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1 **Approximate Solutions for Ideal Dam-Break Sediment-Laden Flows on Uniform Slopes**  
2

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9

10 **Key Points:**

- 11 • Approximate solutions are derived for idealized dam-break sediment-laden flows on  
12 uniform slopes with negligible bed evolution
- 13 • The approximate solutions are shown to agree well with numerical solutions based on  
14 finite volume method
- 15 • This work facilitates benchmark solutions for verifying shallow water hydro-sediment-  
16 morphodynamic models  
17

## 18 **Abstract**

19 Shallow water hydro-sediment-morphodynamic (SHSM) models have been applied increasingly  
20 widely in hydraulic engineering and geomorphological studies over the past few decades.  
21 Analytical and approximate solutions are usually sought to verify such models and therefore  
22 confirm their credibility. Dam-break flows are often evoked because such flows normally feature  
23 shock waves and contact discontinuities that warrant refined numerical schemes to solve. While  
24 analytical and approximate solutions to clear-water dam-break flows have been available for  
25 some time, such solutions are rare for sediment transport in dam-break flows. Here we aim to  
26 derive approximate solutions for ideal dam-break sediment-laden flows resulting from the  
27 sudden release of a finite volume of frictionless, incompressible water-sediment mixture on a  
28 uniform slope. The approximate solutions are presented for three typical sediment transport  
29 scenarios, i.e., pure advection; pure sedimentation; and concurrent entrainment and deposition.  
30 Although the cases considered in this paper are not real, the approximate solutions derived  
31 facilitate suitable benchmark tests for evaluating SHSM models, especially presently when shock  
32 waves can be numerically resolved accurately with a suite of finite volume methods, whilst the  
33 accuracy of the numerical solutions of contact discontinuities in sediment transport remains  
34 generally poorer.

35

## 36 **1. Introduction**

37 Over the past few decades, there have been widespread applications of shallow water  
38 hydro-sediment-morphodynamic (SHSM) models in hydraulic engineering and  
39 geomorphological studies (Cao et al., 2017). A wide range of fluvial processes have been  
40 investigated, including dam-break floods over erodible beds (Cao et al., 2004; Capart & Young,  
41 1998; Huang et al., 2012, 2014; Wu & Wang, 2007; Xia et al., 2010) and morphodynamic  
42 processes (Nicholas et al., 2013; Qian et al., 2017; Wang et al., 2008; Wu, 2007; Xie, 1990). In  
43 fact, as pointed out by Cao et al. (2017), the SHSM equations have been extended to model  
44 coastal processes (Kim, 2015; Xiao et al., 2010; Zhu & Dodd, 2015), watershed erosion  
45 processes (Kim et al., 2013), subaqueous sediment-laden flows and turbidity currents (Hu & Cao,  
46 2009; Hu et al., 2012), and sharply stratified processes (Li et al., 2013; Spinewine & Capart,  
47 2013; Zech et al., 2015). Most recently, shallow water two-phase models have been developed to  
48 better resolve fluvial processes (Cristo et al., 2015; Li, Cao, Qian, et al., 2017) and debris flows  
49 (Li, Cao, Hu, et al., 2017a, b). Pivotal to the success of SHSM models are the recent advances in  
50 both computer technology and numerical methods. Examples include developments in the finite  
51 difference method (Wu, 2007; Xie, 1990), the finite element method (Michoski et al., 2013), and  
52 the finite volume method (FVM) (Tan, 1998; Toro, 2001). Prior to applications, such  
53 mathematical models have to be either verified against analytical and approximate solutions or  
54 validated against experimental datasets for SHSM processes or pure hydrodynamic processes  
55 without involving sediment transport. However, laboratory experiments are time-consuming and  
56 measurement accuracy is often limited. In contrast, analytical and approximate solutions can be  
57 derived for simplified physical problems, of which the governing equations are mathematically

58 tractable by analytical techniques (Fraccarollo & Capart, 2002; Toro, 2001). Although it is  
59 inevitable to neglect some physical mechanisms to obviate mathematical difficulty, analytical  
60 and approximate solutions are particularly valuable for the systematic assessment of numerical  
61 models describing more complex and realistic problems (Ancey et al., 2008; Berthon et al., 2012;  
62 Fernandez-Feria, 2006; Hogg, 2006; Pritchard & Hogg, 2002; Ritter, 1892; Stoker, 1957; Toro,  
63 2001; Zoppou & Roberts, 2003).

64 Dam-break flows are notable for their destructive power, which can cause catastrophic  
65 loss of life and substantial societal, economic and infrastructural damage (Cao et al., 2004).  
66 Dam-break flows are also of great interest to researchers investigating the ability of  
67 mathematical models to capture shock waves and contact discontinuities (Pritchard & Hogg,  
68 2002; Toro, 2001). Ancey et al. (2008) have summarized progress towards both analytical and  
69 approximate solutions for clear-water dam-break flows. In early stages, a variety of dam-break  
70 problems have been designed, including floods propagating over a frictionless horizontal bed  
71 (Hogg, 2006; Ritter, 1892; Stoker, 1957) and a rough horizontal bed (Dressler, 1952; Hogg &  
72 Pritchard, 2004; Whitham, 1955). However, only asymptotic solutions could be developed for  
73 floods over a rough horizontal bed. Afterwards, cases with frictionless and inclined beds have  
74 been considered. Mangeney et al. (2000) studied debris avalanches and examined the case of a  
75 dam perpendicular to the bed holding back water of constant depth in a reservoir of infinite  
76 length. Ancey et al. (2008) and Antuono and Hogg (2009) considered a flood of finite volume;  
77 the dam was again perpendicular to the bed, rather than vertical, in order to prevent singular  
78 behavior occurring when the method of characteristics was employed. Fernandez-Feria (2006)  
79 attempted to derive analytical solutions for dam-break flow with a vertical dam on a uniform  
80 slope, yet the solutions were actually approximate as numerical realization was required..

81 To date, however, analytical and approximate solutions for sediment-laden dam-break  
82 flows are rare compared to their clear-water counterparts. Fraccarollo and Capart (2002)  
83 presented an approximate theoretical description of the formative stages of erosional dam-break  
84 flows over a horizontal bed, under the assumptions of equilibrium sediment transport and  
85 negligible bed resistance. Pritchard and Hogg (2002) derived exact solutions for suspended  
86 sediment transport under dam-break flows over a frictionless horizontal bed with an infinite  
87 volume of water upstream, assuming that the bed geometry was not altered. Berthon et al. (2012)  
88 provided an analytical solution of the shallow water system coupled to the Exner equation, which  
89 was used to update the bed elevation, presuming a steady state condition of flow. Pritchard (2005)  
90 developed asymptotic solutions for passive suspended sediment transport under a flood surge on  
91 a uniform slope; however, the work was based on a kinematic wave equation. Therefore, the  
92 need for analytical and approximate solutions of sediment-laden dam-break flows in general  
93 conditions is obvious.

94 This paper aims to derive approximate solutions for ideal dam-break sediment-laden  
95 flows resulting from the sudden release of a finite volume of frictionless, incompressible water-  
96 sediment mixture on a uniform slope as opposed to a horizontal bed presumed by Pritchard and  
97 Hogg (2002). On an inclined bed without friction, the flow will accelerate unboundedly as in  
98 previous studies (e.g., Ancey et al. 2008; Fernandez-Feria, 2006), while on a horizontal bed  
99 without friction, the fluid will flow downstream unlimitedly (e.g., Pritchard & Hogg 2002; Ritter,

1892; Stoker, 1957). On the other hand, the absence of basal friction may substantially affect the front dynamics (Ancey et al., 2006; Hogg & Pritchard, 2004). Therefore, both cases are physically unrealistic and the analytical and approximate solutions are not directly applicable to engineering practice. However, analytical and approximate solutions of both cases facilitate benchmark solutions, especially when model developers and end-users are still extremely keen to have them to test numerical models. The present work is based on a simplified version of the full set of SHSM equations (Cao et al., 2017), which includes mass and momentum conservation equations for the water-sediment mixture flow, and a mass conservation equation for sediment. Here it is assumed that sediment particles are uniform in size, bed deformation is negligible, and the effects of sediment transport on the mixture flow can be ignored. Notwithstanding that certain mechanisms are overlooked, the governing equations here are more comprehensive physically than previously considered, representing non-equilibrium rather than equilibrium sediment transport (Fraccarollo & Capart, 2002), adopting a dynamic instead of kinematic wave equation for the mixture flow (Pritchard, 2005), and taking into account unsteady other than steady state condition of flow (Berthon et al., 2012). Succinctly, the present work extends Fernandez-Feria (2006) for clear-water dam-break flows to sediment-laden dam-break flows, and Pritchard and Hogg (2002) for sediment-laden flows over a horizontal bed to an inclined bed. It is significant for verifying numerical SHSM models, as some numerical methods excessively smear waves associated with sediment continuity equation, resulting in rather inaccurate or even incorrect predictions (Toro, 2001). Nevertheless, it is noted that the present work is approximate as parts of the solutions demand numerical realization.

This paper is organized as follows. Section 2 presents the governing equations. In section 3, the solution strategy of Fernandez-Feria (2006) for ideal dam-break flow (without sediment transport) is applied to derive approximate solutions for the water-sediment mixture flow, while sediment concentration evolution is solved based on the ordinary differential equation (ODE) along its characteristic curve. Section 4 outlines the numerical scheme based on FVM. Section 5 presents a comparison between the approximate solutions and numerical predictions for three distinct sediment transport scenarios: pure advection; pure sedimentation; and concurrent entrainment and deposition. Finally, conclusions are drawn in section 6.

129

## 130 **2. Governing Equations**

In order that the SHSM equations can be tractable to obtain approximate solutions, it is necessary to invoke several simplifications (Ancey et al., 2008; Fernandez-Feria, 2006; Fraccarollo & Capart, 2002; Pritchard & Hogg, 2002). First, the bed is taken to be frictionless. Second, it is assumed that bed deformation and the effects of sediment transport on the mixture flow are negligible, which means the mass conservation equation for bed material together with the sediment-related terms in the momentum equation for the mixture flow are decoupled from the SHSM systems and can be eliminated. As a result, the governing equations for the mixture flow here are identical in form to those for clear water flows. Third, uniform sediment transport is considered, which means sediment particles share the same density, shape and size. Under the aforementioned assumptions, the SHSM equations can be written in a conservative form:

140

$$141 \quad \frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} = 0 \quad (1)$$

$$142 \quad \frac{\partial(hu)}{\partial t} + \frac{\partial}{\partial x} \left( hu^2 + \frac{1}{2} g \cos \theta \cdot h^2 \right) = gh \sin \theta \quad (2)$$

$$143 \quad \frac{\partial(hc)}{\partial t} + \frac{\partial(huc)}{\partial x} = E - D \quad (3)$$

144 where  $t$  is time;  $x$  is the downstream coordinate;  $h$  is the flow depth;  $u$  is the depth-averaged  
 145 stream-wise flow velocity;  $g$  is gravitational acceleration;  $\theta$  is the bed inclining angle;  $c$  is the  
 146 depth-averaged sediment concentration;  $E$  represents the sediment entrainment flux; and  $D$   
 147 represents the sediment deposition flux.

148 It is noted that the understanding of the mechanism of sediment exchange with the bed  
 149 remains far from complete to date, which inevitably renders uncertainty in its estimation ( $= E -$   
 150  $D$ ). Comparatively, the uncertainty in the entrainment estimation prevails, whilst the  
 151 determination of the deposition flux is practically more reliable. Here, the conventional  
 152 relationships are introduced for the entrainment and deposition fluxes (Cao & Carling, 2002;  
 153 Huang et al., 2014; Qain et al., 2015; Wu, 2007; Xia et al., 2010)

$$154 \quad E = \alpha \omega_0 c_* \quad (4)$$

$$155 \quad D = \alpha \omega_0 c \quad (5)$$

156 where  $\alpha$  is an empirical parameter representing the ratio of the near-bed sediment concentration  
 157 to the depth-averaged sediment concentration, and is specified to be unity here for simplicity;  $\omega_0$   
 158 denotes the settling velocity of a single sediment particle; and  $c_*$  denotes the sediment transport  
 159 capacity determined by local flow conditions. In this connection, the approximate solutions  
 160 derived here are not limited to specific formulas for  $c_*$ ; rather they are intended to be valid for  
 161 general transport capacity formulas in the context of river dynamics (e.g., Celik & Rodi, 1991;  
 162 Dey, 2014; Wu, 2007; Yang, 1996).

163 Formulated in a Cartesian coordinate system with axes aligned along and perpendicular  
 164 to the bed, equations (1)-(3) are now recast into the following non-conservative form in terms of  
 165 primitive variables, ready for the derivation of approximate solutions.

$$166 \quad \frac{\partial h}{\partial t} + h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x} = 0 \quad (6)$$

$$167 \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \cos \theta \frac{\partial h}{\partial x} = g \sin \theta \quad (7)$$

$$168 \quad \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = \frac{\alpha \omega_0}{h} (c_* - c) \quad (8)$$

169

170 **3. Approximate Solutions**

171 3.1. Normalization and Homogenization

172 We choose the initial flow depth  $H_0$  at the dam wall,  $\sqrt{gH_0}$ , and the initial sediment  
173 concentration  $C_0$  at a given location as the length scale, velocity scale, and sediment  
174 concentration scale. Then the dimensionless variables are deduced as follows:

175 
$$T = \frac{t}{\sqrt{H_0/g}}, X = \frac{x}{H_0}, U = \frac{u}{\sqrt{gH_0}}, H = \frac{h}{H_0}, C = \frac{c}{C_0}, \beta = \frac{\omega_0}{\sqrt{gH_0}}, \text{ and } C_* = \frac{c_*}{C_0} \quad (9)$$

176 Substitution of the dimensionless variables (9) into equations (6)-(8) yields the following  
177 normalized equations

178 
$$\frac{\partial H}{\partial T} + H \frac{\partial U}{\partial X} + U \frac{\partial H}{\partial X} = 0 \quad (10)$$

179 
$$\frac{\partial U}{\partial T} + U \frac{\partial U}{\partial X} + \cos \theta \frac{\partial H}{\partial X} = \sin \theta \quad (11)$$

180 
$$\frac{\partial C}{\partial T} + U \frac{\partial C}{\partial X} = \frac{\alpha \beta}{H} (C_* - C) \quad (12)$$

181 Initial conditions are given as follows:

182 
$$U(0, X) = 0 \quad (13)$$

183 
$$H(0, X) = \begin{cases} 0 & \text{if } X < -1/e \\ eX + 1 & \text{if } -1/e \leq X \leq 0 \\ -X/e + 1 & \text{if } 0 < X \leq e \\ 0 & \text{if } X > e \end{cases} \quad (14)$$

184 
$$C(0, X) = \begin{cases} 0 & \text{if } X < -1/e \\ F(X) & \text{if } -1/e \leq X \leq e \\ 0 & \text{if } X > e \end{cases} \quad (15)$$

185 where  $e \equiv \tan \theta$  for convenience; and  $F(X)$  is the expression of initial sediment concentration  
186 distribution. Figure 1 illustrates the initial configuration dictated by equations (13) and (14).

187 In order to render the governing equations homogeneous and simplify the calculations,  
188 the dimensionless variables are altered to:

189 
$$\tilde{T} = T, \tilde{X} = X - \frac{1}{2} \sin \theta \cdot T^2, \tilde{U} = U - \sin \theta \cdot T, \tilde{\eta} = \sqrt{H \cos \theta}, \tilde{C} = C, \text{ and } \tilde{C}_* = C_* \quad (16)$$

190 After transformation, equations (10)-(15) are mapped onto

191 
$$2 \frac{\partial \tilde{\eta}}{\partial \tilde{T}} + \tilde{\eta} \frac{\partial \tilde{U}}{\partial \tilde{X}} + 2\tilde{U} \frac{\partial \tilde{\eta}}{\partial \tilde{X}} = 0 \quad (17)$$

$$\frac{\partial \tilde{U}}{\partial \tilde{T}} + \tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{X}} + 2\tilde{\eta} \frac{\partial \tilde{\eta}}{\partial \tilde{X}} = 0 \quad (18)$$

$$\frac{\partial \tilde{C}}{\partial \tilde{T}} + \tilde{U} \frac{\partial \tilde{C}}{\partial \tilde{X}} = \frac{\alpha\beta \cdot \cos\theta}{\tilde{\eta}^2} (\tilde{C}_* - \tilde{C}) \quad (19)$$

$$\tilde{U}(0, \tilde{X}) = 0 \quad (20)$$

$$\tilde{\eta}(0, \tilde{X}) = \begin{cases} 0 & \text{if } \tilde{X} < -1/e \\ \sqrt{\cos\theta \cdot (e\tilde{X} + 1)} & \text{if } -1/e \leq \tilde{X} \leq 0 \\ \sqrt{\cos\theta \cdot (-\tilde{X}/e + 1)} & \text{if } 0 < \tilde{X} \leq e \\ 0 & \text{if } \tilde{X} > e \end{cases} \quad (21)$$

$$\tilde{C}(0, \tilde{X}) = \begin{cases} 0 & \text{if } \tilde{X} < -1/e \\ F(\tilde{X}) & \text{if } -1/e \leq \tilde{X} \leq e \\ 0 & \text{if } \tilde{X} > e \end{cases} \quad (22)$$

Figure 2 illustrates the mapped initial configuration expressed by equations (20) and (21).

198

### 199 3.2. Eigenstructure

By virtue of the chain rule, equations (17)-(19) are written in a quasi-linear form as

$$\frac{\partial \mathbf{W}}{\partial \tilde{T}} + \mathbf{A} \frac{\partial \mathbf{W}}{\partial \tilde{X}} = \mathbf{R} \quad (23)$$

in which

$$\mathbf{W} = \begin{bmatrix} \tilde{\eta} \\ \tilde{U} \\ \tilde{C} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \tilde{U} & \tilde{\eta}/2 & 0 \\ 2\tilde{\eta} & \tilde{U} & 0 \\ 0 & 0 & \tilde{U} \end{bmatrix}, \text{ and } \mathbf{R} = \begin{bmatrix} 0 \\ 0 \\ \alpha\beta \cos\theta \cdot (\tilde{C}_* - \tilde{C}) / \tilde{\eta}^2 \end{bmatrix} \quad (24)$$

where  $\mathbf{W}$  is the vector of primitive variables;  $\mathbf{A}$  is the Jacobian matrix; and  $\mathbf{R}$  is the vector of source terms. According to the method of characteristics, the characteristic curves and the corresponding compatibility equations are given below as:

$$\begin{cases} \frac{d\tilde{X}}{d\tilde{T}} = \lambda_1 = \tilde{U} + \tilde{\eta} \\ \frac{d(\tilde{U} + 2\tilde{\eta})}{d\tilde{T}} = 0 \end{cases} \quad (25a, b)$$



$$\left\{ \begin{array}{l} \frac{d\tilde{X}}{d\tilde{T}} = \lambda_2 = \tilde{U} - \tilde{\eta} \\ \frac{d(\tilde{U} - 2\tilde{\eta})}{d\tilde{T}} = 0 \end{array} \right. \quad (26a, b)$$

$$\left\{ \begin{array}{l} \frac{d\tilde{X}}{d\tilde{T}} = \lambda_3 = \tilde{U} \\ \frac{d\tilde{C}}{d\tilde{T}} = \frac{\alpha\beta \cdot \cos\theta}{\tilde{\eta}^2} (\tilde{C}_* - \tilde{C}) \end{array} \right. \quad (27a, b)$$

210

### 211 3.3. Solutions for the Mixture Flow

212 As mentioned in section 2, the governing equations (6) and (7) for the mixture flow here  
 213 are identical in form to those for clear water flow. Hence, the solution strategy of Fernandez-  
 214 Feria (2006) for ideal dam-break flow (without sediment transport) is applied herein. Succinctly,  
 215 the strategy features an integration of the adoption of exact solutions and the method of  
 216 characteristics. Given that both the left and right fronts of the flood wave are initially singular  
 217 points, namely  $\tilde{X} = -1/e$  and  $\tilde{X} = e$  with  $\tilde{\eta} = \tilde{U} = 0$ , they are unsuitable as starting points for the  
 218 method of characteristics. Therefore, the exact solutions would be useful to serve as the starting  
 219 condition near these singular points as well as provide the spreading of the two fronts at the  
 220 initial stages. In order to adopt the exact solutions, which had explicit expressions originally  
 221 derived for dam-break flows caused by an initial semi-infinite mass of water (Thacker, 1981),  
 222 Fernandez-Feria (2006) divided the initial fluid (Figure 2) vertically around the origin into two  
 223 portions and presumed a semi-infinite fluid behind each respectively. Actually, according to the  
 224 method of characteristics, the exact solutions are invalid in the domain of influence of the  
 225 imaginary fluid. Correspondingly, the regions of validity of the exact solutions in space and time  
 226 are specified near both fronts by the characteristic curves starting at the origin (supporting  
 227 information Figure S1). Outside these regions, the method of characteristics is employed by  
 228 virtue of equations (25) and (26).

229 A brief summary of the explicit exact solutions mentioned above is provided below, with  
 230 detailed developments available in the supporting information Text S1. The exact solution that  
 231 expresses the propagation of the left-hand flow front is

$$232 \quad \tilde{U}(\tilde{T}, \tilde{X}) = -\sin\theta \cdot \tilde{T} \quad (28)$$

$$233 \quad \tilde{\eta}(\tilde{T}, \tilde{X}) = \begin{cases} 0 & \text{if } \tilde{X} < l_1(\tilde{T}) \equiv -1/e - \sin\theta \cdot \tilde{T}^2/2 \\ \sqrt{\sin\theta} \sqrt{\tilde{X} + 1/e + \sin\theta \cdot \tilde{T}^2/2} & \text{if } \tilde{X} \geq l_1(\tilde{T}) \end{cases} \quad (29)$$

234 where  $l_1(\tilde{T})$  represents the trajectory of the left front. The corresponding characteristic curve  
 235 starting at the point  $\tilde{X} = 0$  for  $\tilde{T} = 0$  is given by

$$236 \quad A_1(\tilde{T}) = -\sqrt{\cos\theta} \cdot \tilde{T} - \sin\theta \cdot \tilde{T}^2/4 \quad (30)$$

237 This determines the region of the plane  $(\tilde{X}, \tilde{T})$  corresponding to the range of validity of the  
 238 exact solution (28) and (29) as follows

$$239 \quad l_1(\tilde{T}) \leq \tilde{X} \leq A_1(\tilde{T}) \quad (31)$$

240 within which the left-hand flow is independent of any influence from the right portion of the  
 241 initial fluid before the instant

$$242 \quad \tilde{T}_1 \equiv 2 \frac{\sqrt{\cos \theta}}{\sin \theta} \quad (32)$$

243 at which the curves  $l_1(\tilde{T})$  and  $A_1(\tilde{T})$  cross each other.

244 Likewise, the initial stage of the advance of the right flow front is given by

$$245 \quad \tilde{U}(\tilde{T}, \tilde{X}) = \frac{\cos \theta}{e} \cdot \tilde{T} \quad (33)$$

$$246 \quad \tilde{\eta}(\tilde{T}, \tilde{X}) = \begin{cases} 0 & \text{if } \tilde{X} > l_2(\tilde{T}) \equiv e + \frac{\cos \theta}{e} \frac{\tilde{T}^2}{2} \\ \sqrt{\frac{\cos \theta}{e}} \sqrt{-\tilde{X} + e + \frac{\cos \theta}{e} \frac{\tilde{T}^2}{2}} & \text{if } \tilde{X} \leq l_2(\tilde{T}) \end{cases} \quad (34)$$

247 where  $l_2(\tilde{T})$  represents the trajectory of the right front. The corresponding characteristic curve  
 248 starting at the point  $\tilde{X} = 0$  is expressed by

$$249 \quad A_2(\tilde{T}) = \sqrt{\cos \theta} \cdot \tilde{T} + \frac{\cos \theta}{e} \cdot \frac{\tilde{T}^2}{4} \quad (35)$$

250 which now determines the region of the plane  $(\tilde{X}, \tilde{T})$  where the exact solution (33) and (34)  
 251 remains valid for the right flow front without any influence from the left portion of the initial  
 252 fluid. Thus, the exact solution (33) and (34) is valid on the right hand side of the origin point  
 253 over the range

$$254 \quad A_2(\tilde{T}) \leq \tilde{X} \leq l_2(\tilde{T}) \quad (36)$$

255 until the time

$$256 \quad \tilde{T}_2 \equiv 2 \frac{e}{\sqrt{\cos \theta}} \quad (37)$$

257 at which the curves  $l_2(\tilde{T})$  and  $A_2(\tilde{T})$  cross each other.

258

### 259 3.4. Solutions for Sediment Transport

260 The sediment transport compatibility equation (27b) is an ODE with a non-zero term on  
 261 the right hand side, and has a general solution (supporting information Text S2, Figures S2 and  
 262 S3)

$$\tilde{C} - \tilde{C}_* = -Y_1^{-1} \cdot \left( \int \frac{d\tilde{C}_*}{d\tilde{T}} \cdot Y_1 d\tilde{T} + M \right) \quad (38a)$$

$$Y_1 = \exp \left( \int \frac{\alpha\beta \cdot \cos \theta}{\tilde{\eta}^2} d\tilde{T} \right) \quad (38b)$$

where  $M$  is an integral constant. Note that  $\tilde{C}_*$  is determined by the local flow conditions and requires update in each time step. The lower limit of the integrals in equation (38) is set to  $\tilde{T} = \tilde{T}_0$  when the initial sediment concentration  $\tilde{C} = \tilde{C}_0$  and the initial sediment transport capacity  $\tilde{C}_* = \tilde{C}_{*0}$ . Setting the upper limit of the integrals to  $\tilde{T} = \tilde{T}_1$ , the particular solution is given by

$$\tilde{C} = \tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3 \quad (39a)$$

$$\tilde{Z}_1 = \tilde{C}_* \quad (39b)$$

$$\tilde{Z}_2 = (\tilde{C}_0 - \tilde{C}_{*0}) \cdot Y_2^{-1}, \quad Y_2 = \exp \left( \int_{\tilde{T}_0}^{\tilde{T}_1} \frac{\alpha\beta \cdot \cos \theta}{\tilde{\eta}^2} d\tilde{T} \right) \quad (39c)$$

$$\tilde{Z}_3 = -Y_2^{-1} \cdot \int_{\tilde{T}_0}^{\tilde{T}_1} \left( \frac{d\tilde{C}_*}{d\tilde{T}} \cdot Y_3 \right) d\tilde{T}, \quad Y_3 = \exp \left( \int_{\tilde{T}_0}^{\tilde{T}} \frac{\alpha\beta \cdot \cos \theta}{\tilde{\eta}^2} d\tilde{T}' \right) \quad (39d)$$

273

### 274 3.5. Numerical Realization

275 Although analytical solutions (25) and (26) are not fully explicit expressions, they can be  
276 manipulated numerically readily by utilizing the method of characteristics as equations (25b) and  
277 (26b) are homogeneous. On the other hand, the solution (39) for the evolution of sediment  
278 concentration field involves integrals, which also require numerical realization. This is why the  
279 solutions presented in this paper are approximate.

280 Briefly, the exact solutions (28) and (29) are adopted in the region prescribed by equation  
281 (31) until the time instant expressed by equation (32), and the exact solutions (33) and (34) are  
282 employed in the region dictated by equation (36) until the time instant given by equation (37).  
283 Over the remaining region of space and time, the method of characteristics is deployed to resolve  
284 the mixture flow according to equations (25) and (26), with a predictor-corrector method and a  
285 parabolic interpolation to achieve second-order accuracy in both time and space. Meanwhile, the  
286 sediment concentration along the characteristic curve is predicted by equation (39), with the  
287 integrations evaluated using the trapezoidal rule. Following mesh independence tests, a time step  
288  $\Delta\tilde{T} = 0.005$  and a spatial step  $\Delta\tilde{X} = 0.002$  are selected.

289

## 290 4. Finite Volume Method

291 The governing equations (1)-(3) are simplified from the full SHSM equations (Cao et al.,  
292 2017). Numerical solutions of these equations can be readily attained using a scheme based on

293 FVM, given the plenty of computational studies for a hierarchy of complicated flow and  
 294 sediment transport processes over erodible beds as stated by Cao et al. (2017). The numerical  
 295 solutions can then be compared with the approximate solutions given above.

296 Equations (1)-(3) are rearranged in a hyperbolic vector-matrix form as

$$297 \quad \frac{\partial \mathbf{V}}{\partial t} + \frac{\partial \mathbf{G}}{\partial x} = \mathbf{S} \quad (40)$$

298 in which

$$299 \quad \mathbf{V} = \begin{bmatrix} h \\ hu \\ hc \end{bmatrix}, \mathbf{G} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} g \cos \theta \cdot h^2 \\ huc \end{bmatrix}, \text{ and } \mathbf{S} = \begin{bmatrix} 0 \\ gh \sin \theta \\ E - D \end{bmatrix} \quad (41)$$

300 where  $\mathbf{V}$  is the vector of conservative variables;  $\mathbf{G}$  is the vector of fluxes; and  $\mathbf{S}$  is the vector  
 301 of source terms. Under the framework of FVM, an explicit discretization of equation (40) gives

$$302 \quad \mathbf{V}_i^p = \mathbf{V}_i^n - \frac{\Delta t}{\Delta x} [\mathbf{G}_{i+1/2} - \mathbf{G}_{i-1/2}]^n \quad (42a)$$

$$303 \quad \mathbf{V}_i^{n+1} = \mathbf{V}_i^p + \Delta t \cdot \mathbf{S}^{RK}(\mathbf{V}_i) \quad (42b)$$

304 where the superscript  $n$  denotes the time step index; the subscript  $i$  denotes the spatial node  
 305 index;  $\Delta t$  is the time step;  $\Delta x$  is the spatial increment;  $\mathbf{G}_{i+1/2}$  and  $\mathbf{G}_{i-1/2}$  represent the inter-cell  
 306 fluxes; and  $\mathbf{S}^{RK}$  represents the source terms which are solved by the second-order Runge-Kutta  
 307 method as follows:

$$308 \quad \mathbf{S}^{RK}(\mathbf{V}_i) = \frac{1}{2} [\mathbf{S}(\mathbf{V}_i^p) + \mathbf{S}(\mathbf{V}_i^*)] \quad (43a)$$

$$309 \quad \mathbf{V}_i^* = \mathbf{V}_i^p + \Delta t \cdot \mathbf{S}(\mathbf{V}_i^p) \quad (43b)$$

310 The inter-cell fluxes are calculated by the Slope-Limiter Centered (SLIC) method, in  
 311 which the total variation diminishing (TVD) version of the Monotonic Upstream-Centered  
 312 Scheme for Conservation Laws (MUSCL) is employed for data reconstruction. Accordingly,  
 313 numerical predictions here achieve second-order accuracy in both space and time (Toro, 2001).

314 The spatial step is set to  $\Delta x = 0.01$ , satisfying the criteria of mesh independence.  
 315 Numerical stability is controlled by the Courant-Friedrichs-Lewy (CFL) criterion (Toro, 2001)

$$316 \quad \Delta t = \frac{\Delta x}{\max(|u_i| + \sqrt{gh_i})} \cdot Cr \quad (44)$$

$$317 \quad 0 < Cr \leq 1.0 \quad (45)$$

318 where  $Cr$  represents the Courant number and a value of 0.95 is adopted.

319

## 320 5. Results

321 This study is primarily concerned with the evolution of the sediment concentration field  
 322 in shallow water-sediment flow over a bed with uniform slope. Three typical sediment transport  
 323 scenarios are considered, i.e., pure advection; pure sedimentation; and concurrent entrainment  
 324 and deposition. Although an accelerating, high velocity dam-break flow would be likely to  
 325 produce substantial sediment entrainment as time progresses, the essentially academic cases  
 326 selected here examine different forms of the source terms in the sediment continuity equation  
 327 that relate to key physical mechanisms (see Table 1) and are very useful for model verification.

328 While sediment transport capacity can be calculated by one of a plethora of formulas in  
 329 general (Celik & Rodi, 1991; Dey, 2014; Wu, 2007; Yang, 1996), a specific formula is  
 330 demanded here to test the proposed approximate solutions. The Zhang formula is based on the  
 331 theory of energy balance along with laboratory flume and field observed datasets and has seen  
 332 wide applications in engineering practice (Guo, 2002; Wu, 2007; Zhang & Xie, 1993). By virtue  
 333 of logarithmic matching method, Guo (2002) gave an explicit expression of the Zhang formula  
 334 as follows

$$335 \quad c_* = \gamma \frac{1}{20\rho_s} \frac{[u^3 / gh\omega_0]^{1.5}}{1 + [u^3 / 45gh\omega_0]^{1.15}} \quad (46)$$

336 where  $\gamma = 0.1$  is a modification coefficient introduced herein to prevent the calculated sediment  
 337 transport capacity from being excessively large (e.g. for high velocity dam-break flows over a  
 338 frictionless bed with steep slope); and  $\rho_s$  denotes sediment density. Notwithstanding that the  
 339 Zhang formula applies to suspended load (Guo, 2002; Wu, 2007; Zhang & Xie, 1993),  
 340 alternative formulas for either bed load or total bed-material load can be employed instead, for  
 341 which the present approximate solutions continue to hold.

342 The study parameters are as follows. The bed slope has angle  $\theta = 30^\circ$ . The sediment  
 343 particle diameter is  $d = 0.2\text{mm}$  with a dimensionless settling velocity of about  $\beta = 0.006$   
 344 according to the formula of Zhang and Xie (1993). The density of sediment is  $\rho_s = 2650 \text{ kg/m}^3$ .  
 345 In each scenario, two distinct initial sediment distributions are considered: one is a uniform  
 346 distribution such that

$$347 \quad F(X) = 1 \quad (47)$$

348 and the other a linear distribution such that

$$349 \quad F(X) = \frac{eX + 1}{e^2 + 1} \times 0.9 + 0.1 \quad (48)$$

350 Note that  $F(\tilde{X}) = F(X)$  initially. The scaled sediment concentration is set to  $C_0 = 0.01$ . For  
 351 convenience, the approximate solutions and the FVM numerical predictions presented in Figures  
 352 3-18 are all in terms of dimensionless variables.

353

## 354 5.1. Mixture Flow

355 Obviously, since bed deformation and the effects of sediment transport on the mixture  
 356 flow are neglected, all the cases mentioned above share the same solutions for the mixture flow.  
 357 Figure 3 shows the approximate solutions and corresponding numerical predictions by FVM for  
 358 the advancing flow fronts with time. Note that the left front remains at rest until the backward  
 359 wave reaches the upstream boundary. Figure 4 illustrates the approximate solutions and FVM  
 360 numerical predictions for the mixture flow at four instants of time. Figure 4a shows the gradual  
 361 attenuation of the mixture flow with time, which is consistent with Figure 4b where it can be  
 362 observed that the downstream portion of flow consistently propagates faster than the upstream  
 363 portion. Moreover, the flow velocity appears to be increased progressively along the domain  
 364 with time, in keeping with mass conservation (Figure 4). Apparently, the numerical predictions  
 365 are indistinguishable from the approximate solutions (Figures 3 and 4). On the one hand, the  
 366 FVM velocity profiles manifest sudden steps at the beginning and end of the flow regions in  
 367 keeping with the shock-capturing capability of the FVM. On the other hand, the results underline  
 368 the significance of the present approximate solutions for verifying SHSM models.

369

## 370 5.2. Pure Advection

371 Here sediment transport is assumed to be in an equilibrium state, such that  $E = D = 0$ .  
 372 Now the corresponding governing equations can also describe the pure advection of a passive  
 373 scalar, such as the concentration of solute pollutant, neutrally buoyant particles or wash load that  
 374 barely exchanges with the bottom boundary (Dey, 2014; Yang, 1996; Zhang & Xie, 1993).  
 375 Accordingly, the particular solution of the ODE (27b) along the characteristic curve (27a) is

$$376 \quad \tilde{C} = \tilde{C}_0 \quad (49)$$

377 It is easy to infer from equation (49) that the sediment concentration remains constant along the  
 378 characteristic curve. Specifically, the sediment concentration profiles become stretched in time  
 379 and space with invariable extreme values compared to the initial profile. Results for two different  
 380 initial sediment distributions are discussed separately below.

381 Figure 5 shows the approximate solutions obtained for the evolution of sediment  
 382 concentration field in  $X$ - $T$  space. In accordance with the uniformity of the initial distribution, the  
 383 magnitude of sediment concentration remains constant. Figure 6 compares the approximate  
 384 solutions and numerical predictions by FVM for sediment transport at four instants of time.  
 385 These two solutions match rather well with each other and illustrate the stretching of the  
 386 concentration profiles with time, which demonstrates the ability of the FVM adopted here to  
 387 capture contact discontinuity in sediment transport.

388 Figure 7 illustrates the approximate solutions for the evolution of an initially linear  
 389 sediment concentration field in  $X$ - $T$  space under pure advection. The sediment concentration  
 390 field sustains an almost linear distribution throughout the flow region with the extreme values  
 391 unaltered. Figure 8 shows the remarkable agreement between the approximate solutions and  
 392 FVM numerical predictions of the sediment concentration profiles at four instants of time. The

393 results again demonstrate the remarkable ability of the FVM to capture discontinuity in sediment  
 394 transport and also the usefulness of the approximate solutions for model verification purposes.

395

### 396 5.3. Pure Sedimentation

397 Absolute deposition of sediment is now assumed throughout the flow region, without any  
 398 entrainment from the bed (such that  $E = 0$ ). Hence, the particular solution (39) for sediment  
 399 concentration along the characteristic curve can be simplified to give

$$400 \quad \tilde{C} = \tilde{C}_0 \cdot Y_2^{-1}, \quad Y_2 = \exp\left(\int_{\tilde{T}_0}^{\tilde{T}_1} \frac{\alpha\beta \cdot \cos\theta}{\tilde{\eta}^2} d\tilde{T}\right) \quad (50)$$

401 In light of equation (50), the sediment concentration is expected to attenuate along the  
 402 characteristic curve. Results obtained for uniform and linear initial sediment distributions are  
 403 presented below.

404 Figure 9 shows the approximate solutions obtained for the evolution of an initially  
 405 uniform distribution of sediment in  $X$ - $T$  space. As expected, the concentration profiles diminish  
 406 with time as sedimentation occurs. Figure 10 compares the approximate solutions and FVM  
 407 numerical predictions of the sediment concentration profiles at four different times. Again, the  
 408 numerical predictions reproduce the approximate solutions accurately. It can be seen that the  
 409 initial profile flattens out and almost disappears by  $T = 30$ .

410 Figure 11 illustrates the approximate solutions obtained for an initially linear distribution  
 411 of sediment as it evolves in  $X$ - $T$  space under pure sedimentation. The sediment concentration  
 412 profiles again exhibit a progressive decline with time and eventually wash out as sediment is  
 413 deposited, the process being completed by  $T = 30$ . Figure 12 confirms the close agreement  
 414 between the approximate solutions and FVM numerical predictions of the sediment  
 415 concentration profiles at four different times.

416

### 417 5.4. Concurrent Entrainment and Deposition

418 In general, entrainment and deposition take place synchronously in fluvial processes,  
 419 which are closer to practice than the scenarios evaluated above (pure advection and  
 420 sedimentation). Yet, since the flow depth is contained in the denominators of the Zhang formula  
 421 (46), the sediment transport capacity so calculated would become numerically excessively large  
 422 at both left and right fronts featuring small flow depths. For numerical considerations, a  
 423 constraint is introduced herein: sediment transport capacity is set to zero if the dimensionless  
 424 flow depth is below a sufficiently small value of 0.001. Results obtained for the two types of  
 425 initial sediment distribution are presented below.

426 Figure 13 shows the evolution of sediment concentration field in  $X$ - $T$  space for the  
 427 initially uniform distribution of sediment undergoing both deposition and entrainment. There is a  
 428 monotonic growth in sediment concentration with peaks at the ends and a depression in between,

429 which becomes increasingly evident with time. There is almost perfect agreement between the  
430 approximate solutions and FVM predictions as shown in Figure 14.

431 Figure 15 shows the approximate solutions obtained for the sediment concentration field  
432 in  $X$ - $T$  space commencing from an initially linear distribution of sediment. Again, there is a  
433 monotonic increase in sediment concentration with time, and the end profiles are qualitatively  
434 very similar to those obtained for the initially uniform distribution, indicating that the exact  
435 shape of the initial profile is unimportant when simultaneous entrainment and deposition are  
436 occurring. Close agreement is again obtained between the approximate and numerically  
437 predicted sediment concentration profiles in Figure 16.

438 At first glance, the upward concavity of the sediment concentration profiles in Figures  
439 13-16 may be attributed to the use of the Zhang formula (46), as a smaller flow depth would lead  
440 to a higher sediment transport capacity. To clarify, an alternative formula is applied in lieu of the  
441 Zhang formula (46). Following Bohorquez and Fernandez-Feria (2008) and Pritchard and Hogg  
442 (2002), we consider the following form for the entrainment flux, replacing equations (4) and (46):

$$443 \quad E = \begin{cases} \frac{m_e}{\rho_s} \left( \frac{u^2}{u_e^2} - 1 \right)^q & \text{for } |u| \geq u_e \\ 0 & \text{for } |u| < u_e \end{cases} \quad (51)$$

444 Apparently, sediment transport capacity  $c_*$  can be expressed by  $E/\alpha\omega_0$  as per equation (4).  
445 Equation (51) is often used to estimate sediment entrainment rate and no sediment will move  
446 until some critical shear stress, corresponding to a velocity  $u_e$ , is exceeded (Dyer & Soulsby,  
447 1988; Pritchard & Hogg, 2002; Sanford & Maa, 2001; Teisson et al., 1993). The quantity  $m_e$   
448 represents a constant mass flux per unit area that usually ranges between  $5 \times 10^{-5}$  and  
449  $5 \times 10^{-3} \text{ kg} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$ , and a value of  $5 \times 10^{-4} \text{ kg} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$  is specified here;  $q$  is a dimensionless  
450 exponent that is often taken to be unity; and  $u_e$  denotes the critical velocity for incipient motion  
451 of sediment particles and is given by

$$452 \quad u_e = k \cdot \omega_0 \sqrt{8/f} \quad (52)$$

453 where  $k$  is a dimensionless constant, and a value of 1.2 is specified;  $f$  is the Darcy-Weisbach  
454 friction factor and a value of 0.03 is utilized. Since the denominators in equation (51) do not  
455 contain the flow depth, the aforementioned constraint as per small flow depth is no longer  
456 demanded.

457 Figure 17 shows the corresponding sediment concentration profiles for an initially  
458 uniform distribution of sediment. Obviously, upward concave curves are obtained qualitatively  
459 similar to those in Figures 13 and 14, and also there is satisfactory agreement between the  
460 approximate solutions and the FVM numerical predictions. It follows that the upward concave  
461 sediment concentration profiles do not result from the specific sediment transport capacity or  
462 entrainment formulations.

463



## 464 5.5. Decomposition of Concentration Field

465 In order to explain the upward concavity of the sediment concentration profiles, the  
 466 constituents determining sediment concentration distribution are elaborated. As per equation (39),  
 467 the sediment concentration at the present time level consists of three parts:  $\tilde{Z}_1$  represents the  
 468 sediment transport capacity at the present time level;  $\tilde{Z}_2$  means the difference between sediment  
 469 concentration and sediment transport capacity at the previous time level as modified by the flow  
 470 conditions; and  $\tilde{Z}_3$  indicates the deviation in sediment transport capacity due to the change of  
 471 flow conditions. This is in accordance with the physical mechanisms incorporated here in the  
 472 mass conservation equation (3) for sediment. Sediment transport tends to be at its capacity in line  
 473 with the local flow scenario due to the upward entrainment process. On the other hand, sediment  
 474 transport also involves streamwise advection and downward deposition (gravitational action)  
 475 processes. In this connection, the sediment concentration field would deviate from its capacity  
 476 state and be modified corresponding to the local flow and sediment conditions.

477 Figure 18 shows the sediment concentration profiles and the distributions of  $Z_1$ ,  $Z_2$ , and  
 478  $Z_3$  for an initially uniform distribution of sediment concentration, with the sediment transport  
 479 capacity evaluated using formulas (46) and (51), respectively. It can be seen from Figure 18 that  
 480 the sediment concentration  $C$  is mainly determined by  $Z_1$  and  $Z_2$  since the value of  $Z_3$  is  
 481 around zero, which means the deviation in sediment transport capacity along the characteristic  
 482 curve (27a) is very small. Comparison between either Figures 18a and 18c or Figures 18b and  
 483 18d demonstrates that the distributions of sediment transport capacity ( $Z_1 = C_*$ ) vary  
 484 significantly owing to the distinct formulas adopted, while all the profiles of  $Z_2$  are rather  
 485 similar in curvature. This is because the term  $Y_2$  in equation (39c) for  $Z_2$  is determined by the  
 486 flow depth as  $H = \tilde{\eta}^2 / \cos \theta$  according to equation (16). Consequently, the upward concavity  
 487 characterizes the distribution of  $Z_2$  while the flow depth profiles are concave downwards (Figure  
 488 4a). Finally, the superposition of the profiles of  $Z_1$  and  $Z_2$  leads to the upward concave curves  
 489 of sediment concentration in Figures 13-17.

490 Bohorquez and Fernandez-Feria (2008) investigated sediment transport under dam-break  
 491 flows on a fixed, rough and inclined bed. Their results featured an initial increase and subsequent  
 492 depression in velocity as well as the formation of roll waves in the long term. Moreover, the  
 493 recent work by Cao et al. (2015) demonstrates that the turbulent Reynolds stress, expressed by a  
 494 second-order term in the shallow water equations, plays a considerable role in roll waves  
 495 modeling. Unfortunately, it remains unrealistic to derive analytical solutions if bed resistance and  
 496 the second-order term for the Reynolds stress are incorporated. Nevertheless, with bed resistance  
 497 accounted for, the present FVM model can readily solve the governing equations in Bohorquez  
 498 and Fernandez-Feria (2008) (supporting information Text S3). Notably, the FVM numerical  
 499 predictions are consistent with Bohorquez and Fernandez-Feria (2008), in which the sediment  
 500 concentration profiles do not feature upward concavity as opposed to those in Figures 13-17  
 501 (supporting information Figures S4 and S5). Therefore, the upward concave sediment  
 502 concentration profiles in Figures 13-17 are essentially dictated by the particular flow and  
 503 sediment conditions, rather than the specific sediment transport capacity or entrainment  
 504 formulations.

505

## 506 **6. Conclusions**

507         Approximate solutions have been derived for ideal dam-break sediment-laden flows  
508 initiated by the sudden release of a finite volume of frictionless, incompressible water-sediment  
509 mixture on a uniform slope. In order that the reduced, governing SHSM equations are tractable  
510 for approximate solutions, it is assumed that sediment transport is uniform and does not affect  
511 the mixture flow, and there is negligible bed deformation. The solution strategy of Fernandez-  
512 Feria (2006) for ideal dam-break flow (without sediment transport) is used to resolve the mixture  
513 flow, while the sediment concentration evolution is solved based on the compatibility equation  
514 along its characteristic curve.

515         Quantitative results are given for a water-sediment dam-break flow over a bed sloping at  
516 an angle of  $\theta=30^\circ$ . The mixture flow profiles feature a gradual depression and expansion  
517 accompanied by a persistent increase in velocity. The approximate solutions for sediment  
518 transport are the highlights of this work. Under the case of pure advection, the sediment  
519 concentration remains constant along the characteristic curve. Under the case of pure  
520 sedimentation, the sediment concentration profiles diminish progressively with time and  
521 eventually disappear when all sediment in the water column has been deposited on the bed. In  
522 the case of concurrent entrainment and deposition, upward concavity characterizes the sediment  
523 concentration profiles irrespective of the formulas adopted for sediment transport capacity. This  
524 is due to the particular flow and sediment conditions resolved here by virtue of the simplified  
525 SHSM equations (1)-(3). In all cases, very close agreement is achieved between the approximate  
526 solutions and the FVM numerical predictions, confirming the reliability of FVM model and the  
527 potential of the present approximate solutions for testing the ability of SHSM models to capture  
528 contact discontinuities in sediment transport.

529         Extensions of the present work are warranted, albeit challenging. Notably, friction and  
530 bed evolution need to be incorporated, non-uniform sediment transport to be accounted for, and  
531 the interactions between flow, sediment transport and morphological evolution accommodated so  
532 that analytical models contain as nearly a complete representation of the mechanisms of fluvial  
533 processes as possible, thereby facilitating minimization of model uncertainty.

534

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542

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- 688

689 **Table 1.** *Summary of Sediment Transport Scenarios*

Scenarios	Source terms in equation (3)
Pure advection	0 ( $E = D = 0$ )
Pure sedimentation	$-D$
Concurrent entrainment and deposition	$E - D$

690

691

692 **List of Figure captions**

693

694 **Figure 1.** Initial configuration in terms of dimensionless variables.

695 **Figure 2.** Initial configuration in terms of transformed variables.

696 **Figure 3.** Approximate solutions and FVM numerical predictions of the locations of the  
697 advancing flow fronts with time.

698 **Figure 4.** Approximate solutions and FVM numerical predictions of the water-sediment mixture  
699 flow at four instants of time: (a) depth profiles; and (b) velocity profiles. The magenta lines in (b)  
700 are plotted outside the flow region to delineate more clearly the approximate solutions obtained  
701 for the velocity of the flow fronts.

702 **Figure 5.** Approximate solutions for the evolution of sediment concentration field in  $X$ - $T$  space  
703 for an initially uniform distribution of sediment under pure advection.

704 **Figure 6.** Approximate solutions and FVM numerical predictions of the sediment concentration  
705 profiles at four instants of time for an initially uniform distribution of sediment under pure  
706 advection.

707 **Figure 7.** Approximate solutions for the evolution of sediment concentration field in  $X$ - $T$  space  
708 for an initially linear (triangular) distribution of sediment under pure advection.

709 **Figure 8.** Approximate solutions and FVM numerical predictions of the sediment concentration  
710 profiles at four instants of time for an initially linear (triangular) distribution of sediment under  
711 pure advection.

712 **Figure 9.** Approximate solutions for the evolution of sediment concentration field in  $X$ - $T$  space  
713 for an initially uniform distribution of sediment under pure sedimentation.

714 **Figure 10.** Approximate solutions and FVM numerical predictions of the sediment concentration  
715 profiles at four instants of time for an initially uniform distribution of sediment under pure  
716 sedimentation.

717 **Figure 11.** Approximate solutions for the evolution of sediment concentration field in  $X$ - $T$  space  
718 for an initially linear (triangular) distribution of sediment under pure sedimentation.

719 **Figure 12.** Approximate solutions and FVM numerical predictions of the sediment concentration  
720 profiles at four instants of time for an initially linear (triangular) distribution of sediment under  
721 pure sedimentation.

722 **Figure 13.** Approximate solutions for the evolution of sediment concentration field in  $X$ - $T$  space  
723 for an initially uniform distribution of sediment when entrainment and deposition occur  
724 concurrently.



725 **Figure 14.** Approximate solutions and FVM numerical predictions of the sediment concentration  
726 profiles at four instants of time for an initially uniform distribution of sediment when  
727 entrainment and deposition occur concurrently.

728 **Figure 15.** Approximate solutions for the evolution of sediment concentration field in  $X-T$  space  
729 for an initially linear (triangular) distribution of sediment when entrainment and deposition occur  
730 concurrently.

731 **Figure 16.** Approximate solutions and FVM numerical predictions of the sediment concentration  
732 profiles at four instants of time for an initially linear (triangular) distribution of sediment when  
733 entrainment and deposition occur concurrently.

734 **Figure 17.** Approximate solutions and FVM numerical predictions of the sediment concentration  
735 profiles at four instants of time for an initially uniform distribution of sediment when  
736 entrainment and deposition occur concurrently. Sediment transport capacity is calculated  
737 according to formula (51).

738 **Figure 18.** Sediment concentration profiles along with the distributions of dimensionless values  
739  $Z_1$ ,  $Z_2$ , and  $Z_3$  in equation (39) at two instants of time for an initially uniform distribution of  
740 sediment concentration. Sediment transport capacity is calculated by the Zhang formula (46) in  
741 (a) and (b), and by formula (51) in (c) and (d).

742

743

Figure 1.

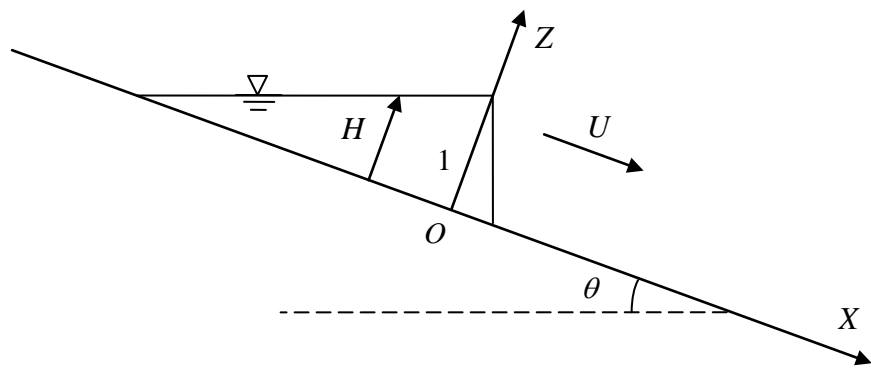


Figure 2.

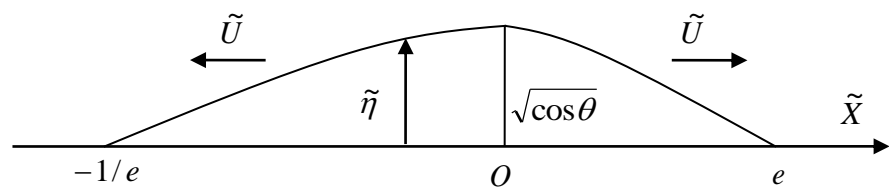
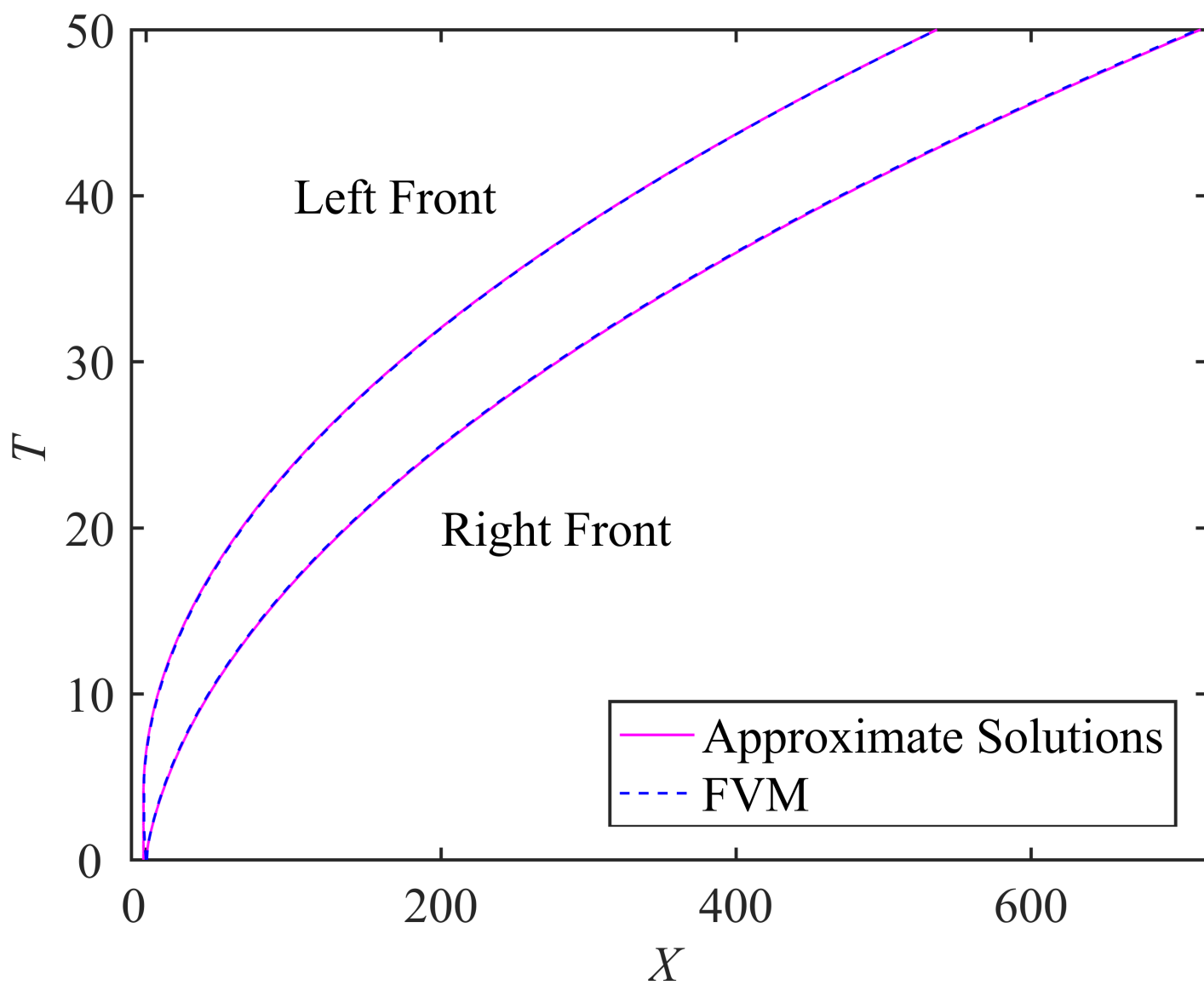


Figure 3.



**Figure 4.**



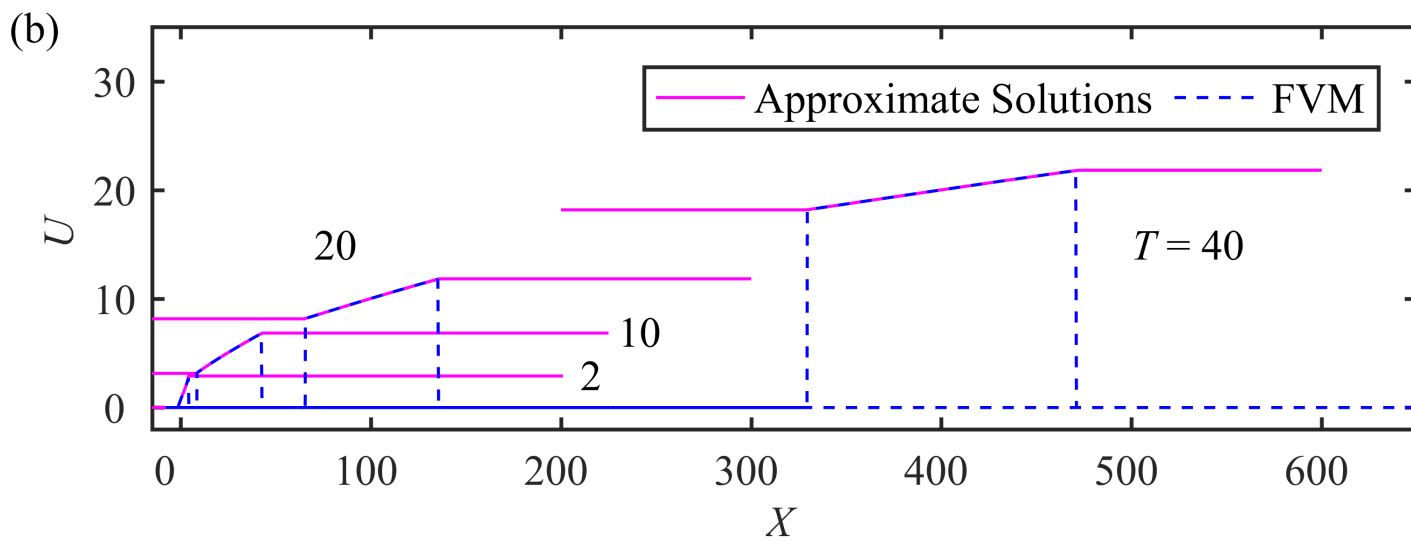
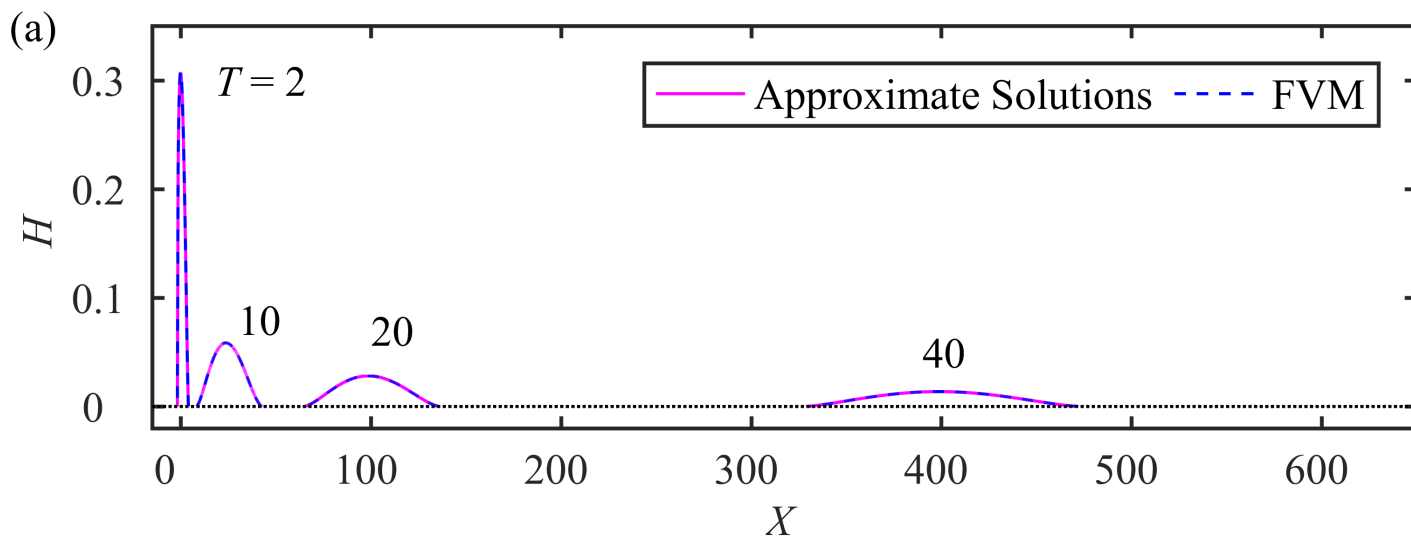


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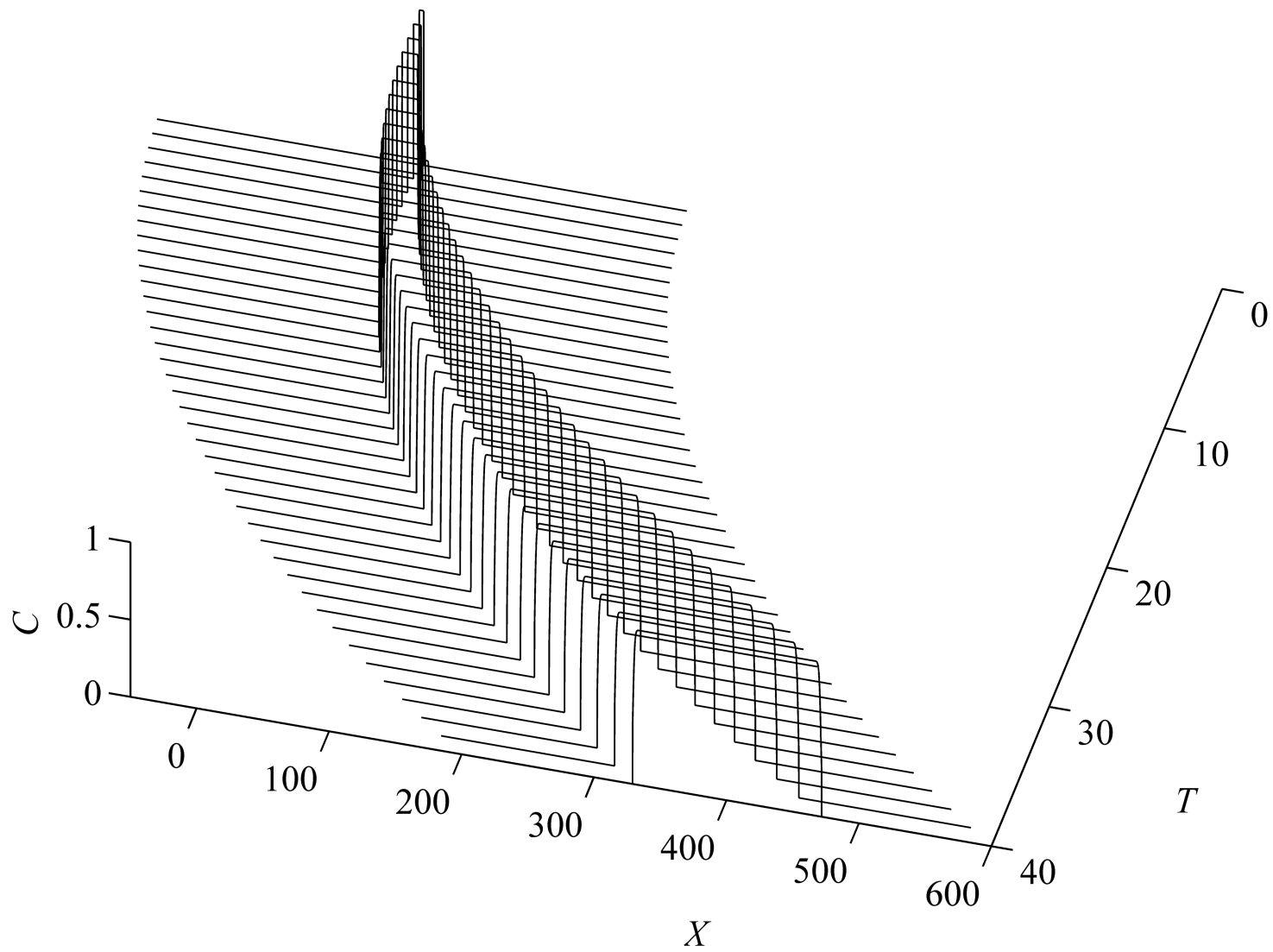
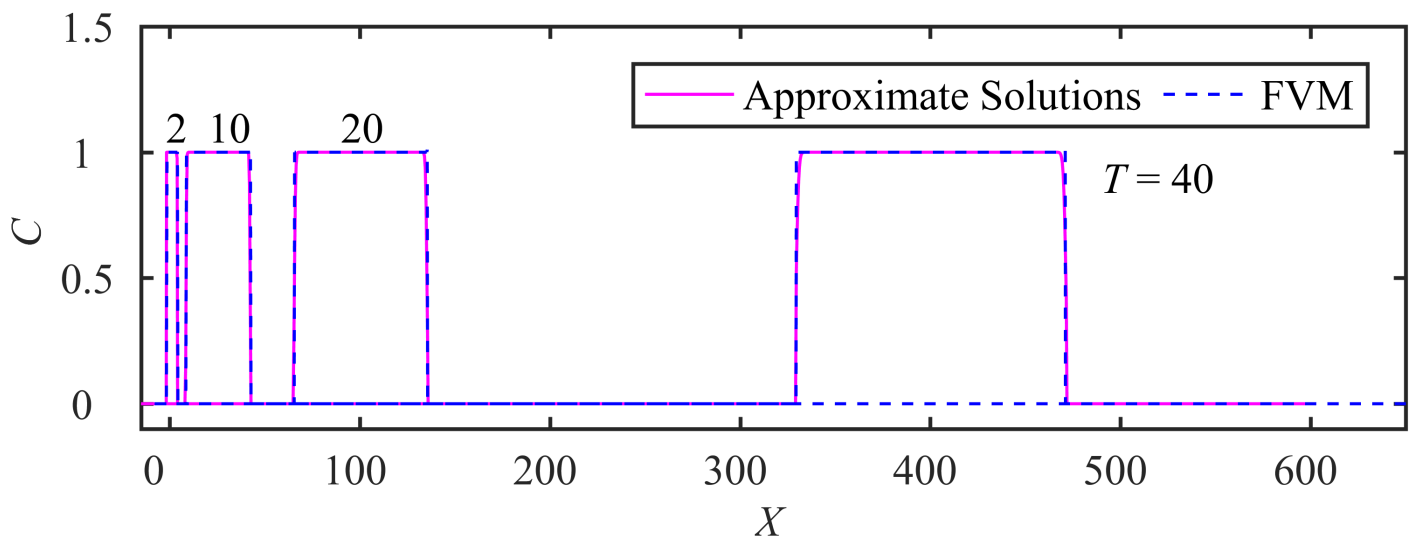


Figure 6.



**Figure 7.**

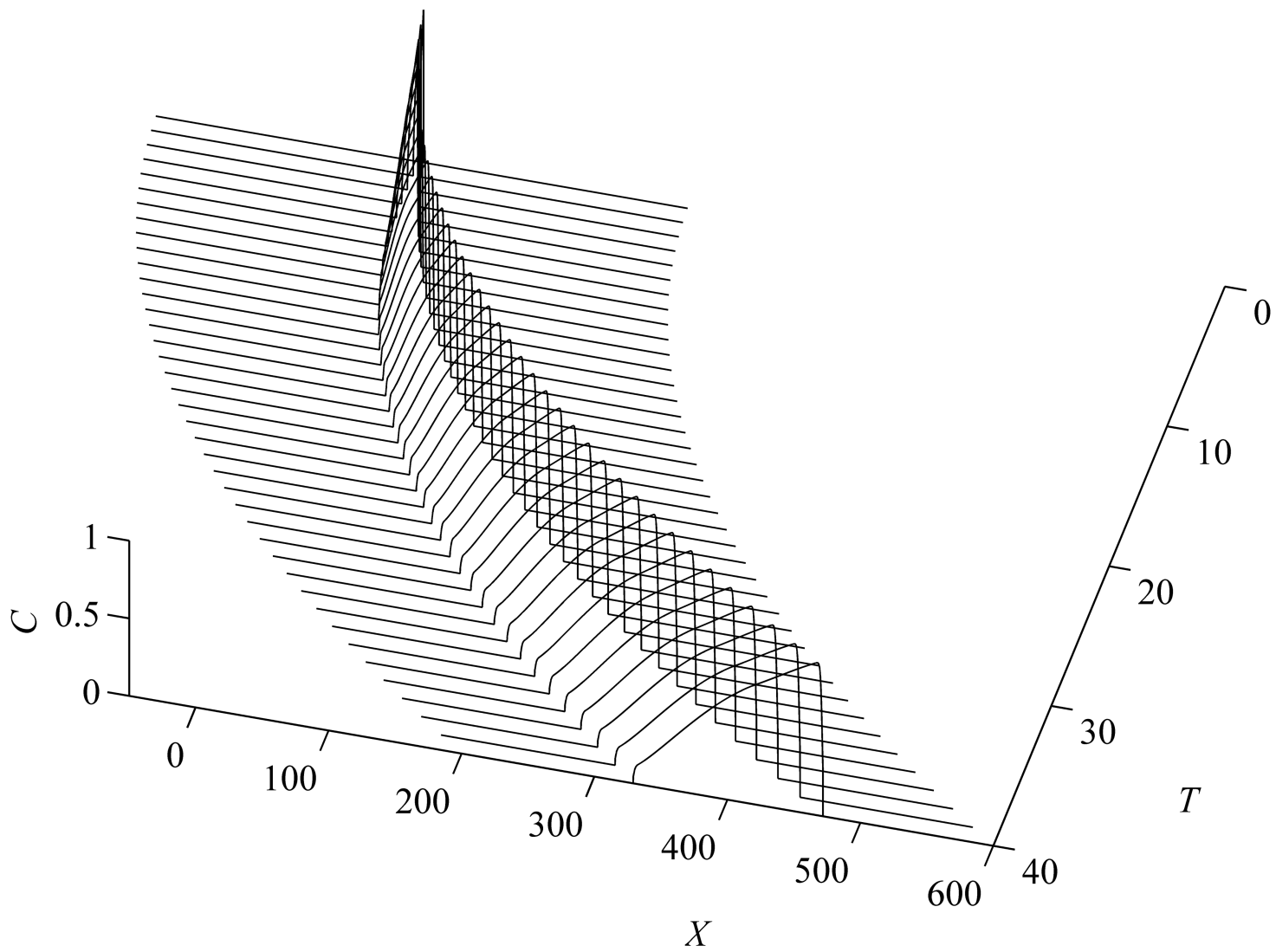


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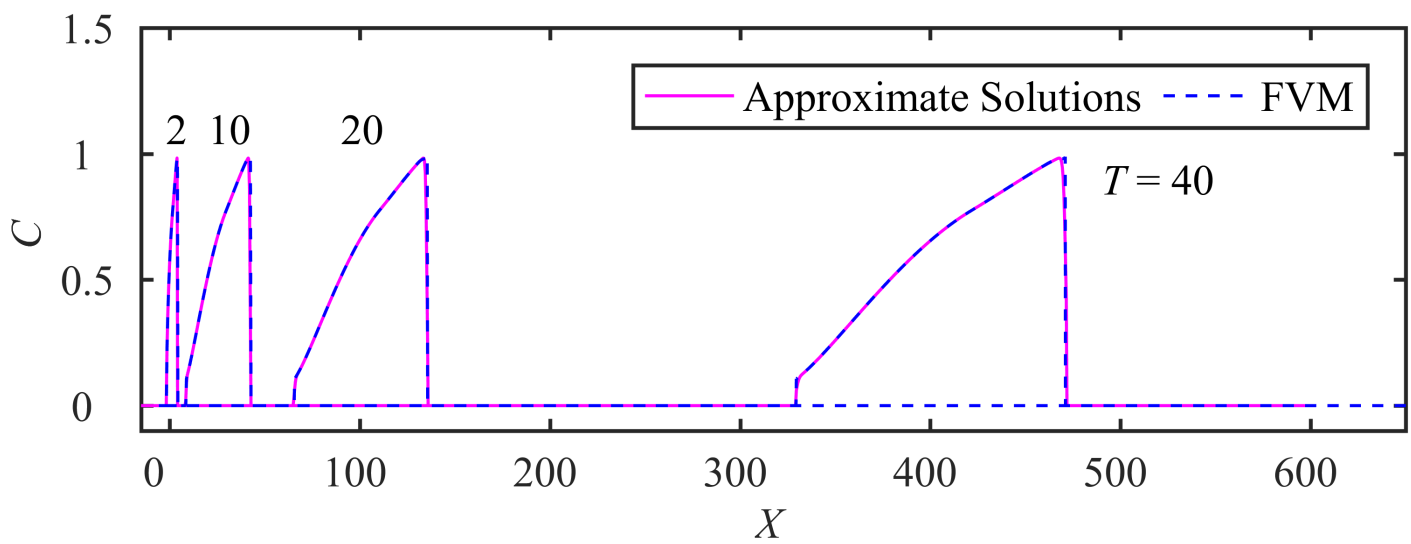


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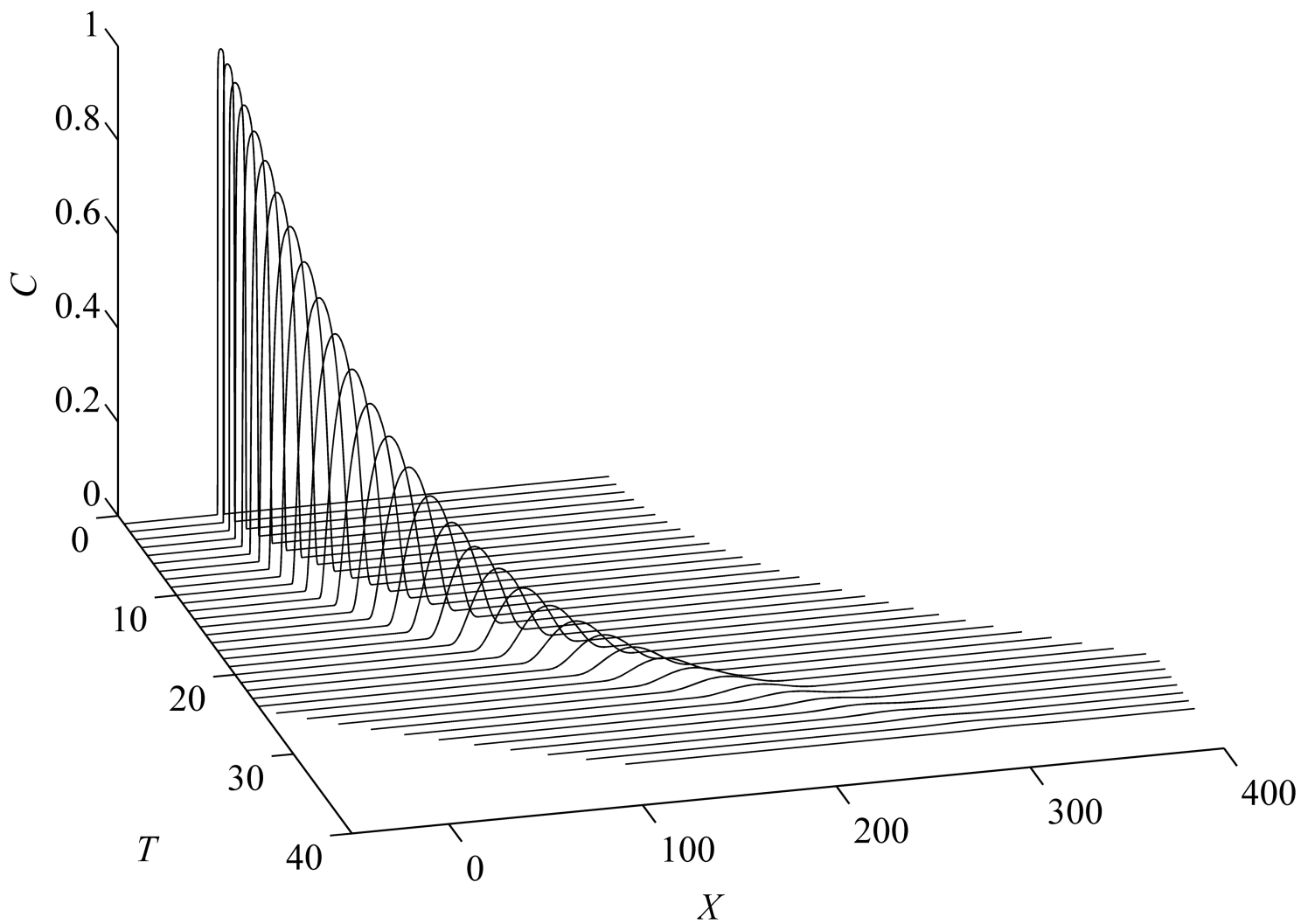


Figure 10.

— Approximate Solutions    - - - FVM

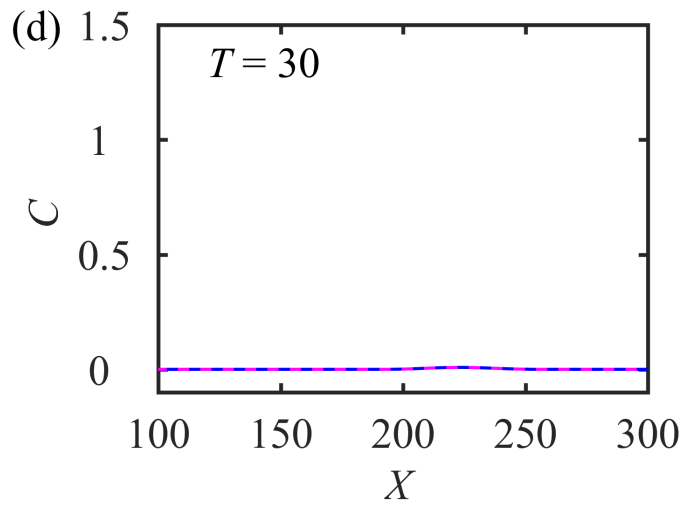
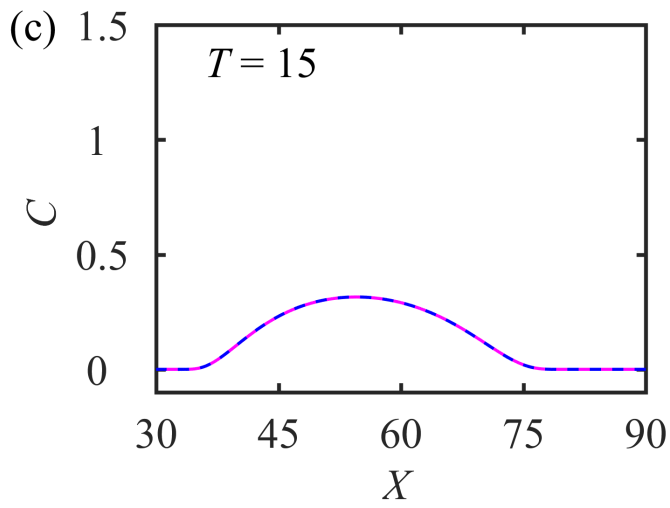
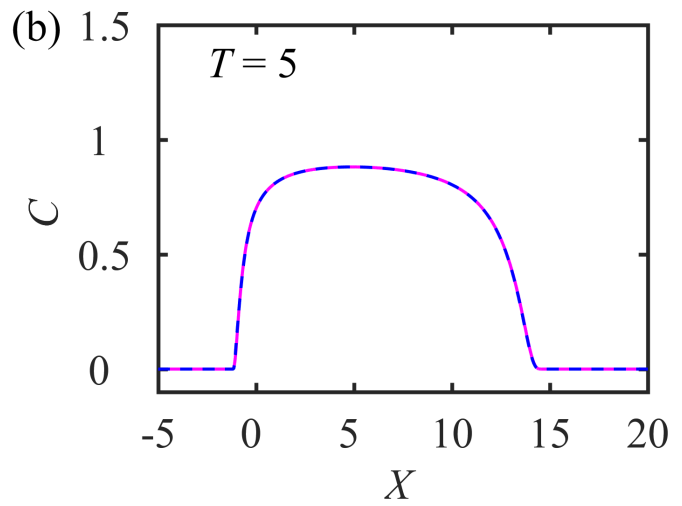
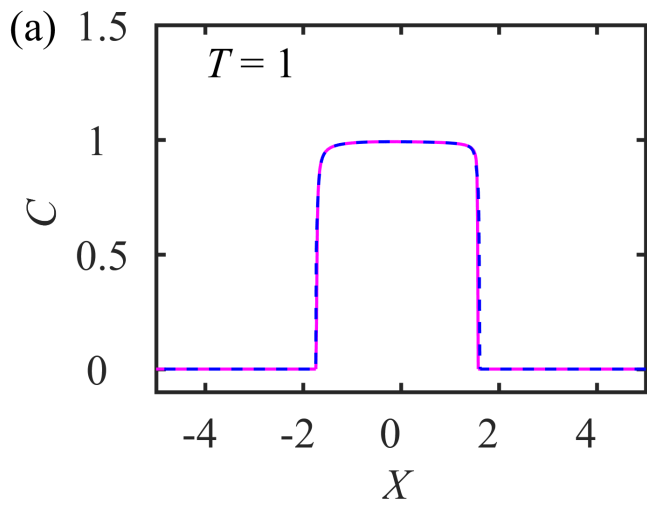


Figure 11.

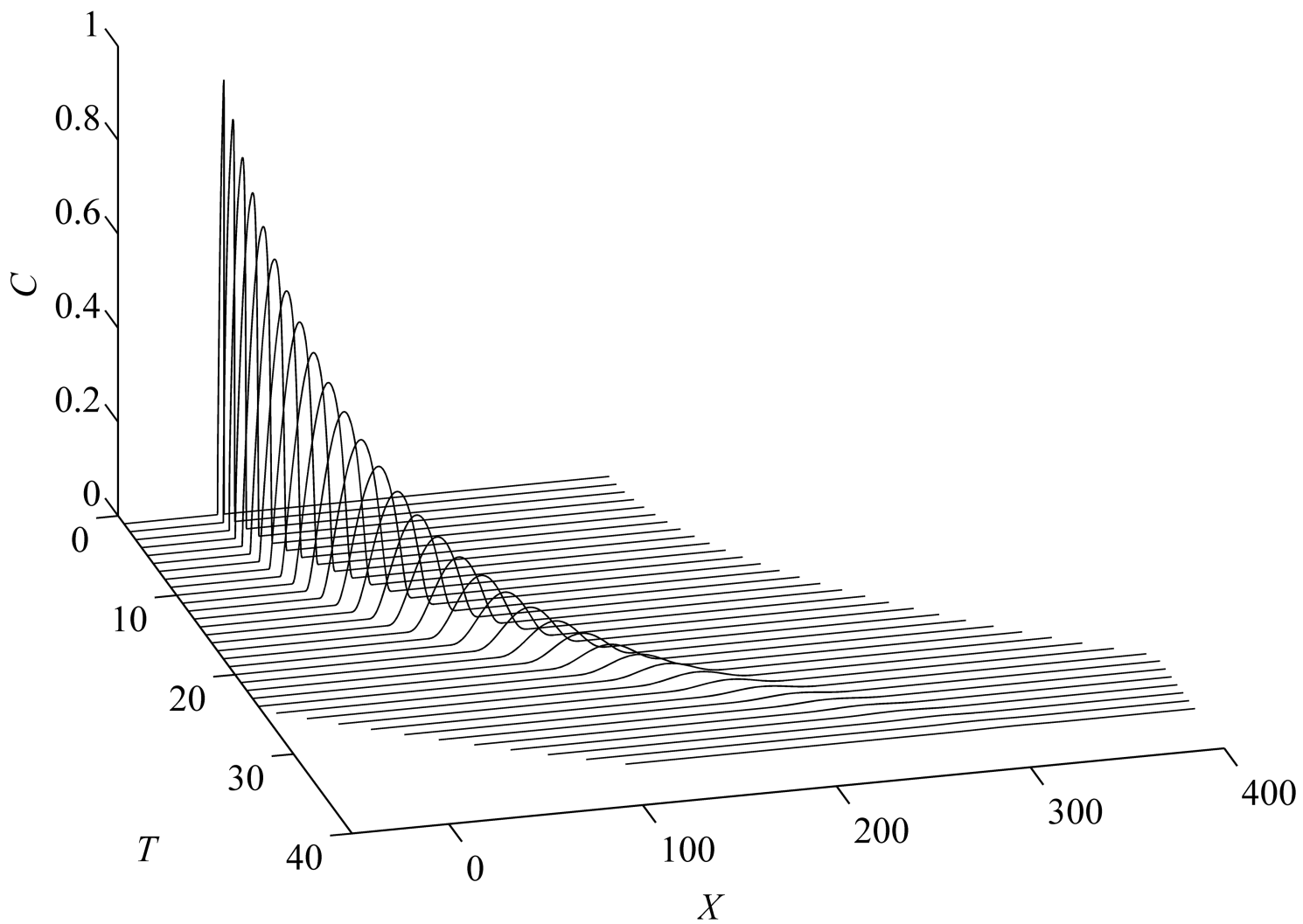


Figure 12.



— Approximate Solutions    - - - FVM

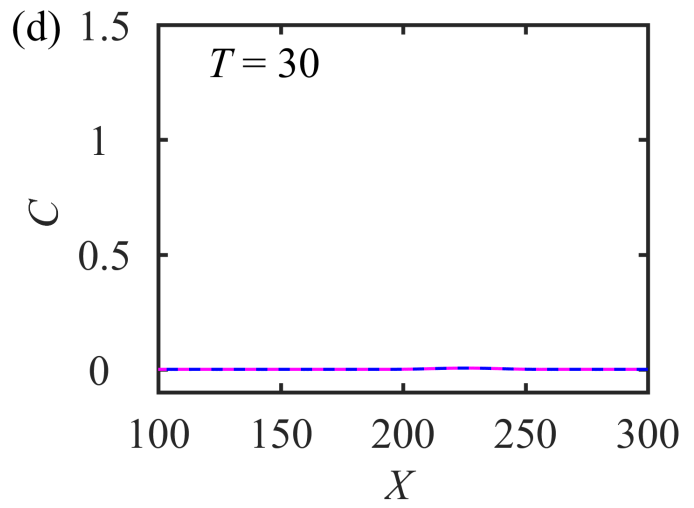
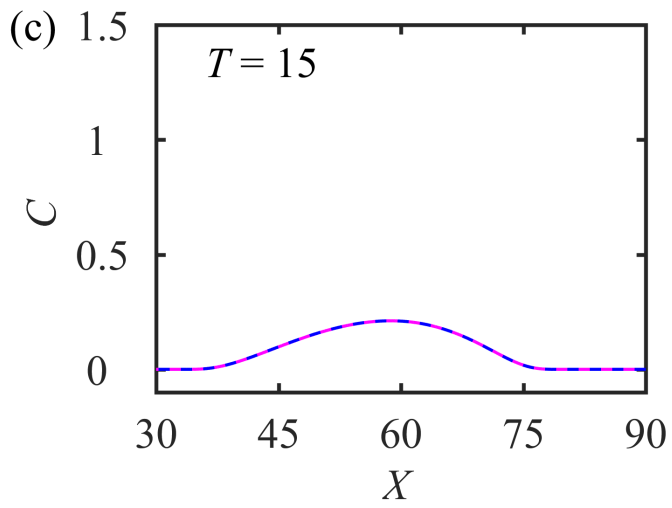
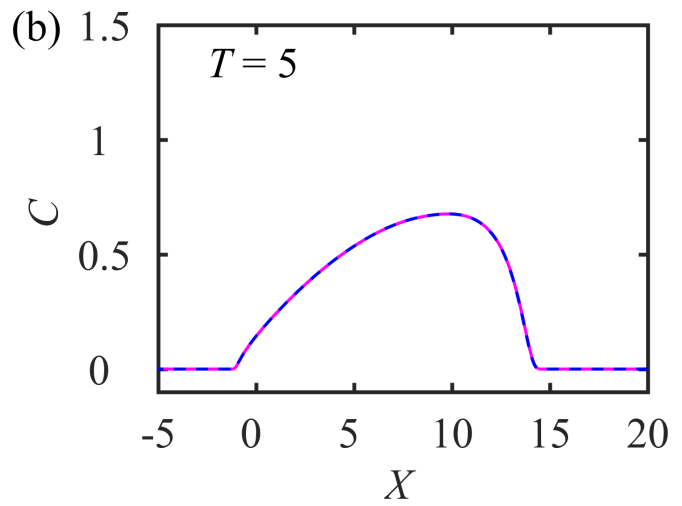
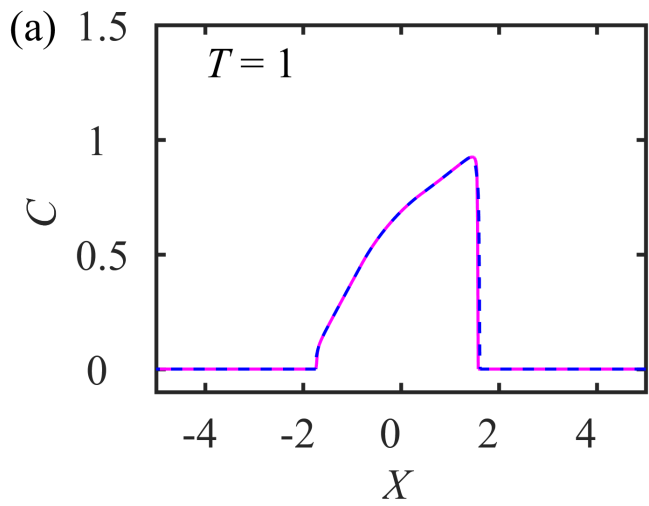


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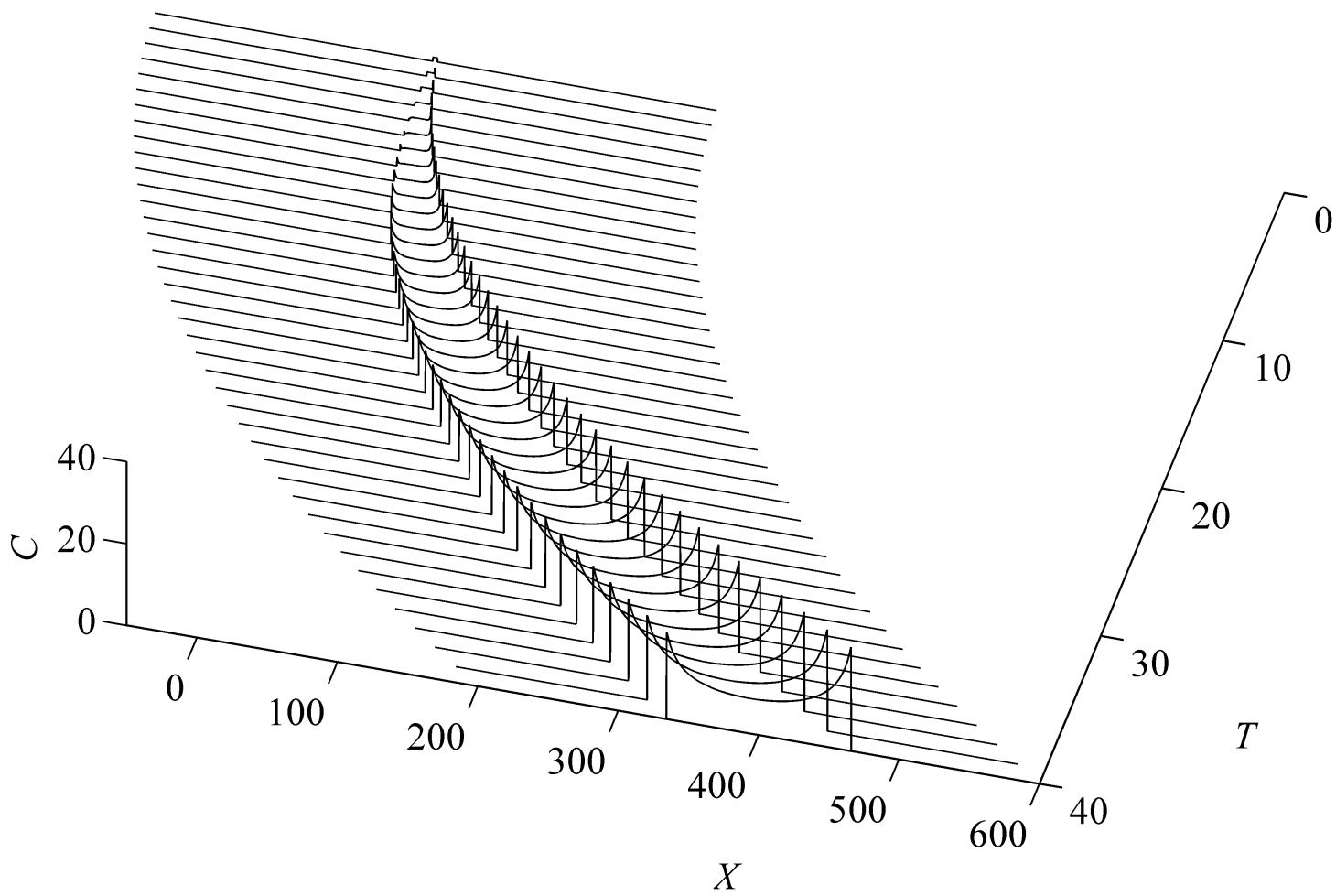


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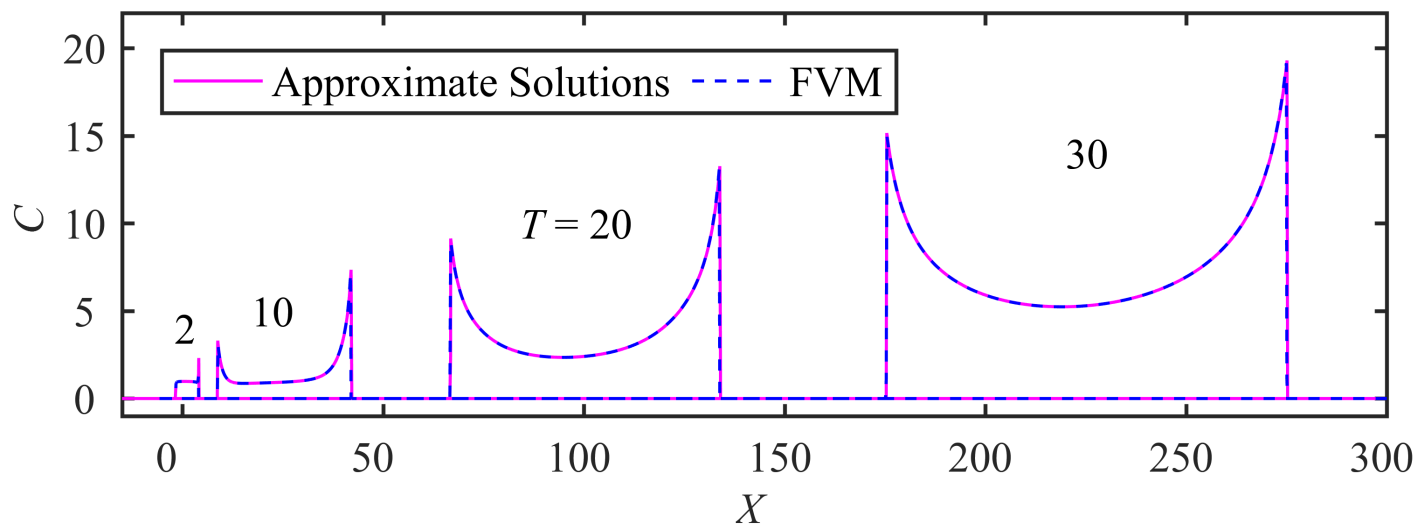
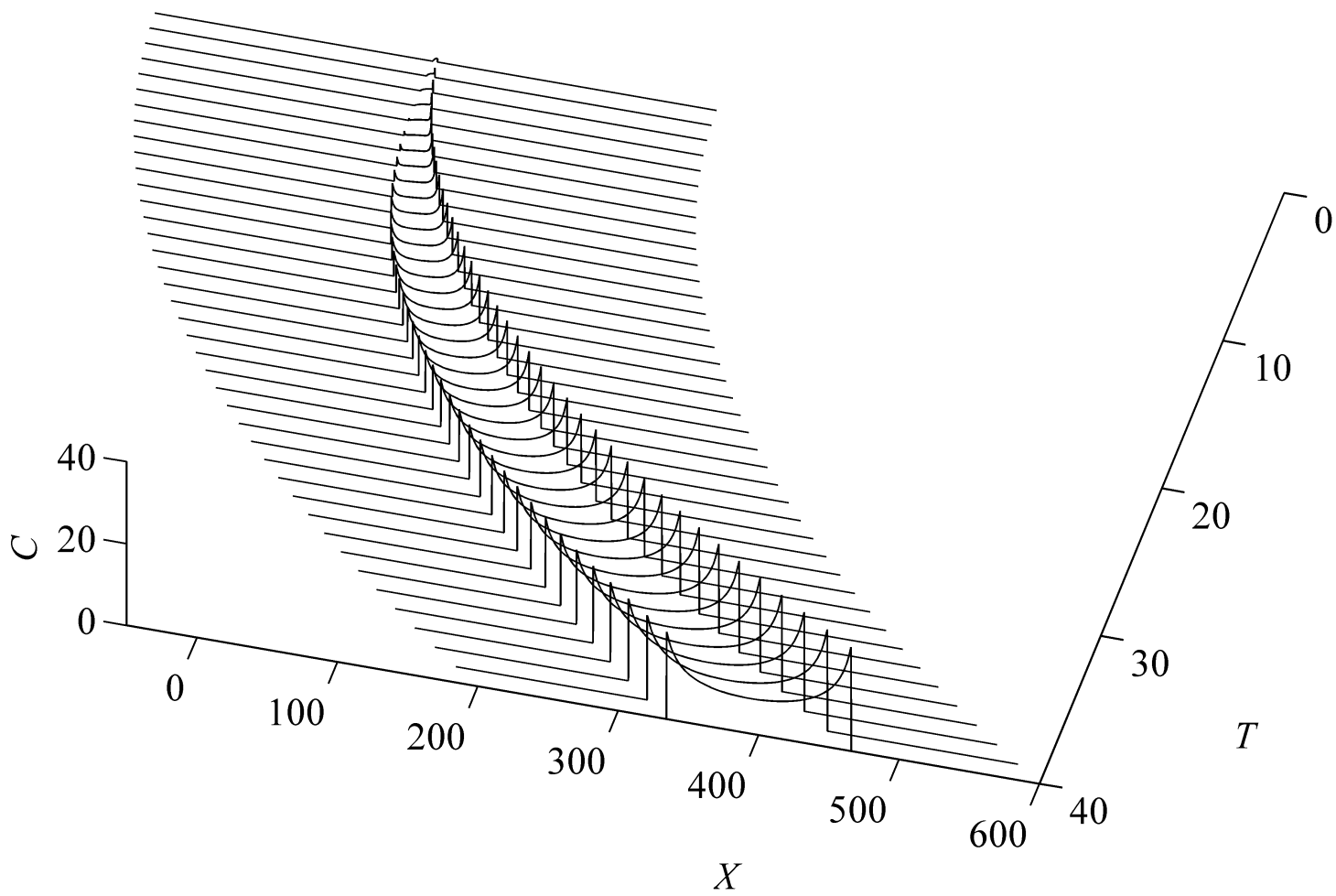


Figure 15.



**Figure 16.**



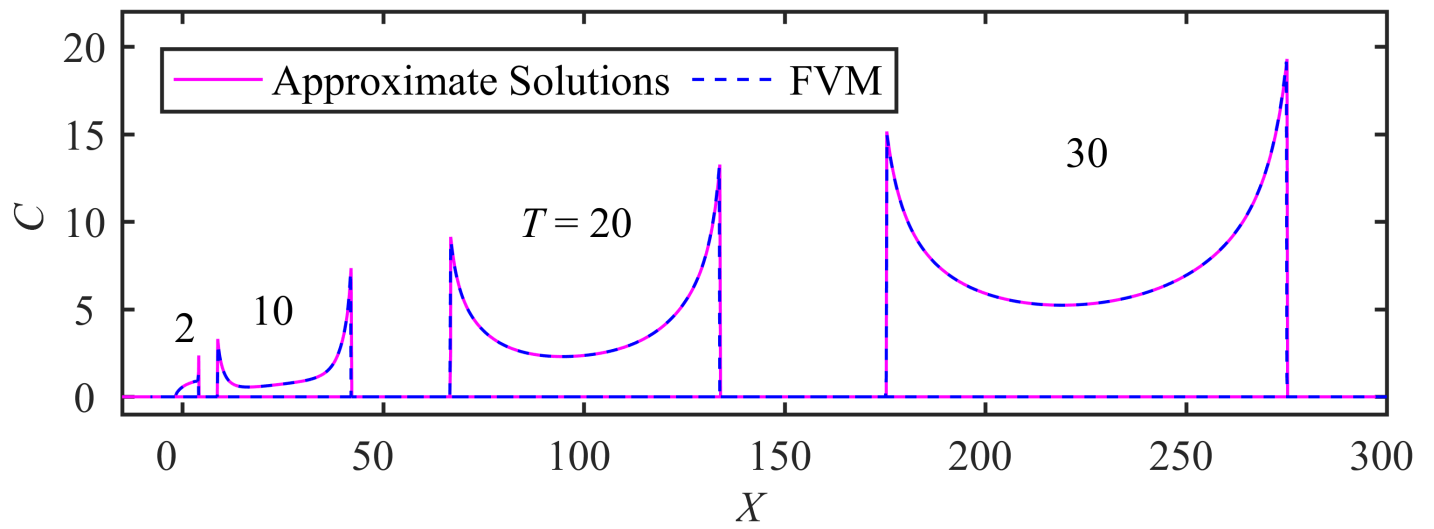


Figure 17.

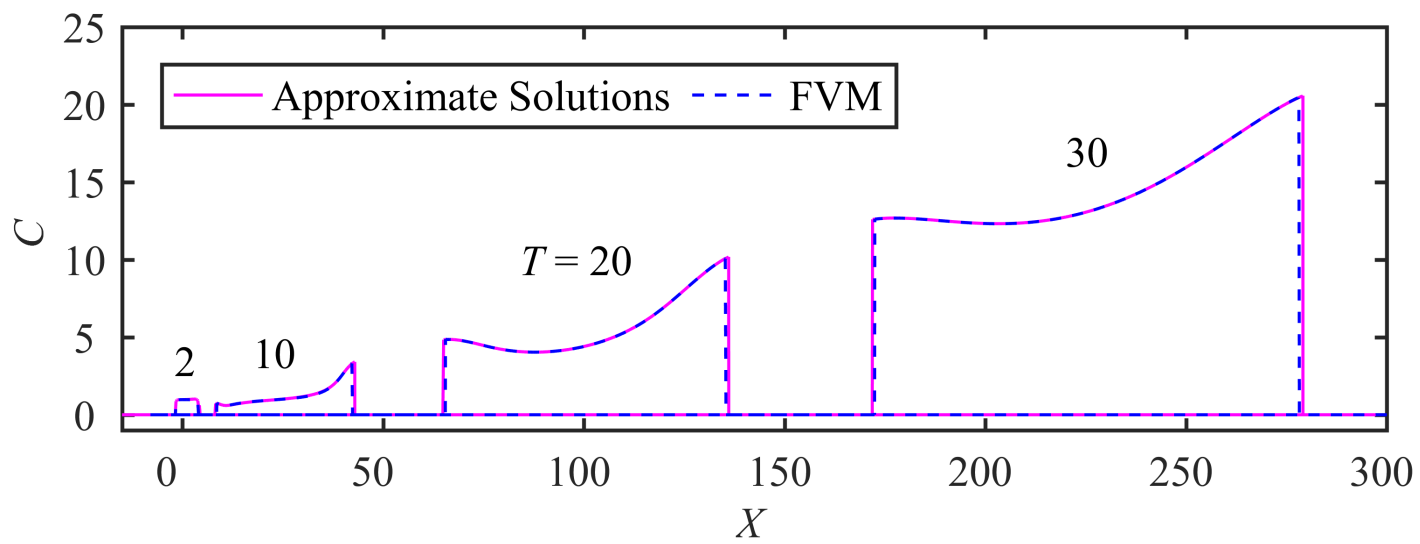


Figure 18.

