2011-10

Alpha-Level Aggregation: A Practical Approach to Type-1 OWA Operation for Aggregating Uncertain Information with Applications to Breast Cancer Treatments

Zhou, Shang-Ming

http://hdl.handle.net/10026.1/20370

10.1109/tkde.2010.191
IEEE Transactions on Knowledge and Data Engineering
Institute of Electrical and Electronics Engineers (IEEE)

All content in PEARL is protected by copyright law. Author manuscripts are made available in accordance with publisher policies. Please cite only the published version using the details provided on the item record or document. In the absence of an open licence (e.g. Creative Commons), permissions for further reuse of content should be sought from the publisher or author.
Alpha-Level Aggregation: A Practical Approach to Type-1 OWA Operation for Aggregating Uncertain Information with Applications to Breast Cancer Treatments

Shang-Ming Zhou, Member, IEEE, Francisco Chiclana, Robert I. John, Senior Member, IEEE, and Jonathan M. Garibaldi

Abstract—Type-1 OWA operator provides us with a new technique for directly aggregating uncertain information with uncertain weights via OWA mechanism in soft decision making and data mining, in which uncertain objects are modelled by fuzzy sets. The Direct Approach to performing type-1 OWA operation involves high computational overhead. In this paper, we define a type-1 OWA operator based on the \( \alpha \)-cuts of fuzzy sets. Then we prove a Representation Theorem of type-1 OWA operators, by which type-1 OWA operators can be decomposed into a series of \( \alpha \)-level type-1 OWA operators. Furthermore, we suggest a fast approach, called Alpha-Level Approach, to implementing the type-1 OWA operator. A practical application of type-1 OWA operators to breast cancer treatments is addressed. Experimental results and theoretical analyses show that: (i) the Alpha-Level Approach with linear order complexity can achieve much higher computing efficiency in performing type-1 OWA operation than the existing Direct Approach, and (ii) the type-1 OWA operators exhibit different aggregation behaviours from the existing fuzzy weighted averaging (FWA) operators. (iii) the type-1 OWA operators demonstrate the ability to efficiently aggregate uncertain information with uncertain weights in solving real-world soft decision making problems.

Index Terms—OWA operators, aggregation, fuzzy sets, type-1 OWA operators, Alpha-cuts, Alpha level, uncertain information, soft decision making, breast cancer treatments.

1 INTRODUCTION

A G G R E G A T I O N operation is not only an important research topic in knowledge and data engineering [1]–[5], but also one of the most important steps in dealing with multi-expert decision making, multi-criteria decision making and multi-expert multi-criteria decision making [6]–[8]. The objective of aggregation is to combine individual sources of information into an overall one in a proper way so that the final result of aggregation can take into account all the individual contributions [9]. Currently, at least 90 different families of aggregation operators have been studied [9]–[19]. Amongst them, the Ordered Weighted Averaging (OWA) operator proposed by Yager [18] is one of the most widely used, with many successful applications achieved in areas such as: decision making [6], [8], [12], [21], [22], fuzzy control [23], [24], market analysis [25], image compression [26]. However, the majority of the existing aggregation operators, including the OWA one, focus exclusively on aggregating crisp numbers. As a matter of fact, inherent subjectivity, imprecision and vagueness in the articulation of opinions in real world decision applications make human experts exhibit remarkable capability to manipulate perceptions without any measurements [20]. In these cases, the use of linguistic terms instead of precise numerical values seems to be more adequate in dealing with vague or imprecise information or to express experts’ opinions on qualitative aspects that cannot be assessed by means of quantitative values [6], [21]. Thus, techniques for aggregating uncertain information rather than precise crisp values are in high demand, which motivated us to suggest a new OWA operator, called type-1 OWA operator [27]. The type-1 OWA operator is able to aggregate linguistic terms represented as fuzzy sets via OWA mechanism, and a Direct Approach has been suggested to perform type-1 OWA operation [27]. Interestingly, some well-known existing aggregation operators, such as Yager’s OWA operator, the join and the meet operators of fuzzy sets [41], [42] are special cases of this type-1 OWA operator [28].

Different ways of aggregating linguistic assessments, including the ones that follow the way of fuzzifying Yager’s OWA operators, have been proposed in literature [13], [21], [29]–[35]. A detailed review of the state-of-the-art research in this topic can be found in [27].
and [28]. The type-1 OWA operator is different from these existing methods. For example, an approach to OWA aggregation with interval weights and interval inputs was suggested in [32], in which two definitions of aggregating interval arguments with interval weights based on the rank of intervals via probabilistic measures were given. However, different probabilistic distributions could lead to different re-orderings of the inputs and consequently different outputs could be derived using this approach. Ahn’s method focused on the use of the uniform distribution, although no evidence is provided to support that this type of distribution should always be used [32]. The type-1 OWA operator does not suffer from the aforementioned drawback as it is defined according to Zadeh’s Extension Principle, only the issues of reordering of crisp values are involved and therefore it avoids dealing with the ranking of fuzzy sets/interval aggregations. Moreover, in this paper, we propose an α-level type-1 OWA operator and prove that the Alpha-Level Approach can lead to its equivalence one obtained by the Extension Principle. There is no evidence to support that Ahn’s method has such property.

To the best of our knowledge, the research work by Mitchell and Schaefer [33], and the research on fuzzified Choquet integral [34], [35] may be the most relevant to our research on type-1 OWA operators. Mitchell and Schaefer also applied Zadeh’s Extension Principle to Yager’s OWA operator, but their approach focused on the ordering of fuzzy sets during the aggregation process. The type-1 OWA operator avoids ordering fuzzy sets. Yager’s OWA operator is treated as a non-linear function and is fuzzified to the case of having fuzzy sets as inputs in a type-1 OWA operator. As for the research on fuzzified Choquet integrals, the existing approaches only consider the aggregation of fuzzy sets with crisp weights, while the type-1 OWA operator is able to aggregate fuzzy sets with fuzzy weights as well.

Another widely investigated fuzzified aggregation operators, the fuzzy weighted averaging (FWA) operators [36]–[38], can also be applied to the aggregation of fuzzy sets with fuzzy weights. Noteworthy, Yager’s OWA operator is a non-linear aggregation operator, while the weighted averaging operator is linear. Therefore, the type-1 OWA operator is significantly different from the FWA operator [27], [28].

However, the Direct Approach to performing type-1 OWA operation suggested in [27] involves high computational load, which inevitably curtails further applications of the type-1 OWA operator to real world decision making. This paper focuses on how to achieve a high computing efficiency in performing type-1 OWA operations for aggregating uncertain information with uncertain weights, where these uncertain objects are modelled by fuzzy sets. To this end, the α-level type-1 OWA operator is defined using the α-cuts of fuzzy sets. Moreover, a fast approach to type-1 OWA operation, called Alpha-Level Approach, with detailed theoretical analyses is addressed. Promisingly, the complexity of this Alpha-Level Approach is of linear order, so it can be used in real time soft decision making, database integration and information fusion that involve aggregation of uncertain information.

This paper is organised as follows. Section 2 describes the definition of α-level type-1 OWA operator. Section 3 proposes the fast approach to implementing the type-1 OWA operation. The complexity of the Direct Approach and the fast Alpha-Level Approach are analysed in Section 4. Section 5 extensively evaluates the computing efficiency of the proposed approach including a practical application of type-1 OWA operators to breast cancer treatments. Finally, conclusions and discussion are presented in Section 6.

2 DEFINITION OF TYPE-1 OWA OPERATORS BASED ON α-CUTS OF FUZZY SETS

As a generalisation of Yager’s OWA operator and based on Zadeh’s Extension Principle, the type-1 OWA operator is defined to aggregate uncertain information with uncertain weights, when both are modelled as fuzzy sets.

First, let \( F(X) \) be the set of fuzzy sets with domain of discourse \( X \), a type-1 OWA operator is defined as follows [27], [28]:

**Definition 1.** Given \( n \) linguistic weights \( \{W_i\}_{i=1}^n \) in the form of fuzzy sets defined on the domain of discourse \( U = [0,1] \), a type-1 OWA operator is a mapping, \( \Phi \),

\[
\Phi : F(X) \times \cdots \times F(X) \rightarrow F(X)
\]

\[
(A_1, \cdots, A_n) \mapsto Y
\]

such that

\[
\mu_Y(y) = \sup_{\sum_{k=1}^n \bar{w}_i a_{\sigma(i)} = y} \left( \mu_{W_1}(w_1) \land \cdots \land \mu_{W_n}(w_n) \right)
\]

\[
\land \mu_{A_1}(a_1) \land \cdots \land \mu_{A_n}(a_n)
\]

where \( \bar{w}_i = \frac{w_i}{\sum_{i=1}^n w_i} \), and \( \sigma : \{1, \cdots, n\} \to \{1, \cdots, n\} \) is a permutation function such that \( a_{\sigma(i)} \geq a_{\sigma(i+1)} \), \( \forall i = 1, \cdots, n-1 \), i.e., \( a_{\sigma(i)} \) is the \( i \)th highest element in the set \( \{a_1, \cdots, a_n\} \).

From the above definition, it can be seen that the aggregation result \( \Phi(A_1, \cdots, A_n) = Y \in F(X) \) is a fuzzy set defined on \( X \). However, implementation of type-1 OWA operation in aggregating a group of fuzzy sets is not straightforward and easy. A Direct Approach to performing type-1 OWA operation has been suggested in [27], but it involves high computational load.

In the interests of improving computing efficiency of type-1 OWA aggregation, in this section we describe an alternative way of defining type-1 OWA operators based on α-cuts of fuzzy sets. To do this, we first introduce the concept of the α-level type-1 OWA operator guided by α-cuts of fuzzy weights:
Definition 2. Given the $n$ linguistic weights $\{W^i\}_{i=1}^n$ in the form of fuzzy sets defined on the domain of discourse $U = [0, 1]$, then for each $\alpha \in [0, 1]$, an $\alpha$-level type-1 OWA operator with $\alpha$-level sets $\{W^i_\alpha\}_{i=1}^n$ to aggregate the $\alpha$-cuts of fuzzy sets $\{A^i\}_{i=1}^n$ is given as

$$\Phi_\alpha (A^1, \ldots, A^n) = \bigg\{ \frac{\sum_{i=1}^n \omega_i \sigma^{-1} a_i}{\sum_{i=1}^n \omega_i} \bigg| a_i \in A^i, i = 1, \ldots, n \bigg\}$$

(3)

where $W^i_\alpha = \{ w_i \mid \mu_{W^i}(w) \geq \alpha \}$, $A^i_\alpha = \{ x \mid \mu_{A^i}(x) \geq \alpha \}$, and $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)} \forall i = 1, \ldots, n-1$, i.e., $a_{\sigma(i)}$ is the $i$th largest element in the set $\{a_1, \ldots, a_n\}$.

According to the Representation Theorem of fuzzy set [40], the $\alpha$-level sets $\Phi_\alpha (A^1_\alpha, \ldots, A^n_\alpha)$ obtained via Definition 2 can be used to construct the following fuzzy set

$$G = \bigcup_{0 \leq \alpha \leq 1} \alpha \Phi_\alpha (A^1, \ldots, A^n)$$

(4)

with membership function

$$\mu_G(x) = \bigvee_{\alpha \leq \Phi_\alpha (A^1, \ldots, A^n)} \alpha$$

(5)

From the above definition, it can be seen that the $\alpha$-level type-1 OWA operator is to aggregate the $\alpha$-cuts of fuzzy sets $\{A^i\}_{i=1}^n$ with the $\alpha$-cuts of fuzzy set weights $\{W^i\}_{i=1}^n$. Given the fact that the $\alpha$-cuts of fuzzy numbers (i.e., normal and convex fuzzy sets on the domain of real numbers $\mathbb{R}$) are intervals, the $\alpha$-level type-1 OWA operator actually provides a way of aggregating uncertain arguments with uncertain weights to some extent as Ahn’s method did [32]. However, we proceed further to aggregate uncertain information modelled by fuzzy sets.

First, the two apparently different aggregation operators in (2) and (5), defined according to Zadeh’s Extension Principle and the $\alpha$-cut of fuzzy sets respectively, are equivalent as it is proved in the following:

Theorem 1. Given the $n$ linguistic weights $\{W^i\}_{i=1}^n$ in the form of fuzzy sets defined on the domain of discourse $U = [0, 1]$, and the fuzzy sets $A^1, \ldots, A^n$, then we have that

$$Y = G$$

where $Y$ is the aggregation result defined in (2) and $G$ is the result defined in (4).

Proof: We need to prove that for any fuzzy sets $A^1, \ldots, A^n$ and $\alpha \in [0, 1]$,

$$Y_\alpha = \Phi_\alpha (A^1_\alpha, \ldots, A^n_\alpha)$$

To prove $Y_\alpha \subseteq \Phi_\alpha (A^1_\alpha, \ldots, A^n_\alpha)$, we note that $\forall y \in Y_\alpha$, there exist $w_1, \ldots, w_n \in \mathbb{R}$, and $a_1, \ldots, a_n \in X$ such that $y = \sum_{i=1}^n w_i a_{\sigma(i)}$, where $w_i = \frac{w_i}{\sum_{i=1}^n w_i}$, and

$$\alpha \leq \mu_{W^1}(w_1) \land \cdots \land \mu_{W^n}(w_n) \land \mu_{A^1}(a_1) \land \cdots \land \mu_{A^n}(a_n)$$

Thus, we have that $\alpha \leq \mu_{Y_\alpha}(w_i)$ and $\alpha \leq \mu_{A^i}(a_i)$ $\forall i$, i.e. $w_i \in W^i_\alpha, a_i \in A^i_\alpha$, $i = 1, \ldots, n$. As a result, $y \in \Phi_\alpha (A^1_\alpha, \ldots, A^n_\alpha)$ according to Definition 2.

To prove that $\Phi_\alpha (A^1_\alpha, \ldots, A^n_\alpha) \subseteq Y_\alpha$, we note that $\forall y \in \Phi_\alpha (A^1_\alpha, \ldots, A^n_\alpha)$, there exist $\hat{w}_1 \in W^i_\alpha, \hat{w}_n \in W^n_\alpha$ and $\hat{a}_1 \in A^1_\alpha, \ldots, \hat{a}_n \in A^n_\alpha$ such that $y = \sum_{i=1}^n \hat{w}_i \hat{a}_{\sigma(i)}$, where $\hat{w}_i = \frac{\hat{w}_i}{\sum_{i=1}^n \hat{w}_i}$. Because $\alpha \leq \mu_{W^i}(\hat{w}_i)$ and $\alpha \leq \mu_{A^i}(\hat{a}_i)$ $\forall i$, then

$$\alpha \leq \mu_{Y_\alpha}(\hat{w}_i) \land \cdots \land \mu_{Y_\alpha}(\hat{w}_n) \land \mu_{A^1}(\hat{a}_1) \land \cdots \land \mu_{A^n}(\hat{a}_n)$$

As a result,

$$\alpha \leq \mu_{Y_\alpha}(\hat{w}_1) \land \cdots \land \mu_{Y_\alpha}(\hat{w}_n) \land \mu_{A^1}(\hat{a}_1) \land \cdots \land \mu_{A^n}(\hat{a}_n)$$

Hence, $y \in Y_\alpha$.

Theorem 1 is called the Representation Theorem of type-1 OWA operators. According to this Representation Theorem, type-1 OWA operators can be decomposed into a series of $\alpha$-level type-1 OWA operators. It provides an effective tool for performing type-1 OWA operations.

It is noted that in fuzzy sets based soft decision making, linguistic terms are commonly modelled by fuzzy numbers. In what follows, we will focus on these type of fuzzy sets, unless otherwise stated.

When the linguistic weights and the aggregated objects are fuzzy number, the $\alpha$-level type-1 OWA operator produces closed intervals, as the following theorem states:

Theorem 2. Let $\{W^i\}_{i=1}^n$ be fuzzy numbers on $U = [0, 1]$ and $\{A^i\}_{i=1}^n$ be fuzzy numbers on $\mathbb{R}$. Then for each $\alpha \in [0, 1]$, $\Phi_\alpha (A^1_\alpha, \ldots, A^n_\alpha)$ is a closed interval.

Proof: Firstly, we have that

$$y(w_1, \ldots, w_n, a_1, \ldots, a_n) = \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i}$$

is a continuous function of $w_1, \ldots, w_n, a_1, \ldots, a_n$. Because

$$a_{\sigma(i)} \geq \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i} \geq a_{\sigma(n)}$$

we have that $y(w_1, \ldots, w_n, a_1, \ldots, a_n)$ is also a bounded function.

Secondly, because $\{W^i\}_{i=1}^n$ and $\{A^i\}_{i=1}^n$ are fuzzy numbers on $U = [0, 1]$, their $\alpha$-level sets are of the form $W^i_\alpha = [W^i_\alpha, W^i_{\alpha+}]$, $A^i_\alpha = [A^i_\alpha, A^i_{\alpha+}]$ ($i = 1, \ldots, n$), and therefore compact sets of $\mathbb{R}$ (closed and bounded). The Cartesian product of $W^n_\alpha$ and $A^n_\alpha$ is a compact subset of $\mathbb{R}^{2n}$. Function $y(w_1, \ldots, w_n, a_1, \ldots, a_n)$ is continuous
and therefore the image of the Cartesian product of $W_i^\alpha$ and $A_i^\alpha$ is also a compact subset of $\mathbb{R}$.

It is well known that a closed interval of $\mathbb{R}$ is a connected set, and that the Cartesian product of two closed intervals of $\mathbb{R}$ is a connected set of $\mathbb{R}^2$. Consequently, the Cartesian product of $W_i^\alpha$ and $A_i^\alpha$ is a connected subset of $\mathbb{R}^{2n}$. As a result, the image of the Cartesian product of $W_i^\alpha$ and $A_i^\alpha$ is a connected subset of $\mathbb{R}$. Because the only connected subsets of $\mathbb{R}$ are intervals, we conclude that the image of the Cartesian product of $W_i^\alpha$ and $A_i^\alpha$ by the continuous function $y(w_1, \ldots, w_n, a_1, \ldots, a_n)$ is a closed interval [39]. Hence $\Phi_\alpha (A_1^\alpha, \ldots, A_n^\alpha)$ is a closed interval.

Based on this theorem, the computation of the type-1 OWA output according to (4), $G$, reduces to compute the left end-points and right end-points of the intervals $\Phi_\alpha (A_1^\alpha, \ldots, A_n^\alpha)$:

$$\Phi_\alpha (A_1^\alpha, \ldots, A_n^\alpha)_- \text{ and } \Phi_\alpha (A_1^\alpha, \ldots, A_n^\alpha)_+,$$

where $A_i^\alpha = [A_{i1}^\alpha, A_{i2}^\alpha]$, $W_i^\alpha = [W_{i1}^\alpha, W_{i2}^\alpha]$.

For the left end-points, we have

$$\Phi_\alpha (A_1^\alpha, \ldots, A_n^\alpha)_- = \min_{W_{i1}^\alpha \leq w_i \leq W_{i2}^\alpha} \sum_{i=1}^n w_i a_{\sigma(i)} / \sum_{i=1}^n w_i \tag{6}$$

while for the right end-points, we have

$$\Phi_\alpha (A_1^\alpha, \ldots, A_n^\alpha)_+ = \max_{W_{i1}^\alpha \leq w_i \leq W_{i2}^\alpha} \sum_{i=1}^n w_i a_{\sigma(i)} / \sum_{i=1}^n w_i \tag{7}$$

It can be seen that (6) and (7) are programming problems. In the next section, we will address how to solve these problems so that the type-1 OWA aggregation operation can be performed efficiently.

## 3 Fast Implementation of Type-1 OWA Operation

The objective of type-1 OWA operators is to aggregate uncertain information modelled as fuzzy sets. In this section, we propose a fast algorithm for type-1 OWA operations, which can be used in real-time applications. The idea behind this algorithm hails from the above $\alpha$-level type-1 OWA aggregations. For the type-1 OWA operations, we only need to calculate all the necessary $\alpha$-level aggregations in (6) and (7), then based on the Representation Theorem of fuzzy set, the final aggregation result can be constructed as shown in (4). This fast algorithm is called the Alpha-Level Approach in this paper.

First, in the following lemma, we list some basic inequalities as described in some textbooks that will be used later in the paper.

**Lemma 1.** If $a \geq 0, c \geq 0, b \geq d$, then

$$\frac{b}{a} \geq \frac{b+d}{a+c} \geq \frac{d}{c}$$

2) If $a \geq c, \frac{b}{a} \geq \frac{d}{c}$, then

$$\frac{b-d}{a-c} \geq \frac{b}{a}$$

3) If $a \geq c, \frac{b}{a} \leq \frac{d}{c}$, then

$$\frac{b-d}{a-c} \leq \frac{b}{a}$$

Note that for the left end-points in (6), the function

$$f(w_1, a_1) = \sum_{i=1}^n w_i a_{\sigma(i)} / \sum_{i=1}^n w_i \tag{8}$$

is a monotonically non-decreasing function of $a_i$. So

$$\Phi_\alpha (A_1^\alpha, \ldots, A_n^\alpha)_- = \min_{W_{i1}^\alpha \leq w_i \leq W_{i2}^\alpha} \sum_{i=1}^n w_i A_{\alpha}^{\sigma(i)} / \sum_{i=1}^n w_i \tag{9}$$

where $A_{\alpha}^{\sigma(i)} \geq \cdots \geq A_{\alpha}^{\sigma(n)}$, and

$$h(w_1, \ldots, w_n) = \sum_{i=1}^n w_i A_{\alpha}^{\sigma(i)} \sum_{i=1}^n w_i \tag{10}$$

Now we construct a new function of end-points of intervals $W_{i}^\alpha$ as follows,

$$\rho_{i-1} = \frac{\Delta}{J_{i-1}} \sum_{i=1}^{i-1} W_{i1}^\alpha - A_{\alpha}^{\sigma(i)} + \sum_{i=1}^{n} W_{i1}^\alpha + A_{\alpha}^{\sigma(i)} \tag{11}$$

where

$$J_{i-1} \Delta \sum_{i=1}^{i-1} W_{i1}^\alpha + \sum_{i=1}^{n} W_{i1}^\alpha \tag{12}$$

In particular, we have

$$\rho_{i-1} = \frac{\Delta}{J_{1}} \sum_{i=1}^{n} W_{i1}^\alpha + A_{\alpha}^{\sigma(i)} \tag{13}$$

where

$$J_{1} \Delta \sum_{i=1}^{n} W_{i1}^\alpha \tag{14}$$

Then we have the following theorem:

**Theorem 3.** 1) If $\rho_{i-1} \geq A_{\alpha}^{\sigma(i)}$, then

$$\rho_{i-1} \geq A_{\alpha}^{\sigma(i)} \geq A_{\alpha}^{\sigma(i)}$$

2) If $\rho_{i-1} \leq A_{\alpha}^{\sigma(i)}$, then

$$A_{\alpha}^{\sigma(i)} \geq A_{\alpha}^{\sigma(i)} \geq \rho_{i-1}$$

**Proof:** Denoting

$$E = \sum_{i=1}^{i-1} W_{i1}^\alpha - A_{\alpha}^{\sigma(i)}$$


and

\[ F = \sum_{i=0}^{n} W_i A^{(i)}_{\alpha-} \]

then

\[ \rho_{\alpha-}^i = \frac{E + F}{J_i} \]

and

\[ \rho_{\alpha-}^{i+1} = \frac{E + W_i A^{(i)}_{\alpha-} + F - W_i A^{(i)}_{\alpha-}}{J_i} \]

\[ = \frac{E + F - W_i A^{(i)}_{\alpha-}}{J_i} \]

Because

\[ J_i \geq W_{i0} \geq W_{i0} + W_{i0} \]

then according to statements 2) and 3) in Lemma 1, results 1) and 2) can be derived.

The solution to problem (9) and thus (6) is given in the following theorem:

**Theorem 4.** Let \( i_0 \) be the minimum number in \( \{1, \ldots, n\} \) satisfying \( \rho_{\alpha-}^{i_0} \geq A^{(i_0)}_{\alpha-} \), then \( \rho_{\alpha-}^{i_0} \) is the minimum of (9).

**Proof:** Starting with \( i_0 = 1 \) we check the relation between \( \rho_{\alpha-}^{i_0} \) and \( A^{(i_0)}_{\alpha-} \) until the first pair \( \{ \rho_{\alpha-}^{i_0}, A^{(i_0)}_{\alpha-} \} \) satisfying \( \rho_{\alpha-}^{i_0} \geq A^{(i_0)}_{\alpha-} \) is found. This search process is guaranteed to produce such a first pair because

\[ \rho_{\alpha-}^{n-1} = \sum_{i=1}^{n-1} W_i A^{(i)}_{\alpha-} + W_n A^{(n)}_{\alpha-} = \Lambda_{i_0}^{\alpha-} \]

Next we prove that \( \rho_{\alpha-}^{i_0} \) is the minimum of (9).

According to the above search process, for any \( j \in \{1, \ldots, i_0 - 1\} \) we have that \( \rho_{\alpha-}^j \leq A^{(j)}_{\alpha-} \). Theorem 3 implies that

\[ \rho_{\alpha-}^{i_0} \leq \rho_{\alpha-}^{i_0-1} \leq \cdots \leq \rho_{\alpha-}^1 \leq \rho_{\alpha-}^0 \]

On the other hand, the application of Theorem 3 to \( \rho_{\alpha-}^{i_0} \geq A^{(i_0)}_{\alpha-} \) leads to

\[ \rho_{\alpha-}^{i_0+1} \geq \rho_{\alpha-}^{i_0} \geq A^{(i_0)}_{\alpha-} \]

Because \( A^{(i_0)}_{\alpha-} \geq A^{(i_0+1)}_{\alpha-} \) then we have that \( \rho_{\alpha-}^{i_0+1} \geq A^{(i_0+1)}_{\alpha-} \), and therefore

\[ \rho_{\alpha-}^{i_0+2} \geq \rho_{\alpha-}^{i_0+1} \geq A^{(i_0+1)}_{\alpha-} \]

Following a similar reasoning, we get

\[ \rho_{\alpha-}^{n-1} \geq \rho_{\alpha-}^{n-2} \geq \cdots \geq \rho_{\alpha-}^0 \geq A^{(n-1)}_{\alpha-} \]

So,

\[ \rho_{\alpha-}^0 \geq \cdots \geq \rho_{\alpha-}^0 \geq \rho_{\alpha-}^0 \]

and therefore \( \rho_{\alpha-}^{i_0} \) is the minimum of \( \{ \rho_{\alpha-}^1, \ldots, \rho_{\alpha-}^n \} \). In the following, we prove the minimum of \( h(w_1, \ldots, w_n) \) is in the form of \( \rho_{\alpha-}^{i_0} \).
Theorem 5. 1) If $\rho_{\alpha+}^{i_0} \geq A_{\alpha+}^{\sigma(i_0)}$, then

$$\rho_{\alpha+}^{i_0} \geq \rho_{\alpha+}^{i_0+1} \geq A_{\alpha+}^{\sigma(i_0)}$$

2) If $\rho_{\alpha+}^{i_0} \leq A_{\alpha+}^{\sigma(i_0)}$, then

$$A_{\alpha+}^{\sigma(i_0)} \geq \rho_{\alpha+}^{i_0+1} \geq \rho_{\alpha+}^{i_0}$$

Proof: Let

$$C = \sum_{i=1}^{i_0-1} W_i A_{\alpha+}^{\sigma(i)}$$

and

$$D = \sum_{i=i_0}^{n} W_i A_{\alpha+}^{\sigma(i)}$$

then

$$\rho_{\alpha+}^{i_0} = \frac{C + D}{H_{i_0}}$$

and

$$\rho_{\alpha+}^{i_0+1} = \frac{C + W_{i_0} A_{\alpha+}^{\sigma(i_0)} + D - W_{i_0} A_{\alpha+}^{\sigma(i_0)}}{H_{i_0} + (W_{i_0} - W_{i_0})}$$

Because $H_{i_0} \geq 0$, then according to the statement 1) in Lemma 1, results 1) and 2) can be derived.

The solution to problems (7) and (16) is given in the following theorem:

Theorem 6. Let $i_0^*$ be the minimum number in $\{1, \ldots, n\}$ satisfying $\rho_{\alpha+}^{i_0^*} \geq A_{\alpha+}^{\sigma(i_0^*)}$, then $\rho_{\alpha+}^{i_0^*}$ is the maximum of (17), and thus the solution of (7).

Proof: Starting with $i_0 = 1$ we check the relation between $\rho_{\alpha+}^{i_0}$ and $A_{\alpha+}^{\sigma(i_0)}$ until the first pair $\{\rho_{\alpha+}^{i_0^*}, A_{\alpha+}^{\sigma(i_0^*)}\}$ satisfying $\rho_{\alpha+}^{i_0^*} \geq A_{\alpha+}^{\sigma(i_0^*)}$ is found. This search process is guaranteed to produce such a pair because

$$\rho_{\alpha+}^{i_0} = \sum_{i=1}^{n-1} W_i A_{\alpha+}^{\sigma(i)} + W_n A_{\alpha+}^{\sigma(n)} \geq A_{\alpha+}^{\sigma(n)}$$

Next we prove $\rho_{\alpha+}^{i_0^*}$ is the maximum of (17).

According to the above search process, for any $j \in \{1, \ldots, i_0^*-1\}$, we have that $\rho_{\alpha+}^{i_0} \leq A_{\alpha+}^{\sigma(j)}$. Theorem 5 implies

$$\rho_{\alpha+}^{j} \leq \rho_{\alpha+}^{j+1} \leq A_{\alpha+}^{\sigma(j)}$$

So

$$\rho_{\alpha+}^{i_0^*} \leq \rho_{\alpha+}^{i_0^*+1} \leq \cdots \leq \rho_{\alpha+}^{i_0^*}$$

On the other hand, the application of Theorem 5 to $\rho_{\alpha+}^{i_0} \geq A_{\alpha+}^{\sigma(i_0^*)}$ leads to

$$\rho_{\alpha+}^{i_0} \geq \rho_{\alpha+}^{i_0+1} \geq A_{\alpha+}^{\sigma(i_0^*)}$$

Because $A_{\alpha+}^{\sigma(i_0^*)} \geq A_{\alpha+}^{\sigma(i_0^*+1)}$, then we have that $\rho_{\alpha+}^{i_0+1} \geq A_{\alpha+}^{\sigma(i_0^*+1)}$, and therefore

$$\rho_{\alpha+}^{i_0} \geq \rho_{\alpha+}^{i_0+1} \geq \cdots \geq \rho_{\alpha+}^{i_0^*}$$

and therefore $\rho_{\alpha+}^{i_0^*}$ is the maximum of $\{\rho_{\alpha+}^{1}, \ldots, \rho_{\alpha+}^{n}\}$. In the following, we prove the maximum of $g(w_1, \ldots, w_n)$ is in the form of (18).

An analysis of function $g(w_1, \ldots, w_n)$ similar to the one applied to function $h(w_1, \ldots, w_n)$ in Theorem 3 produces the following: (i) If $A_{\alpha+}^{\sigma(i)} \geq g(w_1, \ldots, w_n)$ then function $g(w_1, \ldots, w_n)$ is monotonically non-decreasing on each of its arguments, $w_i$, and the maximum of $g(w_1, \ldots, w_n)$ in the direction of $w_i$ is achieved at $W_{i+}^{i}$:

$$g(w_1, \ldots, w_{i-1}, W_{i+}^{i}, w_{i+1}, \ldots, w_n) \geq g(w_1, \ldots, w_n).$$

(ii) If $A_{\alpha+}^{\sigma(i)} \leq g(w_1, \ldots, w_n)$ then function $g(w_1, \ldots, w_n)$ is monotonically non-increasing on each of its arguments, $w_i$, and the maximum of $g(w_1, \ldots, w_n)$ in the direction of $w_i$ is achieved at $W_{i-}^{i}$:

$$g(w_1, \ldots, w_{i-1}, W_{i-}^{i}, w_{i+1}, \ldots, w_n) \geq g(w_1, \ldots, w_n).$$

Assume that $A_{\alpha+}^{\sigma(i) \geq g(w_1, \ldots, w_n)}$ and $A_{\alpha+}^{\sigma(i) \leq g(w_1, \ldots, w_n)}$, then function $g(w_1, \ldots, w_n)$ reaches the maximum at $w_1 = W_{1+}^{1}, \ldots, w_{i-1} = W_{i-1}^{i-1}, w_i = W_{i-}^{i}, \ldots, w_n = W_{n-}^{n}$, that is to say, this maximum can be expressed in the form of (18). Hence $\rho_{\alpha+}^{i_0^*}$ is the maximum of $g(w_1, \ldots, w_n)$, i.e., the solution of (7) and (16).

Theorem 4, Theorem 6, and their proofs actually indicate the procedures for finding the values $\rho_{\alpha+}^{i_0^*}$ and $\rho_{\alpha+}^{i_0^*}$ respectively. Given $n$ linguistic weights $(W_i)^n_{i=1}$, the procedure to aggregate $\{A^i\}_{i=1}^n$ by a type-1 OWA operator via the $\alpha$-level aggregation scheme is given in Figure 1, in which the $\alpha$ values are required to cover all the available membership grades in $\{\mu_W(w_i)\}$ and in $\mu_A(w_i)$. 

Example 1. Assume the following numerical domains $U = \{0, 0.5, 1.0\}$ and $X = \{0, 0.5, 1.0, 2.0\}$. Let the given linguistic weights $W = \left( \begin{array}{c} u_i \\ \mu_W(u_i) \end{array} \right)_{u_i \in U}$ be

$$W^1 = \left( \begin{array}{c} 0.0 \\ 0.0 \end{array} \right),$$

$$W^2 = \left( \begin{array}{c} 0.0 \\ 0.0 \end{array} \right),$$

$$W^3 = \left( \begin{array}{c} 0.0 \\ 0.0 \end{array} \right),$$

and the aggregated objects on $X$ be

$$A^1 = \left( \begin{array}{c} 0.0 \\ 0.0 \end{array} \right),$$

$$A^2 = \left( \begin{array}{c} 0.0 \\ 0.0 \end{array} \right),$$

$$A^3 = \left( \begin{array}{c} 0.0 \\ 0.0 \end{array} \right).$$

To calculate the $\alpha$-cuts of $W^i$ and $A^i(i = 1, 2, 3)$, the following set of $\alpha$ values will be used: $\{0, 0.5, 1.0\}$. We use
Step 1. To set up the α-level resolution in [0, 1].

Step 2. For each $\alpha \in [0, 1]_\rho$,

Step 2.1. To calculate $\rho_{\alpha+}^{i_0}$

1) Let $i_0 = 1$;
2) If $\rho_{\alpha+}^{i_0} \geq A_\alpha^{(i_0)}$, stop, $\rho_{\alpha+}^{i_0}$ is the solution; otherwise go to Step 2.1-3.
3) $i_0 \leftarrow i_0 + 1$, go to Step 2.1-2.

Step 2.2. To calculate $\rho_{\alpha-}^{i_0}$

1) Let $i_0 = 1$;
2) If $\rho_{\alpha-}^{i_0} \geq A_\alpha^{(i_0)}$, stop, $\rho_{\alpha-}^{i_0}$ is the solution; otherwise go to Step 2.2-3.
3) $i_0 \leftarrow i_0 + 1$, go to step Step 2.2-2.

Step 3. To construct the aggregation resulting fuzzy set $G$ based on all the available intervals $[\rho_{\alpha-}^{i_0}, \rho_{\alpha+}^{i_0}]$:

$$\mu_G(x) = \bigvee_{\alpha; x \in \left[\rho_{\alpha-}^{i_0}, \rho_{\alpha+}^{i_0}\right]} \alpha$$

Fig. 1: Procedure of the Alpha-Level Approach to type-1 OWA operation

the type-1 OWA operator $\Phi_{W_1,W_2,W_3}$ to aggregate the sets $A^1, A^2, A^3$ according to the procedure in Figure 1:

$$G = \Phi_{W_1,W_2,W_3}(A^1, A^2, A^3)$$

So, we get the $\alpha$-levels of $G$ at $\alpha = 0, 0.5$ and $1.0$ respectively.

Case I. $\alpha = 0.0$

Obviously, the $\alpha$-levels of $A^i$ and $W^i (i=1,2,3)$ are

$$A^1_\alpha = A^2_\alpha = A^3_\alpha = \{0.0, 1.0, 2.0\}$$

and

$$W^1_\alpha = W^2_\alpha = W^3_\alpha = \{0.0, 0.5, 1.0\},$$

respectively. Thus, we have

$$A^1_{\alpha-} = A^2_{\alpha-} = A^3_{\alpha-} = 2.0;$$

$$A^1_{\alpha+} = A^2_{\alpha+} = A^3_{\alpha+} = 0.0;$$

$$W^1_{\alpha-} = W^2_{\alpha-} = W^3_{\alpha-} = 0.0;$$

$$W^1_{\alpha+} = W^2_{\alpha+} = W^3_{\alpha+} = 1.0$$

- Computation of $\rho_{\alpha-}^{i_0}$.
  Because $A^i_{\alpha-} = A^i_{\alpha+} = A^i_{\alpha-} = \alpha$, the permutation operator is $\sigma = (1, 2, 3)$. Then
  
  1) $i_0 = 1$. According to the equation (13), we have

  $$\rho_{\alpha-}^{i_0} = \frac{W^1_{\alpha-}A^{(1)}_{\alpha-} + W^2_{\alpha-}A^{(2)}_{\alpha-} + W^3_{\alpha-}A^{(3)}_{\alpha-}}{W^1_{\alpha-} + W^2_{\alpha-} + W^3_{\alpha-}}$$

  $$= 0.0$$

  $$\geq A_\alpha^{(i_0)}$$

  $$= A_{\alpha-}$$

  So, we get $\rho_{\alpha-}^{i_0} = 0.0$.

- Computation of $\rho_{\alpha+}^{i_0}$.
  Because $A^i_{\alpha+} = A^i_{\alpha+} = A^i_{\alpha+}$, the permutation operator is $\sigma = (1, 2, 3)$. Then
  
  1) $i_0 = 1$. According to the equation (20), we have

$$\rho_{\alpha+}^{i_0} = \frac{W^1_{\alpha+}A^{(1)}_{\alpha+} + W^2_{\alpha+}A^{(2)}_{\alpha+} + W^3_{\alpha+}A^{(3)}_{\alpha+}}{W^1_{\alpha+} + W^2_{\alpha+} + W^3_{\alpha+}}$$

$$= 0.0$$

$$< A_\alpha^{(i_0)}$$

$$= A_{\alpha+}$$

So, we should continue this procedure by letting $i_0 = 2$.

2) $i_0 = 2$. According to the equation (18), we have

$$\rho_{\alpha+}^{i_0} = \frac{W^1_{\alpha+}A^{(1)}_{\alpha+} + W^2_{\alpha+}A^{(2)}_{\alpha+} + W^3_{\alpha+}A^{(3)}_{\alpha+}}{W^1_{\alpha+} + W^2_{\alpha+} + W^3_{\alpha+}}$$

$$= 0.0$$

$$\geq A_\alpha^{(i_0)}$$

$$= A_{\alpha+}$$

So, we get $\rho_{\alpha+}^{i_0} = 2.0$. As a result, $G_\alpha = [0.0, 2.0] \cap X = \{0.0, 1.0, 2.0\}$.

Case II. $\alpha = 0.5$

The $\alpha$-levels of $A^i$ and $W^i (i=1,2,3)$ are

$$A^1_\alpha = \{1.0, 2.0\}, A^2_\alpha = \{0.0, 1.0\}, A^3_\alpha = \{1.0\}$$

and

$$W^1_\alpha = \{0.0, 0.5\}, W^2_\alpha = \{0.5\}, W^3_\alpha = \{0.5, 1.0\},$$

respectively. Thus, we have

$$A^1_{\alpha-} = 1.0, A^1_{\alpha+} = 2.0;$$

$$A^2_{\alpha-} = 0.0, A^2_{\alpha+} = 1.0;$$

$$A^3_{\alpha-} = 1.0, A^3_{\alpha+} = 1.0;$$

$$W^1_{\alpha-} = 0.0, W^1_{\alpha+} = 0.5;$$

$$W^2_{\alpha-} = 0.5, W^2_{\alpha+} = 0.5;$$

and
$W_{a-}^3 = 0.5, W_{a+}^3 = 1.0$

- Computation of $\rho_{a-}^i$.
  Because $A_{a-}^1 \geq A_{a-}^3 \geq A_{a+}^2$, the permutation operator is $\sigma = (1, 3, 2)$. Then

1) $i_0 = 1$. According to the equation (13), we have

$$\rho_{a-}^i = \frac{W_{a-}^1 - A_{a-}^1 + W_{a-}^2 - A_{a-}^2 + W_{a-}^3 - A_{a-}^3}{0.5 \times 1.0 + 0.5 \times 1.0 + 0.5 \times 1.0}$$

$$\rho_{a-}^i = 0.5$$

$$A_{a-}^{\sigma(i_0)} = A_{a-}^2$$

So, we should continue this procedure by letting $i_0 = 2$.

2) $i_0 = 2$. According to the equation (11), we have

$$\rho_{a-}^i = \frac{W_{a-}^1 - A_{a-}^1 + W_{a-}^2 - A_{a-}^2 + W_{a-}^3 - A_{a-}^3}{0.0 \times 1.0 + 0.5 \times 1.0 + 0.5 \times 1.0}$$

$$\rho_{a-}^i = \frac{1}{3}$$

$$A_{a-}^{\sigma(i_0)} = A_{a-}^1$$

So, we should continue this procedure by letting $i_0 = 3$.

3) $i_0 = 3$. According to the equation (11), we have

$$\rho_{a-}^i = \frac{W_{a-}^1 - A_{a-}^1 + W_{a-}^2 - A_{a-}^2 + W_{a-}^3 - A_{a-}^3}{0.0 \times 1.0 + 0.5 \times 1.0 + 0.5 \times 1.0}$$

$$\rho_{a-}^i = \frac{1}{3}$$

$$A_{a-}^{\sigma(i_0)} = A_{a-}^1$$

So, we get $\rho_{a-}^i = \frac{1}{3}$.

- Computation of $\rho_{a+}^i$.
  Because $A_{a+}^1 > A_{a+}^2 \geq A_{a+}^3$, the permutation operator is $\sigma = (1, 2, 3)$. Then

1) $i_0 = 1$. According to the equation (20), we have

$$\rho_{a+}^i = \frac{W_{a+}^1 - A_{a+}^1 + W_{a+}^2 - A_{a+}^2 + W_{a+}^3 - A_{a+}^3}{0.0 \times 2.0 + 0.5 \times 1.0 + 0.5 \times 1.0}$$

$$\rho_{a+}^i = 1.0$$

$$A_{a+}^{\sigma(i_0)} = A_{a+}^3$$

So, we should continue this procedure by letting $i_0 = 2$.

2) $i_0 = 2$. According to the equation (18), we have

$$\rho_{a+}^i = \frac{W_{a+}^1 - A_{a+}^1 + W_{a+}^2 - A_{a+}^2 + W_{a+}^3 - A_{a+}^3}{0.5 \times 2.0 + 0.5 \times 1.0 + 0.5 \times 1.0}$$

$$\rho_{a+}^i = \frac{4}{3}$$

$$A_{a+}^{\sigma(i_0)} = A_{a+}^2$$

So, we get $\rho_{a+}^i = \frac{4}{3}$. As a result, $G_{\alpha} = \left[ \frac{1}{3}, \frac{4}{3} \right] \cap X = \{1.0\}$.

Case III. $\alpha = 1.0$

The $\alpha$-levels of $A^i$ and $W^i (i = 1, 2, 3)$ are

$A_{a-}^1 = \{2.0\}, A_{a+}^1 = \{0.0\}, A_{a-}^3 = \{1.0\}$

and

$W_{a-}^1 = \{0.0\}, W_{a+}^1 = \{0.5\}, W_{a+}^3 = \{1.0\}$.

respectively. Thus, we have

$A_{a-}^1 = A_{a+}^1 = 2.0$;

$A_{a-}^2 = A_{a+}^2 = 0.0$;

$A_{a-}^3 = A_{a+}^3 = 1.0$;

and

$W_{a-}^1 = W_{a+}^1 = 0.0$;

$W_{a-}^2 = W_{a+}^2 = 0.5$;

$W_{a-}^3 = W_{a+}^3 = 1.0$.

Following a similar computation process as in the two previous cases, we get $\rho_{a+}^i = \rho_{a+}^i = \frac{1}{3}$. As a result, $G_{\alpha} = \left( \frac{1}{3}, \frac{4}{3} \right) \cap X = \{0\}$.

Now we proceed to compute the membership grades of $G$ according to the equation (5):

$$\mu_G(0) = \bigvee_{\alpha=0:0 \in G_{\alpha}} \alpha = 0.0$$

$$\mu_G(1.0) = \bigvee_{\alpha=0:0 \in G_{\alpha}} \alpha = 0.0 \lor 0.5 = 0.5$$

$$\mu_G(2.0) = \bigvee_{\alpha=0:0 \in G_{\alpha}} \alpha = 0.0$$

Hence, the result of aggregating the fuzzy sets $A^1, A^2, A^3$ by the type-1 OWA operator $\Phi_{W^1, W^2, W^3}$ is

$$G = \left( \begin{array}{ccc} 0.0 & 1.0 & 2.0 \ 0.0 & 0.5 & 0.0 \end{array} \right)$$.

4 Complexity Analyses of the Direct Approach and the Proposed Alpha-Level Approach to Type-1 OWA Operations

Given $n$ fuzzy set $\{A^i\}_{i=1}^n$ to be aggregated by a type-1 OWA associated with $n$ uncertain weights $\{W^i\}_{i=1}^n$, in this section we analyse the complexity of the Direct Approach [27] and Alpha-Level Approach to type-1 OWA operations, which was not addressed yet in [27].

In the Direct Approach, assume the domain $U = [0, 1]$ be discretised with $n_a$ points and the domain $X$ with $n_x$ points. For each combination of $w_1 \in U, \cdots, w_n \in U, a_1 \in X, \cdots, a_n \in X$, the type-1 OWA aggregation in the Direct Approach will involve $2(n-1)$ additions, $n$ multiplications, 1 division, $2n - 1$ t-norm operations and 1 maximum operation. Hence the total operations for each combination of $w_1, \cdots, w_n, a_1, \cdots, a_n$ is

$$2(n - 1) + n + 1 + 2(n - 1) + 1 = 5n - 1$$

(22)

Then $(n_a)^n(n_x)^n$ combinations of $w_1, \cdots, w_n, a_1, \cdots, a_n$ lead to the number of operations involved in a Direct
Approach to type-1 OWA operator to aggregate \( \{A_i\}_{i=1}^{n} \) to be
\[
(n_u n_x)^n (5n - 1) = O (K^n)
\] (23)
where \( K \) is a constant. Hence the complexity of the Direct Approach to type-1 OWA operation is in exponential order.

In the proposed Alpha-Level Approach, assume the number of \( \alpha \) values in \([0, 1]\) be \( n_\alpha \), and the domain \( X \) be discretised with \( n_x \) points. For each \( \alpha \) value, the operations in each round of the total \( i_0^\alpha \) involved in the computation of each right end-point \( \rho_{\alpha+}^i \) of an \( \alpha \)-cut include \( 2(n - 1) \) additions, \( n \) multiplications, and 1 division. So, the total number of operations to compute the right end-point \( \rho_{\alpha+}^i \) is
\[
i_0^\alpha (2(n - 1) + n + 1) = i_0^\alpha (3n - 1)
\] (24)
Similarly, the total number of operations to compute the left end-point \( \rho_{\alpha-}^i \) is \( i_0^\alpha (3n - 1) \). Therefore, the computation of each \( \alpha \)-cut \([\rho_{\alpha-}^i, \rho_{\alpha+}^i]\) involves \( (i_0^\alpha + i_0^\alpha) (3n - 1) \) times of operations. Considering there exist \( n_x (n_\alpha - 1) \) operations to obtain the membership grades of the \( n_x \) points in \( X \), the total number of operations involved in the Alpha-Level Approach is
\[
n_\alpha (i_0^\alpha + i_0^\alpha) (3n - 1) + n_x (n_\alpha - 1) = O(n)
\] (25)
That is to say, the complexity of the Alpha-Level Approach is in linear order. Hence the Alpha-Level Approach achieves much higher computing efficiency than the Direct Approach.

5 EXPERIMENTAL RESULTS

In this section, we first evaluate the computing efficiency of the proposed scheme in comparison with the Direct Approach [27], in which eight different kinds of type-1 OWA operators are designed to aggregate a group of fuzzy sets. Then we provide a practical example for breast cancer treatment in which type-1 OWA operators are used. In these examples, the proposed type-1 OWA operators are compared with another widely investigated aggregation operator, the FWA operator [36]–[38].

5.1 Evaluation of computing efficiency and comparisons with Direct Approach

As Yager’s OWA operators do, type-1 OWA operators also depend on the choices of linguistic weights \( \{W^i\}_{i=1}^{n} \). By choosing appropriate uncertain weights modelled as fuzzy sets, we can obtain a type-1 OWA operator with desired properties. In this subsection, eight different type-1 OWA operators are designed to aggregate the fuzzy sets shown in Figure 2. These eight type-1 OWA operators are the meet operator, two meet-like operators, the join operator, two join-like operators, the mean operator, and a mean-like operator.

The meet and join operators of fuzzy sets were proposed by Zadeh [41] and named in [42]. Interestingly, as indicated in [27] and [28], the meet and join operations of fuzzy sets can be performed by type-1 OWA operators with singleton weights. For example, a type-1 OWA operator of dimension 3 becomes a meet operator if the following singleton weights are used: \( W^1 = 0 \) \((i \neq 3)\), \( W^3 = 1 \), i.e.,
\[
\mu_{W^3}(w) = \begin{cases} 1 & w = 1 \\ 0 & \text{others} \end{cases} \] (26)
\[
\mu_{W^1}(w) = \begin{cases} 1 & w = 0 \\ 0 & \text{others} \end{cases} \] (27)
whilst the singleton weights \( W^1 = 0 \) \((i \neq 1)\), \( W^1 = 1 \) make the type-1 OWA operator into a join operator. As a matter of fact, the meet of fuzzy sets yields the fuzzified minimum whilst the join of fuzzy sets yields the fuzzified maximum [27].

The traditional mean operator is a particular type of Yager’s OWA operator with weights all equal to \(1/n\). Therefore, the type-1 OWA operator with all weights in the form of singleton fuzzy sets \(1/n\)
\[
\mu_G(y) = \sup \frac{1}{n} \sum_{i=1}^{n} \mu_{A_i}(a_i) = \mu_{\frac{1}{n} \sum_{i=1}^{n} a_i = y} \] (28)
can be seen as an extended mean operation on fuzzy sets [27], [28].

Meet-like type-1 OWA (MLT1OWA) operators [27], [28] can be obtained by selecting appropriate linguistic weights: the last linguistic weight is to approach to the singleton fuzzy set \(1\), and the rest of linguistic weights are to approach to the singleton fuzzy set \(0\) in turn. The MLT1OWA operator of dimension 3 with linguistic weights \( W^1 = W^2 = L_0, W^3 = L_1 \) depicted in Figure 3 is denoted as MLT1OWA 1. Figure 4 shows linguistic weights \( \{W^i\}_{i=1}^{3} \) that guide another meet-like type-1 OWA operation, which is denoted as MLT1OWA2.
Join-like type-1 OWA (JL1OWA) operators can also be obtained by selecting appropriate linguistic weights [27], [28]. Indeed, this is the case when the first linguistic weight is close to the singleton fuzzy set \( 1 \), and the rest are close to the singleton fuzzy set \( 0 \) in turn. One example of linguistic weights chosen for JL1OWA operator is to set \( W^1 = L_1, W^2 = W^3 = L_0 \), in which the \( L_0 \) and \( L_1 \) are depicted in Figure 3. This JL1OWA is denoted as JL1OWA1, whereas Figure 5 illustrates another case of linguistic weights chosen for JL1OWA operator, which is denoted as JL1OWA2.

Fig. 5: Linguistic weights for JL1OWA2 (from right to left): \( W^1, W^2, \) and \( W^3 \)

Mean-like type-1 OWA (MA11OWA) operators can be obtained by selecting the linguistic weights appropriately. For example, Figure 6 shows three linguistic weights in the forms of triangular fuzzy numbers whose cores locate at 1/3 as follows,

\[
\mu_{W^i}(u) = \max\{0, \min(3u, 2 - 3u)\} 
\]  

(29)

Fig. 6: Linguistic weights with cores locating at 1/3: \( W^i \) (\( i = 1, 2, 3 \))

After choosing the above associated weights respectively, we can use the proposed Alpha-Level Approach to
implement these eight type-1 OWA operators for aggregating the fuzzy sets depicted in Figure 2, and compare with the Direct Approach [27] in terms of computing efficiency respectively. Table 1 shows the corresponding time costs of the proposed Alpha-Level Approach and the Direct Approach in completing these operations. It can be seen that the computing efficiency achieved by the Alpha-Level Approach is much higher than the one achieved by the Direct Approach.

5.2 Comparisons of the type-1 OWA operators with the FWA operators

In this subsection, we further compare type-1 OWA operators using the proposed \(\alpha\)-level approach with FWA operators [36]–[38] in aggregating fuzzy sets. In our experiments, the type-1 OWA operators and FWA operators use the same uncertain weights to aggregate the same groups of fuzzy sets, then we evaluate what different aggregation results can be achieved.

In the first example, a FWA operator with linguistic weights \(W_1, W_2, W_3\) being the fuzzy sets from right to left given in Figure 5 is used to aggregate the three fuzzy sets depicted in Figure 2. Figure 7 illustrates the aggregation results obtained with the FWA and the corresponding type-1 OWA operator for the same set of weights, the JLTIOWA2 operator.

In the second example, Figure 9 shows the corresponding aggregation results obtained using the FWA and type-1 OWA operator associated with the same linguistic weights depicted in Figure 8b to aggregate the same group of fuzzy sets shown in Figure 8a.

From the above examples it can be seen that type-1 OWA operators and the FWA operators exhibit different aggregation behaviours, which resembles the different behaviours Yager’s OWA operators and the weighted averaging operators have associated when data is crisp.

5.3 Type-1 OWA based fuzzy inferences for breast cancer treatments

In this subsection, we further apply type-1 OWA operators to the aggregation of non-stationary fuzzy sets for diagnoses of breast cancer patients.

Non-stationary fuzzy sets [43], [44] have been proposed to model intra-expert variability and inter-expert variability exhibited in multi-expert decision making, in which the membership function of a non-stationary fuzzy set may alter over time. As a result, given a problem, a non-stationary fuzzy system may generate different output fuzzy sets in different runs [45]. This means that some additional components become necessary besides the commonly used in the standard fuzzy system: fuzzifier, rule base, rule engine, defuzzifier. Among them, an important additional component is to aggregate these output sets into an overall one. In the following, we use the type-1 OWA operator as uncertain operator to aggregate the output sets, which leads to a type-1 OWA based non-stationary fuzzy system (T1ONFS) as depicted in Figure 10.

Generally speaking, the T1ONFS works as follows. In each run, crisp input values first feed into the system through the fuzzifier by which the fuzzification of these inputs is carried out in a singleton or non-singleton way. The fuzzified non-stationary fuzzy sets then activate the inference engine and rule base to yield an output set by performing the union and intersection operations of fuzzy sets and compositions of relations. This process repeats \(n\) times. So \(n\) output sets are generated. Then a type-1 OWA operator is applied to these output sets to generate an overall set. Finally, this overall fuzzy set is defuzzified to produce a crisp output.

In our study towards the design of a non-stationary fuzzy expert system for breast cancer treatments, 12 initial fuzzy rules are acquired [46] according to the professional clinical guidelines provided by Nottingham University Hospitals NHS Trust Breast Directorate, i.e., the fuzzy rule base is obtained from human experts’ knowledge, which is different from the scheme of in-
TABLE 1: Comparison of computing efficiency of Alpha-Level Approach and Direct Approach to type-1 OWA operations

<table>
<thead>
<tr>
<th>Type-1 OWA operators</th>
<th>Alpha-Level Approach</th>
<th>Direct Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meet</td>
<td>0.13 seconds</td>
<td>200.81 seconds</td>
</tr>
<tr>
<td>MELT1OWA1</td>
<td>0.16 seconds</td>
<td>8313.72 seconds</td>
</tr>
<tr>
<td>MELT1OWA2</td>
<td>0.16 seconds</td>
<td>10824.67 seconds</td>
</tr>
<tr>
<td>Join</td>
<td>0.13 seconds</td>
<td>208.61 seconds</td>
</tr>
<tr>
<td>JLT1OWA1</td>
<td>0.14 seconds</td>
<td>7671.46 seconds</td>
</tr>
<tr>
<td>JLT1OWA2</td>
<td>0.14 seconds</td>
<td>11270.19 seconds</td>
</tr>
<tr>
<td>Mean</td>
<td>0.12 seconds</td>
<td>52.75 seconds</td>
</tr>
<tr>
<td>MALT1OWA</td>
<td>0.17 seconds</td>
<td>11552.68 seconds</td>
</tr>
</tbody>
</table>

Fig. 10: Type-1 OWA based non-stationary fuzzy system

TABLE 2: Confusion matrix obtained by MALT1OWA3 based fuzzy decision

<table>
<thead>
<tr>
<th>Confusion Matrix</th>
<th>Clinician Decision</th>
<th>Model Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No</td>
<td>Maybe</td>
</tr>
<tr>
<td>Clinician Decision</td>
<td>79%</td>
<td>4.1%</td>
</tr>
<tr>
<td>Model Decision</td>
<td>0.2%</td>
<td>0.0%</td>
</tr>
<tr>
<td>Yes</td>
<td>1.8%</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

TABLE 3: Confusion matrix obtained by FWA based fuzzy decision

<table>
<thead>
<tr>
<th>Confusion Matrix</th>
<th>Clinician Decision</th>
<th>Model Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No</td>
<td>Maybe</td>
</tr>
<tr>
<td>Clinician Decision</td>
<td>75%</td>
<td>3.8%</td>
</tr>
<tr>
<td>Model Decision</td>
<td>4.5%</td>
<td>0.3%</td>
</tr>
</tbody>
</table>

6 DISCUSSION AND CONCLUSIONS

This paper first defined the $\alpha$-level type-1 OWA operator to aggregate the $\alpha$-cuts of fuzzy sets. The Representation Theorem of type-1 OWA operators was proved. According to the Representation Theorem, type-1 OWA operators can be decomposed into its $\alpha$-level type-1 OWA operators, which led to the proposal and development of a fast approach to implementing type-1 OWA operations. Promisingly, the complexity of the Alpha-Level Approach is in linear order, it can achieve much higher computing efficiency in performing type-1 OWA operation than the Direct Approach, and therefore it provides an efficient way of aggregating uncertain information via OWA mechanism in real time applications.

It is known that in Yager’s OWA aggregation, the identification of appropriate OWA weights is a very active research topic [47]–[51]. We have a similar issue in
the case of the type-1 OWA operators, i.e., how to determine type-1 OWA weights to reflect the decision makers’ desired agenda for aggregating the criteria/preferences. Type-2 linguistic quantifiers have been proposed for this purpose [27], although further schemes are worth investigating for different situations. Other interesting issues include the possibility of applying type-1 OWAs to the merging of similar fuzzy sets for improving fuzzy model interpretability/transparency and parsimony [52]–[54], as well as their applications to multi-expert decision making and multi-criteria decision making.

**ACKNOWLEDGMENT**

The authors would like to thank the anonymous reviewers very much for their excellent comments that have helped us to improve the quality of this paper. This work has been supported by the EPSRC Research Grant EP/C542215/1.

**REFERENCES**


Shang-Ming Zhou (M’01) received the BSc degree in mathematics from Liaocheng University, China, the MSc degree in applied mathematics from Beijing Normal University, China, and the PhD degree in computer science from the University of Essex, UK respectively. Currently he is with the Health Information Research Unit (HIRU) at School of Medicine, Swansea University, UK. His research interests include data mining and modelling in public health, uncertain information aggregation, system modelling with inconsistent/incomplete information, soft decision making using type-1/type-2 fuzzy logics, interpretable knowledge based system modelling, machine learning (via kernel machines, artificial neural networks) under uncertainty, pattern recognition. He has published extensively on these topics.

Francisco Chiclana received the B.Sc. and Ph.D. degrees in Mathematics, both from the University of Granada (Spain) in 1989 and 2000, respectively.

In August 2003, he joined De Montfort University, Leicester, UK. Since August 2006, he is a Principal Lecturer and currently holds a Readership in Computational Intelligence. He has published in international journals such as: IEEE Transactions on Fuzzy Systems; IEEE Transactions on Systems, Man and Cybernetics (Part A/Part B); European Journal of Operational Research; Fuzzy Sets and Systems; International Journal of Intelligent Systems; Information Sciences; and International Journal of Uncertainty, Fuzziness and Knowledge Based Systems. He serves as member of the editorial board of The Open Cybernetics and Systemics Journal.

His research interests include fuzzy preference modelling, decision making problems with heterogeneous fuzzy/uncertain information, decision support systems, the consensus reaching process, recommender systems, social networks, modelling situations with missing/incomplete information, rationality/consistency and aggregation of information.

Robert I. John received the B.Sc. (Hons.) 1st class degree in Mathematics from Leicester Polytechnic, Leicester, U.K., the M.Sc. degree in Statistics from UMIST, Manchester, U.K., and his Ph.D in type-2 fuzzy logic from De Montfort University, Leicester, U.K., in 1979, 1981, and 2000, respectively.

Currently, he is a Professor in Computer Science and the Director of the Centre for Computational Intelligence at De Montfort University. He is a Member of the Editorial Boards of International Journal of Cognitive Neurodynamics, International Journal of Computational Intelligence, International Journal for Computational Intelligence and Information and Systems Sciences. He was a co-general chair of the IEEE International Conference on Fuzzy Systems in London (2007). His paper with Simon Coupland (Geometric Type-1 and Type-2 Fuzzy Logic Systems) won the best paper award in IEEE Transactions on Fuzzy Systems in 2007. Professor John has been awarded research funding to the value of over 1m from a variety of sources including the UK Government, the European Union and Venture Capital.

He has published over 150 papers in the area of type-2 fuzzy logic and his research interests are in the general field of modelling human decision making using type-2 fuzzy logic.

Jonathan M. Garibaldi received the B.Sc (Hons) degree in Physics from Bristol University, U.K., and the M.Sc. degree in Intelligent Systems and the Ph.D. degree in uncertainty handling in immediate neonatal assessment from the University of Plymouth, U.K., in 1984, 1990, and 1997, respectively.

He is currently an Associate Professor and Reader within the Intelligent Modelling and Analysis (IMA) Research Group in the School of Computer Science at the University of Nottingham, U.K. The IMA research group undertakes research into intelligent modelling, utilising data analysis and transformation techniques to enable deeper and clearer understanding of complex problems. Dr Garibaldi has published over 40 papers on fuzzy expert systems and fuzzy modelling, including three book chapters, and has edited two books. His main research interests are modelling uncertainty in human reasoning and especially in modelling the variation in normal human decision making, particularly in medical domains. He has created and implemented fuzzy expert systems, and developed methods for fuzzy model optimisation.