Correct generation of the bound set-down for surface gravity wave groups in laboratory experiments of intermediate to shallow depth

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\textbf{A R T I C L E   I N F O}

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\textbf{A B S T R A C T}

Using linear (first-order) wave generation theory in laboratory experiments, leads to significant contamination of the wave field by free non-linear (second-order) error waves, increasingly so at shallower depths. Second-order wave generation theory has previously been established, and so has correct generation of the bound set-down, made up from second-order bound waves in the sub-harmonic part of the spectrum, for bichromatic and irregular wave fields in shallow to intermediate depth. In the present work, different from previous studies, we validate second-order wave theory explicitly for isolated wave groups, which provide a demanding test on the correct generation of sub-harmonic bound waves and the stroke length of the wavemaker. We do so for shallow to intermediate water depth, where some previous attempts at full elimination of sub-harmonic error waves have been hampered by limited paddle stroke. We overcome these limitations by applying second-order wavemaker theory to a piston-type paddle with an extended paddle stroke that can thence generate the bound set-down correctly. We show that sub-harmonic error waves are eliminated by considering wave groups in relative depths $k_0d = 0.6-1.1$, with important applications in coastal engineering experiments, such as run-up and overtopping.

1. \textbf{Introduction}

Occurrence of wave-induced coastal flooding is expected to increase due to the combined effects of sea level rise (Taherkhani et al., 2020) and more frequent occurrence of large transient waves in the coastal environment (Cattrell et al., 2019; Young and Ribal, 2019). Large transient waves events can pose significant threat to coastal assets, primarily due to their ability to damage and overtop coastal flood defences (see Dawson et al. (2016)). Such waves are highly nonlinear and highly transient, yet they are a key design condition for coastal engineering schemes (Van der Meer et al., 2018) and therefore require accurate estimation by coastal engineers.

Focused wave groups provide an efficient approach to approximate large transient wave events using a compact wave group of select component frequencies. They have been widely used in coastal engineering research (Longuet-Higgins, 1974; Chan and Melville, 1988; Drazen et al., 2008; Tian et al., 2011; Fernández et al., 2014; Antonini et al., 2017; Abroug et al., 2020; Fang et al., 2020) and have been suggested as a design wave in industry-recommended practice (Det Norske Veritas, 2016; ISO:19 901-1:2015).

In general, wave-driven structural responses of interest to coastal engineers are related to the most severe waves of a sea state. Focused wave groups offer the ability to recreate these wave conditions through judicious selection of the underlying energy spectrum, permitting assessment of the associated structural responses in a time efficient and highly repeatable manner (e.g., Borthwick et al. (2006); Holland et al. (2014)), when compared with irregular wave tests. Focused wave groups are used within a framework known both as NewWave theory (Tromans et al., 1991; Jonathan and Taylor, 1997; Walker and Taylor, 2004; Taylor and Williams, 2004; Borthwick et al., 2006; Whittaker et al., 2016; Chen et al., 2018) and the theory of quasi-determinism (QD) (Boccotti, 1983, 1989, 2000), based on the statistical underpinnings of Lindgren (1970). In this framework, the average shape of an extreme wave crest in a linear random Gaussian sea (i.e., the linear surface
Whittaker et al. (2016) demonstrated that a NewWave/QD group published, and application in shallower-water scenarios is becoming common.

Studies that have implemented second-order wave generation.

Table 1

<table>
<thead>
<tr>
<th>Authors of study</th>
<th>Method Used</th>
<th>Wave types used in study</th>
<th>Validated in freq.</th>
<th>Validated in time</th>
<th>Assessment of agreement between theory and experiments in the time and frequency.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barthel et al. (1983)</td>
<td>0.6</td>
<td>Bichromatic</td>
<td>✓</td>
<td>✓</td>
<td>Excellent agreement in time. Reasonable agreement in frequency with some under-prediction of magnitude.</td>
</tr>
<tr>
<td>Barthel et al. (1983)</td>
<td>1.1</td>
<td>Bichromatic</td>
<td>✓</td>
<td>✓</td>
<td>Partial agreement in time and poor agreement in frequency with significant under-prediction of magnitude.</td>
</tr>
<tr>
<td>Barthel et al. (1983)</td>
<td>0.7</td>
<td>Irregular</td>
<td>✓</td>
<td>✓</td>
<td>Excellent agreement in time and frequency.</td>
</tr>
<tr>
<td>Barthel et al. (1983)</td>
<td>0.9</td>
<td>Irregular</td>
<td>✓</td>
<td>✓</td>
<td>No improvement in agreement in time and worse agreement in frequency with second-order generation.</td>
</tr>
<tr>
<td>Schäffer (1996)</td>
<td>Includes correction of Barthel et al. (1983)</td>
<td>0.6–1.2</td>
<td>Bichromatic</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Schäffer (1996)</td>
<td>Includes correction of Barthel et al. (1983)</td>
<td>0.7 and 0.9</td>
<td>Irregular</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Van Leeuwen and Klopman (1996)</td>
<td>1.0–1.2</td>
<td>Bichromatic</td>
<td>✓</td>
<td>✓</td>
<td>Good agreement shown in comparison between amplitudes of bound components.</td>
</tr>
<tr>
<td>Van Leeuwen and Klopman (1996)</td>
<td>1.0</td>
<td>Irregular</td>
<td>✓</td>
<td>×</td>
<td>Reasonable agreement in frequency with some remaining free waves present probably due to noise, also for broad-banded spectra. No comparison in time.</td>
</tr>
<tr>
<td>Boers (1996); Battjes et al. (2004)</td>
<td>Van Leeuwen and Klopman (1996)</td>
<td>0.5–0.9</td>
<td>Irregular</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>Baldock et al. (2000)</td>
<td>Barthel et al. (1983)</td>
<td>1.4–3.4</td>
<td>Bichromatic</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Sriman et al. (2015)</td>
<td>Schäffer (1996)</td>
<td>1.5</td>
<td>Isolated groups</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>Whitaker et al. (2017)</td>
<td>Schäffer (1996)</td>
<td>0.7</td>
<td>Isolated groups</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>Martins et al. (2021) (GLOBEX)</td>
<td>Van Leeuwen and Klopman (1996)</td>
<td>0.9–1.5</td>
<td>Irregular &amp; Bichromatic</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>Present study</td>
<td>Van Leeuwen and Klopman (1996)</td>
<td>0.6–1.1</td>
<td>Isolated groups</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Elevation follows a Gaussian distribution) can be approximated by the scaled autocorrelation function of the underlying free surface.

Use of focused wave groups in deep-water scenarios is well established, and application in shallower-water scenarios is becoming common. Whitaker et al. (2016) demonstrated that a NewWave/QD group that is corrected for second-order bound waves can reproduce the average shape of the largest wave crests in shallow water (k_d < 0.5). Borthwick et al. (2006); Orszaghova et al. (2014); Whittaker et al. (2017), and Judge et al. (2019) used similar groups to investigate flow kinematics and wave run-up on plane beaches. Hunt (2003); Orszaghova et al. (2014); Hoffland et al. (2014), and Antonini et al. (2017) examined wave overtopping and Whitaker et al. (2018) and Chen et al. (2018) wave forcing on inclined to vertical coastal structures and surface-piercing columns. Karmpadakis and Swan (2020) found that when assessing storm-sea time series for the largest (e.g., 1%) wave crests, there is a likelihood that the breaking status of the waves is neglected. Therefore, comparing a focused wave group profile to the average shape of the largest crests could reach an inappropriate conclusion. They found that a focused wave group, specifically in the form of the N ewWave/QD framework, provides good approximation of large wave shapes in finite depth. Yet, they note that such wave crest statistics are significantly amplified by high-order nonlinear wave-wave interactions (above second-order), where the nonlinear amplification of wave crests in very steep sea states has poor agreement with the NewWave/QD framework.

Nonlinear free error waves are an inherent product of first-order wave generation (Barthel et al., 1983; Schäffer, 1996). Error waves are widely discussed in literature, where they are also referred to as parasitic or spurious waves (e.g., Hunt (2003); Orszaghova et al. (2014); Akinin and Spinneken (2017); Vyzikas et al. (2018), and Pierella et al. (2021)). They are created through a disparity between bound sub- and super-harmonics present in the wave field and the first-order boundary condition at the wavemaker’s face (e.g., Schäffer (1996)). In multi-frequency wave fields, such as focused wave groups, error waves at sub- and super-harmonic frequencies are produced as an instantaneous correction for the absence of bound nonlinearities in the wavemaker displacement. Free error waves satisfy the linear dispersion relation, so freely disperse at a celerity defined by their frequency. Generally, sub-harmonic error waves travel faster than the first-order wave group and are first to arrive at the domain of interest, whereas super-harmonic error waves travel more slowly and trail behind (e.g., Hunt (2003)). Therefore, due to their faster speeds and long wave lengths, sub-harmonic error waves present the most persistent challenge to experimentalists.

Orszaghova et al. (2014) found major discrepancy in run-up and overtopping induced by wave groups due to the presence of sub-harmonic error waves. Their Boussinesq-type numerical model showed sub-harmonic error waves increased run-up by 18–57% and overtopping volumes by 25–83%. Borthwick et al. (2006); Hunt-Raby et al. (2011); Buldakov et al. (2017), and Calvert et al. (2019) also
includes the key contributions to the development and validation of (sub-harmonic) second-order wave generation from various studies. The table includes the relative water depth, k0d, and the wave types used (bichromatic, irregular or isolated groups) in the study. Furthermore, we note whether validation has been carried out in the frequency and/or the time domain. This distinction is important for isolated groups as time-domain comparisons allow a comparison of the amplitude, shape and alignment of the set-down, whereas frequency-domain comparison allow assessment of whether all wave periods are correctly represented, including the longer waves, which may suffer from paddle stroke limitations the most.

Summarising, Barthel et al. (1983) were able to obtain good agreement between the theory they developed and experiments for both bichromatic and irregular waves for shallow water (k0d = 0.6–0.7), but not in intermediate water (k0d = 0.9–1.1). The theory by Barthel et al. (1983) was corrected by Schäffer (1996), who derived a full multi-chromatic theory for wave generation. Schäffer found good agreement for bichromatic and irregular wave fields in both shallow and intermediate water depths (k0d = 0.6–1.2). By considering narrow-banded packets, Van Leeuwen and Klopman (1996) were able to validate their results for both bichromatic and irregular waves in intermediate water depths (k0d = 1.0–1.2). Furthermore, Van Leeuwen and Klopman note that their method is also applicable to large bandwidths in practice despite the theoretical narrow-bandwidth limit their results rely on (a conclusion findings in the present paper will support). Only Sriram et al. (2015) (for their shallowest case) and Whittaker et al. (2017) (partially) have validated second-order wave generation for isolated wave groups, with Sriram et al. (2015) examining wave groups in intermediate to deep water and Whittaker et al. (2017) only achieving a 60% reduction of the sub-harmonic free error wave amplitude in shallow water depths.

Schäffer (1996), builds upon the work of Barthel et al. (1983), and derived a formulation for the complete (i.e., no bandwidth limitations) second-order paddle displacement signals for both piston and pivot-type wavemakers, which is based on a traditional, multi-frequency Stokes-type perturbation expansion. Schäffer’s theory dictates that the paddle’s (sub-harmonic) backward stroke length increases as k0d decreases. This means that in shallow water depths, not only is the effect of error waves greater, but also the required paddle sub-harmonic stroke length is greater too. Schäffer presents experimental results in water depths of, k0d = 0.6 up to 1.1.

Herein, we focus on sub-harmonic generation, yet we note for completeness that Spinneken and Swan (2009a,b); Aknin and Spinneken (2017) derived and successfully implemented second-order theory using force-feedback wavemaker control. The complexity of the method meant it is only applicable to mitigating super-harmonic error waves in regular wave fields. Moreover, we do not examine wave absorption. We note that for irregular or long-duration cases absorption is essential and has been successfully implemented.

For isolated focused wave groups, such as those given by the NewWave/QD framework, the time signal of the wave group corresponds to an infinite repeat period and an infinitesimal frequency discretization,
In the laboratory, group generation. The periodic paddle displacement resembles a saw-tooth displacement profile (displayed in Fig. 1). In the laboratory, an infinite number of Fourier components is used, always periodic, and relatively long signals (much longer than the wave group itself) need to be created in order for the finite repeat period (as an arbitrary time scale is determined by frequency dependent linear dispersion ($\omega$ = $\omega_n^2 = gk_n\tanh (k_n d)$, where $d$ is water depth). The desired shape of the wave representation of the wavemaker and its operating system, (c) photograph of the wavemaker.

### Table 2
Experimental input matrix.

<table>
<thead>
<tr>
<th>Expt.</th>
<th>$f_0$ (Hz)</th>
<th>$a_0$ [m]</th>
<th>$\sigma$ [m]</th>
<th>$c_{p,0}$ [m/s]</th>
<th>$\Delta x_{p,0}$ [m]</th>
<th>$k_{p,d}$</th>
<th>$\epsilon - \epsilon_{k_{p,d}}$</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6</td>
<td>0.034</td>
<td>4.70</td>
<td>1.27</td>
<td>-0.57</td>
<td>0.61</td>
<td>0.09</td>
<td>0.10</td>
</tr>
<tr>
<td>2</td>
<td>0.6</td>
<td>0.034</td>
<td>2.51</td>
<td>1.27</td>
<td>-0.31</td>
<td>0.61</td>
<td>0.09</td>
<td>0.19</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>0.034</td>
<td>1.51</td>
<td>1.27</td>
<td>-0.18</td>
<td>0.61</td>
<td>0.09</td>
<td>0.32</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
<td>0.024</td>
<td>2.37</td>
<td>1.11</td>
<td>-0.11</td>
<td>0.85</td>
<td>0.09</td>
<td>0.09</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
<td>0.024</td>
<td>1.80</td>
<td>1.11</td>
<td>-0.07</td>
<td>0.85</td>
<td>0.09</td>
<td>0.17</td>
</tr>
<tr>
<td>6</td>
<td>0.8</td>
<td>0.024</td>
<td>1.08</td>
<td>1.11</td>
<td>-0.04</td>
<td>0.85</td>
<td>0.09</td>
<td>0.29</td>
</tr>
<tr>
<td>7</td>
<td>1.0</td>
<td>0.040</td>
<td>2.53</td>
<td>0.94</td>
<td>-0.03</td>
<td>1.14</td>
<td>0.20</td>
<td>0.08</td>
</tr>
<tr>
<td>8</td>
<td>1.0</td>
<td>0.040</td>
<td>1.35</td>
<td>0.94</td>
<td>-0.02</td>
<td>1.14</td>
<td>0.20</td>
<td>0.16</td>
</tr>
<tr>
<td>9</td>
<td>1.0</td>
<td>0.040</td>
<td>0.81</td>
<td>0.94</td>
<td>-0.01</td>
<td>1.14</td>
<td>0.20</td>
<td>0.26</td>
</tr>
</tbody>
</table>

where the signals are constructed from periodic Fourier components, such as in the theory of Schaffer (1996). This means that the resulting paddle displacements in Schaffer (1996) are, in practice (unless an infinite number of Fourier components is used), always periodic, and the first-order group generation. The periodic paddle displacement resembles a saw-tooth displacement profile (displayed in Fig. 1). In the laboratory, relatively long signals (much longer than the wave group itself) need to be created in order for the finite repeat period (as an arbitrary time scale to be chosen by the experimentalist) not to result in additional sub-harmonic energy being created and interfering with the correct generation of the sub-harmonic bound waves.

Finer frequency discretization can be used to mitigate this unwanted response in the sub-harmonic wavemaker displacement. In the narrow-banded theory of Van Leeuwen and Klopman (1996), which makes use of the method of multiple scales, isolated wave groups with an infinite repeat period are readily obtained in the time domain. Second-order wavemaker theory then encompasses a net (sub-harmonic) backwards movement of the paddle during wave group generation. The sub-harmonic wavemaker signal is readily evaluated in the form of an integral of the (square of the) wave group envelope in the time domain, which is inexpensive to compute.

In this paper, we apply the second-order wave generation theory derived by Van Leeuwen and Klopman (1996) to a prototype, piston-type wavemaker to generate isolated wave groups. The piston-type wavemaker to generate isolated wave groups. The wavemaker has a long stroke length, allowing second-order (sub-harmonic) wave generation to be applied to shallow to intermediate water depths of $k_d = 0.6 - 1.1$. We present surface elevation measurements of a range of focused wave groups, comparing cases with first-order and second-order accurate wave generation. Our work aims to provide a simple methodology that can be employed by coastal engineers in future experimental campaigns concerned with wave-structure interaction studies using focused wave groups or to validate the efficacy of existing wavemakers with built-in second-order generation.

### 2. Review of the wavemaker theory

We begin by reviewing the second-order wavemaker theory we implement to control our new long-stroke, piston-type wavemaker. A piston-type paddle operates with its paddle face moving horizontally through the entire water column in response to a predetermined displacement time series. This second-order accurate paddle signal is denoted as

$$X_p(t) = X^{(1)}_p + X^{(2)}_p,$$

where the superscript corresponds to the order in steepness. The second-order displacement is the sum of sub and super-harmonic displacements ($X^{(2)}_p = X^{(-2)}_p + X^{(2)}_p$).

#### 2.1. First-order wave generation $X^{(1)}_p$

The surface elevation of a linearly focused wave group, composed of $N$ frequencies, is given as

$$\eta^{(1)}(x, t) = \sum_{n=1}^{N} a_n \cos(k_n x - \omega_n t - \phi_n),$$

where $x$ denotes location in a flume away from the wave generation origin at $x = 0$, $t$ is time, $x$ and $t$ note the desired spatial and temporal focus locations, $a_n$ is the amplitude, $k_n$ the wavenumber, and $\omega_n$ the angular frequency of the $n$th component. Phasing of the $n$th component is determined by frequency dependent linear dispersion ($a_n^2 = gk_n\tanh (k_n d)$, where $d$ is water depth). The desired shape of the wave
group at focus dictates the phasing of the component waves. A crest-focused group is produced when \( \phi_f = 0^\circ \), a trough-focused group when \( \phi_f = 180^\circ \) and up or down crossings at focus when \( \phi_f = 90^\circ \) or \( 270^\circ \), respectively. The paddle displacement for a first-order wave group is \( 90^\circ \) out of phase with the surface elevation field and given as

\[
X_p^{(1)} = \sum_{n=1}^{N} a_n \sin(k_n(x-x_f) - \omega_n(t-t_f) + \phi_f),
\]

where \( a_{n,\text{offset}} \) is the amplitude of paddle displacement related to \( a_n \) through the paddle transfer function. The first-order paddle transfer function for a piston-type wavemaker is given as (e.g., Biésel and Suquet (1951); Ursell et al. (1960); Flick and Guza (1980); Sand and Donslund (1985))

\[
a_{n,\text{offset}} = k_d D_n(k_d) \frac{\tanh(k_d)}{\sinh(k_d) \cosh(k_d)} + 1 \]

where \( D_n(k_d) = \frac{k_d}{2} \frac{\sinh(k_d) \cosh(k_d)}{\sinh(k_d) \cosh(k_d)} + 1 \).

The first-order wave field comprises a progressive wave, matching the intended surface elevation, eq. (2), away from \( x = 0 \), and evanescent modes, which are products of the disparity between the paddle face’s flat geometry and the local depth-varying velocity profile. Evanescent modes decay rapidly away from the paddle, typically becoming negligible at a distance of \( x = 3d \) (e.g., Dean and Dalrymple (1991)).

### 2.2. Second-order wave generation \( X_p^{(2)} \)

To apply the wavemaker theory of Van Leeuwen and Klopman (1996), we must rewrite eq. (2) using the narrow-bandwidth approximation made in Van Leeuwen and Klopman (1996). To do so, we first express eq. (2) using complex notation

\[
\eta^{(1)}(x,t) = \text{Re} \left[ \sum_{n=1}^{N} a_n e^{i(k_n(x-x_f) - \omega_n(t-t_f) + \phi_f)} \right].
\]

Approximating the linear dispersion relationship as \( \omega_n \approx \omega_0 + c_{g,0}(k_n - k_0) \) with \( c_{g,0} = \partial \omega / \partial k \big|_{k_0} \), consistent with the narrow-bandwidth approximation, eq. (5) can be rewritten as

\[
\eta^{(1)}(x,t) = \text{Re} \left[ A(x-c_{g,0}t)e^{i(k_n(x-x_f) - \omega_0(t-t_f) + \phi_f)} \right]
\]

with

\[
A = \sum_{n=1}^{N} a_n e^{i(k_n(x-x_f) - \omega_0(t-t_f) + \phi_f)}.
\]
where \( \omega_0 \) and \( k_0 \) are now the carrier wave’s angular frequency and wavenumber, respectively, chosen to correspond to the spectral peak, and the (complex) envelope \( A(x - c_\text{g} \cdot t) \) travels with the group velocity (without focusing, consistent with the narrow-bandwidth approximation). The group velocity is given by

\[
c_\text{g},0 = n c_p,0,
\]

where \( n = \frac{1}{2} + \frac{k_0 d}{\sinh(2k_0 d)} \). \( \text{(8)} \)

The envelope (or modulation) \( A \) varies on much greater spatial and temporal scales than the carrier wave.

Mei (1989) gives the compatible (i.e., narrow-banded) second-order sub-harmonic surface elevation as

\[
\eta^{(-2)} = -\frac{g|A|^2}{2(gc_p,0 - gd)} \left( \frac{2c_\text{g},0}{c_p,0} \frac{1}{2} \right).
\]

where \( c_{p,0} = \omega_0/k_0 \) is the phase speed of the carrier wave. The second-order sub-harmonic (or ‘wave-averaged’) surface elevation eq. (9) has the appearance of a wide trough and is known as the set-down (Longuet-Higgins and Stewart, 1962). The set-down is forced by and slaved to the first-order wave group \( A \) and can therefore be described as bound, propagating at the group celerity \( c_\text{g},0 \) with the first-order wave group \( A \).

The occurrence of a set-down can be readily explained, using the unsteady Bernoulli equation, as the surface manifestation of the wave-induced return flow running beneath the wave group in the negative \( x \)-direction (e.g., Calvert et al. (2019)) or in terms of radiation stresses (Longuet-Higgins and Stewart, 1962). In shallower water depths, the return flow, which is driven by the divergence of the Stokes transport on the scale of the wave group, must be ‘returned’ or ‘funneled’ through a shallower depth, increasing the magnitudes of the (negative) return flow velocity and thus of the set-down.

To generate this set-down (i.e., eq. (9)), Van Leeuwen and Klopman (1996) (see also Klopman and Van Leeuwen (1990)) show that the sub-harmonic paddle displacement signal (for a piston wavemaker) is given as

\[
X_{\text{p, periodic}} = -\frac{gc_{p,0}}{2d(c_p,0 - gd)} \left( 2\eta - \frac{1}{2} \right) \int_{\frac{T}{2}}^{T/2} \left(A(t)^2 - |A|^2\right) dt,
\]

where we have set \( x = 0 \) (cf. the wavemaker’s location) in the envelope \( A \) and we have chosen a time-periodic signal \( t = (-T/2, T/2) \) with repeat period \( T \). In eq. (10), the term \( \overline{|A|^2} = \frac{1}{T} \int_{-T/2}^{T/2} |A|^2 dt \) ensures the sub-harmonic paddle signal is periodic, and the paddle returns to its original position after generating the wave. For isolated wave groups, which

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Fig. 4. Measured first-order surface elevations at wave gauge 2 (black lines) for the nine experiments with the group envelopes (blue lines) obtained using a Hilbert transform.
correspond to the limit of an infinite repeat period (i.e., $T \to \infty$), this term can be ignored, and the non-periodic sub-harmonic paddle signal is given by

$$X_p^{(-2)} = \frac{-g \epsilon d}{2d (C_{g0} - gd)} \int_{-\infty}^{\infty} |A(\tau)|^2 d\tau.$$  \hspace{1cm} (11)

We can obtain the net (backward) paddle displacement accumulated over the course of generating an isolated wave group by letting $t \to \infty$ in eq. (11).

Fig. 1 illustrates the first and second-order paddle displacement times series for a Gaussian wave group. Fig. 1a shows the first-order paddle displacement $X_p^{(1)}$ as a function of time, which shows the isolated wave group. Fig. 1b compares the periodic $X_p^{(-2)}_{\text{periodic}}$ (for $T = 32$ s) and non-periodic $X_p^{(-2)}$ sub-harmonic paddle displacement. Although it is evident that the periodic and non-periodic sub-harmonic signals both display a backward motion at the same speed at the centre of the group (at $t = 0$), the periodic signal readily returns to zero on both sides of the group, giving rise to a saw-tooth-like profile. Such a periodic signal can create the set-down correctly for isolated groups (Orszaghova et al., 2014), but requires a long repeat period for the paddle’s restoration to zero not to generate significant sub-harmonic motion of its own. In our experiments, we used the non-periodic sub-harmonic paddle signal, and then restored the paddle position to zero after each experiment. Finally, Fig. 1c shows the summation of first and second-order paddle displacement. Also shown in Fig. 1 are the second-order super-harmonic paddle displacements $X_p^{(-2)}$, which we include in all our experiments, but do not focus on in this paper (see Van Leeuwen and Klopman (1996) for details).

2.3. Error waves

If the correct sub-harmonic paddle displacement is not included, the physical response to the disparity in the boundary condition at second order is the formation of an error wave in the form of a sub-harmonic hump, which cancels out the set-down at the wave maker. Unlike the set-down, the error wave satisfies the linear dispersion relation, and propagates at the shallow-water wave speed ($\sqrt{gd}$). This results in the error wave being ‘free’ to propagate ahead of the linear wave group. In practice, the finite length of experimental flumes means that the sub-harmonic error wave typically does not have time to separate out from the linear group (e.g., Calvert et al. (2019)) and instead appears superimposed on the group set-down.

For completeness, we note that, as well as a low-frequency sub-harmonic error waves, a high-frequency super-harmonic error is formed in case of first-order generation. The super-harmonic error wave propagates slower than the linear group, and is therefore typically of much less concern in coastal engineering experiments that use wave groups.

3. Experimental methodology

3.1. Experimental set-up

The present experiments were conducted in a wave flume in the COAST (Coastal, Ocean And Sediment Transport) Laboratory, at the University of Plymouth, UK. The flume is 20 m in length, 0.6 m in width, with a constant water depth of 0.23 m. Fig. 2a shows the experimental set-up. The wavemaker (photograph in Fig. 2c) is situated at one end of the flume and a 3 m-long energy dissipating beach with a 1 : 2.5 uniform slope at the other end. Current circulation ducts at each end of the flume were completely sealed off to avoid unwanted flow. All experimental locations are referenced from the paddle resting position at $x = 0$ m. In between the wavemaker and beach, eight resistance-type wave gauges measured the free surface elevation at 128 Hz. The gauges were located along the central line of the flume, and their $x$-locations are denoted in the table in Fig. 2. A three-point gauge calibration, over a vertical range of 10 cm, was performed each morning prior to tests. Data acquisition from all eight wave gauges was triggered simultaneously with the wave paddle displacement time series.

3.2. Prototype wavemaker

The wavemaker (photograph in Fig. 2c) was designed and built by Edinburgh Designs Ltd (EDL) and has an $x$-displacement range of $\pm 0.8$ m. It is wet-back and comprises a carriage with four independent wheels connected to two stainless steel rails mounted on the sides of the flume. The carriage is driven by an electrical motor with a rack and pinion system. The paddle is operated through displacement control according to a pre-defined paddle displacement time series. This allows for
generation of non-periodic paddle signals, such as used in our experiments for isolated wave groups.

Fig. 2b shows the impermeable seal placed around the edges and base of the paddle face. The seal was designed to retain a head of water across the paddle face for an extended period of time. A key finding from preliminary calibration experiments is that to generate the sub-harmonic surface elevation correctly, it is critical to maintain the volume of water behind the paddle, preventing water leaking forwards around the paddle face, which can result in additional sub-harmonic error waves at late times. The final seal design comprised a dual layer of semi-rigid plastic film with petroleum grease lubricating the boundary with the flume wall. After each experiment, the paddle position was restored to $x = 0 \text{ m}$. A 10 min settling period was held between each experiment to let residual energy dissipate. A small foam dissipative beach installed behind the paddle face reduced seiche effects in the wet-back area.

### 3.3. Experimental matrix

Table 2 shows the experimental matrix of the nine wave group experiments we have conducted. Each experimental case was generated twice, once using purely first-order paddle displacement and again with the additional second-order correction applied, where we use a non-periodic sub-harmonic signal eq. (11). We chose signals with Gaussian wave groups of the form

$$A = a_0 \exp\left(- \left(\frac{(x-x_f) - c_g(t-t_f)}{\sigma^2}\right)^2\right),$$

where $x$ is the characteristic length scale of the wave group. Three non-dimensional numbers characterise each experiment: the non-dimensional water depth $k_0d$, the steepness $\epsilon = k_0 a_0$ and the bandwidth $\nu = \nu_{1/2} = \frac{\sigma_{1/2}}{\epsilon} = \frac{\sqrt{\nu_{1/2}}}{\epsilon}$, where $m_n$ is the $n$th moment of the energy spectrum. For Gaussian groups, the energy spectrum $S(\omega)$ of the surface elevation can be readily evaluated in closed form:

$$S(\omega) = \frac{\sigma_{(n)}^2}{4\pi^2} \exp\left(-\frac{(\omega - \omega_0)^2}{4\sigma^2}\right).$$

from which the bandwidth parameter $\nu$ can be obtained: $\nu = \sqrt{2\sigma_{(0)}} / (\langle \omega_o \rangle = \sqrt{2\pi} / (k_0 \sigma))$. 

Fig. 6. Sub-harmonic surface elevation at wave gauge 2 for the nine experiments showing results for first-order generation (dotted black lines), second-order generation (solid black lines), and the theoretically predicted set-down of Mei (1989) (dashed red lines) computed using eq. (9).
Fig. 7. Frequency spectra for the sub-harmonic contribution to all nine experiments showing results for first-order generation (dotted black lines), second-order generation (solid black lines), and the theoretically predicted set-down of Mei (1989) (dashed red lines) computed using eq. (9).

Fig. 8. The sub-harmonic measured surface elevation with second-order generation after filtering with the same high-pass limits as in Janssen et al. (2003).
Our experimental matrix consisted of three peak frequencies ($f_0 = 0.6, 0.8, 1.0$ Hz), corresponding to shallow to intermediate relative water depths ($k_0d = 0.6, 0.9, 1.1$). For each frequency, we considered three bandwidths. The bandwidths were chosen so that the groups remained quasi-monochromatic, but sufficiently compact in time and space for any sub-harmonic error waves to separate out from the first-order group. Each experimental run had a duration of $T = 128$ s, $x_f = 4.5$ m, and $t_f = T/2$.

3.4. Separation of harmonics

To separate the second-order sub-harmonic waves from the first-order waves, we use the so-called two-phase harmonic extraction (or phase inversion) technique (Baldock et al. (1996), see also Hunt (2003)). To do so, we carry out each experiment in duplicate form, with a group focusing to a crest ($\eta_0(t)$) and a group focusing to a trough ($\eta_{180}(t)$). Specific combinations of the two inverted group time-series can yield the odd and even powers in amplitude, such that:

$$\eta_{\text{odd}} = \frac{\eta_0 - \eta_{180}}{2} \quad \text{and} \quad \eta_{\text{even}} = \frac{\eta_0 + \eta_{180}}{2} \quad (14)$$

To leading order, $\eta_{\text{odd}}$ is dominated by the first-order (in steepness) signal, and $\eta_{\text{even}}$ by the second-order (in steepness) signal in an underlying Stokes expansion. The latter is made up from bound waves and free error waves at second-order.

Fig. 3 illustrates the separation of harmonics method. Each harmonic contribution is displayed in both the time and the frequency domain. The top left (Fig. 3a) shows a crest-focused ($\eta_0$) and trough-focused ($\eta_{180}$) wave group measured within the flume. The top right (Fig. 3b) shows the decomposition of the amplitude spectrum into odd ($\eta_{\text{odd}}$) and even ($\eta_{\text{even}}$) components. The subsequent panels (Fig. 3c–j) separately display the different components, which have been isolated using a band-pass filter. The components are divided up into: second-order sub-harmonic components ($\eta^{(2)}$) obtained from band-pass filtering $\eta_{\text{even}}$. 

![Fig. 9. Sub-harmonic surface elevation for experiment 9 at all 8 wave gauges (WG) showing the two different generation methods: first-order generation (dotted black lines) and second-order generation (solid black lines). Also shown is the theoretically predicted set-down of Mei (1989) (dashed red lines) computed using eq. (9).](image-url)
between 0 and 0.5 Hz; first-order components \( \eta(1) \) obtained from band-pass filtering \( \eta_{\text{odd}} \) between 0.5\( f_0 \) and 1.5\( f_0 \); second-order super-harmonic components \( \eta(2) \) obtained from band-pass filtering \( \eta_{\text{even}} \) between 1.5\( f_0 \) and 2.5\( f_0 \); and third- (or higher) order components \( \eta(3) \) obtained from band-pass filtering \( \eta_{\text{odd}} \) between 2.5\( f_0 \) and 3.5\( f_0 \). For the sub-harmonic time series thus obtained, the error wave crest is labelled, preceding ahead of the set-down trough in Fig. 3c for this first-order generated group. Fig. 3i–j illustrate the negligible contribution at third order.

3.5. First-order wave groups

Fig. 4 shows the linearised wave groups at wave gauge 2 for all nine experimental cases with the envelopes obtained using a Hilbert transform also shown. Moving from left to right, the groups have increasing bandwidth, evident in the time domain as a group that is made up from fewer waves. The first-order surface elevations measured are equivalent for both the first and second-order generation methods.

3.6. Maximum paddle displacement

As shown in Equation (11), we obtain the net (backward) paddle displacement over the course of generating an isolated wave group by letting \( t \to \infty \) (i.e., \( \Delta X_{p, \text{total}}^{(-2)} = X_{p, \text{total}}^{(-2)}(t \to \infty) \)):

\[
\frac{\Delta X_{p, \text{total}}^{(-2)}}{a_0} = -\sqrt{\frac{\pi}{2}} \frac{en}{\nu} \left( \frac{2n - 1}{\mu} \right)^{1/2} \left( \frac{\text{anh}^{-1}(\text{anh}^{1/2} - 1)}{a_0} \right)
\]

where we have used the properties of a Gaussian group to evaluate the integral in eq. (11) in closed form. Equation (15) gives the non-dimensional total backward paddle displacement \( \Delta X_{p, \text{total}}^{(-2)}/a_0 \) as a function of three non-dimensional groups (nb. \( n = n(k_0 d) \) through eq. (8)): the relative depth \( k_0 d \), the steepness \( \varepsilon \), and the bandwidth parameter \( \nu \).

Table 2 gives the total backward paddle displacement for the different experiments, demonstrating the paddle displacement is greatest for the shallowest (smallest \( k_0 d \)) and most narrow-banded (smallest \( \nu \)) experiments. Fig. 5 shows the non-dimensional total backward paddle displacement \( X_{p, \text{total}}^{(-2)}/a_0 \) as a function of relative water depth \( k_0 d \) and...
steepness $\varepsilon$ for fixed bandwidth $\nu$ (Fig. 5a) as a function of relative water depth $k_0d$ and bandwidth $\nu$ for fixed steepness $\varepsilon$ (see, Fig. 5b). It is evident from Fig. 5 that the total net backward paddle displacement can be significantly larger than the linear wave amplitude, up to seventeen times observed in experiment 1, and increases rapidly as the relative water depth becomes shallower or the bandwidth decreases.

Within both panels of Fig. 5, values of the total backward displacement are not shown for Ursell numbers greater than 40, with the Ursell number defined as

$$Ur = \frac{H\lambda_0^2}{d^2} = \frac{8\varepsilon^2\nu}{(k_0d)^3},$$

(16)

where $\lambda_0 = 2\pi/k_0$. The Ursell number indicates the relative importance of nonlinearity in shallow water depths. $Ur > 40$ typically indicates regions where nonlinearity beyond second order becomes important and cnoidal wave theories need to be considered, which is beyond the scope of the present paper.

4. Results

4.1. Bound second-order set-down

Fig. 6 shows the measured sub-harmonic surface elevations for both first-order and second-order accurate wave generation compared to the theoretically predicted set-down according to Mei (1989) given in eq. (9). It is worth noting that the theoretical set-down given by Mei’s expression is congruent with the set-down given by the full second-order (broad-banded) theory for all our experimental cases (shown in fig. B14, included as an appendix). The measurements in Fig. 6 are recorded at wave gauge 2 ($x = 4.5$ m). The second-order generated sub-harmonics appear to match the theoretical set-down by Mei (1989) very well, especially at the centre of the group. The second-order generated cases do not show the preceding free error wave, which is clearly present as a set-up for the first-order generated cases. The first-order generated sub-harmonics shows poor agreement with the theoretical set-down. The free error wave is shown to be contaminating the first-order set-down in all nine cases, appearing as a set-up between $t = 4.0$ and $2.0$ s. After the group has passed, the second-order generated sub-harmonic surface elevation returns to the original still water level. The slight
increase in the water level after the group has passed is likely caused by a slow leaking forwards of the water held behind the wave paddle. Fig. 7 shows the sub-harmonic frequency contributions of the same nine experimental cases as displayed in Fig. 6. Again, the figure compares first-order generated, second-order generated, and theoretically derived sub-harmonic for of the nine wave group cases. The figure shows that, just as in time (e.g., Fig. 6) there is good agreement in frequency between the second-order generated groups and the theoretical prediction given by Mei (1989). Whereas, the first-order generated groups simply requires a much greater stroke length (see Fig. 5).

Through analysis of the harmonic structure of a range of wave groups, varying in peak frequency and spectral bandwidth, we have validated second-order wave theory explicitly for isolated wave groups in shallow to intermediate water depth (c_e = \sqrt{gd}) (continuous red line). These two speeds (i.e., c_g and c_e = \sqrt{gd}) are similar in magnitude for the shallower cases, so that the error wave no longer separates out. The reflections of the error wave and set-down are indicated by dashed lines in their respective colours. The absence of error waves and their reflections in Fig. 11 confirms second-order accurate generation has been successful in all nine experiments.

Fig. 12 compares the measured set-down amplitude \( \eta_{\text{min, Measured}}^{(1-2)} \) of both first-order (red dots) and second-order (black dots) generated wave groups with theoretically predicted second-order correct amplitude \( \eta_{\text{min, Theory}}^{(1-2)} \) all normalized by \( k_d d_g^2 \). The relative water depths of our experiments are indicated on the plot as well as a one to one agreement between measurements and theory (red dashed line). Error bars mark ±2 standard deviations around the mean obtained from five repeats.

5. Conclusion

Second-order wave generation theory has been established for nearly three decades, and numerous studies have partially or fully implemented the theory for bichromatic and irregular wave experiments. Different from previous studies, we validate second-order wave theory explicitly for isolated wave groups in shallow to intermediate water depth (k_d = 0.6–1.1). Previously this has only been achieved in deeper water with Sriram et al. (2015) only examining wave groups in intermediate to deep water (k_d = 1.5–3.4) and Whittaker et al. (2017) only achieving a 60% reduction of the sub-harmonic free error wave amplitude in shallow water (k_d = 0.7).

As water depth is reduced, stroke-length of the wavemaker becomes an important limiter in applying second-order wave theory, as noted by Whittaker et al. (2017). This is due to the required backward displacement of the wavemaker to correctly reproduce the set-down. We show that this backward displacement can readily exceed seventeen times the amplitude of the first-order group in shallow cases. In the present work, we have implemented a prototype wavemaker with an extended paddle stroke to enable us to reproduce the correct set-down for isolated groups in shallow water depths. To do so, we have applied the narrow-banded second-order wavemaker theory of Van Leeuwen and Klopper (1996), which has proven to be simple and efficient to compute in closed form, making its implementation to a laboratory wavemaker highly suitable to a broad range of users for different coastal engineering applications. Particularly, studies using isolated wavepackets as a design conditions for extreme wave-structure interactions.

Through analysis of the harmonic structure of a range of wave groups, varying in peak frequency and spectral bandwidth, we have
shown that the measured sub-harmonics are in excellent agreement with the theoretically predicted sub-harmonic free surface elevation (e.g., Mei (1989)). As further evidence of this agreement, the sub-harmonic error wave that normally travels ahead of the first-order group and contaminates experimental wave fields using first-order generation (e.g., Orszaghova et al. (2014)) are entirely eliminated. During our application of second-order correct paddle displacement, we have found it to be essential to have a seal around the paddle face, to maintain a water head during the paddle’s movement and prevent leakage of water around the paddle face.

Future work will use the newly implemented paddle and theory to quantify the implications of second-order correct focused wave groups, free of second-order error waves, for wave-structure interaction studies, in particular, run-up and wave loading on vertical structures.

CRediT authorship contribution statement


Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Repeatability

To assess repeatability and obtain an estimate of experimental error, fig. A13 shows the mean sub-harmonic free-surface elevation taken from five repeats of second-order generated wave groups, measured at wave gauge 2 (black lines), a confidence band of ±2 standard deviations around the mean (blue lines) and the theoretically predicted set-down according to Mei (1989) (red lines) for experiment 1–6. The six experimental cases show excellent repeatability, even when the sub-harmonic amplitude is < 3 mm. The confidence bands of all six experimental cases capture the magnitude of theoretical set-down maximum well with minor differences in shape in the time-domain between the measured and predicted set-down.

Fig. A.13. Sub-harmonic second-order generation repeatability in experiments 1–6. The mean of five repeats (black line), 2 standard deviations (±2std) around the mean (blue lines), the theoretical set-down according to Mei (1989) (red dashed line).
Appendix B. Narrow and broad-banded set-down comparison

Figure B14 is similar to the previously seen Fig. 6, but includes the broad-banded second-order free surface, which is given as the wave-averaged free-surface of a broad banded group in eq. (2.4) in McAllister et al. (2018). Figure B14, shows that there is a high degree of congruence between the broad-banded and narrow-banded second-order subharmonic.

![Fig. B.14. Similar to Fig. 6 but with the broad-banded second-order sub-harmonic surface elevation also included (blue dashed lines).](image-url)

References


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