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Color dependence of the topological susceptibility in Yang-Mills theories

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For Yang-Mills theories in four dimensions, we propose to rescale the ratio between topological susceptibility and string tension squared in a universal way, dependent only on group factors. We apply this suggestion to $SU(N_c)$ and $Sp(N_c)$ groups, and compare lattice measurements performed by several independent collaborations. We show that the two sequences of (rescaled) numerical results in these two families of groups are compatible with each other. We hence perform a combined fit, and extrapolate to the common large-N_c limit.

I. INTRODUCTION

Lattice studies provide numerical evidence that, at zero temperature, four-dimensional Yang-Mills theories with compact non-Abelian gauge group G confine. This statement can be made precise, for instance by formulating it in terms of the expectation values of either the Polyakov loop or the Wilson loop, and then extracting the string tension σ from suitable correlation functions. It is of general interest to identify other observables that characterise the long-distance behaviour of Yang-Mills theories, for all choices of group G. By doing so, one can relate lattice results to alternative approaches based on the large-N_c expansion. A resurgence of interest in the latter, motivated by gauge-gravity dualities [1–4], led to much effort being focused on the glueballs, as the results of lattice calculations of their spectra [5–17] can be compared to those of gravity calculations [18–27]—or other semi-analytical calculations [28–30].

The topological susceptibility, χ, is a non-perturbative quantity that plays a central role in our understanding of strong nuclear forces—see for instance the review in Ref. [31]. It enters the Witten-Veneziano formula [32, 33] for the mass of the η' particle, and the solution of the $U(1)_A$ problem. Being related to the θ-dependence of the free energy, χ also enters the electric dipole moment of hadrons, the strong-CP problem, and its putative solutions (the axion). Being topological in nature, χ is intrinsically difficult to compute on the lattice; yet, modern lattice techniques are mature enough that increasingly precise and reliable measurements have been published in the past two decades for $SU(N_c)$ Yang-Mills theories [5, 15, 34–36]—see also Refs. [37–49]. Our collaboration has just completed the calculation of χ in the $Sp(N_c)$ Yang-Mills theories [50]. In this paper we propose a way to compare χ in different sequences of gauge groups, and perform a combined large-N_c extrapolation.

II. YANG-MILLS THEORIES

The Yang-Mills theory with gauge group G, in four-dimensional Minkowski space, has the classical action:

$$\mathcal{S}_{YM} = -\frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}\,, \qquad (1)$$

with g the coupling, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu\,,\,A_\nu]$ the field-strength tensor, and $A_\mu \equiv \sum_A A_\mu^A T^A$ the gauge field. The matrices T^A, with $A = 1, \cdots, d_G$, are the generators in the fundamental representation, normalised by the relation $\mathrm{Tr}\,T^AT^B = \frac{1}{2}\delta^{AB}$.

Yang-Mills theories are asymptotically free at short distance, hence can be interpreted as conformal theories admitting a marginally relevant deformation: the gauge coupling. Long distance physics is not accessible to perturbative calculations; its numerical treatment is implemented by discretising the Euclidean spacetime on

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a lattice. The discretised action and range of its parameters are chosen so that Monte Carlo numerical studies are performed within the basin of attraction of a fixed point belonging to the universality class of the aforementioned conformal theory. By doing so, it is possible to suppress non-universal features of the lattice formulation and study the universal properties of the gauge dynamics characterising the continuum, four-dimensional physical system of interest. Observable quantities are measured as ensemble averages of appropriately chosen operators, and extrapolated towards the continuum limit, where the lattice spacing a vanishes, by changing the lattice parameters so as to approach the fixed point in a controlled way.

We do not report the details of the lattice theories of interest here, except for highlighting the fact that in comparing measurements with different ensembles, and extrapolating towards the continuum limit, one measures the dimensional observables of interest in units of a physical scale, hence introducing a scale setting procedure. We compare measurements in different theories, performed by different collaborations, with different lattice algorithms, but all of them adopting the same scale-setting procedure, based upon the string tension σ.

A. String Tension

On the lattice, to extract the string tension σ one measures the correlation functions between non-contractible path-ordered loops, separated by Euclidean distance L. The resulting fluxtubes are described by effective string theory when $L/a \gg 1$, and the mass $am(L)$ (in lattice units) of the lightest (torelon) state is

$$am(L) \;=\; (\sigma a^2)\frac{L}{a}\left(1+\sum_{k=1}^{+\infty}\frac{d_k}{(\sigma L^2)^k}\right)\,. \tag{2}$$

The effective string theory [51] is characterised by the values of d_k, dimensionless coefficients that capture the dynamics at large distances; $d_1 = -\pi/3$ is the universal Lüscher term [52]. One estimates σa^2 by repeating lattice measurements for different L/a, and curve-fitting the results. For further details on the measurements of σa^2, we refer the reader to Ref. [12], for example.

Lattice measurements are affected by both statistical and systematic uncertainties that are difficult to reduce below the few percent level. Furthermore, one intrinsic limiting factor in the adoption of σ as a universal scale setting procedure in non-Abelian gauge theories is that σ is not well defined for asymptotically large L, if string-breaking effects are present, as is the case with dynamical matter fields. Yet many lattice collaborations report their results in terms of σ, because of the simplicity of its extraction and its intuitive meaning. We adopt this strategy for the purposes of this paper, and in this work we do not attempt to compare with results that use a different scale setting method, such as the gradient flow, as done, e.g., in Ref. [45].

B. Topological Susceptibility

The topological charge Q of a gauge configuration is

$$Q \equiv \int \mathrm{d}^4x\; q(x)\,, \tag{3}$$

where

$$q(x) \equiv \frac{1}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\; F_{\mu\nu}(x)F_{\rho\sigma}(x)\,, \tag{4}$$

with $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol. The topological susceptibility is defined as

$$\chi \equiv \int \mathrm{d}^4x\;\langle q(x)q(0)\rangle. \tag{5}$$

The inclusion of a θ term yields the action $\tilde{\mathcal{S}}$, which extends Eq. (1):

$$\tilde{\mathcal{S}} \;=\; -\frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} - \frac{\theta}{32\pi^2}\int \mathrm{d}^4x\,\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma}\,. \tag{6}$$

The vacuum (free) energy (density) $F(\theta)$ is defined by the path integral

$$e^{-V_4F(\theta)} \;\equiv\; \int \mathcal{D}A_\mu e^{-\tilde{\mathcal{S}}_E}\,, \tag{7}$$

where V_4 is the four-dimensional volume, and $\tilde{\mathcal{S}}_E$ the Euclidean version of Eq. (6). The topological susceptibility is then computed as

$$\chi \;=\; \left.\frac{\partial^2F(\theta)}{\partial\theta^2}\right|_{\theta=0}\,. \tag{8}$$

In the continuum theory, the charge $Q\in\mathbb{Z}$ is quantised. Lattice artefacts spoil the discreteness of the topological charge and prevent Q from taking integer values on configurations generated in numerical simulations. The assignment of integer topological charge on the lattice is affected by an ambiguity, though this is expected to be irrelevant in the continuum limit.

Other factors that affect the accuracy of the results stem from the practical limitations of Monte Carlo updating algorithms and of the finite range of lattice spacings that can be simulated. Among them, we mention the existence of (auto)correlation between configurations, (partial) topological freezing, and numerical noise due to short-distance fluctuations, as well as the appearance of other uncertainties in the continuum limit extrapolation. We refer to the original literature for details [5, 15, 34–36, 43, 44, 50], and for a survey of the advanced strategies that the lattice collaborations implement in order to minimise the statistical error and the systematic effects in the measurement of χ. Under the reasonable assumption that the identified errors have been evaluated correctly, a direct comparison of the results from the measurements of the different groups is a way to assess the size of any potentially remaining systematic effects.

III. TOWARDS LARGE N_c

Since the θ term is topological, it does not affect the local dynamics of the gauge fields, such as the running coupling. It is therefore widely believed that at low energy Yang-Mills theories confine even in the presence of a non-vanishing θ, at least as long as θ is small. The θ-dependent vacuum is gapped, and all the excitations (glueballs) are color-singlets. In order for CP to be a well defined symmetry, we also expect the vacuum energy to be an even function of θ, minimised at $\theta = 0$, by consequence of the Schwarz inequality applied to the Euclidean partition function [53, 54]:

$$F(0) \leq F(\theta) = F(-\theta)\,. \tag{9}$$

By defining the 't Hooft coupling $\lambda \equiv g^2 N_c$, because the trace of any $N_c \times N_c$ matrix is proportional to N_c, while the couplings are proportional to λ/N_c, Yang-Mills theories can be analysed in a $1/N_c$ expansion in which one holds λ fixed. For consistency at the quantum level, the θ term must be scaled holding θ/N_c fixed as well, and physical observables are multi-valued functions of θ with periodicity 2π [55]. For example, the vacuum energy is expected to take the form

$$F(\theta) = f_G \min_k h\left(\frac{\theta + 2\pi k}{N_c}\right)\,, \tag{10}$$

with $k = 0, \cdots, N_c - 1$, and the pre-factor $f_G = \mathcal{O}\left(N_c^2\right)$ for large N_c. h is smoothly dependent on θ/N_c for small θ, and is determined by G in a way that admits a finite limit as $N_c \to \infty$. For $\theta = 0$, the minimum is expected for $k = 0$ [55], and the large-N_c limit of the topological susceptibility is finite:

$$\lim_{N_c \to \infty} \chi = \chi_\infty\,, \tag{11}$$

with $\chi_\infty = h''(0)$. As each gauge field contributes equally, one expects that

$$f_G \propto d_G\,, \tag{12}$$

where d_G is the dimension of the group; $d_G = N_c^2 - 1$ for $SU(N_c)$ and $d_G = (N_c+1)N_c/2$ for $Sp(N_c)$. The proportionality factor must be finite in the large-N_c limit.

The string tension is the energy density per unit length of a fluxtube, the limiting case of a fermion-antifermion pair in the fundamental representation, separated by an asymptotically large distance. We hence expect σ to be proportional to the strength of the coupling between the fermions, which can be measured by the quadratic Casimir of the fundamental representation [28]:

$$\sigma \propto C_2(F) = \begin{cases} \frac{N_c^2-1}{2N_c} & \text{for } SU(N_c) \\ \frac{N_c+1}{4} & \text{for } Sp(N_c) \end{cases}. \tag{13}$$

The proportionality factor is itself a function of N_c, and encodes non-perturbative dynamics in such a way that the string tension has a finite large-N_c limit, σ_∞, as expected because the coupling of fundamental fermions scales as $1/\sqrt{N_c}$, while there are N_c components to them.

The topological susceptibility inherits its group-dependence from the vacuum energy. Hence, we expect the following ratio to capture universal features:

$$\eta_\chi \equiv \frac{\chi C_2(F)^2}{\sigma^2 d_G} = \frac{\chi}{\sigma^2}\cdot \begin{cases} \frac{N_c^2-1}{4N_c^2} & \text{for } SU(N_c) \\ \frac{N_c+1}{8N_c} & \text{for } Sp(N_c) \end{cases}. \tag{14}$$

Furthermore, we expect the ratio η_χ to be finite and universal in the limit $N_c \to \infty$:

$$\lim_{N_c\to\infty} \frac{\chi C_2(F)^2}{\sigma^2 d_G} = b\,\frac{\chi_\infty}{\sigma_\infty^2} = \eta_\chi(\infty) < \infty\,, \tag{15}$$

where $b = 1/4$ for $SU(N_c)$, while $b = 1/8$ for $Sp(N_c)$.

IV. NUMERICAL RESULTS

We summarise in Table I lattice measurements for the quantity χ/σ^2 taken from Refs. [5, 15, 34–36, 43, 44, 50], extrapolated to the continuum limit. The same results are graphically displayed in Fig. 1, where we organise the measurements in terms of (the inverse of) the number of colors N_c in the gauge groups $SU(N_c)$ and $Sp(N_c)$, respectively. In the table, we show also the group factor C_F^2/d_G, which we use in Fig. 2 to rescale the measurements of χ/σ^2, as described in Section III. In this second plot we also change the abscissa to display $1/d_G$; for large N_c, $d_G \propto N_c^2$, and this more physical choice removes conventional ambiguities in comparing across different sequences of groups within Cartan's classification. The data of Tab. I and the analysis code used to prepare Figs. 1 and 2, as well as the numbers quoted later in this Section, are available at Ref. [56, 57].

TABLE I: Summary table of measurements used in this study.

Group	Reference	χ/σ^2	$C_2(F)^2/d_G$
$Sp(2)$	Bennett et al. [50]	0.0519(27)	0.1875
$Sp(4)$	Bennett et al. [50]	0.0424(27)	0.1562
$Sp(6)$	Bennett et al. [50]	0.0396(49)	0.1458
$Sp(8)$	Bennett et al. [50]	0.0424(40)	0.1406
$SU(2)$	Lucini et al. [5]	0.0507(24)	0.1875
$SU(3)$	Lucini et al. [5]	0.0355(32)	0.2222
$SU(4)$	Lucini et al. [5]	0.0224(39)	0.2344
$SU(5)$	Lucini et al. [5]	0.0224(49)	0.2400
$SU(3)$	Del Debbio et al. [34]	0.0282(12)	0.2222
$SU(4)$	Del Debbio et al. [34]	0.0257(10)	0.2344
$SU(6)$	Del Debbio et al. [34]	0.0236(10)	0.2431
$SU(4)$	Bonati et al. [35]	0.02480(80)	0.2344
$SU(6)$	Bonati et al. [35]	0.02300(80)	0.2431
$SU(3)$	Bonanno et al. [36, 43, 44]	0.0289(13)	0.2222
$SU(4)$	Bonanno et al. [36]	0.02499(54)	0.2344
$SU(6)$	Bonanno et al. [36]	0.02214(69)	0.2431
$SU(2)$	Athenodorou et al. [15]	0.05565(64)	0.1875
$SU(3)$	Athenodorou et al. [15]	0.0325(11)	0.2222
$SU(4)$	Athenodorou et al. [15]	0.02469(67)	0.2344
$SU(5)$	Athenodorou et al. [15]	0.0213(13)	0.2400

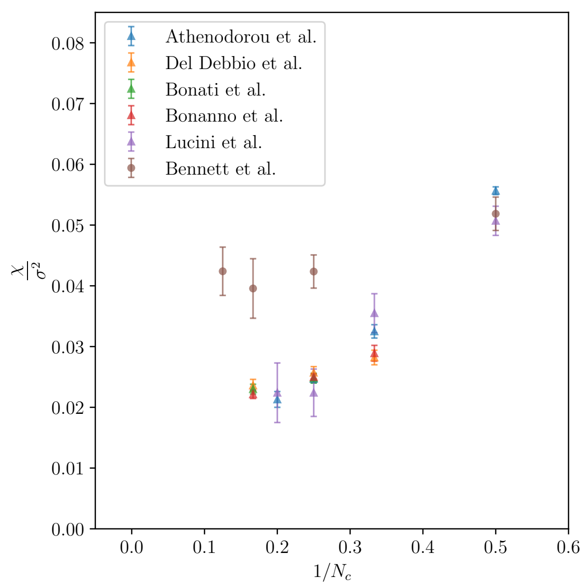

FIG. 1: Topological susceptibility χ, in units of the string tension σ, in the continuum limit, for various groups $SU(N_c)$ and $Sp(N_c)$, and as a function of the parameter $1/N_c$. The measurements reported here are labelled by the collaboration that published them, and are also summarised in Table I.

Before proceeding, we comment on some subtleties about the numerical results we quote, which have been obtained with heterogeneous treatments of systematic effects. The topological charge in pure gauge theories can be computed in different ways [46], from ensembles of gauge configurations generated with Monte Carlo algorithms, all converging towards the same continuum limit.

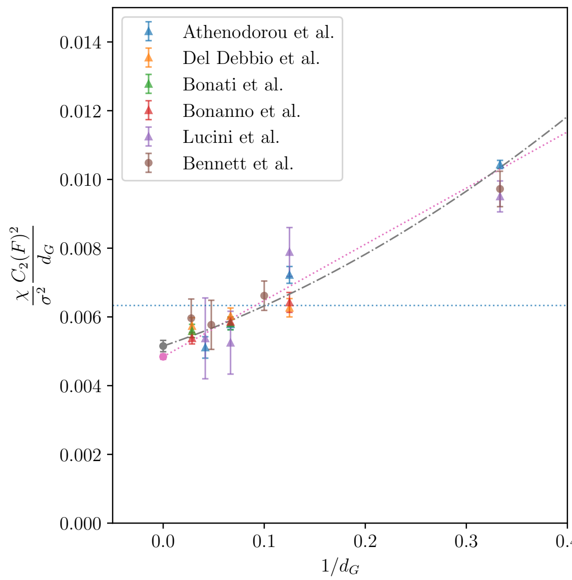

FIG. 2: Ratio of topological susceptibility and string tension squared, rescaled by the group factor $C_2(F)^2/d_G$, as a function of $1/d_G$. We also show the best-fit results of a 2-parameter fit (dotted line) and of a 3-parameter fit including $O(1/d_G^2)$ corrections (dashed line), as explained in the main text. The horizontal dashed line is the NDA estimate $1/(4\pi)^2$.

Two technical aspects deserve special attention. Firstly, the continuum χ is related to the lattice χ_L by both additive and multiplicative renormalisation. Second, the lattice discretisation renders the lattice topological charge, Q_L, non-integer.

All quoted calculations of χ make use of the definition of Q_L that employs the clover-leaf plaquette [58, 59] on ensembles of configurations generated with the Cabibbo-Marinari implementation of the heat bath algorithm [60]. In order to circumvent the noisy signal resulting from ultraviolet fluctuations of Q_L, one exploits the stability of the topological charge under smooth deformations of the fields, and computes it after a smoothing process such as cooling or Wilson flow. An integer value of Q_L on the lattice can then be assigned either by *small-instanton-correction* [5], or by *correction-and-rounding* [34]. The former consists of rounding the lattice topological charge to one of its neighbouring integer values, chosen with the sign of the net contribution of small instantons. The latter comprises rescaling Q_L by minimising the average deviation of the lattice topological charge from integer multiples.

For $SU(N_c)$ gauge theories, Ref. [34] assigns integer values to Q_L by correction-and-rounding on cooled configurations and computes the continuum limit of χ for $N_c = 3, 4, 6$. The same strategy is used in Ref. [35], which reports the continuum limits for $N_c = 4, 6$. With respect to these two works, Ref. [36] differs because the configurations are obtained by an algorithm that considers a larger ensemble of systems with boundary conditions interpolating from periodic to open to soften the effects of topological freezing (see the quoted work for details); the continuum limits are then obtained for $N_c = 3$, 4, 6 although for $SU(3)$ the numerical results are taken from Refs. [43, 44]. By contrast, in Refs. [5, 15] small-instanton-correction is applied to Q_L, obtained from cooled configurations, and the continuum χ is then extrapolated for $N_c = 2$, 3, 4, 5.

In the case of $Sp(N_c)$ gauge theories, we borrow the results from a companion publication, Ref. [50], which is part of the ongoing programme of study of $Sp(N_c)$ lattice gauge theories [11, 13, 61–63], and uses the HiRep code [64], adapted to $Sp(N_c)$ groups [11]. The lattice topological charge is obtained from Wilson-flowed configurations [65, 66], and correction-and-rounding is used to assign integer topological charge. The topological susceptibility χ is obtained in the continuum limit for $N_c = 2$, 4, 6, 8.

By comparing Figs. 1 and 2, we observe two interesting facts. Firstly, the two sequences of measurements of χ/σ^2 are clearly dissimilar, yet they share interesting properties at the extrema: measurements by different collaborations for $Sp(2) \sim SU(2)$ are in broad agreement, and going to large N_c the two sequences show a tendency to converge towards two different constants for $N_c \gtrsim 4$. Second, once we apply the rescaling by the group factor, C_F^2/d_G, the two sequences can no longer be distinguished, the measurements for $Sp(N_c)$ and $SU(N_c)$ theories agreeing with one another, given current uncertainties. A rough estimate, based upon naive dimensional analysis (NDA) [67], yields:

$$\eta_\chi \;=\; \frac{\chi C_2(F)^2}{\sigma^2 d_G} \;=\; \mathcal{O}\left(\frac{1}{(4\pi)^2}\right) . \tag{16}$$

This estimate falls straight in the middle of the range of measurements, possibly by mere numerical coincidence. Yet, it is remarkable that no more than a factor of 2 separates existing measurements, for all groups G, and that this estimate yields the correct order of magnitude.

The scaling procedure allows us to perform a simple global fit of the whole set of measurement, in the form

$$\eta_\chi \;=\; \frac{\chi C_2(F)^2}{\sigma^2 d_G} \;=\; a + \frac{c}{d_G} . \tag{17}$$

The result of the fit, which has reduced $\tilde{\mathcal{X}}^2 \equiv \mathcal{X}^2/N_{\rm d.o.f.} = 1.58$, is $a = 0.004842(77)$ and $c = 0.01635(46)$. Visual inspection of Fig. 2 and Table I highlights some modest tension between measurements performed by different collaborations for $SU(2)$, as well as for $SU(3)$, suggesting that for these two groups the systematic uncertainty is not negligible, compared to the statistical uncertainty. To quantify this effect, we repeat the same fitting procedure, but by omitting the $Sp(N_c)$ measurements, and obtain as a result that $\tilde{\mathcal{X}}^2 = 1.83$, hence demonstrating that the combination of measurements taken in theories with the two families of groups does not affect the goodness of the fit.

We also performed alternative fits, by including corrections $\mathcal{O}(1/\sqrt{d_G})$ or $\mathcal{O}(1/d_G^2)$, to test the scaling hypothesis we made; these additional terms do not change appreciably the results of the maximum likelihood analysis. Our final result is

$$\lim_{N_c\to\infty} \eta_\chi = (48.42 \pm 0.77 \pm 3.31) \times 10^{-4} , \tag{18}$$

where the first error is the statistical one from the 2-parameter fit in the form Eq. (17), while the second is the systematic error of the fitting procedure. The latter is conservatively estimated as the difference between using in the extrapolation either the 2-parameter fit or a 3-parameter fit including an additional term proportional to $1/d_G^2$—we show the result of both fits in Fig. 2.

For $SU(N_c)$, $C_2(F)^2 \to d_G/4$ in the large-N_c limit, hence our combined result in Eq. (18) can be recast as $\chi/\sigma^2 \to 0.01937 \pm 0.00136$. This is $\simeq 1.4$ standard deviations lower than the result $\chi/\sigma^2 \to 0.0221(14)$ from Ref. [34], but in excellent agreement with Ref. [36], which quotes $\chi/\sigma^2 \to 0.0199(10)$, and with Ref. [15], from which one deduces that $\chi/\sigma^2 \to 0.01836(56)$.

V. OUTLOOK

We proposed a rescaling by group-theoretical factors of the dimensionless quantity χ/σ^2, the ratio of topological susceptibility and square of the string tension, to yield η_χ, a quantity that can be meaningfully compared across different (four-dimensional) Yang-Mills theories. We collected from the literature the results of the continuum limit extrapolation of several independent lattice measurements of η_χ in theories with groups $SU(N_c)$ and $Sp(N_c)$. All measurements of η_χ are of the order of magnitude indicated by a rough NDA estimate. The two sequences of groups display the same functional dependence of η_χ on the dimension d_G of the group, in support of the proposed rescaling. We assessed this statement by performing a combined fit of all the measurements, and by extrapolating towards the large-N_c limit.

We conclude by highlighting a number of open questions, deserving of further future investigation. The numerical evidence we collected suggests that the group-theoretical scaling we proposed allows to combine measurements of χ within the sequences of $SU(N_c)$ and $Sp(N_c)$ Yang-Mills theories. It would be fascinating to extend this analysis to other choices of gauge group. After rescaling, there remains clearly visible a non-trivial (though mild) dependence on the group dimension; the precise functional form of the quantity $\chi C_2(F)^2/\sigma^2 d_G$ remains a subject for non-perturbative studies. It would be interesting to reassess these statements with future higher precision measurements.

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