easy to establish its truth, and, in fact, a quarter of a century elapsed before any one published a demonstration of it. [Rouse Ball, 1890, p.123]

Thus, the discovery that all the cubics can be generated by projecting the five divergent parabolas was essentially algebraic.

In his annotated translation of Newton's 1704 *Enumeratio*, Talbot preferred the view that Newton might have followed a geometrical procedure. He argued that Newton generated all the cubic curves by projection of the five divergent parabolas, using a method in which he began by noting that the position of the horizon line determined the nature of the asymptotes of the projected line [Talbot, 2007, pp.72–83].

There is no real evidence for either hypothesis in Newton's work. Guicciardini and Whiteside both seem to favour Talbot's geometrical explanation. We agree: Newton may well have used Lemma 22 to test specific cases, but the general result must surely have been perceived by him as a geometrical insight.

So far in this chapter we have observed various distinctions and differences between the methods of Newton and Descartes. First, Newton had specifically used the ancient Pappus' problem to attack the new Cartesian methods by developing an entirely synthetic construction. This construction, as with his work on the porisms, led naturally to the organic description of curves. This was significant for two reasons. Not only did it fulfil Descartes' criteria of single continuous motion, it allowed Newton to describe general curves, which had not been achieved before. Secondly, Newton was able to study singularities of curves, especially cubics, by using the organic construction either to generate or to resolve these singularities. Further, we believe that it was likely Newton actually made and implemented these rulers, unlike Descartes' various instruments which were both less practicable and less powerful.

In addition to the above comparisons, in this section we focused on the development of the enumeration of the cubics. As we have seen, Newton returned to this topic several times over four decades. Here we observed the evolution of his methods. In particular, Newton improved his algebraic procedures as well as developing a remarkable projective

method after refocusing on classical geometry in the 1690s. Examples of Newton's ease of use of projective methods are evident both here and in work on locus problems, in spite of there being no formal subject of projective geometry at that time.

As we have identified in the previous chapters, Viète and Descartes had attempted to define and restrict geometry. We propose that Newton was much less focused on foundational aspects of the subject, which meant that he had the freedom to repeatedly return to topics, such as loci problems and the cubics. Newton's achievements in geometry were defined by an approach whereby he gradually improved his results, working towards the geometrical certainty that he required. In doing so we can perceive that he was able to take steps that his predecessors were not able to because they had been governed by a perception of geometry as bounded and finite.

We now take a final opportunity to observe the impacts of Newton's geometrical thought. In his work in physics we see that geometry gave Newton a certain clarity that he could not obtain through algebraic methods.

4.6 Geometry in Newton's physics

Some of the most extraordinary examples of Newton's geometrical power arose in the exposition of his physical discoveries. Here we take the opportunity to briefly note three such cases which exemplify the importance of Newton's understanding, appreciation of, and approach to geometry.

In Newton's *Principia*, Section 6 of Book 1 is called *To find motions in given orbits*, and it includes Lemma 28 on algebraically integrable ovals:

No oval figure exists whose area, cut off by straight lines at will, can in general be found by means of equations finite in the number of their terms and dimensions. [Newton, 1999, p.511]

Newton's proof (figure 4.12) simply takes a straight line rotating indefinitely about a pole inside the oval, and a point moving along the line in such a way that its distance from the pole is directly proportional to the area swept out by the line. This point describes a spiral, which intersects any fixed straight line infinitely many times.

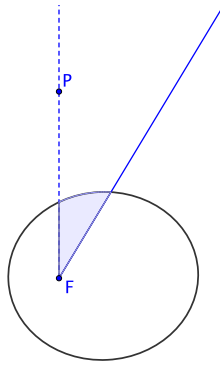


Figure 4.12: Lemma 28

Then, noting almost as an aside what is essentially Bézout’s Theorem (1779) on the intersections of algebraic curves, the proof is completed by the observation that if the spiral were given by a polynomial then it would intersect any fixed straight line finitely many times.

At the end of his proof Newton applies the result to ellipses (which were of course the original motivation), and defines “geometrically rational” curves, noting casually that spirals, quadratrices, and cycloids are geometrically irrational. Thus he leapt to the modern demarcation of algebraic curves, while demonstrating that a restriction to these curves (following Descartes) would not be enough for a description of orbital motion.

This is how Arnol’d puts it:

Comparing today the texts of Newton with the comments of his successors, it is striking how Newton’s original presentation is more modern, more understandable and richer in ideas than the translation due to commentators of his geometrical ideas into the formal language of the calculus of Leibnitz.[Arnol’d, 1990, p.94]

Unfortunately, Newton did not make explicit what he meant by an oval, which has led to considerable controversy⁸². Although in later editions of the *Principia* Newton inserted a note excluding ovals “touched by conjugate figures extending out to infinity”, he never made clear his assumptions on the smoothness of the oval. Nor did the statement of the

⁸²Whiteside’s own counter-example (which he gave in [MP, 1967–1981, 6, p.302–303, note 121]) was elegantly ruled out in [Pesic, 2001].

to get $\frac{dF}{d\phi}$. Finally, this is integrated to get $F = \frac{4\pi a^2}{r^2}$, as required. We compare this with Newton's proof, as Chandrasekhar has done [Chandrasekhar, 1995, p.272].

Here is a sketch of Newton's geometrical argument (Figure 4.13). The spherical surface attracts "corpuscles" at P and p , and we wish to find the ratio of the attractive forces on these two corpuscles. Draw lines PHK and phk such that $HK = hk$, and draw infinitesimally close lines PIL and pil with $IL = il$. (These are not shown in our figure.) Rotate the segments HI and hi about the line Pp to obtain two "annular" slices of the sphere. Compare the attractions of these slices at P and p respectively. In other words, calculate the ratio

$$\frac{\delta F_p}{\delta F_P},$$

where δF_P and δF_p are the forces acting on P and p , respectively, by each of the two "annular" slices. This calculation merely uses the many similar triangles in the construction, which arise as a direct consequence of the choice of equal chords HK and hk . Finally, obtain

$$\frac{\delta F_p}{\delta F_P} = \frac{PS^2}{ps^2}.$$

Littlewood felt that the proof's key geometrical construction (of the lines PHK and phk cutting off equal chords HK and hk) "must have left its readers in helpless wonder". We agree, but unlike Littlewood we would argue that the integration Newton is supposed to have performed would in no way have suggested the key geometrical construction. In other words, there is absolutely no link between the supposed analysis and the synthesis.

As Chandrasekhar says, "his physical and geometrical insights were so penetrating that the proofs emerged whole in his mind" [Chandrasekhar, 1995, p.273]⁸³.

Our last example is a little more philosophical. Newton clearly and explicitly understood the Galilean relativity principle [DiSalle, 2006, p.28], and as was pointed out by [Penrose, 1987, pp.21–22] Newton even considered⁸⁴ adopting it as one of his fundamental principles. But in what framework was this principle to operate? We agree with DiSalle,

⁸³Compare Penrose's discussion of this feature of inspirational thought, and his remarks on Mozart's similar ability to seize an entire composition in his mind [Penrose, 1989, p.423].

⁸⁴This is in *De motu corporum in mediis regulariter cedentibus*. See [MP, 1967–1981, 6, pp.188–194]

who argues that Newton's absolute space and time shares with special and general relativity that space-time is an objective geometrical structure which expresses itself in the phenomena of motion [DiSalle, 2006, p.16].

We note that Newton is not alone in regarding geometry as yielding deeper insights. A striking modern example comes from . In the "Prologue" to his book Chandrasekhar says:

The manner of my study of the *Principia* was to read the enunciations of the different propositions, construct proofs for them independently *ab initio*, and then carefully follow Newton's own demonstrations. [Chandrasekhar, 1995, p.xxiii]

In his review of this book, Penrose describes Chandrasekhar's discovery that

In almost all cases, he found to his astonishment that Newton's "archaic" methods were not only shorter and more elegant [than those using the standard procedures of modern analysis] but more revealing of the deeper issues. [Penrose, 1995]

The examples above represent Newton's approach to geometry, and in particular, his change in thought from Cartesian methods to a fierce rejection of these methods, especially as he adopted a more classical view. Newton marked the end of a period of questioning of the meaning of geometry. The next great period of discovery and variation in approaches to geometry, it could be argued, was not until the 19th century when new geometries were developed, once again highlighting a divide in algebraic and synthetic approaches.

However Newton's work continued to have an impact on his successors in the 18th century. In the next chapter we pursue some of the key geometrical ideas which were further developed by Newton's successors in the first half of 18th century. As Guicciardini notes,

The Newtonian heritage branches into many different schools and styles of thought, which frequently developed in different directions distant from the great master's intentions. [Guicciardini, 2004a, p.219]

In light of our re-evaluation of Newton's geometry we will begin to examine and discuss the current literature concerning what became of Newton's geometrical methods.

Here we will speculate on the opportunity that remains for a more detailed and thorough examination of the continuation of Newtonian geometry in Britain when it is considered outside of the shadow of the calculus which dominates the historical view of geometrical development in the 18th century.

Chapter 5

The reception of Newton's geometry in the 18th century

Throughout our work we have concentrated on important questions surrounding the nature of geometry that were of concern during the 16th and 17th centuries, before focusing on Newton's approach to geometry.

In our examination of Newton's predecessors we saw attempts to confine and bound geometry by restricting to certain types of curves or instruments. In contrast we observed and demonstrated that Newton was less interested in applying these boundaries to geometry, and more interested in utilising its "tools", including some early projective ideas. In the 1680s Newton had resolved to write a great treatise of geometry in which he would express all of his mathematics in the style of the ancients. However, the *Geometria Curvilinear* [MP, 1967–1981, 4, pp.407–518] was never published, and in its most finished form contains much of Newton's own original work including the classification of the cubics and the method of fluxions, rather than being an explicit expression of ancient analysis and synthesis [Galuzzi, 2010, p.548]. By 1684 the work was abandoned after Halley had encouraged Newton to consider some problems on planetary motion, which would eventually form results in the *Principia* [Guicciardini, 2009, p.217–219].

Furthermore, and as we shall see, Newton was reluctant to write down his methods of discovery, which he believed adhered to the ancient methods of analysis and synthesis. This contributed to the accusation that his work was inaccessible (see [Guicciardini, 2009, pp.115–117]). Newton published with great caution and hesitation. However he was

eventually forced to publish more of his work as a result, in part, of his disputes with other mathematicians and scientists. The most famous example is the calculus priority dispute with Leibniz, but there were also the criticisms by Hooke over Newton's ideas on the theory of light and colour.

In this chapter we focus on the reception and influence of Newton's geometrical work, both during his life and into the eighteenth century after his death in 1727. Guicciardini helpfully notes that the era of mathematical study and response after Newton's death may be divided roughly into three periods [Guicciardini, 2004a, p.242]. Firstly, a continued adherence to, and study of, the ancients with the same reverence Newton had afforded them. Secondly, a divided period when favour for geometric methods began to give way to more algorithmic and algebraic practices. And finally, in the second half of the eighteenth century, a complete decline in Newtonian geometry, as well as British mathematics, compared with a fruitful period of development by the Continental mathematicians, especially in the area of calculus and analysis. We limit our discussion to the first period in order to open up areas for further discussion and research. In the context of our work we seek to highlight some of Newton's most direct influences, and to demonstrate the type of geometry being studied in Britain in the early 18th century whilst calculus was taking hold and being developed in the Continent.

During the middle part of his life, Newton had gained great recognition and fame, and had many advocates such as James Gregory and his nephew David, James Stirling, Colin Maclaurin, Roger Cotes, and Edmond Halley, who had instigated the publication of the *Principia*. However, Newton's efforts (and his infamously difficult character) were not universally well received. For example, Newton had well known disputes with astronomer John Flamsteed, Robert Hooke, and of course, Wilhelm Leibniz.

In the first half of the eighteenth century, some of Newton's followers continued to contribute in a direct way. For example, James Stirling worked on plane cubic curves in his *Lineae Tertii Ordinis Neutoniana* (1717) [Stirling, 1717], in which he also used the method of fluxions, as well as identifying four more species of curve [Guicciardini, 2004a,

p.233]. Colin Maclaurin produced a treatise on the organic construction [Maclaurin, 1720], as well as his highly significant *Treatise on Fluxions* (1742) [Maclaurin, 1742].

Maclaurin took an interest in many aspects of Newton's work, and was one of the most important characters in the continuation of his work. Guicciardini states that Maclaurin's four part *Account of Sir Isaac Newton's Philosophical Discoveries* (1748) [Maclaurin, 1748] proved to be "a popular, but extremely well informed, presentation of Newton's methodology and natural philosophy" [Guicciardini, 2004a, p.236]. As testament to its popularity a second edition of the work appeared in 1750, and a third in 1775.

5.1 The publication of Newton's work

In advance of any formal publication Newton corresponded with fellow mathematicians, as was usual in the 17th century. However, he still failed to disclose many of his results, especially details of proofs. Many of Newton's ideas were only uncovered in relation to the priority dispute with Leibniz. Guicciardini notes that the innovation of those early discoveries, and the impact they might have had had they been made public, cannot be overestimated [Guicciardini, 2004c, p.456]. For example, by 1666 Newton had already written an accomplished treatise on his method of fluxions. Two years later Mercator's *Logarithmotechnia* (1668) [Mercator, 1668] was published, containing results close to Newton's own on infinite series and the binomial theorem.

At this time, the only person Newton allowed his work to be passed to was the president of the Royal Society, William Brouncker, after which Newton requested its return. However, Newton's mentor, Isaac Barrow, had recognised potential in Newton. In 1669 Barrow communicated Newton's *De analysi per aequationes numero terminorum infinitas*, a treatise on infinite series, to John Collins. Collins was an accountant, publisher, and librarian for the Royal Society and did much to publicise the work of English and Scottish mathematicians. However, both Barrow and Collins had failed to convince Newton to allow the *De analysi* to be published. It finally appeared in 1711 during the priority dispute with Leibniz. As Guicciardini points out, this was a typical fate for Newton's work [Guicciardini, 2004c, p.457]. And as Newton expressed to Henry Oldenburg in a letter (26

October, 1676) “Pray let none of my mathematical papers be printed without my special licence” [Turnbull et al., 1977, 2, p.163]. Evidence and commentary such as this clearly demonstrates that Newton was secretive about his work.

Newton did however allow a close circle of friends and supporters to read, and sometimes reproduce, his manuscripts [Guicciardini, 1998, p.311]. This method of circulating manuscripts, which Guicciardini describes as “scribal publication”, meant that the works were only available to a purposefully selective group. However, it became more difficult for Newton to control the dissemination of his work as his popularity grew and monetary gain was to be made from unauthorised copies [Brewster, 1855, pp.440–443] in [Guicciardini, 2004c, p.461]. This also had the effect that many of the manuscripts were inaccurately copied or even broken up so as not be recognised as complete works. In spite of recent speculation, it remains unclear why Newton went to such extreme lengths to limit access to his work.

Newton was equally inconsistent with the publication of his work in areas other than mathematics. Whereas he was sometimes happy to present his optical findings in provisional form, he had to take political considerations into account with his theological researches [Guicciardini, 2003, p.411]. He surveys the various reasons which have been suggested as to Newton’s general reluctance to publish. For example, the cost of book printing after the Great Fire [MP, 1967–1981, 3, pp.5–6], and Newton’s neurotic character [Christianson, 1984, pp.137–143] have been given as possible reasons. Others have suggested that his interest in mathematics shifted to other subjects after 1670 [Mamiani, 1998, pp.39–48]. Whilst we now know that Newton had a continued and consistent interest in mathematics after this time, this does coincide with a significant shift in Newton’s attitude towards geometry in particular. Guicciardini makes the following remark about Newton’s “scribal publication” of his method of fluxions.

[Newton] found it convenient to avoid print publication of a method that appeared to him not well-founded from a logical point of view and distant from the rigor attained by the ancient geometrical synthesis. [Guicciardini, 2003, p.412]

However, speaking of the *Principia*, Newton says how he had used analysis as his

method of discovery, but that he had translated his results into synthetic geometrical form, achieving the certainty which would have satisfied the ancients. He speaks here in the third person.

By the help of this new Analysis Mr Newton found out most of the Propositions in his *Principia Philosophiae*. But because the Ancients for making things certain admitted nothing into Geometry before it was demonstrated synthetically, he demonstrated the Propositions synthetically that the systeme of the heavens might be founded upon good Geometry. And this makes it now difficult for unskillful men to see the Analysis by which those Propositions were found out. [MP, 1967–1981, 8, p.599]

But Guicciardini warns us, rightly so, that we should read with caution Newton's own words, and not to accept them as "faithful historical accounts" [Guicciardini, 2003, p.413]. Newton had spoken these words in light of the dispute with Leibniz and likely sought to distance himself from the modern methods. Furthermore, the timing is consistent with his rejection more generally of the modern analytic methods in favour of ancient synthetic geometry. Newton had believed that, in line with the ancient tradition, the only part of the geometrical process that was suitable for presentation was the synthesis. It was necessary to hide the analysis from view, and to share the methods of discovery with only select disciples. For example, on the quadrature of curves Newton wrote:

After the area of some curve has thus been found, careful considerations should be given to fabricating a demonstration of the construction which as far as permissible has no algebraic calculation, so that the theorem embellished with it may turn out worthy of public utterance. [MP, 1967–1981, 3, p.279]

The change that took place in Newton's relationship with the analytical methods after 1670 almost certainly had an effect on his publication. In fact, Newton did begin to rework some of his earlier discoveries into synthetic geometrical terms, which he assembled into his *Geometria curvilinear* of the 1680s [MP, 1967–1981, 4, pp.420–521]. In this work he used no infinitesimals and no algebraic symbolic representation, opting instead for geometric limit procedures. This is the method which appeared in the *Principia*. As Guicciardini notes

It is somewhat astonishing to see one of the most creative algebraists of the history of mathematics spend so much time and effort in reformulating his analytical results in geometric terms, but Newton had compelling reasons to do so. [Guicciardini, 2006, pp.1735–1736]

He cites, in particular, Newton's anti-Cartesianism combined with a belief of the superiority of the ancients.

In addition to Newton's rapidly changing view of geometry, another conflict may also have contributed to his reluctance to publish more generally. In 1671 Newton published his first paper on light and colour in the *Philosophical Transactions of the Royal Society* [Newton, 1671]. The paper contained results which he had obtained during his time away from Cambridge during the Plague a few years earlier. The reception of this paper was generally good, but Hooke and Huygens objected to Newton's experimental methods. Newton wrote to Collins on the matter (25 May, 1672).

I have now determined otherwise of them; finding already by that little use I have made of the Presse, that I shall not enjoy my former serene liberty till I have done with it [Turnbull et al., 1977, 1, p.161].

Newton's troubled relationships with his contemporaries continued, and in 1675 Hooke accused Newton of stealing some of his own results in optics [Shapiro, 2001, p.9]. Newton's full account of his researches, *Opticks*, did not appear until 1704, one year after the death of Hooke.

Only after the publication of *Opticks* did some of Newton's other works begin to appear in print, many of which he had developed decades earlier. For example, his *Arithmetica universalis* (1707) [Newton, 1707], which contained work on the relationship between algebra and geometry that Newton had developed during the 1660s, was published by William Whiston. Two more editions appeared, in 1720 (an English translation, by Joseph Raphson) and 1722 (a further Latin edition by John Machin), neither of which had Newton's name on it. An edition crediting Newton with the work was eventually published in 1732 by the Dutch mathematician, Willem 'sGravesande.

In the preface to the *Opticks*, Newton explained why he had appended the works *Enumeratio linearum tertii ordinis* and *De Quadratura Curvarum*, saying "Some Years ago I

lent out a Manuscript containing such Theorems, and having since met with some Things copied out of it, I have on this Occasion made it public [...] And I have joined with it another small Tract conceding the Curvilinear Figures of the Second Kind, which was also written many Years ago, and made known to some Friends, who have solicited the making it publick” [MP, 1967–1981, 8, p.92] in [Rouse Ball, 1890, p.105].

5.2 The Newtonians

Next we look at the continuation of Newton’s work in the early part of the 18th century, focusing on the direct contributions to the extension of Newton’s geometrical work. Many of Newton’s followers also sought to preserve a more classical, and synthetically geometrical, way of thinking. This inspired some continuation of Newton’s geometrical work in a more direct way, especially in the work of Stirling and Maclaurin which we discuss below. But, it also meant the bridging of a gap between the classical and modern methods. For example, Maclaurin’s *Treatise of Fluxions* (1742) served to mediate between the Newtonian geometry and the new analysis of the Continent by retaining a synthetic style in the first volume and a more analytical one in the second.

Guicciardini identifies four distinct “lines of research” within the Newtonian school: common analysis (algebra); the new analytical method of series and fluxions; the synthetic method of fluxions; and pure geometry, which he claims is “an anticipation of projective geometry” [Guicciardini, 2004a, p.232]. Whilst it is difficult to claim a relationship to the development of the formal subject of projective geometry, the early Newtonians did contribute many novel results, as we see below. Here Guicciardini also distinguishes an early period or activity, spanning the early to mid-eighteenth century, from a later period. We here focus on this early period, noting that much of the purely geometrical work in this period has been greatly overshadowed by the calculus. Guicciardini reinforces this idea by suggesting that “[c]ontrary to the received view, in the early period Newtonian mathematics flourished” [Guicciardini, 2004a, p.232]. He has identified, for example, Brook Taylor, James Stirling, Roger Cotes, Abraham de Moivre, and Colin Maclaurin, who all contributed greatly to geometry.

In his survey of *The development of Newtonian calculus in Britain: 1700–1800* (1989), Guicciardini comments further that whilst good mathematicians such as Cotes, Taylor,

Maclaurin, and Stirling were capable of mastering “every aspect” of Newton’s work on fluxions, more attention was given to areas such as the geometry of higher order curves and the method of series. He suggests that “[t]he reason why the calculus of fluxions was not considered a fruitful area of research is that it appeared to have been developed by Newton to the highest level of perfection” [Guicciardini, 1989, p.28]¹.

The classification of cubics in Newton’s *Enumeratio* (1704) attracted more attention. In a letter to Campbell (3 February, 1721, cited in the above footnote) Maclaurin said of the work:

You’ll [sic] find too new scenes of Learning opened up in the treatise of the lines of the third order. How poor do the old Geometricians seem now within their three conick sections we have got a new order of lines that contain no less than 72 kinds and are enriched with an innumerable multitude to those of the other orders. [Mills, 1982, pp.13–14]

The interest in Stirling’s contribution to the subject, *Lineae tertii ordinis newtonianae* (1717), is evident from the extensive list of subscriptions to the work, including two copies for Newton [Stirling, 1717]². Guicciardini here chooses to concentrate on the “project of extending the calculus of fluxions [by] Taylor and Cotes”, that being the subject of his book. Instead, we will focus here on the furthering of the geometrical aspects of Newton’s work, namely the study of cubic curves by Stirling, and the organic description of higher order curves by Maclaurin.

5.2.1 Continued study of geometry

James Stirling

James Stirling was born in 1692 in Garden, Scotland to a landed family. Little is known of his childhood or his undergraduate studies, which he completed in Scotland. In 1710 he travelled to Oxford to study there, probably with the intention of entering the church. Stirling’s first published work was *Lineae tertii ordinis newtonianae* (1717). Stirling applied

¹Guicciardini cites, for example, Colin Maclaurin who, in a letter to Colin Campbell (3 February, 1721), says “The *Quadratures* [(1704)] brought to such generall theorems that little further seems left to be done in that vast feild [sic] of Invention [Mills, 1982, p.13]” [Guicciardini, 1989, p.28].

²Page numbers are not given for the front matter, but a “Catalogue of Subscribers” is given in alphabetical order by surname before the main body of work.

the new analytical method of fluxions to the study of cubic curves, which had previously been classified by Newton (section 4.5), adding four new species not listed by Newton in his *Enumeratio*³. A copy of the work was received by Newton but it is not clear if Newton commented on the work⁴.

Later that same year Stirling travelled to Venice where he continued his mathematical researches and submitted a paper to the Royal Society entitled *Methodus differentialis Newtoniana illustrata* (1719). During this time he made the acquaintance of Nicolaus Bernoulli.

Stirling returned to Glasgow in 1722 with the intention of becoming a teacher. He expressed this wish to Newton who assisted Stirling by telling his plan to Colin Maclaurin. In 1724 Stirling travelled to London where he did become a teacher. During the next ten years Stirling corresponded with many mathematicians, but was particularly close to Newton. In a letter to his brother, John (5 June, 1725), Stirling comments on his relationship with Newton.

S Isaac Newton lives a little way of in the country. I go frequently to see him, and find him extremely kind and serviceable in every thing I desire but he is much failed and not able to do as he has done. [Tweedie, 1922, p.13]

Soon after, Newton nominated Stirling for a fellowship of the Royal Society, to which Stirling was elected in November 1726. This is likely where he met Maclaurin, he too being a close acquaintance of Newton at that time. Maclaurin frequently consulted Stirling during the writing of his *Treatise of Fluxions*. According to Tweedie's account they were both "intimate friends of Newton, and fervent admirers of his genius, and both eagerly followed in his footsteps" [Tweedie, 1922, p.15].

The following year Stirling formed a friendship with Gabriel Cramer, who visited the Royal Society in 1727. Tweedie notes that "[a] copy, kept by Stirling of a letter to Cramer furnishes interesting information regarding his own views of his *Methodus Differentialis*,

³We noted earlier that Newton had in fact identified these additional species of cubic by his projective method, but he had not included them in the final version of his *Enumeratio* for reasons which are not clear.

⁴Stirling may have first been brought to Newton's attention in 1715. John Keill wrote to Newton (24 February, 1715) concerning the problem of orthogonal trajectories, which had been proposed by Leibniz, stating that it had recently been solved by "Mr. Stirling, an undergraduate here" John Keill to Newton (24 February, 1715), [MS Add. 9597/2/18/69, Cambridge University Library, Cambridge].

and also regarding the date at which the *Supplement* to De Moivre's *Miscellanea Analytica* was printed" [Tweedie, 1922, p.15].

Stirling sent out two copies of his *Supplement*, one to Cramer, and one to Nicolaus Bernoulli via Cramer. Cramer had requested to be the intermediary between Stirling and Bernoulli so that he might benefit from their mathematical discussions. In the letters from Bernoulli, he pointed out several errata in the works of Stirling, including the "omission" by both Stirling and Newton of a species in their enumeration of cubic curves. In his *Lineae tertii ordinis newtonianae* Stirling had added four to Newton's seventy-two.

In the preface to his survey of the life and works of Stirling, Tweedie comments that "Stirling's influence as a mathematician of profound analytical skill has been a notable feature within the inner circle of mathematics" [Tweedie, 1922, viii]. He notes that Stirling was the first of three Scottish mathematicians who "earned for themselves a permanent reputation by their commentaries on Newton's work" [Tweedie, 1922, p.23], the other two being Colin Maclaurin [Maclaurin, 1720] and Maclaurin's student, Patrick Murdoch [Murdoch, 1746]. Murdoch, who had adopted a geometric approach similar to that of Newton's, had given a proof that all curves of third order can be obtained by suitable projection from one of the five divergent parabolas given by $y^2 = ax^3 + bx^2 + cx + d$. In their commentaries on Newton's *Enumeratio* the method of fluxions was not used by either Maclaurin or Murdoch⁵.

***Lineae tertii ordinis newtonianae* (1717)**

In his 1922 description of this work, Tweedie comments that the book is very scarce⁶. He notes that [Edleston, 1850, p.235] refers to a letter from Brook Taylor to John Keill (17 July, 1717), which gives a critique of Stirling's book [Tweedie, 1922, p.23]. However, neither a copy of the letter, nor any further details are given here. At the time of its printing the work must have been in reasonable circulation due to the number of subscriptions, as we noted above.

In the *Enumeratio* Newton had given no proofs of his statements. He had stated that

⁵ A short entry on the contributions of Murdoch can be found in [Andersen, 2007, pp.592–594].

⁶ Now freely available on the web!

the equation for the general cubic could be reduced to one of four forms, and had examined these cases one by one. Stirling, whilst following Newton's line of enquiry, had applied the method of fluxions to the study of cubic curves, adding four new ones which Newton had not included in his final copy of the *Enumeratio*. According to Tweedie, Stirling had based his analytical discussion Newton's *Series*⁷ and gave an account of Newton's *Parallelogram*⁸ for expanding y in ascending or descending powers [Tweedie, 1922, p.24].

Lineae tertii ordinis newtonianae also contains an appendix on three topics: the study of the brachistochrone, the catenary, and orthogonal trajectories to a family of hyperbolas. As Guicciardini notes these were "routine exercises" by 1720 [Guicciardini, 1989, p.36].

The first part (around two-thirds) of Stirling's book is given over to a substantial introduction to the method of fluxions, Newton's method of finding power series representations of fluents, and comments on Newton's *De analysi* (1711). Next he explains how the calculus of fluxions could be applied to the study of curves, for example, finding zeros, asymptotes, and cusps. Stirling then considered Newton's *Enumeratio*. Stirling had used the method of fluxions in his work on the cubics. He explained how the analytical methods of the calculus of fluxions could be successfully applied where Newton had preferred geometry [Guicciardini, 1989, p.36].

Stirling also gives a demonstration of how to trace rational curves. He explains that the manner by which a curve approaches its asymptotes is explained by means of series. Tweedie gives the following summary:

In the curves given by $y = a + bx + \dots + kx^n$ there are only two infinite branches which are on the same, or opposite sides of the x -axis, according as n is even or odd. When x is large the lower terms in x may be neglected as compared with kx^n . Then follows the graphical discussion of quadratic, cubic, and quartic equations in x . The graph of $y = x^2 + ax + b$ shows that the roots of the corresponding quadratic equation in x are real or imaginary according as the turning value of y is negative or positive. [Tweedie, 1922, pp.25–26]

⁷*De methodis serierum et fluxionum* (c.1671) [MP, 1967–1981, 3, pp.32–372].

⁸Newton's *Parallelogram* (or *Newton's Polygon*) approximates the possible values of y in terms of x if

$$\sum_{i,j=0}^n a_{ij}x^i y^j = 0.$$

He notes also that

For the cubic $x^3 + ax^2 + bx + c = 0$ he gives the excellent rule, which has recently been resuscitated, that the three roots are real and distinct only when the graph of the corresponding function has two real turning values opposite in sign. A similar test is applied to discuss the reality of the roots of a quartic. These results are required later in the enumeration of cubic curves. On p.69 he gives the important theorem that a curve of degree n is determined by $\frac{1}{2}n(n+3)$ points on it. The demonstrations of Newton's general theorems in higher plane curves are then given in detail. An indication of some of these is interesting, and the modern geometer will note the entire absence of trigonometry"[Tweedie, 1922, pp.25–26].

Stirling first gives an enumeration of the conics [Stirling, 1717, pp.80–83], before finally moving on to deal with the enumeration of the cubics starting with the reduction of the equation of a cubic to one of the four forms given by Newton [Stirling, 1717, p.83]. Each species is given in turn by Stirling who refers to the figures in Newton's work rather than giving his own. He gave four more in addition to Newton's seventy-two⁹. Tweedie notes that of the two remaining species to be found both arise from the form $xy^2 = ax^2 + bx + c$. One was found by François Nicole in 1731, and the second communicated by Nicolaus Bernoulli in a letter to Stirling in 1733 (1 April, 1733) in [Tweedie, 1922, pp.141–150].

As we noted above, Stirling was one of three Scots to have continued Newton's work on cubics. Stirling sought to validate Newton's work on the cubics, which Newton had famously left lacking in rigorous proofs. In contrast, Murdoch and Maclaurin had not used fluxions in their contributions to the study of cubic curves, in spite of their competence with these methods. Stirling's contribution to this work provides a valuable insight into the struggle between the new methods and the adherence to classical geometry which remained popular in Britain during this period. Stirling was a capable mathematician and did not limit himself to geometry. For example, he made significant contributions to the subject

⁹Species 11 on page 99, 15 on page 100, and 25 and 25 on page 102 of [Stirling, 1717]. As we noted in the previous chapter it is not known why Newton omitted six species from his *Enumeratio*, having identified all 78 by his method of projecting the divergent parabolas. Further details of the missing species and those added by Stirling are given by Whiteside. See especially, [MP, 1967–1981, 7, p.426, note 54; p.431, note 65].

of numerical analysis with his *Methodus differentialis* (1730) [Stirling, 1730]¹⁰, which included material on transformations of series and limiting processes. As we shall see, Maclaurin would take similar steps in bridging this gap in his *Treatise of fluxions* (1742), balancing the synthetic Newtonian methods with the analytical style of the Continental mathematicians.

Colin Maclaurin

Colin Maclaurin was born in 1698 in Kilmodan, Argyllshire, where his father was the minister of the parish. His father died when he was just a few weeks old. A few years later Colin's mother took him and his older brother John to Dumbarton so that they could go to school there. Colin's mother died when Colin was seven, leaving him and John in the care of their uncle. In 1709 Colin entered the University of Glasgow. Although he was only eleven years old, at the time it was usual for good students to enter university at a young age.

Maclaurin showed promise as a young mathematician. He studied Euclid's *Elements* independently during his second year at university. Maclaurin wrote his undergraduate thesis, aged just fourteen, *On the power of gravity*, in which he developed Newton's ideas. According to Murdoch "in his sixteenth year, he had already invented many of the propositions afterwards published under the title of *Geometria Organica*" [Maclaurin, 1748, pp.ii–iii]. At Glasgow he also met Robert Simson who was a professor of mathematics at the university and had a great enthusiasm for ancient geometry [Turnbull, 1947, p.318]. Maclaurin and Simson would continue to correspond of matters of geometry for several decades after their first meeting¹¹.

Maclaurin had initially intended to enter the church, but on deciding not to he returned to live with his uncle, where he studied mathematics independently until 1717 when he was appointed as professor of mathematics at Marischal College, Aberdeen. In 1719 Maclaurin visited London, where he became acquainted with Newton. During this visit he was elected a Fellow of the Royal Society.

¹⁰The work has been translated with commentary by Tweddle [Tweddle, 1991].

¹¹See, for example, [Tweddle, 1991].

From 1722 to 1724 Maclaurin neglected his duties at Aberdeen in favour of a trip to France, acting as tutor to the son of Lord Polwarth. During this time, Maclaurin remained active in mathematics and was awarded a prize by the *Académie des Sciences* in Paris for his paper [Maclaurin, 1724]. However, Polwarth's son died suddenly of illness, prompting Maclaurin to return to his post at Aberdeen.

Maclaurin later sought a position at the University of Edinburgh for which Newton offered his support¹². Maclaurin took up his post at Edinburgh in November 1725 and remained there until his death in 1746 [Scott, 1981, 8, pp.609–612].

Guicciardini argues that “one of the main objectives of Maclaurin's research in the 1730s was to systematise in a comprehensive theory the various aspects of Newton's mathematical work” [Guicciardini, 1989, p.37]. In fact, Maclaurin had already begun to consider Newton's work in detail. In the early 1720s Maclaurin had studied in particular the *Enumeratio* (1704), during the preparation of his first works in geometry, *Geometria organica* (1720) [Maclaurin, 1720] and *De Linearum* (1721) [Maclaurin, 1748a]. Maclaurin took a keen interest in the organic description of curves, and as Bruneau comments, Maclaurin's undertaking of this early work had been supported by Newton:

At the end of the *Enumeratio*, the author proposes an “organic” construction of curves, in particular of conic sections [Newton, 1704, p.158]. Maclaurin ignores this completely in the *De Linearum* [Maclaurin, 1721] but dedicates an important part of the *Geometria Organica* to this problem [Maclaurin, 1720, pp.2–78]. This is one of the reasons why Newton insisted in 1719 that this work should be published. [Bruneau, 2011a, p.15]

We focus next on the *Geometria organica*¹³.

Geometria organica (1720)

Three years after Stirling's first contribution furthering the work of Newton, Colin Maclaurin's treatise on the organic description of curves, *Geometria organica* (1720) [Maclaurin, 1720],

¹²Edleston remarks on the following letters: Newton to Maclaurin (*date unknown*, 1725) glad that he has a prospect of being joined to James Gregory in the Professorship of Mathematics at Edinburgh, and heartily wishes him good success. Newton to Lord Provost of Edinburgh (*date unknown*, 1725) is ready to contribute £20 per annum towards a provision for Maclaurin, if he will act as assistant to Gregory [Edleston, 1850, p.xl].

¹³Detailed accounts of the development of both the *Geometria organica* and *De Linearum* can be found in [Bruneau, 2011, ch. 3, ch. 4].

was published. Whereas Stirling had favoured the method of fluxions in his work, Maclaurin did not do so in the *Geometria organica*.

Maclaurin's *Geometria organica* extended Newton's organic description of curves. Shortly after its publication, an account showing the significance of the work was given in the *Philosophical Transactions of the Royal Society*, which describes Maclaurin's treatise as follows.

The design of this treatise, is to examine the various methods proposed by Mathematicians, for describing geometric curves; and at the same time to demonstrate a new one, infinitely more general than any hitherto published; built on those theorems proposed by our illustrious President, at the end of his Enumeration of the Lines of the Third Order. [Account, 1721, p.38]

Although its author is not known, the account gives something of an insight into firstly, the impressiveness of Newton's organic description, and secondly, the ever present scepticism of the focus on "Mechanic and Exponential" curves. The account continues

The great improvements that have been made by most of the other modern Geometricians, have related chiefly to the lines of the infinite order; they have been so fond of applying their new methods to mechanic and exponential curves, (which undoubtedly ought to give place to those that are more strictly Geometrical) that they have neglected to cultivate Geometry after the most regular manner. [Account, 1721, pp.38–39]

The criticism is that little was known of geometric curves of higher order, except in a few rare instances. Further praise is given to Newton for his enumeration of lines of the third order with the accolade that he had "enlarged the Bounds of Geometry". It is then stated that the aim of Maclaurin's treatise was to remedy this neglect by giving "an universal description of all geometric lines of the third, or any order whatsoever" [Account, 1721, p.39]. In addition to the description of the curves themselves, Maclaurin also demonstrated methods for determining asymptotes and other properties. He also provided demonstrations of theorems that had previously been published by Newton in the *Enumeratio*. The account in the *Philosophical Transactions* outlines impressive examples of the use of the organic description to describe lines of the 1024th order ("by making angles move on seven conick

section”) and “by three Conick Sections more, lines may be described above the 11,000th order” [Account, 1721, p.40].

Maclaurin used no fluxional techniques in this part of his work, but he did include a section on theorems on central forces to show the use of curves in natural philosophy [Maclaurin, 1720, pp.120–135]. Here Maclaurin used Newton’s dot notation to represent infinitesimal quantities. Guicciardini notes that “[i]t is interesting that a use of infinitesimals is found in the early work of Maclaurin. He later became a great adversary of “infinites”. Maclaurin’s shift is typical of the development of the fluxional calculus in the first half of the century” [Guicciardini, 1989, p.37, note 25]¹⁴. He notices also that this section is quite separate from the rest of the work, and no attempt is made to link Newton’s *Enumeratio* (1704), where the organic description appears, with the analysis of *De quadratura* (1704) [Guicciardini, 1989, p.37].

It would be a great understatement to suggest that Maclaurin’s treatise simply developed the work of Newton. He contributed substantial further material worthy of further study. For example, Maclaurin made a study of “pedal” curves¹⁵ in this paper as well as providing a mechanical method for describing them. Many further details of the work are given by Tweedie in his paper on Maclaurin’s *Geometria organica*. He points out, for example, that Maclaurin had anticipated what is now known as Cramer’s paradox [Tweedie, 1916, p.148]. Through his study of Maclaurin’s work, Tweedie has also identified that the transformations effected by Newton’s organic rulers are in fact the Cremona transformations [Tweedie, 1916, pp.94–95]¹⁶.

Treatise of Fluxions (1742)

Attacks on Newton’s *Principia* did not only come from the Continent and the Leibnizian school. It was also a subject of criticism for the Irish bishop and philosopher George

¹⁴See also [Guicciardini, 1989, Chapter 3, §3.4] and [Guicciardini, 1989, p.47] on Maclaurin’s rejection of infinitesimals.

¹⁵The pedal of a curve C with respect to a “pedal” point P is the locus of the foot of the perpendicular from P to the tangent to the curve. For example, the “pedal” curve of a parabola is the “Maclaurin trisectrix”, which can be used for the resolution of the classical problem of trisecting an angle. The name “pedal” does not originate with Maclaurin, but was given by Olry Terquem in the 19th century.

¹⁶We believe Tweedie to have been the first person to recognise this, although Shkolenok is usually credited with this observation [Shkolenok, 1972].

Berkeley. In 1734 Berkeley's work *The Analyst* [Berkeley, 1734] was published. Berkeley argued that whilst there was no question that the results of the calculus are true, it was not founded upon logical reasoning any more than religion. He questioned the status of fluxions:

And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities? [Berkeley, 1734, p.59]

Berkeley's criticisms did inspire attempts by De Moivre, Taylor, Maclaurin and others to make the foundations of the calculus more rigorous by using classical arguments. Maclaurin's *Treatise of Fluxions*, which Guicciardini calls "the true manifesto of the fluxionists" [Guicciardini, 1989, p.x], is recognised to have been the best response to Berkeley¹⁷.

Maclaurin's *Treatise of Fluxions* (1742) [Maclaurin, 1742] served to mediate between the Newtonian geometry and the new analysis of the Continent. Guicciardini notes that "[i]n the years in which Maclaurin had composed the *Treatise*, Continental mathematicians were in the process of "de-geometrizing" the Leibnizian calculus" [Guicciardini, 2004a, p.241]¹⁸ The first half of Maclaurin's treatise was geometric in style, whilst the second part of the work balanced the geometric with the analytic¹⁹. Maclaurin stressed the importance of combining the new methods with the rigorous nature of the geometry of the ancients. He expresses this in the opening paragraph of the introduction to the *Treatise*:

Geometry is valued for its extensive usefulness, but has been most admired for its evidence; mathematical demonstration being such as has been always supposed to put an end to dispute, leaving no place for doubt or cavil. It acquired this character by the great care of the old writers, who admitted no principles but a few self-evident truths, and no demonstrations but such as were accurately deduced from them. [...] it has been objected on several

¹⁷On Berkeley's criticisms see, for example, [Grattan-Guinness, 1969]. See also Murdoch's brief commentary of the dispute and Maclaurin's achievement in [Maclaurin, 1748, pp.viii–ix]. A thorough account of both Maclaurin's response to Berkeley's *Analyst* and the subsequent development of *Treatise of Fluxions* is given in [Bruneau, 2005, Part C].

¹⁸See also [Bos, 2001, p.10].

¹⁹See [Bruneau, 2005, pp.317–327] for a comparison of the two parts of Maclaurin's work.

occasions, that the modern improvements have been established for the most part upon new and exceptional maxims, of too abstruse a nature to deserve a place amongst the plain principles of the ancient geometry: And some have proceeded so far as to impute false reasoning to those authors who have contributed most to the late discoveries, and have at the same time been most cautious in their manner of describing them. [Maclaurin, 1742, p.1]

Guicciardini notes the significance of the *Treatise* which he claims is “the most influential work on the method of fluxions written in the eighteenth century and can be taken as representative of the Newtonian tradition” [Guicciardini, 2004a, p.221]²⁰. Maclaurin’s continued belief in the value of the rigorous nature of classical geometry was, like Newton, defined by his need for a clarity and certainty that could not be obtained from algebra. He was able to see the progressive nature of calculus and its role within geometry, but saw the classical methods as necessary to provide it with firm and solid foundations. At the same time, Maclaurin was also able to see the potential of an analytical point of view and that there was progress to be made. In doing so, Maclaurin’s *Treatise* publicised Newton’s calculus in a way that could be accepted on the Continent. In fact, it was translated into French in 1749 and, according to Guicciardini, was read by “some of the foremost Continental mathematicians”. He stresses, however, that Maclaurin did not represent the Continental school, and a comparison between the *Treatise* and contemporary Continental works “reveals more conspicuous differences than agreements” [Guicciardini, 2004a, p.240]. Maclaurin remained a staunch defender of Newton’s work and the geometric style (see, e.g., [Bruneau, 2010]).

5.2.2 Restoration of ancient texts

In addition to an interest in Newton’s original work in the early 18th century there was also a sustained enthusiasm for classical texts. As we saw in the previous chapter, Newton developed an interest in ancient analysis, especially in the 1690s, and we highlighted in particular his work on porisms. Whereas Newton had gained most of his information on this topic from Pappus’ *Collectio*, Book 7, in the 18th century Halley and Robert Simson had attempted restorations of Euclid’s *Porisms*—the original work from which Pappus claims to have taken his information.

²⁰See also [Grabner, 1997, p.394–395].

The subject was still not fully understood, and as we have seen was believed to be of utmost importance to the geometrical analysis of the ancients. Guicciardini suggests that they “were motivated by the genuine belief that the geometrical analysis of the ancients was superior to the modern techniques of the calculus” [Guicciardini, 1989, p.37]. The topic was continued well into the 18th century with additions to the subject given by Matthew Stewart under the title *Some General Theorems of Considerable Use in the Higher Parts of Mathematics* (1746) [Stewart, 1746], although he avoided the use of the word “porism”, reportedly to avoid anticipation of Simson’s work [Porism, 1842, p.442]. Further discussions on porisms were given towards the end of the century by Playfair [Playfair, 1794] and Wallace [Wallace, 1798].

Simson’s first results were presented in [Simson, 1723]. It was arguably the most illuminating work on porisms of the time, since Simson had succeeded in explaining the only three complete propositions which Pappus had indicated in Book 7 of the *Collectio* (two of which we gave in chapter 2), including those given by Newton (chapter 4). Further and more general work was published posthumously in 1776 [Simson, 1776, pp.315–594]. In particular, Simson had stated that Pappus’ definition was too general. He also identifies the locus as a species of porism (See [Heath, 1981, p.435]). Recall that in the previous chapter we speculated on the possibility that porisms need not always be akin to locus problems as they are often thought to be today. The topic remained of interest and further work on the subject was also undertaken in the mid-19th century by Chasles [Chasles, 1860], who also makes several references to porisms and the work of Simson and Stewart in his *Aperçu historique* (1837) [Chasles, 1837].

In the context of our study, it is intriguing that the porisms continue to offer a point of interest. As we have observed above, particular attention was given to the porisms by the Newtonians. We identified in the previous chapter that the nature and role of porisms had always been elusive, due in no small part to the lack of available classical evidence. Most of the information obtained on the subject during the early modern period, and even for Newton, came from Pappus’ limited extractions of Euclid’s original work.

The porisms remained a subject of study in the 18th and even into the 19th centuries,

notably inspiring results in projective geometry. Guicciardini comments on the influence of Newton's projective work on his successors, and in particular, his "mathematical classicism". This is especially evident in the restored classical works of Apollonius' *Conics* by David Gregory and Edmond Halley, and of Euclid's *Porisms* by Robert Simson and Matthew Stewart. He points out that whilst these were restorations of ancient texts they also "produced innovative results in projective geometry that were still praised in the nineteenth century by Michel Chasles" [Guicciardini, 2004a, p.235]. There remains an opportunity to develop a further, more detailed, study of the transition of ideas surrounding this topic between the 16th and 19th centuries. We discuss this opportunity later in the further research section of our concluding chapter.

5.3 Other commentaries on and later references to Newton's work

5.3.1 The calculus priority dispute with Leibniz

It would be impossible to consider properly the reception of Newton's work in the 18th century without acknowledging the impact of various disputes Newton had with his contemporaries.

We noted above, for example, a dispute with Hooke over some results in Optics, which may have led to Newton postponing his publication of *Opticks* until after Hooke's death in 1703. Also, the foundations of Newton's method of fluxions were undermined by Berkeley, the most notable response to which being Maclaurin's *Treatise of Fluxions*. Guicciardini also points out the methodological tensions between Newton and the naturalists at the Royal Society [Guicciardini, 2004c, p.463, note 34]²¹.

The main source of criticism of Newton in the 18th century was undoubtedly the famous calculus priority dispute with Leibniz. The dispute itself, and evidence in favour of either side has been well documented by many historians. Details are given in both [MP, 1967–1981, 8] and in [Hall, 1980]. A brief summary of facts established according to manuscript evidence is given in [Guicciardini, 2006, p.1737, note 20]²².

In line with his preference for classical mathematics and his rejection of Cartesian and

²¹On this issue see also [Feingold, 2001].

²²See also [Guicciardini, 2009, pp.331–332].

algebraic methods, Newton did not want to show publicly that he had used such methods. As Guicciardini notes, under its façade of classical geometry, the *Principia* “hides a panoply of mathematical methods: series, infinitesimals, quadratures, geometric limit procedures, classical theories of conic sections and higher curves, projective geometry, interpolation techniques, and much more” [Guicciardini, 2003, p.407]. In light of the priority dispute the Leibnizians were critical of Newton’s “translation” of propositions into synthetic style, suggesting it “unreadable” [Guicciardini, 1998, p.308].

As we commented above, Newton retained a close group of disciples and contacts, among whom he shared his ideas and unpublished manuscripts. Guicciardini notes that the ideas Newton promoted were “in sharp conflict with the values of the Leibnizian school” [Guicciardini, 2004a, p.231]. The anti-Cartesian ideas which Newton had developed in the 1670s were maintained by him, and even strengthened over the following decades. The dispute with Leibniz was about more than priority, Leibniz favoured a completely different view of mathematics.

Newton’s influence on the use and development of the calculus is usually judged less favourably than that of Leibniz. In particular, the modern use of Leibnizian notation is often referred to. However, we point out that in modern differential geometry Newton’s dot notation is used. Nonetheless, the calculus developed much more rapidly in Europe (especially France) where it was divorced from any geometrical roots, and given a more abstract interpretation, which is more familiar to the nineteenth and twentieth centuries [Guicciardini, 2004a, p.221]. By the second half of the eighteenth century a divide had started to form between the Newtonian geometric style and the analysis of Europe.

5.3.2 Further commentaries

There is a common perception that British mathematics was in steep decline in the 18th century due in no small part to a preference for geometrical thinking [Guicciardini, 2004a, p.218]. As we have already commented, a divide was forming between the geometrical traditions of Newtonian Britain and the modern analytic style of the Continent. This can be seen especially towards the mid-18th century in the works of the disciples of Newton and Leibniz.

The reasons for this divide between the two communities after the mid-eighteenth century remain an unanswered historiographic problem. [Guicciardini, 2004a, p.248]

In his *Treatise* (1742), Maclaurin negotiated this difference by presenting his work in two halves appealing to two different audiences. However just a few years later Euler published his celebrated *Introductio in analysin infinitorum* (1748) representing a completely different style which Bos describes as “de-geometrization” [Bos, 2001, p.10]. As Guicciardini informs us

The first volume is entirely devoted to defining, classifying, and manipulating “functions” of one or more variables, defined as symbolic expressions involving variable and constant quantities. Such an approach can not be found in either Newton or Leibniz [...] In the second volume the algorithmic techniques so apprehended are applied to geometric topics such as the study of cubics, quartics, asymptotes, curvatures, and surfaces.”[Guicciardini, 2004a, p.243–244]

This represented a shift from the consideration of geometry as the resolution of geometrical problems to the study of geometrical objects. The symbolic representation which began with Viète, but who had taken little interest in it, had evolved sufficiently to allow easier manipulation of geometrical expressions. Note also that the idea of a “function” was a particularly modern one, and is essential to contemporary geometry.

In his earlier *Mechanica* (1736), Euler applied calculus to mechanical problems. Referring to [Euler, 1736, p.8] Guicciardini comments that “Euler was aware of the fact that he was departing not only from the Newtonian tradition of the *Principia*, but also from the Leibnizian one exemplified by Jacob Hermann’s *Phoronomia* (1716)” [Guicciardini, 2004a, p.245]. In the passage quoted by Guicciardini Euler is critical of the *Principia* since the analytical methods are not clear, and this leaves the reader with little understanding of the problem at hand. Furthermore, the texts (the *Principia* and Hermann’s *Phoronomia*) do not teach one how to deal with problems that are “slightly different”. However, Euler says that he spent some years trying to understand for himself the “analysis behind those synthetic methods”, and in doing so he perceived a “remarkable improvement of my knowledge”.

5.4 Concluding comments

In this chapter we have observed the difficulties faced by Newton as a result of his reluctance to publish. It is not absolutely clear why this happened, and as we have seen, many ideas have been suggested. It may well be that these all contributed in some way. However, in light of our research and examination of Newton's approach to geometry, we propose an additional observation. We believe it is reasonable to suppose that Newton appears to have rarely viewed his work as being complete. In the previous chapter, we pointed out that Newton had returned to many topics time and time again, revising and redrafting his manuscripts. This approach was in stark contrast to his predecessors who sought to claim the completion of geometry.

Newton's publication strategy made for a broken and fragmented dissemination of his work, as well as publication under difficult circumstances which left him open to criticism. What he did publish was often incomplete and lacking in proofs. Both of these aspects had a significant impact on his successors who tried to continue his work, but presented them with an opportunity to reinvigorate interest in Newton's approach whilst allowing for progress and development. For example, the contributions made by Stirling and Maclaurin which we discussed in section 3, show that they were more open to taking elements of both synthetic and analytical geometry. In this sense, it is quite striking that their ideas were not taken up more readily.

Whilst Stirling, Maclaurin, and their contemporaries were edging towards geometrical progress it was not nearly as rapid as that of their Continental counterparts. There remained a large emphasis on trying to uncover and understand what the ancients had done, and this perhaps held them back. Again, this is quite surprising, since they were still able to come up with novel results, in particular projective ideas (even though no firm subject had yet been formalised)²³. As we have noted, many of these results were praised in the 19th century by Chasles, but in a historical context.

As we have discovered, there remain many unanswered questions surrounding the geometry of this period. A divide between British and Continental geometry from at least

²³See, for example, the paper by Tweddle [Tweddle, 1991].

the mid-18th century has been widely acknowledged, but this does not necessarily indicate a cessation in the study of geometry in Britain altogether. An opportunity is presented here for further research to understand what geometrical research and education took place after 1750. In particular, one could try to understand the view of geometry as a subject in this period before the discovery of non-Euclidean geometries changed the whole landscape.

Chapter 6

Conclusion

Our original project set out undertake critical and explorative research in order that we might better understand Newton's work and thought as a geometer. We limited the scope of this project to the period between the 16th and 18th centuries. This period was defined by a rapidly changing view of geometry in light of the rereading of Pappus' *Collectio*.

Throughout our research we uncovered repeated themes and connections that provide a framework in which the approaches to geometry of our main protagonists—Viète, Kepler, Descartes, and Newton—can begin to be compared, contrasted, and examined. As we were to discover, here a particularly sharp distinction can be drawn between the approaches of Descartes and Newton.

In order to draw such comparisons, it was first necessary for us to understand the questions and challenges facing geometers of this period. Many of the questions—such as the definition of geometrical curves, the appropriate use of geometry, and the exploration of various construction methods—were inspired by the reading of the Latin translation of Pappus' *Collectio* (1588).

Building on this foundation we next examined the specific reaction of Descartes to the changing nature of geometry. Descartes, too, had responded directly to the words of Pappus and his view of classical geometry. However, he was greatly dismissive of the geometry of the ancients, and went to great lengths to redefine geometry from a *tabula rasa*. More than any of his predecessors Descartes was to have a fundamental influence on the introduction of algebra into geometry.

Our exploration of these various responses to a changing view of geometry set the context in which we began our examination of the geometrical work of Newton. In particular, we observed the progression of his geometrical thought away from a Cartesian view towards classically synthetic methods. Here we arrived at the pivotal point of our thesis in our examination of Newton's work on the organic construction. We observed that his achievements here emerged from his critique of Descartes' algebraic solution of the Pappus problem. We argued that these achievements have not been given the value they deserve, and in particular we introduced a number of new interpretations in relation to Newton's organic construction, notably in contrast to those of Whiteside.

Finally, we saw that, contrary to conventional opinion, geometry in Britain did not end with Newton and the introduction of the calculus. Whilst it was undoubtedly overshadowed by the analytical work being undertaken on the Continent, several key geometers continued to contribute to furthering the work of Newton. We examined a number of examples of Newtonians, and we speculated on what opportunities remain for further exploration of this somewhat overlooked period in the history of geometry.

In the following discussion we provide provide a précis of the achievements and critical observations drawn from each of our primary chapters. This brief critical summary of the research provides a synthesis of the observations made in this thesis, and makes explicit our original work. We reiterate our views on the themes and connections leading up to and concerning the organic construction, which is the pivotal point in our study, and where we make our most significant contribution. Then we highlight the further potential research questions which arise as a consequence of our research, specifically the work of the Newtonians in the 18th century.

6.1 Chapter summary

The Early Modern Geometers

In this chapter we provided a foundational examination of the challenges faced by geometers of the early modern period in response to the interest in classical civilisation of the Renaissance, and in particular Commandino's recent Latin translation of Pappus' *Collectio* (1588). We focused specifically on the exploration of and reflection upon Pappus'

commentary on the ancient methods of analysis and synthesis. We looked at two highly contrasting points of view.

Here we noted how, by redefining algebraic operations in order to apply them to geometrical processes, Viète's method had retained a connection to the classical approach. We observed the considerable effort made towards finding a geometrically acceptable means of construction, and how in contrast to Viète, Kepler sought to retain a much closer attachment to a Platonic view of geometry. In spite of their differences it is clear from our study here that Kepler and Viète both demonstrated a desire to refine formal geometric parameters.

Finally we saw a retained emphasis on construction and on looking back to reflect on what the ancients had done. However, now there was potentially a new method for the classification of geometric problems, namely an algebraic one. Whilst considerable developments had been made the questions and preoccupations of the early modern period persisted, but the focus and resolutions evolved over the next century.

Descartes

Whilst Descartes also took an algebraic approach to the analysis of geometrical problems, we saw that it is still unclear how much of Viète's work was known to him. We identified key differences in their work, such as their interpretation of dimension and Descartes' preference for the use of instruments in geometrical construction, compared with Viète who had somewhat avoided the issue. Furthermore, in the context of Descartes' geometrical work we observed the implications of the rapid development of geometrical thought at this key period in mathematical history. Our examination of these developments provides a context from which to compare Newton's exploration of and approach to geometry, in particular, his response to the Cartesian methods.

We noted how in relation to Descartes' geometrical thought, there seems to have been a shift in opinion over the last thirty or forty years from the idea that Descartes' had simply applied algebra to geometry to the "geometrization of algebra". Boyer has claimed that in this way Descartes was "returning in thought to the ancient geometrical algebra, while at the same time he encouraged the development of symbolic forms of

expression”[Boyer, 1959, p.391]. Similarly we noted that Molland agrees that it is not so straight forward to characterise the mathematical work of Descartes. He states that the view of Descartes as “inventor of analytical geometry” is “not satisfactory” [Molland, 1976, p.22]

Finally, based on our reading of Descartes’ work and of secondary commentaries, we concluded that Descartes can be perceived as having approached geometry as a means to an end. It appears that for him geometry was something to be completed and “tidied” in order to be put to use in other sciences and the “mechanical arts”.

Newton’s geometry

Having prepared the context, in chapter 4 we focused on the main subject of our thesis, namely the geometry and geometrical thought of Isaac Newton. By examining Newton’s mathematical development from his undergraduate days and early career right through to the 18th century, we specifically observed an abrupt change in his response to the Cartesian methods.

After an initially positive response to the algebraic analysis of Descartes, we observed how Newton came to criticise and reject these methods. His rejection strengthened with his growing admiration for the classical geometers. Newton’s rejection of the Cartesian methods in favour of a more synthetic approach is perhaps nowhere more evident than in his solution to the Pappus problem, which had played a central role in Descartes’ *Géométrie*.

Newton used the Pappus problem to specifically attack Descartes. Here we noted how Newton’s interest in such locus problems, and indeed the related porisms, came from his interest in classical geometry. Both of these aspects led naturally to the organic description of curves. This was significant for two reasons. Not only did it fulfil Descartes’ criteria of single continuous motion, it allowed Newton to describe general curves, which had not been achieved before. Secondly, Newton was able to study singularities of curves by their resolution. Further, we believe that it was likely Newton actually made and implemented these rulers, unlike Descartes’ various instruments which were both less practicable and less powerful.

Here we argued that Newton’s work on the description of curves has been undervalued.

Specifically, we challenged Whiteside on his treatment of Newton's organic construction on two counts. First, we introduced evidence that led us to refute that Newton was strongly inspired by van Schooten. In contrast to acknowledged commentaries by Whiteside, we demonstrated that Newton's treatments were completely unprecedented in various ways. Second, we noted that Whiteside's "explanation" of the transformation, which results in a preservation of degree between the directing and described curves, does not work. We emphasised instead that it was the standard quadratic transformation. Because of this interpretation, we claim that Whiteside had failed to see the full significance of the organic construction, such as Newton's use of it to study singularities of curves by their resolution.

We also focused on the development of the enumeration of the cubics as an example of a topic that Newton had returned to many times over a period of forty years. In his returning anew to questions of geometry he perceived as unfinished we observe the evolutionary nature of Newton's approach. In particular we observed how Newton improved his algebraic procedures as well as developing a remarkable projective method after refocusing on classical geometry in the 1690s. Examples of Newton's ease of use of projective methods are evident both here and in work on locus problems, in spite of there being no formal subject of projective geometry at this time.

We also suggested that it was perhaps because Newton had been far less focused on foundational aspects of geometry that that he had the freedom to return repeatedly to topics such as loci problems and the cubics. In doing so we can perceive that he was able to take steps that his predecessors were not able to because they had been governed by a perception of geometry as bounded and finite.

Finally we saw Newton's geometrical thought exemplified in his work in physics. This allowed us to demonstrate that Newton's geometrical thought was not restricted to his work on geometry, but that it also permeated his scientific endeavours, where we saw that geometry gave Newton a certain clarity that he could not obtain through algebraic methods.

After Newton and The Newtonians

Having examined Newton's geometry and perceived that his work in many ways marked the end of a period of questioning of the meaning of geometry, here we moved on to discuss

how Newton's work had continued to affect his successors in the 18th century.

We began by briefly discussing the current literature concerning what became of Newton's geometrical methods. Here we observed the difficulties faced by Newton as a result of his reluctance to publish, remarking that it remains unclear why this happened. Newton's publication strategy made for a broken and fragmented dissemination of his work, as well as publication under difficult circumstances which left him open to criticism.

Furthermore, we noted that what Newton did publish was often incomplete and lacking in proofs, and that this left an opportunity for his successors to reinvigorate interest in his synthetic approach to geometry whilst allowing for progress and development. Here we identified, in particular, the contributions of Stirling and Maclaurin who had both undertaken work developing the main examples from our previous chapter. We noted, specifically, that they had been open to using elements of both synthetic and analytical geometry. In doing so they made significant contributions to the development of geometry, and yet we noted that it is quite striking that their ideas were not taken up more readily. We suggested that they had perhaps been restricted by a continued emphasis on understanding what the ancients had achieved.

In our study of the Newtonians we observed that there remain many unanswered questions surrounding the geometry of the 18th century. Whilst the divide between British and Continental geometry has been widely acknowledged, we observed that this does not reflect a cessation in the study of geometry in Britain altogether. Thus we were able to speculate on opportunities for further research to understand what geometrical research took place after 1750, and outside of the shadow of the calculus which dominates the historical view of geometrical development in the 18th century.

6.2 Themes

As we noted in the introduction the chapters in this thesis are structured around a historical progression from the foundations of the early modern period, through to the work of Newton and beyond. However, we also noted that the chapters are connected by various themes which recur throughout the period and provide distinct contrasts between the work of Newton and his predecessors. In this section we briefly return to these themes, noting

the key points of comparison we have drawn in our work and how they have been crucial in developing our narrative and understanding of Newton's geometry, and especially his work on the organic construction.

Geometry and Mechanics. One of the main distinctions that has been made by all of our key protagonists is that between geometry and mechanics, or geometrical and mechanical. In particular, the distinction has been used to separate geometrical from non-geometrical.

In chapter 2 we noted that Kepler had adopted a Hellenistic definition of mechanical, meaning a procedure of minor adjustments not restricted to instruments—a cone, for example, could be gradually aligned. Kepler criticised what he saw as mechanical procedures in the *Collectio*, which he believed had a lower intellectual status.

In his *Various replies on mathematical matters*, Viète had commented that shifting rulers were mechanical, but not geometrical [Viète, 1646, p.359]. Whilst he made the distinction between geometrical and mechanical, Viète also allowed for a compromise between what could physically be constructed and ideal geometrical objects. In this sense we noted that he retained the classical character of wanting to produce something which is exact and ideal, but acknowledges that where this is not possible we may still gain knowledge from an accurate description.

In contrast, we noted that Descartes criticised what he saw as the ancient distinction between mechanical and geometrical. For Descartes, instruments were not of lower status than ruler and compasses simply because they were mechanical. Instead, he applied a criteria of acceptability to the type of motion produced by an instrument. We suggested that Descartes' use of the word "mechanical" pertained directly to mechanics, rather than to a description of motion.

Finally we observed how in the preface to the *Principia* Newton distinguishes between what he viewed as rational and practical mechanics. In particular, we note that he saw a relationship between rational mechanics and geometry. Newton saw the subjects of mechanics and geometry as being closely related, with geometry depending upon mechanics. Newton had also completely rejected Descartes' distinction between "geometrical" and

“mechanical” curves. In particular, he objected to the idea that geometry could somehow be restricted to curves that could be constructed by ruler and compasses or, in Descartes’ case, generalised compasses.

Hidden analysis. In addition to the wider thematic connections of mechanical or geometric methods, we also note a particular contrast between Newton and Descartes in their responses to the analysis of the ancients. Descartes had viewed the ancients as purposefully deceptive, whereas Newton valued the elegance and certainty of their geometrical methods.

Here we highlighted the significance of the ancient methods of analysis, as described in Pappus’ *Collectio*, and as understood by Newton. In particular, we studied the role of porisms, often dismissed as akin to locus problems, which remained a large part of the British study of geometry in the 18th century. We noted how in spite of the significance of porisms in the ancient method of analysis, this aspect of the subject has received relatively little attention.

Instruments. Since ancient Greek times, geometrical construction had often meant the use of instruments of various types, and their use had been closely related to the distinctions made between mechanical and geometrical. In the *Collectio*, Pappus had included many different ancient instruments in the context of the three classical problems, although he had offered little in the way of guidance on the appropriate use of the various methods.

In chapter 2 we saw that Viète had adopted neusis as an additional postulate, and by restricting his view of geometry, he had avoided much of the need to justify particular construction methods. On the other hand, Kepler had preferred strictly Euclidean constructions, believing that knowledge could only be obtained when these methods could be applied.

Next we saw that Descartes had defined geometrical curves as those curves which could be drawn by an instrument that embodied single continuous motion. However, his “instruments” in the *Géométrie* were sometimes implied rather than practical. Descartes’

emphasis on the motion of instruments meant that for him the subject of geometry was also restricted, although it was broader than either Viète or Kepler had allowed. Outside of the subject of geometry, Descartes had been less concerned about the methods of curve tracing he employed.

In chapter 4 we saw that Newton's powerful organic rulers had allowed him to go much further in the field of curve description than anyone before him. He had been able to think of curve description as a transformation. He was able to generate an extremely wide class of curves, and he was able to study and resolve singularities.

Newton's rulers, and indeed his wider view of "appropriate" construction methods was in great contrast to his predecessors. He rejected assumptions of correctness in favour of exploration of what was possible and what was most elegant for the problem at hand. He had stipulated that geometry should not be restricted by what can be drawn based on what he saw as arbitrary decisions, and that particular instruments did not even need to be defined and were only limited to what a geometer could imagine.

The "domain" of geometry. Another recurring theme that we have identified, particularly in the early modern period, is the idea that geometry is somehow restricted. Again, the concept of what did and did not belong to the subject of geometry was not consistent throughout this period. Whilst Pappus placed a strict classification on geometric problems, this did not necessarily restrict geometry. However, the early modern geometers had interpreted this as placing a preference on the first two classes, plane and solid.

In chapter two we noted that Viète had chosen to restrict geometry to those problems that resulted in equations of degree four or less, that is, the plane and solid classes. By accepting neusis as a postulate Viète had expanded his geometry. Most importantly, with a firm boundary placed upon geometry, we noted how Viète believed it was possible to imagine that one could "finish" geometry in the sense that all "geometrical" problems could be reduced to a standard construction.

Similarly, we saw that Descartes had also viewed geometry as something that could be limited, and therefore "finished" in this way. However, for Descartes, this did not

mean restriction to a particular degree of curve or problem, instead he had included all the algebraic curves. Descartes' constraint on geometry had been a consequence of his criterion of singular continuous motion applied to curve tracing instruments. Further, he saw this relationship as extending to the generalised Pappus problem which, although it reinforced the idea for Descartes, was shown to be false by Newton.

Here our examples in this study demonstrate that Newton had been much more open minded about geometry, seeing it as a subject that was free from limits, or anyway certainly those that Descartes had prescribed. We note that he hardly focused on the foundational aspects of geometry in this sense, and certainly did not see it as something to be "finished". Newton was much more interested in utilising the tools of geometry and exploring what it could be used for rather than defining its limits. This is perhaps most clearly demonstrated by our observations of Newton's ease of use of projective methods in his geometry. We can begin to suggest that had the subject of projective geometry been more fully established at this time, Newton would have been much more receptive to the possibilities of a new geometry than his predecessors.

6.3 Conclusion

We initially sought to better understand and make better known not just Newton's geometry, but also his geometrical thought and development. We did this in the context of a rapidly changing view of geometry between the late 16th and early 18th centuries. Having identified some of the major challenges facing geometers of this period in both defining and practicing geometry, we were able to compare the approach of Newton. In particular, we focused on Newton's strong rejection of the new Cartesian methods and geometrical philosophies. In doing so, we identified an opportunity to more fully explore some of Newton's most astonishing geometrical contributions.

In summary of the achievements of this thesis we highlight two clear aspects of our work. Firstly we note the clarity, depth and detail demonstrated in our exploration of key geometers of the early modern period in specific contrast to Newton. Secondly, we note the specific outcome of this exploration in our original contribution to the understanding of the significance of Newton's organic construction. Here we note in particular the limitations

of Whiteside's observations on this subject.

We propose that Newton's organic rulers were genuinely original. We disagree with Whiteside that they were inspired by van Schooten, except in the loosest sense. Further, we argue that Newton's study of singular points by their resolution was new, and that it has been misunderstood by Whiteside in his interpretation of the transformation effected by the rulers. We instead emphasise that it was the standard quadratic transformation.

In exploring Newton's work in contrast to that of Descartes, we identify the significance of his connection with the geometry of the ancients coupled with a detachment from the foundational aspects of mathematics. It was not just that Newton did geometry, he thought in a geometrical way, and it provided him with a standard of certainty that he could not obtain from the new algebraic methods.

Overall we wish to make better known the importance of geometry in Newton's scientific thought, as well as highlighting the mathematical and historical importance of his organic description of curves as an example of his synthetic approach to geometry. In this context, we hope that our thesis has added to contemporary discourse surrounding Newton's geometry, and specifically provides a foundation for further research into the implications of Newton's geometrical methods for his successors.

Further work

In Newton's geometrical work there remain many aspects which are not well understood. For example, and as we noted in chapter 4, Newton had "missed" six species of cubic curves in his analytical enumeration, but had identified them by projection of the divergent parabolas. A closer examination of his 1690s draft of this work along with Whiteside's careful notes may help us to speculate on the reasons for this apparent omission.

In chapter 5 we started to explore the contributions of the 18th century geometers to furthering Newton's work. We also identified a continued interest in classical geometry; in particular there were many attempts to understand the porisms, which also continued into the 19th century. Since so little was known of the nature of porisms, at least not with any consistency, it would be of interest to study more closely the various reconstructions and commentaries between the 16th and 19th centuries to identify how the definitions altered

during this period, particularly as so little of the subject seems to have been treated with any authority in modern literature.

We have provided a select sample of the geometrical work being undertaken in Britain during this period, highlighting something of a balance between the classical synthetic methods and the new analytical methods favoured by the Continental mathematicians. Due to the contemporary view that geometry in Britain was all but non-existent in the later part of the 18th century, we propose that further work could be done here to examine the diffusion of the work of the Newtonians.

These areas of further research remain for us opportunities for further discoveries. We believe the opportunity to critically reinterpret and revalue Newton's geometry outside of the shadow of the calculus is of significant value. We hope that having contributed a new layer and new details to contemporary study of Newton's geometry we have provided a platform from which to explore, examine and pursue the influence of his work.

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Appendix A

Names and Dates

Pappus of Alexandria (c.290–350), Greek geometer and commentator of ancient works.

Proclus Diadochus (c.411–485), Greek philosopher and commentator of mathematical works.

Federico Commandino (1506–1575), Italian mathematician and translator of many ancient works.

François Viète (1540–1603), French mathematician and astronomer.

Johannes Kepler (1571–1630), German mathematician and astronomer.

Frans van Schooten (1615–1660), Dutch mathematician and translator of many mathematical works.

Isaac Beeckman (1588–1637), Dutch philosopher and natural philosopher.

René Descartes (1596–1650), French philosopher.

Isaac Barrow (1630–1677), English mathematician.

Isaac Newton (1643–1727), English mathematician and natural philosopher.

Robert Simson (1687–1768), Scottish mathematician.

James Stirling (1692–1770), Scottish mathematician.

Colin Maclaurin (1698–1746), Scottish mathematician.

Appendix B

Neusis

Neusis (or verging) is a method equivalent to sliding a marked ruler between two given lines or curves. For example, given a line with the segment AB marked on it, to be able to rotate this line about O and slide it through O until A lies on the fixed line and B lies on the fixed circle (figure B.1).

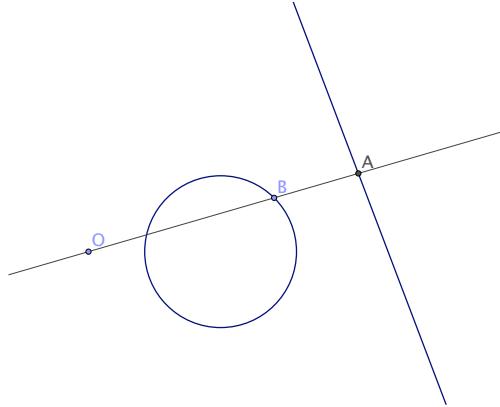


Figure B.1: Neusis

Using a neusis construction cube duplication, angle trisection, and construction of the regular heptagon are soluble, and this was known to the classical geometers. In the *Supplementum Geometriae*, Viète referred to the “conchoid of Nicomedes” [Viète, 2006, p.388], which may be used to solve all of these problems by neusis. It is described as follows (figure B.2).

Given a line l , a point O not on l , and a distance a , draw a line m passing through O and any point P on l . Mark points P_1 and P_2 on m such that $PP_1 = PP_2 = a$. The conchoid is then the locus of points P_1 and P_2 as P moves along l .

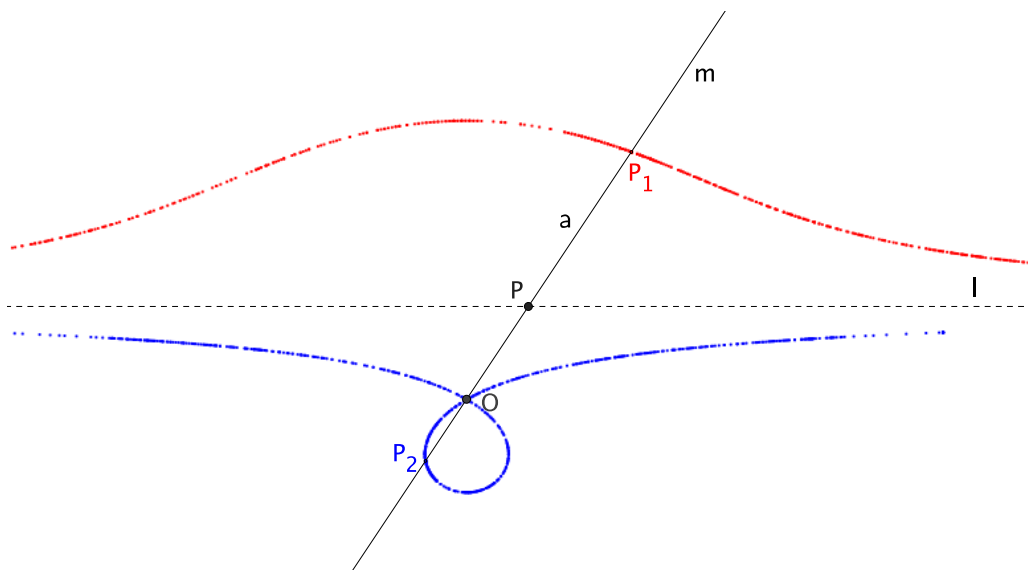


Figure B.2: Conchoid of Nicomedes

The conchoid may be used to trisect an angle as follows (figure B.3).

Given an angle AOB we wish to find AOC such that $AOB = 3AOC$. Draw l perpendicular to AO , and intersecting it in D and intersecting OB in E . Describe the conchoid as above with pole O and directrix l , such that $a = 2OE$. Draw a line m through E and perpendicular to l . Let C be the intersection of m and the conchoid. Draw OC . Angle AOC is $\frac{1}{3}AOB$.

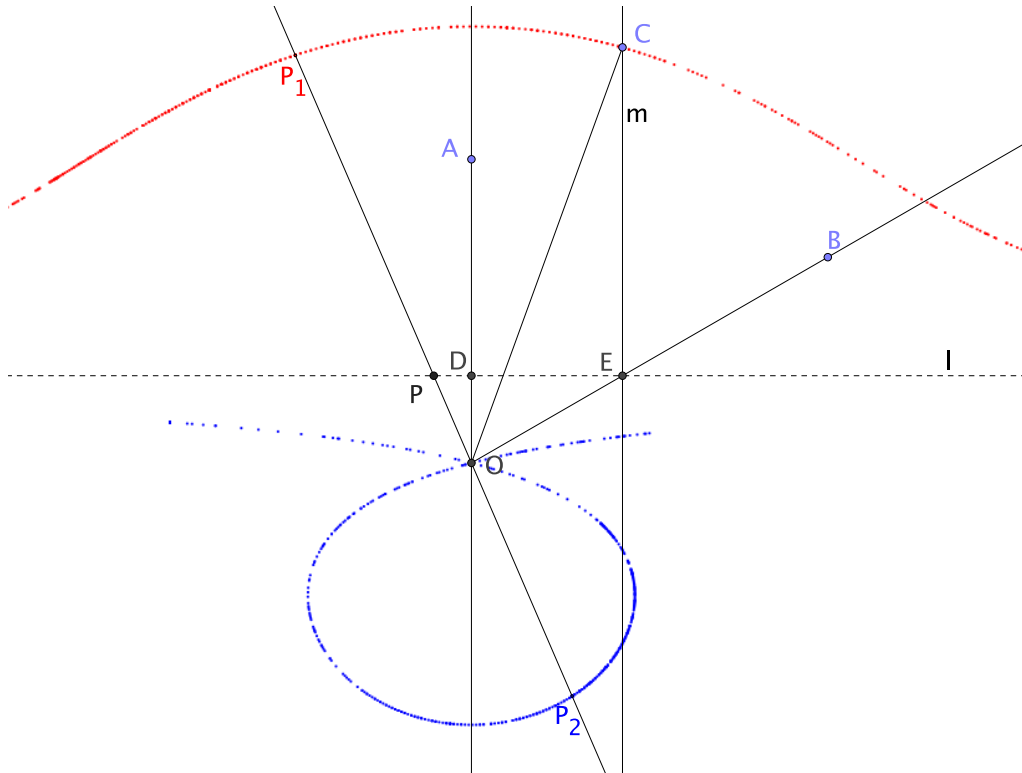


Figure B.3: Angle trisection

Appendix C

Cartesian instruments

In Figure C.1 let the angle BOA be θ , so that

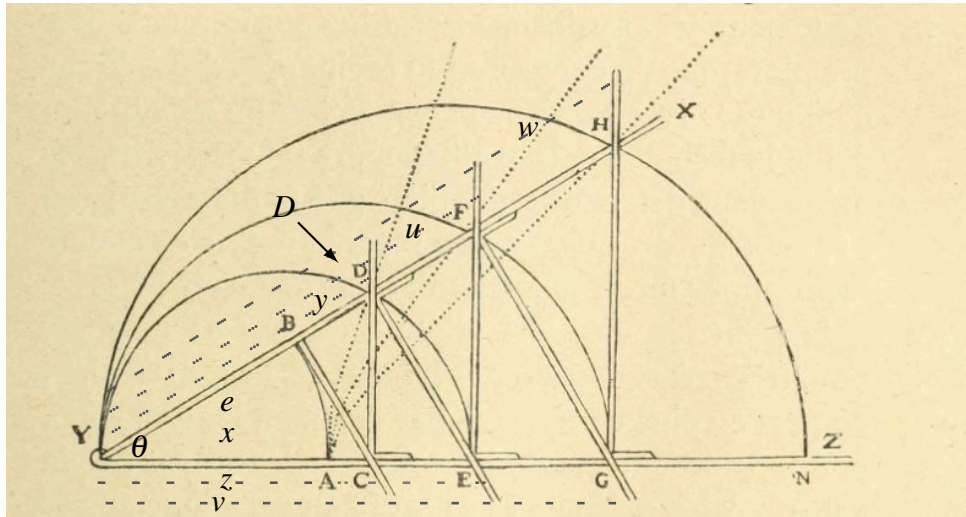
$$x = \frac{e}{\cos \theta}.$$


Figure C.1: Descartes’ “mesolabum” [AT, 1964–1974, 6, p.391]

$$y = \frac{x^2}{e} \sqrt{1 - e^2/x^2}$$
$$e^2 y^2 = x^4 - e^2 x^2.$$
$$y = \frac{x^4}{e} \sqrt{1 - e^2/x^2}$$
$$e^2 y^2 = x^8 - e^2 x^6.$$
$$y = \frac{x^6}{e} \sqrt{1 - e^2/x^2}$$
$$e^2 y^2 = x^{12} - e^2 x^{10}.$$

The pattern is clear.

C.2 The “turning ruler and sliding curve” procedure (Instrument 3.3.1)

Suppose the ruler is hinged at the origin O (figure C.2). Then it has equation

$$y = x \tan \theta.$$

The curve slides up and down, linked to the ruler, and so it has equation

$$y + h \tan \theta = p(x)$$

where h is constant and p is a function, chosen by Descartes to be a polynomial.

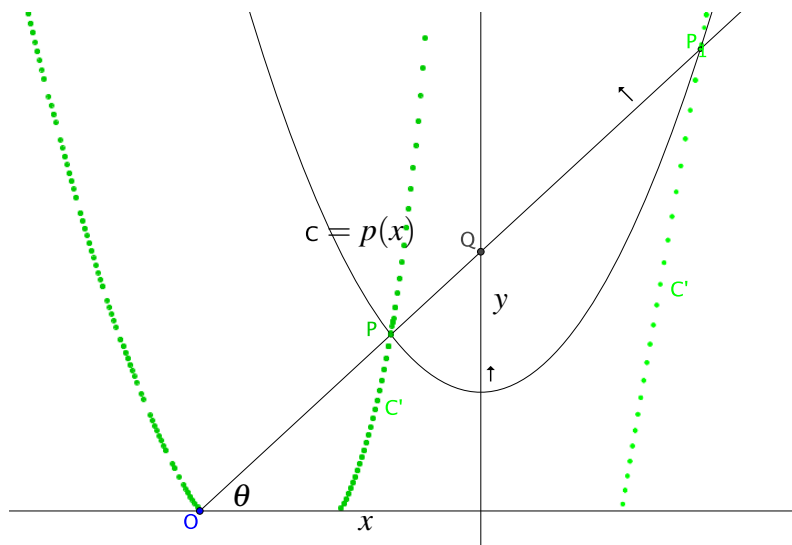


Figure C.2: “Turning ruler and sliding curve” procedure

To find the intersection between the ruler and the curve we solve

$$(x + h) \tan \theta = p(x)$$

and choose a root. This root will be a function of θ , and thus we obtain parametric equations of the new curve (with parameter θ).

Descartes used this procedure to construct a hyperbola (from a straight line) and a “Cartesian parabola” (for the Pappus problem).

Appendix D

Newton, the geometer
[Bloye and Huggett, 2011]

Newton, the Geometer

Nicole Bloye and Stephen Huggett

1 Introduction

Isaac Newton was a geometer. Although he is much more widely known for the calculus, the inverse square law of gravitation, and the optics, geometry lay at the heart of his scientific thought. Geometry allowed Newton the creative freedom to make many of his astounding discoveries, as well as giving him the mathematical exactness and certainty that other methods simply could not.

In trying to understand what geometry meant to Newton we will also discuss his own geometrical discoveries and the way in which he presented them. These were far ahead of their time. For example, it is well-known that his classification of cubic curves anticipated projective geometry, and thanks to Arnol'd [1] it is also now widely appreciated that his lemma on the areas of oval figures was an extraordinary leap 200 years into Newton's future.

Less well-known is his extraordinary work on the *organic construction*, which allowed him to perform what are now referred to as Cremona transformations to resolve singularities of plane algebraic curves.

Geometry was not a branch of mathematics; it was a way of doing mathematics and Newton defended it fiercely, especially against Cartesian methods. We will ask why Newton was so sceptical of what most mathematicians regarded as a powerful new development. This will lead us to consider Newton's methods of curve construction, his affinity with ancient mathematicians and his wish to uncover the mysterious analysis supposedly underlying their work.

These were all hot topics in early modern geometry. Great controversy surrounded the questions of which problems were to be regarded as geometric and which methods might be allowable in their solution. The publication of Descartes' *Géométrie* [7] was largely responsible for the introduction of algebraic methods and criteria, in spite of Descartes' own wishes. This threw into sharp relief the demarcation disputes which arose, originally, from the ancient focus on allowable rules of construction, and we will discuss Newton's challenge to Cartesian methods.¹

It is important to note that Descartes' *Géométrie* was to some extent responsible for Newton's own early interest in mathematics, and geometry in particular.² It was not until the 1680s that he focused his attention on ancient geometrical methods and became dismissive of Cartesian geometry.

This will not be a review of Guicciardini's excellent book [11] but we will refer to it more than to any other. We find in this book compelling arguments for a complete reappraisal of the core of Newton's work.

We would like to thank June Barrow-Green, Luca Chiantini and Jeremy Gray for their help and encouragement.

2 Analysis and synthesis

As Guicciardini³ argues, the certainty Newton sought was "guaranteed by geometry" and Newton "believed that only geometry could provide a certain and therefore publishable demonstration". But how, precisely, was geometry to be defined? In order to obtain this certainty, it was necessary to *know* and *understand* precisely what it was that was to be demonstrated. This had been a difficult question for the early modern predecessors of Newton. What did it really mean to have *knowledge* of a geometrical entity? Was it simply enough to postulate it or to be able to deduce its existence from postulates, or should it be physically constructed, even when this is merely a representation of the object?

If it should be physically constructed then by what means? For example, Kepler (1619) took the view⁴ that only the strict Euclidean tools should be used. He therefore regarded the heptagon as "unknowable", although he was happy to discuss properties that it would have were it to exist. On the other hand, Viète (1593) believed that the ancient *neusis* construction should be adopted as an additional postulate and showed that one could thereby solve problems involving third and fourth degree equations.⁵

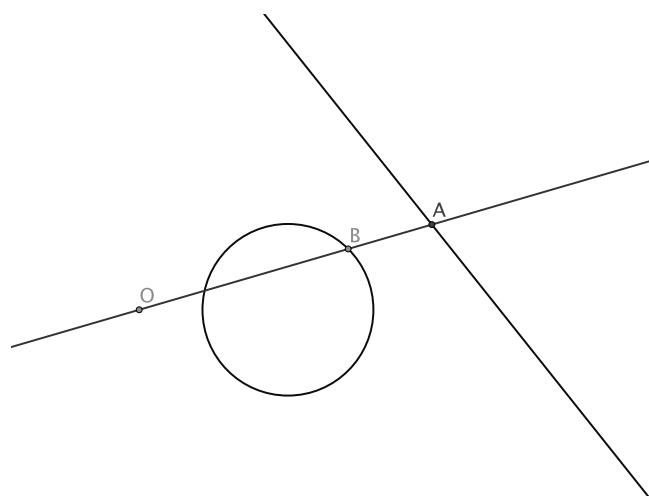


Figure 1. Neusis – given a line with the segment AB marked on it, to be able to rotate this line about O and slide it through O until A lies on the fixed line and B lies on the fixed circle.

We defer until later a discussion of Newton's preferred construction methods. [Given that *neusis* was used in ancient times, it is striking that Euclid chose arguably the most restrictive set of axioms. We are attracted by the hypothesis in [9] that these were chosen because the anthyphairctic sequences which are eventually periodic are precisely those which come from ratios with ruler and compass constructions. In other words, those ratios for which the Euclidean algorithm gives a finite description have ruler and compass constructions. However, this is a digression here as there is no evidence that Newton was aware of this property of Euclidean constructions.]

The early modern mathematicians followed the ancients in dividing problem solving into two stages. The first stage, *analysis*, is the path to the discovery of a solution. Bos⁶ explores in great depth the various types of analysis that may have been performed. The main distinction we shall make here is between algebraic and geometric analyses. We shall see that in the mid 1670s Newton became sceptical of algebraic methods and the idea of an algebraic analysis no less so.

The second stage, *synthesis*, is a demonstration of the construction or solution. This was a crucial requirement before a geometrical problem could be considered solved. Indeed, following the ancient geometers, early modern mathematicians usually removed all traces of the underlying analysis, leaving only the geometrical construction.

Of course, in many cases this geometrical construction was simply the reverse of the analysis and Descartes tried to maintain this link between analysis and synthesis even when the analysis, in his case, was entirely algebraic. Newton argued, however, that this link was broken:

Through algebra you easily arrive at equations, but always to pass therefrom to the elegant constructions and demonstrations which usually result by means of the method of porisms is not so easy, nor is one's ingenuity and power of invention so greatly exercised and refined in this analysis.⁷

There are two points being made here. One is that the constructions arising from Cartesian analysis were anything but elegant and that one should instead use the method of porisms, about which we will say more in a moment. The other is that the Cartesian procedures are algorithmic and allow no room for the imagination.

In spite of the methods in Descartes' *Géométrie* having become widely accepted, Newton believed that there not only could but should be a geometrical analysis. Early in his studies he mastered the new algebraic methods and only later turned his attention to classical geometry, reading the works of Euclid and Apollonius and Commandino's Latin translation of the *Collectio* (1588) by the fourth century commentator Pappus. According to his friend Henry Pemberton (1694–1771), editor of the third edition of *Principia Mathematica*, Newton had a high regard for the classical geometers:⁸

Of their taste, and form of demonstration Sir Isaac always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret for his mistake at the beginning of his mathematical studies, in applying himself to the works of Des Cartes and other algebraic writers, before he had considered the elements of Euclide with that attention, which so excellent a writer deserves.

It was from Pappus' work that Newton learned of what he believed to be the ancient method of analysis: the porisms. Guicciardini explores the possibility that Newton may have been trying to somehow recreate Euclid's work on porisms in order to identify ancient geometrical analysis.⁹ Agreement on precisely what the classical geometers meant by a porism is still elusive but as the early modern geometers understood it, the porisms required the construction of a locus satisfying set conditions, such as the ancient problem that came to be known as Pappus' Problem.

3 Pappus' Problem

The contrast between Newton and Descartes is perhaps nowhere more evident than in their approaches to *Pappus' problem*. This was thought to have been introduced by Euclid and studied by Apollonius but it is often attributed to Pappus because the general problem, extending to any number of given lines, appeared in his *Collection* (in the fourth century). The classic case, however, is the *four-line locus*:¹⁰

Given four lines and four corresponding angles, find the locus of a point such that the angled distances d_i from the point to each line maintain the constant ratio $d_1 d_2 : d_3 d_4$.

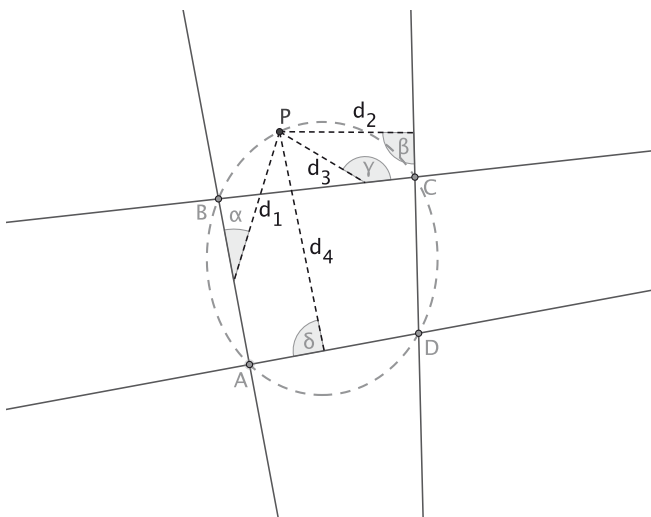


Figure 2. The four-line locus problem

Descartes dedicated much time to the problem, reconstructing early solutions in the case with five lines.¹¹ In his extensive study of the problem in the *Géométrie*, Descartes introduces a coordinate system along two of the lines and points on the locus are described by coordinates in that system. He was able to reduce the four-line problem to a single quadratic equation in two variables. Bos argues¹² that the study of Pappus' problem convinced Descartes more than anything else of the power of algebraic methods.

Indeed, Descartes claimed that every algebraic curve¹³ is the solution of a Pappus problem of n lines, which Newton shows to be false. Newton considered the case $n = 12$. He noted¹⁴ that 6th degree curves have 27 parameters, whilst the corresponding Pappus problem would involve 11 or 12 lines. But the 12 line problem requires that

$$d_1 d_2 d_3 d_4 d_5 d_6 = k d_7 d_8 d_9 d_{10} d_{11} d_{12},$$

which has 22 parameters in determining the position of 11 lines with respect to the 12th, and the factor k , making 23 parameters. So, there must exist algebraic curves that are not solutions of Pappus problems.

He then develops a completely synthetic solution, in his manuscript *Solutio problematis veterum de loco solido*,¹⁵ a version of the first section of which was later included in the first edition of the *Principia*¹⁶ (1687), Book 1 Section V, as Lemmas 17–22.

Guicciardini [11] describes how Newton's solution is in two steps. Firstly, from Propositions 16–23 of Book 3 of

the *Conics* of Apollonius,¹⁷ he shows that (in the words of Lemma 17):

If four straight lines PQ, PR, PS, PT are drawn at given angles from any point P of a given conic to the four infinitely produced sides AB, CD, AC, DB of some quadrilateral $ABCD$ inscribed in the conic, one line being drawn to each side, the rectangle $PQ \cdot PR$ of the lines drawn to two opposite sides will be in a given ratio to the rectangle $PS \cdot PT$ of the lines drawn to the other two opposite sides.¹⁸

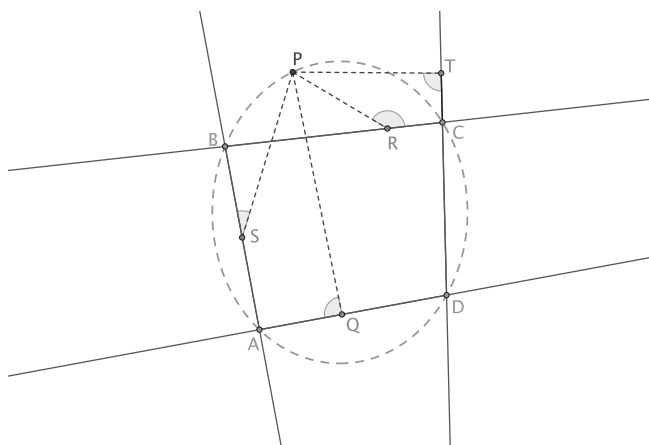


Figure 3. Lemma 17

The converse is Lemma 18: if the ratio is constant then P will be on a conic. Then Lemma 19 shows how to *construct* the point P on the curve.

Newton's second step is to show how the locus which solves the problem – a conic through five given points – can be constructed. Commenting that this was essentially given by Pappus, Newton then introduces the startling *organic construction*. We will discuss this in much more detail later but the essence is this. Newton chose two fixed points B and C called poles and around each pole he allowed to rotate a pair of rulers, each pair at a fixed angle (the two angles not having to be equal). In each pair he designated one ruler the directing “leg” and the other the describing “leg”.

There is a third special point: when the directing legs are chosen to coincide then the point of intersection of the describing legs is denoted A .

In general, of course, the directing legs do not coincide and as their point M of intersection moves, it determines the movement of the point D of intersection of the describing legs. Newton showed that if M is constrained to move along a straight line then D describes a conic through A, B , and C , and conversely that any such conic arises in this way.

This beautiful result appears in the *Principia* as Lemma 21 of Book 1 Section V:

If two movable and infinite straight lines BM and CM , drawn through given points B and C as poles, describe by their meeting-point M a third straight line MN given in position, and if two other infinite straight lines BD and CD are drawn, making given angles MBD and MCD with the first two lines at those given points B and C ; then I say that the point D , where these two lines BD and CD meet, will describe a conic passing through points B and C . And conversely, if the point D , where the straight lines BD and CD meet, describes a conic passing through the given points B, C, A , and the angle DBM is always equal to the

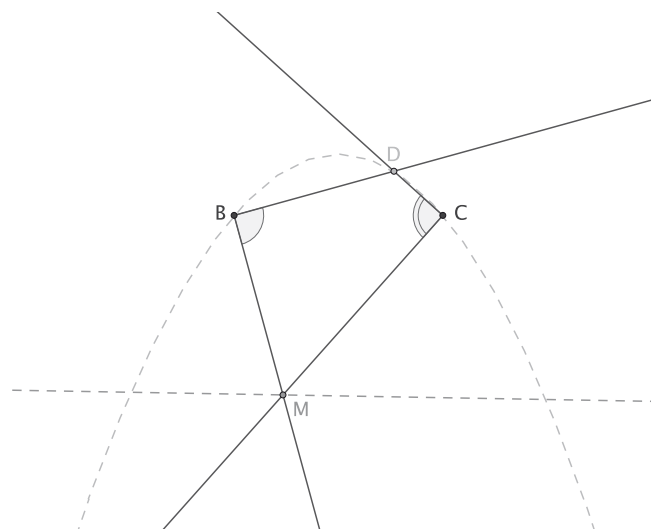


Figure 4. The organic construction

given angle ABC , and the angle DCM is always equal to the given angle ACB ; then point M will lie in a straight line given in position.

Newton's proofs of both the result and its converse are elegant and clear.¹⁹ They follow from the anharmonic property of conics (his Lemma 20) and the fact that two conics do not intersect in more than four points (his Lemma 20, Corollary 3). Guicciardini [11] argues that this sequence of ideas came from an extension of the “main porism” of Pappus to the case of conics and Newton had indeed been determined to restore this ancient method.

Newton's description of conics was in a fairly strong sense what we would now refer to as the projective description. In Proposition 22 he shows how to construct the conic through five given points. In fact he gives two constructions. White-side and others interpret the first as evidence that Newton had at least an intuitive if not explicit grasp of Steiner's Theorem.²⁰ The second uses the organic construction but this should not be taken as indicating any reserve about this construction on Newton's part, as he also published it in the *Enumeratio* (1704) and the *Arithmetica Universalis* (1707).

However, in the *Principia* Newton's solution of the classical Pappus problem appears as a corollary to Lemma 19, after which he cannot resist the following comments:

And thus there is exhibited in this corollary not an [analytical] computation but a geometrical synthesis, such as the ancients required, of the classical problem of four lines, which was begun by Euclid and carried on by Apollonius.

4 Rules for construction

Among geometers it is in a way considered to be a considerable sin when somebody finds a plane problem by conics or line-like curves and when, to put it briefly, the solution of the problem is of an inappropriate kind.²¹

The influence of this remark by Pappus was very great in the early modern period. Bos²² gives three examples, from Descartes, Fermat and Jacob Bernoulli, in which this passage on sin was explicitly quoted. Mathematicians wishing to extend geometrical knowledge struggled to formulate precise

definitions of the subject itself and of the simplicity of the various types (“plane”, “solid” and “linear”) of geometrical constructions.

It was accepted that straight lines and circles formed a basis for classical geometry and that the way to construct them in practice was by straight edge and compasses. In addition, it was also well-known that the ancients had studied other curves, such as conic sections, conchoids, the Archimedian spiral and Hippias’ quadratrix, and other means of construction, such as *neusis*. However, these wider ideas were somehow less well defined than the strict Euclidean ones and hence the focus on demarcation.

Indeed, some of these constructions were dismissed as being “mechanical” but for Descartes this did not make sense: circles and straight lines were also mechanical, in fact, and yet they were perfectly acceptable. He introduced his own “new compasses”²³ for solving the trisection problem and wrote that the precision with which a curve could be understood should be the criterion in geometry, not the precision with which it could be traced by hand or by instruments.²⁴

From our point of view, Descartes’ extension of the geometrical boundaries to include all algebraic curves was a dramatic and important one. Bos [4] argues that although Descartes’ attempts to define the constructions which would generate all algebraic curves were neither explicit nor conclusive, they were nevertheless the deepest part of the *Géométrie*. We describe them very briefly and then consider Newton’s fierce criticisms of them.

Descartes started by claiming that:

nothing else need be supposed than that two or several lines can be moved one by the other and that their intersections mark other lines

and in the interpretation by Bos these curves satisfied the four criteria:

1. The moving objects were themselves straight or curved lines.
2. The tracing point was defined as the intersection of two such moving lines.
3. The motions of the lines were continuous.
4. They were strictly coordinated by one initial motion.

For example, Descartes objected to the quadratrix on the grounds that it required both circular and linear motions,²⁵ which could not be strictly coordinated by one motion because this would amount to a rectification of the circumference of a circle, which he believed “would never be known to man”.²⁶

This is also why Descartes rejected methods of construction in which a string is sometimes straight and sometimes curved, such as the device generating a spiral described by Huygens.²⁷ In contrast, he accepted pointwise constructions but was careful to distinguish those in which generic points on the curve could be constructed from those in which only a special subset of points on the curve could be reached. He argued that curves with these generic pointwise constructions could also be obtained by a continuous motion so that their intersections with other similar curves could be regarded as constructible.

Having shown how to reduce the analysis of a geometrical problem to algebra and having decided that algebraic curves

were precisely those acceptable in geometry, Descartes still had to demonstrate how to perform the *synthesis*.

Descartes was faced with the task of providing the standard constructions that were to be used once the algebra had been performed. He divided problems into classes according to the degree of their equation. In each case a standard form of the equation was given and this was to be accompanied by a standard construction. For the plane problems Descartes simply referred to the standard ruler and compass constructions, while for problems involving third and fourth degree equations he gave his own constructions using the parabola and circle. He then claimed that analogous constructions in the higher degree cases “are not difficult to find”, thus dismissing the subject.

Pappus’ remark depends upon having a clear criterion for the simplicity of a construction. Here Descartes adopted an unequivocally algebraic view: simplicity was defined by the degree of the equation. Guicciardini argues that Newton was in a weak position when he criticised this criterion because Newton’s arguments were based on aesthetic judgments, while Descartes’ criterion was at least precise, whether right or wrong.

It is ironic that Newton’s organic construction satisfied Descartes’ criteria for allowable constructions, given that Newton so explicitly distanced himself from Descartes on construction methods. Newton was scornful of pointwise constructions because one has to complete the curve by “a chance of the hand” and he also rejected, in an argument reminiscent of Kepler’s,²⁸ the “solid” constructions involving intersections of planes and cones. The underlying difference, though, was that (in modern terminology) to Descartes only algebraic curves were geometrical, the others being “mechanical”, while to Newton *all* curves were mechanical:

But these descriptions, insofar as they are achieved by manufactured instruments, are mechanical; insofar, however, as they are understood to be accomplished by the geometrical lines which the rulers in the instruments represent, they are exactly those which we embrace ... as geometrical.²⁹

Of course, before one reaches the stage of construction, one has to perform an analysis of the problem and here the distinction between Newton and Descartes is even clearer. For Newton, the link between analysis and construction was extremely important:

Whence it comes that a resolution which proceeds by means of appropriate porisms is more suited to composing demonstrations than is common algebra.³⁰

But it was not merely a question of adopting a method which would lead to clear and elegant constructions. Newton also felt that mechanical (that is, geometrical) constructions had another crucial feature:

[I]n definitions [of curves] it is allowable to posit the reason for a mechanical genesis, in that the species of magnitude is best understood from the reason for its genesis.³¹

We note that Newton is not alone in regarding geometry as yielding deeper insights. A striking modern example comes from [5]. In the “Prologue” to his book Chandrasekhar says:

The manner of my study of the *Principia* was to read the enunciations of the different propositions, construct proofs for them independently *ab initio*, and then carefully follow Newton's own demonstrations.

In his review [20] of this book, Penrose describes Chandrasekhar's discovery that

In almost all cases, he found to his astonishment that Newton's "archaic" methods were not only shorter and more elegant [than those using the standard procedures of modern analysis] but more revealing of the deeper issues.

5 The Organic construction

Exercitationum mathematicarum libre quinque (1656–1657), by the Dutch mathematician and commentator Frans van Schooten, includes some 'marked ruler' constructions and a reconstruction of some of Apollonius' work *On Plane Loci*. According to Whiteside [27], it was through a study of the fourth book, *Organica conicarum sectionum*, together with *Elementa curvarum linearum* by Schooten's student Jan de Witt,³² that Newton learnt of the organic construction.

We have seen Newton's brilliant use of the organic construction of a conic in his solution of the Pappus problem and indeed Whiteside notes that the organic construction can, in fact, be derived almost as a corollary of Newton's work on that problem. But Newton knew that these rotating rulers could do much more: he thought of them as giving a transformation of the plane.

It was therefore natural for him to think of the construction in Lemma 21 as a transformation taking the straight line (on which the directing legs intersect) to the conic (on which the describing legs intersect). This is clear from his manuscript³³ of about 1667:

And accordingly as the situation or nature of the line PQ varies from one place to another, so will a correspondingly varying line DE be described. Precisely, if PQ is a straight line, DE will be a conic passing through A and B ; if PQ is a conic through A and B , then DE will be either a straight line or a conic (also passing through A and B). If PQ is a conic passing through A but not B and the legs of one rule lie in a straight line [...], DE will be a curve of the third degree [...]³⁴

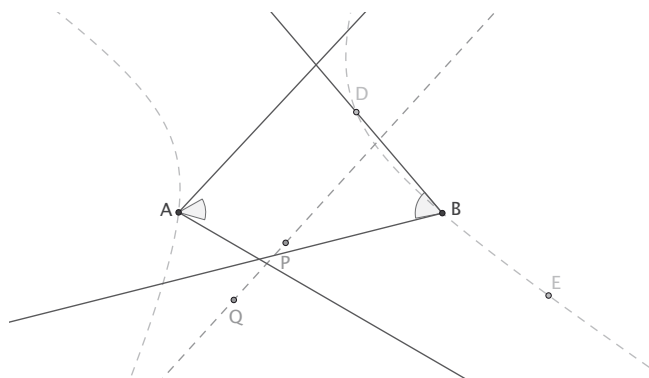


Figure 5. Another view of the organic construction

In fact Newton went much further than this, as is evident for example in his lovely construction³⁵ of the 7-point cubic in the *Enumeratio* (1704). In this extract, note that "curves of

second kind" are cubics and that the letters do not correspond to those in our figure.

All curves of second kind having a double point are determined from seven of their points given, one of which is that double point, and can be described through these same points in this way. In the curve to be described let there be given any seven points A, B, C, D, E, F, G , of which A is the double point. Join the point A and any two other of the points, say B and C , and rotate both the angle \widehat{CAB} of the triangle ABC round its vertex A and either one, \widehat{ABC} , of the remaining angles round its vertex, B . And when the meeting point C of the legs AC, BC is successively applied to the four remaining points D, E, F, G , let the meet of the remaining legs AB and BA fall at the four points P, Q, R, S . Through those four points and the fifth one A describe a conic, and then so rotate the before-mentioned angles $\widehat{CAB}, \widehat{CBA}$ that the meet of the legs AB, BA traverses that conic, and the meet of the remaining legs AC, BC will by the second Theorem describe the curve proposed.

Even in his earlier manuscript (1667), Newton studied various types of singular point and indeed he went so far as to devise a little pictorial representation of them. He also gave a long list of examples, up to and including quintics and sextics. Finally, we note that just after the construction of the 7-point cubic he considers the case in which the double point A is at infinity, as he often did elsewhere, thus in effect working in the projective plane.

As noted by Shkolenok [25], the transformations effected by the organic construction are in fact birational maps from the projective plane to itself, now known as Cremona transformations.³⁶ (We give a short technical account of this in the Appendix.)

Of course one wonders how Newton could possibly have discovered such extraordinary results, so far ahead of their time, and it seems clear at least (as Guicciardini argues) that Newton actually made a set of organic rulers. For example, in the 1667 manuscript referred to above Newton uses terms such as *manufactured*, *steel nail* and *threaded to take a nut*. Guicciardini also draws our attention to Newton's choice of language in his letter (20 August 1672) to Collins explaining his constructing instrument:

And so I dispose them that they may turne freely about their poles A & B without varying the angles they are thus set at.³⁷

Finally, Guicciardini also notes that the drawing accompanying this letter is quite realistic. We return to this point in the next section.

6 Cubics, and projective geometry

In the early 17th century very little was known about cubic curves. Newton revealed the potential complexities of these curves, which, to quote Guicciardini³⁸ "reinforced his conviction that Descartes' criteria of simplicity were foreign to geometry". Newton's first manuscript on the subject, *Enumeratio Curvarum Trium Dimensionum*, thought to have been written around 1667, contained an equation for the general cubic

$$ay^3 + bxy^2 + cx^2y + dx^3 + ey^2 + fxy + gx^2 + hy + kx + l = 0$$

which he was able to reduce to four cases by clever choices of axes.

$$Axy^2 + By = Cx^3 + Dx^2 + Ex + F,$$

$$xy = Ax^3 + Bx^2 + Cx + D,$$

$$y^2 = Ax^3 + Bx^2 + Cx + D,$$

$$y = Ax^3 + Bx^2 + Cx + D.$$

He then divided the curves into 72 species by examining the roots of the right-hand side. It is often remarked that there are in fact 78 species, Newton failing to identify six of them. However, as Guicciardini points out, Newton had in fact identified the remaining six but had chosen to omit them from his paper for some unknown reason.³⁹ Newton returned to his classification of cubic curves in the late 1670s with a second paper⁴⁰ *Enumeratio Linearum Tertii Ordinis* appearing as an appendix to his *Opticks* (1704).

The 1704 *Enumeratio* contained Newton's astonishing discovery that every cubic can be generated by centrally projecting one of the five divergent parabolas (encompassed by the equation $y^2 = Ax^3 + Bx^2 + Cx + D$), starting with the evocative phrase:⁴¹

If onto an infinite plane lit by a point-source of light there should be projected the shadows of figures ...

This remained unproven until 1731 and was first demonstrated by François Nicole (1683–1758) and Alexis Clairaut (1713–1765).

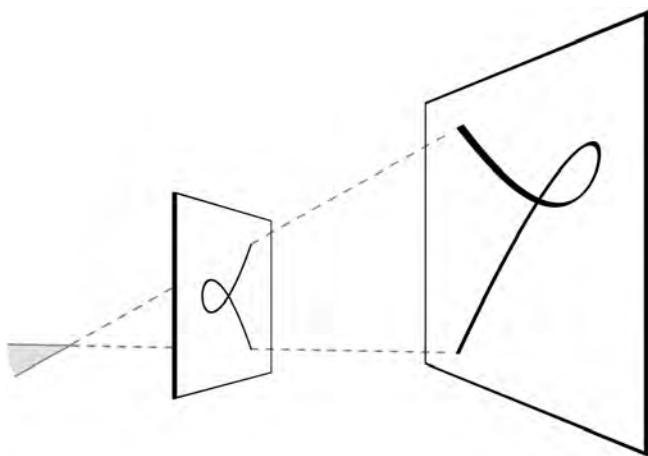


Figure 6. Projection of cubics

Here again, it seems extremely plausible that Newton's intuition was supported by his use of an actual projection from a point source of light but Guicciardini notes that there have been differing views on this question. Rouse Ball⁴² argued that the result was obtained using the projective transformations given in the *Principia*, Book 1 Section V, Lemma 22. Thus, the discovery that all the cubics can be generated by projecting the five divergent parabolas was essentially algebraic.

Talbot⁴³ preferred the view that Newton might have followed a geometrical procedure. He argued that Newton generated all the cubic curves by projection of the five divergent parabolas, using a method in which he began by noting that the position of the horizon line determined the nature of the asymptotes of the projected line.

There is no real evidence for either hypothesis in Newton's work. Guicciardini and Whiteside both seem to favour Talbot's geometrical explanation. We agree: Newton may well have used Lemma 22 to test specific cases but the general result must surely have been perceived by him as a geometrical insight.

7 Physics

Some of the most extraordinary examples of Newton's geometrical power arose in the exposition of his physical discoveries. In this section we note, rather briefly, three such cases, starting with a question in the foundations of the subject. Newton clearly and explicitly understood the Galilean relativity principle⁴⁴ and, as was pointed out by Penrose [22], Newton even considered⁴⁵ adopting it as one of his fundamental principles. But in what framework was this principle to operate? We agree with DiSalle, who argues⁴⁶ that Newton's absolute space and time shares with special and general relativity that space-time is an objective *geometrical* structure which expresses itself in the phenomena of motion.

Our second example comes from Section 6 of Book 1 of *Principia*, which is called *To find motions in given orbits*. Lemma 28 is on algebraically integrable ovals:

No oval figure exists whose area, cut off by straight lines at will, can in general be found by means of equations finite in the number of their terms and dimensions.

Newton's proof simply takes a straight line rotating indefinitely about a pole inside the oval and a point moving along the line in such a way that its distance from the pole is directly proportional to the area swept out by the line. This point describes a spiral, which intersects any fixed straight line infinitely many times.

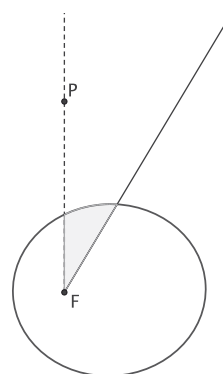


Figure 7. Lemma 28

Then, after noting almost as an aside what is essentially Bézout's Theorem (1779) on the intersections of algebraic curves, the proof is completed by the observation that if the spiral were given by a polynomial then it would intersect any fixed straight line finitely many times.

At the end of his proof Newton applies the result to ellipses (which were of course the original motivation) and defines "geometrically rational" curves, noting casually that spirals, quadratics and cycloids are geometrically irrational. Thus, he leapt to the modern demarcation of algebraic curves, while demonstrating that a restriction to these curves (follow-

ing Descartes) would not be enough for a description of orbital motion.

This is how Arnol'd puts it:⁴⁷

Comparing today the texts of Newton with the comments of his successors, it is striking how Newton's original presentation is more modern, more understandable and richer in ideas than the translation due to commentators of his geometrical ideas into the formal language of the calculus of Leibnitz.

Unfortunately, Newton did not make explicit what he meant by an oval, which has led to considerable controversy.⁴⁸ Although in later editions of the *Principia* Newton inserted a note excluding ovals "touched by conjugate figures extending out to infinity", he never made clear his assumptions on the smoothness of the oval. Nor did the statement of the Lemma distinguish between local and global integrability. There is therefore a family of possible interpretations of Newton's work, which has been elegantly dissected in [24], where it is concluded that:

... Newton's argument for the algebraic nonintegrability of ovals in Lemma 28 embodies the spirit of Poincaré: a concern for existence or nonexistence over calculation, for global properties over local, for topological and geometric insights over formulaic manipulation ...

Our final example comes from Section 12 of Book 1, which has the title *The attractive forces of spherical bodies*. Here Newton shows that the inverse square law of gravitation is not an approximation when the attracting body is a sphere instead of a point, and one of the key results is Proposition 71:

a corpuscle placed outside the spherical surface is attracted to the centre of the sphere by a force inversely proportional to the square of its distance from that same centre.

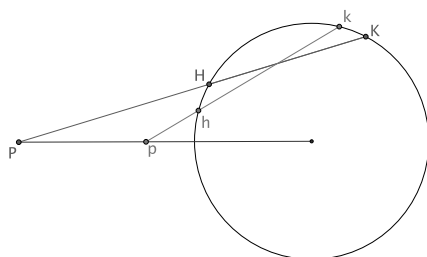


Figure 8. Gravitational attraction of a spherical shell

Newton's proof is utterly geometrical and utterly beautiful.⁴⁹ Here is a sketch of the argument. The spherical surface attracts "corpuscles" at P and p and we wish to find the ratio of the two attractive forces. Draw lines PHK and phk such that $HK = hk$ and draw infinitesimally close lines PIL and pil with $IL = il$. (These are not shown in our figure.) Rotate the segments HI and hi about the line Pp to obtain two ring-shaped slices of the sphere. Compare the attractions of these slices at P and p respectively, merely using the many similar triangles in the construction, and obtain the result.

Littlewood [15] felt that the proof's key geometrical construction (of the lines PHK and phk cutting off equal chords HK and hk) "must have left its readers in helpless wonder" but conjectured that Newton had first proved the result using calculus, only later to give his geometrical proof. We agree with [5] that this is highly implausible. As Chandrasekhar

says: "his physical and geometrical insights were so penetrating that the proofs emerged whole in his mind."⁵⁰ We would argue, further, that the integration Newton is supposed to have performed would in no way have suggested the key geometrical construction. In other words, there is absolutely no link between the supposed analysis and the synthesis.

8 Concluding remarks

In focusing on Newton's geometry we do not mean to imply that he was not also a brilliant algebraist, of which there is ample evidence in the *Principia*, and as we noted in our introduction he is of course widely known for his calculus.

However, it is unfortunate, to say the least, that Newton claimed that he had first found the results in the *Principia* by using the calculus, a claim for which there is no evidence at all.⁵¹

On the contrary, many scholars have given clear and convincing arguments that Newton's claim is simply false. Guicciardini [11] rehearses these, as do Cohen [6] and Needham [17], for example. The claim was made during the row with Leibnitz over priority and simply does not make sense.

Of course the calculus was another profound achievement of Newton's but just because the calculus came to dominate mathematics it should not be assumed that Newton must always have used it in this way. Why ever *should* he?

Newton was one of the most gifted geometers mathematics has ever seen and this allowed him to see further, much further, than others and to express this extraordinary insight with precision and certainty.

Appendix: Cremona transformations

In [18] Book 1 Section 5 Lemma 21 it is shown that the organic transformation maps a line to a conic through the poles B and C , and conversely that any conic through the three points B , C and A will be mapped to a line.

The crucial part of this is that the conic goes through the point A (as well as the two poles B and C). This point A is special: it is the third of the three points which are needed for the Cremona transformations.⁵²

Note also that it is clear from this Lemma that the organic transformation is generically one-one and self-inverse. It can be shown by a short analytical argument that organic transformations are rational maps.⁵³ But a rational map is birational if and only if it is generically one-to-one.⁵⁴ So the organic transformation is a birational map from \mathbf{P}^2 to itself, and hence a *Cremona transformation*.

Without loss of generality we can take the points A , B and C to have homogeneous coordinates $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. Conics in \mathbf{P}^2 through these three points have the form

$$axy + byz + czx = 0.$$

Consider the *standard quadratic transformation* $\phi : \mathbf{P}^2 \rightarrow \mathbf{P}^2$

$$\phi(x, y, z) = (yz, zx, xy),$$

which is a special case of a Cremona transformation. Let L be a line in the codomain. Then L is

$$axy + byz + czx = 0,$$

which is a conic through $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in the domain. So the space of lines in the codomain is the same as this linear system of conics in the domain and $\phi^{-1}(L)$ is one of these conics.

In fact, the organic transformation is this standard quadratic transformation. To see this we use Hartshorne's argument,⁵⁵ as follows.

Let \mathcal{S} be the subsheaf of $\mathcal{O}(2)$ consisting of those elements which vanish at the three base points and let

$$s_0, s_1, s_2 \in \Gamma(\mathbf{P}^2, \mathcal{S})$$

be global sections which generate \mathcal{S} . In other words s_0, s_1 and s_2 are three conics which generate the linear system of conics through the three base points. Also, let

$$x_0, x_1, x_2 \in \Gamma(\mathbf{P}^2, \mathcal{O}(1))$$

be global sections which generate $\mathcal{O}(1)$. Then x_0, x_1 and x_2 are simply lines generating the linear system of lines in \mathbf{P}^2 .

Note that we are thinking of the conics as being in the domain \mathbf{P}^2 and the lines as being in the codomain \mathbf{P}^2 , as in the diagram below:

$$\begin{array}{ccc} \mathcal{S} & & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ \mathbf{P}^2 & \xrightarrow{\phi} & \mathbf{P}^2 \end{array}$$

Then there is a *unique* rational map

$$\phi : \mathbf{P}^2 \rightarrow \mathbf{P}^2$$

such that

$$\mathcal{S} = \phi^*(\mathcal{O}(1)),$$

with $s_i = \phi^*(x_i)$. In other words there is a *unique* rational map from \mathbf{P}^2 to itself with the property that for any line L in the codomain, $\phi^{-1}(L)$ is a conic in the domain through the three base points. So the organic transformation is the same as the standard quadratic transformation.

Notes

1. According to David Gregory, Newton referred to people using Cartesian methods as the “bunglers of mathematics”! See page 42 of [13].
2. Newton studied van Schooten's second Latin edition of the *Géométrie*.
3. See page 13 of [11].
4. See Section 11.3 of [4].
5. See pages 167–168 of [4].
6. See Chapter 5 of [4].
7. This dates from the 1690s. See page 102 of [11] and page 261 of Volume 7 of [19].
8. See page 378 of [26].
9. See page 82 of [11].
10. The three-line problem occurs when two of these four given lines are coincident. In the general case of many lines, the angled distances must maintain the constant ratio $d_1 \dots d_k : d_{k+1} \dots d_{2k}$ for $2k$ lines or $d_1 \dots d_{k+1} : \alpha d_{k+2} \dots d_{2k+1}$ for $2k + 1$ lines.
11. The general solution to this is the Cartesian parabola. See Sections 19.2 and 19.3 in [4].
12. See Chapters 19 and 23 of [4].
13. He thought of these as the geometrical curves.
14. This dates from the late 1670s. See page 343 in Volume 4 of [19].
15. See page 282 in Volume 4 of [19].

16. See [18]. Newton needed these results in this part of the *Principia* in order to find orbits of comets but in the 1690s he considered removing them from the second edition and publishing them separately. Sections IV and V are also discussed in [16].
17. Approximate dates for Apollonius are (260–190).
18. Whiteside [27] observes that this is equivalent to Desargues' *Conic Involution Theorem* and also notes that the condition amounts to the constancy of a cross-ratio.
19. The point A is crucial to the construction and it may be helpful to the reader to note that in his thesis [27] Whiteside did not appear to grasp its importance and drew the conclusion that the proof of the converse was flawed. He corrected this misunderstanding on page 298 of Volume 4 of [19].
20. Steiner's Theorem (1833) states that if p and p' are pencils of lines through vertices A and B respectively and if there is a correspondence between the lines of p and p' having the property that the cross-ratio of any four lines in p is equal to the cross-ratio of the corresponding four lines in p' then the locus of the point of intersection of corresponding lines is a conic through A and B .
21. Here, “finds” means “solves” and the strong language – *sin* – comes from the Latin translation of Pappus' *Collection* published by Commandino in 1588. See page 49 of [4].
22. See note 31 on page 50 of [4].
23. In Descartes' *Cogitationes Privatae* (1619–1620) he sketched three such instruments, one for angle trisection and two others for solving particular cubic equations. The first was an assembly of four hinged rulers OA, OB, OC, OD , extending from a single point O . These rulers were connected by a further four rulers of fixed length, also hinged, such that the three inner angles, AOB, BOC, COD , would always be equal. These instruments certainly fulfilled Descartes' criteria for curve tracing (see below). See also Section 16.4 of [4].
24. See page 338 of [4].
25. Bos shrewdly observes that “it is not necessary to pre-install a special ratio of velocities to draw a quadratrix. The ratio ... arises only because the square in which the quadratrix is to be drawn is supposed as given”. See note 15 on pages 42–43 of [4].
26. See page 342 of [4].
27. This is from a manuscript of 1650 and Bos suggests that Huygens may have learned about this device from Descartes. See page 347 of [4].
28. See page 188 of [4].
29. See page 104 of [11].
30. See page 102 of [11].
31. See page 72 of [11].
32. This appeared in the second edition of Schooten's translation of Descartes' *Géométrie* (1659–1661).
33. See pages 106 and 135 of Volume 2 of [19].
34. In this context it is interesting to note that the general problem of constructing algebraic curves by linkages was solved in [14].
35. See page 639 of Volume 7 of [19].
36. These were published by Luigi Cremona in *Introduzione ad una teoria geometrica delle curve piane* Tipi Gamberini e Parmegiani, Bologna, 1862.
37. See page 94 of [11].
38. See page 112 of [11].
39. See note 8 on page 111 of [11].
40. See Volume 2 of [28].
41. See page 635 in Volume 7 of [19].
42. See Sections 6.4.2 and 6.4.3 of [11].
43. C R M Talbot (1803–1890) published a translation of Newton's 1704 *Enumeratio* in 1860, with notes and examples.
44. See page 28 of [8].
45. This is in *De motu corporum in mediis regulariter cedentibus*.

- See pages 188–194 in Volume 6 of [19].
46. See page 16 of [8].
 47. See page 94 of [1].
 48. Whiteside's own counter-example (which he gave in note 121 on pages 302–303 in Volume 6 of [19]) was elegantly ruled out in [23].
 49. It certainly meets Whitehead's criterion of style! See page 19 of A N Whitehead, *The Aims of Education and Other Essays*, New York: Macmillan, 1929
 50. Compare Penrose's discussion of this feature of inspirational thought and his remarks on Mozart's similar ability to seize an entire composition in his mind, on page 423 of [21].
 51. See page 123 of [6].
 52. Newton only refers to the third base point A in the converse. In fact it is easy to see that if CA, BC and AB intersect the line in Q, R and S , respectively, then the organic transformation maps Q to B , R to A and S to C .
 53. We would prefer a synthetic argument for this but have not yet found one.
 54. See page 493 of [10], for example.
 55. See page 150 of [12].

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