A panorama of positivity. II: Fixed dimension

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To Thomas Ransford, on his 60th birthday

Abstract. This survey contains a selection of topics unified by the concept of positive semidefiniteness (of matrices or kernels), reflecting natural constraints imposed on discrete data (graphs or networks) or continuous objects (probability or mass distributions). We put emphasis on entrywise operations which preserve positivity, in a variety of guises. Techniques from harmonic analysis, function theory, operator theory, statistics, combinatorics, and group representations are invoked. Some partially forgotten classical roots in metric geometry and distance transforms are presented with comments and full bibliographical references. Modern applications to high-dimensional covariance estimation and regularization are included.

Contents

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This is the second part of a two-part survey; we include on p. 41 the table of contents for the first part [**9**]. The survey in its unified form may be found online; see [**8**]. The abstract, keywords, MSC codes, and introduction are the same for both parts.

1. Introduction

Matrix positivity, or positive semidefiniteness, is one of the most wide-reaching concepts in mathematics, old and new. Positivity of a matrix is as natural as positivity of mass in statics or positivity of a probability distribution. It is a notion which has attracted the attention of many great minds. Yet, after at least two centuries of research, positive matrices still hide enigmas and raise challenges for the working mathematician.

The vitality of matrix positivity comes from its breadth, having many theoretical facets and also deep links to mathematical modelling. It is not our aim here to pay homage to matrix positivity in the large. Rather, the present survey, split for technical reasons into two parts, has a limited but carefully chosen scope.

Our panorama focuses on entrywise transforms of matrices which preserve their positive character. In itself, this is a rather bold departure from the dogma that canonical transformations of matrices are not those that operate entry by entry. Still, this apparently esoteric topic reveals a fascinating history, abundant characteristic phenomena and numerous open problems. Each class of positive matrices or kernels (regarding the latter as continuous matrices) carries a specific toolbox of internal transforms. Positive Hankel forms or Toeplitz kernels, totally positive matrices, and group-invariant positive definite functions all possess specific *positivity preservers*. As we see below, these have been thoroughly studied for at least a century.

One conclusion of our survey is that the classification of positivity preservers is accessible in the dimension-free setting, that is, when the sizes of matrices are unconstrained. In stark contrast, precise descriptions of positivity preservers in fixed dimension are elusive, if not unattainable with the techniques of modern mathematics. Furthermore, the world of applications cares much more about matrices of fixed size than in the free case. The accessibility of the latter was by no means a sequence of isolated, simple observations. Rather, it grew organically out of distance geometry, and spread rapidly through harmonic analysis on groups, special functions, and probability theory. The more recent and highly challenging path through fixed dimensions requires novel methods of algebraic combinatorics and symmetric functions, group representations, and function theory.

As well as its beautiful theoretical aspects, our interest in these topics is also motivated by the statistics of big data. In this setting, functions are often applied entrywise to covariance matrices, in order to induce sparsity and improve the quality of statistical estimators (see [**33, 34, 53**]). Entrywise techniques have recently increased in popularity in this area, largely because of their low computational complexity, which makes them ideal to handle the ultra high-dimensional datasets arising in modern applications. In this context, the dimensions of the matrices are fixed, and correspond to the number of underlying random variables. Ensuring that positivity is preserved by these entrywise methods is critical, as covariance matrices must be positive semidefinite. Thus, there is a clear need to produce characterizations of entrywise preservers, so that these techniques are widely applicable and mathematically justified. We elaborate further on this in the second part of the survey.

We conclude by remarking that, while we have tried to be comprehensive in our coverage of the field of matrix positivity and the entrywise calculus, there are very likely to be some inadvertent omissions. Even if our survey is not complete in terms of results and connections, we hope that it serves to impress upon the reader the depth and breadth, the classical history and modern applications, and the influence and beauty of the many facets of positivity.

2. A selection of classical results on entrywise positivity preservers

We begin by mentioning some results from the first part of the survey ([**9**] or [**8**]), which are used or referred to in this part.

2.1. From metric geometry to matrix positivity. As discussed in the first part of the survey, the study of entrywise positivity preservers naturally emerged out of considerations of metric geometry. We recall here some early results of Schoenberg, beginning with the following connection between metric geometry and matrix positivity.

Theorem 2.1 (Schoenberg [**55**]). *Let $d \geq 1$ be an integer and let (X, ρ) be a metric space. An $(n+1)$-tuple of points x_0, x_1, $\ldots$, x_n in X can be isometrically embedded into Euclidean space $\mathbb{R}^d$, but not into $\mathbb{R}^{d-1}$, if and only if the matrix*

$$[\rho(x_0, x_j)^2 + \rho(x_0, x_k)^2 - \rho(x_j, x_k)^2]_{j,k=1}^n \tag{2.1}$$

is positive semidefinite with rank equal to d.

The positivity of the matrix (2.1) is equivalent to the statement that the associated $(n+1) \times (n+1)$ matrix

$$[-\rho(x_j, x_k)^2]_{j,k=0}^n$$

is *conditionally positive semidefinite*: recall that a real symmetric matrix A is conditionally positive semidefinite if $\mathbf{u}^T A \mathbf{u} \geq 0$ whenever the coordinates of the real vector $\mathbf{u}$ sum to zero.

Schoenberg's Theorem 2.1 was perhaps the first time that positive and conditionally positive matrices appeared in the analysis literature. It says that applying the function $-x^2$ entrywise transforms Euclidean-distance matrices into conditionally positive semidefinite matrices. A natural next step to remove the word "conditionally" and ask which functions transform distance matrices, from a given metric space (X, ρ), into positive matrices. This is precisely the definition of *positive definite functions* on (X, ρ).

Schoenberg showed [**56**] that Euclidean spaces are characterized by the property that Gaussian kernels with arbitrary variances are positive definite on them. He similarly showed [**55**] that among metric spaces of diameter no more than π, the unit spheres $S^{d-1} \subset \mathbb{R}^d$ and $S^\infty \subset \ell^2_{\mathbb{R}}$ admit a similar characterization in terms of just one function, cosine. Following this result, and the work of Bochner [**14, 15**] in classifying positive definite functions on Euclidean and compact homogeneous spaces, Schoenberg was interested in understanding classes of positive definite functions on these spheres.

THEOREM 2.2 (Schoenberg [**57**]). *Let $f : [-1,1] \to \mathbb{R}$ be continuous.*

(1) *For a given dimension $d \geq 2$, the function $f \circ \cos$ is positive definite on the unit sphere S^{d-1} if and only if it has a distinguished Fourier-series decomposition with non-negative coefficients. That is,*

$$f(\cos\theta) = \sum_{k=0}^{\infty} c_k P_k^{(\lambda)}(\cos\theta) \qquad (\theta \in \mathbb{R}), \tag{2.2}$$

where $P_k^{(\lambda)}$ are the ultraspherical orthogonal polynomials with $\lambda = (d-2)/2$ and the coefficients $c_k \geq 0$ for all $k \geq 0$ with $\sum_{k=0}^{\infty} c_k < \infty$.

(2) *The function $f(\cos\theta)$ is positive definite on all finite-dimensional spheres, or, equivalently, is positive definite on S^{∞}, if and only if*

$$f(\cos\theta) = \sum_{k=0}^{\infty} c_k \cos^k \theta, \tag{2.3}$$

where $c_k \geq 0$ for all $k \geq 0$ and $\sum_{k=0}^{\infty} c_k < \infty$.

By freeing the previous result from the spherical context, Schoenberg obtained his celebrated result on positivity preservers.

THEOREM 2.3 (Schoenberg [**57**]). *Let $f : [-1,1] \to \mathbb{R}$ be continuous. If the matrix $[f(a_{jk})]_{j,k=1}^n$ is positive semidefinite for all $n \geq 1$ and all positive semidefinite matrices $[a_{jk}]_{j,k=1}^n$ with entries in $[-1,1]$, then, and only then,*

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \qquad (x \in [-1,1]),$$

where $c_k \geq 0$ for all $k \geq 0$ and $\sum_{k=0}^{\infty} c_k < \infty$.

2.2. Entrywise functions preserving positivity in all dimensions. Theorem 2.3 provides a definitive answer to one version of the following central question, which is the driving idea throughout this survey.

Which functions, when applied entrywise to certain classes of matrices, preserve positive semidefiniteness?

The fundamental result for answering this question is the Schur product theorem [**58**]: if A and B are positive semidefinite matrices of the same size, then their entrywise product is positive semidefinite too. As observed by Pólya and Szegö [**51**], the fact that the set of positive matrices forms a closed convex cone immediately implies, by Schur's result, that every power series with non-negative Maclaurin coefficients is a positivity preserver; they asked if there are any other functions with this property. It follows from Schoenberg's Theorem 2.3 that there are no additional continuous functions, and Rudin [**54**] subsequently removed the continuity hypothesis for real-valued functions on $(-1,1)$.

A similar variant was proved by Vasudeva [**60**], for a different domain. To state this result, and for later, we recall some notation from the first part of the survey [**9**].

Definition 2.4. Fix a domain $I \subset \mathbb{C}$ and integers $m, n \geq 1$. Let $\mathcal{P}_n(I)$ denote the set of $n \times n$ Hermitian positive semidefinite matrices with entries in I, with $\mathcal{P}_n(\mathbb{C})$ abbreviated to $\mathcal{P}_n$. A function $f : I \to \mathbb{C}$ acts *entrywise* on a matrix

$$A = [a_{jk}]_{1 \leq j \leq m,\ 1 \leq k \leq n} \in I^{m \times n}$$

by setting

$$f[A] := [f(a_{jk})]_{1 \leq j \leq m,\ 1 \leq k \leq n} \in \mathbb{C}^{m \times n}.$$

Below, we allow the dimensions m and n to vary, while keeping the uniform notation $f[-]$. We also let $\mathbf{1}_{m \times n}$ denote the $m \times n$ matrix with each entry equal to one. Note that $\mathbf{1}_{n \times n} \in \mathcal{P}_n(\mathbb{R})$.

Now we can state Vasudeva's result.

Theorem 2.5 (Vasudeva [**60**]). *Let $f : (0, \infty) \to \mathbb{R}$. Then $f[-]$ preserves positivity on $\mathcal{P}_n\big((0, \infty)\big)$ for all $n \geq 1$, if and only if $f(x) = \sum_{k=0}^{\infty} c_k x^k$ on $(0, \infty)$, where $c_k \geq 0$ for all $k \geq 0$.*

A final variant is for matrices with possibly complex entries. This result was conjectured by Rudin in [**54**] and proved four years later.

Theorem 2.6 (Herz [**37**]). *Let $D(0, 1)$ denote the open unit disc in $\mathbb{C}$, and suppose $f : D(0, 1) \to \mathbb{C}$. The entrywise map $f[-]$ preserves positivity on $\mathcal{P}_n\big(D(0, 1)\big)$ for all $n \geq 1$, if and only if*

$$f(z) = \sum_{j,k \geq 0} c_{jk} z^j \overline{z}^k \qquad \text{for all } z \in D(0, 1),$$

where $c_{jk} \geq 0$ for all j, $k \geq 0$.

2.3. The Horn–Loewner theorem and its variants. The first part of this survey [**9**] focuses on various refinements of our central question when the matrices under consideration are of arbitrary dimension (the "dimension-free" setting). Here, we consider the situation where the dimension N of the test matrices is fixed. This turns out to be highly challenging, and remains open to date for each $N \geq 3$. The following necessary condition was first published by R. Horn (who in [**40**] attributes it to his PhD advisor C. Loewner), and is essentially the only general result known.

Theorem 2.7 (Horn–Loewner [**40**]). *Let $f : (0, \infty) \to \mathbb{R}$ be continuous. Fix a positive integer n and suppose $f[-]$ preserves positivity on $\mathcal{P}_n\big((0, \infty)\big)$. Then $f \in C^{n-3}(I)$,*

$$f^{(k)}(x) \geq 0 \qquad \text{whenever } x \in (0, \infty) \text{ and } 0 \leq k \leq n - 3,$$

and $f^{(n-3)}$ is a convex non-decreasing function on $(0, \infty)$. Furthermore, if $f \in C^{n-1}\big((0, \infty)\big)$, then $f^{(k)}(x) \geq 0$ whenever $x \in (0, \infty)$ and $0 \leq k \leq n - 1$.

This theorem has produced several variants: the arguments are purely local, they involve low-rank matrices, and continuity need not be assumed. Another possibility involves working with real-analytic functions, and we use this below.

Lemma 2.8 (Belton–Guillot–Khare–Putinar [**6**] and Khare–Tao [**43**]). *Let n be a positive integer, suppose $0 < \rho \leq \infty$ and let $f(x) = \sum_{k \geq 0} c_k x^k$ be a convergent power series on $I = [0, \rho)$ that preserves positivity entrywise for all rank-one matrices in $\mathcal{P}_n(I)$. Suppose further that $c_{m'} < 0$ for some m'.*

(1) *If $\rho < \infty$, then $c_m > 0$ for at least n values of $m < m'$. (Thus, the first n non-zero Maclaurin coefficients of f, if they exist, must be positive.)*
(2) *If $\rho = \infty$, then $c_m > 0$ for at least n values of $m < m'$ and at least n values of $m > m'$. (Thus, if f is a polynomial, then the first n non-zero coefficients and the last n non-zero coefficients of f, if they exist, are all positive.)*

These results, and others in the literature for smooth functions, admit a common generalization that was recently obtained.

THEOREM 2.9 (Khare [**42**]). *Let $a \in \mathbb{R}_+$ and $\epsilon \in (0,\infty)$, and supppose $f : [a, a+\epsilon) \to \mathbb{R}$ is smooth. Fix integers n, p, q such that $n \geq 1$ and $0 \leq p \leq q \leq n$, with $p = 0$ if $a = 0$, and such that f has $q - p$ non-zero derivatives at a of order at least p; let*

$$m_p < m_{p+1} < \cdots < m_{q-1}$$

be the orders of these derivatives.

If there exists $\mathbf{u} := (u_1, \ldots, u_n)^T \in (0,1)^n$ with distinct entries and such that $f[a\mathbf{1}_{n\times n} + t\mathbf{u}\mathbf{u}^T] \in \mathcal{P}_n(\mathbb{R})$ for all $t \in [0,\epsilon)$, then the derivative $f^{(k)}(a)$ is non-negative whenever $0 \leq k \leq m_{q-1}$.

The proof of Theorem 2.9 involves a determinant computation that generalizes one by Horn and Loewner, and leads to an unexpected connection to symmetric function theory. See Theorem 3.22 for more details.

2.4. Preservers of positive Hankel matrices. Finally for this chapter, we consider entrywise maps preserving the set of positive Hankel matrices. A distinguished subset of these matrices arise as moment matrices for measures on the real line; we collect some concepts from the first part of the survey.

DEFINITION 2.10. A measure μ with support in $\mathbb{R}$ is said to be *admissible* if μ is non-negative and all its moments are finite:

$$s_k(\mu) := \int_{\mathbb{R}} x^k \, \mathrm{d}\mu(x) < \infty \qquad (k \in \mathbb{Z}_+).$$

The sequence $\mathbf{s}(\mu) := \big(s_k(\mu)\big)_{k=0}^{\infty}$ is the *moment sequence* of μ, and the *moment matrix* of μ is the semi-infinite Hankel matrix

$$H_\mu := \begin{bmatrix} s_0(\mu) & s_1(\mu) & s_2(\mu) & \cdots \\ s_1(\mu) & s_2(\mu) & s_3(\mu) & \cdots \\ s_2(\mu) & s_3(\mu) & s_4(\mu) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

A function $f : \mathbb{R} \to \mathbb{R}$ acts entrywise on moment sequences, so that

$$f[\mathbf{s}(\mu)] := (f\big(s_0(\mu)\big), \ldots, f\big(s_k(\mu)\big), \ldots),$$

and $f[H_\mu] = H_\sigma$ if $f[\mathbf{s}(\mu)] = \mathbf{s}(\sigma)$ for some admissible measure σ.

Working with positive moment matrices and their entrywise preservers provides a route to proving stronger versions of Vasudeva's and Schoenberg's theorems. We conclude this section by stating these results.

THEOREM 2.11 (Belton–Guillot–Khare–Putinar [**7**]). *Suppose $I = (0,\infty)$ and $f : I \to \mathbb{R}$. The following are equivalent.*

(1) *The entrywise map $f[-]$ preserves positivity on $\mathcal{P}_n(I)$ for all $n \geq 1$.*
(2) *There exists $u_0 \in (0,1)$ such that $f[-]$ preserves positivity for all moment matrices of the form H_μ, where $\mu = a\delta_1 + b\delta_{u_0}$ and a, $b \in I$.*
(3) *The function f has a power-series representation $\sum_{k=0}^\infty c_k x^k$ valid for all $x \in I$, where the Maclaurin coefficients $c_k \geq 0$ for all $k \geq 0$.*

THEOREM 2.12 (Belton–Guillot–Khare–Putinar [**7**]). *Suppose $0 < \rho \leq \infty$, let $I = (-\rho, \rho)$ and suppose $f : I \to \mathbb{R}$. The following are equivalent.*
(1) *The entrywise map $f[-]$ preserves positivity on $\mathcal{P}_n(I)$, for all $n \geq 1$.*
(2) *The entrywise map $f[-]$ preserves positivity on the set of Hankel matrices in $\mathcal{P}_n(I)$ of rank at most 3, for all $n \geq 1$.*
(3) *The function f is real analytic, and absolutely monotonic on $(0, \rho)$, so that $f(x) = \sum_{k=0}^\infty c_k x^k$ for all $x \in I$, with $c_k \geq 0$ for all $k \geq 0$.*

3. Entrywise polynomials preserving positivity in fixed dimension

Having discussed at length the dimension-free setting, we now turn our attention to functions that preserve positivity in a fixed dimension $N \geq 2$. This is a natural question from the standpoint of both theory as well as applications. This latter connection to applied fields and to high-dimensional covariance estimation will be explained below in Chapter 5.

Mathematically, understanding the functions f such that $f[-] : \mathcal{P}_N \to \mathcal{P}_N$ for fixed $N \geq 2$, is a non-trivial and challenging refinement of Schoenberg's 1942 theorem. A complete characterization was found for $N = 2$ by Vasudeva [**60**]:

THEOREM 3.1 (Vasudeva [**60**]). *Given a function $f : (0,\infty) \to \mathbb{R}$, the entrywise map $f[-]$ preserves positivity on $\mathcal{P}_2\big((0,\infty)\big)$ if and only f is non-negative, non-decreasing, and multiplicatively mid-convex:*

$$f(x)f(y) \geq f\big(\sqrt{xy}\big)^2 \qquad \text{for all } x, y > 0.$$

In particular, f is either identically zero or never zero on $(0,\infty)$, and f is also continuous.

On the other hand, if $N \geq 3$, then such a characterization remains open to date. As mentioned above, perhaps the only known result for general entrywise preservers is the Horn–Loewner theorem 2.7 or its more general variants, some of which are stated above.

In light of this challenging scarcity of results in fixed dimension, a strategy adopted in the literature has been to further refine the problem, in one of several ways:

(1) Restrict the class of functions, while operating entrywise on all of $\mathcal{P}_N$ (over some given domain I, say $(0,\rho)$ or $(-\rho,\rho)$ for $0 < \rho \leq \infty$). For example, in this survey we consider possibly non-integer power functions, polynomials and power series, and even linear combinations of real powers.
(2) Restrict the class of matrices and study entrywise functions over this class in a fixed dimension. For instance, popular sub-classes of matrices include positive matrices with rank bounded above, or with a given sparsity pattern (zero entries), or classes such as Hankel or Toeplitz matrices; or intersections of these classes. For instance, in discussing the Horn–Loewner and Schoenberg–Rudin results, we encountered Toeplitz and Hankel matrices of low rank.

(3) Study the problem under both of the above restrictions.

In this chapter we begin with the first of these restrictions. Specifically, we will study polynomial maps that preserve positivity, when applied entrywise to $\mathcal{P}_N$. Recall from the Schur product theorem that if the polynomial f has only non-negative coefficients then $f[-]$ preserves positivity on $\mathcal{P}_N$ for every dimension $N \geq 1$. It is natural to expect that if one reduces the test set, from all dimensions to a fixed dimension, then the class of polynomial preservers should be larger. Remarkably, until 2016 not a single example was known of a polynomial positivity preserver with a negative coefficient. Then, in quick succession, the two papers [**6, 43**] provided a complete understanding of the sign patterns of entrywise polynomial preservers of $\mathcal{P}_N$. The goal of this chapter is to discuss some of the results in these works.

3.1. Characterizations of sign patterns. Until further notice, we work with entrywise polynomial or power-series maps of the form

$$f(x) = c_0 x^{n_0} + c_1 x^{n_1} + \cdots, \quad \text{with } 0 \leq n_0 < n_1 < \cdots, \tag{3.1}$$

and $c_j \in \mathbb{R}$ typically non-zero, which preserve $\mathcal{P}_N(I)$ for various I. Our goal is to try and understand their sign patterns, that is, which c_j can be negative. The first observation is that as soon as I contains the interval $(0, \rho)$ for any $\rho > 0$, by the Horn–Loewner type necessary conditions in Lemma 2.8, the lowest N non-zero coefficients of $f(x)$ must be positive.

The next observation is that if $I \not\subset \mathbb{R}_+$, then, in general, there is no structured classification of the sign patterns of the power series preservers on $\mathcal{P}_N(I)$. For example, let k be a non-negative integer; the polynomials

$$f_{k,t}(x) := t(1 + x^2 + \cdots + x^{2k}) - x^{2k+1} \qquad (t > 0)$$

do not preserve positivity entrywise on $\mathcal{P}_N\big((-\rho, \rho)\big)$ for any $N \geq 2$. This may be seen by taking $\mathbf{u} := (1, -1, 0, \dots, 0)^T$ and $A := \eta \mathbf{u}\mathbf{u}^T$ for some $0 < \eta < \rho$, and noting that

$$\mathbf{u}^T f_{k,t}[A] \mathbf{u} = -4\eta^{2k+1} < 0.$$

Similarly, if one allows complex entries and uses higher-order roots of unity, such negative results (vis-a-vis Lemma 2.8) are obtained for complex matrices.

Given this, in the rest of the chapter we will focus on $I = (0, \rho)$ for $0 < \rho \leq \infty$.[1] As mentioned above, if f as in (3.1) entrywise preserves positivity even on rank-one matrices in $\mathcal{P}_N\big((0, \rho)\big)$ then its first N non-zero Maclaurin coefficients are positive. Our goal is to understand if any other coefficient can be negative (and if so, which of them). This has at least two ramifications:

(1) It would yield the first example of a polynomial entrywise map (for a fixed dimension) with at least one negative Maclaurin coefficient. Recall the contrast to Schoenberg's theorem in the dimension-free setting.
(2) This also yields the first example of a polynomial (or power series) that entrywise preserves positivity on $\mathcal{P}_N(I)$ but not $\mathcal{P}_{N+1}(I)$. In particular it would imply that the Horn–Loewner type necessary condition in Lemma 2.8(1) is "sharp".

[1] That said, we also briefly discuss the one situation in which our results do apply more generally, even to $I = D(0, \rho) \subset \mathbb{C}$ (an open complex disc).

These goals are indeed achieved in the particular case $n_0 = 0, \dots, n_{N-1} = N-1$ in [**6**], and subsequently, for arbitrary $n_0 < \cdots < n_{N-1}$ in [**43**]. (In fact, in the latter work the n_j need not even be integers; this is discussed below.) Here is a 'first' result along these lines. Henceforth we assume that $\rho < \infty$; we will relax this assumption midway through Section 3.5 below.

Theorem 3.2 (Belton–Guillot–Khare–Putinar [**6**] and Khare–Tao [**43**]). *Suppose $N \geq 2$ and $n_0 < \cdots < n_{N-1}$ are non-negative integers, and ρ, c_0, ..., c_{N-1} are positive scalars. Given $\epsilon_M \in \{0, \pm 1\}$ for all $M > n_{N-1}$, there exists a power series*

$$f(x) = c_0 x^{n_0} + \cdots + c_{N-1} x^{n_{N-1}} + \sum_{M > n_{N-1}} d_M x^M$$

such that f is convergent on $(0, \rho)$, the entrywise map $f[-]$ preserves positivity on $\mathcal{P}_N\big((0,\rho)\big)$ and d_M has the same sign (positive, negative or zero) as ϵ_M for all $M > n_{N-1}$.

Outline of proof. The claim is such that it suffices to show the result for exactly one $\epsilon_M = -1$. Indeed, given the claim, for each $M > n_{N-1}$ there exists $\delta_M \in (0, 1/M!)$ such that $\sum_{j=0}^{N-1} c_j x^{n_j} + d x^M$ preserves positivity entrywise on $\mathcal{P}_N\big((0,\rho)\big)$ whenever $|d| \leq \delta_M$. Now let $d_M := \epsilon_M \delta_M$ for all $M > n_{N-1}$, and define

$$f_M(x) := \sum_{j=0}^{N_1} c_j x^{n_j} + d_M x^M \quad \text{and} \quad f(x) := \sum_{M > n_{N-1}} 2^{n_{N-1} - M} f_M(x).$$

Then it may be verified that $|f(x)| \leq \sum_{j=0}^{N-1} c_j x^{n_j} + 2^{n_{N-1}} e^{x/2}$, and hence f has the desired properties. $\square$

Thus it suffices to show the existence of a polynomial positivity preserver on $\mathcal{P}_N\big((0,\rho)\big)$ with precisely one negative Maclaurin coefficient, the leading term. In the next few sections we explain how to achieve this goal. In fact, one can show a more general result, for real powers as well.

Theorem 3.3 (Khare–Tao [**43**]). *Fix an integer $N \geq 2$ and real exponents $n_0 < \cdots < n_{N-1} < M$ in the set $\mathbb{Z}_+ \cup [N-2, \infty)$. Suppose ρ, c_0, ..., $c_{N-1} > 0$ as above. Then there exists $c' < 0$ such that the function*

$$f(x) = c_0 x^{n_0} + \cdots + c_{N-1} x^{n_{N-1}} + c' x^M \qquad \big(x \in (0,\rho)\big)$$

preserves positivity entrywise on $\mathcal{P}_N\big((0,\rho)\big)$. [Here and below, we set $0^0 := 1$.]

The restriction of the n_j lying in $\mathbb{Z}_+ \cup [N-2, \infty)$ is a technical one that is explained in a later chapter on the study of entrywise powers preserving positivity on $\mathcal{P}_N\big((0,\infty)\big)$; see Theorem 4.1.

Remark 3.4. A stronger result, Theorem 3.15, which also applies to real powers, is stated below. We mention numerous ramifications of the results in this chapter following that result.

The proofs of the preceding two theorems crucially use type-A representation theory (specifically, a family of symmetric functions) that naturally emerges here via generalized Vandermonde determinants. These symmetric homogeneous polynomials are introduced and used in the next section.

For now, we explain how Theorem 3.3 helps achieve a complete classification of the sign patterns of a family of generalised power series, of the form

$$f(x) = \sum_{j=0}^{\infty} c_j x^{n_j}, \qquad n_j \in \mathbb{Z}_+ \cup [N-2, \infty) \text{ for all } j \geq 0,$$

but without the requirement that that exponents are non-decreasing. In this generality, one first notes that the Horn–Loewner-type Lemma 2.8 still applies: if some coefficient $c_{j_0} < 0$, then there must be at least N indices j such that $n_j < n_{j_0}$ and $c_j > 0$. The following result shows that once again, this necessary condition is best possible.

THEOREM 3.5 (Classification of sign patterns for real-power series preservers, Khare–Tao [**43**]). *Fix an integer $N \geq 2$, and distinct real exponents n_0, n_1, ... in $\mathbb{Z}_+ \cup [N-2, \infty)$. Suppose $\epsilon_j \in \{0, \pm 1\}$ is a choice of sign for each $j \geq 0$, such that if $\epsilon_{j_0} = -1$ then $\epsilon_j = +1$ for at least N choices of j such that $n_j < n_{j_0}$. Given any $\rho > 0$, there exists a choice of coefficients c_j with sign ϵ_j such that*

$$f(x) := \sum_{j=0}^{\infty} c_j x^{n_j}$$

is convergent on $(0, \rho)$ and preserves positivity entrywise on $\mathcal{P}_N\big((0,\rho)\big)$.

Notice this result is strictly more general than Theorem 3.2, because the sequence $n_0, n_1, \ldots$ can contain an infinite decreasing sequence of positive non-integer powers, for example, all rational elements of $[N-2, \infty)$. Thus Theorem 3.5 covers a larger class of functions than even Hahn or Puiseux series.

Theorem 3.5 is derived from Theorem 3.3 in a similar fashion to the proof of Theorem 3.2, and we refer the reader to [**43**, Section 1] for the details.

3.2. Schur polynomials; the sharp threshold bound for a single matrix. We now explain how to prove Theorem 3.3. The present section will discuss the case of integer powers, and end by proving the theorem for a single 'generic' rank-one matrix. In the following section we show how to extend the results to all rank-one matrices for integer powers. The subsequent section will complete the proof for real powers, and then for matrices of all ranks.

The key new tool that is indispensable to the following analysis is that of *Schur polynomials*. These can be defined in a number of equivalent ways; we refer the reader to [**16**] for more details, including the equivalence of these definitions shown using ideas of Karlin–Macgregor, Lindström, and Gessel–Viennot. For our purposes the definition of Cauchy is the most useful:

DEFINITION 3.6. Given non-negative integers $N \geq 1$ and $n_0 < \cdots < n_{N-1}$, let

$$\mathbf{n} := (n_0, \ldots, n_{N-1})^T, \qquad \text{and} \quad \mathbf{n}_{\min} := (0, 1, \ldots, N-1)^T,$$

and define $V(\mathbf{n}) := \prod_{0 \leq i < j \leq N-1} (n_j - n_i)$.

Given a vector $\mathbf{u} = (u_1, \ldots, u_N)^T$ and a non-negative integer k, let $\mathbf{u}^{\circ k} := (u_1^k, \ldots, u_N^k)^T$, and let $\mathbf{u}^{\circ \mathbf{n}}$ be the $N \times N$ matrix with (j,k) entry $\mathbf{u}_j^{n_{k-1}}$.

The *Schur polynomial* in variables $u_1, \ldots, u_N$ of *degree* $\mathbf{n}$ is given by

$$s_{\mathbf{n}}(\mathbf{u}) := \frac{\det \mathbf{u}^{\circ \mathbf{n}}}{\det \mathbf{u}^{\circ \mathbf{n}_{\min}}}. \tag{3.2}$$

Notice that the numerator is a generalized Vandermonde determinant, so a homogeneous and alternating polynomial, while the denominator is the usual Vandermonde determinant in the indeterminates u_j. Hence their ratio $s_{\mathbf{n}}(\mathbf{u})$ is a homogeneous symmetric polynomial in $\mathbb{Z}[u_1, \dots, u_N]$. It follows that Schur polynomials are well defined when working over any commutative unital ring.

Schur polynomials are an extremely well-studied family of symmetric functions. Their appeal lies in the important observation that they are the characters of all irreducible (finite-dimensional) polynomial representations of the complex Lie group $GL_n(\mathbb{C})$ (or of the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$). In this setting, the definition of Cauchy is a special case of the Weyl character formula. Thus, its specialization yields the corresponding Weyl dimension formula, which will be of use below:

$$s_{\mathbf{n}}((1, \dots, 1)^T) = \prod_{0 \le i < j \le N-1} \frac{n_j - n_i}{j - i} = \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})}. \tag{3.3}$$

An alternate proof of (3.3) comes from the *principal specialization formula*: for a variable q, one has that

$$s_{\mathbf{n}}\big((1, q, \dots, q^{N-1})^T\big) = \prod_{0 \le i < j \le N-1} \frac{q^{n_j} - q^{n_i}}{q^j - q^i}; \tag{3.4}$$

this follows from (3.2) because now the numerator is also a standard Vandermonde determinant. We also refer the reader to [**48**] for many more results and properties of Schur polynomials.

Returning to polynomial positivity preservers, we wish to consider functions of the form

$$f(x) = c_0 x^{n_0} + \cdots + c_{N-1} x^{n_{N-1}} + c' x^M,$$

with non-negative integers $n_0 < \cdots < n_{N-1} < M$ and positive coefficients $c_0, \dots, c_{N-1}$. We are interested in characterizing those $c' \in \mathbb{R}$ for which the entrywise map $f[-]$ preserve positivity on $\mathcal{P}_N\big((0, \rho)\big)$. By the Schur product theorem, this is equivalent to finding the smallest c' such that $f[-]$ is a preserver. We may assume that $c' < 0$, so we rescale by $t := |c'|^{-1}$ and define

$$p_t(x) := t \sum_{j=0}^{N-1} c_j x^{n_j} - x^M. \tag{3.5}$$

The goal now is to find the smallest $t > 0$ such that $p_t[-]$ preserves positivity on $\mathcal{P}_N\big((0, \rho)\big)$. We next achieve this goal for a single rank-one matrix.

Proposition 3.7. *With notation as above, define*

$$\mathbf{n}_j = (n_0, \dots, n_{j-1}, \widehat{n_j}, n_{j+1}, \dots, n_{N-1}, M)^T$$

for $0 \le j \le N-1$. Given a vector $\mathbf{u} \in (0, \infty)^N$ with distinct coordinates, the following are equivalent.

1. *The matrix $p_t[\mathbf{u}\mathbf{u}^T]$ is positive semidefinite.*
2. $\det p_t[\mathbf{u}\mathbf{u}^T] \ge 0$.
3. $t \ge \displaystyle\sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_j s_{\mathbf{n}}(\mathbf{u})^2}$.

In particular, this shows that for a generic rank-one matrix in $\mathcal{P}_N\big((0, \rho)\big)$, there does exist a positivity-preserving polynomial with a negative leading term.

In essence, the equivalences in Proposition 3.7 hold more generally; this is distilled into the following lemma.

LEMMA 3.8 (Khare–Tao [**44**][2]). *Fix $\mathbf{w} \in \mathbb{R}^N$ and a positive-definite matrix H. Fix $t > 0$ and define $P_t := tH - \mathbf{w}\mathbf{w}^T$. The following are equivalent.*

(1) *P_t is positive semidefinite.*
(2) $\det P_t \geq 0$.
(3) $t \geq \mathbf{w}^T H^{-1} \mathbf{w} = 1 - \dfrac{\det(H - \mathbf{w}\mathbf{w}^T)}{\det H}$.

We refer the reader to [**44**] for the detailed proof of Lemma 3.8, remarking only that the equality in assertion (3) follows by using Schur complements in two different ways to expand the determinant of the matrix $\begin{bmatrix} H & \mathbf{w} \\ \mathbf{w}^T & 1 \end{bmatrix}$.

Now Proposition 3.7 follows directly from Lemma 3.8, by setting

$$H = \sum_{j=0}^{N-1} c_j \mathbf{u}^{\circ n_j} (\mathbf{u}^{\circ n_j})^T \quad \text{and} \quad \mathbf{w} = \mathbf{u}^{\circ M},$$

where H is positive definite because of the following general matrix factorization (which is also used below).

PROPOSITION 3.9. *Let $f(x) = \sum_{k=0}^M f_k x^k$ be a polynomial with coefficients in a commutative ring R. For any integer $N \geq 1$ and any vectors $\mathbf{u} = (u_1, \dots, u_N)^T$ and $\mathbf{v} = (v_1, \dots, v_N)^T \in R^N$, it holds that*

$$\begin{aligned} f[t\mathbf{u}\mathbf{v}^T] &= \sum_{k=0}^M f_k t^k \mathbf{u}^{\circ k} (\mathbf{v}^{\circ k})^T \\ &= \begin{bmatrix} 1 & u_1 & \cdots & u_1^M \\ 1 & u_2 & \cdots & u_2^M \\ \vdots & \vdots & \ddots & \vdots \\ 1 & u_N & \cdots & u_N^M \end{bmatrix} \begin{bmatrix} f_0 & 0 & \cdots & 0 \\ 0 & f_1 t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_M t^M \end{bmatrix} \begin{bmatrix} 1 & v_1 & \cdots & v_1^M \\ 1 & v_2 & \cdots & v_2^M \\ \vdots & \vdots & \ddots & \vdots \\ 1 & v_N & \cdots & v_N^M \end{bmatrix}^T, \end{aligned} \tag{3.6}$$

where 1 is a multiplicative identity which is adjoined to R if necessary.

Now to adopt Lemma 3.8(3), this same equation and the Cauchy–Binet formula allow one to compute $\det(H - \mathbf{w}\mathbf{w}^T)$ in the present situation, and this yields precisely that $t \geq \displaystyle\sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_j s_{\mathbf{n}}(\mathbf{u})^2}$, as desired.

3.3. The threshold for all rank-one matrices: a Schur positivity result. We continue toward a proof of Theorem 3.3. The next step is to use Proposition 3.7 to achieve an intermediate goal: a threshold bound for c' that works for all rank-one matrices in $\mathcal{P}_N\big((0,\rho)\big)$, still working with integer powers. Clearly, to do so one has to understand the supremum of each ratio $R_j := s_{\mathbf{n}_j}(\mathbf{u})^2 / s_{\mathbf{n}}(\mathbf{u})^2$, as $\mathbf{u}$ runs over vectors in $(0, \sqrt{\rho})^N$ with distinct coordinates. More precisely, one has to understand the supremum of the weighted sum $\sum_j R_j / c_j$.

---

[2]The work [**44**] is an extended abstract of the paper [**43**], but some of the results in it have different proofs from [**43**].

This observation was first made in the work [**6**] for the case $n_j = j$, that is, $\mathbf{n} = \mathbf{n}_{\min}$. It led to the first proof of Theorem 3.3, with all of the denominators being the same: $s_{\mathbf{n}_{\min}}(\mathbf{u}) = 1$. We now use another equivalent definition of Schur polynomials, by Littlewood, realizing them as sums of monomials corresponding to certain Young tableaux. Every monomial has a non-negative integer coefficient. It follows by the continuity and homogeneity of $s_{\mathbf{n}_j}$ and the Weyl Dimension Formula (3.3), that the supremum in the previous paragraph equals the value at $(\sqrt{\rho}, \dots, \sqrt{\rho})^T$, namely

$$\sup_{\mathbf{u} \in (0,\sqrt{\rho})^N} s_{\mathbf{n}_j}(\mathbf{u})^2 = \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \rho^{M-n_j}.$$

Since all of these suprema are attained at the same point $\sqrt{\rho}(1, \dots, 1)^T$, the weighted sum in Proposition 3.7(3) also attains its supremum at the same point. Thus, we conclude using Proposition 3.7 that

$$f(x) = \sum_{j=0}^{N-1} c_j x^{n_j} + c' x^M$$

preserves positivity entrywise on all rank-one matrices $\mathbf{u}\mathbf{u}^T \in \mathcal{P}_N\big((0,\rho)\big)$ if and only if

$$c' \geq -\bigg(\sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{c_j V(\mathbf{n}_{\min})^2} \rho^{M-n_j} \bigg)^{-1}.$$

In fact, if $\mathbf{n} = \mathbf{n}_{\min}$ then the entire argument above goes through even when one changes the domain to the open complex disc $D(0,\rho)$, or any intermediate domain $(0,\rho) \subset D \subset D(0,\rho)$. This is precisely the content of the main result in [**6**].

Theorem 3.10 (Belton–Guillot–Khare–Putinar [**6**]). *Fix $\rho > 0$ and integers $M \geq N \geq 2$. Let*

$$f(z) = \sum_{j=0}^{N-1} c_j z^j + c' z^M, \qquad \text{where } c_0, \dots, c_{N-1}, c' \in \mathbb{R},$$

and let $I := \overline{D}(0,\rho)$ be the closed disc in the complex plane with centre 0 and radius ρ. The following are equivalent.

(1) *The entrywise map $f[-]$ preserves positivity on $\mathcal{P}_N(I)$.*
(2) *The entrywise map $f[-]$ preserves positivity on rank-one matrices in $\mathcal{P}_N\big((0,\rho)\big)$.*
(3) *Either $c_0, \dots, c_{N-1}, c'$ are all non-negative, or $c_0, \dots, c_{N-1}$ are positive and*

$$c' \geq -\bigg(\sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{c_j V(\mathbf{n}_{\min})^2} \rho^{M-j} \bigg)^{-1},$$

where $\mathbf{n}_j := (0, 1, \dots, j-1, \widehat{j}, j+1, \dots, N-1, M)^T$ for $0 \leq j \leq N-1$.

This theorem provides a complete understanding of which polynomials of degree at most N preserve positivity entrywise on $\mathcal{P}_N\big((0,\rho)\big)$ and, more generally, on any subset of $\mathcal{P}_N\big(\overline{D}(0,\rho)\big)$ that contains the rank-one matrices in $\mathcal{P}_N\big((0,\rho)\big)$.

Remark 3.11. Clearly (1) $\implies$ (2) here, and the proof of (2) $\iff$ (3) was outlined above via Proposition 3.7. We defer mentioning the proof strategy for (2) $\implies$ (1), because we will later see a similar theorem over $I = (0,\rho)$ for more general powers n_j. The proof of that result, Theorem 3.15, will be outlined in some detail.

Having dealt with the base case of $\mathbf{n} = \mathbf{n}_{\min}$, as well as $\mathbf{n} = (k, k+1, \ldots, k+N-1)$ for any $k \in \mathbb{Z}_+$, which holds by the Schur product theorem, we now turn to the general case. In general, $s_{\mathbf{n}}(\mathbf{u})$ is no longer a monomial, and so it is no longer clear if and where the supremum of each ratio $s_{\mathbf{n}_j}(\mathbf{u})^2/s_{\mathbf{n}}(\mathbf{u})^2$, or of their weighted sum, is attained for $\mathbf{u} \in (0, \sqrt{\rho})^N$. The threshold bound for all rank-one matrices itself is not apparent, and the bound for all matrices in $\mathcal{P}_N\big((0,\rho)\big)$ is even more inaccessible.

By a mathematical miracle, it turns out that the same phenomena as in the base case hold in general. Namely, the ratio of each $s_{\mathbf{n}_j}$ and $s_{\mathbf{n}}$ attains its supremum at $\sqrt{\rho}(1, \ldots, 1)^T$. Hence one can proceed as above to obtain a uniform threshold for c', which works for all rank-one matrices in $\mathcal{P}_N\big((0,\rho)\big)$.

Example 3.12. To explain the ideas of the preceding paragraph, we present an example. Suppose

$$N = 3, \qquad \mathbf{n} = (0,2,3), \qquad M = 4, \quad \text{and} \quad \mathbf{u} = (u_1, u_2, u_3)^T.$$

Then

$$\begin{aligned} \mathbf{n}_3 &= (0,2,4), \\ s_{\mathbf{n}}(\mathbf{u}) &= u_1u_2 + u_2u_3 + u_3u_1, \\ \text{and} \quad s_{\mathbf{n}_3}(\mathbf{u}) &= (u_1+u_2)(u_2+u_3)(u_3+u_1). \end{aligned}$$

The claim is that $s_{\mathbf{n}_3}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ is coordinatewise non-decreasing for $\mathbf{u} \in (0,\infty)^3$; the assertion about its supremum on $(0, \sqrt{\rho})^N$ immediately follows from this. It suffices by symmetry to show the claim only for one variable, say u_3. By the quotient rule,

$$s_{\mathbf{n}}(\mathbf{u})\partial_{u_3} s_{\mathbf{m}}(\mathbf{u}) - s_{\mathbf{m}}(\mathbf{u})\partial_{u_3} s_{\mathbf{n}}(\mathbf{u}) = (u_1+u_2)(u_1u_3 + 2u_1u_2 + u_2u_3)u_3,$$

and this is clearly non-negative on the positive orthant, proving the claim. As we see, the above expression is, in fact, monomial positive, from which numerical positivity follows immediately.

In fact, an even stronger fact holds. Viewed as a polynomial in u_3, every coefficient in the above expression is in fact *Schur positive*. In other words, the coefficient of each u_3^j is a non-negative combination of Schur polynomials in u_1 and u_2:

$$(u_1+u_2)(u_1u_3 + 2u_1u_2 + u_2u_3)u_3 = \sum_{j \geq 0} p_j(u_1,u_2)u_3^j,$$

where

$$p_j(u_1,u_2) = \begin{cases} 2s_{(1,3)}(u_1,u_2) & \text{if } j = 1, \\ s_{(0,3)}(u_1,u_2) + s_{(1,2)}(u_1,u_2) & \text{if } j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this implies that each coefficient is monomial positive, whence numerically positive. We recall here that the monomial positivity of Schur polynomials follows from the definition of $s_{\mathbf{n}}(\mathbf{u})$ using Young tableaux.

The miracle to which we alluded above, is that the Schur positivity in the preceding example in fact holds in general.

THEOREM 3.13 (Khare–Tao [**43**]). *If $n_0 < \cdots < n_{N-1}$ and $m_0 < \cdots < m_{N-1}$ are N-tuples of non-negative integers such that $m_j \geq n_j$ for $j = 0, \ldots, N-1$, then the function*

$$f_{\mathbf{m},\mathbf{n}} : (0,\infty)^N \to \mathbb{R};\ \mathbf{u} \mapsto \frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})}$$

is non-decreasing in each coordinate. Furthermore, if

$$s_{\mathbf{n}}(\mathbf{u})\partial_{u_N} s_{\mathbf{m}}(\mathbf{u}) - s_{\mathbf{m}}(\mathbf{u})\partial_{u_N} s_{\mathbf{n}}(\mathbf{u}) \tag{3.7}$$

is considered as a polynomial in u_N, then the coefficient of every monomial u_N^j is a Schur-positive polynomial in $u_1, \ldots, u_{N-1}$.

The second, stronger part of Theorem 3.13 follows from a deep and highly non-trivial result in symmetric function theory (or type-A representation theory) by Lam, Postnikov, and Pylyavskyy [**45**], following earlier results by Skandera. We refer the reader to this paper and to [**43**] for more details. Notice also that the first assertion in Theorem 3.13 only requires the numerical positivity of the expression (3.7). This is given a separate proof in [**43**], using the method of condensation due to Charles Lutwidge Dodgson [**18**].[3] In this context, we add for completeness that in [**43**] the authors also show a log-supermodularity (or FKG, or MTP_2) phenomenon for determinants of totally positive matrices.

3.4. Real powers; the threshold works for all matrices. We now return to the proof of Theorem 3.3, which holds for real powers. Our next step is to observe that the first part of Theorem 3.13 now holds for all real powers. Since one can no longer define Schur polynomials in this case, we work with generalized Vandermonde determinants instead:

COROLLARY 3.14. *Fix N-tuples of real powers $\mathbf{n} = (n_0 < \cdots < n_{N-1})$ and $\mathbf{m} = (m_0 < \cdots < m_{N-1})$, such that $n_j \leq m_j$ for all j. Letting $\mathbf{u}^{\circ \mathbf{n}} := [u_j^{n_{k-1}}]_{j,k=1}^N$ as above, the function*

$$f : \{\mathbf{u} \in (0,\infty)^N : u_i \neq u_j \text{ if } i \neq j\} \to \mathbb{R};\ \mathbf{u} \mapsto \frac{\det \mathbf{u}^{\circ \mathbf{m}}}{\det \mathbf{u}^{\circ \mathbf{n}}}$$

is non-decreasing in each coordinate.

We sketch here one proof. The version for integer powers, Theorem 3.13, gives the version for rational powers, by taking a "common denominator" $L \in \mathbb{Z}$ such that Lm_j and Ln_j are all integers, and using a change of variables $y_j := u_j^{1/L}$. The general version for real powers then follows by considering rational approximations and taking limits.

Corollary 3.14 helps prove the real-power version of Theorem 3.3, just as Theorem 3.13 would have shown the integer powers case of Theorem 3.3. Namely, first note that Proposition 3.7 holds even when the n_j are real powers; the only changes are (a) to assume that the coordinates of $\mathbf{u}$ are distinct, and (b) to rephrase the last assertion (3) to the following:

$$t \geq \sum_{j=0}^{N-1} \frac{(\det \mathbf{u}^{\circ \mathbf{n}_j})^2}{c_j (\det \mathbf{u}^{\circ \mathbf{n}})^2}.$$

[3] This article by Dodgson immediately follows his better-known 1865 publication, *Alice's Adventures in Wonderland*.

These arguments help prove the first part of the following result, which is the culmination of these ideas.

THEOREM 3.15 (Khare–Tao [**43**]). *Fix an integer $N \geq 1$ and real exponents $n_0 < \cdots < n_{N-1} < M$, as well as scalars $\rho > 0$ and $c_0, \ldots, c_{N-1}, c'$. Let*

$$f(x) := \sum_{j=0}^{N-1} c_j x^{n_j} + c' x^M.$$

The following are equivalent.

(1) *The function f preserves positivity entrywise on all rank-one matrices in $\mathcal{P}_N\big((0,\rho)\big)$.*
(2) *The function f preserves positivity entrywise on all Hankel rank-one matrices in $\mathcal{P}_N\big((0,\rho)\big)$.*
(3) *Either the coefficients $c_0, \ldots, c_{N-1}$ and c' are non-negative, or $c_0, \ldots, c_{N-1}$ are positive and*

$$c' \geq -\left(\sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{c_j V(\mathbf{n})^2} \rho^{M-n_j} \right)^{-1},$$

where $V(\mathbf{n})$ and $\mathbf{n}_j$ are as defined above.

If, moreover, the exponents n_j all lie in $\mathbb{Z}_+ \cup [N-2, \infty)$, then these assertions are also equivalent to the following.

(4) *The function f preserves positivity entrywise on $\mathcal{P}_N\big((0,\rho)\big)$.*

Before sketching the proof, we note several ramifications of this result.

(1) The theorem completely characterizes linear combinations of up to $N+1$ powers that entrywise preserve positivity on $\mathcal{P}_N\big((0,\rho)\big)$. The same is true for any subset of $\mathcal{P}_N\big((0,\rho)\big)$ that contains all rank-one positive semidefinite Hankel matrices.
(2) As discussed above, Theorem 3.15 implies Theorem 3.5, which helps in understanding which sign patterns correspond to countable sums of real powers that preserve positivity entrywise on $\mathcal{P}_N\big((0,\rho)\big)$ (or on the subset of rank-one matrices). In particular, the existence of sign patterns which are not all non-negative shows the existence of functions which preserve positivity on $\mathcal{P}_N$ but not on $\mathcal{P}_{N+1}$.
(3) Theorem 3.15 bounds $A^{\circ M}$ in terms of a multiple of $\sum_{j=0}^{N-1} c_j A^{\circ n_j}$. More generally, one can do this for an arbitrary convergent power series instead of a monomial, in the spirit of Theorem 3.2. Even more generally, one may work with Laplace transforms of measures; see Corollary 3.17 below.

For completeness, we also mention two developments related (somewhat more distantly) to the above results.

- A refinement of a conjecture of Cuttler, Greene, and Skandera (2011) and its proof; see [**43**] for more details. In particular, this approach assists with a novel characterization of weak majorization, using Schur polynomials.
- A related "Schubert cell-type" stratification of the cone $\mathcal{P}_N(\mathbb{C})$; see [**6**] for further details.

We conclude this section by outlining the proof of Theorem 3.15.

PROOF. Clearly, (4) $\implies$ (1) $\implies$ (2). If (2) holds, then, by Lemma 2.8, either all the c_j and c' are non-negative, or c_j is positive for all j. Thus, we suppose that $c_j > 0 > c'$.

Note that if $\mathbf{u}(u_0) := (1, u_0, \ldots, u_0^{N-1})^T$ for some $u_0 \in (0,1)$, then

$$A(u_0) := \rho u_0^2 \mathbf{u}(u_0)\mathbf{u}(u_0)^T$$

is a rank-one Hankel matrix and hence in our test set. Repeating the analysis in Section 3.2, using generalized Vandermonde determinants instead of Schur polynomials and rank-one Hankel matrices of the form $A(u_0)$,

$$\begin{aligned} |c'|^{-1} &\geq \sup_{u_0 \in (0,1)} \sum_{j=0}^{N-1} \frac{(\det[\sqrt{\rho} u_0 \mathbf{u}(u_0)]^{\circ \mathbf{n}_j})^2}{c_j (\det[\sqrt{\rho} u_0 \mathbf{u}(u_0)]^{\circ \mathbf{n}})^2} \\ &= \sum_{j=0}^{N-1} \lim_{u_0 \to 1^-} \sum_{j=0}^{N-1} \frac{(\det \mathbf{u}(u_0)^{\circ \mathbf{n}_j})^2}{c_j (\det \mathbf{u}(u_0)^{\circ \mathbf{n}})^2} (\rho u_0^2)^{M - n_j}, \end{aligned}$$

where the equality follows from Corollary 3.14 above. The real-exponent version of (3.4) holds if $q \in (0,\infty) \setminus \{1\}$ and the exponents n_j are real and non-decreasing:

$$\det \mathbf{u}(q)^{\circ \mathbf{n}} = \prod_{0 \leq i < k \leq N-1} (q^{n_k} - q^{n_i}) = V(q^{\circ \mathbf{n}}).$$

Applying this identity, the above computation yields

$$|c'|^{-1} \geq \lim_{u_0 \to 1^-} \sum_{j=0}^{N-1} \frac{V(u_0^{\circ \mathbf{n}_j})^2}{V(u_0^{\circ \mathbf{n}})^2} \frac{(\rho u_0^2)^{M-n_j}}{c_j} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{c_j V(\mathbf{n})^2} \rho^{M-n_j}.$$

Thus (2) $\implies$ (3). Conversely, that (3) $\implies$ (1) follows by a similar analysis to that given above, using Corollary 3.14 and the density of matrices $\mathbf{u}\mathbf{u}^T$, where $\mathbf{u} \in \big(0, \sqrt{\rho}\big)^N$ has distinct entries, in the set of all rank-one matrices in $\mathcal{P}_N\big((0,\rho)\big)$.

It remains to show that (1) $\implies$ (4) if all the exponents $n_j \in \mathbb{Z}_+ \cup [N-2, \infty)$. We proceed by induction on N. The case $N = 1$ is immediate. For the inductive step, we apply the extension principle of the following Proposition 3.16 with $h = f$, which requires verification that $f'[-]$ preserves positivity on $\mathcal{P}_{N-1}$. This is a straightforward calculation via the induction hypothesis. □

The following extension principle was inspired by work of FitzGerald and Horn [**25**].

PROPOSITION 3.16 (Khare–Tao [**43**]). *Suppose $0 < \rho \leq \infty$, and $I = (0, \rho)$, $(-\rho, \rho)$ or the closure of one of these sets. Let $h : I \to \mathbb{R}$ be a continuously differentiable function on the interior of I. If $h'[-]$ preserves positivity entrywise on $\mathcal{P}_{N-1}(I)$ and $h[-]$ does so on the rank-one matrices in $\mathcal{P}_N(I)$, then $h[-]$ in fact preserves positivity on all of $\mathcal{P}_N(I)$.*[4]

Proposition 3.16 relies on two arguments found in [**25**]: (a) every matrix in $\mathcal{P}_N$ may be written as the sum of a rank-one matrix in $\mathcal{P}_N$, and a matrix in $\mathcal{P}_{N-1}$ with its last row and column both zero, and (b) applying the integral identity

$$h(x) - h(y) = \int_x^y h'(t)\,\mathrm{d}t = \int_0^1 (x-y) h'(\lambda x + (1-\lambda) y)\,\mathrm{d}\lambda$$

[4]An analogous version of this results holds for $I = D(0,\rho)$ or its closure in $\mathbb{C}$, with $h : I \to \mathbb{C}$ analytic. This is used to prove the corresponding implication in Theorem 3.10 above.

entrywise to this decomposition. See [**43**, Section 3] for more details. The original use of these arguments was when h is a power function; this is explained in Chapter 4 below.

3.5. Power series preservers and beyond; unbounded domains. In the remainder of this chapter, we use Theorem 3.15 to derive several corollaries; thus, we retain and use the notation of that theorem. As discussed following Theorem 3.15, the first consequence extends the theorem from bounding monomials $A^{\circ M} = (x^M)[A]$ by a multiple of $\sum_{j=0}^{N-1} c_j A^{\circ n_j}$, to bounding $f[A]$ for more general power series. Even more generally, one can work with Laplace transforms of real measures on $\mathbb{R}$.

Corollary 3.17 (Khare–Tao [**43**]). *Let the notation be as for Theorem 3.15, with $c_j > 0$ for all j. Suppose μ is a real measure supported on $[n_{N-1} + \epsilon, \infty)$ for some $\epsilon > 0$, and let*

$$g_\mu(x) := \int_{n_{N-1}+\epsilon}^{\infty} x^t \, \mathrm{d}\mu(t). \tag{3.8}$$

If g_μ is absolutely convergent at ρ, then there exists a finite threshold $t_\mu > 0$ such that, for all $A \in \mathcal{P}_N\big((0,\rho)\big)$, the matrix

$$t_\mu \sum_{j=0}^{N-1} c_j A^{\circ n_j} - g_\mu[A]$$

is positive semidefinite.

Proof. By Theorem 3.15 and the fact that $\mathcal{P}_N(\mathbb{R})$ is a closed convex cone, it suffices to show the finiteness of the quantity

$$\int_{n_{N-1}+\epsilon}^{\infty} \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{c_j V(\mathbf{n})^2} \rho^{M-n_j} \, \mathrm{d}\mu_+(M),$$

where μ_+ is the positive part of μ. This follows from the hypotheses. $\square$

We now turn to the $\rho = \infty$ case, which was briefly alluded to above. In other words, the domain is now unbounded: $I = (0, \infty)$. As in the bounded-domain case, the question of interest is to classify all possible sign patterns of polynomial or power-series preservers on $\mathcal{P}_N(I)$ for a fixed integer N.

Similar to the above discussion for bounded I, the crucial step in classifying sign patterns of power series (or more general functions, as in Theorem 3.5) is to work with integer powers and precisely one coefficient that can be negative. Thus, one first observes that Lemma 2.8(2) holds in the unbounded-domain case $I = (0, \infty)$. Hence given a polynomial

$$f(x) = \sum_{j=0}^{2N-1} c_j x^{n_j} + c' x^M,$$

where

$$0 \leq n_0 < \cdots < n_{N-1} < M < n_N < n_{N+1} \cdots < n_{2N-1},$$

if $f[-]$ preserves positivity on $\mathcal{P}_N\big((0,\infty)\big)$, then either all the coefficients $c_0, \ldots, c_{2N-1}, c'$ are non-negative, or $c_0, \ldots, c_{2N-1}$ are positive and c' can be negative. In this case, an explicit threshold is not known as it is in Theorem 3.15, but we now explain why such a threshold exists.

We start from (3.6) and repeat the subsequent analysis via the Cauchy–Binet formula. To find a uniform threshold for c' that works for all rank-one matrices in $\mathcal{P}_N\big((0,\infty)\big)$, it suffices to bound, uniformly from above, certain ratios of sums of squares of Schur polynomials. This may be done because of the following tight bounds.

PROPOSITION 3.18 (Khare–Tao [**43**]). *If* $\mathbf{n} := (n_0, \dots, n_{N-1})$ *and* $\mathbf{u} := (u_1, \dots, u_N)$, *where* $n_0 < \cdots < n_{N-1}$ *are non-negative integers and* $u_1 \leq \cdots \leq u_N$ *are non-negative real numbers, then*

$$\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}} \leq s_{\mathbf{n}}(\mathbf{u}) \leq \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})} \mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}, \tag{3.9}$$

where $\mathbf{n}_{\min} := (0, \dots, n_{N-1})$. *The constants* 1 *and* $V(\mathbf{n})/V(\mathbf{n}_{\min})$ *on each side of (3.9) cannot be improved.*

We refer the reader to [**43**, Section 4] for further details, including how Proposition 3.18 implies the existence of preservers f as above for rank-one matrices with $c' < 0$. The extension from rank-one matrices to all of $\mathcal{P}_N\big((0,\infty)\big)$ is carried out using the extension principle in Proposition 3.16.

In a sense, Proposition 3.18 isolates the 'leading term' of every Schur polynomial. This calculation can be generalized to the case of non-integer powers,[5]which helps extend the above results for the unbounded domain $I = (0,\infty)$ to real powers. This yields the desired classification, similar to Theorem 3.5 in the bounded-domain case.

THEOREM 3.19 (Khare–Tao [**43**]). *Let* $N \geq 2$, *and let* $\{\alpha_j : j \geq 0\} \subset \mathbb{Z}_+ \cup [N-2,\infty)$ *be a set of distinct real numbers. For each* $j \geq 0$, *let* $\epsilon_j \in \{0, \pm 1\}$ *be a sign and suppose that, whenever* $\epsilon_{j_0} = -1$, *then* $\epsilon_j = +1$ *for at least* N *choices of* j *such that* $\alpha_j < \alpha_{i_0}$ *and also for at least* N *choices of* j *such that* $\alpha_j > \alpha_{i_0}$. *There exists a series with real coefficients,*

$$f(x) = \sum_{j=0}^{\infty} c_j x^{\alpha_j}$$

which converges on $(0,\infty)$, *preserves positivity entrywise on* $\mathcal{P}_N\big((0,\infty)\big)$, *and is such that* c_j *has the same sign as* ϵ_j *for all* $j \geq 0$.

Note that, in particular, Theorem 3.19 reaffirms that the Horn–Loewner-type conditions in Lemma 2.8(2) are sharp.

3.6. Digression: Schur polynomials from smooth functions, and new symmetric function identities. Before proceeding to additional applications of Theorem 3.15 and related results, we take a brief detour to explain how Schur polynomials arise naturally from any sufficiently differentiable function.

THEOREM 3.20 (Khare [**42**]). *Fix non-negative integers* $m_0 < m_1 < \cdots < m_{N-1}$, *as well as scalars* $\epsilon > 0$ *and* $a \in \mathbb{R}$. *Let* $M := m_0 + \cdots + m_{N-1}$ *and suppose*

[5] We refer the reader again to [**43**, Section 5] for the details, which use additional concepts from type-A representation theory: the Harish-Chandra–Itzykson–Zuber integral and Gelfand–Tsetlin patterns.

the function $f : [a, a+\epsilon) \to \mathbb{R}$ is M-times differentiable at a. Given vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^N$, define $\Delta : [0, \epsilon') \to \mathbb{R}$ for a sufficiently small $\epsilon' \in (0, \epsilon)$ by setting

$$\Delta(t) := \det f[a\mathbf{1}_{N\times N} + t\mathbf{u}\mathbf{v}^T].$$

Then,

$$\Delta^{(M)}(0) = \sum_{\mathbf{m} \vdash M} \binom{M}{m_0, m_1, \dots, m_{N-1}} V(\mathbf{u})V(\mathbf{v}) s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}) \prod_{k=0}^{N-1} f^{(m_k)}(a), \tag{3.10}$$

where the first factor in the summand is a multinomial coefficient, and we sum over all partitions $\mathbf{m} = (m_0, \dots, m_{N-1})$ of M with unequal parts, that is, $M = m_0 + \cdots + m_{N-1}$ and $0 \le m_0 < \cdots < m_{N-1}$.

In particular, $\Delta(0) = \Delta'(0) = \cdots = \Delta^{(\binom{N}{2}-1)}(0) = 0$.

Remark 3.21. As a special case, if $f : \mathbb{R} \to \mathbb{R}$ is smooth at a, and $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^N$, then defining $\Delta(t) := \det f[a\mathbf{1}_{N\times N} + t\mathbf{u}\mathbf{v}^T]$ gives a function Δ which is smooth at 0, and Theorem 3.20 gives all of these derivatives via the formula (3.10). The general version of Theorem 3.20 is a key ingredient in showing Theorem 2.9, which subsumes all known variants of Horn–Loewner-type necessary conditions in fixed dimension.

The key determinant computation required to prove the original Horn–Loewner necessary condition in fixed dimension (see Theorem 2.7) is the special case of Theorem 3.20 where $\mathbf{u} = \mathbf{v}$ and $m_j = j$ for all j. In this situation, $s_{\mathbf{m}}(\mathbf{u}) = s_{\mathbf{m}}(\mathbf{v}) = 1$, so Schur polynomials do not appear. The general version of Theorem 3.20 decouples the vectors $\mathbf{u}$ and $\mathbf{v}$, and holds for all $M > 0$ if f is smooth (as in Loewner's setting). Moreover, it reveals the presence of Schur polynomials in every other case than the ones studied by Loewner, that is, when $M > \binom{N}{2}$.

While Theorem 3.20 involves derivatives of a smooth function, the result and its proof are, in fact, completely algebraic, and valid over any commutative ring. To show this, an algebraic analogue of the differential operator is required, with more structure than is given by a derivation. The precise statement and its proof may be found in [**42**, Section 2].

We conclude this section by applying Theorem 3.20 and its algebraic avatar to symmetric function theory. We begin by recalling the famous *Cauchy summation identity* [**48**, Example I.4.6]: if $f_0(x) := 1 + x + x^2 + \cdots$ is the geometric series, viewed as a formal power series over a commutative unital ring R, and $u_1, \dots, u_N$, $v_1, \dots, v_N$ are commuting variables, then

$$\det f_0[\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \sum_{\mathbf{m}} s_{\mathbf{m}}(\mathbf{u}) s_{\mathbf{m}}(\mathbf{v}), \tag{3.11}$$

where the sum runs over all partitions $\mathbf{m}$ with at most N parts.[6]

A natural question is whether similar formulae hold when f_0 is replaced by other formal power series. Very few such results were known; this includes one due to Frobenius [**26**], for the function $f_c(x) := (1 - cx)/(1 - x)$ with c an scalar. (This

[6]Usually one uses infinitely many indeterminates in symmetric function theory, but given the connection to the entrywise calculus in a fixed dimension, we will restrict our attention to u_j and v_j for $1 \le j \le N$.

is also connected to theta functions and elliptic Frobenius–Stickelberger–Cauchy determinant identities.) For this function,

$$\begin{aligned}\det f_c[\mathbf{u}\mathbf{v}^T] &= \det\Big[\frac{1-cu_jv_k}{1-u_jv_k}\Big]_{j,k=1}^N\\ &= V(\mathbf{u})V(\mathbf{v})(1-c)^{N-1}\\ &\quad\times\Big(\sum_{\mathbf{m}:m_0=0} s_{\mathbf{m}}(\mathbf{u})s_{\mathbf{m}}(\mathbf{v}) + (1-c)\sum_{\mathbf{m}:m_0>0} s_{\mathbf{m}}(\mathbf{u})s_{\mathbf{m}}(\mathbf{v})\Big).\end{aligned} \tag{3.12}$$

A third, obvious identity is if f is a 'fewnomial' with at most $N-1$ terms. In this case, $f[\mathbf{u}\mathbf{v}^T]$ is a sum of at most $N-1$ rank-one matrices, and so its determinant vanishes.

The following result extends all three of these cases to an arbitrary formal power series over an arbitrary commutative ring R, and with an additional $\mathbb{Z}_+$-grading.

Theorem 3.22 (Khare [**42**]). *Fix a commutative unital ring R and let t be an indeterminate. Let $f(t) := \sum_{M\geq 0} f_M t^M \in R[[t]]$ be an arbitrary formal power series. Given vectors $\mathbf{u}$, $\mathbf{v} \in R^N$, where $N \geq 1$, we have that*

$$\det f[t\mathbf{u}\mathbf{v}^T] = V(\mathbf{u})V(\mathbf{v}) \sum_{M\geq\binom{N}{2}} t^M \sum_{\mathbf{m}=(m_{N-1},\dots,m_0)\vdash M} s_{\mathbf{m}}(\mathbf{u})s_{\mathbf{m}}(\mathbf{v}) \prod_{k=0}^{N-1} f_{m_k}. \tag{3.13}$$

The heart of the proof involves first computing, for each $M \geq 0$, the coefficient of t^M in $\det f[t\mathbf{u}\mathbf{v}^T]$, over the "universal ring"

$$R' := \mathbb{Q}[u_1,\dots,u_N,v_1,\dots,v_N,f_0,f_1,\dots],$$

where u_j, v_k and f_m are algebraically independent over $\mathbb{Q}$. These coefficients are seen to equal $\Delta^{(M)}(0)/M!$, by the algebraic version of Theorem 3.20. Thus, (3.13) holds over R'. Then note that both sides of (3.13) lie in the subring $R_0 := \mathbb{Z}[u_1,\dots,u_N,v_1,\dots,v_N,f_0,f_1,\dots]$, so the identity holds in R_0. Finally, it holds as claimed by specializing from R_0 to R.

An alternate approach to proving Theorem 3.22 is also provided in [**42**]. The identity (3.6) is applied, along with the Cauchy–Binet formula, to each truncated Taylor–Maclaurin polynomial $f_{\leq M}$ of $f(x)$. The result follows by taking limits in the t-adic topology, using the t-adic continuity of the determinant function.

3.7. Further applications: linear matrix inequalities, Rayleigh quotients, and the cube problem. This chapter ends with further ramifications and applications of the above results. First, notice that Theorem 3.15 implies the following linear matrix inequality version that is 'sharp' in more than one sense:

Corollary 3.23. *Fix $\rho > 0$, real exponents $n_0 < \cdots < n_{N-1} < M$ for some integer $N \geq 1$, and scalars $c_j > 0$ for all j. Then,*

$$A^{\circ M} \leq \mathcal{C}\Big(c_0 A^{\circ n_0} + \cdots + c_{N-1}A^{\circ n_{N-1}}\Big),$$

$$\textit{where } \mathcal{C} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{c_j V(\mathbf{n})^2}\rho^{M-n_j},$$

for all $A \in \mathcal{P}_N\big((0,\rho)\big)$ of rank one, or of all ranks if $n_0, \ldots, n_{N-1} \in \mathbb{Z}_+ \cup [N-2,\infty)$. Moreover, the constant $\mathcal{C}$ is the smallest possible, as is the number of terms N on the right-hand side.

Seeking a uniform threshold such as $\mathcal{C}$ in the preceding inequality can also be achieved (as explained above) by first working with a single positive matrix, then optimizing over all matrices. The first step here can be recast as an extremal problem that involves Rayleigh quotients:

PROPOSITION 3.24 (see [**6, 43**]). *Fix an integer $N \geq 2$ and real exponents $n_0 < \cdots < n_{N-1} < M$, where each $n_j \in \mathbb{Z}_+ \cup [N-2,\infty)$. Given positive scalars $c_0, \ldots, c_{N-1}$, let*

$$h(x) := \sum_{j=0}^{N-1} c_j x^{n_j} \qquad \big(x \in (0,\infty)\big).$$

Then, for $0 < \rho < \infty$ and $A \in \mathcal{P}_N\big([0,\rho]\big)$,

$$t\, h[A] \succeq A^{\circ M} \qquad \text{if and only if} \quad t \geq \varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2}), \tag{3.14}$$

where $\varrho[B]$ and $B^\dagger$ denote the spectral radius and the Moore–Penrose pseudo-inverse of a square matrix B, respectively. Moreover, for every non-zero matrix $A \in \mathcal{P}_N\big([0,\rho]\big)$, the following variational formula holds:

$$\varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2}) = \sup_{\mathbf{u} \in (\ker h[A])^\perp \setminus \{\mathbf{0}\}} \frac{\mathbf{u}^T A^{\circ M} \mathbf{u}}{\mathbf{u}^T h[\mathbf{u}\mathbf{u}^T] \mathbf{u}} \leq \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_j}.$$

Proposition 3.24 is shown using the Kronecker normal form for matrix pencils; see the treatment in [**27**, Section X.6]. When the matrix A is a generic rank-one matrix, the above generalized Rayleigh quotient has a closed-form expression, which features Schur polynomials for integer powers. This reveals connections between Rayleigh quotients, spectral radii, and symmetric functions.

PROPOSITION 3.25. *Notation as in Proposition 3.24; but now with n_j not necessarily in $\mathbb{Z}_+ \cup [N-2,\infty)$. If $A = \mathbf{u}\mathbf{u}^T$, where $\mathbf{u} \in (0,\infty)^N$ has distinct coordinates, then $h[A]$ is invertible, and the threshold bound*

$$\varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2}) = (\mathbf{u}^{\circ M})^T h[\mathbf{u}\mathbf{u}^T]^{-1} \mathbf{u}^{\circ M} = \sum_{j=0}^{N-1} \frac{(\det \mathbf{u}^{\circ \mathbf{n}_j})^2}{c_j (\det \mathbf{u}^{\circ \mathbf{n}})^2}. \tag{3.15}$$

In fact, the proof of the final equality in (3.15) is completely algebraic, and reveals new determinantal identities that hold over any field $\mathbb{F}$ with at least N elements.

PROPOSITION 3.26 (Khare–Tao [**43**]). *Suppose $N \geq 1$ and $0 \leq n_0 < \cdots < n_{N-1} < M$ are integers, and $\mathbf{u}, \mathbf{v} \in \mathbb{F}^N$ each have distinct coordinates. Let $c_j \in \mathbb{F}^\times$ and define $h(t) := \sum_{j=0}^{N-1} c_j t^{n_j}$. Then $h[\mathbf{u}\mathbf{v}^T]$ is invertible, and*

$$(\mathbf{v}^{\circ M})^T h[\mathbf{u}\mathbf{v}^T]^{-1} \mathbf{u}^{\circ M} = \sum_{j=0}^{N-1} \frac{\det \mathbf{u}^{\circ \mathbf{n}_j} \det \mathbf{v}^{\circ \mathbf{n}_j}}{c_j \det \mathbf{u}^{\circ \mathbf{n}} \det \mathbf{v}^{\circ \mathbf{n}}}.$$

The final result is a variant of the matrix-cube problem [**49**], and connects to spectrahedra [**13, 61**] and modern optimization theory. Given two or more real symmetric $N \times N$ matrices $A_0, \ldots, A_{M+1}$ for the corresponding matrix cube of size $2\eta > 0$ is

$$\mathcal{U}[\eta] := \big\{ A_0 + \sum_{m=1}^{M+1} u_m A_m : u_m \in [-\eta, \eta] \big\}.$$

The matrix-cube problem is to find the largest $\eta > 0$ such that $\mathcal{U}[\eta] \subset \mathcal{P}_N(\mathbb{R})$. In the present setting of the entrywise calculus, the above results imply asymptotically matching upper and lower bounds for the size of the matrix cube.

Theorem 3.27 (see [**6, 43**]). *Suppose $M \geq 0$ and $0 \leq n_0 < n_1 < \cdots$ are integers. Fix positive scalars $\rho > 0$, $0 < \alpha_1 < \cdots < \alpha_{M+1}$, and $c_j > 0 \ \forall j \geq 0$, and define for each $N \geq 1$ and each matrix $A \in \mathcal{P}_N\big([0, \rho]\big)$, the cube*

$$\mathcal{U}_A[\eta] := \left\{ \sum_{j=0}^{N-1} c_j A^{\circ n_j} + \sum_{m=1}^{M+1} u_m A^{\circ(n_{N-1}+\alpha_m)} : u_m \in [-\eta, \eta] \right\}. \tag{3.16}$$

Also define for $N \geq 1$ and $\alpha > 0$:

$$\mathcal{K}_\alpha(N) := \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j(\alpha, N))^2}{V(\mathbf{n}(N))^2} \frac{\rho^{\alpha - n_j}}{c_j}, \tag{3.17}$$

where $\mathbf{n}(N) := (n_0, \ldots, n_{N-1})^T$, and

$$\mathbf{n}_j(\alpha, N) := (n_0, \ldots, n_{j-1}, n_{j+1}, \ldots, n_{N-1}, n_{N-1} + \alpha).$$

Then for each fixed $N \geq 1$, we have the uniform upper and lower bounds:

(3.18)

$$\eta \leq \big(\mathcal{K}_{\alpha_1}(N) + \cdots + \mathcal{K}_{\alpha_{M+1}}(N)\big)^{-1} \implies \mathcal{U}_A[\eta] \subset \mathcal{P}_N \text{ for all } A \in \mathcal{P}_N\big([0,\rho]\big)$$
$$\implies \eta \leq \mathcal{K}_{\alpha_{M+1}}(N)^{-1}.$$

Moreover, if the n_j grow linearly, in that

$$\alpha_{M+1} - \alpha_M \geq n_{j+1} - n_j \qquad \text{for all } j \geq 0,$$

then the lower and upper bounds for $\eta = \eta_N$ in (3.18) are asymptotically equal as $N \to \infty$:

$$\lim_{N \to \infty} \mathcal{K}_{\alpha_{M+1}}(N)^{-1} \sum_{m=1}^{M+1} \mathcal{K}_{\alpha_m}(N) = 1.$$

3.8. Entrywise preservers of totally non-negative Hankel matrices. The first part of this survey discusses entrywise preservers of totally positive and totally non-negative matrices; these turn out to be very rigid in nature. If, instead, we consider the subfamily of totally non-negative matrices which are Hankel, then a richer class of preservers emerges, as well as a parallel story to that of entrywise positivity preservers on all matrices.

Definition 3.28. A real matrix A is said to be *totally non-negative* or *totally positive* if every minor of A is non-negative or positive, respectively. We will denote these matrices, as well as the property, by TN and TP.

In the recent article [**22**] by Fallat, Johnson, and Sokal, the authors study when various classes of totally non-negative (TN) matrices are closed under taking sums or Schur products. As they observe, the set of all TN matrices is not closed under these operations; for example, the 3×3 identity matrix and the all-ones matrix $\mathbf{1}_{3\times 3}$ are both TN but their sum is not.

It is of interest to isolate a class of TN matrices that is a closed convex cone, and is furthermore closed under taking Schur products. Indeed, it is under these conditions that the observation of Pólya–Szegö (see Section 2.2) holds, leading to large classes of TN preservers.

Such a class of matrices has been identified in both the dimension-free as well as fixed-dimension settings. It consists of the *TN Hankel matrices*. In a fixed dimension, there is the following classical result from 1912.

LEMMA 3.29 (Fekete [**24**]). *Let A be a possibly rectangular real Hankel matrix such that all of its contiguous minors are positive. Then A is totally positive.*

Recall that a minor is said to be *contiguous* if it is obtained from successive rows and successive columns of A.

If A is a square Hankel matrix, let $A^{(1)}$ be the square submatrix of A obtained by removing the first row and the last column. Notice that every contiguous minor of A is a principal minor of either A or $A^{(1)}$. Combined with Fekete's lemma, these observations help show another folklore result.

THEOREM 3.30. *Let A be a square real Hankel matrix. Then A is TN or TP if and only if both A and $A^{(1)}$ are positive semidefinite or positive definite, respectively.*

Theorem 3.30 is a very useful bridge between matrix positivity and total non-negativity. A related dimension-free variant (see [**2, 28**]) concerns the Stieltjes moment problem: a sequence $(s_0, s_1, \ldots,)$ is the moment sequence of an admissible measure on $\mathbb{R}_+$ (see Definition 2.10) if and only if the Hankel matrices $H := (s_{j+k})_{j,k\geq 0}$ and $H^{(1)}$ (obtained by excising the first row of H, or equivalently, the first column) are both positive semidefinite. By Theorem 3.30, this is equivalent to saying that H is totally non-negative.

With Theorem 3.30 in hand, one can easily show several basic facts about Hankel TN matrices; we collect these in the following result for convenience.

LEMMA 3.31. *For an integer $N \geq 1$ and a set $I \subset \mathbb{R}_+$, let $HTN_N(I)$ denote the set of $N \times N$ TN Hankel matrices with entries in I. For brevity, we let $HTN_N := HTN_N(\mathbb{R}_+)$.*

(1) *The family HTN_N is closed under taking sums and non-negative scalar multiples, or more generally, integrals against non-negative measures (as long as these exist).*
(2) *In particular, if μ is an admissible measure supported on $\mathbb{R}_+$, then its moment matrix $H_\mu := \big(s_{j+k}(\mu)\big)_{j,k=0}^{\infty}$ is totally non-negative.*
(3) *HTN_N is closed under taking entrywise products.*
(4) *If the power series $f(x) = \sum_{k\geq 0} c_k x^k$ is convergent on $I \subset \mathbb{R}_+$, with $c_k \geq 0$ for all $k \geq 0$, then the entrywise map $f[-]$ preserves total non-negativity on $HTN_N(I)$, for all $N \geq 1$.*

Given Lemma 3.31(4), which is identical to the start of the story for positivity preservers, it is natural to expect parallels between the two settings. This does in fact occur, in both the dimension-free and the fixed-dimension settings, and we now elaborate on both of these. For example, one can ask if a Schoenberg-type phenomenon also holds for preservers of total non-negativity on $\bigcup_{N\geq 1} HTN_N\big([0,\rho)\big)$ with

$0 < \rho \leq \infty$. This is indeed the case; we set $\rho = \infty$ for ease of exposition. From Theorem 2.12 and the subsequent discussion, it follows via Hamburger's theorem that the class of functions $\sum_{k \geq 0} c_k x^k$ with all $c_k \geq 0$ characterizes the entrywise maps preserving the set of moment sequences of admissible measures supported on $[-1, 1]$. By the above discussion, in considering the family of matrices HTN_N for all $N \geq 1$, we are studying moment sequences of admissible measures supported on $I = \mathbb{R}_+$, or the related Hausdorff moment problem for $I = [0, 1]$. In this case, one also has a Schoenberg-like characterization, outside of the origin.

THEOREM 3.32 (Belton–Guillot–Khare–Putinar [**7**]). *Let $f : \mathbb{R}_+ \to \mathbb{R}$. The following are equivalent.*

(1) *Applied entrywise, the map f preserves the set HTN_N for all $N \geq 1$.*
(2) *Applied entrywise, the map f preserves positive semidefiniteness on HTN_N for all $N \geq 1$.*
(3) *Applied entrywise, the map f preserves the set of moment sequences of admissible measures supported on $\mathbb{R}_+$.*
(4) *Applied entrywise, the map f preserves the set of moment sequences of admissible measures supported on $[0, 1]$.*
(5) *The function f agrees on $(0, \infty)$ with an absolutely monotonic entire function, hence is non-decreasing, and $0 \leq f(0) \leq \lim_{\epsilon \to 0^+} f(\epsilon)$.*

REMARK 3.33. If we work only with $f : (0, \infty) \to \mathbb{R}$, then we are interested in matrices in HTN_N with positive entries. Since the only matrices in HTN_N with a zero entry are scalar multiples of the elementary square matrices E_{11} or E_{NN} (equivalently, the only admissible measures supported in $\mathbb{R}_+$ with a zero moment are of the form $c\delta_0$), the test set does not really reduce, and hence the preceding theorem still holds in essence: we must replace HTN_N by $HTN_N\big((0, \infty)\big)$ in (1) and (2), reduce the class of admissible measures to those that are not of the form $c\delta_0$ in (3) and (4), and end (5) at 'entire function'. These five modified statements are, once again, equivalent, and provide further equivalent conditions to those of Vasudeva (Theorems 2.5 and 2.11).

In a similar vein, we now present the classification of sign patterns of polynomial or power-series functions that preserve TN entrywise in a fixed dimension on Hankel matrices. This too turns out to be exactly the same as for positivity preservers.

THEOREM 3.34 (Khare–Tao [**43**]). *Fix $\rho > 0$ and real exponents $n_0 < \cdots < n_{N-1} < M$. For any real coefficients $c_0, \ldots, c_{N-1}, c'$, let*

$$f(x) := \sum_{j=0}^{N-1} c_j x^{n_j} + c' x^M. \tag{3.19}$$

The following are equivalent.

(1) *The entrywise map $f[-]$ preserves TN on the rank-one matrices in $HTN_N\big((0, \rho)\big)$.*
(2) *The entrywise map $f[-]$ preserves positivity on the rank-one matrices in $HTN_N\big((0, \rho)\big)$.*
(3) *Either all the coefficients $c_0, \ldots, c_{N-1}, c'$ are non-negative, or $c_0, \ldots, c_{N-1}$ are positive and $c' \geq -\mathcal{C}^{-1}$, where*

$$\mathcal{C} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_j}. \tag{3.20}$$

If $n_j \in \mathbb{Z}_+ \cup [N-2, \infty)$ for $j = 0, \ldots, N-1$, then conditions (1), (2) and (3) are further equivalent to the following.

(4) *The entrywise map $f[-]$ preserves TN on $HTN_N\big([0,\rho]\big)$.*

In particular, this produces further equivalent conditions to Theorem 3.15. Notice that assertion (2) here is valid because the rank-one matrices used in proving Theorem 3.15 are of the form $c\mathbf{u}\mathbf{u}^T$, where $\mathbf{u} = (1, u_0, \ldots, u_0^{N-1})^T$, $u_0 \in (0,1)$, and $c \in (0,\rho)$, so that $c\mathbf{u}\mathbf{u}^T \in HTN_N\big((0,\rho)\big)$.

The consequences of Theorem 3.15 also carry over for TN preservers. For instance, one can bound Laplace transforms analogously to Corollary 3.17, by replacing the words "positive semidefinite" by "totally non-negative" and the set $\mathcal{P}_N\big((0,\rho)\big)$ by $HTN_N\big((0,\rho)\big)$. Similarly, one can completely classify the sign patterns of power series that preserve TN entrywise on Hankel matrices of a fixed size:

Theorem 3.35 (Khare–Tao [**43**]). *Theorems 3.5 and 3.19 hold upon replacing the phrase "preserves positivity entrywise on $\mathcal{P}_N\big((0,\rho)\big)$" with "preserves TN entrywise on $HTN_N\big((0,\rho)\big)$", for both $\rho < \infty$ and for $\rho = \infty$.*

We point the reader to [**43**, End of Section 9] for details.

To conclude, it is natural to seek a general result that relates the positivity preservers on $\mathcal{P}_N(I)$ and TN preservers on the set $HTN_N(I)$ for domains $I \subset \mathbb{R}_+$. Here is one variant which helps prove the above theorems, and which essentially follows from Theorem 3.30.

Proposition 3.36 (Khare–Tao [**43**]). *Fix integers $1 \leq k \leq N$ and a scalar $0 < \rho \leq \infty$. Suppose $f : [0,\rho) \to \mathbb{R}$ is such that the entrywise map $f[-]$ preserves positivity on $\mathcal{P}_N^k\big([0,\rho)\big)$, the set of matrices in $\mathcal{P}_N\big([0,\rho)\big)$ with rank no more than k. Then $f[-]$ preserves total non-negativity on $HTN_N\big([0,\rho)\big) \cap \mathcal{P}_N^k\big([0,\rho)\big)$.*

4. Power functions

A natural approach to tackle the problem of characterizing entrywise preservers in fixed dimension is to examine if some natural simple functions preserve positivity. One such family is the collection of power functions, $f(x) = x^\alpha$ for $\alpha > 0$. Characterizing which fractional powers preserve positivity entrywise has recently received much attention in the literature. One of the first results in this area reads as follows.

Theorem 4.1 (FitzGerald and Horn [**25**, Theorem 2.2]). *Let $N \geq 2$ and let $A = [a_{jk}] \in \mathcal{P}_N\big(\mathbb{R}_+\big)$. For any real number $\alpha \geq N-2$, the matrix $A^{\circ\alpha} := [a_{jk}^\alpha]$ is positive semidefinite. If $0 < \alpha < N-2$ and α is not an integer, then there exists a matrix $A \in \mathcal{P}_N\big((0,\infty)\big)$ such that $A^{\circ\alpha}$ is not positive semidefinite.*

Theorem 4.1 shows that every real power $\alpha \geq N-2$ entrywise preserves positivity, while no non-integers in $(0, N-2)$ do so. This surprising "phase transition" phenomenon at the integer $N-2$ is referred to as the "critical exponent" for preserving positivity. Studying which powers entrywise preserve positivity is a very natural and interesting problem. It also often provides insights to determine which general functions preserve positivity. For example, Theorem 4.1 suggests that functions that entrywise preserve positivity on $\mathcal{P}_N$ should have a certain number of non-negative derivatives, which is indeed the case by Theorem 2.7.

Outline of the proof. The first part of Theorem 4.1 relies on an ingenious idea that we now sketch. The result is obvious for $N = 2$. Let us assume it holds for some $N - 1 \geq 2$, let $A \in \mathcal{P}_N(\mathbb{R}_+)$, and let $\alpha \geq N - 2$. Write A in block form,

$$A = \begin{bmatrix} B & \xi \\ \xi^T & a_{NN} \end{bmatrix},$$

where B has dimension $(N-1) \times (N-1)$ and $\xi \in \mathbb{R}^{N-1}$. Assume without loss of generality that $a_{NN} \neq 0$ (as the case where $a_{NN} = 0$ follows from the induction hypothesis) and let $\zeta := (\xi^T, a_{NN})^T / \sqrt{a_{NN}}$. Then $A - \zeta\zeta^T = (B - \xi\xi^T)/a_{NN} \oplus 0$, where $(B - \xi\xi^T)/a_{NN}$ is the Schur complement of a_{NN} in A. Hence $A - \zeta\zeta^T$ is positive semidefinite. By the fundamental theorem of calculus, for any x, $y \in \mathbb{R}$,

$$x^\alpha = y^\alpha + \alpha \int_0^1 (x - y)(\lambda x + (1 - \lambda)y)^{\alpha - 1} \, \mathrm{d}\lambda.$$

Using the above expression entrywise, we obtain

$$A^{\circ\alpha} = \zeta^{\circ\alpha}(\zeta^{\circ\alpha})^T + \int_0^1 (A - \zeta\zeta^T) \circ (\lambda A + (1 - \lambda)\zeta\zeta^T)^{\circ(\alpha-1)} \, \mathrm{d}\lambda.$$

Observe that the entries of the last row and column of the matrix $A - \zeta\zeta^T$ are all zero. Using the induction hypothesis and the Schur product theorem, it follows that the integrand is positive semidefinite, and therefore so is $A^{\circ\alpha}$.

The converse implication in Theorem 4.1 is shown by considering a matrix of the form $a\mathbf{1}_{N\times N} + t\mathbf{u}\mathbf{u}^T$, where a, $t > 0$, the coordinates of $\mathbf{u}$ are distinct, and $t1$ is small. Recall this is the exact same class of matrices that was useful in proving the Horn–Loewner theorem 2.7 as well as its strengthening in Theorem 2.9. The original proof, by FitzGerald and Horn [**25**], used $\mathbf{u} = (1, 2, \dots, N)^T$, while a later proof by Fallat, Johnson and Sokal [**22**] used the same argument, now with $\mathbf{u} = (1, u_0, \dots, u_0^{N-1})^T$; the motivation in [**22**] was to work with Hankel matrices, and the matrix $a\mathbf{1}_{N\times N} + t\mathbf{u}\mathbf{u}^T$ is indeed Hankel. That said, the argument of FitzGerald and Horn works more generally than both of these proofs, to show that, for any non-integral power $\alpha \in (0, N-2)$, $a > 0$, and vector $\mathbf{u} \in (0, \infty)^N$ with distinct coordinates, there exists $t > 0$ such that $(a\mathbf{1}_{N\times N} + t\mathbf{u}\mathbf{u}^T)^{\circ\alpha}$ is not positive semidefinite. □

In her 2017 paper [**41**], Jain provided a remarkable strengthening of the result mentioned at the end of the previous proof, which removes the dependence on t entirely.

Theorem 4.2 (Jain [**41**]). *Let*

$$A := [1 + u_j u_k]_{j,k=1}^N = \mathbf{1}_{N\times N} + \mathbf{u}\mathbf{u}^T,$$

where $N \geq 2$ and $\mathbf{u} = (u_1, \dots, u_N)^T \in (0, \infty)^N$ has distinct entries. Then $A^{\circ\alpha}$ is positive semidefinite for $\alpha \in \mathbb{R}$ if and only if $\alpha \in \mathbb{Z}_+ \cup [N-2, \infty)$.

Jain's result identifies a family of rank-two positive semidefinite matrices, every one of which encodes the classification of powers preserving positivity over all of $\mathcal{P}_N\big((0,\infty)\big)$. In a sense, her rank-two family is the culmination of previous work on positivity preserving powers for $\mathcal{P}_N\big((0,\infty)\big)$, since for rank-one matrices, every entrywise power preserves positivity: $(\mathbf{u}\mathbf{u}^T)^{\circ\alpha} = \mathbf{u}^{\circ\alpha}(\mathbf{u}^{\circ\alpha})^T$.

An immediate consequence of these results is the classification of the entrywise powers preserving positivity on the $N \times N$ Hankel TN matrices. Recall from the results in Section 3.8 (including Lemma 3.31(4)) that there is to be expected a strong correlation between this classification and the one in Theorem 4.1.

COROLLARY 4.3. *Given $N \geq 2$, the following are equivalent for an exponent $\alpha \in \mathbb{R}$.*

1. *The entrywise power function $x \mapsto x^\alpha$ preserves total non-negativity on HTN_N (see Lemma 3.31).*
2. *The entrywise map $x \mapsto x^\alpha$ preserves positivity on HTN_N.*
3. *The entrywise map $x \mapsto x^\alpha$ preserves positivity on the matrices in $HTN_N\big((0,\infty)\big)$ of rank at most two.*
4. *The exponent $\alpha \in \mathbb{Z}_+ \cup [N-2, \infty)$.*

PROOF. That (4) $\implies$ (2) and (2) $\implies$ (1) follow from Theorems 4.1 and 3.30, respectively. That (1) $\implies$ (2) and (2) $\implies$ (3) are obvious, and Jain's theorem 4.2 shows that (3) $\implies$ (4). $\square$

A problem related to the above study of entrywise powers preserving positivity, is to characterize infinitely divisible matrices. This problem was also considered by Horn in [**40**]. Recall that a complex $N \times N$ matrix is said to be *infinitely divisible* if $A^{\circ\alpha} \in \mathcal{P}_N$ for all $\alpha \in \mathbb{R}_+$. Denote the *incidence matrix* of A by $M(A)$:

$$M(A)_{jk} = m_{jk} := \begin{cases} 0 & \text{if } a_{jk} = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Also, let

$$L(A) := \{\mathbf{x} \in \mathbb{C}^N : \sum_{j,k=1}^N m_{jk} x_j \overline{x_k} = 0\},$$

and note that $L(A)$ is the kernel of $M(A)$ if $M(A)$ is positive semidefinite.

Assuming the arguments of the entries are chosen in a *consistent way* [**40**], we let

$$\log^\# A := M(A) \circ \log[A] = [\mu_{jk} \log a_{jk}]_{j,k=1}^N,$$

with the usual convention $0 \log 0 = 0$.

THEOREM 4.4 (Horn [**40**, Theorem 1.4]). *An $N \times N$ matrix A is infinitely divisible if and only if (a) A is Hermitian, with $a_{jj} \geq 0$ for all j, (b) $M(A) \in \mathcal{P}_N$, and (c) $\log^\# A$ is positive semidefinite on $L(A)$.*

4.1. Sparsity constraints. Theorem 4.1 was recently extended to more structured matrices. Given $I \subset \mathbb{R}$ and a graph $G = (V, E)$ on the finite vertex set $V = \{1, \ldots, N\}$, we define the cone of positive-semidefinite matrices with *zeros according to G*:

$$\mathcal{P}_G(I) := \{A = [a_{jk}] \in \mathcal{P}_N(I) : a_{jk} = 0 \text{ if } (j,k) \notin E \text{ and } i \neq j\}. \tag{4.1}$$

Note that if $(j,k) \in E$, then the entry a_{jk} is unconstrained; in particular, it is allowed to be 0. Consequently, the cone $\mathcal{P}_G := \mathcal{P}_G(\mathbb{R})$ is a closed subset of $\mathcal{P}_N$.

A natural refinement of Theorem 4.1 involves studying powers that entrywise preserve positivity on $\mathcal{P}_G$. In that case, the flavor of the problem changes significantly, with the discrete structure of the graph playing a prominent role.

DEFINITION 4.5 (Guillot–Khare–Rajaratnam [**30**]). Given a simple graph $G = (V, E)$, let

$$\mathcal{H}_G := \{\alpha \in \mathbb{R} : A^{\circ\alpha} \in \mathcal{P}_G \text{ for all } A \in \mathcal{P}_G(\mathbb{R}_+)\}. \tag{4.2}$$

Define the *Hadamard critical exponent of* G to be

$$CE(G) := \min\{\alpha \in \mathbb{R} : [\alpha, \infty) \subset \mathcal{H}_G\}. \tag{4.3}$$

Notice that, by Theorem 4.1, for every graph $G = (V, E)$, the critical exponent $CE(G)$ exists, and lies in $[\omega(G) - 2, |V| - 2]$, where $\omega(G)$ is the size of the largest complete subgraph of G, that is, the clique number. To compute such critical exponents is natural and highly non-trivial.

FitzGerald and Horn proved that $CE(K_n) = n-2$ for all $n \geq 2$ (Theorem 4.1), while it follows from [**31**, Proposition 4.2] that $CE(T) = 1$ for every tree T. For a general graph, it is not *a priori* clear what the critical exponent is or how to compute it. A natural family of graphs that encompasses both complete graphs and trees is that of chordal graphs. Recall that a graph is *chordal* if it does not contain an induced cycle of length 4 or more. Chordal graphs feature extensively in many areas, such as the theory of graphical models [**46**], and in problems involving positive-definite completions (see [**59**]). Examples of important chordal graphs include trees, complete graphs, Apollonian graphs, band graphs, and split graphs.

Recently, Guillot, Khare, and Rajaratnam [**30**] were able to compute the complete set of entrywise powers preserving positivity on $\mathcal{P}_G$ for all chordal graphs G. Here, the critical exponent can be described purely combinatorially.

THEOREM 4.6 (Guillot–Khare–Rajaratnam [**30**]). *Let $K_r^{(1)}$ denote the complete graph with one edge removed, and let G be a finite simple connected chordal graph. The critical exponent for entrywise powers preserving positivity on $\mathcal{P}_G$ is $r-2$, where r is the largest integer such that K_r or $K_r^{(1)}$ is an induced subgraph of G. More precisely, the set of entrywise powers preserving $\mathcal{P}_G$ is $\mathcal{H}_G = \mathbb{Z}_+ \cup [r-2, \infty)$, with r as before.*

The set of entrywise powers preserving positivity was also computed in [**30**] for cycles and bipartite graphs.

THEOREM 4.7 (Guillot–Khare–Rajaratnam [**30**]). *The critical exponent of cycles and bipartite graphs is* 1.

Surprisingly, the critical exponent does not depend on the size of the graph for cycles and bipartite graphs. In particular, it is striking that any power greater than 1 preserves positivity for families of dense graphs such as bipartite graphs. Such a result is in sharp contrast to the general case, where there is no underlying structure of zeros. That small powers can preserve positivity is important for applications, since such entrywise procedures are often used to regularize positive definite matrices, such as covariance or correlation matrices, where the goal is to minimally modify the entries of the original matrix (see [**47, 63**] and Chapter 5 below).

For a general graph, the problem of computing the set $\mathcal{H}_G$ or the critical exponent $CE(G)$ remains open. We now outline some other natural open problems in the area.

Problems.

(1) In every currently known case (Theorems 4.6, 4.7), $CE(G)$ is equal to $r-2$, where r is the largest integer such that K_r or $K_r^{(1)}$ is an induced subgraph of G. Is the same true for every graph G?
(2) Is $CE(G)$ always an integer? Can this be proved without computing $CE(G)$ explicitly?
(3) Recall that every chordal graph is perfect. Can the critical exponent be calculated for other broad families of graphs such as the family of perfect graphs?

4.2. Rank constraints and other Loewner properties. Another approach to generalize Theorem 4.1 is to examine other properties of entrywise functions such as monotonicity, convexity, and super-additivity (with respect to the Loewner semidefinite ordering) [**38, 29**]. Given a set $V \subset \mathcal{P}_N(I)$, recall that a function $f : I \to \mathbb{R}$ is

- *positive on* V with respect to the Loewner ordering if $f[A] \geq 0$ for all $0 \leq A \in V$;
- *monotone on* V with respect to the Loewner ordering if $f[A] \geq f[B]$ for all A, $B \in V$ such that $A \geq B \geq 0$;
- *convex on* V with respect to the Loewner ordering if $f[\lambda A + (1-\lambda)B] \leq \lambda f[A] + (1-\lambda) f[B]$ for all $\lambda[0,1]$ and all A, $B \in V$ such that $A \geq B \geq 0$;
- *super-additive on* V with respect to the Loewner ordering if $f[A+B] \geq f[A] + f[B]$ for all A, $B \in V$ for which $f[A+B]$ is defined.

The following relations between the first three notions were obtained by Hiai.

THEOREM 4.8 (Hiai [**38**, Theorem 3.2]). *Let $I = (-\rho, \rho)$ for some $\rho > 0$.*

(1) *For each $N \geq 3$, the function f is monotone on $\mathcal{P}_N(I)$ if and only if f is differentiable on I and f' is positive on $\mathcal{P}_N(I)$.*
(2) *For each $N \geq 2$, the function f is convex on $\mathcal{P}_N(I)$ if and only if f is differentiable on I and f' is monotone on $\mathcal{P}_N(I)$.*

Power functions satisfying any of the above four properties have been characterized by various authors. In recent work, Hiai [**38**] has extended Theorem 4.1 by considering the odd and even extensions of the power functions to $\mathbb{R}$. For $\alpha > 0$, the even and odd extensions to $\mathbb{R}$ of the power function $f_\alpha(x) := x^\alpha$ are defined to be $\phi_\alpha(x) := |x|^\alpha$ and $\psi_\alpha(x) := \operatorname{sign}(x)|x|^\alpha$. The first study of powers $\alpha > 0$ for which ϕ_α preserves positivity entrywise on $\mathcal{P}_N(\mathbb{R})$ was carried out by Bhatia and Elsner [**10**]. Subsequently, Hiai studied the power functions ϕ_α and ψ_α that preserve Loewner positivity, monotonicity, and convexity entrywise, and showed for positivity preservers that the same phase transition occurs at $n-2$ for ϕ_α and ψ_α, as demonstrated in [**25**]. The work was generalized in [**29**] to matrices satisfying rank constraints.

DEFINITION 4.9. Fix non-negative integers $n \geq 2$ and $n \geq k$, and a set $I \subset \mathbb{R}$. Let $\mathcal{P}_n^k(I)$ denote the subset of matrices in $\mathcal{P}_n(I)$ that have rank at most k, and let

$$\begin{aligned}
&\mathcal{H}_{\mathrm{pos}}(n,k) := \{\alpha > 0 : x^\alpha \text{ preserves positivity on } \mathcal{P}_n^k(\mathbb{R}_+)\},\\
&\mathcal{H}^\phi_{\mathrm{pos}}(n,k) := \{\alpha > 0 : \phi_\alpha \text{ preserves positivity on } \mathcal{P}_n^k(\mathbb{R})\},\\
&\mathcal{H}^\psi_{\mathrm{pos}}(n,k) := \{\alpha > 0 : \psi_\alpha \text{ preserves positivity on } \mathcal{P}_n^k(\mathbb{R})\}.
\end{aligned} \tag{4.4}$$

Similarly, let $\mathcal{H}_J(n,k)$, $\mathcal{H}_J^{\phi}(n,k)$ and $\mathcal{H}_J^{\psi}(n,k)$ denote sets of the entrywise powers preserving Loewner properties on $\mathcal{P}_n^k(\mathbb{R}_+)$ or $\mathcal{P}_n^k(\mathbb{R})$, where $J \in \{\text{monotonicity, convexity, super-additivity}\}$.

The set of entrywise powers preserving the above notions are given in the table below (see [**29**, Theorem 1.2]).

J	$\mathcal{H}_J(n,k)$	$\mathcal{H}_J^{\phi}(n,k)$	$\mathcal{H}_J^{\psi}(n,k)$
Positivity			
$k=1$	$\mathbb{R}$ G–K–R	$\mathbb{R}$ G–K–R	$\mathbb{R}$ G–K–R
$2 \leq k \leq n$	$\mathbb{N} \cup [n-2,\infty)$ FitzGerald–Horn	$2\mathbb{N} \cup [n-2,\infty)$ Hiai, Bhatia–Elsner, G–K–R	$(-1+2\mathbb{N}) \cup [n-2,\infty)$ Hiai, G–K–R
Monotonicity			
$k=1$	$\mathbb{R}_+$ G–K–R	$\mathbb{R}_+$ G–K–R	$\mathbb{R}_+$ G–K–R
$2 \leq k \leq n$	$\mathbb{N} \cup [n-1,\infty)$ FitzGerald–Horn	$2\mathbb{N} \cup [n-1,\infty)$ Hiai, G–K–R	$(-1+2\mathbb{N}) \cup [n-1,\infty)$ Hiai, G–K–R
Convexity			
$k=1$	$[1,\infty)$ G–K–R	$[1,\infty)$ G–K–R	$[1,\infty)$ G–K–R
$2 \leq k \leq n$	$\mathbb{N} \cup [n,\infty)$ Hiai, G–K–R	$2\mathbb{N} \cup [n,\infty)$ Hiai, G–K–R	$(-1+2\mathbb{N}) \cup [n,\infty)$ Hiai, G–K–R
Super-additivity			
$1 \leq k \leq n$	$\mathbb{N} \cup [n,\infty)$ G–K–R	$2\mathbb{N} \cup [n,\infty)$ G–K–R	$(-1+2\mathbb{N}) \cup [n,\infty)$ G–K–R

TABLE 1. Summary of real Hadamard powers preserving Loewner properties, with additional rank constraints. See Bhatia–Elsner [**10**], FitzGerald–Horn [**25**], Guillot–Khare–Rajaratnam [**29**], and Hiai [**38**].

5. Motivation from statistics

The study of entrywise functions preserving positivity has recently attracted renewed attraction due to its importance in the estimation and regularization of covariance/correlation matrices. Recall that the covariance between two random variables X_j and X_k is given by

$$\sigma_{jk} = \operatorname{Cov}(X_j, X_k) = E\big[(X_j - E[X_j])(X_k - E[X_k])\big],$$

where $E[X_j]$ denotes the expectation of X_j. In particular, $\operatorname{Cov}(X_j, X_j) = \operatorname{Var}(X_j)$, the variance of X_j. The *covariance matrix* of a random vector $\mathbf{X} := (X_1, \dots, X_m)$, is the matrix $\Sigma := [\operatorname{Cov}(X_j, X_k)]_{j,k=1}^m$. Covariance matrices are a fundamental tool that measure linear dependencies between random variables. In order to discover relations between variables in data, statisticians and applied scientists need to obtain estimates of the covariance matrix Σ from observations $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^m$ of $\mathbf{X}$. A traditional estimator of Σ is the *sample covariance matrix* S given by

$$S = [s_{jk}]_{j,k=1}^m = \frac{1}{n-1}\sum_{i=1}^n (\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})^T, \tag{5.1}$$

where $\overline{\mathbf{x}} := \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i$ is the average of the observations. In the case where the random vector $\mathbf{X}$ has a multivariate normal distribution with mean μ and covariance matrix Σ, one can show that $\overline{\mathbf{x}}$ and $\frac{n-1}{n}S$ are the maximum likelihood estimators of μ and Σ, respectively [**3**, Chapter 3]. It is not difficult to show that S is an unbiased estimator of Σ. More generally, under weak assumptions, one can show that the distribution of $\sqrt{n}(S-\Sigma)$ is asymptotically normal as $n \to \infty$. The exact description of the limiting distribution depends on the moments and the cumulants of $\mathbf{X}$ (see [**12**, Chapter 6.3]). For example, in the two-dimensional case, we have the following result.

Let $N_m(\mu, \Sigma)$ denote the m-dimensional normal distribution with mean μ and covariance matrix Σ.

PROPOSITION 5.1 (see [**12**, Example 6.4]). *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^2$ be an independent and identically distributed sample from a bivariate vector $\mathbf{X} = (X_1, X_2)$ with mean $\mu = (\mu_1, \mu_2)$ and finite fourth-order moments, and let S be as in Equation (5.1). Then*

$$\sqrt{n}\left[\begin{bmatrix} s_1^2 \\ s_{12} \\ s_2^2 \end{bmatrix} - \begin{bmatrix} \sigma_1^2 \\ \sigma_{12} \\ \sigma_2^2 \end{bmatrix}\right] \xrightarrow{\mathrm{d}} N_3(\mathbf{0}, \Omega),$$

where Ω is the symmetric 3×3 matrix

$$\Omega = \begin{bmatrix} \mu_4^1 - (\mu_2^1)^2 & \mu_{31}^{12} - \mu_{11}^{12}\mu_2^1 & \mu_{22}^{12} - \mu_2^1\mu_2^2 \\ \mu_{31}^{12} - \mu_{11}^{12}\mu_2^1 & \mu_{22}^{12} - (\mu_{11}^{12})^2 & \mu_{13}^{12} - \mu_{11}^{12}\mu_2^2 \\ \mu_{22}^{12} - \mu_2^1\mu_2^2 & \mu_{31}^{12} - \mu_{11}^{12}\mu_2^1 & \mu_4^2 - (\mu_2^2)^2 \end{bmatrix},$$

and $\mu_k^i = E[(X_i - \mu_i)^k]$ and $\mu_{kl}^{ij} = E[(X_i - \mu_i)^k (X_j - \mu_j)^l]$.

In traditional statistics, one usually assumes the number of samples n is large enough for asymptotic results such as the one above to apply. In covariance estimation, one typically requires a sample size at least a few times the number of variables m for that to apply. In such a case, the sample covariance matrix provides a good approximation of the true covariance matrix Σ. However, this ideal setting is rarely seen nowadays. Indeed, our systematic and automated way of collecting data today yields datasets where the number of variables is often orders of magnitude larger than the number of instances available for study [**19**]. Classical statistical methods were not designed and are not suitable to analyze data in such settings. Developing new methodologies that are adapted to modern high-dimensional problems is the object of active research. In the case of covariance estimation, several strategies have been proposed to replace the traditional sample covariance matrix estimator S. These approaches typically leverage low-dimensional structures in the data (low rank, sparsity, ...) to obtain reasonable covariance estimates, even when the sample size is small compared to the dimension of the problem (see [**52**] for a detailed description of such techniques). One such approach involves applying functions to the entries of sample covariance matrices to improve their properties (see e.g. [**5, 11, 21, 35, 36, 47, 53, 63**]). For example, *hard thresholding* a matrix entails setting to zero the entries of the matrix that are smaller in absolute value than a prescribed value $\epsilon > 0$ (thinking the corresponding variables are independent, for example). Letting

$$f_\epsilon^H(x) = \begin{cases} x & \text{if } |x| > \epsilon, \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

thresholding is equivalent to applying the function f_ϵ^H entrywise to the entries of the matrix. Another popular example that was first studied in the context of wavelet shrinkage [**20**] is *soft thresholding*, where f_ϵ^H is replaced by

$$f_\epsilon^S : x \mapsto \operatorname{sign}(x)\big(|x| - \epsilon\big)_+ \qquad \text{with } y_+ := \max\{y, 0\}.$$

Soft thresholding not only sets small entries to zero, it also shrinks all the other entries continuously towards zero. Several other thresholding and shrinkage procedures were also recently proposed in the context of covariance estimation (see [**23**] and the references therein).

Compared to other techniques, the above procedure has several advantages. Firstly, the resulting estimators are often significantly more precise than the sample covariance matrices. Secondly, applying a function to the entries of a matrix is very simple and not computationally intensive. The procedure can therefore be performed in very high dimensions and in real-time applications. This is in contrast to several other techniques that require solving optimization problems and often become too intensive to be used in modern applications. A downside of the entrywise calculus, however, is that the positive definiteness of the resulting matrices is not guaranteed. As the parameter space of covariance matrices is the cone of positive definite matrices, it is critical that the resulting matrices be positive definite for the technique to be useful and widely applicable. The problem of characterizing positivity preservers thus has an immediate impact in the area of covariance estimation by providing useful functions that can be applied entrywise to covariance estimates in order to regularize them.

Several characterizations of when thresholding procedures preserve positivity have recently been obtained.

5.1. Thresholding with respect to a graph. In [**33**], the concept of thresholding with respect to a graph was examined. In this context, the elements to threshold are encoded in a graph $G = (V, E)$ with $V = \{1, \dots, p\}$. If $A = (a_{jk})$ is a $p \times p$ matrix, we denote by A_G the matrix with entries

$$(A_G)_{jk} = \begin{cases} a_{jk} & \text{if } (j,k) \in E \text{ or } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

We say that A_G is the matrix obtain by thresholding A with respect to the graph G. The main result of [**33**] characterizes the graphs G for which the corresponding thresholding procedure preserves positivity. Denote by $\mathcal{P}_N^+$ the set of real symmetric $N \times N$ positive definite matrices and by $\mathcal{P}_G^+$ the subset of positive definite matrices contained in $\mathcal{P}_G$ (see Equation 4.1).

THEOREM 5.2 (Guillot–Rajaratnam [**33**, Theorem 3.1]). *The following are equivalent:*

1. $A_G \in \mathcal{P}_N^+$ *for all* $A \in \mathcal{P}_N^+$;
2. $G = \bigcup_{i=1}^d G_i$, *where* $G_1, \dots, G_d$ *are disconnected and complete components of* G.

The implication (2) $\implies$ (1) of the theorem is intuitive and straightforward, since principal submatrices of positive definite matrices are positive definite. That (1) $\implies$ (2) may come as a surprise though, and shows that indiscriminate or arbitrary thresholding of a positive definite matrix can quickly lead to loss of positive definiteness.

Theorem 5.2 also generalizes to matrices that already have zero entries. In that case, the characterization of the positivity preservers remains essentially the same.

THEOREM 5.3 (Guillot–Rajaratnam [**33**, Theorem 3.3]). *Let $G = (V, E)$ be an undirected graph and let $H = (V, E')$ be a subgraph of G, so that $E' \subset E$. Then A_H is positive definite for every $A \in \mathcal{P}_G^+$ if and only if $H = G_1 \cup \cdots \cup G_k$, where $G_1, \ldots, G_k$ are disconnected induced subgraphs of G.*

5.2. Hard and soft thresholding. Theorems 5.2 and 5.3 address the case where positive definite matrices are thresholded with respect to a given pattern of entries, regardless of the magnitude of the entries of the original matrix. The more natural case where the entries are hard or soft-thresholded was studied in [**33, 34**]. In applications, it is uncommon to threshold the diagonal entries of estimated covariance matrices, as the diagonal contains the variance of the underlying variables. Hence, for a given function $f : \mathbb{R} \to \mathbb{R}$ and a real matrix $A = [a_{jk}]$, we let the matrix $f^*[A]$ be defined by setting

$$f^*[A]_{jk} := \begin{cases} f(a_{jk}) & \text{if } j \neq k, \\ a_{jk} & \text{otherwise.} \end{cases}$$

THEOREM 5.4 (Guillot–Rajaratnam [**33**, Theorem 3.6]). *Let G be a connected undirected graph with $n \geq 3$ vertices. The following are equivalent.*

(1) *There exists $\epsilon > 0$ such that, for every $A \in \mathcal{P}_G^+$, we have $(f_\epsilon^H)^*[A] \in \mathcal{P}_n^+$.*
(2) *For every $\epsilon > 0$ and every $A \in \mathcal{P}_G^+$, we have $f_\epsilon^H[A] \in \mathcal{P}_n^+$.*
(3) *G is a tree.*

The case of soft-thresholding was considered in [**34**]. Surprisingly, the characterization of the thresholding levels that preserve positivity is exactly the same as in the case of hard-thresholding.

THEOREM 5.5 (Guillot–Rajaratnam [**34**, Theorem 3.2]). *Let $G = (V, E)$ be a connected graph with $n \geq 3$ vertices. Then the following are equivalent:*

(1) *There exists $\epsilon > 0$ such that for every $A \in \mathcal{P}_G^+$, we have $(f_\epsilon^S)^*[A] \in \mathcal{P}_n^+$.*
(2) *For every $\epsilon > 0$ and every $A \in \mathcal{P}_G^+$, we have $f_\epsilon^S[A] \in \mathcal{P}_n^+$.*
(3) *G is a tree.*

An extension of Schoenberg's theorem (Theorem 2.3) to the case where the function f is only applied to the off-diagonal entries of the matrix was also obtained in [**34**].

THEOREM 5.6 (Guillot–Rajaratnam [**34**, Theorem 4.21]). *Let $0 < \rho \leq \infty$ and $f : (-\rho, \rho) \to \mathbb{R}$. The matrix $f^*[A]$ is positive semidefinite for all $A \in \mathcal{P}_n\big((-\rho, \rho)\big)$ and all $n \geq 1$ if and only if $f(x) = xg(x)$, where*

(1) *g is analytic on the disc $D(0, \rho)$;*
(2) *$\|g\|_\infty \leq 1$;*
(3) *g is absolutely monotonic on $(0, \rho)$.*

When $\rho = \infty$, the only functions satisfying the above conditions are the affine functions $f(x) = ax$ for $0 \leq a \leq 1$.

5.3. Rank and sparsity constraints. An explicit and useful characterization of entrywise functions preserving positivity on $\mathcal{P}_N$ for a fixed N still remains out of reach as of today. Motivated by applications in statistics, the authors in [**31, 32**] examined the cases where the matrices in $\mathcal{P}_N$ satisfy supplementary rank and sparsity constraints that are common in applications.

Observe that the sample covariance matrix (Equation (5.1)) has rank at most n, where n is the number of samples used to compute it. Moreover, as explained in Chapter 5, it is common in modern applications that n is much smaller than the dimension p. Hence, when studying the regularization approach described in Chapter 5, it is natural to consider positive semidefinite matrices with rank bounded above.

An immediate application of Schoenberg's theorem on spheres (see Equation (2.2)) provides a characterization of entrywise positivity preservers of correlation matrices of all dimensions, with rank bounded above by n. Recall that a correlation matrix is the covariance matrix of a random vector where each variable has variance 1, so is a positive semidefinite matrix with diagonal entries equal to 1. As in Equation (2.2), we denote the ultraspherical orthogonal polynomials by $P_k^{(\lambda)}$.

Theorem 5.7 (Reformulation of [**57**, Theorem 1]). *Let $n \in \mathbb{N}$ and let $f : [-1,1] \to \mathbb{R}$. The following are equivalent.*

1. $f[A] \in \mathcal{P}_N$ *for all correlation matrices* $A \in \mathcal{P}_N\big([-1,1]\big)$ *with rank no more than n and all $N \geq 1$.*
2. $f(x) = \sum_{j=0}^{\infty} a_j P_j^{(\lambda)}(x)$ *with $a_j \geq 0$ for all $j \geq 0$ and $\lambda = (n-1)/2$.*

Proof. The result follows from [**57**, Theorem 1] and the observation that correlation matrices of rank at most n are in correspondence with Gram matrices of vectors in S^{n-1}. □

In order to approach the case of matrices of a fixed dimension, we introduce some notation.

Definition 5.8. Let $I \subset \mathbb{R}$. Define $\mathcal{S}_n(I)$ to be the set of $n \times n$ symmetric matrices with entries in I. Let $\operatorname{rank} A$ denote the rank of a matrix A. We define:

$$\mathcal{S}_n^k(I) := \{A \in \mathcal{S}_n(I) : \operatorname{rank} A \leq k\},$$
$$\mathcal{P}_n^k(I) := \{A \in \mathcal{P}_n(I) : \operatorname{rank} A \leq k\}.$$

The main result in [**32**] provides a characterization of entrywise functions mapping $\mathcal{P}_n^l$ into $\mathcal{P}_n^k$.

Theorem 5.9 (Guillot–Khare–Rajaratnam [**32**, Theorem B]). *Let $0 < R \leq \infty$ and $I = [0,R)$ or $(-R,R)$. Fix integers $n \geq 2$, $1 \leq k < n-1$, and $2 \leq l \leq n$. Suppose $f \in C^k(I)$. The following are equivalent.*

1. $f[A] \in \mathcal{S}_n^k$ *for all* $A \in \mathcal{P}_n^l(I)$*;*
2. $f(x) = \sum_{k=1}^{r} c_t x^{i_t}$ *for some $c_t \in \mathbb{R}$ and some $i_t \in \mathbb{N}$ such that*

$$\sum_{t=1}^{r} \binom{i_t + l - 1}{l-1} \leq k. \tag{5.3}$$

Similarly, $f[-] : \mathcal{P}_n^l(I) \to \mathcal{P}_n^k$ if and only if f satisfies (2) and $c_t \geq 0$ for all t. Moreover, if $I = [0,R)$ and $k \leq n-3$, then the assumption that $f \in C^k(I)$ is not required.

Notice that Theorem 5.9 is a fixed-dimension result with rank constraints. This may be considered a refinement of a similar, dimension-free result with rank constraints shown in [**4**], in which the authors arrive at the same conclusion as in part (2) above. We compare the two settings: in [**4**], (a) the hypotheses held for all dimensions N rather than in a fixed dimension; (b) the test matrices were a larger set in each dimension, compared to just the positive matrices considered in Theorem 5.9; (c) the test matrices did not consist only of rank-one matrices, similar to Theorem 5.9; and (d) the test functions f in the dimension-free case were assumed to be measurable, rather than C^k as in the fixed-dimension case. Thus, Theorem 5.9 is (a refinement of) the fixed-dimension case of the first main result in [**4**].[7]

The (2) $\implies$ (1) implication in Theorem 5.9 is clear. Indeed, let $i \geq 0$ and $A = \sum_{j=1}^{l} u_j u_j^T \in \mathcal{P}_n^l(I)$. Then

$$A^{\circ i} = \sum_{m_1+\cdots+m_l=i} \binom{i}{m_1, \dots, m_l} \mathbf{w_m} \mathbf{w_m}^T \qquad \text{where } \mathbf{w_m} := u_1^{\circ m_1} \circ \cdots \circ u_l^{\circ m_l}$$

and $\binom{i}{m_1, \dots, m_l}$ is a multinomial coefficient. Note that there are exactly $\binom{i+l-1}{l-1}$ terms in the previous summation. Therefore rank $A^{\circ i} \leq \binom{i+l-1}{l-1}$, and so (1) easily follows from (2). The proof that (1) $\implies$ (2) is much more challenging; see [**32**] for details.

In [**31**], the authors focus on the case where sparsity constraints are imposed to the matrices instead of rank constraints. Positive semidefinite matrices with zeros according to graphs arise naturally in many applications. For example, in the theory of Markov random fields in probability theory ([**46, 62**]), the nodes of a graph G represent components of a random vector, and edges represent the dependency structure between nodes. Thus, absence of an edge implies marginal or conditional independence between the corresponding random variables, and leads to zeros in the associated covariance or correlation matrix (or its inverse). Such models therefore yield parsimonious representations of dependency structures. Characterizing entrywise functions preserving positivity for matrices with zeros according to a graph is thus of tremendous interest for modern applications. Obtaining such characterizations is, however, much more involved than the original problem considered by Schoenberg as one has to enforce and maintain the sparsity constraint. The problem of characterizing functions preserving positivity for sparse matrices is also intimately linked to problems in spectral graph theory and many other problems (see e.g. [**39, 1, 50, 17**]).

As before, for a given graph $G = (V, E)$ on the finite vertex set $V = \{1, \dots, N\}$, we denote by $\mathcal{P}_G(I)$ the set of positive-semidefinite matrices with entries in I and zeros according to G, as in (4.1). Given a function $f : \mathbb{R} \to \mathbb{R}$ and $A \in \mathcal{S}_{|G|}(\mathbb{R})$,

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[7] We also point out the second main result in *loc. cit.*, that is, [**4**, Theorem 2], which classifies all continuous entrywise maps $f : \mathbb{C} \to \mathbb{C}$ that obey similar rank constraints in all dimensions. Such maps are necessarily of the form $g(z) = \sum_{j=1}^{p} \beta_j z^{m_j} (\overline{z})^{n_j}$, where the exponents m_j and n_j are non-negative integers. This should immediately remind the reader of Rudin's conjecture in the 'dimension-free' case, and its resolution by Herz; see Theorem 2.6.

denote by $f_G[A]$ the matrix such that

$$f_G[A]_{jk} := \begin{cases} f(a_{jk}) & \text{if } (j,k) \in E \text{ or } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

The first main result in [**31**] is an explicit characterization of the entrywise positive preservers of $\mathcal{P}_G$ for any collection of trees (other than copies of K_2). Following Vasudeva's classification for $\mathcal{P}_{K_2}$ in Theorem 3.1, trees are the only other graphs for which such a classification is currently known.

THEOREM 5.10 (Guillot–Khare–Rajaratnam [**31**, Theorem A]). *Suppose $I = [0, R)$ for some $0 < R \leq \infty$, and $f : I \to \mathbb{R}_+$. Let G be a tree with at least 3 vertices, and let A_3 denote the path graph on 3 vertices. The following are equivalent.*

(1) *$f_G[A] \in \mathcal{P}_G$ for every $A \in \mathcal{P}_G(I)$;*
(2) *$f_T[A] \in \mathcal{P}_T$ for all trees T and all matrices $A \in \mathcal{P}_T(I)$;*
(3) *$f_{A_3}[A] \in \mathcal{P}_{A_3}$ for every $A \in \mathcal{P}_{A_3}(I)$;*
(4) *The function f satisfies*

$$f\big(\sqrt{xy}\big)^2 \leq f(x)f(y) \qquad \text{for all } x, y \in I \tag{5.4}$$

and is super-additive on I, that is,

$$f(x+y) \geq f(x) + f(y) \qquad \text{whenever } x, y, x+y \in I. \tag{5.5}$$

The implication (4) $\implies$ (1) was further extended to all chordal graphs: it is the following result with $c = 2$ and $d = 1$.

THEOREM 5.11 (Guillot–Khare–Rajaratnam [**30**]). *Let G be a chordal graph with a perfect elimination ordering of its vertices $\{v_1, \ldots, v_n\}$. For all $1 \leq k \leq n$, denote by G_k the induced subgraph on G formed by $\{v_1, \ldots, v_k\}$, so that the neighbors of v_k in G_k form a clique. Define $c = \omega(G)$ to be the clique number of G, and let*

$$d := \max\{\deg_{G_k}(v_k) : k = 1, \ldots, n\}.$$

If $f : \mathbb{R} \to \mathbb{R}$ is any function such that $f[-]$ preserves positivity on $\mathcal{P}_c^1(\mathbb{R})$ and $f[M+N] \geq f[M] + f[N]$ for all $M \in \mathcal{P}_d$ and $N \in \mathcal{P}_d^1$, then $f[-]$ preserves positivity on $\mathcal{P}_G(\mathbb{R})$. [Here, $\mathcal{P}_d^1$ denotes the matrices in $\mathcal{P}_d$ of rank at most one.]

See [**30**] for other sufficient conditions for a general entrywise function to preserve positivity on $\mathcal{P}_G$ for G chordal.

To state the final result in this section, recall that Schoenberg's theorem (Theorem 2.3) shows that entrywise functions preserving positivity for all matrices (that is, according to the family of complete graphs K_n for $n \geq 1$) are absolutely monotonic on the positive axis. It is not clear if functions satisfying (5.4) and (5.5) in Theorem 5.10 are necessarily absolutely monotonic, or even analytic. As shown in [**31**, Proposition 4.2], the critical exponent (see Definition 4.5) of every tree is 1. Hence, functions satisfying (5.4) and (5.5) do not need to be analytic. The second main result in [**31**] demonstrates that even if the function is analytic, it can in fact have arbitrarily long strings of negative Taylor coefficients.

THEOREM 5.12 (Guillot–Khare–Rajaratnam [**31**, Theorem B]). *There exists an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that*

(1) *$a_n \in [-1, 1]$ for every $n \geq 0$;*

(2) *The sequence* $(a_n)_{n\geq 0}$ *contains arbitrarily long strings of negative numbers;*
(3) *For every tree* G, $f_G[A] \in \mathcal{P}_G$ *for every* $A \in \mathcal{P}_G(\mathbb{R}_+)$.

In particular, if $\Delta(G)$ *denotes the maximum degree of the vertices of* G*, then there exists a family* G_n *of graphs and an entire function* f *that is not absolutely monotonic, such that*

(1) $\sup_{n\geq 1} \Delta(G_n) = \infty$*;*
(2) $f_{G_n}[A] \in \mathcal{P}_{G_n}$ *for every* $A \in \mathcal{P}_{G_n}(\mathbb{R}_+)$.

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