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# Master formulas for $N$ -photon tree level amplitudes in plane wave backgrounds

Patrick Copinger<sup>1,\*</sup>, James P. Edwards<sup>1,†</sup>, Anton Ilderton<sup>2,‡</sup> and Karthik Rajeev<sup>2,§</sup>

<sup>1</sup>*Centre for Mathematical Sciences, University of Plymouth, Plymouth PL4 8AA, United Kingdom*

<sup>2</sup>*Higgs Centre, School of Physics and Astronomy, University of Edinburgh,  
Edinburgh EH9 3FD, United Kingdom*

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The presence of strong electromagnetic fields adds huge complexity to QED Feynman diagrams, such that new methods are required to calculate higher-loop and higher-multiplicity scattering amplitudes. Here we use the worldline formalism to present “master formulas” for all tree level amplitudes of two massive particles and an arbitrary number of photons, in a plane wave background, in both scalar and spinor QED. The plane wave is treated without approximation throughout, meaning in particular that our formulas are valid in the strong-field regime of current theoretical and experimental interest. We check our results against literature expressions obtainable at low multiplicity via direct Feynman diagram calculations.

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## I. INTRODUCTION

Strong fields can generate nonlinear and nonperturbative effects in particle interactions. Strong electromagnetic fields may be generated terrestrially by several means, including by ultraintense lasers [1,2]. QED processes in the presence of these fields acquire an intensity dependence characterized by a coupling which typically exceeds unity, and which must therefore be treated without recourse to perturbation theory. Several upcoming experiments aim to observe nonlinear effects in the scattering of electrons [3–5] and photons [6,7] on intense lasers.

The standard theory approach to “strong field QED” is based on the Furry expansion, or background field perturbation theory. The strong (e.g., laser) field is described as a fixed background, the coupling of which to matter is treated exactly. Interactions between particles scattering on this background are then treated in perturbation theory as usual, see [8] for a recent review. There are, however, several topics in strong field QED which require the development of new theoretical methods.

First, the majority of progress to date has been made for the special, highly symmetric laser model of a plane wave

background, for which the Furry expansion can be practically realized. It is a long-standing challenge to account analytically for realistic pulse geometry, and the new phenomenology this brings [8]. Second, while plane wave results can be extended to realistic fields via local approximations (e.g., [9–11]), and so implemented in numerical codes, those codes must still be benchmarked against theory. This has been performed for first-order (i.e. low multiplicity) processes, but benchmarking higher-order processes is made challenging by, in part, a lack of analytic results; the state of the art in the plane wave model is, at tree level, only *four*-point scattering. Third, if we consider higher-loop corrections, it has been conjectured [12–14] that at very high background field strengths the loop expansion must be resummed in order to provide reliable physical predictions (at least in the low frequency, “constant crossed field” limit). Doing so is a formidable challenge [15–17].

To attack these problems one can use approximations that do not rely on weak coupling [18], develop exactly solvable models which capture some physics of interest [19], or use alternative methods to simplify Furry-picture quantities. One potential method is the worldline formalism, which casts quantum field theory (QFT) in terms of path integrals over relativistic point particle trajectories. Its roots can be traced back to Feynman [20,21], though its use as a serious alternative to the standard QFT formalism was first advocated by Strassler [22], following [23,24]. One of the main advantages of the worldline approach is that it automatically sums over all Feynman diagrams which contribute at fixed multiplicity and loop order, thus greatly simplifying the combinatorics which comes with higher numbers of scatterers and/or loops.

\*patrick.copinger@plymouth.ac.uk

†james.p.edwards@plymouth.ac.uk

‡anton.ilderton@ed.ac.uk

§karthik.rajeev@ed.ac.uk

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The worldline formalism was initially developed for one-loop (and then higher loop) processes in vacuum and in background fields, and a common output of the approach is “master formulas”; these are *all*-multiplicity formulas for correlation functions of a chosen set of fields, at fixed loop order. Such master formulas, which would be extremely challenging to reproduce using Feynman diagrams, have been obtained for processes in vacuum [22,25,26], in constant electromagnetic backgrounds [27–32], and in plane wave backgrounds [33,34]. The worldline approach has also been applied to the calculation of effective actions in background fields via numerical implementations [35], the Casimir effect [36], vacuum birefringence [37], tadpole corrections [38–40], and nonlinear Breit-Wheeler pair production [41]. A long-standing focus of the approach has been the investigation of nonperturbative effects via worldline instantons [42–48]. For reviews see [49,50].

Only recently has much attention been paid to worldline master formulas for processes with external matter lines, or processes at tree level [51–56]. Furthermore, while external photon lines typically appear in the worldline formalism already Lehmann-Symanzik-Zimmermann (LSZ) amputated, matter lines do not, and it has not yet been fully established how one should perform the required LSZ amputation which turns correlation functions into amplitudes.

We fill in some missing pieces of this puzzle in this paper, which is organized as follows. In Sec. II we construct worldline master formulas for all tree level  $(N + 2)$ -point correlation functions describing the emission of  $N$  photons from a massive particle in a background plane wave, in both scalar and spinor QED. In Sec. III we turn to the LSZ amputation of the master formula, converting it into an all-multiplicity formula for the corresponding  $N$ -photon emission/absorption amplitudes from a massive particle in a plane wave background. Example calculations in which we compare with known literature results at low multiplicity are presented in Sec. IV. We conclude in Sec. V. The Appendix contains additional checks on our results.

*Conventions.* We set  $\hbar = c = 1$ . We work throughout in Minkowski space with light front coordinates, so that  $ds^2 = dx^+dx^- - dx^\perp dx^\perp$  where  $x^\perp = (x^1, x^2)$  are the “transverse” directions. We introduce a null vector  $n_\mu$  which projects onto the “light front time” direction, that is  $n \cdot x = x^+$ . The covariant derivative is  $D_\mu = \partial_\mu + ieA_\mu$ .



F1:1 FIG. 1. We consider tree level scattering amplitudes of two massive charges and  $N$  photons, as illustrated on the right (for scalar QED).  
 F1:2 The double line represents the presence of a plane wave background, the coupling to which is treated exactly. Amplitudes are obtained  
 F1:3 by LSZ reduction of the corresponding correlation functions. In the worldline approach, a natural starting pointing is the partially  
 F1:4 amputated correlator, or “dressed propagator,” in which the photons are already reduced out, but the matter fields are not. This is  
 F1:5 illustrated on the left. Thus LSZ reduction is still required for the external matter lines.

## II. MASTER FORMULAS FOR $(2 + N)$ -POINT TREE LEVEL CORRELATORS IN PLANE WAVE BACKGROUNDS

The goal of this section is to write down and evaluate the worldline path integral master formulas for tree level *correlation functions* of  $N$  photons and two charged particles in the presence of a plane wave background, valid for arbitrary  $N$ . We will do this in both scalar and spinor QED.

Our plane wave background may be described by the potential  $eA_\mu(x) = a_\mu(x^+) = \delta_\mu^\perp a_\perp(x^+)$ , a transverse function of light front time  $x^+$ . We may always choose  $a_\perp(-\infty) = 0$ , but then  $a_\perp(\infty) =: a_\perp^\infty$  is in general nonzero (and carries an electromagnetic memory effect [57–59]). The corresponding field strength is  $f_{\mu\nu}(x^+) = n_\nu a'_\mu(x^+) - n_\mu a'_\nu(x^+)$ , where a prime denotes an  $x^+$  derivative.

### A. Scalar QED

In the master formulas we derive in this section, the  $N$  external photons will be LSZ-amputated, but the matter lines not, and thus our correlation functions carry spacetime indices  $x$  and  $x'$ , as well as a dependence on the  $N$ -photon momenta  $\{k_i\}$  and polarizations  $\{\epsilon_i\}$ . We hide the latter dependencies, denoting the partially reduced correlators, or dressed propagators as they are called in the worldline literature, by  $\mathcal{D}_N^{x'x}$ ; see Fig. 1. We take all photons to be *outgoing*; other configurations are trivially obtained by sending  $k \rightarrow -k$ .

The worldline representation of such correlation functions is given in terms of a path integral over relativistic point particle trajectories, denoted  $x^\mu(\tau)$  with  $\tau$  the proper time of the trajectory. The trajectories obey Dirichlet boundary conditions  $x^\mu(T) = x'^\mu$ ,  $x^\mu(0) = x^\mu$ , corresponding to the spacetime dependence of the dressed propagator. The trajectories have length  $T$ , which is ultimately also integrated out, respecting reparametrization invariance of the path integral [60,61]. To write down this path integral, we start from the worldline action that minimally couples a relativistic point particle to an arbitrary gauge field  $A_\mu$ , namely

$$S_{\text{WL}}[x(\tau), A] = - \int_0^T d\tau \left[ \frac{\dot{x}^2}{4} + eA(x(\tau)) \cdot \dot{x}(\tau) \right], \quad (1)$$

155 where overdots denote proper-time derivatives, and where  
 156 the unusual normalization of the kinetic term has become  
 157 standard in the worldline literature, so we preserve it here.  
 158  $S_{\text{WL}}$  enters the path integral for the scalar field propagator,  
 159 call it  $\mathcal{D}^{x'x}$ , via

$$\mathcal{D}^{x'x} = \int_0^\infty dT e^{-im^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) e^{iS_{\text{WL}}[x(\tau), A]}. \quad (2)$$

160 Note that  $A_\mu$  is not integrated over, rather it appears as a  
 162 given field—it is well known (see, for example [62]) that  
 163 correlation functions with  $N$  external photons *in vacuum*  
 164 can be extracted from (2) by fixing  $A_\mu$  to be a sum over  
 165 asymptotic photon wave functions with momenta  $k_i$  and  
 166 polarizations  $\varepsilon_i$ :

$$A_\mu(x) \rightarrow A_\mu^\gamma(x) = \sum_{i=1}^N \varepsilon_{\mu i} e^{ik_i \cdot x}, \quad (3)$$

168 and then expanding the dressed propagator (2) to multi-  
 169 linear order in the polarization vectors. The additional  
 170 complication here is the presence of the background gauge  
 171 potential in (6). This is, however, easily included; we  
 172 simply split the gauge field into a semiclassical part  
 173 representing the plane wave background and a “quantized”  
 174 part representing scattering photons:

$$eA_\mu(x) \rightarrow a_\mu(x) + eA_\mu^\gamma(x). \quad (4)$$

176 Inserting this into (2) and expanding to multilinear order,  
 177 the path integral to be performed is

$$\begin{aligned} \mathcal{D}_N^{x'x} &= (-ie)^N \int_0^\infty dT e^{-im^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) e^{iS_{\text{B}}[x(\tau), a]} \\ &\times \prod_{i=1}^N V^{x'x}[\varepsilon_i, k_i], \end{aligned} \quad (5)$$

178 in which the weight is now given by the reduced action

$$S_{\text{B}}[x(\tau), a] = - \int_0^T d\tau \left[ \frac{\dot{x}^2}{4} + a(x(\tau)) \cdot \dot{x}(\tau) \right], \quad (6)$$

180 while the  $N$  external photons appear (following the  
 182 expansion to multilinear order) through the vertex functions

$$V^{x'x}[\varepsilon, k] := \int_0^T d\tau \varepsilon \cdot \dot{x}(\tau) e^{ik \cdot x(\tau)}. \quad (7)$$

183 [We leave implicit a causal and IR convergence factor  
 185  $\exp(-\varepsilon T)$  under the  $dT$  integral in (5).]

186 Our task is to evaluate the integrals in (5). Let us first  
 187 consider the  $x^\mu$  integrals, and in particular the Dirichlet  
 188 boundary conditions (BCs). To deal with these we follow  
 189 the standard procedure used for the evaluation of such

integrals in vacuum, and expand  $x^\mu(\tau)$  into a straight line  
 trajectory and a fluctuation  $q(\tau)$  according to

$$x^\mu(\tau) = x^\mu + z^\mu \frac{\tau}{T} + q^\mu(\tau), \quad z^\mu := x'^\mu - x^\mu. \quad (8)$$

The fluctuation must satisfy the homogeneous Dirichlet BCs  
 $q(0) = q(T) = 0$  [with measure  $\mathcal{D}x(\tau) \rightarrow \mathcal{D}q(\tau)$ ]. For the  
 analog problem in vacuum ( $a(x^+) \rightarrow 0$ ) the path integral is  
 Gaussian in  $q_\mu$  and can thus be computed analytically.<sup>1</sup> Here,  
 however, the fluctuation appears *inside* the background field  
 $a(x^+(\tau)) = a(x^+ + z^+ \tau/T + q^+)$ , and this has an arbitrary  
 functional form. At first glance this seems to destroy the  
 Gaussianity of the path integral, and prohibit its evaluation.  
 However, it has been shown for one-loop photon-scattering  
 processes (meaning no external matter lines, and a path  
 integral with periodic rather than Dirichlet BCs) that the  
 properties of the plane wave background mean the integral  
 is still effectively Gaussian [33,37]. It is thus crucial to  
 demonstrate that the hidden Gaussianity of the path integral  
 is also present here.

To do so we follow the approach of [34], introducing a  
 Lagrange multiplier  $\chi(\tau)$  and auxiliary field  $\xi(\tau)$  into the  
 path integral through the equality

$$\begin{aligned} e^{-i \int d\tau a(x^+(\tau)) \cdot \dot{q}} &= e^{-i \int d\tau a(x^+ + z^+ \frac{\tau}{T} + q^+) \cdot \dot{q}} \\ &= \int \mathcal{D}\xi \mathcal{D}\chi e^{i \int d\tau [\chi(\xi - q^+) - a(x^+ + z^+ \frac{\tau}{T} + \xi)] \cdot \dot{q}}. \end{aligned} \quad (9)$$

These auxiliary integrals render that over  $q(\tau)$  to be  
 Gaussian. The crucial point, as we show below, is that  
 after evaluating the  $q$  integral, the remaining integrals  
 over  $\xi$  and  $\chi$  can still be evaluated, for a plane wave  
 background.

We now compute the fluctuation integral. As is usual in  
 this “string-inspired” approach, it is convenient to manipu-  
 late the vertex operators as follows. We exponentiate the  
 polarization-dependent factor, so that it appears linearly in  
 an exponent in the operator, with the understanding that the  
 result should later be expanded to linear order in (each of)  
 the  $\varepsilon_i$ , so we write

$$V^{x'x}[\varepsilon, k] \rightarrow \int_0^T d\tau e^{ik \cdot x + \varepsilon \cdot \dot{x}} \Big|_{\text{lin. } \varepsilon}. \quad (10)$$

The result of this is that all dependence on the particle  
 trajectory  $x(\tau)$ , or rather the fluctuation  $q(\tau)$  to be inte-  
 grated out, now appears *linearly* under the path integral.  
 The integrals to be evaluated are now

<sup>1</sup>This is also the case for a constant background in Fock-  
 Schwinger gauge [54].

$$\begin{aligned} \mathcal{D}_N^{X_i} &= (-ie)^N \int_0^\infty dT e^{-im^2 T - i\frac{T^2}{4T}} \\ &\times \prod_{i=1}^N \int_0^T d\tau_i e^{i \sum_{j=1}^N ik_j \cdot (x + z\frac{\tau_j}{T}) + \varepsilon_j \cdot \tilde{\tau}} \int \mathcal{D}\xi \mathcal{D}\chi \\ &\times \int_{q(0)=0}^{q(T)=0} \mathcal{D}q(\tau) e^{i \int_0^T d\tau [-\frac{q^2}{4} - \mathcal{J} \cdot q]} \Big|_{\text{lin.}\varepsilon_1 \dots \varepsilon_N}, \end{aligned} \quad \int_{q(0)=0}^{q(T)=0} \mathcal{D}q(\tau) e^{i \int_0^T d\tau [-\frac{q^2}{4} - \mathcal{J} \cdot q]} = -i(4\pi T)^{-2} \times \exp \left[ -i \int_0^T d\tau_i d\tau_j \mathcal{J}_\mu(\tau_i) \Delta_{ij} \mathcal{J}^\mu(\tau_j) \right]. \quad (13)$$

239 in which  $\mathcal{J}^\mu(\tau)$  is an effective (operator valued) source

$$\begin{aligned} \mathcal{J}^\mu(\tau) &:= a^\mu(x^+ + z^+ \tau/T + \xi) \frac{d}{d\tau} + \chi(\tau) n^\mu \\ &+ i \sum_{i=1}^N \left( ik_i^\mu - \varepsilon_i^\mu \frac{d}{d\tau} \right) \delta(\tau - \tau_i). \end{aligned} \quad (11)$$

232 Since the fluctuation integral is now Gaussian, it is easily  
233 computed in terms of the worldline Green function  
234  $\Delta(\tau_i, \tau_j)$ , that is the inverse of  $2d^2/d\tau^2$  with Dirichlet  
235 BCs, which is found to be

$$\Delta_{ij} := \Delta(\tau_i, \tau_j) = \frac{1}{2} |\tau_i - \tau_j| - \frac{1}{2} (\tau_i + \tau_j) + \frac{\tau_i \tau_j}{T}. \quad (12)$$

236 It is easily checked that Dirichlet BCs hold:  $\Delta(0, \tau_i) =$   
237  $\Delta(T, \tau_i) = \Delta(\tau_j, 0) = \Delta(\tau_j, T) = 0$ . With this, the fluc-  
238 tuation integral becomes  
239  
254

$$\begin{aligned} \int \mathcal{J} \cdot \Delta \cdot \mathcal{J} &= \int d\tau_i d\tau_j a_i \cdot a_j \dot{\Delta}_{ij} + 2i \sum_{j=1}^N \int d\tau_i (\dot{\Delta}_{ij} a_i \cdot \varepsilon_j + i \dot{\Delta}_{ij} a_i \cdot k_j) \\ &+ 2i \sum_{j=1}^N \int d\tau_i \chi_i [\Delta_{ij} \varepsilon_j^+ + i \Delta_{ij} k_j^+] - \sum_{i,j=1}^N [i \dot{\Delta}_{ij} \varepsilon_i \cdot \varepsilon_j + 2i \dot{\Delta}_{ij} \varepsilon_i \cdot k_j - \Delta_{ij} k_i \cdot k_j]. \end{aligned} \quad (16)$$

255 The trivial dependence on  $\chi$  means that this field can now be integrated out, yielding a  $\delta$ -functional:

$$\int \mathcal{D}\xi \mathcal{D}\chi e^{i \int d\tau \chi [\xi - 2i \sum_{j=1}^N (\Delta_{ij} \varepsilon_j^+ + i \Delta_{ij} k_j^+)]} = \int \mathcal{D}\xi \delta \left[ \xi(\tau) - 2 \sum_{j=1}^N (i \dot{\Delta}_{ij} \varepsilon_j^+ - \Delta_{ij} k_j^+) \right]. \quad (17)$$

258 This  $\delta$ -functional has the effect of shifting the argument of the background field, such that from here on we have  
260

$$a_i^\mu \equiv a^\mu(\tau_i) \equiv a^\mu \left( x^+ + z^+ \frac{\tau_i}{T} + 2 \sum_{j=1}^N [-\Delta_{ij} k_j^+ + i \dot{\Delta}_{ij} \varepsilon_j^+] \right). \quad (18)$$

261 The dynamical fluctuation is thus replaced by a coupling of  
262 the plane wave to the  $N$  scattering photons [33,37]. This  
263 is particular to plane wave backgrounds because (a) for  
264  $n^2 \neq 0$  Eq. (16) picks up a contribution quadratic in  $\chi$ ,  
265 while (b) for  $n \cdot a \neq 0$  there is an additional term linear in  $\chi$   
266 that depends on the background; instead of (18) one would  
267 have obtained via (17) only an implicit equation for  $a^\mu$ .

268 All remaining background-dependent terms in (17)  
269 may be expressed in terms of just two worldline

This defines the fundamental contraction for the fluctuation  
240 variable,  
242

$$\langle q^\mu(\tau) q^\nu(\tau') \rangle = 2i \eta^{\mu\nu} \Delta(\tau, \tau'), \quad (14)$$

and the free path integral normalization is recovered by  
243 setting  $\mathcal{J} = 0$ . To proceed, we wish to write out the  
244 exponent in (13) explicitly. Note, though, that  $\Delta_{ij}$  is not  
245 proper time-translation invariant due to the boundary  
246 conditions [51], hence left and right proper-time derivatives  
247 must be distinguished. We denote these as follows:  
248  
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$$\begin{aligned} \bullet \Delta_{ij} &:= \frac{d}{d\tau_i} \Delta_{ij}, & \Delta_{ij}^\bullet &:= \frac{d}{d\tau_j} \Delta_{ij}, \\ \bullet\bullet \Delta_{ij} &:= \frac{d^2}{d\tau_i^2} \Delta_{ij}, & \text{etc.} & \end{aligned} \quad (15)$$

With this, we write out the exponent of (13), using that the  
250 background is transverse and on-shell ( $n \cdot a = 0$  and  $n^2 = 0$ )  
252 to simplify. We find, writing  $a_i \equiv a(x^+ + z^+ \tau_i/T + \xi(\tau_i))$ ,  
253

structures, namely the worldline average and the periodic  
270 integral  
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$$\langle\langle f \rangle\rangle := T^{-1} \int_0^T d\tau f(\tau), \quad I_\mu(\tau) := \int_0^\tau d\tau' [a_\mu(\tau') - \langle\langle a_\mu \rangle\rangle], \quad (19)$$

respectively. These would have to be computed for a given  
273 background once the functional form of  $a_\mu$  has been fixed.  
274

275 At this stage the path integral has (at least formally) been  
 276 computed. Gathering everything together we obtain our  
 277 master formulas for the  $N$ -photon dressed propagator

$$\begin{aligned} \mathcal{D}_N^{x'x} &= i(-e)^N \int_0^\infty dT (4i\pi T)^{-2} e^{-i\frac{z^2}{4T}} \prod_{i=1}^N \int_0^T d\tau_i \\ &\times e^{-iM^2(a)T} \bar{\mathfrak{P}}^{x'x}(\varepsilon_1, \dots, \varepsilon_N) \\ &\times e^{-iz \cdot \langle a \rangle + i \sum_{j=1}^N (x + \frac{z}{T} \tau_j - 2I(\tau_j)) \cdot k_j - i \sum_{i,j=1}^N \Delta_{ij} k_i \cdot k_j} \Big|_{\text{lin.} \varepsilon_1 \dots \varepsilon_N}, \end{aligned} \quad (20)$$

278 in which  $M^2(a) := m^2 - \langle a^2 \rangle + \langle a \rangle^2$  is analogous to the  
 280 Kibble “mass” [63] which typically appears in pulsed plane  
 281 waves [64], while  $\bar{\mathfrak{P}}^{x'x}$  is defined by

$$\begin{aligned} \bar{\mathfrak{P}}^{x'x}(\varepsilon_1, \dots, \varepsilon_N) \\ := i^N e^{\sum_{i=1}^N \varepsilon_i \cdot \frac{z}{T} + 2 \sum_{i=1}^N (\langle a \rangle - a_i) \cdot \varepsilon_i + i \sum_{i,j=1}^N [2i^* \Delta_{ij} \varepsilon_i \cdot k_j + \varepsilon_i \cdot \varepsilon_k^* \Delta_{ij}^*]} \end{aligned} \quad (21)$$

283 We emphasize that this master formula holds for any  
 284 multiplicity  $N \geq 0$ ; it would be extremely challenging to  
 285 obtain this starting from the Feynman rules. Evaluating in  
 286 specific cases we can check against the literature; for  $N = 0$ ,  
 287 for example, we recover a one-parameter (proper-time)  
 288 representation of the scalar Volkov propagator:

$$\mathcal{D}_0^{x'x} = ie^{-iz \cdot \langle a \rangle} \int_0^\infty dT (4i\pi T)^{-2} e^{-iM^2(a)T} e^{-i\frac{z^2}{4T}}. \quad (22)$$

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$$\mathcal{D}_N^{x'x} = i(-e)^N \int_0^\infty dT (4i\pi T)^{-2} e^{-i\frac{z^2}{4T}} \prod_{i=1}^N \int_0^T d\tau_i e^{-iM^2(a)T} \bar{\mathfrak{P}}_N^{x'x} e^{-iz \cdot \langle a \rangle + i \sum_{i=1}^N (x + \frac{z}{T} \tau_i - 2I(\tau_i)) \cdot k_i - i \sum_{i,j=1}^N \Delta_{ij} k_i \cdot k_j}, \quad (24)$$

318 where the polynomial  $\bar{\mathfrak{P}}_N^{x'x}$  is defined by the expansion of the polarization-dependent terms to multilinear order:

$$\bar{\mathfrak{P}}_N^{x'x} := i^N e^{\sum_{i=1}^N \varepsilon_i \cdot \frac{z}{T} + 2 \sum_{i=1}^N (\langle a \rangle - a_i) \cdot \varepsilon_i + i \sum_{i,j=1}^N [2i^* \Delta_{ij} \varepsilon_i \cdot k_j + \varepsilon_i \cdot \varepsilon_j^* \Delta_{ij}^*]} \Big|_{\text{lin.} \varepsilon_1 \dots \varepsilon_N}. \quad (25)$$

319 These polynomials generalize those defined for closed worldlines in vacuum ( $P_N$ ) in [49], for open lines in vacuum ( $\bar{P}_N$ ) in  
 321 [31], and for the closed loop in a background field ( $\mathfrak{P}_N$ ) in [33] (in position space for the time being). For convenience let us  
 322 write out the first few terms:

$$\bar{\mathfrak{P}}_0^{x'x} = 1, \quad (26)$$

323

$$\bar{\mathfrak{P}}_1^{x'x} = i \left[ \frac{z}{T} + 2(\langle a \rangle - a_1) - 2^* \Delta_{11} k_1 \right] \cdot \varepsilon_1, \quad (27)$$

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$$\begin{aligned} \bar{\mathfrak{P}}_2^{x'x} &= - \left[ \frac{z}{T} + 2(\langle a \rangle - a_1) - 2^* \Delta_{11} k_1 - 2^* \Delta_{12} k_2 \right] \cdot \varepsilon_1 \\ &\times \left[ \frac{z}{T} + 2(\langle a \rangle - a_2) - 2^* \Delta_{21} k_1 - 2^* \Delta_{22} k_2 \right] \cdot \varepsilon_2 - 2i^* \Delta_{12} \cdot \varepsilon_1 \cdot \varepsilon_2. \end{aligned} \quad (28)$$

Observe that in this case  $a_\mu(\tau) \equiv a_\mu(x^+ + z^+ \frac{\tau}{T})$  so that, changing variables to  $u = \frac{\tau}{T}$ , the worldline average becomes  $T$ -independent and can be taken outside the  $T$  integral. It may be written as a *spacetime* average (see [37]),

$$\begin{aligned} \langle\langle a_\mu \rangle\rangle &= \int_0^1 du a_\mu(x^+ + z^+ u) \\ &= \frac{1}{x'^+ - x^+} \int_{x^+}^{x'^+} dy a_\mu(y) \equiv \langle a_\mu \rangle, \end{aligned} \quad (23)$$

and as such  $M^2(a) = m^2 - \langle a^2 \rangle + \langle a \rangle^2$  now corresponds exactly to the Kibble mass.

Equation (22) is equivalent to the standard momentum-integral representation of the Volkov propagator, and offers a concise version of the position-space propagator in [65,66]. For  $N = 1$  we recover the (two-scalar one-photon) three-point function, and so on. Since the correlators themselves are not of immediate interest, we will present these checks later, implicitly, as part of our checks on the corresponding formula for *scattering amplitudes*.

The actual computation of the dressed propagator (and, later, the amplitudes) is greatly simplified by observing that we can choose the gauge  $n \cdot \varepsilon = \varepsilon^+ = 0$ . This removes the polarization vectors from the argument of  $a_\mu$ , and thus extraction of the multilinear piece of (24) reduces to the expansion of  $\bar{\mathfrak{P}}(\varepsilon_1, \dots, \varepsilon_N)$  alone. We adopt this gauge from here on in order to present the simplest possible expressions and also match to the strong-field QED literature, where this gauge is common. Doing so, then, we can write the master formula in this gauge as

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## B. Spinor QED

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We now turn to the computation of the analogous  $N$ -photon dressed propagators in spinor QED, denoting these by  $\mathcal{S}_N^{x'x}$ . Due to the spin degrees of freedom this is a Dirac matrix-valued function, but we suppress the corresponding indices for brevity. Referring the reader to [51,67] for details, we begin by writing down the analog of the “propagator” (2) in an arbitrary background, but now accounting for the spin of the fermion:

$$\mathcal{S}^{x'x} = (-i\mathcal{D}_{x'} - m)\mathcal{K}^{x'x}(a), \quad (29)$$

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$$\mathcal{K}^{x'x}(a) = \int_0^\infty dT e^{-im^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) e^{iS_{\text{WL}}[x(\tau), A]} 2^{-\frac{D}{2}} \text{symb}^{-1} \oint_{\text{A/P}} \mathcal{D}\psi(\tau) e^{i\tilde{S}_{\text{WL}}[\psi(\tau), x(\tau), A]}, \quad (30)$$

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$$\tilde{S}_{\text{WL}}[\psi(\tau), x(\tau), A] = \int_0^T d\tau \left[ \frac{i}{2} \psi \cdot \dot{\psi} + ie(\psi(\tau) + \eta) \cdot F(x(\tau)) \cdot (\psi(\tau) + \eta) \right]. \quad (31)$$

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The kernel  $\mathcal{K}^{x'x}$  contains an integral over relativistic particle trajectories, as for the scalar case, and also a path integral over Grassmann-valued fields  $\psi(\tau)$ , obeying antiperiodic (A/P) BCs  $\psi(0) = -\psi(T)$ . These represent the spin degrees of freedom of the fermion and are minimally coupled to  $A$  through its field strength  $F(x(\tau))$  appearing in the action  $\tilde{S}_{\text{WL}}$ . An additional Grassmann variable  $\eta$  also appears; the Dirac-matrix structure of the propagator is produced by acting on this variable by the (inverse of the) *symbolic map*, defined by

$$\text{symb}\{\gamma^{[\mu_1 \gamma^{\mu_2} \dots \gamma^{\mu_n}]}\} = (-i\sqrt{2})^n \eta^{\mu_1} \eta^{\mu_2} \dots \eta^{\mu_n}. \quad (32)$$

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This map converts between antisymmetric combinations of Dirac matrices (a combinatorial factor of  $1/n!$  factor is assumed) and products of Grassmann variables  $\eta$ . Use of the symbol map avoids lengthy Dirac-matrix algebra as it automatically produces the kernel in the (even subalgebra of the) Clifford basis of the Dirac algebra. Note that all  $\eta$ -dependence in (30) and (31) or any of our expressions vanishes after evaluation of the inverse map; it is therefore pragmatic to state once and for all the results relevant to us in  $(3+1)$  dimensions as

$$\begin{aligned} \text{symb}^{-1}\{1\} &= \mathbb{I}_4, & \text{symb}^{-1}\{\eta^\mu \eta^\nu\} &= -\frac{1}{2} \gamma^{[\mu} \gamma^{\nu]} = -\frac{1}{4} [\gamma^\mu, \gamma^\nu], \\ \text{symb}^{-1}\{\eta^\mu \eta^\nu \eta^\alpha \eta^\beta\} &= \frac{1}{4!} [\{\gamma^{[\mu} \gamma^{\nu]}, \gamma^{[\alpha} \gamma^{\beta]}\} - \{\gamma^{[\mu} \gamma^{\alpha]}, \gamma^{[\nu} \gamma^{\beta]}\} + \{\gamma^{[\mu} \gamma^{\beta]}, \gamma^{[\nu} \gamma^{\alpha]}\}] = i\gamma_5 \epsilon^{\mu\nu\alpha\beta}. \end{aligned} \quad (33)$$

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Now, taking  $A$  as in (3) to introduce both our background plane wave and the  $N$  external photons, we expand (29) to multilinear order in the photon polarizations to obtain the  $N$ -photon dressed propagator

$$\begin{aligned} \mathcal{S}_N^{x'x} &= (-i\mathcal{D}_{x'} + a(x'^+) - m)\mathcal{K}_N^{x'x}(a) + eA^\gamma(x')\mathcal{K}_{N-1}^{x'x}(a), \\ \mathcal{K}_N^{x'x}(a) &= (-ie)^N \int_0^\infty dT e^{-im^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) e^{iS_{\text{B}}[x(\tau), a]} 2^{-\frac{D}{2}} \text{symb}^{-1} \oint_{\text{A/P}} \mathcal{D}\psi(\tau) e^{i\tilde{S}_{\text{B}}[\psi(\tau), x(\tau), a]} \prod_{i=1}^N V_\eta^{x'x}[\varepsilon_i, k_i], \end{aligned} \quad (34)$$

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where  $\tilde{S}_{\text{B}}[\psi(\tau), x(\tau), a]$  is given by replacing  $eF(x(\tau))$  in  $\tilde{S}_{\text{WL}}[\psi(\tau), x(\tau), A]$  with  $f(x(\tau))$ . In the “ $N$ -photon kernel”  $\mathcal{K}_N^{x'x}(a)$ , the proper time and bosonic integrals are the same as in the scalar case—these represent the orbital degrees of freedom which remain unchanged. In the so-called sub-leading term involving  $\mathcal{K}_{N-1}^{x'x}$ , for each term in the sum in  $A^\gamma(x')$  we remove the corresponding photon from the kernel to maintain the projection onto the multilinear sector. Finally, writing  $\tilde{f}_{i\mu\nu} = k_{i\mu}\varepsilon_{i\nu} - k_{i\nu}\varepsilon_{i\mu}$  for the linearized field strength associated with the  $i$ th photon, the vertex operator is now given by

$$\begin{aligned} V_\eta^{x'x}[\varepsilon_i, k_i] &:= \int_0^T d\tau [\varepsilon_i \cdot \dot{x}(\tau_i) \\ &\quad + (\psi(\tau_i) + \eta) \cdot \tilde{f}_i \cdot (\psi(\tau_i) + \eta)] e^{ik_i x(\tau_i)}, \end{aligned} \quad (35)$$

in which the second term represents the spin coupling of the external photons to the particle trajectories. 366

Despite the obvious added complexity from the spin coupling to the photon fields, we stress that the same hidden Gaussianity is present here as in the scalar case. Consider again the path integral over  $x^\mu$ ; we treat it as we did above, introducing auxiliary fields to yield a Gaussian 367

373 path integral in the fluctuation  $q^\mu$ . While there is now an  
 374 additional dependence on the background  $f_{\mu\nu}$  introduced  
 375 by the spin factor, this behaves in the same way as above  
 376 when integrating out the auxiliary fields, i.e.  $f$  in the spin  
 377 factor is ultimately evaluated at a shifted argument,

$$f_i^{\mu\nu} \equiv f^{\mu\nu}(\tau_i) \equiv f^{\mu\nu}\left(x^+ + z^+ \frac{\tau_i}{T} - 2 \sum_{j=1}^N \Delta_{ij} k_j^+\right), \quad (36)$$

378 just for  $a_\mu$  earlier (recall we have gauged  $\varepsilon_i^\dagger = 0$  for  
 380 convenience). In short, and as is natural, the only real  
 381 difference compared to the scalar case lies in the evaluation  
 382 of the Grassmann path integral, which is the focus of the  
 383 remainder of this section.

384 Observe that the vertex operators (35) introduce factors  
 385 of  $\psi_\eta(\tau) \equiv (\psi(\tau) + \eta)$  under the Grassmann integral. This  
 386 motivates us to introduce the following functions,

$$\mathfrak{B}_\eta(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_s}) := \langle \psi_\eta(\tau_{i_1}) \cdot \tilde{f}_{i_1} \cdot \psi_\eta(\tau_{i_1}) \dots \psi_\eta(\tau_{i_s}) \cdot \tilde{f}_{i_s} \cdot \psi_\eta(\tau_{i_s}) \rangle \quad (37)$$

$$= 2^{-\frac{D}{2}} \oint_{A/P} \mathcal{D}\psi(\tau) \psi_\eta(\tau_{i_1}) \cdot \tilde{f}_{i_1} \cdot \psi_\eta(\tau_{i_1}) \dots \psi_\eta(\tau_{i_s}) \cdot \tilde{f}_{i_s} \cdot \psi_\eta(\tau_{i_s}) e^{i \int_0^T d\tau [\frac{1}{2} \psi \cdot \dot{\psi} + i \psi_\eta(\tau) \cdot f(\tau) \cdot \psi_\eta(\tau)]}, \quad (38)$$

388 which generalize the expectation values of the spin part of  
 389 the vertex operator introduced in vacuum [ $W(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_s})$   
 390 on the loop in [49] and  $W_\eta(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_s})$  for open lines in  
 391 [51]] and for one-loop amplitudes in the plane wave  
 392 background [ $\mathfrak{B}(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_s})$  in [33]]. We generate the  
 393 insertions under the path integral by derivatives with  
 394 respect to a fictitious Grassmann source  $\theta$  (anticommuting  
 395 with  $\psi$  and  $\eta$ ), from which follows

$$\mathfrak{B}_\eta(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_s}) = \frac{\delta}{\delta \theta_{i_1}} \cdot \tilde{f}_{i_1} \cdot \frac{\delta}{\delta \theta_{i_1}} \dots \frac{\delta}{\delta \theta_{i_s}} \cdot \tilde{f}_{i_s} \cdot \frac{\delta}{\delta \theta_{i_s}} 2^{-\frac{D}{2}} \times \oint_{A/P} \mathcal{D}\psi(\tau) e^{i \int_0^T d\tau [\frac{1}{2} \psi \cdot \dot{\psi} + i \psi_\eta \cdot f \cdot \psi_\eta + i \theta \cdot \psi_\eta]} \Big|_{\theta=0}, \quad (39)$$

396 and the corresponding spin factor is produced through

$$\text{Spin}(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_s}) := \text{symb}^{-1} \mathfrak{B}_\eta(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_s}). \quad (40)$$

400 To compute the integral in (39) we require the (spinor)  
 402 worldline propagator in the field,  $\mathfrak{G}^{\mu\nu}(\tau, \tau')$ . This will define  
 403 the fundamental contraction between the Grassmann fields,

$$\langle \psi^\mu(\tau) \psi^\nu(\tau') \rangle = \frac{1}{2} \mathfrak{G}^{\mu\nu}(\tau, \tau'). \quad (41)$$

404 From the quadratic part of the operator appearing in the path  
 406 integral action,  $\mathfrak{G}$  must obey

$$\left( \frac{1}{2} \eta_{\mu\sigma} \frac{d}{d\tau} + f_{\mu\sigma}(\tau) \right) \mathfrak{G}^{\sigma\nu}(\tau, \tau') = \eta_\mu{}^\nu \delta(\tau - \tau'), \quad (42)$$

as well as antiperiodic boundary conditions  $\mathfrak{G}(0, \tau') = -\mathfrak{G}(T, \tau')$  and  $\mathfrak{G}(\tau, 0) = -\mathfrak{G}(\tau, T)$ . Observe that  $\mathfrak{G}$  has the antisymmetric property  $\mathfrak{G}^{\mu\nu}(\tau, \tau') = -\mathfrak{G}^{\nu\mu}(\tau', \tau)$ . The general homogeneous solution of (42) for arbitrary  $f(\tau)$  is written conveniently in terms of an auxiliary function  $\mathcal{O}(\tau, \tau')$ , which takes care of the ordering of  $\tau$  and  $\tau'$ , defined by

$$\mathcal{O}(\tau, \tau') = \mathcal{P}^* e^{-2 \int_{\tau'}^{\tau} d\sigma f(\sigma)}, \quad (43)$$

where  $\Theta$  is the Heaviside step function,  $\mathcal{P}^* \equiv \mathcal{P}^*(\tau, \tau') = \Theta(\tau - \tau') \mathcal{P} + \Theta(\tau' - \tau) \bar{\mathcal{P}}$  with  $\mathcal{P}$  ( $\bar{\mathcal{P}}$ ) denoting (anti)path ordering in proper time and we have made use of a matrix form for the Lorentz indices (with respect to which  $\mathcal{O}$  is orthogonal). With the homogeneous solution, we can then find the general solution to (42) with appropriate antiperiodic boundary conditions as

$$\mathfrak{G}(\tau, \tau') = \text{sgn}(\tau - \tau') \mathcal{O}(\tau, \tau') + \mathcal{O}(\tau, 0) \frac{1 - \mathcal{O}(T, 0)}{1 + \mathcal{O}(T, 0)} \mathcal{O}(0, \tau'). \quad (44)$$

423 However, there are notable simplifications in our particular  
 424 case that  $f$  is a plane wave because, as is well known, the field  
 425 strength is then nilpotent of order 3. Further,  $f$  evaluated at  
 426 different  $\tau$  commute. The Green function thus reduces to<sup>2</sup>  
 427

$$\mathfrak{G}(\tau, \tau') = e^{-2 \int_{\tau'}^{\tau} d\sigma f(\sigma)} \left[ \text{sgn}(\tau - \tau') + \tanh \left( \int_0^T d\sigma f(\sigma) \right) \right] \quad (45)$$

$$= \text{sgn}(\tau - \tau') \left[ 1 - 2 \int_{\tau'}^{\tau} d\sigma f(\sigma) + 2 \left( \int_{\tau'}^{\tau} d\sigma f(\sigma) \right)^2 \right] + T \langle \langle f \rangle \rangle \left[ 1 - 2 \int_{\tau'}^{\tau} d\sigma f(\sigma) \right]. \quad (46)$$

428 Equipped with the Green function, we compute the integral  
 429 in (39) by completing the square, using the shift  $\tilde{\psi}(\tau) = \psi(\tau) + \int d\tau' \mathfrak{G}(\tau, \tau') \cdot (f(\tau') \cdot \eta + \frac{1}{2} \theta(\tau'))$ . The integral over  $\tilde{\psi}$  then generates the determinant  $\text{Det}(\frac{1}{2} \frac{d}{d\tau} + f)$  (for antiperiodic boundary conditions) which because of the nilpotency of  $f$  simply gives a factor of  $2^{\frac{D}{2}}$ , being the number of degrees of freedom of the fermion in  $D$  (even) spacetime dimensions (this should be contrasted with the constant field case, where the normalization picks up a nontrivial field dependence [27,29]).

<sup>2</sup>This is an alternative way of writing the Green function given in Eq. (45) of [33], with the advantage of being manifestly gauge invariant. There  $\mathfrak{G}^{\mu\nu}$  was written in terms of periodic integrals of the derivative of  $a(\tau)$  which made its antiperiodicity easier to see.



441 Gathering all of the above together, the Grassmann integral as defined in (39) becomes

$$\mathfrak{B}_\eta(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_s}) = \frac{\delta}{\delta\theta_{i_1}} \cdot \tilde{f}_{i_1} \cdot \frac{\delta}{\delta\theta_{i_1}} \cdots \frac{\delta}{\delta\theta_{i_s}} \cdot \tilde{f}_{i_s} \cdot \frac{\delta}{\delta\theta_{i_s}} e^{-\int_0^T d\tau[\eta \cdot f(\tau) \cdot \eta + \theta(\tau) \cdot \eta] - \int_0^T d\tau d\tau' [\eta \cdot f(\tau) \cdot \mathfrak{G}(\tau, \tau') \cdot \theta(\tau') + \frac{1}{4}\theta(\tau) \cdot \mathfrak{G}(\tau, \tau') \cdot \theta(\tau')]} \Big|_{\theta=0}. \quad (47)$$

443 The Grassmann path integral is therefore formally computed. In particular,

$$\mathfrak{B}_\eta(\emptyset) = e^{-\int_0^T d\tau \eta \cdot f(\tau) \cdot \eta}, \quad (48)$$

$$\mathfrak{B}_\eta(\tilde{f}_{i_1}) = \left\{ -\frac{1}{2} \text{tr}[\tilde{f}(\tau_{i_1}) \cdot \mathfrak{G}(\tau_{i_1}, \tau_{i_1})] + \eta \cdot \mathfrak{G}^\top(\tau_{i_1}) \cdot \tilde{f}(\tau_{i_1}) \cdot \mathfrak{G}(\tau_{i_1}) \cdot \eta \right\} e^{-\int_0^T d\tau \eta \cdot f(\tau) \cdot \eta}, \quad (49)$$

$$\begin{aligned} \mathfrak{B}_\eta(\tilde{f}_{i_1}; \tilde{f}_{i_2}) = & \left\{ \left[ -\frac{1}{2} \text{tr}[\tilde{f}(\tau_{i_2}) \cdot \mathfrak{G}(\tau_{i_2}, \tau_{i_2})] + \eta \cdot \mathfrak{G}^\top(\tau_{i_2}) \cdot \tilde{f}(\tau_{i_2}) \cdot \mathfrak{G}(\tau_{i_2}) \cdot \eta \right] \times [\tau_{i_2} \rightarrow \tau_{i_1}] \right. \\ & - \frac{1}{2} \text{tr}[\tilde{f}(\tau_{i_1}) \cdot \mathfrak{G}(\tau_{i_1}, \tau_{i_2}) \cdot \tilde{f}(\tau_{i_2}) \cdot \mathfrak{G}(\tau_{i_2}, \tau_{i_1})] \\ & \left. + 2\eta \cdot \mathfrak{G}^\top(\tau_{i_2}) \cdot \tilde{f}(\tau_{i_2}) \cdot \mathfrak{G}(\tau_{i_2}, \tau_{i_1}) \cdot \tilde{f}(\tau_{i_1}) \cdot \mathfrak{G}(\tau_{i_1}) \cdot \eta \right\} e^{-\int_0^T d\tau \eta \cdot f(\tau) \cdot \eta}, \quad (50) \end{aligned}$$

449 where  $\mathfrak{G}_{\mu\nu}(\tau_i) := \eta_{\mu\nu} - \int_0^T d\tau [\mathfrak{G}(\tau_i, \tau) \cdot f(\tau)]_{\mu\nu}$  and  $\top$  denotes the transpose in Lorentz indices—in particular we  
451 have  $\mathfrak{G}^\top_{\mu\nu}(\tau_i) = \eta_{\mu\nu} - \int_0^T d\tau [f(\tau) \cdot \mathfrak{G}(\tau, \tau_i)]_{\mu\nu}$ .

452 Putting all of this together, the  $N$ -photon dressed propagator can be written in a “spin-orbit decomposition” by summing  
453 over assignment of the  $N$  external photons to either the spin or bosonic part of the vertex [33], as follows:

$$\mathcal{S}_N^{x'x} = (-i\partial_{x'} + a(x'^+) - m)\mathcal{K}_N^{x'x}(a) + eA^\gamma(x')\mathcal{K}_{N-1}^{x'x}(a), \quad (51)$$

$$\mathcal{K}_N^{x'x}(a) = \sum_{S=0}^N \sum_{\{i_1: i_S\}} \mathcal{K}_{NS}^{\{i_1: i_S\}x'x}(a), \quad (52)$$

$$\begin{aligned} \mathcal{K}_{NS}^{\{i_1: i_S\}x'x}(a) = & i(-e)^N \int_0^\infty dT (4\pi iT)^{-2} e^{-iM^2(a)T - i\frac{a^2}{4T} - iz \cdot \langle a \rangle} \\ & \times \prod_{i=1}^N \int_0^T d\tau_i \text{Spin}(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_S}) \bar{\mathfrak{P}}_{NS}^{\{i_1: i_S\}x'x} e^{i \sum_{i=1}^N [x + \frac{z}{T}\tau_i - 2I(\tau_i)] \cdot k_i - i \sum_{i,j=1}^N \Delta_{ij} k_i \cdot k_j}. \quad (53) \end{aligned}$$

459 The sum on the second line runs over the allocation of  $S$ , out of the  $N$ , photons to the spin part of the vertex operator,  $V_\eta^{x'x}$ ,  
460 which subsequently appear in  $\text{Spin}(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_S})$ . Then the remaining  $N - S$  photons appear in the polynomial  $\bar{\mathfrak{P}}_{NS}^{\{i_1: i_S\}x'x}$ ,  
461 defined by

$$\bar{\mathfrak{P}}_{NS}^{\{i_1: i_S\}x'x} := i^{N-S} e^{\sum_{i=1}^N \varepsilon_i \cdot \frac{z}{T} + 2 \sum_{i=1}^N [(\langle a \rangle - a_i) \cdot \varepsilon_i] + i \sum_{i,j=1}^N [\varepsilon_i \cdot \varepsilon_j \cdot \Delta_{ij}^* + 2i \cdot \Delta_{ij} \varepsilon_i \cdot k_j]} \Big|_{\substack{\varepsilon_{i_1} \dots \varepsilon_{i_S} = 0 \\ \varepsilon_{i_{S+1}} \dots \varepsilon_{i_N}}}, \quad (54)$$

463 where the notation on the far right means that the  
464 polarization vectors  $\varepsilon_{i_1}$  to  $\varepsilon_{i_S}$  should be put to zero before  
465 the remaining expression is expanded to multilinear order  
466 in the  $\varepsilon_{i_{S+1}}$  to  $\varepsilon_{i_N}$ . These polynomials generalize those  
467 introduced in vacuum ( $\bar{P}_{NS}^{\{i_1: i_S\}}$ ) in [51] and satisfy

$$\bar{\mathfrak{P}}_{N0}^{\{i_1: i_S\}x'x} = \bar{\mathfrak{P}}_{N0}^{x'x}, \quad \bar{\mathfrak{P}}_{NN}^{\{1: N\}x'x} = 1. \quad (55)$$

468 Again, these are position-space expressions, but below we  
470 shall transform to momentum space for the purpose of

evaluating scattering amplitudes. Although this master  
471 formula appears lengthy, it is important to emphasize that  
472 it represents a formal evaluation of the path integral for an  
473 arbitrary number of photons inserted along the background-  
474 dressed propagator, conveniently split into contributions  
475 from the vertex function representing orbital interactions  
476 (in  $\bar{\mathfrak{P}}_{NS}^{\{i_1: i_S\}x'x}$ ) and spin interactions [in  $\text{Spin}(\tilde{f}_{i_1}; \dots; \tilde{f}_{i_S})$ ].  
477 All of these insertions are integrated along the particle  
478 trajectories, so that the master formula represents a sum  
479 over all Feynman diagrams contributing to the dressed  
480

481 propagator that differ by permutation of the external  
 482 photons. Obtaining such a formula from the standard  
 483 formalism (Furry picture, say) of strong-field QED would  
 484 be a significantly more complicated task.

485 For completeness, we note that the  $N = 0$  case provides  
 486 a worldline representation of the well-known Volkov  
 487 propagator as a one-parameter integral

$$\begin{aligned} \mathcal{S}^{x'x} &= i(-i(\partial_{x'} + ia(x'^+)) - m)e^{-iz \cdot (a)} \\ &\times \int_0^\infty dT (4\pi iT)^{-2} e^{-iM^2(a)T - i\frac{z^2}{4T} + \frac{T}{z^+} [n a(x'^+) + a(x^+)n]}, \end{aligned} \quad (56)$$

488 where we used  $\text{Spin}(\theta) = 1 + \frac{T}{2}\gamma \cdot \langle f \rangle \cdot \gamma = 1 + Tn \langle a' \rangle$ ,  
 489 computed the integral in the average explicitly, and  
 490 reexponentiated using  $n^2 = 0$ . This is again equivalent to  
 491 other representations of the Volkov propagator [8,65,66].

### 493 III. LSZ FOR SCATTERING AMPLITUDES

494 The objective of this section is to take the master  
 495 formulas for the dressed propagators  $D_N^{x'x}$  and  $S_N^{x'x}$  above  
 496 and produce from them equivalent master formulas for  
 497 (2-scalar)  $N$ -photon scattering amplitudes (for  $N \geq 1$ ). To  
 498 do so we must perform LSZ reduction on the two massive,  
 499 external legs of the dressed propagators.

500 In previous worldline literature, amputation was often  
 501 done “by hand,” by obtaining the  $N$ -point correlation  
 502 functions in momentum space and then—once the  
 503 proper-time integral had been computed—removing exter-  
 504 nal legs with the appropriate inverse matter propagators  
 505 [51,52]. Only then could the external particles be taken on-  
 506 shell—the proper-time integral produces the pole structure  
 507 of the correlation functions with respect to external matter  
 508 legs and so is divergent in the on-shell limit. This is a  
 509 notable example where the Feynman diagram prescription  
 510 to omit external propagators had appeared less trivial from  
 511 a worldline perspective. Recently, however, [68,69] showed  
 512 how amputation can be achieved under the proper-time  
 513 integral for scalar matter legs, with the inverse propagators  
 514 simply modifying the bounds on the proper-time and  
 515 parameter integrals. This exposes the on-shell residue of  
 516 the correlation functions without the need to carry out  
 517 amputation by hand. We will here generalize this approach  
 518 to spinor theories, and also show it is unspoiled by the plane  
 519 wave background.

520 To perform LSZ we draw the external legs out to  
 521 asymptotic times and Fourier transform. Alternatively,  
 522 we can Fourier transform to momentum space and find  
 523 the residues of the dressed propagator as the momenta are  
 524 taken onto the mass-shell. Starting with scalar QED, the  
 525 amplitude takes the form

$$\begin{aligned} \mathcal{A}_N^{p'p} &= - \lim_{p'^2, p^2 \rightarrow m^2} \int d^4x' d^4x e^{i(p'+a^\infty) \cdot x' - ip \cdot x} [(\partial_{x'} \\ &+ ia^\infty)^2 + m^2][\partial_x^2 + m^2] \mathcal{D}_N^{x'x} \end{aligned} \quad (57)$$

$$= \lim_{p'^2, p^2 \rightarrow m^2} - (p'^2 - m^2)(p^2 - m^2) \mathcal{D}_N^{\tilde{p}'p}, \quad (58) \quad 52\text{B}$$

where in the second line we defined  $\tilde{p}' = p' + a^\infty$  and  
 introduced the momentum-space propagator  $\mathcal{D}_N^{\tilde{p}'p}$ , defined  
 by

$$\mathcal{D}_N^{\tilde{p}'p} := \int d^4x' d^4x e^{i\tilde{p}' \cdot x' - ip \cdot x} \mathcal{D}_N^{x'x}. \quad (59)$$

The expression (57) is (almost) textbook-standard LSZ  
 position space but to compensate for the fact that our  
 potential becomes pure gauge in the far future, the on-shell,  
 outgoing momentum  $p'$  in the Fourier kernel is shifted to  
 $\tilde{p}' = p' + a^\infty$  [57,63]. The expression (58) makes it clear  
 that the amplitude  $\mathcal{A}_N^{p'p}$  is the residue of  $\mathcal{D}_N^{\tilde{p}'p}$  at on-shell  
 momenta. In our conventions  $\mathcal{A}_N^{p'p}$  describes  $N$ -photon  
*emission* from a particle traversing the plane wave.  
 Absorption and pair-production/annihilation amplitudes  
 are of course obtained by crossing.

Similarly for the spinor case, starting from the master  
 formula for the dressed propagator (51), we can extract the  
 spin-polarized amplitude  $\mathcal{M}_{Ns's}^{p'p}$  as

$$\begin{aligned} \mathcal{M}_{Ns's}^{p'p} &= i \lim_{p'^2, p^2 \rightarrow m^2} \int d^4x' d^4x e^{i\tilde{p}' \cdot x' - ip \cdot x} \bar{u}_{s'}(p') \\ &\times (i\partial_{x'} - a^\infty - m) S_N^{x'x} (-i\tilde{\partial}_x - m) u_s(p), \end{aligned} \quad (60)$$

in which  $\bar{u}_{s'}(p')$  and  $u_s(p)$  are free Dirac spinors. We now  
 proceed to perform the LSZ reduction explicitly, starting  
 with scalar QED.

#### A. Scalar QED

We begin by evaluating the momentum-space propagator  
 via direct Fourier transform of the master formula (24):

$$\mathcal{D}_N^{\tilde{p}'p} = \int d^4x' d^4x e^{i\tilde{p}' \cdot x' - ip \cdot x} \mathcal{D}_N^{x'x}. \quad (61)$$

The integrals over  $x'^{\perp-}$  and  $x^{\perp-}$  generate,<sup>3</sup> as in the vacuum  
 case, four  $\delta$ -functions, explicitly  $\delta_{\perp-}^3(\tilde{p}' + K - p) \times$   
 $\delta(x^+ - x'^+ + 2g^+ + 2p'^+ T)$ , where we write  $K =$   
 $\sum_{i=1}^N k_i$  to compactify notation. The first three  $\delta$ -functions  
 describe the (expected) conservation of light front three-  
 momentum in the plane wave background. The final

<sup>3</sup>To evaluate similar integrals in the existing literature it was  
 found to be convenient to change variables to end-point center of  
 mass and relative separation ( $z$ ). However, for our later LSZ  
 amputation of the external legs it is more useful to integrate  
 separately with respect to the end-point coordinates.

560  $\delta$ -function allows us to trivially perform, e.g., the  $x'^+$  integral,  
 561 so that we can replace  $x'^+ \rightarrow x^+ + 2g^+ + 2p'^+T$  in what  
 562 remains; in particular, the classical trajectory on which the  
 563 gauge field depends throughout  $\mathcal{D}_N^{x',x}$ , as in (18), is modified  
 564 to, where  $g \equiv g(\{\tau_i\}) := \sum_{i=1}^N (k_i \tau_i - i \varepsilon_i)$ ,  
 568

$$x_{cl}^+(\tau) = x^+ + g^+ + (p' + p)^+ \tau - \sum_{i=1}^N k_i^+ |\tau - \tau_i|. \quad (62)$$

Thus we can do all but one of the Fourier integrals, which 566  
 eventually yield 567

$$\begin{aligned} \mathcal{D}_N^{\tilde{p},p} &= (-ie)^N (2\pi)^3 \delta_{\pm,-}^3(\tilde{p}' + K - p) \int_0^\infty dT e^{i(p'^2 - m^2 + i0^+)T} \int_{-\infty}^\infty dx^+ e^{i(p'_+ + K_+ - p_+)x^+} \\ &\times \prod_{i=1}^N \int_0^T d\tau_i e^{-2ig \cdot \langle a \rangle - 2iT p' \cdot \langle \delta a \rangle + iT \langle \delta a^2 \rangle - 2i \sum_{i=1}^N [k_i \cdot I(\tau_i) - i \varepsilon_i \cdot I'(\tau_i)]} \\ &\times e^{ig \cdot (2\tilde{p}' + K) - i \sum_{i,j=1}^N \left( \frac{|\tau_i - \tau_j|}{2} k_i \cdot k_j - i \text{sgn}(\tau_i - \tau_j) \varepsilon_i \cdot k_j + \delta(\tau_i - \tau_j) \varepsilon_i \cdot \varepsilon_j \right)} \Bigg|_{\text{lin.} \varepsilon_1 \dots \varepsilon_N}, \end{aligned} \quad (63)$$

570 in which we have defined  $\delta a(x^+) := a(x^+) - a^\infty$  and  
 571  $a(\tau) \equiv a(x_{cl}^+(\tau))$ . Note that in the vacuum limit  $a_\mu \rightarrow 0$   
 572 we can carry out the  $\hat{x}^+$  integral to complete the con-  
 573 servation of 4-momentum and so recover one version of the  
 574 master formula given in [27,51].

575 To convert (63) into a master formula for the amplitudes,  
 576 we have to perform LSZ on each massive scalar leg (these  
 577 are produced by the parameter and proper-time integrals).  
 578 To do so we observe that (58) has, using (63), the following  
 579 form, writing down only the relevant structures:

$$-i(p'^2 - m^2 + i0^+) \int_0^\infty dT e^{i(p'^2 - m^2 + i0^+)T} F(T). \quad (64)$$

580 The on-shell limit  $p'^2 \rightarrow m^2 - i0^+$  therefore returns the  
 582 residue of the mass-shell pole of the function defined by the  
 583 integral. To isolate this pole we proceed as in [68–70]  
 584 where LSZ was considered for, e.g., the  $N$ -graviton-dressed  
 585 propagator in vacuum.<sup>4</sup> We integrate by parts (off-shell) in  
 586 order to expose the residue, as so:  
 598

$$\begin{aligned} \lim_{p'^2 \rightarrow m^2} -i(p'^2 - m^2 + i0^+) \mathcal{D}_N^{\tilde{p},p} &= (-ie)^N (2\pi)^3 \delta_{\pm,-}^3(\tilde{p}' + K - p) \int_{-\infty}^\infty dx^+ e^{i(p'_+ + K_+ - p_+)x^+} \prod_{i=1}^N \int_0^\infty d\tau_i \\ &\times e^{-i \int_0^\infty [2p' \cdot \delta a(\tau) - \delta a^2(\tau)] d\tau - 2i \sum_{i=1}^N [\int_0^{\tau_i} k_i \cdot a(\tau) d\tau - i \varepsilon_i \cdot a(\tau_i)] + ig \cdot (2\tilde{p}' + K) - i \sum_{i,j=1}^N \left( \frac{|\tau_i - \tau_j|}{2} k_i \cdot k_j - i \text{sgn}(\tau_i - \tau_j) \varepsilon_i \cdot k_j + \delta(\tau_i - \tau_j) \varepsilon_i \cdot \varepsilon_j \right)} \Bigg|_{\text{lin.} \varepsilon_1 \dots \varepsilon_N}. \end{aligned} \quad (67)$$

599 We note that all terms with worldline averages have  
 600 ultimately been replaced with (convergent) integrals over  
 601  $\mathbb{R}^+$ . This was the advantage of having computed the  
 602 Fourier integrals with respect to the individual end points

$$\begin{aligned} &-i(p'^2 - m^2 + i0^+) \int_0^\infty dT e^{i(p'^2 - m^2 + i0^+)T} F(T) \\ &= F(0) + \int_0^\infty dT e^{i(p'^2 - m^2 + i0^+)T} \frac{d}{dT} F(T). \end{aligned} \quad (65)$$

We can now take  $p'^2 \rightarrow m^2$  and  $0^+ \rightarrow 0$  (in either order), 588  
 upon which the integral becomes exact, and we have 589

$$\begin{aligned} \lim_{p'^2 \rightarrow m^2} -i(p'^2 - m^2 + i0^+) \int_0^\infty dT e^{i(p'^2 - m^2 + i0^+)T} F(T) \\ = F(\infty). \end{aligned} \quad (66)$$

Ultimately, then, performing the first amputation on (63) is 590  
 equivalent to dropping the integral over proper time  $T$  and 592  
 its accompanying mass-shell exponent, and taking the limit 593  
 $T \rightarrow \infty$  of what remains (this is the same argument as in 594  
 vacuum, which we comment on further after performing the 595  
 second amputation, below). We thus find 596

as discussed above. Equation (67) is the one-side ampu- 603  
 tated propagator. 604

Turning to the amputation with respect to  $p$ , at this stage 605  
 it is advantageous to introduce the mean and deviation 606  
 proper-time variables as follows: 607

$$\tau_0 := \frac{1}{N} \sum_{i=1}^N \tau_i, \quad \bar{\tau}_i := \tau_i - \tau_0. \quad (68)$$

<sup>4</sup>We note in passing that the same “trick” is useful in exposing  
 the connection between gauge invariance and infrared behavior of  
 amplitudes in background plane waves [71].

609 The reason for this change of variable is that it allows us to  
 610 reexpress (67) in a form which renders the *second* LSZ  
 611 amputation immediate. To achieve this, we first rewrite the  
 612 proper-time integrals appearing in (67) in terms of the new  
 613 variables as [note the factor of  $\frac{1}{N}$  in the  $\delta$ -function is missing  
 614 in (3.18) of [69]]

$$\prod_{i=1}^N \int_0^\infty d\tau_i = \int_0^\infty d\tau_0 \prod_{i=1}^N \int_{-\infty}^\infty d\bar{\tau}_i \delta\left(\sum_{j=1}^N \frac{\bar{\tau}_j}{N}\right). \quad (69)$$

616 We also make a change of variable for the  $x^+$ -integration,  
 617  $\bar{x}^+ := x^+ + (p' + p + K)^+ \tau_0 + g^+(\{\bar{\tau}_i\})$ , and it is conven-  
 618 619 620 621 622 623 624 625 626

$$a(\bar{\tau}) \equiv a\left(\bar{x}^+ + (p' + p)^+ \bar{\tau} - \sum_{i=1}^N k_i^+ |\bar{\tau} - \bar{\tau}_i|\right). \quad (70)$$

In terms of the shifted variables  $\{\bar{x}^+, \tau_0, \bar{\tau}_i\}$ , the once-  
 amputated propagator (67) takes the form

$$\begin{aligned} & (-ie)^N (2\pi)^3 \delta_{\perp,-}(\tilde{p}' + K - p) \int_{-\infty}^\infty d\bar{x}^+ e^{i(K+p'-p)_+ \bar{x}^+} \\ & \times \int_0^\infty d\tau_0 e^{i(p^2 - m^2)\tau_0} \int_{-\infty}^\infty \prod_{i=1}^N d\bar{\tau}_i \delta\left(\sum_{i=1}^N \frac{\bar{\tau}_i}{N}\right) G(\tau_0), \end{aligned} \quad (71)$$

in which the function appearing in the factor is

$$\begin{aligned} G(\tau_0) = & e^{-i(2p' + a^\infty) \cdot a^\infty \tau_0 - i \int_{-\tau_0}^\infty d\bar{\tau} [2p' \cdot \delta a(\bar{\tau}) - \delta a^2(\bar{\tau})] - 2i \sum_{i=1}^N \left[ \int_{-\tau_0}^{\bar{\tau}_i} d\bar{\tau} k_i \cdot a(\bar{\tau}) - i \epsilon_i \cdot a(\bar{\tau}_i) \right]} \\ & \times e^{i(\tilde{p}' + p) \cdot g - i \sum_{i,j=1}^N \left( \frac{|\bar{\tau}_i - \bar{\tau}_j|}{2} k_i \cdot k_j - i \text{sgn}(\bar{\tau}_i - \bar{\tau}_j) \epsilon_i \cdot k_j + \delta(\bar{\tau}_i - \bar{\tau}_j) \epsilon_i \cdot \epsilon_j \right)} \Bigg|_{\text{lin. } \epsilon_1, \dots, \epsilon_N} \end{aligned} \quad (72)$$

628 Note that the factor  $-i(2p' + a^\infty) \cdot a^\infty \tau_0$  in the exponential diverges in the  $\tau_0 \rightarrow \infty$  limit, but can be absorbed into  
 630 the Volkov-like term, also divergent in the same limit, to yield the convergent factor  $-i \int_{-\tau_0}^\infty [2\tilde{p}' \cdot a(\bar{\tau}) -$   
 631  $a^2(\bar{\tau})] d\bar{\tau} - i \int_0^\infty [2p' \cdot \delta a(\bar{\tau}) - \delta a^2(\bar{\tau})] d\bar{\tau}$ . After this rearrangement, one finds that the dependence on  $\{p^2 - m^2, \tau_0\}$  in  
 632 (71) and (72) exactly mirrors the dependence on  $\{p'^2 - m^2, T\}$  in the original expression, before the first amputation. Thus  
 633 we can simply repeat the previous LSZ argument but applied to  $\{p^2 - m^2, \tau_0\}$  in order to extract the pole at the *incoming*  
 634 mass-shell; effectively this removes the integral over  $\tau_0$  and takes  $\tau_0 \rightarrow \infty$  in the remainder, yielding our final master  
 635 formula for the 2-scalar  $N$ -photon scattering amplitudes:

$$\begin{aligned} \mathcal{A}_N^{p',p} = & (-ie)^N (2\pi)^3 \delta_{\perp,-}(\tilde{p}' + K - p) \int_{-\infty}^\infty dx^+ e^{i(K+p'-p)_+ x^+} \int_{-\infty}^\infty \prod_{i=1}^N d\tau_i \delta\left(\sum_{j=1}^N \frac{\tau_j}{N}\right) \\ & \times e^{-i \int_{-\infty}^0 [2\tilde{p}' \cdot a(\tau) - a^2(\tau)] d\tau - i \int_0^\infty [2p' \cdot \delta a(\tau) - \delta a^2(\tau)] d\tau - 2i \sum_{i=1}^N \left[ \int_{-\infty}^{\tau_i} k_i \cdot a(\tau) d\tau - i \epsilon_i \cdot a(\tau_i) \right]} \\ & \times e^{i(\tilde{p}' + p) \cdot g - i \sum_{i,j=1}^N \left( \frac{|\tau_i - \tau_j|}{2} k_i \cdot k_j - i \text{sgn}(\tau_i - \tau_j) \epsilon_i \cdot k_j + \delta(\tau_i - \tau_j) \epsilon_i \cdot \epsilon_j \right)} \Bigg|_{\text{lin. } \epsilon} \end{aligned} \quad (73)$$

637 where  $a(\tau)$  is as in (70), and we have simply relabeled  
 638  $\bar{x}^+ \rightarrow x^+$ , and  $\bar{\tau}, \bar{\tau}_i \rightarrow \tau, \tau_i$ .

639 There are several features of this all-orders formula  
 640 worth discussing. First, as a consistency check, it is  
 641 straightforward to check that in the vacuum limit  
 642 ( $a \rightarrow 0$ ) the  $x^+$  integral can again be performed and one  
 643 recovers the known results in [54,69,72]. Second, similarly  
 644 to [69], a short set of rules summarizes the LSZ reduction.  
 645 The first three are shared with the vacuum case [69]:  
 646 (i) drop the  $T$  integral, (ii) insert  $\delta(\sum_{j=1}^N \tau_j/N)$ , and  
 647 (iii) take the  $d\tau_i$  and  $d\tau$  integrals over  $\mathbb{R}$ . Here, beyond  
 648 the vacuum case, there are additional rules: (iv) drop all  
 649 worldline averages and (v) “introduce” the divergent factor  
 650  $\int_{-\infty}^0 -2i\tilde{p}' \cdot a^\infty d\tau$  into the exponential, which ensures that

the proper-time integral is convergent in the asymptotic  
 past—we stress that this by hand addition only occurs at the  
 level of these rules, it emerges naturally as part of LSZ  
 reduction, as described above.

Third, the change in integration range for the  $d\tau_i$   
 integrals can be understood as manifesting the fact that  
 $\mathcal{A}_N^{p',p}$  is an asymptotic quantity, while the purpose of  
 $\delta(\sum_{j=1}^N \tau_j/N)$  is to “gauge” the proper-time translational  
 symmetry of the system. Clearly neither of these features  
 should be particular to any choice of background that tends  
 to at most a constant asymptotically, and indeed they are the  
 same in our plane wave background as in vacuum.

Finally, we observe that  $x_{cl}^+(\tau)$  in (70) solves the classical  
 worldline equation of motion with the boundary conditions

665  $\frac{1}{4}\dot{x}^+(-\infty) = p_-$  and  $\frac{1}{4}\dot{x}^+(\infty) = p'_-$ . It is natural for this  
 666 solution to appear in the amplitudes because, although it  
 667 may not be obvious, the stated boundary conditions are  
 668 (particular components of) those in play for the momen-  
 669 tum-space propagator, from which the amplitude is con-  
 670 structed. We will show this in the following subsection, in  
 671 which we briefly digress from the master formula in order  
 672 to investigate how the Volkov wave functions arise from  
 673 worldline path integrals.

### 674 **B. Mixed boundary conditions** 675 **and the Volkov wave function**

676 Before moving on to the spinor case, we remark that one  
 677 can, in fact, compute the momentum-space propagator  
 678 without going *explicitly* via the position-space representa-  
 679 tion. Returning to the original expression (5) for  $\mathcal{D}_N^{x'x}$ , we  
 680 immediately perform the Fourier transform (59). Now, the  
 681 exponent  $p' \cdot x' - p \cdot x$  in the Fourier kernel is, under  
 682 the path integral, the same as  $p' \cdot x(T) - p \cdot x(0)$ , and  
 683 the spacetime integrals  $d^4x' d^4x$  can be interpreted as  
 684  $d^4x(T) d^4x(0)$ . Hence, taking the Fourier transform of (5)  
 685 is equivalent to performing a path integral with a free  
 686 boundary, i.e. no *apparent* restriction on the end points of  
 687 the worldline. There is though an alternative, but equiv-  
 688 alent, perspective; consider the change of the total action,  
 689  $\delta S$ , under the variations of the end points of the worldline,  
 690  $x(0) \rightarrow x(0) + \delta x_0$  and  $x(T) \rightarrow x(T) + \delta x_T$ :

$$\begin{aligned} \delta S &\equiv \delta S_B + \delta(p' \cdot x(T) - p \cdot x(0)) \\ &= \left[ \frac{1}{2}\dot{x}(0) + a(x(0)) - p \right] \cdot \delta x_0 \\ &\quad - \left[ \frac{1}{2}\dot{x}(T) + a(x(T)) - p' \right] \cdot \delta x_T. \end{aligned} \quad (74)$$

692 Integrating over  $\delta x_T$  and  $\delta x_0$  therefore returns delta  
 693 functions which impose the vanishing of the terms in  
 694 square brackets of (74); these are *Robin* boundary con-  
 695 ditions which relate the worldline end-point momenta  $\dot{x}$  to  
 696 the end-point positions  $x$  and the external asymptotic  
 697 momenta. It follows that the momentum-space propagator  
 698 can be computed alternatively from the path integral  
 699 expression

$$\begin{aligned} \mathcal{D}_N^{p'p} &= (-ie)^N \int_0^\infty dT e^{-im^2 T} \\ &\quad \times \int_{\dot{x}(0)+2a(x(0))=2p}^{\dot{x}(T)+2a(x(T))=2p'} \mathcal{D}x(\tau) e^{iS_B[x(\tau)]} \prod_{i=1}^N V[\varepsilon_i k_i]. \end{aligned} \quad (75)$$

700 In the previous section we carried out the Fourier transform  
 702 of  $\mathcal{D}_N^{x'x}$  literally, to obtain  $\mathcal{D}_N^{p'p}$ . Expression (75) shows a  
 703 more “direct” approach to deriving the master formula in  
 704 (63), through a modification of the boundary conditions on

the path integral. This fits in more naturally with the  
 “worldline philosophy” of incorporating all information  
 into the worldline path integral. Note that evaluation of (74)  
 requires a worldline propagator with different boundary  
 conditions. Indeed, this helps explain a puzzle arising  
 in [26] (Section 3, footnote 3), where a version of the  
 momentum space master formula was given that involves a  
 Green function with mixed boundary conditions: by  
 expanding about a suitable reference trajectory, (75) can  
 be cast into a path integral for the fluctuation variable  
 that must satisfy the mixed boundary conditions  
 $\dot{q}(0) = 0 = q(T)$ .

This discussion prompts us to study the propagator  $\mathcal{D}_N^{xp}$   
 with mixed boundary conditions which, examining (75), is  
 given by the integral

$$\begin{aligned} \mathcal{D}_N^{xp} &= (-ie)^N \int_0^\infty dT e^{-im^2 T} \int_{\dot{x}(0)+2a(x(0))=2p}^{x(T)=x} \mathcal{D}x(\tau) e^{iS_B[x(\tau)]} \\ &\quad \times \prod_{i=1}^N V[\varepsilon_i k_i]. \end{aligned} \quad (76)$$

To see the significance of the mixed propagator, consider  
 the case  $N = 0$ , that is the tree level two-point function for  
 the scalar field, with mixed boundary conditions. In  
 Feynman diagram language, this is just an external leg,  
 Fourier transformed at one end. Taking the momentum at  
 this end onto the mass-shell, i.e. performing LSZ reduction,  
 we must recover the scalar Volkov wave functions. These  
 are solutions of the Klein-Gordon equation in a plane wave  
 background which reduce to  $e^{\pm ip \cdot x}$  in the asymptotic past/  
 future and thus represent incoming and outgoing particles  
 in scattering amplitudes.

To confirm this, we first compute the path integral in (76)  
 for  $N = 0$  (we drop the product of vertex operators). We  
 do not dwell on this step; the entire integral turns out,  
 unsurprisingly given the nature of the Volkov solutions and  
 hidden Gaussianity of the worldline path integral, to be  
 equal to its semiclassical value  $\exp[iS_{cl}(T)]$ , i.e. the  
 exponential of the classical action evaluated on the classical  
 path obeying the mixed boundary conditions, which is

$$\begin{aligned} S_{cl}(T) &= (p^2 - m^2 + i0^+)T - p \cdot x \\ &\quad - \int_{x^+ - 4p_- T}^{x^+} ds \frac{2p \cdot a(s) - a^2(s)}{4p_-}. \end{aligned} \quad (77)$$

The final step is to take  $p^2 \rightarrow m^2$  and identify the on-shell  
 residue via

$$\lim_{p^2 \rightarrow m^2} -i(p^2 - m^2 + i0^+) \int_0^\infty dT e^{-im^2 T} e^{iS_{cl}(T)}. \quad (78)$$

Of course it is clear from the preceding calculations how to  
 proceed; we perform the same manipulations as for the  
 master formula, in particular taking the  $T \rightarrow \infty$  limit,  
 immediately finding

$$\begin{aligned} & \lim_{p^2 \rightarrow m^2} -i(p^2 - m^2) \mathcal{D}^{xp} \\ &= \exp \left[ -i p \cdot x - i \int_{-\infty}^{x^+} ds \frac{2p \cdot a(s) - a(s)^2}{4p_-} \right] \equiv \varphi_p^{\text{in}}(x). \end{aligned} \quad (79)$$

749 The right-hand side is precisely the incoming scalar Volkov  
 750 wave function  $\varphi_p^{\text{in}}(x)$  which reduces to  $e^{-ip \cdot x}$  in the  
 751 asymptotic past. A similar amputation of the propagator  
 752  $\mathcal{D}_0^{p'x}$  (where the boundary conditions are swapped) yields  
 753 the outgoing Volkov wave functions, i.e. those which  
 754 reduce to  $e^{+i\tilde{p}' \cdot x}$  in the asymptotic future. Of course the  
 755 same procedure can be applied to the spinor propagator,  
 756 wherein the path integral with mixed boundary conditions  
 757 produces the spinor Volkov wave functions. Worldline path  
 758 integrals analogous to (76), with mixed boundary con-  
 759 ditions, have also been used before, in a similar context, to  
 760 recover the exact solutions of the Klein-Gordon equation in  
 761 a constant external electromagnetic field [73]. For numeri-  
 762 cal studies of open line instantons see [41].

### 763 C. Spinor QED

764 Turning to LSZ reduction in spinor QED, we proceed  
 765 from (60), writing  $\mathcal{S}_N^{x'x}$  in terms of the kernels appearing in  
 766 (51) and evaluating the  $\partial_{x'}$ ,  $\partial_x$  derivatives (using integration  
 767 by parts) in (60) to find

$$\begin{aligned} \mathcal{M}_{Ns's}^{p'p} &= i \lim_{p'^2, p^2 \rightarrow m^2} \int d^4x' d^4x e^{i\tilde{p}' \cdot x' - ip \cdot x} \bar{u}_{s'}(p') (\not{p}' - m) \\ &\times \left\{ (-\not{p}' + \delta a(x'^+) - m) \mathcal{K}_N^{x'x} + e \sum_{i=1}^N \varepsilon_i e^{ik_i \cdot x'} \mathcal{K}_{N-1}^{x'x} \right\} \\ &\times (\not{p} - m) u_s(p). \end{aligned} \quad (80)$$

768 Next, following [52] we use the *on-shell* relation  
 770  $\bar{u}_{s'}(p') (\not{p}' + m)^{-1} = \bar{u}_{s'}(p') (2m)^{-1}$ , (which is allowed  
 771 since it does not remove the associated pole, or affect the  
 772 final expression), and likewise for  $(\not{p} + m)^{-1} u_s(p)$  to find

$$\begin{aligned} \mathcal{M}_{Ns's}^{p'p} &= i \lim_{p'^2, p^2 \rightarrow m^2} \frac{1}{2m} \int d^4x' d^4x e^{i\tilde{p}' \cdot x' - ip \cdot x} \bar{u}_{s'}(p') (p'^2 - m^2) \\ &\times \left\{ \left[ -1 + \frac{1}{2m} \delta a(x'^+) \right] \mathcal{K}_N^{x'x} \right. \\ &\left. + \frac{e}{2m} \sum_{i=1}^N \varepsilon_i e^{ik_i \cdot x'} \mathcal{K}_{N-1}^{x'x} \right\} (p^2 - m^2) u_s(p). \end{aligned} \quad (81)$$

773 Due to the worldline approach being based on the  
 775 second-order formalism of QED, the exponent under the

776 proper-time integral of the spinor amplitude contains  
 777 the same terms as for the scalar amplitude—in particular  
 778 the parameter and proper-time integrals produce (free) *scalar*  
 779 *propagators*. Hence it suffices to revise the scalar case for  
 780 this argument. The difference lies in the spin factor of the  
 781 kernel, the subleading contributions (those proportional to  
 782  $\mathcal{K}_{N-1}$ ), and the  $\delta a(x'^+)$  factor from the covariant derivative.  
 783 However the differences do not impede processing the  $T$ , and  
 784 later  $\tau_0$ , proper time integrals as for scalars. The result is that  
 785 the LSZ amputation is realized in precisely the same way, by  
 786 taking  $T, \tau_0 \rightarrow \infty$  as in Eqs. (64)–(69). Moreover, after  
 787 taking the Fourier transform, the conservation of momenta  
 788 enforced by  $\delta(x^+ - x'^+ + 2g^+ + 2p'^+ T)$  sends

$$a(x'^+) \rightarrow a(2T p'^+ + x^+ + 2g^+). \quad (82)$$

789 The LSZ truncation projects onto asymptotic late time,  
 790 taking  $a(x'^+) \rightarrow a^\infty$  when  $T \rightarrow \infty$ , canceling the field-  
 791 dependent term in square brackets of (81). One may then  
 792 express (81) in terms of the momentum-space kernel  
 793

$$\begin{aligned} \mathcal{M}_{Ns's}^{p'p} &= i \lim_{p'^2, p^2 \rightarrow m^2} \frac{1}{2m} \bar{u}_{s'}(p') (p'^2 - m^2) \\ &\times \left\{ -\mathcal{K}_N^{p'p} + \frac{e}{2m} \sum_{i=1}^N \varepsilon_i \mathcal{K}_{N-1}^{(\tilde{p}'+k_i)p} \right\} \\ &\times (p^2 - m^2) u_s(p). \end{aligned} \quad (83)$$

794 Now we address the subleading terms. These are seen to  
 795 have poles not in the required mass-shell  $p'^2 - m^2$ , but  
 796 rather in  $((p' + k_i)^2 - m^2)$ . Contributions involving these  
 797 shifted poles hence vanish after taking the on-shell limit of  
 798  $(p'^2 - m^2)/((p' + k_i)^2 - m^2)$ . This is a remarkable gener-  
 799 alization of the vacuum case [52]. We can be more precise  
 800 with how this cancellation comes about. In the kernel of the  
 801 subleading terms,  $\mathcal{K}_{N-1}^{(\tilde{p}'+k_i)p}$ , one must first remove an  $\varepsilon_i$  and  
 802  $k_i$ , and then replace  $a^\infty$  with  $a^\infty + k_i$  in (73). This operation  
 803 leaves  $\tilde{p}' + K$  invariant, but it does affect the term  
 804  $\int_0^\infty d\tau p' \cdot \delta a(\tau)$ , which was convergent as  $\tau \rightarrow \infty$ , but  
 805 now produces a rapidly oscillating phase; noting that the  
 806 proper-time integral calculates the Laplace transform of the  
 807 function  $F(T)$  in (64), the Abelian final value theorem can  
 808 be invoked to confirm that the subleading contributions  
 809 must vanish.  
 810

811 Since the manipulations are similar to the scalar case, let  
 812 us simply record the spinor amplitude in its final form as  
 813

$$\mathcal{M}_{Ns's}^{p'p} = \sum_{S=1}^N \sum_{\{i_1: i_S\}} \mathcal{M}_{NSs's}^{\{i_1: i_S\} p' p}, \quad (84)$$

814

816

$$\begin{aligned}
\mathcal{M}_{NSs's}^{\{i_1:i_s\}p'p} &= (-ie)^N (2\pi)^3 \delta_{\perp,-}(\tilde{p}' + K - p) \int_{-\infty}^{\infty} dx^+ e^{i(K+p'-p)_+ x^+} \int_{-\infty}^{\infty} \prod_{i=1}^N d\tau_i \delta\left(\sum_{j=1}^N \frac{\tau_j}{N}\right) \\
&\times e^{-i \int_{-\infty}^0 [2\tilde{p}' \cdot a(\tau) - a^2(\tau)] d\tau - i \int_0^{\infty} [2p' \cdot \delta a(\tau) - \delta a^2(\tau)] d\tau - 2i \sum_{i=1}^N \left[ \int_{-\infty}^{\tau_i} k_i \cdot a(\tau) d\tau - i \varepsilon_i \cdot a(\tau_i) \right]} \\
&\times e^{i(\tilde{p}' + p) \cdot g - i \sum_{i,j=1}^N \left( \frac{|\tau_i - \tau_j|}{2} k_i \cdot k_j - i \text{sgn}(\tau_i - \tau_j) \varepsilon_i \cdot k_j + \delta(\tau_i - \tau_j) \varepsilon_i \cdot \varepsilon_j \right)} \Bigg|_{\substack{\varepsilon_{i_1} \dots \varepsilon_{i_s} = 0 \\ \varepsilon_{i_{s+1}} \dots \varepsilon_{i_N}}} \\
&\times \frac{1}{2m} \bar{u}_{s'}(p') \text{Spin}(\tilde{f}_{i_1:i_s}) u_s(p).
\end{aligned} \tag{85}$$

818 After LSZ reduction, the argument of the exponential in the spin factor, (47), takes the following form

819

$$- \int_{-\infty}^{\infty} d\tau [\eta \cdot f \cdot \eta + \theta \cdot \eta] - \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \left[ \eta \cdot f(\tau) \cdot \mathfrak{G}(\tau, \tau') \cdot \theta(\tau') + \frac{1}{4} \theta(\tau) \cdot \mathfrak{G}(\tau, \tau') \cdot \theta(\tau') \right]; \tag{86}$$

820 the worldline average in the fermion Green function is also  
821 now understood to be  $T\langle\langle f \rangle\rangle = \int_{-\infty}^{\infty} d\tau f(\tau)$ . Also, the  
822 background gauge potential,  $a$ , and field strength,  $f$ , are  
823 understood to be functions of the classical solution  $x_{cl}^+(\tau)$  as  
824 shown in (70). Finally, the sums in the first line of (84)  
825 are—as usual—over the assignment of  $S$  photons out of  $N$   
826 to the spin part of the vertex operator.

827

#### IV. EXAMPLES

828 In this section we provide checks on our amplitude  
829 master formulas (73) and (84), showing by comparison  
830 with the existing literature that they are consistent with  
831 results expected from Furry-picture perturbation theory.

##### A. $N = 1$ , nonlinear Compton scattering in scalar QED

832 The case  $N = 1$  describes single photon emission from a  
833 (scalar) electron in a plane wave background, which is the  
834 well-studied process of “nonlinear Compton scattering.” In  
835 this case, several parts of the master formulas (73) simplify  
836 immediately. First, the delta function fixes  $\tau_1 = 0$ . Next, the  
837 gauge field is evaluated as  
838

$$a(\tau) = \begin{cases} a(x^+ + 2p^+\tau), & \tau < 0, \\ a(x^+ + 2p'^+\tau), & \tau > 0. \end{cases} \tag{87}$$

839 This form facilitates an easy conversion of integrals over  
840 proper time  $\tau$  to integrals over light front time  $x^+$ , which are  
841 expected in the standard formalism (see also [37]).  
842 Specifically, we can conveniently treat the positive and  
843 negative  $\tau$  regions separately. The field-dependent terms in  
844 the exponent of the master formula then reduce to  
845

$$\begin{aligned}
&-i \int_{-\infty}^0 d\tau [2\tilde{p}' \cdot a(\tau) - a^2(\tau)] \\
&-i \int_0^{\infty} d\tau [2p' \cdot \delta a(\tau) - \delta a^2(\tau)] - 2i \int_{-\infty}^0 d\tau k_1 \cdot a(\tau),
\end{aligned} \tag{88}$$

$$\begin{aligned}
&= -i \int_{-\infty}^{x^+} ds^+ \frac{2p \cdot a(s^+) - a^2(s^+)}{2p^+} \\
&-i \int_{x^+}^{\infty} ds^+ \frac{2p' \cdot \delta a(s^+) - \delta a^2(s^+)}{2p'^+},
\end{aligned} \tag{89}$$

846 in which we simply inserted (87) and used momentum  
847 conservation in the transverse directions to eliminate  $k_1$  in  
848 favor of  $p'$  and  $p$ . With this, expanding (73) for  $N = 1$  to  
849 linear order in  $\varepsilon_1$ , and using the Fourier representation of  
850 the momentum conserving  $\delta$ -functions shows that the  
851 amplitude is equivalent to  
852

$$\begin{aligned}
\mathcal{A}_1^{p'p} &= -ie \int d^4x \{ \tilde{p}'_\mu + p_\mu - 2a_\mu(x^+) \} \\
&\times \varepsilon_1^\mu e^{ik_1 \cdot x} \varphi_p^{\text{out}}(x) \varphi_p^{\text{in}}(x),
\end{aligned} \tag{90}$$

853 where  $\varphi_p^{\text{in}}$  is the incoming scalar Volkov wave function  
854 of (79) while  $\varphi_p^{\text{out}}$  is the outgoing wave function,  
855

$$\varphi_p^{\text{out}}(x) = e^{i\tilde{p}' \cdot x} \exp \left[ -i \int_{x^+}^{\infty} ds^+ \frac{2p' \cdot \delta a(s^+) - \delta a^2(s^+)}{2p'^+} \right]. \tag{91}$$

856 Expression (90) is precisely the expected result for non-  
857 linear Compton scattering in scalar QED, providing a  
858 positive check on our master formula.  
859

860 We stress that the method we employed above to process  
861 the worldline integrals was meant only to allow direct  
862 comparison with existing results. It is *not* the approach we  
863 wish to take in future work; instead, we will use the  
864 worldline representation to deal *directly* with the  $\tau$  inte-  
865 grals. Since the major advantages of the worldline approach  
866 include that (a) one does not have to split amplitudes into  
867 sectors according to permutations of external legs, and  
868 (b) internal momentum integrals are recast in terms of the  
869 proper-time integral, we expect this to provide some  
870

872 advantage over the standard formalism, at least in various  
873 physical limits of interest. This will be discussed elsewhere.

### 874 **B. $N = 1$ , nonlinear Compton scattering in spinor QED**

875 Let us now confirm the  $N = 1$  case for spinor QED,  
876 which requires expanding the master formula (84) to linear  
877 order in  $\varepsilon_1$ . Since the field dependence of the exponent in  
878 for spinor QED contains that of scalar QED one may write  
879 the resulting amplitude using the scalar Volkov wave  
880 functions, (91), as

$$\begin{aligned} \mathcal{M}_{1s's}^{p'p} &= -ie \frac{1}{2m} \int d^4x e^{ik_1 \cdot x} \varphi_{p'}^{\text{out}}(x) \varphi_p^{\text{in}}(x) \bar{u}_{s'}(p') \\ &\quad \times [(\tilde{p}' + p - 2a(x^+)) \cdot \varepsilon_1 \text{Spin}(\emptyset) \\ &\quad + \text{Spin}(\tilde{f}_1)] u_s(p), \end{aligned} \quad (92)$$

882 requiring only the evaluation of the spin factor (we have  
883 again used the Fourier representation of the  $\delta$ -functions).  
884 Before embarking upon the comparison to the standard  
885 formalism, we should emphasize that the approach outlined  
886 here, namely writing in terms of spacetime averages with  
887 steps to follow, is necessary to make the connection to the  
888 perturbative Furry picture with Volkov wave functions.  
900

889 However, this would be inefficient for practical worldline  
890 calculations.

891 The spin factors are determined using (48) and (49)  
892 under the LSZ reduction (86) and the inverse symbol  
893 map, (33). Because of the nilpotency of  $f$  one has,  
894 *under the inverse symbol map*,  $\exp(-\int_{-\infty}^{\infty} d\tau \eta \cdot f \cdot \eta) =$   
895  $1 - \int_{-\infty}^{\infty} d\tau \eta \cdot f \cdot \eta$ , and therefore the factor without photon  
896 insertion is readily determined to be

$$\text{Spin}(\emptyset) = \left[ 1 - \frac{1}{2p'^+} n \delta a(x^+) \right] \left[ 1 + \frac{1}{2p^+} n a(x^+) \right], \quad (93)$$

898 where we have already transformed the parameter integral  
899 to a spacetime average and computed its value. This is  
900 simply the Dirac-matrix structure necessary to construct the  
901 spinor Volkov wave functions.

902 Let us next treat the single photon spin factor,  $\text{Spin}(\tilde{f}_1)$ .  
903 Beginning with the Grassmann integral with one photon  
904 insertion, provided in (49) we apply the inverse symbolic  
905 map in (33) and realize the LSZ reduction according  
906 to (86). The various worldline averages are then trans-  
907 formed into their corresponding spacetime averages as was  
908 done in the  $N = 1$  scalar case, to find

$$\begin{aligned} \text{Spin}(\tilde{f}_1) &= -\frac{1}{2} [k_1, \varepsilon_1] + k_1^+ \varepsilon_1 \cdot \left( -\frac{\delta a(x^+)}{2p'^+} + \frac{a(x^+)}{2p^+} \right) + \varepsilon_1 \cdot \left( \frac{\delta a(x^+)}{2p'^+} + \frac{a(x^+)}{2p^+} \right) \frac{1}{2} [k_1, n] \\ &\quad + k_1^+ \frac{1}{2} \left[ \varepsilon_1, \frac{\delta a(x^+)}{2p'^+} + \frac{a(x^+)}{2p^+} \right] + \left[ k_1 \cdot \left( \frac{\delta a(x^+)}{2p'^+} + \frac{a(x^+)}{2p^+} \right) + 2k_1^+ \left( \frac{\delta a(x^+)}{2p'^+} \cdot \frac{a(x^+)}{2p^+} \right) \right] n \varepsilon_1 \\ &\quad + \frac{2k_1^+}{2p'^+ 2p^+} \varepsilon_1 \cdot [a(x^+) \delta a(x^+) + \delta a(x^+) a(x^+)] n + (k_1 + a^\infty)_\mu \varepsilon_{1\nu} n_\alpha \left( \frac{\delta a(x^+)}{2p'^+} - \frac{a(x^+)}{2p^+} \right)_\beta i \gamma_5 \varepsilon^{\mu\nu\alpha\beta}. \end{aligned} \quad (94)$$

912 Next, we express the photon momentum,  $k_1$ , in terms of the electron momenta and asymptotic value of the background  
913 field. For the  $+$ ,  $\perp$  components we can use momentum conservation,  $k_1^{+,\perp} = (p - \tilde{p}')^{+,\perp}$ . The  $k_1^-$  component requires us to  
914 carry out an integration by parts with respect to  $x^+$ . We illustrate this step, to be applied to the various  $k_1$  terms in (94), with  
915 the following manipulation:  
916

$$\int d^4x e^{ik_1 \cdot x} k_1^\mu \varphi_{p'}^{\text{out}}(x) \varphi_p^{\text{in}}(x) = \int d^4x e^{ik_1 \cdot x} \left[ \left( \frac{2p \cdot a(x^+) - a(x^+)^2}{2p^+} - \frac{2p' \cdot \delta a(x^+) - \delta a(x^+)^2}{2p'^+} \right) n^\mu + p^\mu - \tilde{p}'^\mu \right] \varphi_{p'}^{\text{out}}(x) \varphi_p^{\text{in}}(x), \quad (95)$$

917 In fact, if additional factors of  $a(x^+)$  appear under the  
918 above integral, it turns out that the additional derivatives  
919 produced by integrating by parts *always* contract away.  
920 Therefore (95) can be used throughout (94). Moreover,  
921 applying the above procedure to  $k_1$  in the  $\gamma_5$  term of (94),  
922 one can see that in effect  $k_1^\mu \rightarrow p^\mu - \tilde{p}'^\mu$ , since the two  $n^\mu$   
923 contract to zero against the Levi-Civita tensor. In fact the  
924 only term in which the  $n^\mu$  part of (95) survives after these  
925 replacements is the first term on the RHS of (94).

926 Last, since we are taking the on-shell limit we may  
927 use the Dirac equation for the sandwiching spinors so as  
928 to send their corresponding  $\not{p}$  and  $\not{p}'$  to  $m$ , anticomm-  
929 tating where necessary. Again, illustrating this step with  
930 the  $\gamma_5$  term in (94) we rewrite  $\gamma_5$  in terms of products of  
931 four matrices using (33). After acting on the spinor  
932 solutions at most three matrices will remain. After this  
933 process, the  $\gamma_5$  term, as it appears in the amplitude (92),  
934 becomes



$$\begin{aligned}
(k_1 + a^\infty)_\mu \varepsilon_{1\nu} n_\alpha \left( \frac{\delta a(x^+)}{2p'^+} - \frac{a(x^+)}{2p^+} \right)_\beta i\gamma_5 \varepsilon^{\mu\nu\alpha\beta} &= (p^+ + p'^+) \frac{1}{2} \left[ \frac{\delta a(x^+)}{2p'^+} - \frac{a(x^+)}{2p^+}, \varepsilon_1 \right] + (p + p') \cdot \varepsilon_1 n \left( \frac{\delta a(x^+)}{2p'^+} - \frac{a(x^+)}{2p^+} \right) \\
&+ (p + p') \cdot \left( \frac{\delta a(x^+)}{2p'^+} - \frac{a(x^+)}{2p^+} \right) \varepsilon_1 n - m \left\{ \varepsilon_1, n \left( \frac{\delta a(x^+)}{2p'^+} - \frac{a(x^+)}{2p^+} \right) \right\}.
\end{aligned} \tag{96}$$

936 Using the above steps to replace  $k_1^\mu$  in the remaining terms of (94), after some algebra one may gather terms to  
937 find that

$$\bar{u}_{s'}(p') \{ (\tilde{p}' + p - 2a(x^+)) \cdot \varepsilon_1 \text{Spin}(\emptyset) + \text{Spin}(\tilde{f}_1) \} u_s(p) = 2m \bar{u}_{s'}(p') \left\{ \varepsilon_1 - \frac{1}{2p'^+} n \delta a(x^+) \varepsilon_1 + \frac{1}{2p^+} \varepsilon_1 n a(x^+) \right\} u_s(p), \tag{97}$$

939 and hence

$$\mathcal{M}_{1s's}^{p'p} = -ie \int d^4x e^{ik \cdot x} \Psi_{p',s'}^{\text{out}}(x) \varepsilon_1 \Psi_{p,s}^{\text{in}}(x), \tag{98}$$

940 where we have used the spinor Volkov wave functions,  
942 which read

$$\Psi_{p,s}^{\text{in}}(x) = \left[ 1 + \frac{1}{2p^+} n a(x^+) \right] u_s(p) \varphi_p^{\text{in}}(x), \tag{99}$$

943

$$\Psi_{p',s'}^{\text{out}}(x) = \bar{u}_{s'}(p') \left[ 1 - \frac{1}{2p'^+} n \delta a(x^+) \right] \varphi_{p'}^{\text{out}}(x). \tag{100}$$

946 This successfully verifies that the worldline approach  
947 reproduces the known amplitude for the  $N = 1$  process.

### 948 C. $N = 2$ , double nonlinear Compton scattering 949 in scalar QED

950 To complete our discussion of the relevant structures in  
951 scalar QED we must also consider the case  $N = 2$ , where  
952 the so-called seagull vertex (the four-point scalar-photon-  
953 photon-scalar vertex) first appears. We will describe the  
954 way this works briefly here, as the calculations proceed  
955 largely as for  $N = 1$ , leaving the details for the Appendix.  
956 Expanding (73), there are now two  $\tau$  integrals, with one,  
957 say  $\tau_2$ , fixed by the worldline delta function in (73), and the  
958 other, say  $\tau_1$ , remaining. The mapping onto Feynman  
959 diagrams is most natural: the contributions from  $\tau_1 > 0$   
960 and  $\tau_1 < 0$  recover one each of the expected contributions  
961 from the two diagrams with two three-point vertices, with  
962  $\tau_1$  being mapped to the light front time of one vertex. The  
963 seagull contribution is picked up from the term in (73)  
964 which goes like  $\varepsilon_1 \cdot \varepsilon_2$ ; this comes with a delta function  
965 with support at exactly  $\tau_1 = 0$ , hence leaving only a single  
966 unevaluated integral, as expected. The full calculation is  
967 presented in the Appendix.

## V. CONCLUSIONS

968

969 We have presented worldline master formulas for all-  
970 multiplicity tree level scattering amplitudes of two massive  
971 charged particles and  $N$  photons, in a plane wave back-  
972 ground, in both scalar and spinor QED. The background  
973 field may have arbitrary strength and functional profile,  
974 and is treated without approximation throughout. This is  
975 particularly relevant as the target application of our results  
976 is to laser-matter interactions in the *high intensity* regime  
977 where the field is characterized by a dimensionless strength  
978 (the coupling to matter) larger than unity, and hence must  
979 be treated without recourse to perturbation theory.

980 Our master formulas have been derived using the world-  
981 line approach to quantum field theory. While several  
982 previous publications have derived worldline master for-  
983 mulas for various correlation functions in vacuum, or even  
984 at higher loop level in background fields, our focus here has  
985 been on scattering amplitudes involving external matter. As  
986 such it was necessary to identify the *worldline description*  
987 of LSZ reduction in a plane wave background. We found  
988 this to be a fairly direct generalization of the known  
989 worldline prescription for LSZ amplitudes in vacuum  
990 [68,69]. A second notable generalization from known  
991 results in vacuum holds for the spinor case: namely that  
992 in the second-order formalism, which implies a split into  
993 “leading” and “subleading” terms, only the former survives  
994 the on-shell limit once the LSZ prescription is imposed.  
995 Furthermore, the background-field-dependent part of this  
996 leading term *also* drops out in the asymptotic limit. This  
997 allows for a large number of terms to be discarded (and in  
998 the vacuum case allowed for the gauge invariance of the  
999 amplitudes to be manifest).

1000 We have checked our results against the existing liter-  
1001 ature, which contains only *low*-multiplicity amplitudes  
1002 derived using Feynman rules. Explicitly, these are the  
1003 cases  $N = 1$  and  $N = 2$ , or single and double nonlinear  
1004 Compton scattering. Moving beyond scattering amplitudes,  
1005 we have also seen how to recover *off-shell* quantities,  
1006 in particular the scalar and spinor correlation functions

1007 dressed by the background and the Volkov wave functions,  
 1008 from worldline path integrals. The latter is a particularly  
 1009 interesting case as it exposes the relevance of mixed  
 1010 boundary conditions; the relevant path integrals carry  
 1011 Dirichlet conditions at one limit, representing the local  
 1012 spacetime argument of the wave function, and Robin  
 1013 boundary conditions at the other limit, encoding the  
 1014 asymptotic momentum characterizing the Volkov solution.

1015 It is fair to say that the master formulas for amplitudes we  
 1016 have derived here still require, for a chosen number of  
 1017 photons  $N$ , some processing in order to extract all their  
 1018 physical content. In future work we will pursue methods of  
 1019 evaluating the remaining proper-time integrals in an effi-  
 1020 cient manner, or in an approximate manner relevant to  
 1021 interesting physical regimes. Here, benefit should be gained  
 1022 by *not* breaking the parameter integrals into ordered sectors  
 1023 corresponding to photon permutations, which will max-  
 1024 imally exploit the calculational efficiency. Constructing  
 1025 observables from our amplitudes at  $N > 2$  (which are  
 1026 lacking in the literature) will help to benchmark numerical  
 1027 codes which approximate multiphoton processes using  
 1028 sequential single photon emissions. It would be revealing  
 1029 to compare our expressions with the compact all-multi-  
 1030 plicity results of [74,75]. We also plan to generalize our  
 1031 results to higher-loop orders, in order to pursue the Ritus-  
 1032 Narozhny conjecture on the behavior of loop corrections at  
 1033 very high intensity, see [8,14] for reviews.

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### 1039 APPENDIX: MASTER FORMULA CHECK 1040 FOR $N=2$

1041 In this appendix we confirm that the master formula (73)  
 1042 correctly reproduces, at  $N = 2$ , the amplitude for “double  
 1043 nonlinear Compton scattering” [76,77] in scalar QED, that  
 1044 is the emission of two photons from a particle in a plane  
 1045 wave background. (By crossing symmetry this is directly  
 1046 related to the amplitude for the Compton effect in the  
 1047 background.) Recall that in scalar QED, the standard  
 1048 approach would require evaluation of three separate  
 1049 Feynman diagrams—conveniently combined into one cal-  
 1050 culation on the worldline—one of which contains the four-  
 1051 point seagull vertex.

1052 Starting from (73) with  $N = 2$ , the LSZ factor  $\delta(\tau_1/2 +$   
 1053  $\tau_2/2)$  means that we have only one nontrivial proper-time

1054 integral, over, say,  $\tau_1$ . It is convenient to split this integral 1054  
 1055 into three pieces and analyze each separately; we split the 1055  
 1056 integration range into  $-\infty < \tau_1 < 0^-$ ,  $0^- < \tau_1 < 0^+$  and 1056  
 1057  $0^+ < \tau_1 < \infty$ , and refer henceforth to the corresponding 1057  
 1058 contribution to the amplitudes as  $\mathcal{A}_{2^-}^{p'p}$ ,  $\mathcal{A}_{2\delta}^{p'p}$  and  $\mathcal{A}_{2^+}^{p'p}$ , 1058  
 1059 respectively. 1059

#### 1060 1. $\tau_1 \in (0, \infty)$

1061 When  $\tau_1 > 0$ , the field-independent terms in the expo- 1061  
 1062 nential of (73) reduce to 1062

$$1063 \begin{aligned} & i(\tilde{p}' + p) \cdot (k_1 - k_2)\tau_1 + \varepsilon_1 \cdot (\tilde{p}' + p - k_2) \\ & + \varepsilon_2 \cdot (\tilde{p}' + p + k_1) - 2i\tau_1 k_1 \cdot k_2 \\ & + i(K_+ + p'_+ - p_+)x^+. \end{aligned} \quad (A1)$$

1064 The gauge field at the interaction points  $\pm\tau_1$  (indicating the 1064  
 1065 insertion point of photon with momentum  $k_1$ ) takes the 1065  
 1066 values 1066

$$1067 a(\tau_1) = a(x^+ + \tau_1(2p'^+ + k_1^+ - k_2^+)), \quad (A2)$$

$$1068 a(-\tau_1) = a(x^+ - \tau_1(2p^+ + k_1^+ - k_2^+)). \quad (A3)$$

1069 This motivates us to make the change of variable 1069  
 1070  $x^+ \rightarrow x^+ - \tau_1(2p^+ + k_1^+ - k_2^+)$ , such that the field- 1070  
 1071 independent terms (A1) transform to 1071  
 1072

$$1073 \begin{aligned} \mathcal{T}_0 \equiv & i(4(p_+ + k_{1+})q^+ - 2q_1^2 - 2m^2 + i0^+)\tau_1 \\ & + \varepsilon_1 \cdot (2\tilde{p}' + k_1) + \varepsilon_2 \cdot (\tilde{p}' + p + k_1) \\ & + i(K_+ + p'_+ - p_+)x^+ - i(2p' + a^\infty)a^\infty\tau_1, \end{aligned} \quad (A4)$$

1074 where we have defined  $q = p - k_2$  and used the fact the 1074  
 1075 momenta are on-shell to simplify. We shall shortly need the 1075  
 1076 last term  $-i(2p' + a^\infty)a^\infty\tau_1$  to simplify some of the field- 1076  
 1077 dependent terms. Before going into that, we return to the 1077  
 1078 exponent of (73) and note that the following field-depen- 1078  
 1079 dent term is already sufficiently simplified: 1079

$$1080 \begin{aligned} \mathcal{T}_1 \equiv & -2 \sum_{i=1}^N \varepsilon_i \cdot a(\tau_i) \rightarrow -2\varepsilon_1 \cdot a(x^+) \\ & - 2\varepsilon_2 \cdot a(x^+ + 4q^+\tau_1). \end{aligned} \quad (A5)$$

1081 The rest of the field-dependent terms combine with 1080  
 1082  $-i(2p' + a^\infty)a^\infty\tau_1$  from (A4) to yield 1082

1083

$$\begin{aligned}
\mathcal{T}_2 - i(2p' + a^\infty)a^\infty\tau_1 &\equiv -2i \sum_{i=1}^N \int_{-\infty}^{\tau_i} d\tau k_i \cdot a(\tau) - i \int_{-\infty}^0 d\tau [2\tilde{p}' \cdot a(\tau) - a^2(\tau)] \\
&\quad - i \int_0^\infty d\tau [2p' \cdot \delta a(\tau) - \delta a^2(\tau)] - i(2p' + a^\infty) \cdot a^\infty\tau_1 \\
&= -2i \sum_{i=1}^N \int_{-\infty}^{\tau_i} d\tau k_i \cdot a(\tau) - i \int_{-\infty}^{\tau_1} d\tau [2\tilde{p}' \cdot a(\tau) - a^2(\tau)] - i \int_{\tau_1}^\infty d\tau [2p' \cdot \delta a(\tau) - \delta a^2(\tau)]. \quad (\text{A6})
\end{aligned}$$

1084 We now use the dependence of  $a_\mu(x_{cl}(\tau))$  on the classical solution to transform the proper-time integrals into spacetime  
1086 integrals and simplify the above terms as

$$-2i \sum_{i=1}^N \int_{-\infty}^{\tau_i} d\tau k_i \cdot a(\tau) - i \int_{-\infty}^{\tau_1} d\tau [2\tilde{p}' \cdot a(\tau) - a^2(\tau)] - i \int_{\tau_1}^\infty d\tau [2p' \cdot \delta a(\tau) - \delta a^2(\tau)] \quad (\text{A7})$$

1088

$$= -i \int_{-\infty}^{x^+} \frac{2p \cdot a(s) - a^2(s)}{2p^+} ds - i \int_{x^+}^{x^+ + 4q^+ \tau_1} ds \frac{2q \cdot a(s) - a^2(s)}{2q^+} - i \int_{x^+ + 4q^+ \tau_1}^\infty ds \frac{2p' \cdot \delta a(s) - \delta a^2(s)}{2p'^+}, \quad (\text{A8})$$

1089 where we have used momentum conservation to replace  $\tilde{p}_\perp + K_\perp$  with  $p_\perp$ , and  $\tilde{p}_\perp + k_{1\perp}$  with  $q_\perp$ . The contribution  $\mathcal{A}_{2+}^{p'p}$   
1091 to the amplitude from  $\tau_1 > 0$  can then be written as

$$\mathcal{A}_{2+}^{p'p} = 2(-ie)^2 (2\pi)^3 \delta_{\perp,-}(\tilde{p}' + K - p) \int_{-\infty}^\infty dx^+ \int_0^\infty d\tau_1 e^{\mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2} \Big|_{\text{lin.}\varepsilon}. \quad (\text{A9})$$

1093 We are now going to show that the right-hand side of the above expression is equivalent to one of the three Feynman  
1094 diagram contributions to double nonlinear Compton, namely that containing two three-point vertices in which photon  $k_1$  is  
1095 emitted on the outgoing leg. The Feynman rules give this contribution as

$$(-ie)^2 \int d^4x' d^4x e^{ik_1 \cdot x'} [\varphi_{p'}^{\text{out}}(x') (\varepsilon_1 \cdot \overleftrightarrow{D}_{x'}) G(x', x) (\varepsilon_2 \cdot \overleftrightarrow{D}_x) \varphi_p^{\text{in}}(x)] e^{ik_2 \cdot x}, \quad (\text{A10})$$

1096 where  $D$  denotes the background-covariant derivative and  $G(x', x) = \mathcal{D}_0^{x'x}$  is the scalar particle propagator in the plane wave  
1098 background (the double arrow indicates the right-left alternating derivative). We then observe that this is equivalent to

$$\int d^4x' d^4x \varphi_{p'}^{\text{out}}(x' - i\varepsilon_1) e^{ik_1 \cdot x' - 2\varepsilon_1 \cdot a(x')} G(x' + i\varepsilon_1, x - i\varepsilon_2) e^{ik_2 \cdot x - 2\varepsilon_2 \cdot a(x)} \varphi_p^{\text{in}}(x + i\varepsilon_2) \Big|_{\text{lin.}\varepsilon_1 \dots \varepsilon_N}. \quad (\text{A11})$$

1099 Taking this expression, we start by using the Fourier representation of  $G(x', x)$  to rewrite it as

$$\begin{aligned}
&\int d^4x' d^4x \varphi_{p'}^{\text{out}}(x' - i\varepsilon_1) e^{ik_1 \cdot x' - 2\varepsilon_1 \cdot a(x')} G(x' + i\varepsilon_1, x - i\varepsilon_2) e^{ik_2 \cdot x - 2\varepsilon_2 \cdot a(x)} \varphi_p^{\text{in}}(x + i\varepsilon_2) \\
&= \int \frac{d^4r}{(2\pi)^4} d^4x' d^4x \varphi_{p'}^{\text{out}}(x' - i\varepsilon_1) e^{ik_1 \cdot x' - 2\varepsilon_1 \cdot a(x')} \frac{i e^{-ir \cdot (x' - x + i\varepsilon_1 + i\varepsilon_2) - i \int_{x^+}^{x'^+} \frac{2r \cdot a(s) - a^2(s)}{4r_-} ds}}{r^2 - m^2 + i0^+} e^{ik_2 \cdot x - 2\varepsilon_2 \cdot a(x)} \varphi_p^{\text{in}}(x + i\varepsilon_2). \quad (\text{A12})
\end{aligned}$$

1102 We can easily evaluate the  $x'^{\perp}$ ,  $x^{\perp}$ , and  $r^{\perp}$  integrals and rewrite the propagator denominator using a standard  
1103 Schwinger proper-time integral to obtain

$$\begin{aligned}
&(2\pi)^3 \delta_{\perp,-}(\tilde{p}' + K - p) e^{p \cdot \varepsilon_1 + q \cdot \varepsilon_2} \int_{-\infty}^\infty dx'^+ e^{i(p_+ + k_{1+} - r_+)x'^+ - 2\varepsilon_1 \cdot a(x'^+)} e^{-i \int_{x'^+}^\infty \frac{2p' \cdot \delta a(s) - \delta a^2(s)}{2p'^+} ds} \\
&\quad \times 2 \int_{-\infty}^\infty dx^+ e^{-2\varepsilon_2 \cdot a(x^+)} e^{-ix^+ q_+} \int_0^\infty d\tau_1 \int \frac{d\mathbf{r}_+}{2\pi} e^{ir_+ \cdot (x^+ - x'^+ + 4q^+ \tau_1)} e^{-2i\tau_1 [q_\perp^2 + m^2 - i0^+]} e^{-i \int_{x^+}^{x'^+} ds \frac{2q \cdot a(s) - a^2(s)}{2q^+} - i \int_{-\infty}^{x^+} ds \frac{2p \cdot a(s) - a^2(s)}{2p^+}}. \quad (\text{A13})
\end{aligned}$$

1105 The  $r_+$  integral can now be evaluated to give  $2\pi\delta(x^+ -$   
 1106  $x'^+ + 8q_-\tau_1)$ . The remaining  $x'^+$  integral is therefore  
 1107 trivialized and effects the replacement  $x'^+ \rightarrow x^+ + 8q_-\tau_1$ .  
 1108 Taking the multilinear limit, one recovers precisely the  
 1109 right-hand side of (A9) as promised.  
 1116

2.  $\tau_1 \in (-\infty, \mathbf{0}^-)$

For  $\tau_1 < 0$ , one recovers the Feynman diagram contri-  
 1111 bution in which photon  $k_2$  is emitted from the outgoing leg.  
 1112 The proof of this follows exactly the same steps as for  $\mathcal{A}_{2+}^{p'p}$   
 1113 above. Hence we simply state that  
 1114

$$\mathcal{A}_{2-}^{p'p} = (-ie)^2 \int d^4x' d^4x e^{ik_2 \cdot x} [\varphi_{p'}^{\text{out}}(x') (\epsilon_2 \cdot \overleftrightarrow{D}_{x'}) G(x', x) (\epsilon_1 \cdot \overleftrightarrow{D}_x) \varphi_p^{\text{in}}(x)] e^{ik_1 \cdot x}. \quad (\text{A14})$$

3.  $\tau_1 \in (\mathbf{0}^-, \mathbf{0}^+)$

1118 In this range, the field-independent term in the exponent of (73) going like  $\delta(\tau_1)\epsilon_1 \cdot \epsilon_2$  cannot be neglected. Noting that  
 1121 this term is already linear in both  $\epsilon_1$  and  $\epsilon_2$ , the corresponding contribution to the amplitude is immediately seen to be  
 1122 proportional to the  $\tau_1 \rightarrow 0$  and  $\epsilon_{1,2} \rightarrow 0$  limit of the integrand of the proper-time integral:

$$\begin{aligned} \mathcal{A}_{2\delta}^{p'p} &= -2(-ie)^2 (2\pi)^3 \delta_{\perp,-}(\tilde{p}' + K - p) \\ &\times \int_{-\infty}^{\infty} dx^+ (i\epsilon_1 \cdot \epsilon_2) e^{+i(K+p'-p)_+ x^+ - i \int_{-\infty}^0 [2\tilde{p}' \cdot a(\tau) - a^2(\tau)] d\tau - i \int_0^{\infty} [2p' \cdot \delta a(\tau) - \delta a^2(\tau)] d\tau - 2i \int_{-\infty}^0 K \cdot a(\tau) d\tau}. \end{aligned} \quad (\text{A15})$$

1123 By inspection, this is equivalent to

$$\mathcal{A}_{2\delta}^{p'p} = -2i(-ie)^2 \epsilon_1 \cdot \epsilon_2 \int d^4x e^{i(k_1+k_2) \cdot x} \varphi_{p'}^{\text{out}}(x) \varphi_p^{\text{in}}(x), \quad (\text{A16})$$

1126 which is indeed the seagull vertex contribution to double nonlinear Compton scattering. Summing (A9), (A14), and (A16)  
 1127 recovers the full amplitude.

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