Bed-slope-related diffusion of an erodible hump

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ABSTRACT: In certain conditions, the bed-slope has a substantial influence on sediment transport rates and hence morphological evolution. Approaches to account for such influence usually suffer from a high degree of empiricism and/or mathematical complexity. We propose a bed-slope-related diffusivity parameter, derived from a morphodynamic model previously validated against empirical data for bedload transport on horizontal and steep sloping channels. The proposed diffusivity parameter is easy to include in a conventional morphodynamic model via the modification of a typical bedload formula originally developed for sediment transport on a nearly-horizontal channel. A conventional model modified through this parameter proves to yield enhanced results in the case study of a submerged migrating hump/sandbar, by avoiding the generation of unrealistic high-frequency oscillations in the bed profile, yet permitting the expected steepening of its downstream face with time. Other models derived for sloping channels do not satisfy the latter condition. It is also shown that unrealistic oscillations can be avoided through numerical means; however, their use should be interpreted carefully from a phenomenological viewpoint.

1 INTRODUCTION

The earliest studies of sediment transport related to the fluvial environment, where bed-slopes tend to be small (in the order of $< 0.05^\circ$). Thus, pioneering works in the field were all based on steady unidirectional flows over mild bed-slopes. The tendency of considering these types of flows has continued until today and expanded to other environments, some of which can be very different hydrodynamically from typical rivers. Neglect of the potential effect of bed-slope on bedload transport is justifiable in practice for most cases concerned with mature river hydraulics, given the predominance of small bed-slopes in practice. However, in other hydraulic environments, such as beach shores and mountain streams, the bed-slope can be of such magnitude that its effect on bedload transport (and hence, on morphological evolution) may not be negligible.

In mountain streams, beds are usually composed of coarse sediments and have slopes that are sufficiently large to affect the overall flow behaviour, including the sediment transport rate (Bayazit 1983). For coarse-sediment beds, bedload is the primary mode of transport, and occurs at small flow velocities relative to the threshold for sediment motion. It is under these conditions that extra-hydrodynamic factors, in particular the bed-slope, may play an important role in the transport process. This has indeed been confirmed through laboratory experiments (e.g. Smart 1984; Damgaard, Whitehouse, & Soulsby 1997; Dey & Debnath 2001). The present paper is thus solely concerned with bedload (rather than total or suspended) sediment transport.

One way of accounting for the bed-slope influence is by adding a slope-related diffusivity term to a sediment-transport formula, which typically translates into an additional calibration parameter (Johnson & Zyserman 2002). Other alternatives include semi-empirical models based on Bagnoldian ideas (e.g. Bagnold 1963; Bailard & Inman 1981; Kovacs & Parker 1994); formulae explicitly derived for sloping beds, which often imply a significant degree of empiricism or complexity (e.g. Smart 1984; Chiari, Friedl, & Rickenmann 2010; Parker, Seminara, & Solari 2003); and modification of the threshold of motion for sloping beds by inclusion of the gravity contribution of a resting particle.
In this paper, we outline the derivation of an alternative analytical slope-related diffusivity parameter, which is then incorporated into a conventional morphodynamic model and compared against a Bagnoldian model by means of the benchmark case of a migrating hump/sandbar. Inclusion of numerical diffusion is also discussed. Remarks are made regarding the phenomenological importance of including physical (rather than numerical) slope-related diffusion.

2 METHODOLOGY

Conventional Morphodynamic Models (CMMs) often consist of a coupling between a hydrodynamic model and an equation governing the morphological evolution. The coupling is achieved through sediment transport formulae, which are in turn functions of hydrodynamic and sediment-related parameters. A popular example for bedload transport in one-dimension is provided by the set of equations formed by combining the Shallow Water Equations (SWEs) and the Exner morphological equation:

\[
\frac{\partial h}{\partial t} + \frac{\partial (U h)}{\partial x} = 0 \tag{1}
\]

\[
\frac{\partial (U h)}{\partial t} + \frac{\partial}{\partial x} \left( h U^2 + \frac{1}{2} g h^2 \right) = -g h S_b - \frac{\tau}{\rho} \tag{2}
\]

\[
\frac{\partial z_b}{\partial t} + \xi \frac{\partial q_b}{\partial x} = 0, \tag{3}
\]

where \(h\) is the total depth of the flow; \(U\) denotes the horizontal velocity vertically averaged over \(h\); \(S_b\) is the local bed-slope (with negative value in the down-sloping direction); \(\tau\) represents the bed shear stress; \(\rho\) is the density of water; \(q_b\) is the volumetric bedload transport rate; \(z_b\) is the bed level from a datum; \(\xi\) relates to bed porosity; \(g\) is the gravitational acceleration; and \(x\) is horizontal distance and \(t\) is time.

The set of equations (1)–(3) requires estimation of \(\tau\) and \(q_b\) for closure. The bed friction, \(\tau\), can be computed from a Chézy expression, as:

\[
\tau = \rho c_f U |U|, \tag{4}
\]

where \(c_f\) is the bed friction coefficient. Numerous options exist in the literature to estimate \(q_b\). A particularly popular formula to compute bedload transport is that of Meyer-Peter & Müller (1948). For the sheet-flow regime, a transport rate proportional to a power of \(U\) (usually 3) is often employed.

The influence of bed-slope on bedload can be included by adding a term to the sediment transport formula—originally derived for nearly horizontal channels—that promotes (inhibits) sediment motion for down(up)-sloping beds (see e.g. Johnson & Zyserman 2002; Watanabe 1988; Bailard & Inman 1981). Such a term is proportional to the bed-slope, and can be added as follows:

\[
q_{b\beta} = q_{bh} + \varepsilon |q_{bh}| S_b, \tag{5}
\]

where \(q_{b\beta}\) and \(q_{bh}\) represent bedload transport on a sloping and a horizontal bed, respectively; and \(\varepsilon\) denotes the slope-related diffusivity.

The reason for \(\varepsilon\) being referred to as a diffusion term is that, under certain circumstances (see e.g. Hudson & Sweby 2003), it is sensible to write the Exner equation (eq. 3) as an advection-diffusion equation. To illustrate this, consider \(q_b = q_{b\beta}\) and observe that \(S_b \equiv \partial q_b / \partial x\); thus, in unidirectional flow, eq. (3) can be manipulated into:

\[
\frac{\partial z_b}{\partial t} + \xi \frac{\partial q_b}{\partial x} = \frac{\partial}{\partial x} \left( \varepsilon q_{bh} \frac{\partial q_b}{\partial x} \right), \tag{6}
\]

where \(\varepsilon\) is related to the diffusion term on the right-hand side of (6). Such diffusion is responsible for smoothing out perturbations to the bed elevation profile that would otherwise be present, as will be exemplified later. In practice, \(\varepsilon\) typically represents an additional tuning parameter within morphological models, which naturally increases their level of empiricism.

Maldonado-Villanueva (2015) proposed a two-layer-like, Shallow-Water-Equation-based model for sediment transport and morphological evolution in open channels. The model, which has the novelty that it requires no empirical formulae for sediment transport rates (although some empiricism is still necessary for closure of the model), was validated satisfactorily against empirical data for bedload transport rates in horizontal and inclined channels. Fig. 1 illustrates the comparison between the Quasi-2-Layer model proposed by Maldonado-Villanueva (Q2L model) against the laboratory data from Damgaard, Whitehouse, & Soulsby (1997) and the model (originally derived for arbitrary bed-slopes) by Bagnold (1963).

Fig. 1 depicts predicted and measured bedload transport rates versus the bed-slope angle, \(\beta\), for three different values of the non-dimensional bed shear stress, \(\theta \equiv \tau / |\rho g (s - 1) D|\) (where \(s\) is the sediment
relative density; and $D$ is the sediment particle diameter). The model proposed by Maldonado-Villanueva (2015) is seen to agree well with empirical data reported by Damgaard et al. (1996) and against empirical data reported by Damgaard, Whitehouse, & Soulsby (1997). Use of (7) is limited to values of the bed-slope of $S_b < 0.2$ (all negative values included), corresponding to angles of $\lesssim 11^\circ$. This range encompasses most of the applications to be expected in fluvial and coastal environments.

Another estimation of $\varepsilon$, which does not require additional calibration parameters, can be obtained from the Bagnoldian model of Bailard & Inman (1981). Manipulation of Bailard’s work permits us to derive the following expression:

$$\varepsilon_{B\&I} = \frac{1}{\tan \varphi}. \quad (8)$$

It is worth noting that, although obtained following different approaches, both (7) and (8) predict a slope-related diffusivity inversely proportional to the angle of repose. However, $\varepsilon_{B\&I}$ is independent of the flow conditions, and so, unlike (7), it does not vanish at large flow velocities.

In the following section, the performance of the diffusivity parameter proposed in (7) is compared against that derived from Bagnoldian ideas (i.e. $\varepsilon_{B\&I}$) and against the case where no diffusion is included in (5) (i.e. $\varepsilon = 0$). The benchmark case employed is that of a completely submerged erodible hump (sandbar, in two dimensions) subject to a regular, nearly-uniform, unidirectional, subcritical current. The expected qualitative behaviour of the hump under these conditions is well known – it ought to migrate downstream and its downstream-face to steepen with time. Diffusion can also be introduced through numerical solution of the governing equations (see e.g. Johnson & Zyserman 2002) – one example which is also discussed below.

3 RESULTS

A submerged erodible hump (refer to Fig. 2) is located in an otherwise flat, horizontal channel of length, $l = 1000\, \text{m}$. Bed friction is neglected within the hydrodynamic module ($\tau = 0$ in eq. 2), and the hump profile is described by:
Adams-Bashforth time integration. The hydrodynamic part of the governing equations is discretised using a second-order central difference scheme in space and second-order accuracy in time (Harten, Lax, & van Leer 1979; Toro, Spruce, & Speares 1994). The Exner equation is solved by means of a Harten-Lax-van Leer Contact (HLLC) Riemann solver with the MUSCL-Hancock second-order time integration (van Leer 1979). The solution by Hudson & Sweby (2003) considers no diffusion, and so unrealistic steep slopes are permitted. The importance of slope-related diffusion from a phenomenological model more stable (not shown here for brevity). For example, the Exner equation can be solved using a phenomenological model that includes diffusion from Bailard & Inman (1981) is invoked, high-frequency oscillations are prevented; however, the correct migration of the hump is not replicated – the hump attenuates, demonstrating that ε is over-diffusive. On the other hand, the modification of qb through ε from (7) prevents development of oscillations in zb, while predicting a realistic migration of the hump (steepling of its downstream face). The latter modification also allows us to run a much longer simulation, as it renders the model stable (not shown here for brevity).

Similar results to those achieved through the inclusion of ε can be obtained by means of numerical techniques. For example, the Exner equation can be spatially discretised as the arithmetic mean of an upwind and a central finite difference scheme, such that

\[
\frac{\partial q_b}{\partial x} = \frac{1}{2} \left[ \frac{q_b(i+1) - q_b(i-1)}{2\Delta x} + \frac{3q_b(i) - 4q_b(i-1) - q_b(i-2)}{2\Delta x} \right],
\]

where \( \Delta x \) is the length of a grid cell denoted by \( i \). Without any modification to the bedload formula, the development of high-frequency oscillations can also...
be delayed using this numerical technique. This is illustrated in Fig. 4, which compares results obtained using the above numerical technique against the use of $\varepsilon$ from (7). The foregoing numerical technique retards the generation of oscillations (compare against the black thin curves in Fig. 3); these begin to develop at the upstream base of the hump towards the end of this simulation (see top-right panel in Fig. 4). Moreover, this approach yields significantly less diffusion than the use of (7).

It is important to note that, although numerical solvers can be used to avoid unrealistic oscillations in the bed level in some limiting cases, as has been demonstrated here, the primary goal of introducing a bed-slope-related diffusion term is not to avoid numerical oscillations. This term is included to ensure that the model represent more correctly observed physical phenomena such as the diffusion (smoothing effect) reported in the evolution of excavated holes in the surf zone (Moulton, Elgar, & Raubenheimer 2014). As such, the addition of $\varepsilon$ is a more general approach—grounded in the physics of the phenomenon—that avoids development of numerical oscillations in the bed.

The two-dimensional version of the case analysed above can also be studied (i.e. a sandbar). The hump is projected 20 m in the direction orthogonal to $x$ (i.e. $y$) and the flow is kept unidirectional (in the $x$-direction). The rest of the parameters are unchanged from above. Fig. 5 compares the four approaches previously considered. The findings are the same for the two-dimensional case. Lack of modification to the CMM promotes development of unrealistic oscillations in the bed level; use of $\varepsilon_{B&I}$ shrinks the sandbar; and the diffusion parameter yields stable realistic results (which are also similar—although slightly more diffusive—to those achieved through numerical manipulation of the bed-update equation).

To model a fully two-dimensional flow (i.e. with velocity component in the $y$-direction) over a two-dimensional hump (where $\partial z_b/\partial y \neq 0$) it would be necessary to include the influence of the transverse bed-slope. Further research is recommended to address this problem.

4 CONCLUSIONS

In order to account for the influence of bed-slope on bedload (and thus, morphological evolution), we have proposed a physically meaningful diffusivity parameter that is easy to implement within conventional morphodynamic models based on the coupling described by (1)–(3). The parameter is derived from a morphodynamic model previously validated against empirical data for bedload transport rates in horizontal and steep sloping channels. The proposed expression for $\varepsilon$ (eq. 7) does not require additional tuning variables and can enhance the performance of conventional morphodynamic models. This has been demonstrated through the benchmark cases of a submerged migrating hump (in one-dimension) and sandbar (in two-dimensions). The proposed $\varepsilon$ prevents development of spurious oscillations in bed level, while reproducing realistic migration of the erodible hump including steepening of its downstream face; this is not achieved by implementing the model of Bailard & Inman (1981) which leads to over-diffusive results. High-frequency oscillations in $z_b$ could also be avoided through mathematical means (Hudson & Sweby 2003) or numerical techniques (eq. 9; Johnson & Zyserman 2002). However, inclusion of slope-related diffusion has morphodynamic consequences, and so should not be used merely as a tool to stabilize numerical models. The present paper has focused on the effect of streamwise slope on bedload transport; extension of the approach to accommodate the transverse-slope effect is presently under investigation by the authors.

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REFERENCES

Figure 5: Evolution of a two-dimensional sandbar predicted by: a) CMM with no modification of $q_b$ (i.e., $\varepsilon = 0$); b) discretisation of Exner equation according to (9); c) CMM with $q_{b,\varepsilon}$ modified through $\varepsilon_{E&I}$; and d) CMM with $q_{b,\varepsilon}$ modified via the herein proposed expression for $\varepsilon$ (eq. 7). Results shown at $t = 238 \times 10^3$ s.


