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N-photon amplitudes in a plane-wave background

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ABSTRACT

We use the worldline formalism to derive master formulas for the one-loop N-photon amplitudes in a plane-wave background, for both scalar and spinor quantum electrodynamics. This generalises previous work by Ilderton and Torgrimsson for the vacuum polarisation case, although with some change in methodology since, instead of evaluating the path integral on the semi-classical trajectory, we use the special kinematics of the plane-wave background to uncover the crypto-gaussian character of this type of worldline path integral.

1. Introduction

In strong-field QED, there are two external field configurations that play a special rôle: the constant field, and the plane-wave one. Both are not only important from the physics point of view, but also special mathematically, since they allow for an exact solution of the Dirac equation in the field, which makes it possible to perform non-perturbative calculations in such fields (see, e.g., [1,2]). Nevertheless, beyond the simplest special cases such calculations tend to be extremely lengthy and tedious [3–24]. For example, the one-loop QED vertex [25] in a plane-wave field has been calculated only very recently.

For the constant-field case there exists an alternative approach, based on Feynman's worldline path integral formulation of QED [26,27] and concepts originally borrowed from string theory [28,29], that has been shown to offer various technical advantages for closed-loop photonic processes [30–33] and recently also for amplitudes involving open scalar [34] and fermion [35] lines – for reviews of this formalism see [36,37].

The plane-wave case has attracted much attention in recent years because of its relevance for laser physics [38–40]. However, the application of the worldline formalism to this case has turned out to be less straightforward. A calculation of the scalar and spinor QED vacuum polarisations along these lines was achieved by A. Ilderton and G. Torgrimsson [42], but it is not obvious how to extend their approach to the general N-photon amplitudes. Here we will use a slightly different approach, based on a direct rewriting of the worldline path integral as a gaussian one, to construct compact master formulas for the scalar and spinor QED N-photon amplitudes in a plane-wave background.

We start in the following section with a short summary of the worldline representation of the N-photon amplitudes in vacuum (for details see [36]). The following two sections are devoted to the derivation of master formulas for the N-photon amplitudes in a plane-wave background, first for scalar and then for spinor QED. As a check, in section 5 we work out the $N = 2$ cases and recover the results of [42]. In the final section we shortly summarise our results and point out possible generalisations.

2. N-photon amplitudes in the worldline formalism

The starting point for the calculation of the scalar QED N-photon amplitudes in the worldline formalism is Feynman's [26] worldline representation of the one-loop effective action $\Gamma_{\rm scal}[A]$:

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$$\Gamma_{\rm scal}[A] = \int_0^\infty \frac{dT}{T}\, {\rm e}^{-m^2T} \int\limits_{x(T)=x(0)} Dx\, {\rm e}^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4}+ie\,\dot{x}\cdot A(x)\right)} \tag{1}$$

The path integral runs over all closed trajectories in spacetime obeying the periodicity condition $x(T) = x(0)$ in proper time. The N-photon amplitudes are obtained from this by expanding the "interaction exponential" and Fourier transformation, which leads to the "vertex operator representation" of the N-photon amplitude:

$$\Gamma_{\rm scal}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) = (-ie)^N \int_0^\infty \frac{dT}{T}\, {\rm e}^{-m^2T} \int\limits_{x(0)=x(T)} Dx\, {\rm e}^{-\int_0^T d\tau \frac{\dot{x}^2}{4}} V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N] \tag{2}$$

Here each photon is represented by the following *photon vertex operator*, integrated along the trajectory:

$$V^\gamma_{\rm scal}[k,\varepsilon] \equiv \int_0^T d\tau\, \varepsilon\cdot\dot{x}(\tau)\, {\rm e}^{ik\cdot x(\tau)} \tag{3}$$

After a formal exponentiation $\varepsilon_i\cdot\dot{x}_i\,{\rm e}^{ik_i\cdot x_i} = {\rm e}^{\varepsilon_i\cdot\dot{x}_i+ik_i\cdot x_i}\,|_{\varepsilon_i}$ the path integral can be done by gaussian integration using the basic correlator

$$\langle x^\mu(\tau)x^\nu(\tau')\rangle = -G(\tau,\tau')\delta^{\mu\nu}, \quad G(\tau,\tau') \equiv |\,\tau-\tau'\,| - \frac{(\tau-\tau')^2}{T}\,. \tag{4}$$

This results in the following "Bern-Kosower representation" of the N-photon amplitude [43,28,29],

$$\begin{aligned}\Gamma_{\rm scal}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) &= (-ie)^N(2\pi)^D\delta\Big(\sum_i k_i\Big)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\\ &\times\prod_{i=1}^N\int_0^T d\tau_i \exp\Big\{\sum_{i,j=1}^N\Big[\frac{1}{2}G_{ij}k_i\cdot k_j - i\dot{G}_{ij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\Big]\Big\}\Big|_{\varepsilon_1\varepsilon_2\dots\varepsilon_N}\end{aligned} \tag{5}$$

Here a 'dot' denotes a derivative acting on the first variable,

$$\dot{G}(\tau,\tau') = {\rm sign}(\tau-\tau') - 2\frac{(\tau-\tau')}{T}, \quad \ddot{G}(\tau,\tau') = 2\delta(\tau-\tau') - \frac{2}{T} \tag{6}$$

and we abbreviate $G_{ij} \equiv G(\tau_i,\tau_j)$ etc. The factor $(4\pi T)^{-\frac{D}{2}}$ represents the free Gaussian path integral determinant factor, and the $(2\pi)^D\delta(\sum_i k_i)$ factor is produced by the integration over the zero mode $x_0^\mu \equiv \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau)$ of the path integral. The exponential must still be expanded and only the terms that contain each polarisation vector ε_i linearly be retained:

$$\exp\{\cdot\}\big|_{\varepsilon_1\varepsilon_2\dots\varepsilon_N} \equiv (-i)^N P_N(\dot{G}_{ij},\ddot{G}_{ij})\exp\Big[\frac{1}{2}\sum_{i,j=1}^N G_{ij}k_i\cdot k_j\Big] \tag{7}$$

with certain polynomials P_N.

For spinor QED, a generalization of (1) suitable for analytical calculations is given by the Feynman-Fradkin representation [27,44]

$$\Gamma_{\rm spin}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\int Dx\int D\psi\, e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{q}^2+\frac{1}{2}\psi\cdot\dot{\psi}+\frac{1}{2}ie\,\dot{x}\cdot A(x)-ie\psi^\mu F_{\mu\nu}(x)\psi^\nu\right)}\,. \tag{8}$$

Here $\psi^\mu(\tau)$ is a Lorentz vector whose components are Grassmann functions, $\{\psi^\mu(\tau),\psi^\nu(\tau')\}=0$, and the path integral $\int D\psi$ has to be taken over antiperiodic such functions, $\psi^\mu(T) = -\psi^\mu(0)$. Note that it is already gaussian as it stands.

Applying the same procedure as for the scalar case above, one obtains the following generalization of (2) to the spinor QED case:

$$\Gamma_{\rm spin}(k_1,\varepsilon_1;\dots;k_N,\varepsilon_N) = -\frac{1}{2}(-ie)^N\int_0^\infty \frac{dT}{T}\,{\rm e}^{-m^2T}\int\limits_{x(0)=x(T)} Dx\, e^{-\int_0^T d\tau\frac{\dot{x}^2}{4}}\int\limits_A D\psi\,{\rm e}^{-\int_0^T d\tau\frac{1}{2}\psi\cdot\dot{\psi}}V^\gamma_{\rm spin}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm spin}[k_N,\varepsilon_N]\,. \tag{9}$$

The photon vertex operator for spinor QED $V^\gamma_{\rm spin}$ differs from the scalar one (3) by a second term representing the interaction of the fermion spin with the photon,

$$V^\gamma_{\rm spin}[k,\varepsilon] \equiv \int_0^T d\tau\,\Big[\varepsilon\cdot\dot{x}(\tau) - i\psi(\tau)\cdot f\cdot\psi(\tau)\Big]\,{\rm e}^{ik\cdot x(\tau)} \tag{10}$$

with $f^{\mu\nu} \equiv k^\mu\varepsilon^\nu - \varepsilon^\mu k^\nu$ the photon field-strength tensor. Thus the N-photon amplitude is naturally obtained in terms of a spin-orbit decomposition

$$\Gamma_{\text{spin},N} = \sum_{S=0}^{N} \Gamma_{NS}\,, \quad \Gamma_{NS} = \sum_{\{i_1 i_2 \dots i_S\}} \Gamma_{NS}^{\{i_1 i_2 \dots i_S\}}\,, \tag{11}$$

where S denotes the number of spin interactions, and the sum $\sum_{\{i_1 i_2 \dots i_S\}}$ runs over all choices of S out of the N photons as the ones assigned to those interactions. It is then straightforward to arrive at the following master formula for $\Gamma_{NS}^{\{i_1 i_2 \dots i_S\}}$:

$$\Gamma_{NS}^{\{i_1 i_2 \dots i_S\}} = -2(-e)^N \int_0^\infty \frac{dT}{T}\,(4\pi T)^{-\frac{D}{2}}\,\mathrm{e}^{-m^2 T} \prod_{i=1}^{N} \int_0^T d\tau_i\, W(k_{i_1}, \varepsilon_{i_1}; \dots; k_{i_S}, \varepsilon_{i_S}) P_{NS}^{\{i_1 i_2 \dots i_S\}} \mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^{N} G_{ij} k_i \cdot k_j}\,. \tag{12}$$

Here the polynomials $P_{NS}^{\{i_1 i_2 \dots i_S\}}$ are now defined by (compare with (7))

$$\mathrm{e}^{\sum_{i,j=1}^{N}\left(-i\dot{G}_{ij}\varepsilon_i \cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\right)}\Big|_{\varepsilon_{i_1}=\dots=\varepsilon_{i_S}=0}\Big|_{\varepsilon_{i_{S+1}}\cdots\varepsilon_{i_N}} \equiv (-i)^{N-S} P_{NS}^{\{i_1 i_2 \dots i_S\}}\,, \tag{13}$$

where the notation on the left-hand side means that one first sets the polarisation vectors $\varepsilon_{i_1}, \dots, \varepsilon_{i_S}$ equal to zero, and then selects all the terms linear in the surviving polarisation vectors. In particular, one has the extremal cases $P_{N0}^{\{\}} = P_N$, $P_{NN}^{\{12\dots N\}} = 1$. For the Wick-contraction of the spin interaction terms we have introduced the notation

$$W(k_{i_1}, \varepsilon_{i_1}; \dots; k_{i_S}, \varepsilon_{i_S}) \equiv \Big\langle \psi_{i_1} \cdot f_{i_1} \cdot \psi_{i_1} \cdots \psi_{i_S} \cdot f_{i_S} \cdot \psi_{i_S} \Big\rangle \tag{14}$$

to be evaluated with the basic correlator

$$\langle \psi^\mu(\tau)\psi^\nu(\tau')\rangle = \frac{1}{2} G_F(\tau,\tau')\delta^{\mu\nu}, \quad G_F(\tau,\tau') \equiv \mathrm{sgn}(\tau-\tau')\,. \tag{15}$$

This object possesses the following closed-form description [45]. Define a "Lorentz cycle of length n" Z_n by

$$Z_n(i_1 i_2 \dots i_n) \equiv \Big(\frac{1}{2}\Big)^{\delta_{n2}} \mathrm{tr}\Big(\prod_{j=1}^{n} f_{i_j}\Big) \tag{16}$$

and a "fermionic bi-cycle of length n" by

$$G_F(i_1 i_2 \dots i_n) \equiv G_{Fi_1i_2} G_{Fi_2i_3} \cdots G_{Fi_ni_1} Z_n(i_1 i_2 \dots i_n) \qquad (n \geq 2)\,. \tag{17}$$

Then we can write

$$W(k_1, \varepsilon_1; \dots; k_S, \varepsilon_S) = \sum_{\text{partitions}} (-1)^{cy} G_F(i_1 i_2 \dots i_{n_1}) G_F(i_{n_1+1} \dots i_{n_1+n_2}) \cdots G_F(i_{n_1+\dots+n_{cy-1}+1} \dots i_{n_1+\dots+n_{cy}})\,. \tag{18}$$

Here the sum runs over all inequivalent possibilities to distribute the indices $1, \dots, S$ among the arguments of any number cy of bi-cycles, and n_k denotes the length of the bi-cycle k. Working out (18) up to $S = 4$, we find

$$\begin{aligned} W(k_1,\varepsilon_1) &= 0\,,\\ W(k_1,\varepsilon_1;k_2,\varepsilon_2) &= -G_F(12)\,,\\ W(k_1,\varepsilon_1;k_2,\varepsilon_2;k_3,\varepsilon_3) &= -G_F(123)\,,\\ W(k_1,\varepsilon_1;k_2,\varepsilon_2;k_3,\varepsilon_3;k_4,\varepsilon_4) &= -G_F(1234) - G_F(1243) - G_F(1324)\\ &\quad + G_F(12)G_F(34) + G_F(13)G_F(24) + G_F(14)G_F(23)\,. \end{aligned} \tag{19}$$

3. *N*-photon amplitude in a plane-wave background (scalar QED)

In general, a plane-wave field can be defined by a vector potential $A(x)$ of the form

$$eA_\mu(x) = a_\mu(n\cdot x) \tag{20}$$

where n^μ is a null vector,

$$n^2 = 0 \tag{21}$$

and, as is usual, we will further impose the *light-front gauge condition*

$$n \cdot a = 0 \tag{22}$$

Note that we absorb the charge e in the definition of a_μ. Repeating the procedure of the previous section with the addition of the potential a_μ to the worldline Lagrangian, we straightforwardly get a representation of the N-photon amplitude in the plane-wave background that generalises the vacuum formula (2),

$$\Gamma_{\rm scal}(\{k_i,\varepsilon_i\};a) = (-ie)^N \int_0^\infty \frac{dT}{T}\, {\rm e}^{-m^2T} \int Dx\, {\rm e}^{-\int_0^T d\tau\left[\frac{\dot{x}^2}{4}+i\dot{x}^\mu a_\mu(n\cdot x)\right]} V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N] \tag{23}$$

Fixing the zero-mode problem as usual by separating off the average position x_0^μ of the trajectory, $x^\mu(\tau)=x_0^\mu+q^\mu(\tau)$, we note that, differently from the vacuum case, it now appears not only in the exponents of the vertex operators, but also in the argument of $a_\mu(n\cdot x)$. Thus it will now be convenient to introduce (euclidean) light-cone coordinates adapted to the null vector n^μ [41]. Thus we set $n^\mu \equiv \frac{1}{\sqrt{2}}(0,0,1,i)$, and define $x^+ \equiv n\cdot x = \frac{1}{\sqrt{2}}(x^3+ix^4)$ ("light-front time") and $x^- \equiv \frac{1}{\sqrt{2}}(-x^3+ix^4)$. We will further denote $x^\perp \equiv (x^1,x^2)$. Defining also $k^\pm \equiv \frac{1}{\sqrt{2}}(\pm k^3+ik^4)$, and using the decomposition

$$k\cdot x = -k^+x^- - k^-x^+ + k^1x^1 + k^2x^2 \tag{24}$$

allows us to integrate out x_0^μ but for its x_0^+ component:

$$\int Dx\, {\rm e}^{-\int_0^T d\tau\left[\frac{\dot{x}^2}{4}+i\dot{x}^\mu a_\mu(n\cdot x)\right]} V^\gamma_{\rm scal}[k_1,\varepsilon_1]\cdots V^\gamma_{\rm scal}[k_N,\varepsilon_N]$$
$$= (2\pi)^3\delta\Big(\sum_{i=1}^N k_i^1\Big)\delta\Big(\sum_{i=1}^N k_i^2\Big)\delta\Big(\sum_{i=1}^N k_i^+\Big)\int_{-\infty}^{\infty} dx_0^+\, {\rm e}^{-ix_0^+\sum_{i=1}^N k_i^-}\prod_{i=1}^N\int_0^T d\tau_i \int Dq\, {\rm e}^{-\int_0^T d\tau\left[\frac{\dot{q}^2}{4}+i\dot{q}^\mu a_\mu(x_0^+ + n\cdot q(\tau))\right]}\, {\rm e}^{\sum_{i=1}^N(ik_i\cdot q_i+\varepsilon_i\cdot\dot{q}_i)}\Big|_{\varepsilon_1\varepsilon_2\cdots\varepsilon_N}. \tag{25}$$

The calculation of the functional integral at first sight looks like an intractable problem, since the integration variable $q(\tau)$ appears in the argument of the unknown function a_μ. In [42] this problem was solved for the two-point case using the fact that the plane-wave path integral possesses the gaussian property that its semiclassical approximation is exact. For the N-point generalization, we find it more convenient to exhibit the crypto-gaussian nature of the path integral using the relations (21) and (22). In principle, we could do the functional integral by expanding

$${\rm e}^{-\int_0^T d\tau\, i\dot{q}\cdot a(x_0^+ + n\cdot q(\tau))} = 1 - i\int_0^T d\tau\, \dot{q}\cdot a(x_0^+ + n\cdot q(\tau)) + \frac{(-i)^2}{2!}\int_0^T d\tau_1\, \dot{q}_1\cdot a(x_0^+ + n\cdot q_1)\int_0^T d\tau_2\, \dot{q}_2\cdot a(x_0^+ + n\cdot q_2) + \cdots \tag{26}$$

and then Taylor-expanding

$$a_\mu(x_0^+ + n\cdot q_m) = a_\mu(x_0^+) + a'_\mu(x_0^+)(n\cdot q_m) + \frac{1}{2!}a''_\mu(x_0^+)(n\cdot q_m)^2 + \cdots \tag{27}$$

(Note that we use a 'prime' for the derivative of a function with respect to its argument, while the 'dot' will be used for the total derivative with respect to proper time.) Now, we observe that a factor $n\cdot q_m$ can neither be Wick contracted with another such factor because of (21), nor with a factor of $\dot{q}_n\cdot a^{(k)}(x_0^+)$ because of (22). Thus each $n\cdot q_m$ has to be contracted with the exponential ${\rm e}^{\sum_{i=1}^N(ik_i\cdot q_i+\varepsilon_i\cdot\dot{q}_i)}$, and this will convert it into

$$n\cdot q_m \longrightarrow n\cdot\sum_{i=1}^N\big[-iG_{mi}k_i + \dot{G}_{mi}\varepsilon_i\big] \tag{28}$$

We can then resum (27) into

$$a_\mu(x_0^+ + n\cdot q_m) \longrightarrow a_\mu\Big(x_0^+ + n\cdot\sum_{i=1}^N[-iG_{mi}k_i + \dot{G}_{mi}\varepsilon_i]\Big) \tag{29}$$

and subsequently also (26),

$${\rm e}^{-\int_0^T d\tau\, i\dot{q}\cdot a(x_0^+ + n\cdot q(\tau))} \longrightarrow {\rm e}^{-\int_0^T d\tau\, i\dot{q}\cdot a\left(x_0^+ + n\cdot\sum_{i=1}^N[-iG(\tau,\tau_i)k_i + \dot{G}(\tau,\tau_i)\varepsilon_i]\right)} \tag{30}$$

where we can now, with some abuse of notation, replace

$$a_\mu\Big(x_0^+ + n\cdot\sum_{i=1}^N[-iG(\tau,\tau_i)k_i + \dot{G}(\tau,\tau_i)\varepsilon_i]\Big) \equiv a_\mu(\tau) \tag{31}$$

Thus we have removed the functional integration variable from the argument of a_μ, and converted the functional integral (25) into a gaussian one. Now the usual "completing-the-square" procedure can be applied, and yields

$$\int Dq\, {\rm e}^{-\int_0^T d\tau\left[\frac{\dot{q}^2}{4}+i\dot{q}\cdot a(\tau)\right]}\, {\rm e}^{\sum_{i=1}^N(ik_i\cdot q_i+\varepsilon_i\cdot\dot{q}_i)}$$
$$= (4\pi T)^{-\frac{D}{2}}\, {\rm e}^{-\frac{1}{2}\int_0^T d\tau\int_0^T d\tau'\ddot{G}(\tau,\tau')a(\tau)\cdot a(\tau') - \sum_{i=1}^N\int_0^T d\tau\left[\dot{G}(\tau,\tau_i)a(\tau)\cdot k_i + i\ddot{G}(\tau,\tau_i)a(\tau)\cdot\varepsilon_i\right]}\, {\rm e}^{\sum_{i,j=1}^N\left[\frac{1}{2}G_{ij}k_i\cdot k_j - i\dot{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\right]} \tag{32}$$

The first term in the exponent on the right-hand-side can, by introducing the worldline average

$$\langle\langle f\rangle\rangle \equiv \frac{1}{T}\int_0^T d\tau\, f(\tau) \tag{33}$$

and using (6), be rewritten as

$$\frac{1}{2}\int_0^T d\tau \int_0^T d\tau' \ddot{G}(\tau,\tau')a(\tau)\cdot a(\tau') = T\Big(\langle\langle a^2\rangle\rangle - \langle\langle a\rangle\rangle^2\Big) \tag{34}$$

Similarly, we can rewrite

$$\sum_{i=1}^N \int_0^T d\tau\, \ddot{G}(\tau,\tau_i)a(\tau)\cdot\varepsilon_i = 2\sum_{i=1}^N \Big(a(\tau_i) - \langle\langle a\rangle\rangle\Big)\cdot\varepsilon_i \tag{35}$$

For the integral involving $\dot{G}(\tau,\tau_i)$, we introduce the periodic integral function

$$I_\mu(\tau) \equiv \int_0^\tau d\tau'\Big(a_\mu(\tau') - \langle\langle a_\mu\rangle\rangle\Big) \tag{36}$$

Integrating by parts, we get

$$\sum_{i=1}^N \int_0^T d\tau\, \dot{G}(\tau,\tau_i)a(\tau)\cdot k_i = -2\sum_{i=1}^N k_i\cdot\big(I(\tau_i) - \langle\langle I\rangle\rangle\big) \tag{37}$$

Putting the pieces together, we get the following master formula for the scalar QED N-photon amplitude in a plane-wave background[1]

$$\begin{aligned}\Gamma_{\rm scal}(\{k_i,\varepsilon_i\};a) = &(-ie)^N(2\pi)^3\delta\big(\sum_{i=1}^N k_i^1\big)\delta\big(\sum_{i=1}^N k_i^2\big)\delta\big(\sum_{i=1}^N k_i^+\big)\int_{-\infty}^{\infty} dx_0^+\, {\rm e}^{-ix_0^+\sum_{i=1}^N k_i^-}\\
&\times\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\prod_{i=1}^N\int_0^T d\tau_i\, {\rm e}^{\sum_{i,j=1}^N\big[\frac{1}{2}G_{ij}k_i\cdot k_j - i\dot{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\big]}\\
&\times\, {\rm e}^{-\big(m^2+\langle\langle a^2\rangle\rangle - \langle\langle a\rangle\rangle^2\big)T + 2\sum_{i=1}^N k_i\cdot\big(I(\tau_i)-\langle\langle I\rangle\rangle\big) - 2i\sum_{i=1}^N\big(a(\tau_i)-\langle\langle a\rangle\rangle\big)\cdot\varepsilon_i}\Big|_{\varepsilon_1\cdots\varepsilon_N}\end{aligned} \tag{38}$$

Note that the appearance of the polarization vectors in the argument of a_μ makes it still messy to extract the terms linear in all of them. Further substantive simplification can be achieved by choosing the ε_i such as to obey

$$n\cdot\varepsilon_i = 0 \quad (i=1,\dots,N) \tag{39}$$

which is possible for generic momenta by a gauge transformation, and will be assumed for the rest of this paper. This will reduce (31) to

$$a_\mu(\tau) = a_\mu\Big(x_0^+ - i\sum_{i=1}^N G(\tau,\tau_i)k_i^+\Big) \tag{40}$$

The master formula (38) can then be written more explicitly as

$$\begin{aligned}\Gamma_{\rm scal}(\{k_i,\varepsilon_i\};a) = &(-e)^N(2\pi)^3\delta\big(\sum_{i=1}^N k_i^1\big)\delta\big(\sum_{i=1}^N k_i^2\big)\delta\big(\sum_{i=1}^N k_i^+\big)\int_{-\infty}^{\infty} dx_0^+\, {\rm e}^{-ix_0^+\sum_{i=1}^N k_i^-}\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,{\rm e}^{-\big(m^2+\langle\langle a^2\rangle\rangle-\langle\langle a\rangle\rangle^2\big)T}\prod_{i=1}^N\int_0^T d\tau_i\\
&\times\mathfrak{P}_N\,{\rm e}^{\frac{1}{2}\sum_{i,j=1}^N G_{ij}k_i\cdot k_j + 2\sum_{i=1}^N k_i\cdot\big(I(\tau_i)-\langle\langle I\rangle\rangle\big)}\end{aligned} \tag{41}$$

where the polynomials $\mathfrak{P}_N$ are defined by (compare (7))

$${\rm e}^{\sum_{i,j=1}^N\big(-i\dot{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\big) - 2i\sum_{i=1}^N\big(a(\tau_i)-\langle\langle a\rangle\rangle\big)\cdot\varepsilon_i}\Big|_{\varepsilon_1\cdots\varepsilon_N} \equiv (-i)^N\mathfrak{P}_N\,. \tag{42}$$

[1] The reader familiar with the worldline formalism may wonder why we could drop the additive constant $\frac{T}{6}$ from this Green's function, which is customary but relies on momentum conservation. It is easy to verify that here, in light-cone coordinates, the removal of the constant requires only that $\sum_{i=1}^N k_i^{+,\perp} = 0$, not full momentum conservation.

4. N-photon amplitude in a plane-wave background (spinor QED)

Proceeding to the spinor QED case, the same argument that we applied above to $eA_\mu = a_\mu$ can be used to convert also the argument of the $eF_{\mu\nu} = n_\mu a'_\nu - a'_\mu n_\nu$ appearing in the spin part of the worldline Lagrangian in (8) in the same way as in (29). Thus the only new element is that the fermionic Wick-contraction rule (14) now has to be calculated with a generalised worldline Green's function inverting the field-dependent operator

$$\mathcal{O} \equiv \frac{\delta_{\mu\nu}}{2}\frac{d}{d\tau} + ia'_\mu(\tau)n_\nu - in_\mu a'_\nu(\tau) \tag{43}$$

The appropriate generalisation of (15) is

$$\langle \psi^\mu(\tau)\psi^\nu(\tau')\rangle = \frac{1}{2}\mathfrak{G}_F^{\mu\nu}(\tau,\tau'), \tag{44}$$

where

$$\mathfrak{G}_F^{\mu\nu}(\tau,\tau') \equiv \Big\{\delta^{\mu\nu} + 2in^\mu \mathcal{J}^\nu(\tau,\tau') + 2i\mathcal{J}^\mu(\tau',\tau)n^\nu + 2\Big[\mathcal{J}^2(\tau,\tau') - \frac{T^2}{4}\langle\langle a'\rangle\rangle^2\Big]n^\mu n^\nu\Big\}G_F(\tau,\tau') \tag{45}$$

and we have further defined

$$J_\mu(\tau) \equiv \int_0^\tau d\tau'\Big(a'_\mu(\tau') - \langle\langle a'_\mu\rangle\rangle\Big), \tag{46}$$

$$\mathcal{J}_\mu(\tau,\tau') \equiv J_\mu(\tau) - J_\mu(\tau') - \frac{T}{2}\dot{G}(\tau,\tau')\langle\langle a'_\mu\rangle\rangle\,. \tag{47}$$

Note that the modified Green's function has a non-zero coincidence limit,

$$\mathfrak{G}_F^{\mu\nu}(\tau,\tau) = -iT\Big(n^\mu\langle\langle a'^\nu\rangle\rangle - \langle\langle a'^\mu\rangle\rangle n^\nu\Big) \tag{48}$$

and satisfies the anti-symmetry relation $\mathfrak{G}_F(\tau',\tau) = -\mathfrak{G}_F^{\mathsf{T}}(\tau,\tau')$. Proceeding as in the vacuum case, we get a spin-orbit decomposition as in (11) with

$$\Gamma_{NS}^{\{i_1i_2\dots i_S\}} = -2(-e)^N(2\pi)^3\delta\Big(\sum_{i=1}^N k_i^1\Big)\delta\Big(\sum_{i=1}^N k_i^2\Big)\delta\Big(\sum_{i=1}^N k_i^+\Big)\int_{-\infty}^{\infty} dx_0^+\,\mathrm{e}^{-ix_0^+\sum_{i=1}^N k_i^-}\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\,\mathrm{e}^{-(m^2+\langle\langle a^2\rangle\rangle - \langle\langle a\rangle\rangle^2)T}\prod_{i=1}^N\int_0^T d\tau_i$$
$$\times\,\mathfrak{W}(k_{i_1},\varepsilon_{i_1};\dots;k_{i_S},\varepsilon_{i_S})\mathfrak{P}_{NS}^{\{i_1i_2\dots i_S\}}\mathrm{e}^{\frac{1}{2}\sum_{i,j=1}^N G_{ij}k_i\cdot k_j + 2\sum_{i=1}^N k_i\cdot(I(\tau_i)-\langle\langle I\rangle\rangle)}\,. \tag{49}$$

The polynomials $\mathfrak{P}_{NS}$ now are defined by

$$\mathrm{e}^{\sum_{i,j=1}^N\left(-i\dot{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{ij}\varepsilon_i\cdot\varepsilon_j\right) - 2i\sum_{i=1}^N\left(a(\tau_i)-\langle\langle a\rangle\rangle\right)\cdot\varepsilon_i}\Big|_{\varepsilon_{i_1}=\dots=\varepsilon_{i_S}=0}\Big|_{\varepsilon_{i_{S+1}}\cdots\varepsilon_{i_N}} \equiv (-i)^{N-S}\mathfrak{P}_{NS}^{\{i_1i_2\dots i_S\}}\,, \tag{50}$$

and $\mathfrak{W}(k_{i_1},\varepsilon_{i_1};\dots;k_{i_S},\varepsilon_{i_S})$ denotes the correlator (14) evaluated with the modified fermionic Wick contraction (44). For the calculation of this correlator we can still use the cycle decomposition formula (18), only that the fermionic bicycle (17), now must be replaced by

$$\mathfrak{G}_F(i_1i_2\dots i_n) \;\equiv\; \left(\tfrac{1}{2}\right)^{\delta_{n2}}\mathrm{tr}(f_{i_1}\cdot\mathfrak{G}_{Fi_1i_2}\cdot f_{i_2}\cdot\mathfrak{G}_{Fi_2i_3}\cdots f_{i_n}\cdot\mathfrak{G}_{Fi_ni_1}) \tag{51}$$

Note that, differently from the case of a constant external field [31,33], the fermionic path-integral determinant factor is not affected by the presence of the plane-wave field, and remains at its free value $2^{\frac{D}{2}}$.

5. The case $N = 2$

As a check, let us show that the above master formulas correctly reproduce the results of [42] for the $N = 2$ case. We first give the general off-shell results before specialising to the on-shell helicity flip process studied there. We work throughout with the gauge choice (39) for convenience and note that momentum conservation in the $+$ direction gives $k_2^+ = -k_1^+$.

5.1. Off-shell

Using the notation introduced in (11), for Γ_{20} we find from (50)

$$\mathfrak{P}_{20}^{\{\}} = \dot{G}_{12}\dot{G}_{21}\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1 - \ddot{G}_{12}\varepsilon_1\cdot\varepsilon_2 + 2\Big(\dot{G}_{12}\varepsilon_1\cdot k_2\varepsilon_2\cdot\big(a(\tau_2)-\langle\langle a\rangle\rangle\big) + \dot{G}_{21}\varepsilon_1\cdot\big(a(\tau_1)-\langle\langle a\rangle\rangle\big)\varepsilon_2\cdot k_1\Big)$$
$$+\,4\varepsilon_1\cdot\big(a(\tau_1)-\langle\langle a\rangle\rangle\big)\varepsilon_2\cdot\big(a(\tau_2)-\langle\langle a\rangle\rangle\big) \tag{52}$$

$$\mathfrak{W}() = 1\,, \tag{53}$$

which is sufficient to produce the scalar QED result when substituted into (12). Furthermore, for the spinor case we also require

$$\mathfrak{P}_{21}^{\{1\}} = \dot{G}_{21}\varepsilon_2 \cdot k_1 + 2\varepsilon_2 \cdot \big(a(\tau_2) - \langle\langle a \rangle\rangle\big) \tag{54}$$

$$\mathfrak{W}(k_1, \varepsilon_1) = -\frac{1}{2}\mathrm{tr}\big(\mathfrak{G}_F(\tau_1, \tau_1)\cdot f_1\big) = -iTk_1^+\varepsilon_1 \cdot \langle\langle a' \rangle\rangle \tag{55}$$

which, with the corresponding results under $1 \leftrightarrow 2$, can be used to construct Γ_{21}. Finally, we determine Γ_{22} from

$$\mathfrak{P}_{22}^{\{12\}} = 1 \tag{56}$$

$$\mathfrak{W}(k_1, \varepsilon_1; k_2, \varepsilon_2) = -\mathfrak{G}_F(12) + \frac{1}{4}\mathrm{tr}\big(\mathfrak{G}_F(\tau_1, \tau_1)\cdot f_1\big)\mathrm{tr}\big(\mathfrak{G}_F(\tau_2, \tau_2)\cdot f_2\big) \tag{57}$$

where ($\mathcal{J}_{ij} \equiv \mathcal{J}(\tau_i, \tau_j)$, $\sigma_{ij} \equiv \mathrm{sgn}(\tau_i - \tau_j)$)

$$\begin{aligned}-\mathfrak{G}_F(12) = {} & 2k_1^+k_2^+\Big[\varepsilon_1\cdot\varepsilon_2\Big(\mathcal{J}_{12}^2 - \frac{T^2}{4}\langle\langle a'\rangle\rangle^2\Big) + 2\varepsilon_1\cdot\mathcal{J}_{21}\varepsilon_2\cdot\mathcal{J}_{12}\Big]\sigma_{12}\sigma_{21} - 2ik_1^+\Big[\varepsilon_1\cdot k_2\varepsilon_2\cdot\mathcal{J}_{12} - \varepsilon_1\cdot\varepsilon_2 k_2\cdot\mathcal{J}_{12}\Big]\sigma_{12}\sigma_{21} \\ & - 2ik_2^+\Big[\varepsilon_2\cdot k_1\varepsilon_1\cdot\mathcal{J}_{21} - \varepsilon_1\cdot\varepsilon_2 k_1\cdot\mathcal{J}_{21}\Big]\sigma_{12}\sigma_{21} - \frac{1}{2}\mathrm{tr}\big(f_1\cdot f_2\big)\sigma_{12}\sigma_{21}\,.\end{aligned} \tag{58}$$

These lead to a spin-orbit decomposition

$$\begin{aligned}\Gamma_{\mathrm{spin},2}(k_1, \varepsilon_1; k_2, \varepsilon_2) = {} & -2e^2\int_{-\infty}^{\infty} dx_0^+\ \mathrm{e}^{-ix_0^+(k_1+k_2)^-}\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\mathrm{e}^{-\left(m^2+\langle\langle a^2\rangle\rangle - \langle\langle a\rangle\rangle^2\right)T}\int_0^T d\tau_1\int_0^T d\tau_2\, \mathrm{e}^{G_{12}k_1\cdot k_2 + 2\sum_{i=1}^2 k_i\cdot\left(I(\tau_i) - \langle\langle I\rangle\rangle\right)} \\ & \times\Big\{\mathfrak{P}_{20}^{\{\}} + \mathfrak{W}(k_1, \varepsilon_1)\mathfrak{P}_{21}^{\{1\}} + \mathfrak{W}(k_2, \varepsilon_2)\mathfrak{P}_{21}^{\{2\}} + \mathfrak{W}(k_1, \varepsilon_1; k_2\varepsilon_2)\Big\}\,,\end{aligned} \tag{59}$$

where we have omitted the momentum conserving δ-functions in the $+$ and $\perp$ directions. The term in the exponent $\sum_{i=1}^2 k_i\cdot\big(I(\tau_i) - \langle\langle I\rangle\rangle\big)$ at the two-point level could be removed by imposing on a_μ, instead of (22), the stronger gauge condition of full transversality, $a^+ = a^- = 0$. To see this, it is easiest to return to (37). Using the conservation of momentum along the $+$ and transversal directions together with (40), and choosing a function $b_\mu(x)$ such that $b'_\mu = a_\mu$, we have

$$\begin{aligned}\sum_{i=1}^2\int_0^T d\tau\,\dot{G}(\tau, \tau_i)a(\tau)\cdot k_i &= \int_0^T d\tau\,(\dot{G}(\tau, \tau_1) - \dot{G}(\tau, \tau_2))a(\tau)\cdot k_1 \\ &= \frac{i}{k_1^+}\int_0^T d\tau\frac{d}{d\tau}k_1\cdot b\Big(x_0^+ - i\sum_{i=1}^N G(\tau, \tau_i)k_i^+\Big) \\ &= \frac{i}{k_1^+}k_1\cdot(b(T) - b(0)) = 0\end{aligned} \tag{60}$$

5.2. On-shell

In the on-shell case and for $N = 2$ photons we gain additional simplifications due to the mass shell condition which, by conservation of momentum in the $+$ and $\perp$ directions, implies the additional condition $k_2^- = -k_1^-$ so that $k_1 = k = -k_2$ with $k^2 = 0$. This removes the exponent $\mathrm{e}^{G_{12}k_1\cdot k_2}$ from (59). Further imposing the transversality conditions $\varepsilon_1\cdot k = 0 = k\cdot\varepsilon_2$, the components of the spin orbit decomposition reduce to

$$\mathfrak{P}_{20}^{\{\}} \to -\ddot{G}_{12}\varepsilon_1\cdot\varepsilon_2 + 4\varepsilon_1\cdot\big(a_1 - \langle\langle a\rangle\rangle\big)\varepsilon_2\cdot\big(a_2 - \langle\langle a\rangle\rangle\big) \tag{61}$$

$$\mathfrak{P}_{21}^{\{1\}} \to 2\varepsilon_2\cdot\big(a_2 - \langle\langle a\rangle\rangle\big),\quad \mathfrak{P}_{21}^{\{2\}} \to 2\varepsilon_1\cdot\big(a_1 - \langle\langle a\rangle\rangle\big) \tag{62}$$

$$\mathfrak{W}(k_i, \varepsilon_i) \to -iTk^+\varepsilon_i\cdot\langle\langle a\rangle\rangle \tag{63}$$

$$\mathfrak{W}(k_1, \varepsilon_1; k_2\varepsilon_2) \to k^{+2}\Big[2\varepsilon_1\cdot\varepsilon_2\big(\mathcal{J}_{12}^2 - \frac{T^2}{4}\langle\langle a'\rangle\rangle^2\big) + 4\varepsilon_1\cdot\mathcal{J}_{12}\varepsilon_2\cdot\mathcal{J}_{21} + T^2\varepsilon_1\cdot\langle\langle a'\rangle\rangle\varepsilon_2\cdot\langle\langle a'\rangle\rangle\Big] \tag{64}$$

5.3. Helicity flip

For the purpose of comparing with the helicity-flip calculation of [42] we can further put $\varepsilon_1\cdot\varepsilon_2 = 0$, resulting in

$$\begin{aligned}\Gamma_{\mathrm{spin},2}(k_1, \varepsilon_1; k_2, \varepsilon_2) = {} & -2e^2\int_{-\infty}^{\infty} dx_0^+\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\mathrm{e}^{-\left(m^2+\langle\langle a^2\rangle\rangle-\langle\langle a\rangle\rangle^2\right)T}\Big\{4\varepsilon_1\cdot\big(a_1 - \langle\langle a\rangle\rangle\big)\varepsilon_2\cdot\big(a_2 - \langle\langle a\rangle\rangle\big) \\ & - 2iTk^+\Big[\varepsilon_1\cdot\langle\langle a\rangle\rangle\varepsilon_2\cdot\big(a_2 - \langle\langle a\rangle\rangle\big) - \varepsilon_1\cdot\big(a_1 - \langle\langle a\rangle\rangle\big)\varepsilon_2\cdot\langle\langle a\rangle\rangle\Big] \\ & + k^{+2}\Big[4\varepsilon_1\cdot\mathcal{J}_{12}\varepsilon_2\cdot\mathcal{J}_{21} + T^2\varepsilon_1\cdot\langle\langle a'\rangle\rangle\varepsilon_2\cdot\langle\langle a'\rangle\rangle\Big]\Big\}\,.\end{aligned} \tag{65}$$

This indeed correctly reproduces the corresponding parameter integrals determined in [42], (34) - (37), as can be easily seen using the integral representations

$$\mathcal{J}_{12\mu} = -\frac{1}{2}\sigma_{12}\int_0^T d\tau\,\sigma(\tau-\tau_1)\sigma(\tau-\tau_2)a'_\mu(\tau)$$

$$\mathcal{J}_{12}^2 - \frac{T^2}{4}\langle\langle a'\rangle\rangle^2 = -\frac{1}{2}\sigma_{12}\int_0^T d\tau\int_0^T d\tau'\,\sigma(\tau-\tau_1)\sigma(\tau-\tau')\sigma(\tau'-\tau_2)a'(\tau)\cdot a'(\tau')\,. \tag{66}$$

6. Summary and outlook

We have used the worldline formalism to construct a master formula for the N-photon amplitudes in a general plane-wave background field, for both scalar and spinor QED. In the scalar QED case, our approach is not essentially different from the one used in [42] in the vacuum polarisation case, while in spinor QED our use of an appropriately chosen worldline Green's function for the evaluation of the Feynman spin factor should simplify the algebra at higher N. As usual in applications of the worldline formalism to QED, the formalism unifies the scalar and spinor cases in the sense that any spinor QED calculation yields the corresponding scalar QED quantity as a side result. Similarly to the well-tested master formula for the N-photon amplitudes in a constant field, we expect this new master formula to substantially reduce the algebraic part of the work in this type of calculation, although of course the final parameter integrals will be of the same type as the ones encountered by more standard methods, which usually can be computed only by numerical means. Moreover, starting with the four-photon case the more global parametrization of the worldline formalism comes into play, that combines the various propagators in the loop and makes it possible to avoid the break-up of the amplitude into partial amplitudes with a fixed ordering of the photons. In the vacuum and constant-field cases, it has turned out to be possible to further optimise the calculation of the N-photon amplitudes in the worldline formalism using a certain integration-by-parts algorithm that exhibits the underlying worldline supersymmetry [28,29,33,46,47]; further work will be required to see whether this strategy can be extended to the plane-wave case.

In a more extensive publication, we will give a more detailed derivation, including an alternative approach using the worldline super-formalism for the spinor QED case, and explore the $N=3$ and $N=4$ cases. It should also be interesting to generalise the mapping of worldline averages to spacetime averages, introduced in [42], to the N-point case. Also under consideration is the extension of the formalism to the open-line case, i.e. the photon-dressed scalar and spinor propagators in a plane-wave background.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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