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On Aggregating Uncertain Information by Type-2 OWA Operators for Soft Decision Making

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Yager's ordered weighted averaging (OWA) operator has been widely used in soft decision making to aggregate experts' individual opinions or preferences for achieving an overall decision. The traditional Yager's OWA operator focuses exclusively on the aggregation of crisp numbers. However, human experts usually tend to express their opinions or preferences in a very natural way via linguistic terms. Type-2 fuzzy sets provide an efficient way of knowledge representation for modeling linguistic terms. In order to aggregate linguistic opinions via OWA mechanism, we propose a new type of OWA operator, termed type-2 OWA operator, to aggregate the linguistic opinions or preferences in human decision making modeled by type-2 fuzzy sets. A Direct Approach to aggregating interval type-2 fuzzy sets by type-2 OWA operator is suggested in this paper. Some examples are provided to delineate the proposed technique.

1. INTRODUCTION

In the real world, decision making is one of the most significant and omnipresent human activities in business, manufacturing, service etc. Existing decision making paradigms include multiexpert decision making (i.e., group decision making), multicriteria decision making and multiexpert multicriteria decision making. All of these approaches require an aggregation operation. The objective of aggregation is to combine individual experts' preferences or criteria into an overall one in a proper way so that the final result of aggregation takes into account in a given fashion all the individual contributions.[1,2] It has become a subject of intensive research due to

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its practical and academic significance. However, the majority of the existing aggregation operators focus on aggregating crisp numbers, but in real-world decision applications human experts exhibit remarkable capability to manipulate perceptions without any measurements and any computations.[3] For example, human experts perceive the distance, size, weight, likelihood, and other characteristics of physical and mental objects in a very natural way via linguistic terms, such as "very long," "big," "very heavy," "good," and so on, when they can not provide exact numbers for expressing vague and imprecise opinions.[4] So the problem about how to effectively aggregate linguistic judgments for decision makers arises and needs to be addressed.

It is known that linguistic terms can be characterized as linguistic variables by type-1 fuzzy sets or type-2 fuzzy sets, where type-1 fuzzy sets are the traditional fuzzy sets proposed by Zadeh[5] in 1965, type-2 fuzzy sets were proposed by Zadeh[4] later in 1975, and extensively investigated in the recent period.[6–9]

Previous research has proposed various approaches to aggregating linguistic information,[10,11,13–16] in which type-1 fuzzy sets are used to model the uncertain information. To aggregate the uncertain information modeled by type-1 fuzzy sets, two main schemes have been proposed. The first scheme is to work directly on linguistic labels without considering the (mathematical) expression of the linguistic terms. The only requirement of this scheme is that these linguistic labels should satisfy an order relation. Bordogna et al.[11] proposed a linguistic modeling of consensus in group decision making, in which both experts' evaluations of alternatives and degree of consensus are expressed linguistically. In this approach, the overall linguistic performance evaluation is computed by extending Yager's OWA operator.[12] Another method defined in Refs. 13 and 14 integrates the OWA operator[12] and a convex combination method of linguistic labels. One advantage of such a scheme lies in its high computing efficiency due to its symbolic aggregation in nature. However, the precision of the linguistic operations is an issue: in some cases, this scheme may yield a solution set with multiple alternatives for decision makers to choose, rather than a single one. Another issue is that most of the existing methods based on this scheme use the traditional OWA operator in nature which aims at aggregating crisp numbers. The second scheme of aggregating uncertain information is via operations performed on their associated type-1 fuzzy membership functions. Zimmermann and Zysno developed a family of compensatory operators for aggregating type-1 fuzzy sets by combining a t-norm and a t-conorm to produce certain compensation between criteria.[17,18] This family of compensatory operators has been extended to aggregate weighted fuzzy sets in heterogeneous decision making problems,[19] in which different experts were assigned different importance weights in the form of crisp numbers. Meyer and Roubens[20] proposed a fuzzified Choquet integral to aggregate type-1 fuzzy numbers (normal convex type-1 fuzzy sets) based on a Mobious transform of a fuzzy measure, Yang et al.[21] suggested a different version of fuzzified Choquet integral for fuzzy-valued integrands. The major advantage of using the Choquet integral lies in that it can provide a profound theoretical analysis and background, but it suffers from the serious drawback of needing to assign real values to the importance of all possible combinations.[20] Moreover, there is a common problem with the above approaches in the two schemes including the fuzzified Choquet integral approaches: the importance weights for different experts are assumed to be precise numerical values. This assumption implies that uncertain linguistic labels are aggregated in terms of certain precise crisp weights rather than uncertain quantities.

Recently, Zhou et al.[16] suggested a new type of OWA operator, *type-1 OWA operator*, to aggregate the uncertain information with uncertain weights via OWA mechanism, in which the aggregated objects and importance weights are all modeled as type-1 fuzzy sets. Type-1 OWA operators have been used to aggregate non-stationary fuzzy sets for breast cancer decision supports,[22] and have the potential for improving fuzzy model interpretability/transparency and parsimony of fuzzy models.[23–25]

However, few efforts have been made to aggregate type-2 fuzzy sets, even though type-2 fuzzy set is claimed to provide a richer knowledge representation and approximate reasoning for computing with words and modeling human perception than type-1 fuzzy sets do.[26–28] Wu and Mendel extended the fuzzy weighted average to the linguistic weighted average by using interval type-2 fuzzy sets (IT2FS) instead of type-1 fuzzy sets to model the weights in aggregation.[29] In this paper, we suggest another way of aggregating type-2 fuzzy sets, i.e., to generalize Yager's OWA operator as an aggregation operator for type-2 fuzzy sets. To this end, a new type of OWA operator, *type-2 OWA operator*, is proposed in this paper. A Direct Approach to aggregation of interval type-2 fuzzy sets by type-2 OWA operator is suggested, and some open problems induced from type-2 OWA operators are raised.

It is pinpointed that Yager's OWA operator is a nonlinear aggregation operator while weighted averaging operator is linear. As a result, the proposed type-2 OWA operator in aggregating type-2 fuzzy sets is significantly different from the type-2 fuzzy weighted average operator.[29]

2. REVIEW OF YAGER'S OWA OPERATOR AND TYPE-2 FUZZY SETS

2.1. Yager's OWA Operator

Since Yager introduced the order weighted averaging (OWA) scheme,[12] many OWA based aggregation strategies have been widely investigated and achieved many successful applications in a wide variety of fields, such as decision making,[12–15,19,30,31] fuzzy control,[32,33] linguistic summaries,[34,35] market analysis,[36] and image compression.[37]

DEFINITION 1. *A Yager's OWA operator of n dimension space is a function $\phi : R^n \to R$, that is associated with a set of weights in a vector $w = (w_1, \dots, w_n)^T$ with $w_i \in [0,\ 1]$ and $\sum_{i=1}^{n} w_i = 1$, and aims to aggregate a list of values $a_1, \dots, a_n$ in the following way,*

$$\phi(a) = \phi(a_1, \dots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)} \tag{1}$$

where $\sigma : \{ 1, \ldots, n \} \to \{ 1, \ldots, n \}$ *is a permutation function such that* $a_{\sigma(i)} \geq a_{\sigma(i+1)}, \ \forall\, i = 1, \ldots, n-1$, *i.e.,* $a_{\sigma(i)}$ *is the ith highest value in the set* $\{a_1, \ldots, a_n\}$.

Generally speaking, the OWA operator based aggregation consists of three steps:

- The first step is to reorder the input arguments in descending order, in particular the element a_i is not associated with a particular weight w_i but rather w_i is associated with a particular ordered position of an aggregated element.
- The second step is to determine the weights for the operator.
- Finally, the OWA weights are used to aggregate these re-ordered arguments.

Among the three steps, the first step introduces a nonlinearity into the aggregation process by reordering the input arguments, which make the Yager's OWA operator significantly different from the linear aggregation operator-weighted averaging operator. In practice, an OWA operator ϕ is determined by its associated importance weights $w_1, \ldots, w_n$, so strictly speaking, ϕ should be denoted as $\phi_{w_1 \cdots w_n}$, but for convenience, we use the notation ϕ unless otherwise stated.

Example: Assume ϕ is an OWA operator of size $n = 4$ with associated weights $w = (0.2, 0.3, 0.1, 0.4)^T$. The ϕ is to aggregate the elements of $a = (0.8, 1.0, 0.3, 0.4)^T$ as follows:

$$\phi(a) = w^T \cdot (1.0, 0.8, 0.4, 0.3)^T = 0.2 \times 1.0 + 0.3 \times 0.8 + 0.1 \times 0.4 + 0.4 \times 0.3 = 0.6$$

It is not difficult to prove that "min" and "max" operators can be achieved by Yager's OWA operator via setting $w_* = (0, 0, \ldots, 0, 1)^T$ and $w^* = (1, 0, \ldots, 0, 0)^T$ respectively. The "min" and "max" in the family of Yager's OWA operators can represent the connectives "and" and "or." One property of compensative connectives is that a higher degree of satisfaction of one criterion can compensate for a lower degree of satisfaction of another criterion. The Yager's OWA operators can vary continuously from the "and" (min) to "or" (max) aggregation.

2.2. Type-2 Fuzzy Sets

A type-2 fuzzy set, denoted as $\tilde{A}$, can be formally expressed[6] by a two-dimensional membership function $\mu_{\tilde{A}} : X \times [0, 1] \to [0, 1]$,

$$\tilde{A} = \{((x, u),\ \mu_{\tilde{A}}(x, u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0,\ 1]\,\} \tag{2}$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$, $J_x \triangleq \{u \mid \mu_{\tilde{A}}(x, u) > 0\} \subseteq [0, 1]$ is called the primary membership of x. $A_x \triangleq \mu_{\tilde{A}}(x, \cdot)$, called the secondary membership function (slice), is a type-1 fuzzy set defined on J_x with membership function:

$$\mu_{A_x}(u) = \mu_{\tilde{A}}(x, u) \tag{3}$$

where $u \in J_x \subseteq [0, 1]$. A type-2 fuzzy set can be expressed by its slices in the following way:

$$\tilde{A} = \{(x,\ A_x) \mid \forall x \in X\} \tag{4}$$

Type-2 fuzzy sets can be viewed as a way of characterizing higher level of uncertainty.

The union of all primary memberships, $\cup_{x \in X} J_x$, is called the footprint of uncertainty (FOU),[26] i.e., $FOU(\tilde{A}) = \cup_{x \in X} J_x$. The FOU defines a bounded region of uncertainty in the primary memberships of a type-2 fuzzy set $\tilde{A}$.

When $\mu_{\tilde{A}}(x, u) = 1,\ \forall\ u \in J_x, \forall\ x \in X$, the type-2 fuzzy sets are called *interval type-2 fuzzy sets* (IT2FSs). Because all the memberships in an interval type-2 set are unity, an interval type-2 set can be represented just by its $FOU(\tilde{A})$.[6] Actually, the two end-points of $FOU(\tilde{A})$ at each point are associated with two type-1 membership functions, referred to as the upper and lower membership functions, which are bounds for $FOU(\tilde{A})$. Interval type-2 fuzzy sets reflecting uniform uncertainty at the primary memberships of x are the most widely used type-2 fuzzy sets to date.

The secondary membership functions of a type-2 fuzzy set are type-1 fuzzy sets. In type-2 fuzzy modeling, two important operators-*meet* and *join* of type-1 fuzzy sets[38,39] are often used for the operations on the secondary membership functions of type-2 fuzzy sets. Given two type-1 fuzzy sets A and B, the meet and join operations of A and B are defined as follows.

The meet of A and B, $A \sqcap B$, is defined as

$$\mu_{A \sqcap B}(z) = \sup_{\substack{x \wedge y = z \\ x \in D_A, y \in D_B}} \mu_A(x) * \mu_B(y) \tag{5}$$

The join of A and B, $A \sqcup B$, is defined as

$$\mu_{A \sqcup B}(z) = \sup_{\substack{x \vee y = z \\ x \in D_A, y \in D_B}} \mu_A(x) * \mu_B(y) \tag{6}$$

where $D_A \subseteq X$ and $D_B \subseteq X$ represent the domains of A and B, respectively, $*$ is a t-norm operator, $\wedge$ represents the minimum operation and $\vee$ represents the maximum operation.

3. DEFINITIONS OF TYPE-2 OWA OPERATOR FOR AGGREGATING TYPE-2 FUZZY SETS

Type-2 fuzzy sets provide efficient way of modeling uncertain information and experts' preferences in soft decision making. The motivation for suggesting type-2 OWA operators is to aggregate the linguistic variables modeled as type-2 fuzzy sets via an OWA mechanism.

3.1. Definition

Let $\tilde{F}(X)$ be the set of type-2 fuzzy sets defined on the domain of discourse X, *i.e.* $\tilde{F}(X) = \{\tilde{A} | \tilde{A}$ is type-2 fuzzy set on X$\}$. Based on Zadeh's Extension Principle, in the following we extend Yager's OWA operator and define the type-2 OWA operator for aggregating type-2 fuzzy sets.

DEFINITION 2. *Given n linguistic weights $\{\tilde{W}_i\}_{i=1}^n$ in the form of type-2 fuzzy sets defined on the domain of discourse $U = [0,\ 1]$, a type-2 OWA operator is a mapping $\tilde{\Phi}$,*

$$\tilde{\Phi} : \ \tilde{F}(X) \times \cdots \times \tilde{F}(X) \to \tilde{F}(X)$$

$$(\tilde{A}_1, \ldots, \tilde{A}_n) \mapsto \tilde{G}$$

that is associated with $\{\tilde{W}_i\}_{i=1}^n$ to aggregate the type-2 fuzzy sets $\{\tilde{A}_i\}_{i=1}^n \subset \tilde{F}(X)$. Each slice of the aggregating result, $\tilde{G}$, is defined as

$$G_x = \bigsqcup_{\substack{\sum_{i=1}^n \bar{w}_i a_{\sigma(i)} = x \\ w_i \in U, a_i \in X}} W_{1,\ w_1} \otimes \cdots \otimes W_{n,\ w_n} \otimes A_{1,\ a_1} \otimes \cdots \otimes A_{n,\ a_n} \tag{7}$$

in which $W_{i,\ w_i} \triangleq \mu_{\tilde{W}_i}(w_i, \cdot)$ and $A_{i,\ a_i} \triangleq \mu_{\tilde{A}_i}(a_i, \cdot)$ are type-1 fuzzy sets, $\bar{w}_i = w_i / \sum_{i=1}^n w_i$; $\sigma : \{\ 1, \ldots, n\ \} \to \{\ 1, \ldots, n\ \}$ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)},\ \forall\ i = 1, \ldots, n-1$, i.e., $a_{\sigma(i)}$ is the ith largest element in the set $\{a_1, \ldots, a_n\}$; $\sqcup$ is the join operator defined in (6), whereas $\otimes$ is a t-norm operator that applies to type-1 fuzzy sets, for example, $A_{i,\ a_i}$ and $A_{j,\ a_j}$, as follows:

$$\mu_{A_{i,\ a_i} \otimes A_{j,\ a_j}}(r) = \sup_{\substack{s \otimes t = r \\ s \in J_{a_i}, t \in J_{a_j}}} \mu_{\tilde{A}_i}(a_i, s) * \mu_{\tilde{A}_j}(a_j, t) \tag{8}$$

where $*$ is a t-norm operator for crisp numbers and can be different from $\otimes$. Similar operations are performed on $W_{i,\ w_i} \otimes W_{j,\ w_j}$ and $W_{i,\ w_i} \otimes A_{j,\ a_j}$.

It can be seen that the aggregation result of type-2 fuzzy sets by the type-2 OWA (7), $\tilde{G} = \tilde{\Phi}(\tilde{A}_1, \ldots, \tilde{A}_n)$, is a type-2 fuzzy set. However, type-2 OWA operations on general type-2 fuzzy sets are computationally intensive. Fortunately, if the linguistic weights and aggregated objects are IT2FSs, type-2 OWA operations can be greatly simplified. In the following, we derive the IT2FSs-oriented type-2 OWA operator.

3.2. IT2FSs-Oriented Type-2 OWA Operator

First, we have a theorem.

THEOREM 1 . *If the linguistic weights* $\{\tilde{W}_i\}_{i=1}^n$ *and aggregated objects* $\{\tilde{A}_i\}_{i=1}^n$ *are IT2FSs, then the type-2 OWA aggregating result* $\tilde{G} = \Phi\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ *is an IT2FS.*

Proof. Because the linguistic weights $\{\tilde{W}_i\}_{i=1}^n$ and aggregated objects $\{\tilde{A}_i\}_{i=1}^n$ are IT2FSs, so $\mu_{\tilde{W}_i}(w_i, \cdot) \equiv 1$, $\mu_{\tilde{A}_i}(a_i, \cdot) \equiv 1\ \ \forall\ w_i, a_i$. Let C be the type-1 fuzzy set:

$$C_{w_1 \cdots w_n a_1 \cdots a_n} = W_{1,\ w_1} \otimes \cdots \otimes W_{n,\ w_n} \otimes A_{1,\ a_1} \otimes \cdots \otimes A_{n,\ a_n}$$

According to definition (8), for $\sum_{i=1}^n \bar{w}_i a_{\sigma(i)} = x_0$, we have $\mu_{C_{w_1 \cdots w_n a_1 \cdots a_n}}(r) \equiv 1$ $\forall\ r \in J_{x_0}$. Then according to the definition of the join ($\sqcup$) operation (6), $\mu_{\tilde{G}}(x, u) \equiv 1$, where $\tilde{G} = \tilde{\Phi}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ defined in (7), $\ \forall\ u \in J_x$ and $\sum_{i=1}^n \bar{w}_i a_{\sigma(i)} = x$. Hence $\tilde{G}$ is an IT2FS. ■

According to Theorem 1, in the IT2FS-oriented type-2 OWA aggregation we only need to calculate the FOU of $\Phi(\tilde{A}_1, \ldots, \tilde{A}_n)$, i.e. $FOU(\tilde{G}) = \cup_{x \in X} J_x$. Hence given a point x, we need to calculate the primary membership grade J_x of $\tilde{G}$. To this end, the maximum ($\vee$) of two intervals $[b_1,\ c_1]$ and $[b_2,\ c_2]$ is defined as in Ref. 40:

$$[b_1,\ c_1] \vee [b_2,\ c_2] \triangleq \{x_1 \vee x_2 \mid x_1 \in [b_1,\ c_1], x_2 \in [b_2,\ c_2]\} \tag{9}$$

LEMMA 1. [40] *Let* $b_i = [b_i^l,\ b_i^r]$ $(i = 1, \ldots, n)$ *be intervals, then*

$$b_1 \vee \cdots \vee b_n = \left[\vee b_i^l,\ \vee b_i^r\right] \tag{10}$$

It can be seen from (8) that for the IT2FSs $\tilde{A}_i$ and $\tilde{A}_j$, the domain of $A_{i,\ a_i} \otimes A_{j,\ a_j}$ is

$$J_{a_i a_j} = J_{a_i} \otimes\ J_{a_j} \triangleq \left\{s \otimes t\ \middle|\ s \in J_{a_i}, t \in J_{a_j}\right\} \tag{11}$$

then, for the IT2FSs $\tilde{W}_1, \ldots, \tilde{W}_n,\ \tilde{A}_1, \ldots\ \tilde{A}_n$, the domain of $W_{1,\ w_1} \otimes \cdots \otimes W_{n,\ w_n} \otimes A_{1,\ a_1} \otimes \cdots \otimes A_{n,\ a_n}$ is

$$J_{w_1 \cdots w_n a_1 \cdots a_n} = J_{w_1} \otimes \cdots \otimes J_{w_n} \otimes J_{a_1} \otimes \cdots \otimes J_{a_n} \tag{12}$$

Then we have

$$J_x = \bigvee_{\substack{\sum_{i=1}^{n} \bar{w}_i a_{\sigma(i)}=x \\ w_i \in U, a_i \in X}} J_{w_1 \cdots w_n a_1 \cdots a_n} \tag{13}$$

which is called IT2FS-oriented type-2 OWA operator.

Considering the common case of IT2FS, i.e. the primary membership grades of type-2 fuzzy sets are intervals,[26] let $J_{w_i} = [g^l_{i,w_i}, g^r_{i,w_i}]$ and $J_{a_i} = [g^l_{i,a_i}, g^r_{i,a_i}]$. In this case $J_{w_1 \cdots w_n a_1 \cdots a_n}$ is also an interval. Assume

$$J_{w_1 \cdots w_n a_1 \cdots a_n} = \left[J^l_{w_1 \cdots w_n a_1 \cdots a_n}, J^r_{w_1 \cdots w_n a_1 \cdots a_n}\right],$$

then the following Theorem 2 has been proved[26]:

THEOREM 2 . *For t-norm operator $\otimes = min$ or $product$,*

$$J^l_{w_1 \cdots w_n a_1 \cdots a_n} = g^l_{1,w_1} \otimes \cdots \otimes g^l_{n,w_n} \otimes g^l_{1,a_1} \otimes \cdots \otimes g^l_{n,a_n} \tag{14}$$

$$J^r_{w_1 \cdots w_n a_1 \cdots a_n} = g^r_{1,w_1} \otimes \cdots \otimes g^r_{n,w_n} \otimes g^r_{1,a_1} \otimes \cdots \otimes g^r_{n,a_n} \tag{15}$$

Theorem 2 indicates that the left and right end points of the interval $J_{w_1 \cdots w_n a_1 \cdots a_n}$ only depends on the left and right end points of the aggregated intervals separately. Hence, according to Lemma 1, we calculate the left end point and right end point of $J_x = [J^l_x,\ J^r_x]$ as follows respectively:

$$J^l_x = \bigvee_{\substack{\sum_{i=1}^{n} \bar{w}_i a_{\sigma(i)}=x \\ w_i \in U, a_i \in X}} J^l_{w_1 \cdots w_n a_1 \cdots a_n} \tag{16}$$

and

$$J^r_x = \bigvee_{\substack{\sum_{i=1}^{n} \bar{w}_i a_{\sigma(i)}=x \\ w_i \in U, a_i \in X}} J^r_{w_1 \cdots w_n a_1 \cdots a_n} \tag{17}$$

It is noted that if the weights $\tilde{W}_i$ reduce to intervals $\bar{W}_i \subseteq [0, 1]$, the above type-2 OWA aggregation can be simplified further as

$$J^l_x = \bigvee_{\substack{\sum_{i=1}^{n} \bar{w}_i a_{\sigma(i)}=x \\ w_i \in \bar{W}_i, a_i \in X}} J^l_{a_1 \cdots a_n} \tag{18}$$

and

$$J_x^r = \bigvee_{\substack{\sum_{i=1}^n \bar{w}_i a_{\sigma(i)} = x \\ w_i \in \bar{W}_i, a_i \in X}} J_{a_1 \cdots a_n}^r \tag{19}$$

where $J_{a_1 \cdots a_n}^l$ and $J_{a_1 \cdots a_n}^r$ are calculated as

$$J_{a_1 \cdots a_n}^l = g_{1,a_1}^l \otimes \cdots \otimes g_{n,a_n}^l$$

and

$$J_{a_1 \cdots a_n}^r = g_{1,a_1}^r \otimes \cdots \otimes g_{n,a_n}^r$$

3.3. A Special Case: Type-1 OWA Operator

For the IT2FS-oriented type-2 OWA operator, one possible question may arise: what happen to the type-2 OWA operators if the aggregated type-2 fuzzy sets and associated importance weights reduce to type-1 fuzzy sets? Zhou et al. have suggested a named *type-1 OWA operator*[16] to aggregate uncertain information with uncertain weights modeled by type-1 fuzzy sets via OWA mechanism. The following theorem indicates that the type-1 OWA operator is a special case of the proposed IT2FS-oriented type-2 OWA operator.

THEOREM 3 . *In IT2FS-oriented type-2 OWA aggregation, if the linguistic weights $\{\tilde{W}_i\}_{i=1}^n$ and the aggregated objects $\{\tilde{A}_i\}_{i=1}^n$ are type-1 fuzzy sets, then the type-2 OWA operator reduces to a type-1 OWA operator.*

Proof. If the linguistic weights $\{\tilde{W}_i\}_{i=1}^n$ and the aggregated objects $\{\tilde{A}_i\}_{i=1}^n$ are type-1 fuzzy sets, then for $\forall w_1, \ldots, w_n \in U, a_1, \ldots, a_n \in X$, the primary membership grades J_{w_i} and $J_{a_i}, i = 1, \ldots, n$ redcuce to singletons from intervals. As a result, according to equation 13 the primary membership grade of aggregation result at $x = \sum_{i=1}^n \bar{w}_i a_{\sigma(i)}$ reduces to the definition of type-1 OWA operator [16] for aggregating the elements $a_1, \ldots, a_n$ using the weighting points $w_1, \ldots, w_n$. ■

4. A PROCEDURE FOR PERFORMING IT2FSS-ORIENTED TYPE-2 OWA OPERATIONS

Given the linguistic weights $\{\tilde{W}_i\}_{i=1}^n \subset \tilde{F}(U)$, as usual, the domains of X and U needs to be discretized during calculation in order for the associated IT2FSs-oriented type-2 OWA operator to aggregate IT2FSs $\{\tilde{A}_i\}_{i=1}^n \subset \tilde{F}(X)$ on computer. Let the discretized domains be $\hat{X} = \{\hat{x_1}, \ldots, \hat{x_p}\}$ and $\hat{U} = \{\hat{u_1}, \ldots, \hat{u_k}\}$, which are

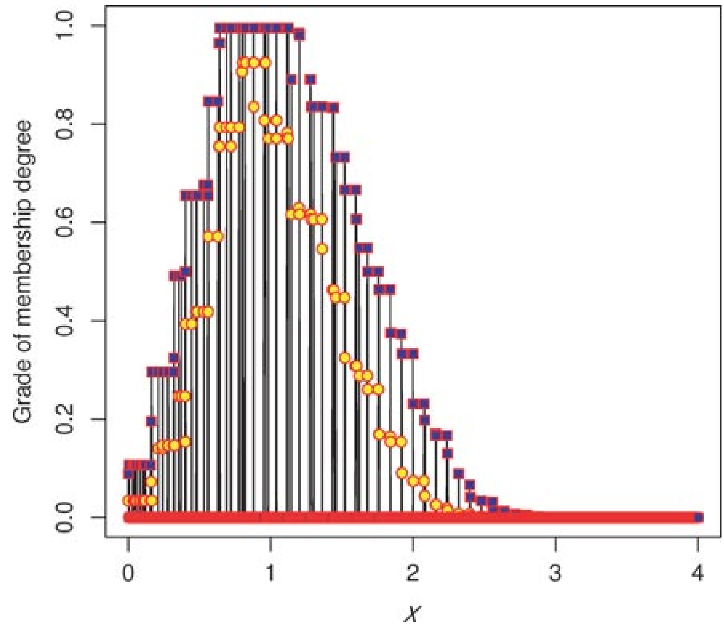

Figure 1. A set generated on the *overpartition* version of discretized X.

partitions of the spaces X and U, respectively. However, $\sum_{k=1}^{n} \bar{w}_i a_{\sigma(i)}$ with all the combinations $(w_1, \ldots, w_n, a_1, \ldots, a_n)$ of weighting points in $\hat{U}$ and aggregating points in $\hat{X}$ may produce another partitioning of X, i.e,

$$\bar{X} = \{\bar{x}_j\} = \left\{ \sum_{k=1}^{n} \bar{w}_i a_{\sigma(i)} \middle| w_i \in \hat{U}, a_i \in \hat{X}, i = 1, \ldots, n \right\} \tag{20}$$

The problem is that $\hat{X} \subseteq \bar{X}$, i.e., the two discretized versions of X may be different. $\bar{X}$ is referred to as *overpartition* given the used $\hat{X}$. As a consequence, the interval type-2 fuzzy set generated on $\bar{X}$, $\bar{G}$, is likely to be unreadable, because for some data points that are in $\bar{X}$ but not in $\hat{X}$, their primary membership grades $J_{\bar{x}_j}$ may not be consistent with the ones of the corresponding nearest points in $\hat{X}$. For example, Figure 1 shows one IT2FS generated on $\bar{X}$. So one should induce the aggregating result, the IT2FS G on $\hat{X}$, from the set on $\bar{X}$. This can be conducted according to the Extension Principle as follows.

The sets $\hat{X}$ and $\bar{X}$ are two partitions of the domain X as shown in Figure 2, in which $\bar{X}$ provides a finer resolution than $\hat{X}$ does. So the data points from the fine partition $\bar{X}$ lying between two neighboring points in the coarse partition $\hat{X}$, for example $\hat{x}_i$ and $\hat{x}_{i+1}$, form one cluster denoted as $\Theta_{\hat{x}_i} \triangleq \{\bar{x}_j | \bar{x}_j \in \bar{X}, \hat{x}_i \leq \bar{x}_j < \hat{x}_{i+1}\}$, in which $\hat{x}_i$ is the cluster prototype. This is analogous to a digital map with different resolutions: by zooming in, we can see a map with fine details, whereas

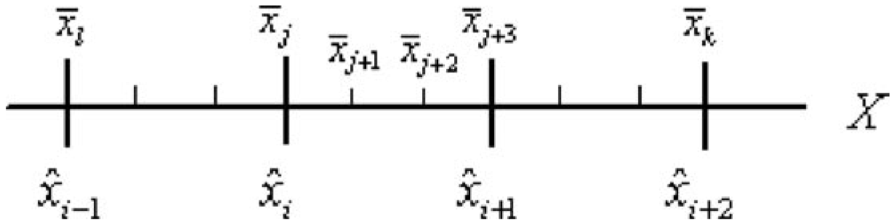

Figure 2. The partitions of the domain X.

by zooming out, all the details are displayed in a point, this point acting as one unit represents all the details behind it. Hence, the whole cluster $\Theta_{\hat{x}_i}$ with the prototype $\hat{x}_i$ is treated as one unit, and all the membership grades of the data points in the $\Theta_{\hat{x}_i}$ are assigned to this unit. Then according to the Extension Principle, the left end point of the primary membership grade of this unit is obtained by maximizing all the available left end points of primary membership grades obtained for this unit, while the right end point of the primary membership grade of this unit is obtained by maximizing all the available right end points of primary membership grades obtained for this unit. Hence, the primary membership grade of the resulting IT2FS G at the prototype point $\hat{x}_i$ is induced as

$$J^l_{\hat{x}_i} = \bigvee_{\bar{x}_j \in \Theta_{\hat{x}_i}} J^l_{\hat{x}_j} \tag{21}$$

and

$$J^r_{\hat{x}_i} = \bigvee_{\bar{x}_j \in \Theta_{\hat{x}_i}} J^r_{\hat{x}_j} \tag{22}$$

Figure 3 shows the resulting fuzzy set induced by applying (21) and (22) to the set depicted in Figure 1. A Direct Approach to IT2FS-oriented type-2 OWA operation is addressed as follows:

Step 1. Initialization

1. Given the linguistic weights $\{\tilde{W}_i\}_{i=1}^n \subseteq \tilde{F}(U)$ in the form of IT2FSs for aggregating IT2FS objects $\{\tilde{A}_i\}_{i=1}^n \subseteq \tilde{F}(X)$.
2. Given the discretized domains of linguistic weights, $\hat{U}$, and that of aggregated objects, $\hat{X}$.
3. Let the initial $\bar{G} = (\bar{X}, \mu_{\bar{G}})$, where $\bar{X} = \emptyset$, $J^l_0 = 0$, $J^r_0 = 0$.

Step 2. Calculate $\bar{G}$.

1. Select $w_1 \in \hat{U}, \ldots, w_n \in \hat{U},\ a_1 \in \hat{X}, \ldots, a_n \in \hat{X}$,
2. Normalize $(w_1, \ldots, w_n)$ as $\bar{w}_i = w_i / \sum_{i=1}^n w_i$

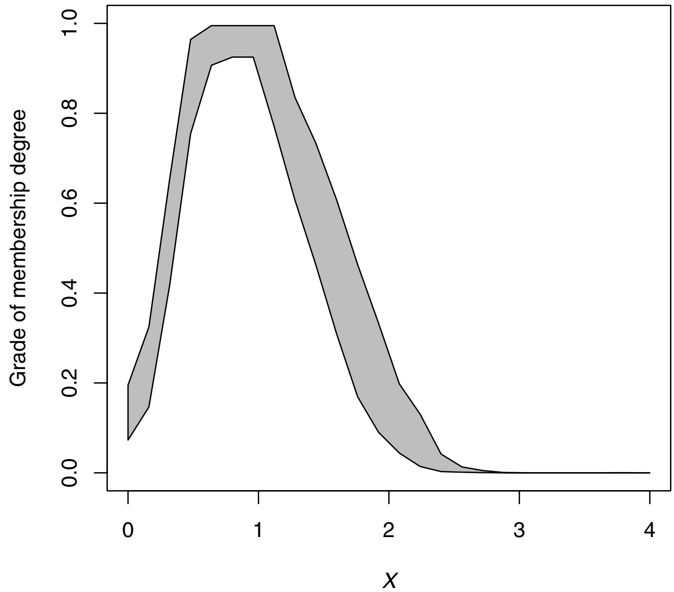

Figure 3. An IT2FS on the $\hat{X}$ induced from the set on the $\bar{X}$ in Figure 1.

3. Perform Yager's OWA operation:

$$\bar{y} = \phi_{\bar{w}_1,\ldots,\bar{w}_n}(a_1, \ldots, a_n)$$

4. Calculate $J^l_{w_1\cdots w_n a_1\cdots a_n}$ and $J^r_{w_1\cdots w_n a_1\cdots a_n}$:

$$J^l_{w_1\cdots w_n a_1\cdots a_n} = g^l_{1,w_1} \otimes \cdots \otimes g^l_{n,w_n} \otimes g^l_{1,a_1} \otimes \cdots \otimes g^l_{n,a_n}$$

$$J^r_{w_1\cdots w_n a_1\cdots a_n} = g^r_{1,w_1} \otimes \cdots \otimes g^r_{n,w_n} \otimes g^r_{1,a_1} \otimes \cdots \otimes g^r_{n,a_n}$$

5. If there exists $\bar{x} \in \bar{X} : \bar{x} = \bar{y}$, then update the potential primary membership grade $J_{\bar{x}}$:

$$J^l_{\bar{x}} \leftarrow \max\left(J^l_{\bar{x}}, J^l_{w_1\cdots w_n a_1\cdots a_n}\right)$$

and

$$J^r_{\bar{x}} \leftarrow \max\left(J^r_{\bar{x}}, J^r_{w_1\cdots w_n a_1\cdots a_n}\right)$$

Otherwise, $\bar{y}$ is added to $\bar{X}$, and the primary membership grade at $\bar{y}$: $J_{\bar{y}} \triangleq J_{w_1\cdots w_n a_1\cdots a_n}$.
6. Go to ***Step 2***-1, and continue until all the weight vectors and aggregating points are selected.

Step 3. Induce the IT2FS G on $\hat{X}$:

$$J^l_{\hat{x}} = \bigvee_{\bar{x}_j \in \Theta_{\hat{x}}} J^l_{\hat{x}_j}$$

and

$$J^r_{\hat{x}} = \bigvee_{\bar{x}_j \in \Theta_{\hat{x}}} J^r_{\hat{x}_j}.$$

5. ILLUSTRATIVE EXAMPLES

In this section, we present some examples of aggregating interval type-2 fuzzy sets by type-2 OWA operators.

The IT2FSs to be aggregated on $X = [0, 4]$ for the first two examples are depicted in Figure 4. In the first example, a type-2 OWA operator with *min* t-norm, $\Phi_{\tilde{W}_1 \tilde{W}_2 \tilde{W}_3}$, is defined by three linguistic weights $\tilde{W}_1$, $\tilde{W}_2$, and $\tilde{W}_3$ in the form of IT2FSs as shown in Figure 5. As usual, the domains of X and U are discretized during calculation. In our case, $\hat{X} = \{0.2 \cdot k | k = 0, \ldots, 20\}$ and $\hat{U} = \{0.05 \cdot k | k = 0, \ldots, 20\}$. Figure 6 shows the overall result of aggregating the IT2FSs depicted in Figure 4 by this type-2 OWA operator. For example, at $x_0 = 0.8$, $J_{0.8} = [0.44, 0.95]$.

In the second example, a type-2 OWA operator is defined by the identical weights with different order as shown in Figure 7, then Figure 8 illustrates the result of applying the Direct Approach to this type-2 OWA operator on aggregating the three IT2FSs depicted in Figure 4.

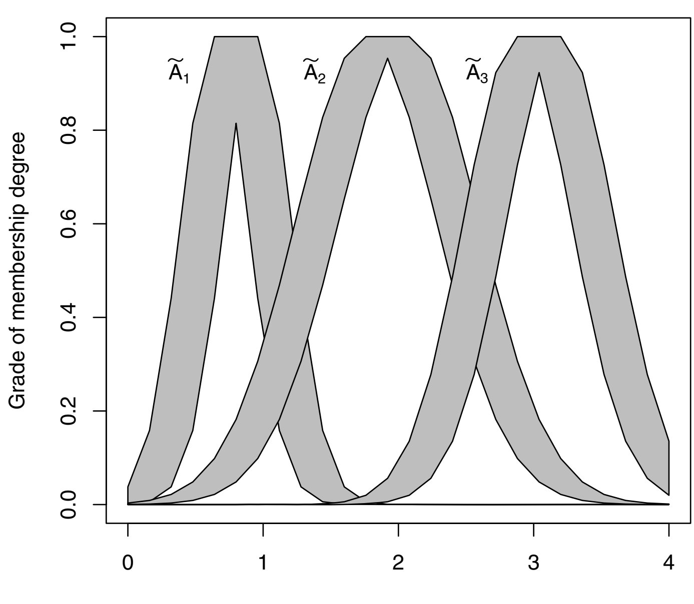

Figure 4. Three interval type-2 fuzzy sets to be aggregated by type-2 OWA operators.

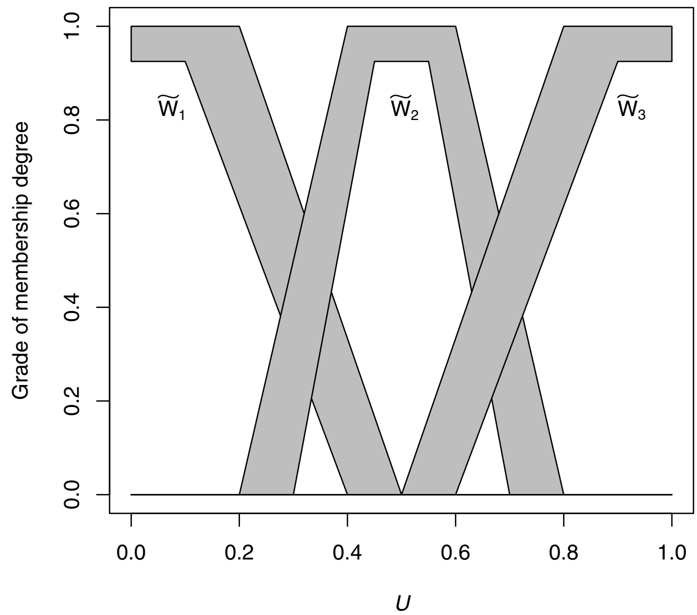

Figure 5. Three linguistic weights defining a type-2 OWA operator.

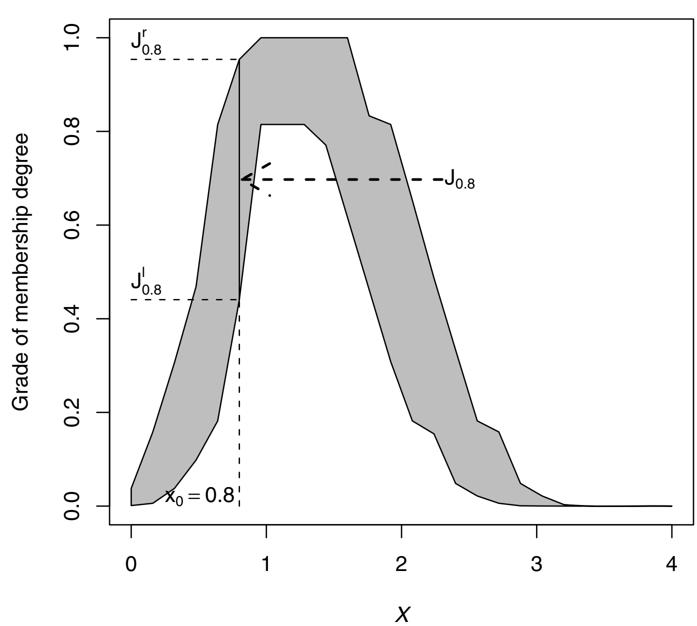

Figure 6. Aggregating result by the type-2 OWA operator with linguistic weights in Figure 5.

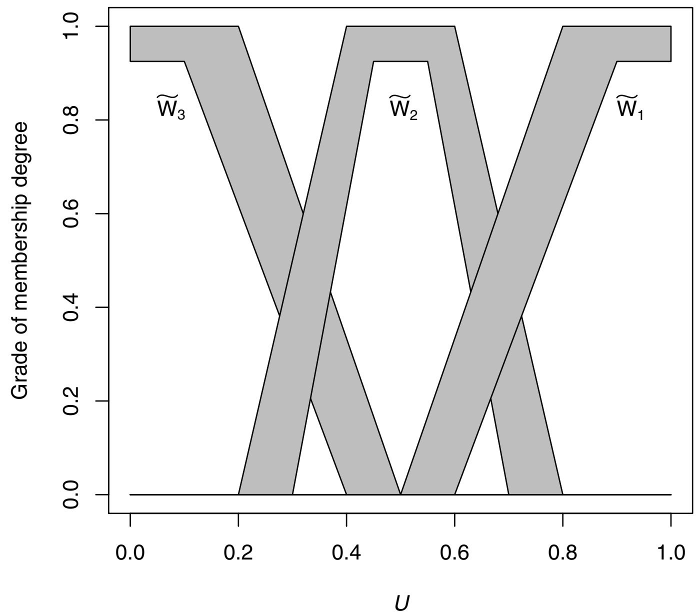

Figure 7. Three linguistic weights defining a type-2 OWA operator.

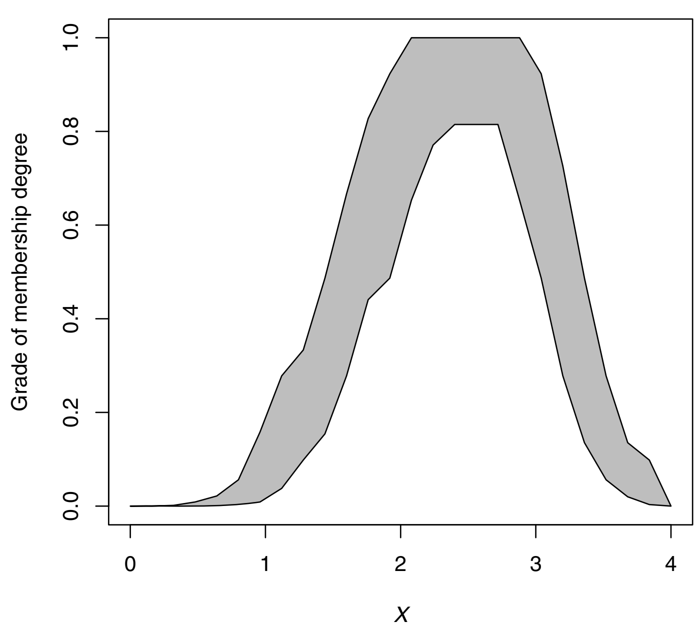

Figure 8. Aggregating result by the type-2 OWA operator with linguistic weights in Figure 7.

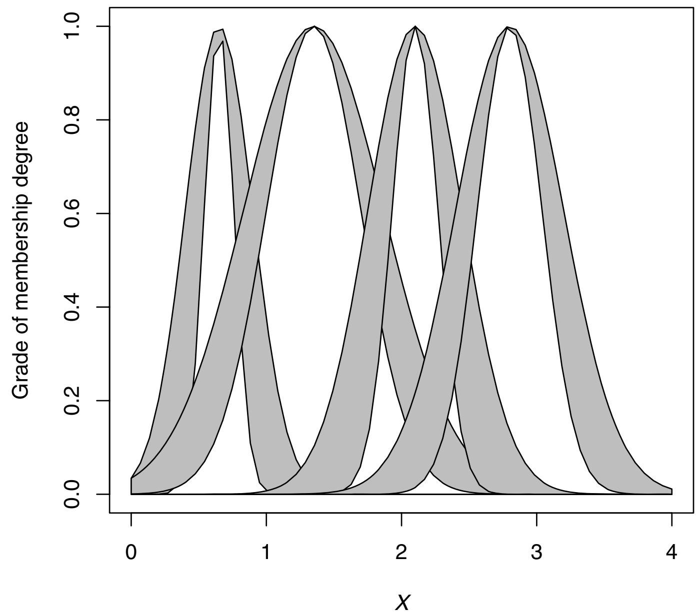

Figure 9. Four aggregated type-2 fuzzy sets.

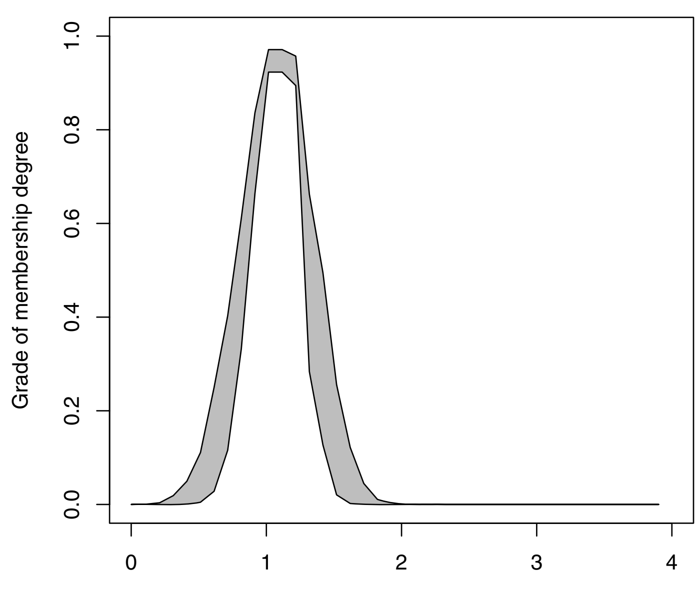

Figure 10. Result of aggregating type-2 fuzzy sets by type-2 OWA operator with interval weights.

Interestingly, in type-2 OWA aggregation the linguistic weights can be used to express semantic meanings. For example, the three IT2FSs shown in the Figure 5 from left to right could be assigned the meanings of "low importance," "fair importance," and "high importance," respectively, producing the result in Figure 6. It can be seen that the resulting IT2FS in Figure 6 is relatively close to the most-left aggregated IT2FS object on the domain X. However, if the first linguistic weight is assigned the meaning "high importance," the second "fair importance" and the last linguistic weight "low importance" as in Figure 7, then the resulting IT2FS obtained via the type-2 OWA operation will move toward to the most-right aggregated IT2FS object on the domain X as indicated in Figure 8. This is not only intuitively plausible but also consistent with the compensative property of Yager's OWA operator[12]: Yager's OWA operators can vary from the "min" (i.e, most-left aggregated object on the domain X) to "max" (i.e, most-right aggregated object on the domain X) aggregation.

In the third example, a type-2 OWA operator defined by four intervals $\bar{W}_1 = [0, 0]$, $\bar{W}_2 = [0.25, 0.4]$, $\bar{W}_3 = [0.875, 0.9]$, and $\bar{W}_4 = [1, 1]$ are used to aggregate four IT2FSs. Figure 10 shows the result of this type-2 OWA operator in aggregating the four IT2FSs depicted in Figure 9.

6. CONCLUSIONS AND DISCUSSIONS

In this paper, a new type of OWA aggregation operator, termed *type-2 OWA operator*, is proposed in the interests of aggregating linguistic information represented by type-2 fuzzy sets in decision making. Moreover, a Direct Approach to type-2 OWA operation on interval type-2 fuzzy sets is suggested.

We believe that the proposed new type of OWA operators will induce some new interesting topics, the immediate ones include how to perform type-2 OWA operation on general type-2 fuzzy sets, how to derive the linguistic weights for type-2 OWA operators etc. Additionally, this new type of OWA aggregation operators would have great potential for being applied to multiexpert decision making and multicriteria decision making.

On the other hand, type 2 fuzzy set provides a richer knowledge representation and approximate reasoning for Zadeh's computing with words paradigm, we believe that type-2 OWA operator would be an efficient technique for computing with words when the words are modeled as type-2 fuzzy sets. These topics certainly merit further research.

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