2021

An Investigation into the Roots of Bernstein via Cohomology of Differential Form

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http://hdl.handle.net/10026.1/17267

http://dx.doi.org/10.24382/1069
University of Plymouth

UNIVERSITY OF PLYMOUTH

An Investigation into the Roots of Bernstein via Cohomology of Differential Form

by

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A thesis submitted to the University of Plymouth
in partial fulfilment for the degree of

DOCTOR OF PHILOSOPHY

School of Engineering, Computing and Mathematics

June 2021

Dedication

To the Iraqi army and police forces ...

To all factions of the Holy Popular Mobilization Forces ..

To all those who participated in the battle of liberating Iraq from the terrorist organization ISIS ...

If it weren't for you, there would be no homeland, no land, or honor..

You are our glory and our pride, and to you I dedicate my humble effort.

Acknowledgements

Completing this work would not have been easy were it not for the support and encouragement that was provided by supervisors, Dr Colin Christopher and Dr Daniel Robertz. So, I must thank them and as I am truly indebted to them for their help.

I would like to thank the Iraqi Ministry of Higher Education and Scientific Research and Wasit University for financing the scholarship that has enable me to complete this work.

Personally, I would also like to thank all my friends, PhD students, in Plymouth and all members of the School of Mathematics and Statistics. Also, I would like to thank all members of Plymouth University.

Finally, I am so thankful for my mother and family who have supported me since my first days of starting my work and until this moment.

To my father, who passed away before this work was finished. I will never forget you. You will always remain in my memory.

Aqeel

Author’s declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award without prior agreement of the Doctoral College Quality Sub-Committee.

Work submitted for this research degree at the University of Plymouth has not formed part of any other degree either at the University of Plymouth or at another establishment.

Word count of main body of thesis: 24000

Signed: AQEEL NOOR ________________

Date: 17 / 12 / 2020

Abstract

An Investigation into the Roots of Bernstein Polynomials Via the Cohomology of Differential Forms.

Aqeel Noor

This work considers the Bernstein polynomial and the methods to calculate it.We consider the polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ and explain how to find the Bernstein polynomial, $b_f(s)$, and the operator $D \in A_n[s]$, which satisfies

$$D f^{s+1} = b_f(s) f^s \,.$$

The calculation of this polynomial depends on the Weyl algebra, A_n. We explain these calculations by a lot of examples for one dimensional, two dimensional and three dimensional polynomials. All algorithms which are developed to calculate the Bernstein polynomial did not gave a method to calculate the operator D above, in our method we give a method to how calculate this operator. The main problem of this work is to find a polynomial which describes the dimension of the first cohomology group, $\tilde{b}_f(s)$, defined using spaces of one forms and two forms, where $f \in \mathbb{C}[x,y]$.

In more detail, starting with a polynomial f, we define two operators

$$d_1 : \mathbb{C} \to \Omega^1 \quad \text{defined by} \quad d_1(h) = f\, d(h) + (s+1)\, d(f)\, h\,,$$
$$d_2 : \Omega^1 \to \Omega^2 \quad \text{defined by} \quad d_2(\omega) = f\, d(\omega) + s\, d(f) \wedge \omega\,.$$

We then calculate the dimension of first cohomology group for specific values of s.

The zeros of the polynomial $\tilde{b}_f(s)$ is chosen to correspond to the values of s where the cohomology is non-zero.

To find a link between our polynomial and the Bernstein polynomial, we compare this with the case when s is the root of the Bernstein polynomial. After a lot of calculation we gave our conjecture that $\tilde{b}_f(s)$ is divisible by $b_f(s)$.
We support our work by a lot of examples when f is a homogeneous polynomial, a quasi-homogeneous polynomial and some more complex cases.

The original motivation for our problem is to study the first integrals of vector fields with Darboux integrating factor in the case where the first integral is not of Darboux form. The roots of $\tilde{b}_f(s)$ are exactly the cases where we find new classes of integrals.

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Chapter 1

Introduction

The story of the Bernstein polynomial started around 1974 when Mikio Sato studied the Zeta function associated with a homogeneous vector space Sato & Shintani (1974). In 1980, Sato called the polynomial $b_f(s)$ which satisfies the equation

$$D f^{s+1} = b_f(s)\, f^s, \tag{1.1}$$

a b-function. Here, D is a differential operator with polynomial coefficients in the variables (including s). The existence of this polynomial was proved by Bernstein in 1971 Bernshtein (1971). From the two authors above this polynomial is now called the Bernstein-Sato polynomial. The idea can be generalised to other situations, for example Sabbah and Gyoja Sabbah (1987), choose a set of polynomial $g_1, g_2, \ldots, g_p$ and prove that there is an operator $D \in A_n$ satisfing

$$D\,(g_1^{s_1+1}, g_2^{s_2+1}, \ldots, g_p^{s_p+1}) = b(s)\,(g_1^{s_1}, g_2^{s_2}, \ldots, g_p^{s_p}). \tag{1.2}$$

In 2006, (Budur et al. (2006*a*)), prove that the equation 1.2 is true for any set of polynomials and give an important result for the roots of $b_f(s)$: they prove that all its roots are negative and rational, thus generalising what was already known for the solution of 1.1.

There are many authors, who have found applications of Bernstein polynomials in their research. It was our hope here to find similar applications to the Darboux integrability of polynomial systems and the multiplicity of invariant algebraic curves. Initial investigations into the latter topic were not continued as they did not appear to give much insight. However, some of the examples tried are given in chapter 3 where we discuss the dimension and multiplicity of modules.

There are many algorithms to calculate the Bernstein polynomial, by using the left ideal $ann(f^s)$.

In our work, we explain a method to find the Bernstein polynomial $b_f(s)$. All algorithms which we considered to calculate the Bernstein polynomial didn't give a way to calculate the operators D which satisfies 1.1, in our work we calculate it.

The main problem in this thesis, is to define a new polynomial, which we call the D$-$Coh polynomial, which is derived from the dimension of the first cohomology group of an exact sequence that we shall define later, we denote our polynomial by $\tilde{b}_f$, where $f \in \mathbb{C}[x,y]$.

We show in this thesis, by many examples, that $\tilde{b}_f$ appears to be always divisible by $b_f(s)$. Although we are not able to prove this, the conjecture would give a concrete way to understand some of the aspects of the Bernstein polynomial and how it relates in particular to the integrability of planar vector fields.

This thesis contains six chapters, In Chapter two, we introduced the definitions of most important concepts with some properties which we used in the main of this thesis such as the Weyl algebra, modules over Weyl algebra and Gröbner bases.

Chapter three contains the explanation of grading and filtration for K-algebras with some example for non commutative rings like the Weyl algebra. In same chapter we give some explanations about how to calculate the dimension and multiplicity in commutative and non commutative cases.

The history of the Bernstein polynomial and the algorithms which are developed by other authors to calculate it will be explained in Chapter four.

In Chapter five, we explain our method to compute the Bernstein polynomial $b_f(s) \in \mathbb{C}[s]$ and the operator $D \in A_n[s]$. We give many examples for this method in both one and two dimensional cases.

In Chapter six, we give our definition for the polynomial derived from the first cohomology group related to the integrability of a one-form, which we call the D$-$Coh polynomial and denote it by $\tilde{b}_f$. Our motivation to study the first cohomology group is to study the first integral of the vector field and it can been seen in section 6.3. We support our method by a lot of examples for both homogeneous and quasi-homogeneous polynomials. Since the definition of this polynomial is new then there are many problems still open for future works.

Chapter 2

Background

In this chapter we will introduce the definitions of the most important concepts, with some of their properties, which we use in this thesis such as the Weyl algebra, direct limits, D-modules and Gröbner bases.

2.1 Weyl algebra

Let K be a field of characteristic 0 and let $K[X] = K[x_1, x_2, \cdots, x_n]$ be the ring of polynomials in n independent commutative variables which are denoted $x_1, x_2, \cdots, x_n$. We can consider $K[X]$ as an infinite dimensional K-vector space.

Let $End_k(K[X])$ denote the algebra of linear operators from $K[X]$ to itself. We can also consider $End_k(K[X])$ as a non-commutative ring with addition and multiplication given by composition.

Now, we will define two types of operators on $K[x_1, x_2, \cdots, x_n]$: these are $\hat{x_i}$, which acts by multiplication by x_i, and ∂_i, which acts by the derivative $\partial/\partial x_i$. For all $f \in K[X]$, we have

$$\hat{x_i} \bullet (f) = x_i \cdot f,$$
$$\partial_i \bullet [f] = \frac{\partial f}{\partial x_i}.$$

The n-th Weyl algebra, A_n, is a K-subalgebra of $End_k(K[X])$ generated by $\hat{x_1}, \hat{x_2}, \cdots, \hat{x_n}$, and $\partial_1, \partial_2, \cdots, \partial_n$. In particular, A_1 is generated by $\hat{x}_1$ and ∂_1. Any element $s \in A_n$ can be written as a finite sum of $s = \sum_{i,j} c_{ij} \hat{x}_i^{\alpha_i} \partial_i^{\beta_i}$, for some $c_{ij} \in K$ and $\alpha_i, \beta_j \in \mathbb{Z}^+$.

One must take care with the Weyl algebra because it is non-commutative:

$$\partial_i \cdot \hat{x}_i(f) = \hat{x}_i \frac{\partial f}{\partial x_i} + f \frac{\partial x_i}{\partial x_i} = \hat{x}_i \frac{\partial f}{\partial x_i} + f = (\hat{x}_i \cdot \partial_i + \hat{1}) f.$$

As a result, $\partial_i \hat{x}_i = \hat{x}_i \partial_i + 1$, where 1 is the identity element.

We will give the definition of the commutator of two operators. For more details see Coutinho (1995).

Definition 1. For any two operators P and Q in A_n, we define the commutator of P and Q, denoted by $[P, Q]$, as follows:

$$[P, Q] = P \cdot Q - Q \cdot P.$$

Proposition 1. *In A_n we have:*

1. $[\hat{x}_i, \hat{x}_j] = [\partial_i, \partial_j] = 0$,
2. $[\hat{x}_i, \partial_i] = \delta_{ij} \cdot 1$,

where $1 \leq i, j \leq n$ and $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ii} = 1$.

Hereafter, we shall write x_i for both the variable and the operator which corresponds to it. Also, we will denote the generators of A_1 by x and ∂.

Proposition 2. *For any $f \in K[X]$, we have $[\partial_i, f] = \partial f / \partial x_i$ in A_n.*

We need to define *multi-index* notation before the next theorem which gives the canonical basis for A_n as a K-vector space. If α is the *multi-index*,

$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$, then we denote by x^α the product $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \cdots \cdot x_n^{\alpha_n}$. The length of α, denoted $|\alpha|$ is the degree of the monomial x^α, $\alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Theorem 1. *A basis of the K-vector space A_n is the set $\boldsymbol{B} = \left\{ x^\alpha \partial^\beta : \alpha, \beta \in \mathbb{N}^n \right\}$.*

Every monomial written using this basis is said to be in *canonical* form. For example $x_1^2 x_3^4 \partial_3$ is in the canonical form in A_3, while $x_1 \partial_2^2 x_2^6$ is not. See Coutinho (1995).

In the polynomial ring $K[X]$, we can define the degree of a polynomial f by the largest power for the non-zero monomials of f (Cohn (1974)). Similarly, we can define the degree of an operator $D \in A_n$ (denoted by $deg(D)$) as the maximum length of multi-index $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ for which $x^\alpha \partial^\beta$ has a non-zero coefficient in D. For example, the degree of $2x_1^2 x_3^4 \partial_2 \partial_3^2 - 5x_1^2 \partial_3^3$ is 9. For more details see Coutinho (1995).

Theorem 2. *For D , $D_1 \in A_n$, the following properties are satisfied:*

1. *$deg(D + D_1) \leq max\{deg(D), deg(D_1)\}$ and if $deg(D) \neq deg(D_1)$ then we have $deg(D + D_1) = max\{deg(D), deg(D_1)\}$.*

2. *$deg(DD_1) = deg(D) + deg(D_1)$.*

3. *$deg[D, D_1] \leq deg(D) + deg(D_1) - 2$.*

A non-zero ring is said to be *simple* if it has no two-sided ideals besides the zero ideal and itself. In a commutative ring, every non-zero simple ring must be a field, but in a non-commutative ring this is no longer true. For example, the Weyl algebra A_n is simple but it is not a field.

Remarks 1.

1. *A_n is a simple non-commutative ring. See Coutinho (1995).*

2. *Every non-zero endomorphism of A_n is injective, since A_n is simple and the kernel of an endomorphism is two-sided ideal, which then must be 0.*

3. *A_n is not a left principal ideal ring, for example the left ideal generated by x_1 and ∂_3 is a left ideal over A_3 but it is not a principal.*

2.2 Ring of differential operators

Definition 2. For an operator, $P \in End_k(R)$, we say that P has an order 0 if $[a,P] = aP - Pa = 0$ for all $a \in R$. Suppose that we have defined operators of order $< n$. An operator $P \in End_k(R)$ has an order n if it does not have order less then n and $[a,P]$ has order less than n for every $a \in R$. We denoted for the set of all operator with order $\leq n$ by $D^n(R)$ and one can easily show $D^n(R)$ is a K-vector space and that, if $P \in D^n(R)$ and $Q \in D^m(R)$, then $PQ \in D^{n+m}(R)$.

A K-derivation of a commutative K-algebra R is a linear operator D which satisfies $D(ab) = aD(b) + bD(a)$ for every $a, b \in R$. The K-vector space $Der_k(R)$ contains all K-derivations of R. It is clear that $Der_k(R) \subseteq End_k(R)$. The ring of differential operators $D(R)$ of the K-algebra R is the set of all operators of $End_K(R)$ of finite order, with the operations of sum and composition of operators. For every derivation D of R and any $a \in R$ we can define a new derivation (aD) by $(aD)(b) = aD(b)$ for every $b \in R$. From the above relation we can consider the K-vector space $Der_k(R)$ as a left R-module.
The following lemmas are very important and we will give them without proof. For more details about the proof of these lemmas. See Coutinho (1995)

Lemma 1. *The operators of order ≤ 1 correspond to the elements of $Der_K(R) + R$, and the elements of order 0 are the elements of R.*

Definition 3. Let C_r be the set below:

$$C_r = \left\{ P \in A_n : P = \sum_{\alpha} f_\alpha \partial^\alpha with\, |\alpha| \leq r \right\}.$$

Proposition 3. *Let P be a derivation of $K[X]$, then P is of the form $\sum_1^n f_i \partial_i$ with $f_i \in K[X]$ for every $i = 1, 2, , \cdots, n$.*

Definition 4. The *ring of differential operators* $D(R)$ of K-algebra R is a subset of $End_K(R)$ which contains all finite order operators with the addition and decomposition of operators. In other words, $D(R) = \bigcup_{n\geq 0} D^n(R)$ for all K-vector space $D^n(R)$. Also, we can show that:

$$C_r = C_{r+1} \bigcap D^r(K[X]).$$

If $R = K[X]$, then it is easy to show that $C_0 = K[X]$, and by Lemma 1 and Proposition 3 we get $C_1 = Der_K(K[X]) + K[X]$.

Definition 5. Let $C^\infty(U)$ be the set of C^∞ functions which are defined on an open set U of $\mathbb{R}^n$. $C^\infty(U)$ is a left $A_n(\mathbb{R})$-module, where $x_i's$ and $\partial_i's$ act by multiplication and differentiation receptively.

Lemma 2. *In particular, if $P \in D(K[X])$. If $[P, x_i] = 0$ for every $i = 1, 2, \cdots, n$, then $P \in K[X]$, where D is defined in Definition 4 above.*

By using the above properties we get the very important following result:

Theorem 3. *A_n can be considered as the ring of differential operators of $K[X]$. Coutinho (1995).*

Definition 6. Let $F : K^n \to K^m$ be a map and p be a point of its domain. If there exist $F_1, F_2, \cdots, F_m \in K[x_1, x_2, \cdots, x_n]$ such that $F(p) = (F_1(p), F_2(p), \cdots, F_m(p))$ for all $p \in K^n$, then we can say that F is a polynomial map. If the F has an inverse and this inverse is also a polynomial, then F is called an *isomorphism* or a *polynomial isomorphism*.

For example, the polynomial $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x^3 + x, y)$ has an inverse

$$G(x, y) = (\frac{x}{2} + \frac{1}{18}\sqrt{81x^2 + 12} - \frac{8}{108x + 12\sqrt{81x^2 + 12}}, y).$$

However, $G(x,y)$ is not polynomial. If K has a characteristic 0 ($\mathbb{C}$ for example) then if F is invertible in $K[x_1,x_2.\cdots,x_m]$, then its inverse is also polynomial. See Bass et al. (1982).

Definition 7. Let $F : K^n \to K^n$ be a polynomial map. We can define the Jacobian matrix for the polynomial F by

$$J(F) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \frac{\partial F_n}{\partial x_3} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}.$$

If F is a polynomial isomorphism, then the Jacobian matrix will be invertible at every $a \in K$. In other words, $\Delta F = det(J(F))$ is an invertible polynomial so must be a non-zero constant in K. See Yan & de Bondt (2013).

Jacobian Conjecture : Let $F : K^n \to K^n$ be a polynomial map. If $\Delta F = 1$ on K^n then F has a polynomial inverse and $K[F_1,F_2,\cdots,F_n] = K[x_1,x_2,\cdots,x_n]$.

Example 1. *Consider the polynomial $F(x,y) = ((x-y)^d + y - 2x, (x-y)^d - x)$, the Jacobian matrix*

$$JF = \begin{bmatrix} d(x-y)^{d-1} - 2 & -d(x-y)^{d-1} + 1 \\ d(x-y)^{d-1} - 1 & -d(x-y)^{d-1} \end{bmatrix}.$$

Clearly, $\Delta F = 1$, and one can easly to show that the polynomial map $G(x,y) = ((y-x)^d - y, (y-x)^d - 2y + x)$ is the inverse of $F(x,y)$.

2.3 Modules over the Weyl Algebra

We know that A_n is a subalgebra of $End_k(K[X])$ generated by $x_1,x_2,\cdots,x_n,\partial_1,\partial_2,\cdots,\partial_n$. We will define the action between the generaters of A_n and any polynomial $f \in K[X]$ to make $K[X]$ as a left A_n-module as follows:

$$x_i \bullet f = x_i f,$$
$$\partial_i \bullet f = \frac{\partial f}{\partial x_i}.$$

We will now define some of the most important properties of such modules. Let R be any ring, an R-module M is *irreducible* (simple) if it does not have any non-zero proper submodules. The element $s \in M$ called *torsion* if $ann_R(s) = \{\forall r \in R : rs = 0\}$ is a non-zero (left) ideal of R. A left module M is *torsion module* if every element of M is *torsion*.

The following Lemma is very important and we give it without proof. For the proof see Coutinho (1995).

Lemma 3. *Let R be a ring and M be an irreducible left R-module.*

1. *For any non-zero element $s \in M$, then $M \cong R/ann_R(s)$.*
2. *If R is not division ring, then M is torsion module.*

Proposition 4. *The ring $K[X]$ is an irreducible torsion A_n-module and we have*

$$K[X] \cong A_n / \sum_1^n A_n \partial_i.$$

Proof. Let f be any non-zero polynomial in $K[X]$ and we consider the submodule $A_n f$. Let $a x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ be a monomial of maximal degree with non-zero coefficient of f and let a be its coefficient. The element $\partial_1^{i_1} \partial_2^{i_2} \cdots \partial_n^{i_n} f = a\, i_1! i_2! \cdots i_n! \in A_n f$. However, it is a constant and $K[X]$ is generated by 1. Thus, $A_n f = K[X]$ and $K[X]$ is an irrducible left A_n-module. Since A_n is not a division ring and by Lemma 3, then $K[X]$ is a torsion module. Let J be the left A_n-ideal generated by $\partial_1, \partial_2, \cdots, \partial_n$. We know that 1 is a non-zero element of $K[X]$ and its annihilator is the left ideal generated by $\partial_1, \partial_2, \cdots, \partial_n$.

Then $J \subseteq ann_{A_n}(1)$. Now let $P \in ann_{A_n}(1)$, then P can be written as $f+Q$, where $Q \in J$ and $f \in K[X]$. Since $P \in ann_{A_n}(1)$, $0 = P.1 = (f+Q)(1) = f.1 = f$. Thus, $P = Q \in J$ and hence $ann_{A_n}(1) \subseteq J$. By Lemma 3, we get $K[X] \cong A_n/\sum_1^n A_n\partial_i$. □

There is another left module $A_n/\sum_1^n A_n x_i$ that is closely related to $K[X]$ which is isomorphic to $K[\partial] = K[\partial_1, \partial_2, \cdots, \partial_n]$ as a vector space.

Definition 8. Let R be a ring and let M be a left R-module. We can define a new module which is called the *twisting module of M by σ*, denoted by M_σ, where σ is an automorphism of R . $M = M_\sigma$ as an abelian group but there is a different action which depends on the action of module M_σ, gives by $a \bullet u = \sigma(a).u$ for every $a \in R$ and $u \in M$.

The following proposition gives some properties of the twisting module.

Proposition 5. *Let R be a ring, σ be an automorphism of R and let M be an R- module. Then:*

1. *M_σ is irreducible if and only if M is irreducible.*

2. *M_σ is a torsion module if and only if M is a torsion module.*

3. *For a submodule, N, of M we get $(M/N)_\sigma \cong M_\sigma/N_\sigma$.*

4. *For a left ideal, J, of R we get $(R/J)_\sigma \cong R/\sigma^{-1}(J)$.*

The most important application of the twisting module is the Fourier transformation .$\mathscr{F} : A_n \to A_n$, which is the automorphism defined by $\mathscr{F}(x_i) = \partial_i$ and $\mathscr{F}(\partial_i) = -x_i$. Let M be any left A_n-module. The twisting module $M_\mathscr{F}$ is called *Fourier transformation* of M.

Proposition 6. *The Fourier transformation of $K[X]$ is $K[\partial]$.*

Proof. By Proposition 4 we get $K[X] \cong A_n/J$, where $J = \sum_1^n A_n.\partial_i$. $\mathscr{F}^{-1}(J) = \{a \in A_n : \mathscr{F}(a) \in J\} = \sum_1^n A_n.x_i$. From Proposition 5(4) we obtain the result $A_n/K[X]$ is $K[\partial]$. □

Since $K[X]_{\mathscr{F}} = K[\partial]$ and, by Proposition 4, $K[X]$ is irreducible, then by Proposition 5(1) we get that $K[\partial]$ is also irreducible. For another example, let $\sigma : A_n \to A_n$ defined by $\sigma(x_i) = x_i$ and $\sigma(\partial_i) = \partial_i + g_i$, where $g_i \in K[x_i]$ for all $i = 1, 2, \cdots, n$. Let J be a left ideal over A_n generated by $\partial_1, \partial_2, \cdots, \partial_n$, then $\sigma^{-1}(J) = \sum_1^n A_n.(\partial_i - g_i)$. The above calculation tells us that the $K[X]_\sigma$ is irreducible. We know that for every left ideal J and every automorphism σ we get is a left ideal $\sigma^{-1}(J)$, that means that the $K[X]_\sigma$ is irreducible for every automorphism σ.

Definition 9. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function. If the complex derivative of f exists at z_0, then f is called *holomorphic* at z_0. If f is holomorphic at every point in open set U of $\mathbb{C}$, then f will be holomorphic on U. For example $f(z) = e^z$ is holomorphic on the whole of $\mathbb{C}$ (Rudin (1987)).

Let $\mathscr{H}(U)$ be the set of all holomorphic functions on an open set, U, of $\mathbb{C}$. It is clear that all polynomials of one variable z of $\mathbb{C}$ are holomorphic, then $\mathbb{C}[z] \subseteq \mathscr{H}(\mathbb{C})$. Since $\mathbb{C}$ is a field of characteristic 0, then $A_1(\mathbb{C})$ is the Weyl algebra generated by z and $\partial = \frac{d}{dz}$. Then $\mathscr{H}(U)$ can be considered as a left $A_1(\mathbb{C})$-module, where z will act by multiplication and ∂ by derivation.

Remarks 2.

1. *$\mathscr{H}(U)$ is not irreducible. Since the submodule generated by e^z is non-zero and a proper ideal of $\mathscr{H}(U)$.*

2. *Every polynomial is a holomorphic function but the converse is not true.*

3. *Since $e^{e^z} \in \mathscr{H}(U)$ is not torsion element of the $A_1(\mathbb{C})$-module $\mathscr{H}(U)$, then $\mathscr{H}(U)$ is not a torsion left $A_1(\mathbb{C})$-module . See Coutinho (1995) Proposition (5.3.2).*

Definition 10. Any operator in A_n can be written in the form $\sum_\alpha g_\alpha \partial^\alpha$, where $g_\alpha \in K[x_1, x_2, \cdot, x_n]$ and $\alpha \in \mathbb{N}^n$. Let P as a differential operator and f is a polynomial or (if $K = \mathbb{R}$, the field of real numbers) a C^∞ function with the same variable of $K[X]$, then f becomes a solution for P if

$$\sum_\alpha g_\alpha \partial^\alpha (f) = 0, \tag{2.1}$$

In general, if $P_1, P_2, \cdots, P_m$ are differential operators in A_n, then f will be a solution for the system of differential operators if

$$P_1(f) = P_2(f) = \cdots = P_m(f) = 0. \tag{2.2}$$

The A_n-module associated to the system of differential system 2.2 is $A_n / \sum_{i=1}^m A_n P_i$. A polynomial solution of the system 2.2 is every polynomial $f \in K[X]$ such that $P_i(f) = 0$ for all $i = 1, 2, \cdots, m$. The set of all polynomial solutions of the system 2.2 is a K-vector space, and this vector space is isomorphic to $Hom_{A_n}(M, K[X])$, where M is the left A_n-module associated to the system 2.2. see Coutinho (1995), Theorem 6.1.2.

The point is that $Hom_{A_n}(M, K[X])$ is not module over $K[X]$ or A_n. It is just a vector space and may be infinite dimensional. For example, the polynomial solutions of the equation $x_1 \partial_2 - x_2 \partial_1 = 0$ over $K[x_1, x_2]$ is an infinite K-vector space. On the other hand, suppose that $P_1, P_2, \cdots, P_m$ are operators in $A_n(\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers. Then, we can see by using the same way above, that C^∞-solutions for the system 2.2 is isomorphic to $Hom_{A_n}(M, C^\infty(U))$.

Definition 11. A set $\mathscr{I}$ with the relation $\leq$ is called *pre-ordered* if $\leq$ is transitive and reflexive. A pre-ordered set $\mathscr{I}$ is called directed if for every $i, j \in \mathscr{I}$ there exists $k \in \mathscr{I}$, such that, $k \leq i$ and $k \leq j$. For examples, the set of natural number $\mathbb{N}$ and the set of real number $\mathbb{R}$ with the relation $\leq$ are directed. while the set

$O = \{\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\}$ with the relation $\subseteq$ is pre-ordered but not directed.

Definition 12. Let R be a ring and $\mathscr{I}$ be a directed set. Assume that for every $i \in \mathscr{I}$ there exists a left R-module M_i. Then the family $\{M_i : i \in \mathscr{I}\}$ is called a *directed family* if for every i, j with $i \leq j$ there exists a module homomorphism

$$\pi_{j,i} : M_j \to M_i,$$

such that, for every $i \leq j \leq k$,

$$\pi_{j,i} \circ \pi_{k,j} = \pi_{k,i}.$$

That means that following diagram commutes

$$\begin{tikzcd} M_k \arrow[r, "\pi_{k,j}"] \arrow[dr, "\pi_{k,i}"'] & M_j \arrow[d, "\pi_{j,i}"] \\ & M_i \end{tikzcd}.$$

Now, consider the set

$$\mathscr{U} = \{(u,i) : u \in M_i\},$$

Which is the disjoint union of the left R-modules M_i, we can define the equivalence relation $\sim$ on $\mathscr{U}$ by $(u,i) \sim (v,j)$ if and only if there exists $k \in \mathscr{I}$ with $k \leq j, k \leq i$ and $\pi_{k,i}(u) = \pi_{k,j}(v)$. The *direct limit* of this family, denoted by $\lim_{\to} M_i$, is the quotient of $\mathscr{U}$ by the equivalence relation $\sim$. We denoted $(u,i) \in \lim_{\to} M_i$ for the equivalence class under the equivalence relation $\sim$.

For example, to explain the definition, let $D(\varepsilon)$ be the open disc in $\mathbb{C}$ centred at 0 of radius ε. Let $\mathscr{H}(\varepsilon)$ be the set of all holomorphic functions defined in $D(\varepsilon)$. $\mathscr{H}(\varepsilon)$ is an $A_1(\mathbb{C})$-module. Let $\mathbb{R}^+$ be the pre-ordered set, then for every $\varepsilon \leq \varepsilon_1 \in \mathbb{R}^+$, we get

$\mathscr{H}(\varepsilon_1) \subseteq \mathscr{H}(\varepsilon)$. Hence, we can define

$$\pi_{\varepsilon_1,\varepsilon} : \mathscr{H}(\varepsilon_1) \to \mathscr{H}(\varepsilon)$$

to be the restriction map of holomorphic function in $\mathscr{H}(\varepsilon_1)$ to $D(\varepsilon)$. The directed family is $\{\mathscr{H}(\varepsilon) : \varepsilon \in \mathbb{R}^+\}$, and the direct limit $\lim_{\to} \mathscr{H}(\varepsilon) = \{(f,\varepsilon) : f \in \mathscr{H}(\varepsilon)\}$. Suppose that $\varepsilon \leq \varepsilon_1$, then $(f,\varepsilon) = (g,\varepsilon_1) \in \lim_{\to} \mathscr{H}(\varepsilon)$ if $f(z) = g(z)$ for every $z \in D(\varepsilon)$. The elements of $\mathscr{H}_0 = \lim_{\to} \mathscr{H}(\varepsilon)$ are called *germs* of holomorphic functions at 0.

We want to make $\lim_{\to} M_i$ into a left R-module. Firstly, we must define the sum (to become abelian group) and scalar multiplication (to become a left R-module) on $\lim_{\to} M_i$. Let $(u,i),(v,j) \in \lim_{\to} M_i$ and $a \in R$. Since $\mathscr{I}$ is pre-ordered, then there exists $k \in \mathscr{I}$ such that $k \leq i$ and $k \leq j$. We can define the sum and scalar multiplication by

$$\begin{aligned} (u,i)+(v,j) &= (\pi_{i,k}(u)+\pi_{j,k}(v),k) \\ a \cdot (u,i) &= (au,i). \end{aligned}$$

However, we must prove that the sum is well defined because its dependency on the element k. Assume that there exist k,k_1 and both are less than or equal to i and j. Claim that $S = (\pi_{i,k}(u)+\pi_{j,k}(v),k) = (\pi_{i,k_1}(u)+\pi_{j,k_1}(v),k_1) = S_1$ in $\lim_{\to} M_i$. Choose $r \in \mathscr{I}$ such that $r \leq k$ and $r \leq k_1$. Thus

$$\pi_{k,r}(\pi_{i,k}(u)+\pi_{j,k}(v)) = \pi_{k_1,r}(\pi_{i,k_1}(U)+\pi_{j,k_1}(v)) = \pi_{i,r}(u)+\pi_{j,r}(v).$$

Hence $S = S_1 = (\pi_{i,r}(u)+\pi_{j,r}(v),r)$ in $\lim_{\to} M_i$. By the explanation above, $\lim_{\to} M_i$ is a left R-module. Now, we will apply that to our example $(f,\varepsilon)+(g,\varepsilon_1) = (f+g,\varepsilon_2)$, where $\varepsilon_2 = min\{\varepsilon, \varepsilon_1\}$ and for every $P \in A_1(\mathbb{C})$, then $P \cdot (f, \varepsilon) = (P(f), \varepsilon)$. Thus $\mathscr{H}_0$ is a left $A_1(\mathbb{C})$-module.

Let $D'(\varepsilon) = D(\varepsilon)\backslash 0$, and $\tilde{D} = \{z \in \mathbb{C} : Re(z) < log(\varepsilon)\}$. Then $\tilde{D}$ with the projection map

$$\pi : \tilde{D}(\varepsilon) \to D'(\varepsilon),$$

defined by $\pi(z) = e^z$ becomes a universal cover for $D'(\varepsilon)$. Let $z = \rho e^{i\theta} \in D'$ with $\rho < \varepsilon$, then $\pi(log(\rho) + i\theta) = z$. that means π is a surjective. The relation of these sets and maps can be shown in the diagram below.

$$\begin{tikzcd}
\tilde{D}(\varepsilon) \arrow[d, "\pi"] & \\
D'(\varepsilon) \arrow[r] & D(\varepsilon).
\end{tikzcd}$$

We want to make the set of holomorphic function in $\mathscr{H}(\tilde{D}(\varepsilon))$ into a left $A_1(\mathbb{C})$-module. Let $h \in \mathscr{H}(\tilde{D}(\varepsilon))$, $f \in \mathbb{C}[x]$ and $\partial = d/d_x$ then the action between the element of ring $A_1(\mathbb{C})$ and h is

$$f \bullet h(z) = f(e^z)h(z), \tag{2.3}$$

$$\partial \bullet h(z) = h'(z)e^{-z}. \tag{2.4}$$

Proposition 7. *The map*

$$\pi^* : \mathscr{H}(D'(\varepsilon)) \to \mathscr{H}(\tilde{D}(\varepsilon)),$$

defined by $\pi^*(h)(z) = h(\pi(z))$ is an injective homomorphism of $A_1(\mathbb{C})$ modules. Let $\mathscr{M}_\varepsilon = \mathscr{H}(\tilde{D}(\varepsilon))/\pi^*(\mathscr{H}(D(\varepsilon)))$. If $\varepsilon_1 \leq \varepsilon$, then $\tilde{D}(\varepsilon_1) \subseteq \tilde{D}(\varepsilon)$ and hence $\mathscr{H}(\tilde{D}(\varepsilon)) \subseteq$

$\mathscr{H}(\tilde{D}(\varepsilon_1))$. This can induce a module homomorphism over $A_1(\mathbb{C})$

$$\tau_{\varepsilon.\varepsilon_1} : \mathscr{M}_\varepsilon \to \mathscr{M}_{\varepsilon_1}.$$

Hence, $\{\mathscr{M}_\varepsilon : \varepsilon \in \mathbb{R}\}$ is a direct family of $A_1(\mathbb{C})$-module. It has a direct limit, denoted by $\mathscr{M}$ and called the module of microfunctions.

The *Canonical projection* from $\mathscr{H}(\tilde{D}(\varepsilon))$ into $\mathscr{M}_\varepsilon$ is compatible with the limit and gives the following homomorphism of $A_1(\mathbb{C})$-modules:

$$can : \mathscr{H}(\tilde{D}(\varepsilon)) \to \mathscr{M}.$$

The function $\frac{e^{-z}}{2\pi i}$ is holomorphic in $\tilde{D}(\varepsilon)$, but $\frac{e^{-z}}{2\pi i} = \pi^*(\frac{1}{2\pi i z})$. However $\frac{1}{2\pi i}z$ is not holomorphic in $D(\varepsilon)$. Hence $can(e^{-z}/2\pi i) =$ is a non-zero element of $\mathscr{M}$ which is called the *Direc delta microfunction*, and denoted by δ.

The equation $xh = 0$ has no non-zero analytic solution, but if we consider the element $x\delta$ and use equation 2.3 we get

$$x\delta = e^z can(\frac{e^{-z}}{2\pi i}) = \frac{1}{2\pi i},$$

But $\frac{1}{2\pi i}$ is a constant, and thus holomorphic in $D(\varepsilon)$. Hence $\frac{1}{2\pi i}$ is a zero in $\mathscr{M}$. That means that $x\delta = 0$. Another important example is the *Heaviside microfunction*, which is $Y = can(\frac{z}{2\pi i})$. But $\frac{z}{2\pi i}$ is the image of $log(\frac{z}{2\pi i})$ under π^*. Since $log(\frac{z}{2\pi i})$ is not holomorphic in $D(\varepsilon)$, then the Hyperfunction Y is non-zero. Moreover, we can consider Y as satifying the equation $\int \delta = Y$, because, using equation 2.4 to differentiate Y, we have

$$\partial \bullet Y = Y'(z)e^{-z} = can(\frac{e^{-z}}{2\pi i}) = \delta.$$

2.4 Gröbner bases

In this section we will introduce the definitions of some important concepts with their properties which has been used it in our report such as term ordering, reduced polynomials modulo F, S- polynomials and Gröbner bases.

Definition 13. Let $K[x_1,x_2.\cdots,x_n]$ be a polynomial ring, we write X^α, where $\alpha \in \mathbb{N}^n$, for the monomials of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$. A monomial order, $>$, defined on $K[x_1,x_2.\cdots,x_n]$ is *a total order of the set of monomials of* $K[x_1,x_2.\cdots,x_n]$ satisfying if $x_i > 1$ for all $i = 1,2,\cdots,n$, and whenever $m_1 > m_2$, then $m_1n > m_2n$, where m_1,m_2,n are monomials.

There are many types of monomial ordering. We will define two of them:

1. The lexicographic order $>_{lex}$ on $K[x_1,x_2,\cdots,x_n]$ with $x_1 >_{lex} x_2 >_{lex} \cdots >_{lex} x_n$ is defined by $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} >_{lex} x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n}$ if $\alpha_i > \beta_i$, where i is the smallest number $1,2,\cdots,n$ for which $\alpha_1 \neq \beta_i$. In the lexicographic order, we are not looking for the total degree of the monomial X^α which is $|\alpha|$. Under this order, the monomial m_1 is bigger than m_2 if the power of x_1 in m_1 is greater than the power of x_1 in m_2. If x_1 has the same the power in m_1 and m_2, then we compare the power of x_2 and so on.
   For example, in $K[x_1,x_2]$, we have $x_1 >_{lex} x_2^5$ and in $K[x_1,x_2,x_3]$ with $x_1 >_{lex} x_2 >_{lex} x_3$, we get $x_1x_2x_3 >_{lex} x_1x_2$. In $K[x,y.z]$ with $x >_{lex} y >_{lex} z$ the lexicographic order would give:

$$1 < z < z^2 < \cdots < y < yz < \cdots < y^2 < y^2z < \cdots < x < xz < \cdots$$

2. The graded lexicographic ordering $>_{glex}$ on $K[x_1,x_2,\cdots,x_n]$ with $x_1 >_{lex} x_2 >_{lex} \cdots >_{lex} x_n$ is defined by $m_1 >_{glex} m_2$ if:
   either $deg(m_1) > deg(m_2)$ or, if $deg(m_1) = deg(m_2)$, $m_1 >_{lex} m_2$.
   For example, in $K[x_1,x_2]$, we have $x_2^2 >_{glex} x_1$ and in $K[x_1,x_2,x_3]$ with

$x_1 >_{lex} x_2 >_{lex} x_3$, we get $x_1x_2x_3 >_{glex} x_1x_2$. In $K[x,y.z]$ with $x >_{lex} y >_{lex} z$ the lexicographic order would give:

$$1 < z < y < x < z^2 < yz < y^2z < xz < xy < x^2 < z^3 < \cdots$$

For more details or more types of term ordering, see Bueso et al. (2003).

From now until the end of this chapter, we will use graded lexicographic ordering ($>_{glex}$), if we do not write a different term ordering.

Definition 14. Let K be a field and $f \in K[x_1,x_2,\ldots x_n]$ be a polynomial of n variable, then the leading term for f, denoted by LT, is the term with highest power of the variables $x_1,x_2,\ldots,x_n$. The leading coefficient, denoted by LC is the coefficient of LT and the leading monomial is ,LM is the monomial which contains the leading term. For example, let $f = 1+3x^2y^2+5y^3$ be a polynomial in $\mathbb{C}[x,y]$, then $LM = 3x^2y^2$, $LC = 3$.

Definition 15. Let K be a field and $F = \{f_1, f_2, \cdots, f_n\} \subseteq K[X]$. A polynomial h can be reduced from another polynomial g modulo F, denoted by $g \longrightarrow_F h$, if $h = g - buf_i$ for some $f_i \in F, u = \frac{LM(g)}{LM(f_i)}, b = \frac{LC(g)}{LC(f_i)}$. Otherwise, g is called irreducible modulo F. Or g is an irreducible polynomial modulo F if the leading monomial of g is not divisible by any leading monomial of $f_i \in F$.

For example if $f_1 = xy^2 - x, f_2 = x - y^3$ and $F = \{f_1, f_2\}$, then the polynomial $g = x^7y^2 + x^3y^2 - y + 1$ reduces to $h = x^7 + x^3y^2 - y + 1$, $g \longrightarrow_F h$, because $h = g - \frac{LM(g)}{LM(f_1)} f_1$.

Definition 16. Let $f, g \in K[X]$, then the S-Polynomial of f and g is a new polynomial in the ideal generated by f and g which is defined by S-Polynomial $(f,g) = \frac{L}{LT(f)} f - \frac{L}{LT(g)} g$, where $L = l.c.m.(LM(f), LM(g))$.

For example, let $f = xy^2z - xyz$ and $g = x^2yz - z^2$, then S-Polynomial $(f,g) = -x^2yz + yz^2$.

Definition 17. Let $F = \{f_1, f_2, \cdots, f_m\}$ be a set of polynomial in $K[X]$, and I be a left ideal generated by the set of polynomials of F. Then F is a Gröbner Basis of I if the S-Polynomial (f_i, f_j) can be reduced to zero for every $1 \leq i, j \leq m$.

According to [Mialebama Bouesso & Sow (2015)], the solution of different Algebra and Algebraic geometry problems could be implemented using Gröbner bases which is an algebraic technique. While [Sturmfels (2005)], defined Gröbner as a set of multivariable polynomials with a certain characteristic so that each set of those polynomials can be transformed into a Gröbner bases.

Algorithm:

Input : Set of polynomials $F = \{f_1, f_2, \cdots, f_m\} \in K[x_1, x_2, \cdots, x_n]$, and the left ideal $I = \langle f_1, f_2, \cdots, f_m \rangle$.

Output : $G = \{g_1, g_2, \cdots, g_s\} \in I$. Which is a Gröbner basis of the left ideal I with respect to the fixed term ordering $>$.

Step 1 : For every $1 \leq i, j \leq m$, calculate the S-polynomial (f_i, f_j), if the result can not be reduced to zero modulo F, then let f_{ij} be the S-polynomial $f_{ij} = (f_i, f_j)$. If it is reduces to zero modulo F, then just ignore it.

Step 2 : We have a new set $F_1 = \left\{f_1, f_2, \cdots, f_m, f_{ij}\right\}$. Calculate the S-polynomial (f_i, f_{kj}), where $1 \leq i \leq m$ and f_{kj} is defined in step 1. Also, we will ignore it if it reduces to zero and if not then we add the S-polynomial to the set F_1

Step 3 : We repeat these steps until we get a set $G = \{f_1, f_2, \cdots, f_m, \cdots, f_{m+k}\}$ which satisfies the property that every S-polynomial (f_i, f_j) can be reduced to zero modulo G. Thus, the set G is a Gröbner bases for I.

Example 2. *Let $F = \{f_1, f_2\} \subseteq \mathbb{C}[x, y]$ and I be a left ideal over $\mathbb{C}[x, y]$ generated by*

f_1 and f_2, where $f_1 = x^2y^2 - xy, f_2 = x^3y^2 - xy^3 + 2$, then the S-polynomial of f_1, f_2 is

$$\begin{aligned} S-Polynomial(f,g) &= \frac{l.c.m.(LM(f),LM(g))}{LT(f)} f - \frac{l.c.m.(LM(f),LM(g))}{LT(g)} g \\ &= x.(f) - (g) = xy^3 - x^2y - 2 = f_3 \end{aligned}$$

We get a new polynomial $f_3 \in I$ which not can be reduced to zero modulo F.
Let $F_1 = \{f_1, f_2, f_3\}$; we then calculate the S-polynomial (f_1, f_3) and the S-polynomial (f_2, f_3). The S-polynomial (f_2, f_3) can be reduced to zero modulo F_1, whilst the S-polynomial (f_1, f_3) cannot. We then let f_4 be the S-polynomial $f_4 = (f_1, f_3) = x^3y - xy^2 + 2x$.
Now, we have $F_2 = \{f_1, f_2, f_3, f_4\}$. The S-polynomial (f_2, f_4) and the S-polynomial (f_3, f_4) reduce to zero Modulo F_2, whilst the S-polynomial $f_5 = (f_1, f_4) = xy - 1$ does not.
We will add the new polynomial, f_5, to the our set to get $F_3 = \{f_1, f_2, f_3, f_4, f_5\}$. Then, we also see that the S-polynomial (f_1, f_5), the S-polynomial (f_2, f_5) and the S-polynomial (f_4, f_5) can be reduced to zero modulo F_3, whilst the S-polynomial $f_6 = (f_3, f_5) = y^2 - x - 2$ does not.
We will continue with this work until we get other three polynomial $f_7 = x^3 + 2x^2 - xy, f_8 = x^2 + 2x - y$ and $f_9 = y^3 - xy - 2y$, giving the set

$$G = \{f_1, f_2, \cdots, f_9\}.$$

One can easily check that S-polynomial (f_i, f_j) can be reduced to zero for all $1 \leq i, j \leq 9$, and hence G is a Gröbner base.

If we have a set of polynomials, how can we determine whenever it is a Gröbner base for some left ideal I or not? The following results will give us an alternative answers to this question.

Definition 18. Let I be a left ideal over $K[X]$ then the initial of the left ideal I is defined by

$$In(I) = \{LM(f), \text{for all } f \in I\}.$$

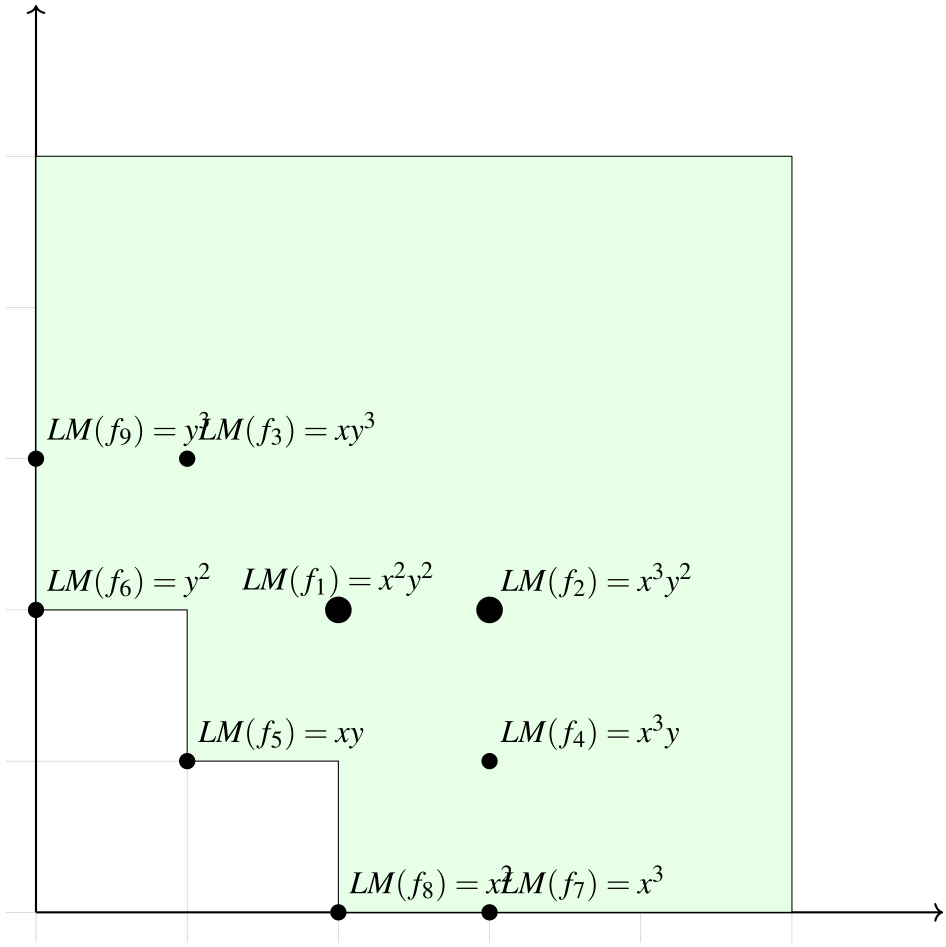
The following definition give us another way to test the set of polynomial is Gröbner basis or not and it is equivalent to Definition 17. See Lovett (2015) Theorem (12.5.14)

Proposition 8. *A set $G = \{g_1, g_2, \cdots, g_s\} \in K[x_1, x_2, \ldots, x_n]$ is a Gröbner basis for left ideal $I = \langle f_1, f_2, \cdots, f_m \rangle$, if the initial left ideal of I, $In(I)$, is the same left ideal as that generated by the leading monomials of $g_i, 1 \leq i \leq s$ (Coutinho (1995))*

$$In(I) = \langle LM(g_1), LM(g_2), \cdots, LM(g_s) \rangle.$$

For example, let $I =< f_1, f_2 >$, where $f_1 = x^2y - 2y^2 + x, f_2 = x^3 - 2xy$, then the set $G = \{f_1, f_2\}$ is not Gröbner bases for I. $x^2 = x.f_1 - y.f_2$, then $x^2 \in I$ and $LT(x^2) = x^2$. Then $x^2 \in In(I)$. But $x^2 \notin \langle LM(f_1), LM(f_2) \rangle$.

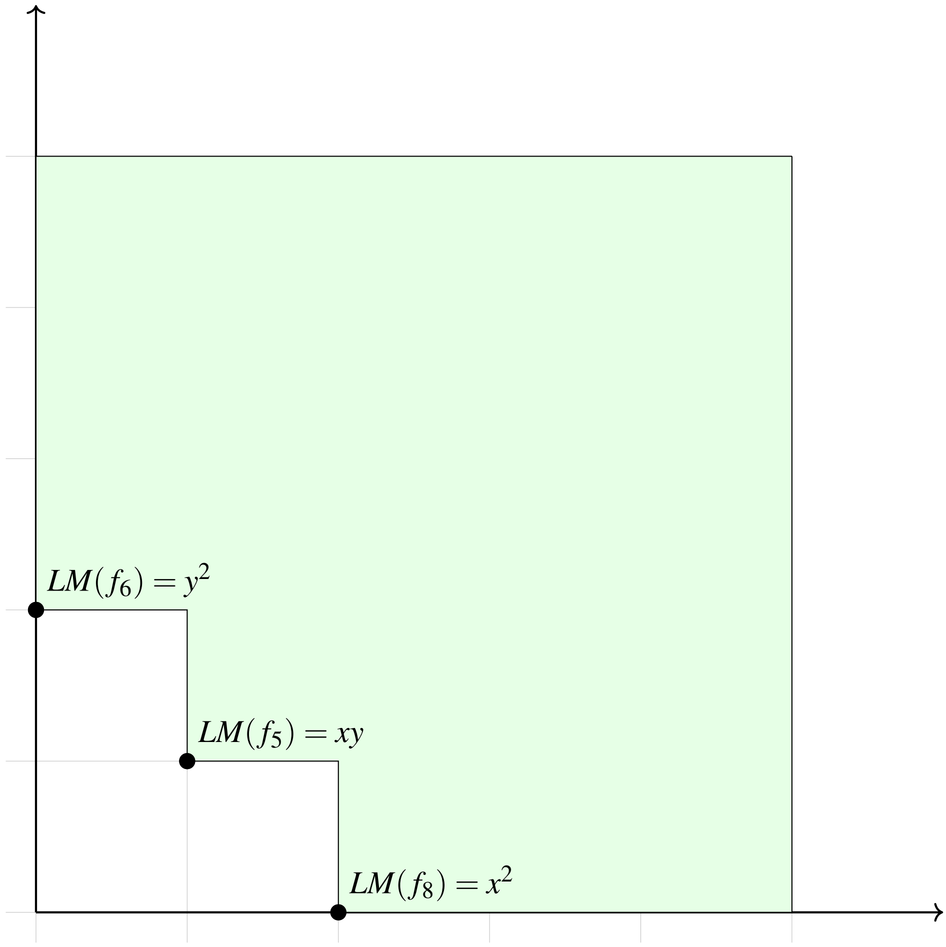
If we return to the Example 2, one can easily check that the G is Gröbner bases by using Proposition 8. This is also clear by looking at the following figure.



The Definition 17 and Example 2, we can see that in the figure above some questions. Is the Gröbner basis calculated in Example 2 the minimal Gröbner bases? Is there another Gröbner bases for the same left ideal? What is the minimal and unique basis for a left ideal I?

In Example 2, we can see that the $LM(f_6)$ can generate $LM(f_9), LM(f_3), LM(f_1), LM(f_2)$. While $LM(f_5)$ can generate $LM(f_4)$ and $LM(f_8)$ can generate $LM(f_7)$ then the set $G^* = \{f_6, f_5, f_8\}$ is a Gröbner basis for the left ideal I as well. The left ideal I contains all monomial in $\mathbb{C}[x,y]$ except $1, x, y$, as we can see in that in the figure.

If $G = \{g_1, g_2, \cdots, g_r\}$ is a Gröbner basis for a left ideal I. The above processes has shown that we can eliminate some of g_i if we find the leading monomial of g_i that a be generated by leading monomial of g_j for some $i \neq j$. The new Gröbner bases become $G^* = G - \{g_i\} : LM(g_i) = nLM(g_j), n$ is a monomial, $i \neq j$. That is the minimal Gröbner basis.



Definition 19. The set $G = \{g_1, g_2, \cdots, g_r\} \subset K[x_1, x_2, \cdots, x_n]$ is a reduced to Gröbner bases for a left ideal I of $K[x_1, x_2, \cdots, x_n]$ if :

1. $LC(g_i) = 1$ for all $1 \le i \le r$.

2. No term of g_i can be divisible by $LT(g_j)$ for all $1 \le i, j \le r$ and $i \ne j$

The next properties are very useful to show the existence and uniqueness of the Gröbner bases. Lovett (2015)

Proposition 9. *Let I be a non-zero left ideal over $K[x_1, x_2, \cdots, x_n]$ then there exists a unique reduced Gröbner bases of I.*

Proposition 10. *Any two left ideals in $K[x_1, x_2, \cdots, x_n]$ are equal if and only if they have the same reduced Gröbner bases.*

One of the main applications for Gröbner bases is to solve systems of polynomial equations. Let $f_1 = 0, f_2 = 0, \cdots, f_m = 0$, be a system of m polynomial equations with n variables, then the solution for this system is the same solution for the system $g_1 = 0, g_2 = 0, \cdots, g_r = 0$, where $G = \{g_1, g_2, \cdots, g_r\}$ is the Gröbner bases of the left ideal generated by $f_1, f_2, \cdots, f_m$. For example , is it difficult to solve the system:

$$\begin{aligned} x^2 + y^2 &= 9 \\ 25x^2 + 4y^2 &= 100 \end{aligned}$$

But, when calculating the Gröbner bases G for the left ideal $I = \langle x^2 + y^2 - 9, 25x^2 + 4y^2 - 100 \rangle$, we get $G = \{221x^2 - 864, 221y^2 - 125\}$.

2.4.1 Gröbner bases for the Weyl algebra

Let $L = \langle d_1, d_2, \cdots, d_s \rangle$ be a finitely generated left left ideal over the $n-th$ Weyl algebra, A_n. Then the subset $G = \{g_1, g_2, \cdots, g_r\}$ of L is a left Gröbner basis for L if the left initial left ideal, $In(L)$, is equal to the left ideal generated by the $LM(g_i)$ for $i = 1, 2, \cdots, r$.

We can use the same constructions used in the commutative algebra case to calculate the left Gröbner basis, but we must take into account that the ring is non-commutative.

Example 3. *Consider the left ideal $I = \langle x\partial_y^3 - y\partial_x^2, x\partial_x^2 + y\partial_y \rangle$ over A_2. Then the set $G = \langle \partial_x^2, \partial_y \rangle$ is the left Gröbner basis for I. This follows from the fact that every element of I can be written as a left linear combination of the elements of G:*

$$\begin{aligned} x\partial_y^3 + y\partial_x^2 &= x\partial_y^2(\partial_y) - y(\partial_x^2), \\ x\partial_x^2 + y\partial_y &= x(\partial_x^2) + y(\partial_y). \end{aligned}$$

Example 4. *consider the left ideal $J = \langle x^3\partial_y^2\partial_z - \partial_x^2\partial_y^2\partial_z - 3x^2y\partial_z, x\partial_x\partial_z + y\partial_y^2 \rangle$. The left Gröbner basis for J is the set*

$G = \left\langle \partial_z^2, x^2\partial_z, x\partial_x\partial_z + y\partial_y^2, xy\partial_y^2 - 2x\partial_z, y\partial_y^4 + 2\partial_y^3, \partial_y^2\partial_z \right\rangle$ *as we can write every element of J as a left linear combination of the leading monomials of the elements of G.*

Chapter 3

Dimension and Multiplicity

3.1 Graded and filtered module

In this section, we will explain grading and filtration for a K-algebra and especially for the Weyl algebra and give some of their properties and examples without assuming commutativity.

Definition 20. For any K-algebra R, R is graded if there exists K-vector subspaces R_i, $i \in \mathbb{N}$, such that

1. $R = \bigoplus_{i\in\mathbb{Z}} R_i$.
2. $R_i \cdot R_j \subseteq R_{i+j}$.

The K-vector subspace R_i is called a homogeneous component of R. Any element $u \in R_i$ called the homogeneous element of degree i. The graded algebra is called positive if $R_i = 0$, when $i < 0$.

A very important example of graded algebra is the polynomial ring $K[x_1, x_2, \cdots, x_n]$. Let R_i be an abelian group containing all monomials on the form $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, where $k_1 + k_2 + \cdots + k_n = i$. If $f \in K[x_1, x_2, \cdots, x_n]$, then f can be written as a linear combination of a finite set of monomials. Therefore, $f \in \bigoplus_{i\in\mathbb{N}} R_i$. Also, for every two monomials, n and m, with degrees i and j respectively, we get $deg(m \cdot n) = i + j$. Hence the two conditions on Definition 20 are satisfied Therefore, $K[x_1, x_2, \cdots, x_n]$ is a graded algebra.

Let R be a graded algebra. A two-side ideal I is called a graded ideal if $I = \bigoplus_{i\geqslant 0}(I \cap R_i)$. Equivalently, the two-sided ideal is graded if and only if it is generated by homogeneous elements. See Mialebama Bouesso & Sow (2015).

Now, let $R = \bigoplus_{i\geqslant 0} R_i$ and $S = \bigoplus_{i\geqslant 0} S_i$ be two graded algebras. A homomorphism of K-algebras $\phi : R \to S$ is graded homomorphism if $\phi(R_i) \subseteq S_i$.

The next proposition will tell us the kernel of graded homomorphism is two-side ideal.

Proposition 11. *Let $\phi : R = \bigoplus_{i\geqslant 0} R_i, S = \bigoplus_{i\geqslant 0} S_i$ be graded K-algebras, then*

1. *The kernel of a graded homomorphism $\phi : R \to S$ is a graded two-side ideal over R.*
2. *For any two-side ideal I of R, the quotient R/I is graded K-algebra.*

Proof. Let $a \in ker(\phi)$, then

$$0 = \phi(a) = \phi(a_1 \oplus a_2 \oplus \cdots \oplus a_s) = \phi(a_1) + \phi(a_2) + \cdots + \phi(a_s),$$

where $a_i \in R_i$, for every $i = 1, 2, \cdots, s$. Since ϕ is graded and the sum is direct, we must have $a_i \in ker(\phi)$, for every $i = 1, 2, \cdots, s$. Thus (1) is proved.

Now, to prove (2), let $I = ker(\phi)$ be a graded two-side ideal over R, we claim:

$$R/I = \bigoplus_{i\geq 0} (R_i/(I \cap R_i))$$

If $a_i \in R_i$ and $a_j \in R_j$, then $(a_i + I)(a_j + I) = a_j a_j + I$ corresponds to the element of $\in (R_{i+j}/(I \cap R_{i+j})$ under this isomophism. Thus (2) is proved as well.

$\square$

Let $g_1, g_2, \cdots, g_r$ be a homogeneous polynomial over $K[X]$. Proposition 11 give us a ways to generate a new graded ring, which is $K[X]/\langle g_1, g_2, \cdots, g_r \rangle$.

We can define a special type of module with respect to the graded algebra.

Definition 21. Let $R = \bigoplus_{i \geq o} R_i$ be a graded algebra, and M be a left R-module. Then M is called graded module if there exists K-vector spaces M_i, for $i \geq 0$, satisfying :

1. $M = \bigoplus_{i \geq 0} M_i$.

2. $R_i \cdot M_j \subseteq M_{i+j}$.

As before, we say that M_i is a homogeneous component for all i, and any element of m_i is a homogeneous element of degree i. It is important to note that the graded module depends on the graded algebra. That means the definition of the graded module depends of the graded structure chosen for the algebra R.
The same processes that were used to define the graded ideal and graded algebra homomorphism, can be used to define a graded submodule and a graded module homomorphism.

Definition 22. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring, and $M = \bigoplus_{i \geq 0} M_i$ and $M' = \bigoplus_{i \geq 0} M'_i$ are graded left R-modules. If a submodule N of M can be written as $N = \bigoplus_{i \geq 0} (N \cap M_i)$, then N becomes a graded submodule. A R-module homomorphism $\phi : M \to M'$ is a graded homomorphism if it satisfies $\phi(M_i) \subseteq M'_i$.

We can generalize Proposition 11 to show that the kernel, $ker(\phi)$, is a graded submodule of M and M/N is graded left R-module.

For example, let $R = \bigoplus_{i \geq 0} R_i$ be a graded algebra and let $R^n = R \oplus R \oplus \cdots \oplus R$ (n-copies) be a free left R- module of rank n. This free left R- module has a natural grading, where $R^n = \bigoplus_{i \geq 0} S_i$, such that:

$$S_j = \sum_{i_1 + i_2 + \cdots + i_n = j} (R_{i_1} \oplus R_{i_2} \oplus \cdots \oplus R_{i_n}).$$

For another example, let $F = K[x,y]$ be a K- algebra with two variables. It is easy to check that $F = \oplus_{i\geq 0} F_i$, where F_i is a K-vector space generated by $x^a y^b$ with $a+b=i$. Thus F is graded K-algebra. Now, let I be the ideal generated by $(xy-\lambda yx)$, where $\lambda \in K$, then I is a two-side ideal and hence F/I is graded K- algebra by Proposition 11.

We want to make the Weyl algebra A_n into a graded algebra by using the degree homogeneous component, but we can not. The element $\partial_1 x_1$ is a homogeneous component of degree 2, but it is equal to $x_1\partial_1 + 1$ and which is not homogeneous. To solve this problem, we must generalize the concept of graded rings.

Definition 23. A family of K- vector spaces $\mathscr{F} = \{F_i\}_{i\geq 0}$ is a filtration of a K-algebra R if

1. $F_0 \subseteq F_1 \subseteq \cdots \subseteq R$.
2. $R = \cup_{i\geq 0} F_i$.
3. $F_i \cdot F_j \subseteq F_{i+j}$.

A filtered algebra is an algebra which has a filtration. Is a graded algebra a filtered algebra? What about the converse?

Proposition 12. *Every Graded algebra is filtered algebra.*

Proof. Let $R = \oplus_{i\geq 0} R_i$ be graded algebra. Take $F_j = \bigoplus_0^j R_i$, then $F_i \subseteq R$ for all i then

$$F_j = \bigoplus_0^j R_i \subseteq \bigoplus_0^j R_i \bigoplus R_{j+1} = F_{j+1},$$

and $\cup_{i\geq 0} F_i = \bigoplus_{i\geq o} R_i = R$. Since $R_i \cdot R_j \subseteq R_{i+j}$, then

$$F_{i+j} = \bigoplus_0^{i+j} R_i \supseteq \bigoplus_{k+m\leq i+j} R_i \cdot R_j = F_i \cdot F_j.$$

Hence $\{F_i\}_{i\geq 0}$ is a filtration of R. $\square$

Conversely, we know that the Weyl algebra A_n is not graded algebra as we explained above, but it has many types of filtration.

The first filtration for A_n we describe is the *Bernstein filtration*, $\mathscr{B} = \{B_k\}_{k\geq 0}$, where B_k is the set of all operators in A_n of degree $\leq k$, where the monomial $x^\alpha \partial^\beta$ has degree $|\alpha|+|\beta|$. We can also consider the B_k as finitely dimensional vector space whose basis is all monomials $x^\alpha \partial^\beta$ with $|\alpha|+|\beta| \leq k$. In particular, $B_0 = K$ and the basis of B_1 is $\{1, x_1, x_2, \cdots, x_n, \partial_1, \partial_2, \cdots, \partial_n\}$.
The second filtration is *order filtration*, denoted by $\mathscr{C}$, which depends on the order of operators. $\mathscr{C} = \{C_k\}_{k\geq 0}$, where C_k is the set of all operators of A_n with order $\leq k$. For more details of these two filtration see Coutinho (1995).

Now, we want to construct a graded algebra associated to a filtration. Suppose that $\mathscr{F} = \{F_i\}_{i\geq 0}$ is a filtration of the K- algebra R. Firstly, we introduce a simple map of order k, which is the vector space canonical projection,

$$\sigma_j : F_j \to F_j/F_{j-1},$$

defined by

$$\sigma_k(a) = \begin{cases} a + F_{k-1} & \text{if } a \notin F_{k-1} \\ F_{k-1} & \text{if } a \in F_{k-1} \end{cases}.$$

Secondly, consider the K-vector space

$$gr^{\mathscr{F}} R = \bigoplus_{i\geq 0} (F_i/F_{i-1}).$$

We want to make the vector space above a graded algebra. Before we prove that $(F_i/F_{i-1}) \cdot (F_j/F_{j-1}) \subseteq (F_{i+j}/F_{i+j-1})$. we must define the product of two homogeneous elements. Let $\sigma_i(a) \in F_i/F_{i-1}$ and $\sigma_j(b) \in F_j/F_{j-1}$, where $a \in F_i$ and $b \in F_j$. Then their product is

$$\sigma_i(a)\sigma_j(b) = \sigma_{i+j}(ab).$$

Since $ab \in F_{i+j}$, then $\sigma_{i+j}(ab) \in F_{i+j}/F_{i+j-1}$. Thus $gr^{\mathscr{F}}R$ is a graded algebra called *the graded algebra of R associated with the filtration* $\mathscr{F}$.

Now, we will apply the above construction for the Weyl algebra A_n. Let $\mathscr{B}$ be a Bernstein filtration for A_n, then B_i/B_{i-1} is isomorphic to vector space of all operators of A_n with degree i. then $S_n = gr^{\mathscr{B}}A_n = \bigoplus_{i\geq 0} B_i/B_{i-1}$ is the graded algebra of A_n associated with the Bernstein filtration $\mathscr{B}$.

The next theorem tells us what the graded algebra S_n looks like. See Coutinho (1995).

Theorem 4. *The graded algebra S_n is isomorphic to the polynomial ring $K[y_1, y_2, \cdots y_{2n}]$, where $y_i = \sigma_1(x_i)$ and $y_{i+n} = \sigma_1(\partial_i)$, for $i = 1, 2, \ldots, n$.*

We now know what is the graded algebra for A_n. We need also to define the filtration for a module over A_n

Definition 24. Let M be a left module over A_n . A family $\{\Gamma_i\}_{i\geq 0}$ of a K-vector spaces of M is a filtration if it satisfies:

1. $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq M$,

2. $M = \cup_{i\geq 0}\Gamma_i$,

3. $B_i\Gamma_j \subseteq \Gamma_{i+j}$,

4. For every i, Γ_i is a finite dimensional K-vector space.

For example, A_n is a filtered A_n-module using the Bernstein filtration. For another example, consider $\Gamma = \{\Gamma_i\}_{i\geq 0}$, where Γ_i is the vector space of all polynomial of $K[X]$

with degree $\leq i$. Thus Γ is a filtration of $K[X]$ module over A_n with Bernstein filtration. Now, we have defined a filtration for the A_n- module M, we can define the graded module associated with a filtered A_n-module M, as we did in the case of algebras. We must first define the symbol map of order k, which is the canonical projection:

$$\mu_k : \Gamma_k \to \Gamma_k/\Gamma_{k-1}.$$

We then put

$$gr^{\Gamma}M = \bigoplus_{i\geq 0}(\Gamma_i/\Gamma_{i-1}).$$

If we want to prove that $(B_i/B_{i-1})(\Gamma_j/\Gamma_{j-1}) \subseteq \Gamma_{i+j}/\Gamma_{i+j-1}$, we must define the action between elements of S_n and elements of $gr^{\Gamma}M$. Let $a \in B_i$ and $b \in \Gamma_j$, then $\sigma_i(a) \in B_i/B_{i-1}$ and $\mu_j(b) \in \Gamma_i/\Gamma_{i-1}$. Thus

$$\sigma_i(a)\mu_j(b) = \mu_{i+j}(ab),$$

and hence $gr^{\Gamma}M$ is a graded S_n-module, called the graded module associated to the filtration Γ.

Let Γ be a filtration of $K[X]$ as a module over A_n with respect to the Bernstein filtration, $\mathscr{B}$. Then Γ_i/Γ_{i-1} is isomorphic to the vector space contains all polynomials of degree i. Thus the vector space $gr^{\Gamma}M$ is isomorphic to $K[X]$. However, by Theorem 4, S_n is isomorphic to the polynomial ring $K[y_1, y_2, \cdots, y_{2n}]$.

We must define the action between the y_i and a homogeneous polynomial f of degree k which is also to be thought of as an element of Γ_k/Γ_{k-1}. For $i = 1, 2, \cdots, n$, then $y_i f = x_i f$. For $i = n+1, n+2, \cdots, 2n$, since $\partial_i(f)$ is a homogeneous element of degree $\leq k-1$, then $\mu_k(\partial_i(f)) = 0$. Thus $y_i f = 0$. In other words, $ann_{s_n}(gr^{\Gamma}M)$ is the ideal generated by $y_{n+1}, y_{n+2}, \cdots, y_{2n}$.

As we did with the graded algebra, we want to define the induced filtration for the submodule N of a left A_n-module M. Suppose that Γ is the filtration for M with respect to $\mathscr{B}$. The filtration of N is $\Gamma' = \{\Gamma_i \cap N\}_{i \geq 0}$. The symbol maps of degree k are the injective linear maps

$$\phi_k : \Gamma_k \cap N / \Gamma_{k-1} \cap N \to \Gamma_k / \Gamma_{k-1}.$$

We can put these maps together to get a new map,

$$\phi : gr^{\Gamma'} N \to gr^{\Gamma} M,$$

making ϕ a module homomorphism. Since for every k, ϕ_k is injective, then so is ϕ. By the same way, let $\Gamma'' = \{\Gamma_i / \Gamma_i \cap N\}_{i \geq 0}$ is a filtration for the quotient module M/N. Note that

$$\Gamma''_k / \Gamma''_k \cong \Gamma_k / (\Gamma_{k-1} + \Gamma_k \cap N).$$

Then we can define the following canonical projection :

$$\pi_k : \Gamma_k / \Gamma_{k-1} \to \Gamma''_k / \Gamma''_{k-1}.$$

Again, when we putting these together, we get the K-linear map,

$$\pi : gr^{\Gamma} M \to gr^{\Gamma''} M/N,$$

making π a surjective homomorphism of S_n-modules. the following lemma gives us an important property for the above construtions. See Coutinho (1995).

Lemma 4. *Let M and N be as defined above. Then the sequence of S_N-modules,*

$$0 \to gr^{\Gamma'} N \xrightarrow{\phi} gr^{\Gamma} M \xrightarrow{\pi} gr^{\Gamma''} M/N \to 0,$$

is exact.

Let us apply the previous lemma. Consider A_n as an A_n-module with Bernstein filtration $\mathscr{B}$, and let $N = A_n d$, where d be an operator d in A_n of degree r. The induced filtration for N is $\mathscr{B}^{'} = \{\mathscr{B}_i\}_{i \geq 0}$, where $\mathscr{B}'_i = B_i \cap A_n d = B_{i-r} d$. Thus

$$\mathscr{B}'_k / \mathscr{B}'_{k-1} = B_{k-r} d / B_{k-r-1} d \cong (B_{k-r} / B_{k-r-1}) \sigma_r(d)$$

It is clear B_k / B_{k-1} is a homogeneous element of degree k of S_n, thus,

$$gr^{\mathscr{B}'}(A_n d) \cong S_n \sigma_r(d).$$

Therefore, applying Lemma 4, we get the following exact sequence

$$0 \to S_n \sigma_r(d) \to S_n \to gr^{\mathscr{B}''}(A_n / A_n d) \to 0,$$

where $\mathscr{B}''$ is the filtration for $(A_n / A_n d)$. From the above exact sequence we get $gr^{\mathscr{B}''}(A_n / A_n d) \cong S_n / S_n \sigma_r(d)$.

Before we define a good filtration for a left A_n-module, M. We need to define a Noetherian module and Noetherian ring. A left R-module M is Noetherian if all its submodules are finitely generated. A ring R is a left Noetherian ring if it is a Noetherian as a left R-module. The ring $K[X]$ is a clear example of a noetheian ring. the following theorems and properties are very useful. For more details see Cohn (1974) and Coutinho (1995).

Theorem 5. *Let M be a left R-module. The following are equivalent:*

1. *M is Noetherian.*
2. *For every infinite ascending chain $N_1 \subseteq N_2 \subseteq \cdots$ of a submodules of M, there exists $k \geq 0$ such that $N_i = N_k$ for every $i \geq k$ (the Ascending Chain Condition (ACC)).*
3. *every set of submodules of M has a maximal element.*

Proposition 13. *Let N be a submodule of a left R-module M. Then M is Noetherian if and only if N and M/N are Noetherian.*

Theorem 6. *Let R be a Noetherian commutative ring. Then the polynomial ring $R[X]$ is Noetherian.*

By using Theorem 6, and the fact $K[x_1, x_2, \cdots, x_n] = K[x_1, x_2, \cdots, x_{n-1}][x_n]$, by induction we get the following corollary.

Corollary 6.1. *For every field K, the polynomial ring $K[x_1, x_2, \cdots, x_n]$ is Noetherian.*

Theorem 7. *Let R be a left Noetherian ring. Then every finitely generated left R-module is Noetherian.*

The following theorems are an application of Noetherian modules to the Weyl algebra.

Theorem 8. *Let Γ be a filtration for a left A_n-module M with respect to $\mathscr{B}$. If $gr^{\Gamma}M$ is a Noetherian S_n-module, then M is also Noetherian.*

Corollary 8.1. *A_n is a left Noetherian ring.*

Definition 25. Let Γ be a filtration for a left A_n-module M with respect to Bernstein filtration $\mathscr{B}$. If $gr^{\Gamma}M$ is finitely generated S_n-module, then Γ is good filtration.

the following properties give us a way to know whenever the filtration is good or not?

Proposition 14. *Let Γ be a filtration for a left A_n-module M with respect to $\mathscr{B}$. Then Γ is good filtration if and only if there exists k_0 such that $B_i\Gamma_k = \Gamma_{i+k}$, for all $k > k_0$.*

For example, let M be a finitely generated left A_n-module generated by $u_1, u_2, \cdots, u_r$, and let $\Gamma_k = \sum_{i=1}^{s} B_k u_i$. It is easy to check that $\Gamma = \{\Gamma_i\}_{i\geq 0}$ is a good filtration for M.

3.2 Hilbert function and Hilbert polynomial

Definition 26. Let $M = \bigoplus_{i\geq 0} M_i$ be a graded $K[x_1, x_2, \cdots, x_n]$-module. The *Hilbert function $HF : \mathbb{N} \to \mathbb{N}$* is defined by

$$HF(n) = dim_k(M_n),$$

for all $n \in \mathbb{N}$. The *Hilbert polynomial* is a polynomial which gives the same value as the Hilbert function for $n >> 0$

Let $I = \langle g_1, g_2. \cdots, g_r\rangle$ be an ideal over $K[x_1, x_2, \cdots, x_n]$, and let $G = \{f_1, f_2, \cdots, f_s\}$ be a Gröbner basis for the generator set I. Let I_m is a vector space generated by all monomial of degree m which divisible by $m_i = LM(f_i)$ for some $i = 1, 2, \cdots, s$. One can easily check that $I = \bigoplus_{j\geq 0} I_j$ be a graded $K[x_1, x_2, \cdots, x_n]$-module. Then the Hilbert function $HF(d)$ is equal to the number of monomials of degree d which are divisible by $m_i = LM(f_i)$ for some $i = 1, 2, \cdots, s$. The Hilbert polynomial for d is equal to the number of all monomials of degree $\leq d$ which can be divisible by m_i for some i.

The next property gives us the way to calculate the Hilbert polynomial for quotient rings or modules. See Carrell (2005)

Proposition 15. *Let V be a finite dimensional K-vector space and W be a subspace of*

V , then V/W is also K-vector space and

$$dim_k V = dim_k W + dim_k (V/W).$$

Example 5. *Let I be an ideal over $\mathbb{C}[x,y]$ generated by $x^3y^2 - 3xy + 5$, with graded lexicographic term ordering $x >_{glex} y$. The Gröbner basis of I is $\{x^3y^2 - 3xy + 5\}$. We want to calculate the Hilbert function and Hilbert polynomial.*
Let Γ be a filtration for $\mathbb{C}[x,y]$, where Γ_i contains all monomials in the form $x^\alpha y^\beta$, with $\alpha + \beta = i$. Then the filtration for the ideal I is $\Gamma' = \{\Gamma_i \cap I\}_{i \geq 0}$.
The table below gives us the dimension of Γ_i which gives the Hilbert function for each i.

i	$dim_{\mathbb{C}}(\Gamma'_i)$
0	0
1	0
2	0
3	0
4	0
5	1
6	2
7	3
8	4
9	5
10	6

Since there is no monomial of degree ≤ 4 that can be divisible by the leading monomial of the polynomial which generates I, then $dim_{\mathbb{C}}\Gamma_i = 0$, for $i = 1, 2, \cdots, 4$. When $j \geq 5$, then $dim_{\mathbb{C}}\Gamma_j = j + 1 - 5$, where $j + 1$ is the number of monomial of degree j. Thus the

Hilbert function $HF(n) = n-4$, where $n \geq 5$. For the Hilbert polynomial we have,

$$HP(j) = \sum_{i=1}^{j-4}(i) = \frac{j^2-7j+12}{2}$$

.

By using Proposition 15 and since the Hilbert polynomial of the ring R with degree filtration is $\binom{t+2}{t} = \frac{1}{2}t^2 + \frac{3}{2}t + 1$. We can show that the Hilbert polynomial for R/I is $5t-5$.

In general, let I be an ideal over $K[x,y]$ and let the Gröbner basis of I is $G = \{g\}$, where the degree of g is s. Then for $n \geq s$, we have

$$HF(n) = n+1-s,$$

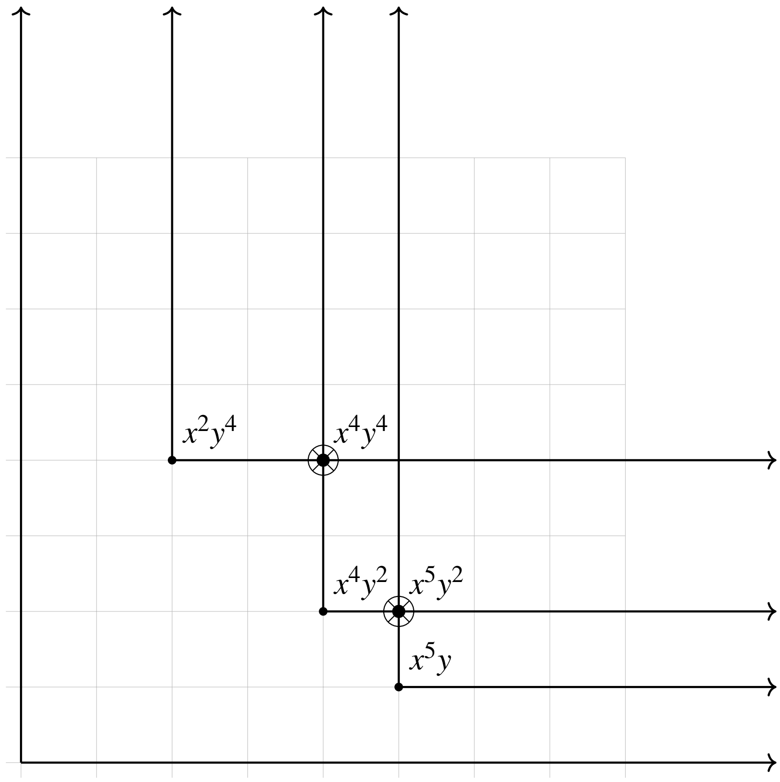
$$HP = \binom{n+1-s}{2}.$$

Example 6. *Let $J = \langle x^2y^4+y^3, x^5y+x^3, x^2y-x^6y^3 \rangle$ be an ideal over $\mathbb{C}[x,y]$, with graded lexicographic term ordering $x >_{glex} y$. The Gröbner basis of J is $G = \{x^2y^4+y^3, x^4y^2+x^2y, x^5y+x^3\}$. Let Γ be the degree filtration for $\mathbb{C}[x,y]$ and Γ' the induced filtration for I. Since the minimum degree for the polynomials of the Gröbner basis for J is 6, then $dim_{\mathbb{C}}\Gamma'_1 = dim_{\mathbb{C}}\Gamma'_2 = \cdots = dim_{\mathbb{C}}\Gamma'_5 = 0$. For $i \geq 6$, the following table gives the results.*

i	$dim_{\mathbb{C}}(\Gamma'_i)$
6	3
7	5
8	6
9	7
10	8

It is easy to show that, for $n \geq 7$,

$$HF(n) \quad = \quad n-2.$$



From the figure above we can calculate the Hilbert polynomial for the ideal I and it is equal to:

$$HP(n) \quad = \quad 3\binom{n-4}{2} - \binom{n-5}{2} - \binom{n-6}{2} = \frac{n^2-3n-12}{2}.$$

Also, the Hilbert polynomial of the ring R/I is $3t+7$.

Example 7. *Let $I = \langle x^4z^2 + x^4y + x^2y^3 + x^2y^2, xz^2 + xy \rangle$ be an ideal over $\mathbf{C}[x,y,z]$, with graded lexicographic term ordering $x >_{glex} y >_{glex} z$. The Gröbner basis for I is $G = \{x^2y^3 + x^2y^2, xz^2 + xy\}$ and the Hilbert polynomial of I is given by*

$$\binom{3+d-3}{3} + \binom{3+d-5}{3} - \binom{3+d-7}{3} = \frac{d^3+3d^2-46d+96}{3!}.$$

While the Hilbert polynomial for R/I is $\frac{1}{2}t^2 + \frac{19}{2}t - 15$.

In the next examples we will denoted by $\chi(t,\Gamma,M)$ the Hilbert polynomial of the graded module $gr^{\Gamma}M$ over the polynomial ring S_n.

Example 8. *Consider A_n as a left A_n-module with respect to Bernstein filtration $\mathscr{B}$. We can consider A_n as a graded algebra $A_n = \bigoplus_{i\geq 0} B_i/B_{i-1}$. Thus, by Definition 26,*

$$\chi(t,\mathscr{B},A_n) = \binom{t+2n}{2n}.$$

Another example is the left A_n-module $K[X]$ with the degree filtration, then we can calculate the Hilbert polynomial as

$$\chi(t,\Gamma,K[X]) = \binom{t+n}{n}.$$

Example 9. *Given the ideal I in Example 3. Since the Gröbner basis for I is $G = \langle \partial_x^2, \partial_y \rangle$, then the induced filtration for I is $\mathscr{B}' = \{B_i \cap G\}_{i\geq 0}$. The dimension of K-vector space $\mathscr{B}'$ is given by the table.*

i	$dim_K(\mathscr{B}'_i)$
0	*0*
1	*1*
2	*6*
3	*12*
4	*26*

Thus the Hilbert polynomial of the ideal I is

$$\chi(t,\mathscr{B}',I) = \frac{t^4+10t^3+11t^2+2t+24}{24},$$

$$\chi(t,\mathscr{B}'',(A_2/I)) = t^2+2t+1.$$

Where $\mathscr{B}''$ is the induced filtration for A_2/I with respect to Bernstein filtration.

Example 10. *We return to the left ideal J of A_3 given in Example 4. By the same construction that we used in Example 3, we can show that*

$$\chi(s,\mathscr{B}',J) = \frac{s^6}{720}+\frac{7\,s^5}{240}+\frac{17\,s^4}{144}-\frac{19\,s^3}{48}+\frac{137\,s^2}{360}-\frac{2s}{15},$$

$$\chi(s,\mathscr{B}'',(A_3/J)) = \frac{S^4}{8}+\frac{17\,s^3}{12}+\frac{15\,s^2}{8}+\frac{31\,s}{12}+1.$$

Where $mathcalB''$ and $mathcalB''$ are induced filtration for J and A_2/I with respect to Bernstein filtration.

3.3 Dimension and Multiplicity

Definition 27. Let M be a left A_n-module and Γ is a good filtration of M with respect to Bernstein filtration, and let $\chi(t,\Gamma,M)$ be the Hilbert polynomial of M. The dimension of M , denoted by $d(M)$, is the degree of $\chi(t,\Gamma,M)$, and the multiplicity of M, denoted by $m(M)$ is given by $m(M) = d(M)!/a_M$, where a_M is the leading coefficient of $\chi(t,\Gamma,M)$.

By using this definition we can calculate the dimension and multiplicity for the examples above for which we calculated the Hilbert polynomial. For example, in Example 8 we have $d(A_n) = 2n$ and $m(A_n) = m(K[X]) = 1$ while $d(K[X]) = n$.

In Examples 9 and 10, we can show that $d(I) = 4$, $d(A_2/I) = 2$ and $m(I) = 1$, $m(A_2/I) = 2$. We also have $d(J) = 6$, $d(A_3/J) = 4$ and $m(J) = 1$, $m(A_3/J) = 3$.

3.3.1 Special cases

The Following examples were motivated by trying to find applications of multiplicity of D-module to multiplicity of algebraic curves.

Unfortunately, we could not make substantial progress in this area, but we include the calculations as relevant examples of D-module. The vector field $x\partial_x + (y+x)\partial_y$ has $x=0$ as invariant algebraic curve of multiplicity 2, where as the vector field $x\partial_x + y\partial_y$ has $x=0$ as invariant algebraic curve of multiplicity 1.

Module 1.

Let $M_1 = A_2/A_2\,(x\partial_x + y\partial_y)$ and let N be the left submodule of M_1 generated by $x\partial_x + y\partial_y$ and $\mathscr{B}'$ the induced filtration of N with respect to the Bernstein filtration. The Gröbner basis for N is $\{x\partial_x + y\partial_y\}$, thus the Hilbert polynomial for N is

$$\chi(s,\mathscr{B}',N) = \frac{s^4+2s^3-s^2-2s}{24}.$$

Thus

$$\chi(s,\mathscr{B}'',M_1) = \chi(s,\mathscr{B},(A_2)) - \chi(s,\mathscr{B}',N) = \frac{2s^3+9s^2+13s+6}{6},$$

where $\mathscr{B}''$ is the filtration of M_1 with respect to $\mathscr{B}$. Thus $d(M_1)=3$ and $m(M_1)=2$.

Module 2.

Let $M_2 = A_2/A_2\,(x\partial_x + y\partial_y + x)$ and let L be the left submodule of M_2 generated by $x\partial_x + y\partial_y + x$ and $\mathscr{B}'$ the induced filtration of L with respect to Bernstein filtration. The Gröbner basis for L is $G_L = \langle x, y\partial_y - 1\rangle$, then the Hilbert polynomial for L is $\chi(s,\mathscr{B}',L) = \frac{s^4+10s^3+11s^2+2s}{24}$. Thus

$$\chi(s,\mathscr{B}'',M_2) = \chi(s,\mathscr{B},(A_2)) - \chi(s,\mathscr{B}',L) = s^2+2s+1,$$

where $\mathscr{B}''$ is the filtration of M_2 with respect to $\mathscr{B}$. Then $d(M_2)=2$ and $m(M_2)=2$.

Module 3.

Let $M_3 = A_2[\frac{1}{x}]$ be the module generated by $[x, y, \partial_x, \partial_y, \frac{1}{x}]$. Let $\Gamma = \{\Gamma_i\}_{i\geq 0}$ be the filtration of M_3, where

$$\begin{aligned} \Gamma_0 &= k, \\ \Gamma_i &= \left\langle [\frac{x^a y^b \partial_x^c \partial_y^d}{x^r}] : a+b+c+d+r \leq i \right\rangle. \end{aligned}$$

we will prove that Γ is a good filtration. It is clear that $B_i\Gamma_j \subseteq \Gamma_{i+j}$. Let $\frac{x^a y^b \partial_x^c \partial_y^d}{x^r} \in \Gamma_{i+j}$, then $x^a y^b \partial_x^c \partial_y^d \in B_{i+j-r}$. Since the Bernstein filtration is good, then $B_{i+j-r} = B_i B_{j-r}$, for r or j sufficiently large. Thus, there exists $x^{a_i} y^{b_i} \partial_x^{c_i} \partial_y^{d_i} \in B_i$ and $x^{a_s} y^{b_s} \partial_x^{c_s} \partial_y^{d_s} \in B_{j-r}$ such that $(x^{a_i} y^{b_i} \partial_x^{c_i} \partial_y^{d_i})(x^{a_s} y^{b_s} \partial_x^{c_s} \partial_y^{d_s}) = x^a y^b \partial_x^c \partial_y^d$ and hence

$$(x^{a_i} y^{b_i} \partial_x^{c_i} \partial_y^{d_i}) \cdot \frac{x^{a_s} y^{b_s} \partial_x^{c_s} \partial_y^{d_s}}{x^r} = \frac{x^a y^b \partial_x^c \partial_y^d}{x^r}.$$

Thus $B_i\Gamma_j = \Gamma_{i+j}$.

One can easily see that $dim_K\Gamma_t = \frac{2t^3+9t^2+13t+6}{6}$. Thus

$$\chi(t,\Gamma,M_3) = \frac{t^4+8t^3+23t^2+28t+12}{12}.$$

By using Definition 27 we get $d(M_3) = 4$ and $m(M_3) = 2$.

Module 4.

Let $M_4 = A_2[\frac{1}{x}]/A_2[\frac{1}{x}][x\partial_x + y\partial_y]$ and let $J = A_2[\frac{1}{x}][x\partial_x + y\partial_y]$ be a submodule of M_3, then $M_4 = M_3/J$. Let $\mathscr{C} = \{\mathscr{C}_i\}_{i\geq 0}$ be the induced filtration from Γ in Module 3. The $dim_K\mathscr{C}_i$ are shown in the table below:

i	$dim_K(\mathscr{C}_i)$
0	0
1	0
2	1
3	4
4	10
5	20
6	35

In general, $dim_K(\mathscr{C}_i) = \frac{i^3-i}{6}$, and

$$\chi(t,\mathscr{C},J) = \frac{t^4+2t^3-t^2-2t}{24}.$$

Let $\mathscr{C}'$ be the induced filtration of M_4 with respect to $\mathscr{C}$, then

$$\chi(t,\mathscr{C}',M_4) = \chi(t,\Gamma,M_3) - \chi(t,\mathscr{C},J) = \frac{t^4+14t^3+27t^2+58t+24}{24}.$$

Thus $d(M_4) = 4$ and $m(M_4) = 1$.

3.4 Spaces of 1−forms and 2−froms

Definition 28. A differential 1-form on an open subset of $\mathbb{R}^n$ is an expression $F_1(x_1,x_2,\dots,x_n)\,dx_1 + F_2(x_1,x_2,\dots,x_n)\,dx_2 + \cdots + F_n(x_1,x_2,\dots,x_n)\,dx_n$, where the F_i are $\mathbb{R}$-valued functions on an open set. If $f(x,y,z)$ is a function of three variables, then its differential, df, is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz. \tag{3.1}$$

It is worth pointing out that a differential form is very similar to a vector field. In fact, we can set up a correspondence:

$$\frac{\partial f}{\partial x}\underline{i}+\frac{\partial f}{\partial y}\underline{j}+\frac{\partial f}{\partial z}\underline{k} \qquad \leftrightarrow \qquad \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz, \tag{3.2}$$

where $\underline{i},\underline{j},\underline{k}$ are the standard unit vectors along the x,y,z axes. For example, if $f = x^2yz^3$, then $df = (2xyz^3)\,dx+(x^2z^3)\,dy+(3x^2yz^2)\,dz$, see Morita (2001) and Edwards (2013)

The differential df above corresponds to the vector field $(2xyz^3)\,\underline{i}+(x^2z^3)\,\underline{j}+(3x^2yz^2)\,\underline{k}$ in $\mathbb{C}^3$.

Definition 29. Let u and v are row vectors in $\mathbb{R}^n$, the wedge product of u and v is an $n\times n$ matrix defined as :

$$u\wedge v = \frac{1}{2}(u^{\intercal}v - uv^{\intercal}). \tag{3.3}$$

For example, let $u = (a,b,c)$ and $v = (d,e,f)$, then

$$u\wedge v = \begin{bmatrix} 0 & ae-bd & af-cd \\ -ae+bd & 0 & bf-ce \\ -af+cd & -bf+ce & 0 \end{bmatrix} \tag{3.4}$$

Now, we can define a differential 2-form for two functions $f,g\in\mathbb{C}^n$ by

$$df\wedge dg = \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}dx_i\wedge dx_j, \tag{3.5}$$

where $dx_i\wedge dx_j$ is define in the next proposition.

Proposition 16. *For every k-dimensional space:*

1. $dx_i \wedge dx_i = 0$.

2. $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

Example 11. *Let* $f = x^3 y^2 z + x y^3 z^3$ *and* $g = x^2 y^2 z^2$ *so*

$$\begin{aligned} df &= (3x^2 y^2 z)\, dx + (2x^3 yz)\, dy + (x^3 y^2)\, dz, \\ dg &= (2xy^2 z^2)\, dx + (2x^2 yz^2)\, dy + (2x^2 y^2 z)\, dz, \\ df \wedge dg &= (2x^4 y^3 z^3)\, dx \wedge dy + (4x^4 y^4 z^2)\, dx \wedge dz + (2x^5 y^3 z^2)\, dy \wedge dz. \end{aligned}$$

For more details and properties about differential k-forms, see Sjamaar (2001)and Victor & Peter (2019).

Chapter 4

Bernstein Sato Polynomial

4.1 The Bernstein-Sato Polynomial

Let $f \in K[x_1, x_2, \cdots x_n]$ be a polynomial with coefficients in a field K of characteristic zero. The Bernstein-Sato polynomial of f, also called the b-function of f, is the non-zero monic polynomial $b_f(s)$ of minimal degree among those $b \in K[s]$ such that:

$$D f^{s+1} = b_f(s) f^s, \tag{4.1}$$

for some operator $D \in A_n[s] = K[x_1, x_2, \cdots x_n, s] < \partial_1, \partial_2, \cdots, \partial_n >$.Yano (1978),Coutinho (1995),Walther (2015),Artal Bartolo et al. (2017).

For example, let $f = x^2 + y^2$ be a polynomial under $\mathbb{C}[x, y]$, then

$$(\frac{\partial_x^2}{4} + \frac{\partial_y^2}{4}) f^{s+1} = (s+1)^2 f^s,$$

so, $D = (\frac{\partial_x^2}{4} + \frac{\partial_y^2}{4})$ and $b_f(s) = (s+1)^2$.

For any polynomial f, we know that $b_f(s) \in \mathbb{Q}[s]$. Since $\mathbb{Q}[s]$ is principle ideal domain and $b_f(s)$ is a monic polynomial, then $b_f(s)$ is unique.

Theorem 9 (The existence of Bernstein polynomials)**.** *For any polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$, there exists an operator $D \in A_n[s]$ and a polynomial $b_f(s) \in \mathbb{Q}[s]$, which satisfy 4.1*

The proof of the theorem can be seen in Bernshtein (1971),Bernshtein (1972).

The existence was proved by Bernstein for a function over a polynomial ring and was extend by Bjork (1974) and Björk (1979) to power series rings.

Malgrange (1975), proved that, when f has an isolated critical point at the origin and $f(0) = 0$, the roots of $b_f(s)$ are rational numbers. He also proved that if $\alpha \neq 1$ is a root of the Berntein polynimial, then $e^{-2\pi\sqrt{-1}\alpha}$ is an eigenvalue of the monodromy of the Gauss-manin connection associated to f. These eigenvalues must be roots of the unity by Briskorn's monodromy theorem Brieskorn (1970).

The next theorem is given in Kashiwara (1976/77) and Guimarães & Hefez (2007) and has important consequences for the Bernstein polynomial.

Theorem 10. *Let g be a holomorphic function, then $b_g(s) \neq 0$ and the roots of it are negative rational numbers.*

Proposition 17. *For any holomorphic function f, we see that $b_f(s)$ is always divisible by $(s+1)$.*

4.2 History of Bernstein polynomial

Sato (1990) and Sato & Shintani (1974), introduced this polynomial in a different context with a view of giving fuctional equations relative invariants of prehomogeneous spaces and also for studying the zeta functions associated with them. In Sato et al. (1980), Sato called these polynomials b-functions. As stated above, Bernstein proved the existence of $b_f(s)$, and hence the name Bernstein-Sato polynomial.

Sabbah (1987) and Gyoja Gyoja (1993) proved a generalization of 4.1 where the generator set of ideal are chosen. Lyubeznik (1997), show that the set of Bernstein polynomials for a set of polynomials is finite. Budur et al. (2006*b*) proved generalization of Bernstein's result for any ideal in a polynomial ring. Then he proved that for any polynomials $f_1, f_2, \ldots, f_k$ there exist a polynomial $b \in K[s_1, s_2, \ldots, s_k]$ and operator $D \in A_n[s_1, s_2, \ldots, s_k]$ such that

$$D(f_1^{s_1+1}, f_2^{s_2+1}, \ldots, f_k^{s_k+1}) = b(f_1^{s_1}, f_2^{s_2}, \ldots, f_k^{s_k}). \tag{4.2}$$

Budur et al. (2006*a*), give a combinatorial description of the roots of the Bernstein-Sato polynomial of a monomial ideal using the Newton polyhedron and some semigroups associated to the ideal. Saito (1993) , exhibits a well determined set of spectral numbers such that the symmetric of each element of this set, when 1 is subtracted, is a root of Bernstein polynomial. Hertling & Stahlke (1999), found the set of spectral number containing Sato's but for a particular case of two variable function, f, with finite monodromy and an isolated critical point at the origin, had the same property. Guimarães & Hefez (2007), proved the same property with just the assumption of an isolated critical point without restriction on the number of variables and the monodromy.

4.3 Algorithms to Compute Bernstein Sato Polynomial

Oaku, in Oaku (1997*a*) and Oaku (1997*b*), introduced an algorithm to compute the generalization of b-function of holomorphic system by using the Gröbner basis for the differential operators as in the Weyl algebra. As a special case of his method, he computed the Bernstein polynomial associated with an arbitrary polynomial. His method depends on the homogenization technique his previous work on the V-filtration. He presented two algorithms to compute the Bernstein polynomial. One of them depends on the theory of D-modules, and the other using a technique to compute the Gröbner basis with respect to a specific term ordering. In both of them, he starts a set of polynomials, $f_1, f_2, \cdots, f_k \in K[x_1, x_2, \ldots x_n]$, then computes the left ideal $ann(f_1^s, f_2^s, \ldots, f_k^s)$ under $A_k[s]$. Then he computes the polynomial $b(s)$ which is a monic generator for the ideal $ann(f_1^s, f_2^s, \ldots, f_k^s) \cap K[s]$. As special case, if he starts with one polynomial f he will compute the Bernstein polynomial $b_f(s)$. He also compares these two algorithms and gives two tables to with the corresponding times needed to compute the Bernstein polynomial by using these two algorithms (see Tables 1 and 2 in Oaku (1997*b*).)

To calculate the left ideal, $ann(f^s)$ of $A_n[s]$, for any polynomial f, we will explain three methods all of these methods depends on a new variables t and ∂_t which are defined as follows.

$$t\,p(x,s)\,f^s = p(x,s+1)\,f^{s+1},$$
$$\partial_t\,p(x,s)\,f^s = -s\,p(x,s-1)\,f^{s-1}.$$

The first method is due to Oaku Oaku (1997*b*) and Saito et. al. Saito et al. (2013). They define an ideal I by

$$I = \left\langle \left\{ t - u\,f, \partial_i + u\frac{\partial f}{\partial x_i}\partial t, u\,v - 1 \right\}, i = 1,2,\ldots,n \right\rangle,$$

which is a left ideal of $A_{n+1}[u,v]$, then they prove that $ann(f^s) = I \cap A_{n+1}$.

Briançon et al. (2002), defined a left ideal J of $A_n < \partial t, s >$ where s is a new variable with property $[\partial t, s] = \partial t$, then

$$J = \left\langle \left\{ s + f\partial_t, \partial_i + \frac{\partial f}{\partial x_i}\partial_t \right\}, i = 1,2,\ldots,n \right\rangle,$$

then $ann(f^s) = I \cap A_n[s]$.

Levandovskyy & Martín Morales (2008), present an algorithm to compute the Bernstein polynomial. They start with a polynomial f and consider the left ideal, I_f, over $A_{n+1}[s] = K[x_1,x_2,\ldots,x_n,t,s] < \partial_1,\partial_2,\ldots,\partial_n,\partial t >$, where s is defined by

$[s,t] = -t$ and $[s,\partial t] = \partial t$:

$$I = \left\langle \left\{ t - f, s + t\,\partial_t, \partial_i + \frac{\partial f}{\partial x_i}\partial_t \right\}, i = 1,2,\ldots,n \right\rangle,$$

then they compute the left ideal, $ann(f^s) = I_f \cap A_n[s]$, by computing the left Gröbner basis for the generators of $ann(f^s)$. The purpose of this algorithm is the roots of Bernstein polynomial with its multiplicity, but they compute the left $A_n[s]$ ideal $ann(f^s)$.

We explained three different method to calculate the annihilator of f^s as a left ideal under $A_n[s]$, this ideal can be use to compute the Bernstein polynomial. We will use the first method to calculate the annihilator for two reasons, it is faster than the others and we use the same strategy to compute the Bernstein polynomial and the operators D which is satisfy 4.1 in the next chapter by using our computer program.

Chapter 5

Calculating the Bernstein–Sato Polynomial

5.1 Introduction

In This chapter we describe in detail how we calculate Bernstein polynomials using the Weyl algebra, the most important calculation in this method ,which we used, is the left ideal $ann(f^s)$. So we will describe how to calculate it and what are the algorithms for this.

5.2 Method

Let $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$, be a polynomial with n independent variables. we want to find an operator $D \in A_n[s]$ ans a polynomial $b_f(s) \in \mathbb{C}[s]$ which satisfy

$$D\, f^{s+1} = b_f(s)\, f^s. \tag{5.1}$$

As a first step, we will calculate the annihilator of f^s, $ann(f^s)$, as ideal of $A_{n+1} = K[x_1, x_2, \ldots, x_n, t, \partial_1, \partial_2, \ldots, \partial_n, \partial_t]$, where t and ∂_t are define as follows:

$$\begin{aligned} t\, p(x,s)\, f^s &= p(x, s+1)\, f^{s+1}, \\ \partial_t\, p(x,s)\, f^s &= -s\, p(x, s-1)\, f^{s-1}. \end{aligned}$$

One can easily check that $t\,\partial_t(f^s) = -(s+1)\,f^s$ and $t^2\,\partial_t^2\,f^s = (s+1)\,(s+2)\,f^s$

We define the ideal I as follows

$$I =< zf - t, \partial_1 + z\frac{\partial f}{\partial x_1}\partial_t, \partial_2 + z\frac{\partial f}{\partial x_2}\partial_t, \dots, \partial_n + z\frac{\partial f}{\partial x_n}\partial_t, -zw + 1 >, \qquad (5.2)$$

under the ring $A_{n+1}[z,w]$,where z,w are new variables such that $zw = 1$

By Theorem **2.1** Berkesch & Leykin (2010) and Theorem **5.3.2** Saito et al. (2013), $ann(f^s) = I \cap A_{n+1}$

Let G be the Gröbner basis for I with respect to graded lexicographic order $(\partial x_1 < \partial x_2 < \cdots < \partial_n < \partial_t < x_1 < x_2 < \cdots < x_n < t < [z] < [w])$, and $G_1 = \{g_1, g_2, \dots, g_j\}$ is the set G, but with removed all terms containing z or w and substitute $t^k\,\partial_t^k$ by $(-1)^k\,(s+1)\,(s+2)\,\dots(s+k)$. So G_1 is the Gröbner basis for the generating set of $ann(f^s)$

The next step is to generate a new ideal L which is generated by f and the elements of G_1. Let G_2 be the Gröbner basis for L with respect to lexicographic order $x_1 > x_2 \cdots > x_n < \partial x_1 > \partial_{x2} > \cdots > x_n > s$. By Levandovskyy & Martín Morales (2008)

$$< b_f(s) >=< G_2 \cap \mathbb{K}[s] > . \qquad (5.3)$$

Since the Bernstein polynomial always exists, then $G_2 \cap \mathbb{K}[s] \neq \phi$ and contain one polynomial which is the Bernstein polynomial $b_f(s)$, let us call it g_1

The last step which is the way to calculate the operator D which satisfy 5.1. Let we consider the free $A_{n+1}[s]$- module

$$A_{n+1}[s] \bigoplus A_{n+1}[s] = \{(r_1, r_2) s.t. r_1, r_2 \in A_{n+1}[s]\}.$$

Let $\hat{L} = \langle (f,1), (g_1,0), (g_2,0), \dots, (g_j,0) \rangle$, then we compute the Gröbner basis for $\hat{L}$ componentwise by using the term ordering $<_L$define as follows:

$m_1\,e_i > m_2\,e_j$ if either $j > i$ or $j = i$ and $m_2 > m_1$ with respect to lexicographic term ordering we used it to compute G_2, where $m_1, m_2 \in A_{n+1}[s]$, $e_1 = (1,0)$ and $e_2 = (0,1)$ Let $G_3 = \{(p_1,d_1),(p_2,d_2),\dots,(p_r,d_r)\}$ is the Gröbner basis for $\hat{L}$. Since we compute the Gröbner basis componentwisely, then $p_1 = b_f(s)$

$$\begin{aligned}
(p_1,d_1) &= c_0\,(f,1) + c_1\,(g_1,0) + \cdots + c_j\,(g_j,0) \\
b_f(s) &= p_1 = c_0\,f + c_0\,g_1 + \cdots + c_0\,g_j \\
d_1 &= c_0
\end{aligned}$$

Now, we want to calculate the operator D which is satisfy 5.1,

$$b_f(s)\,f^s = p_1 = c_0\,f^{s+1} + c_0\,g_1\,f^s + \cdots + c_0\,g_j\,f^s = c_0\,f^{s+1},$$

and hence $d_1 = D$

5.3 Examples

we will apply this method for some examples in various cases of one dimensional polynomials, two dimensional polynomials, and three dimensional polynomials.

5.3.1 One Dimensional Cases

Example 12. *Let us start with an easy example, $f = x^2$. Then the left ideal I which is defined in our method is an ideal of the Weyl algebra $A_2[w,z] = k[x,t,w,z] < \partial_x, \partial_t >$, given by*

$$I =< -x^2 z + t, 2zx\,\partial_t + \partial_x, -wz + 1 > .$$

We calculate the Gröbner bases for the generators of I, which we call G, with respect to the graded lexicographic order $(\partial x < \partial_t < x < t < w < z)$, to give

$$G = \Big\{2\,\partial_t t + \partial_x x + 2, x^2 z - t, \partial_t x z + \tfrac{\partial_x}{2}, z\,\partial_t{}^2 t + \tfrac{3}{2} z\,\partial_t - \tfrac{1}{4}\partial_x{}^2, t\,w - x^2$$
$$2\,\partial_t x + \partial_x w, w z - 1\Big\}.$$

We now calculate the set G_1 where is obtained from the set G by removing all element containing z or w and replacing $\partial_t t$ by $-(s-1)$. Thus, in this case, we have

$$G_1 = \{\partial_x x - 2s\},$$

Since Saito et al. (2013) prove that $ann(f^s) = I \cap K_{n+1}$, then $ann(f^s) =< \partial_x x - 2s >$. For the next step, we generate a new ideal, L, of the weyl algebra $A_1[s] = k[x,s] < \partial_x >$. The ideal L is generated by f and the elements of G_1. Thus, $L =< x^2, \partial_x x - 2s >$. To calculate the Gröbner basis for the generators of L, which we call G_2, by using a term ordering $x < \partial_x < [s]$ this term ordering compares the degree of s first, the leading monomial which have a higher degree of s is bigger than the others. And for any leading monomials that not contain s i we use the graded lexicographic order $x < \partial_x$. In this way, we obtain

$$G_2 = \left\{s^2 + \frac{3}{2}s + \frac{1}{2}, s x + x, \partial_x x - 2s, x^2\right\}.$$

One can easily see that there is just one element of G_2 is a polynomial depending on s only which is the Bernstein polynomial $b_f(s) = (s+1)(s+\frac{1}{2})$.
The last step, is to calculate the operator $D \in A_1[s]$ which satisfies 5.1. In order to do this we use the submodule $\hat{L}$, from $A_{n+1}[s] \bigoplus A_{n+1}[s]$ which is defines as a follows $\hat{L} =< (x^2, 1), (\partial_x x - 2s, 0) >$. To calculate the Gröbner basis for $\hat{L}$, which we call G_3, we use the same term ordering $<_L$Which is defined above. This gives

$$G_3 = \left\{(s^2 + \frac{3}{2}s + \frac{1}{2}, \frac{1}{4}\partial_x{}^2), (s x + x - \frac{\partial_x}{2}, 0), (\partial_x x - 2s, 0), (x^2 - 1, 0)\right\}.$$

Again, we can easily identify that the first element of G_3 gives the desired operator equation:

$$\frac{\partial_x^2}{4}(x^2)^{s+1} = (s+1)(s+\frac{1}{2})(x^2)^s.$$

For any polynomial $f = (x+a)^2$, where a is a real number, we will also get the same $b_f(s)$ and the same operator D.

Example 13. *In this example we will discuss the polynomial $f = x^3$. The ideal I is generated by:*

$$I =< -x^3 z + t, 3\,\partial_t x^2 z + \partial_x, -w z + 1 > .$$

As we did in previous Example 12, the Gröbner basis, G, of $ann(f^s)$, with respect to the graded lexicographic order $(\partial x < \partial_t < x < t < w < z)$, is given by

$$\begin{aligned} G \;=\; & \Big\{3\,\partial_t t + \partial_x x + 3, x^3 z - t, \partial_t x^2 z + \tfrac{\partial_x}{3}, z\partial_t{}^2 x t + \tfrac{4}{3} z\partial_t x - \tfrac{1}{9}\,\partial_x{}^2, z\,\partial_t{}^3 t^2 + 4 z\,\partial_t{}^2 t + \\ & \tfrac{1}{27}\,\partial_x{}^3 + \tfrac{20 z\partial_t}{9}, -x^3 + t\,w, 3\,\partial_t x^2 + \partial_x w, w z - 1\Big\}. \end{aligned}$$

After removing every term involving z or w, and substituting $\partial_t t$ by $-s-1$, we get a new set $G_1 = \{\partial_x x - 3\,s\}$, So $ann(f^s) =< \partial_x x - 3\,s >$.

The ideal $L =< x^3, \partial_x x - 3\,s >$ is an ideal under $A_1[s]$. G_2 is the Gröbner basis for L under the same term ordering which we used to calculate G_2 in Example 12, So

$$G_2 = \left\{s^3 + 2\,s^2 + \frac{11}{9}\,s + \frac{2}{9}, s^2 x + \frac{5}{3}\,s x + \frac{2}{3}\,x, \partial_x x - 3\,s, s x^2 + x^2, x^3\right\}.$$

Since $< b_f(s) >= G_2 \cap \mathbb{K}[s]$, then the Bernstein polynomial is $b_f(s) = (s+1)(s+\frac{1}{3})(s+\frac{2}{3})$. To Calculate the Operator D, we look at the submodule $\hat{L} =< (x^3, 1), (\partial_x x - 3\,s, 0 >$. Also we compute Gröbner basis of $\hat{L}$ with respect to the same term ordering we used for Example 12,

$$\begin{aligned} G_3 &= \Big\{(s^3+2s^2+\frac{11}{9}s+\frac{2}{9},\frac{1}{27}\partial_x{}^3),(s^2x+\frac{5}{3}sx+\frac{2}{3}x-\frac{1}{9}\partial_x{}^2,0),(\partial_x x-3s,0),\\ &\quad (sx^2+x^2-\frac{\partial_x}{3},0),(x^3-1,0)\Big\}. \end{aligned}$$

So the operator $D=\frac{\partial_x^3}{3^3}$ satisfies

$$\frac{\partial_x^3}{3^3}(x^3)^{s+1}=(s+1)(s+\frac{1}{3})(s+\frac{2}{3})(x^3)^s.$$

From the previous two examples, one can guess that for a polynomial in the form $f=(x+a)^n$, the Bernstein polynomial should be $b_f(s)=\prod_{i=0}^{n-1}(s+\frac{n-i}{n})$ and the operator $D=\frac{\partial_x^n}{n^n}$. We have checked this until $n=10$, but it are contain a very big equation and maybe confuse the reader. We will give a more general form after the next two examples.

The next two example are for a polynomial that has more than one root.

Example 14. *Let $f=x^2+3x+2$. Then*

$$I=<-\left(x^2+3x+2\right)z+t,\partial_x+z(2x+3)\,\partial_t,-wz+1>,$$

and the Gröbner basis of I after we ignore all elements containing w or z, and replace $\partial_t t$ by $-(s+1)$, will become

$$G_1=\left\{-(2x+3)(s+1)+\partial_x x^2+3\,\partial_x x+2\,\partial_x+2x+3\right\}.$$

and

$$ann(f^s)=<-(2x+3)(s+1)+\partial_x x^2+3\,\partial_x x+2\,\partial_x+2x+3>.$$

By following the same method as before, the ideal $L = < x^2+3x+2, -(2x+3)(s+1)+\partial_x x^2+3\partial_x x+2\partial_x+2x+3 >$, and the Gröbner basis for L is

$$G_2 = \left\{s+1, x^2+3x+2\right\}.$$

Thus, the Bernstein polynomial is $b_f(s) = s+1$. We follow the same method as before and find an operator $D \in A_1[s]$ which satisfies 5.1,

$$(2\partial_x x+3\partial_x-4s-4)(x^2+3x+2)^{s+1} = (s+1)(x^2+3x+2)^s.$$

Example 15. *Let $f = x^5-8x^4+25x^3-38x^2+28x-8 = (x-1)^2(x-2)^3$. Repeating the previous method, we obtain*

$$b_f(s) = (s+1)(s+\frac{1}{2})(s+\frac{1}{3})(s+\frac{2}{3}).$$

The Operator D which satisfies 4.1 can be found in Appendix 1.

After making a lot of calculation for a different polynomials in one variable, we found that the roots of Bernstein polynomial for $f = f_1 f_2 \dots f_k$, where $f_i = (x+a_i)^{n_i}$, and the a_i are real numbers and the n_i natural numbers, is given by

$$b_f(s) = (s+1)(\prod_{i=1}^{k}\prod_{j=1}^{n_i-1}(s+\frac{n_i-j}{n_i})).$$

Àlvarez Montaner et al. (2017)

5.3.2 Two Dimensional Cases

Example 16. *Let $f = x^2+y^2$, then I is an ideal of the Weyl algebra $A_3[w,z] = k[x,y,t,w,z] < \partial_x, \partial_y, \partial_t >$. By Theorem **5.3.2** Saito et al. (2013), it can be* *written as follows:*

$$I = \left\langle -\left(x^2+y^2\right)z+t, 2\partial_t xz+\partial_x, 2\partial_t yz+\partial_y, -wz+1\right\rangle.$$

The Gröbner basis for I with respect to graded lexicographic order ($\partial x < \partial_y < \partial_t < x < y < t < w < z$), after we ignore every element that has w or z and substitute $\partial_t t$ by $-s-1$, becomes:

$$G_1 = \left\{-\partial_x y + x\partial_y, \partial_x x+\partial_y y-2s, -2\,(s+1)\,\partial_y + y\,\partial_x{}^2+\partial_y{}^2 y+2\,\partial_y\right\}.$$

and

$$ann(f^s) = < -\partial_x y + x\partial_y, \partial_x x+\partial_y y-2s, -2\,(s+1)\,\partial_y + y\,\partial_x{}^2+\partial_y{}^2 y+2\,\partial_y > .$$

As in Example 12, the ideal L of $A_2[s]$ is generated by f and the elements of G_1 and gives

$$L = < x^2+y^2, -\partial_x y + x\,\partial_y, \partial_x x+\partial_y y-2s, -2\,(s+1)\,\partial_y + y\,\partial_x{}^2+\partial_y{}^2 y+2\,\partial_y > .$$

The Gröbner basis for L with respect to the term ordering $x < y < \partial_x, \partial_y < [s]$ is

$$G_2 = \left\{s^2+2s+1, ys+y, sx+x, \partial_x y - x\,\partial_y, \partial_x x+\partial_y y-2s, x^2+y^2\right\},$$

giving, in its first entry, the Bernstein polynomial $b_f(s) = (s+1)^2$. The operator D is in the Gröbner basis of the submodule $\hat{L}$,

$$\hat{L} = < (x^2+y^2,1), (-\partial_x y+x\,\partial_y,0), (\partial_x x+\partial_y y-2s,0), (-2\,(s+1)\,\partial_y+y\,\partial_x{}^2+\partial_y{}^2 y+2\,\partial_y, 0) >,$$

and the Gröbner basis for $\hat{L}$ is

$$G_3 = \Big\{(s^2+2s+1, \frac{1}{4}\partial_y{}^2+\tfrac{1}{4}\partial_x{}^2), (ys+y-\tfrac{1}{2}\partial_y, 0), (sx+x-\tfrac{1}{2}\partial_x, 0), (\partial_x y - x\partial_y, 0), (\partial_x x+\partial_y y-2s, 0), (x^2+y^2-1, 0)\Big\}.$$

The operator $D = \frac{1}{4}\partial_y{}^2 + \frac{1}{4}\partial_x{}^2$ satisfies

$$(\frac{1}{4}\partial_y{}^2+\frac{1}{4}\partial_x{}^2)\,(x^2+y^2)^{s+1} = (s+1)^2\,(x^2+y^2)^s.$$

Whereas all roots of the Bernstein polynomial in the one dimensional case were simple, here we can see that this example has a double root.

Example 17. *We now compute the Bernstein polynomial for $f = x^3+x^2y+xy^2+y^3$. The left Ideal I is given by:*

$$I = <-\left(x^3+x^2y+xy^2+y^3\right)z+t, \partial_x+z\left(3x^2+2xy+y^2\right)\partial_t, \partial_y+z\left(x^2+2xy+3y^2\right)\partial_t, -wz+1>.$$

The Gröbner basis for I after removing every element depending on w or z is

$$G = \big\{3\partial_t t+\partial_x x+\partial_y y+3, \partial_t t x+2\partial_t t y-\partial_x y^2+\partial_y x^2+\partial_y xy+\partial_y y^2+x+2y, 3\,\partial_t^2 t^2-2\partial_t\partial_x t y+3\partial_t\partial_y t x+4\partial_t\partial_y t y+\partial_x{}^2y^2-\partial_x\partial_y y^2+\partial_y{}^2xy+\partial_y{}^2y^2+ 8\partial_t t-2\partial_x y+2\partial_y x+4\partial_y y+2, 3\partial_t{}^2\partial_x t^2-9\partial_t{}^2\partial_y t^2-2\partial_t\partial_x{}^2 t y+4\partial_t\partial_x\partial_y t y- 6\partial_t\partial_y{}^2 t y+y^2\partial_x{}^3-\partial_x{}^2\partial_y y^2+\partial_x\partial_y{}^2y^2-\partial_y{}^3y^2+8\partial_x\partial_t t-24\partial_y\partial_t t-2\partial_x{}^2y+ 4\partial_x\partial_y y-6\partial_y{}^2y+2\partial_x-6\partial_y\big\}.$$

Here, the Gröbner basis not only has terms in $\partial_t t$, but also $\partial_t^2 t^2$ as well, So we will substitute $\partial_t t$ by $-s-1$ and $\partial_t^2 t^2$ by $(s+1)(s+2)$ to give:

$$
\begin{aligned}
G_1 \quad = \quad & \big\{\partial_x x + \partial_y y - 3\,s, -(x+2\,y)\,(s+1) - \partial_x y^2 + \partial_y x^2 + \partial_y x y + \partial_y y^2 + x + 2\,y, \\
& 3\,(s+1)\,(s+2) - (-2\,\partial_x y + 3\,\partial_y x + 4\,\partial_y y + 8)\,(s+1) + {\partial_x}^2 y^2 - \partial_x \partial_y y^2 + \\
& {\partial_y}^2 x y + {\partial_y}^2 y^2 - 2\,\partial_x y + 2\,\partial_y x + 4\,\partial_y y + 2, (3\,\partial_x - 9\,\partial_y)\,(s+1)\,(s+2) - \\
& \Big(-2\,{\partial_x}^2 y + 4\,\partial_x \partial_y y - 6\,{\partial_y}^2 y + 8\,\partial_x - 24\,\partial_y\Big)\,(s+1) + y^2 {\partial_x}^3 - {\partial_x}^2 \partial_y y^2 + \\
& \partial_x {\partial_y}^2 y^2 - {\partial_y}^3 y^2 - 2\,{\partial_x}^2 y + 4\,\partial_x \partial_y y - 6\,{\partial_y}^2 y + 2\,\partial_x - 6\,\partial_y \Big\}.
\end{aligned}
$$

We generate the ideal L from G_1 with the element f added. The Bernstein polynomial of f is then the first element of the Gröbner basis of L, which is

$$
\begin{aligned}
G_2 \quad = \quad & \Big\{s^4 + 4\,s^3 + \tfrac{53\,s^2}{9} + \tfrac{34\,s}{9} + \tfrac{8}{9}, y\,s^3 + \tfrac{10}{3}\,y\,s^2 + \tfrac{11}{3}\,s\,y + \tfrac{4}{3}\,y, x\,s^3 + \tfrac{10}{3}\,x\,s^2 + \tfrac{11}{3}\,s\,x + \\
& \tfrac{4}{3}\,x, y\,\partial_x s^2 - \partial_y s^2 x - 4\,\partial_y s^2 y + 2\,\partial_x s\,y - 2\,\partial_y s\,x - 8\,\partial_y s\,y + 6\,s^3 + \partial_x y - \partial_y x - \\
& 4\,\partial_y y + 12\,s^2 + 6\,s, \partial_x x + \partial_y y - 3\,s, y^2 s^2 + \tfrac{7}{3}\,y^2\,s + \tfrac{4}{3}\,y^2, y\,s\,x + 2\,y^2 s + x\,y + \\
& 2\,y^2, s\,x^2 - y^2\,s + x^2 - y^2, y^2 \partial_y s + \partial_y y^2 + 2\,x s^2 + y s^2 + 4\,s\,x + 4\,s\,y + 2\,x + 3\,y, \\
& \partial_x y^2 - \partial_y x^2 - \partial_y x\,y - \partial_y y^2 + s\,x + 2\,s\,y, y^3\,s + y^3, x^3 + x^2\,y + x\,y^2 + y^3\big\}.
\end{aligned}
$$

Thus, the Bernstein polynomial is $b_f(s) = (s+1)^2\,(s+\dfrac{2}{3})\,(s+\dfrac{4}{3})$ and the operator $D \in A_2[s]$ is

$$
\begin{aligned}
D \quad = \quad & -3\,{\partial_x}^4 y + \partial_y {\partial_x}^3 x + 15\,{\partial_x}^3 \partial_y y - 7\,{\partial_y}^2 x {\partial_x}^2 - 17\,{\partial_x}^2 {\partial_y}^2 y + 13\,{\partial_y}^3 \partial_x x + \\
& 11\,\partial_x {\partial_y}^3 y - 3\,{\partial_y}^4 x - 2\,{\partial_y}^4 y + 12\,{\partial_x}^3 s + 6\,{\partial_x}^2 \partial_y s - 24\,\partial_x {\partial_y}^2 s + 18\,s\,{\partial_y}^3 + \\
& 30\,{\partial_x}^3 - 18\,{\partial_x}^2 \partial_y - 18\,\partial_x {\partial_y}^2 + 30\,{\partial_y}^3,
\end{aligned}
$$

which satisfies (5.1) for the polynomial $b_f(s)$ given above.

Example 18. *Using the same methods, we can show that when $f = x^3 + y^2$, we have $b_f(s) = (s+1)\,(s+\dfrac{5}{6})\,(s+\dfrac{7}{6})$ and $D = 54\,{\partial_y}^2 x\,\partial_x + 24\,{\partial_x}^3 + 162\,{\partial_y}^2 s + 243\,{\partial_y}^2$.*

For the next polynomials, the operator D will be very big, so we will write just the

Bernstein polynomial to avoiding inconveniencing the reader.

Example 19. *We now compute the Bernstein polynomial for* $f = x^4 + x^3y + x^2y^2 + xy^3 + y^4$*. Firstly, we have*

$$\begin{aligned} I &= < -\left(x^4+x^3y+x^2y^2+xy^3+y^4\right)z+t, dx+z\left(4x^3+3x^2y+2xy^2+y^3\right)dt, \\ &\quad dy+z\left(x^3+2x^2y+3xy^2+4y^3\right)dt, -wz+1 > . \end{aligned}$$

The Gröbner basis for I after we ignore all elements that involve have w or z, and substitute $t\,\partial_t$ by $-s-1$ and $t^2\,\partial_t^2$ by $(s+1)(s+2)$ is

$$\begin{aligned} G_1 &= < dx\,x+dy\,y-4s, -\left(x^2+2xy+3y^2\right)(s+1)-dx\,y^3+dy\,x^3+dy\,x^2y+dy\,xy^2+ \\ &\quad dy\,y^3+x^2+2xy+3y^2, (4x+8y)(s+1)(s+2)-\big(-3\,dx\,y^2+4\,dy\,x^2+5\,dy\,xy+ \\ &\quad 6\,dy\,y^2+10x+22y\big)(s+1)+dx^2y^3-dx\,dy\,y^3+dy^2x^2y+dy^2xy^2+dy^2y^3-3\,dx\,y^2+ \\ &\quad 2\,dy\,x^2+4\,dy\,xy+6\,dy\,y^2+2x+6y, -16\,(s+1)(s+2)(s+3)+(-8\,dx\,y+16\,dy\,x+ \\ &\quad 24\,dy\,y+84)(s+1)(s+2)-\big(3\,dx^2y^2-6\,dx\,dy\,y^2+8\,dy^2xy+9\,dy^2y^2-22\,dx\,y+ \\ &\quad 36\,dy\,x+66\,dy\,y+78)(s+1)-dx^3y^3+dx^2dy\,y^3-dx\,dy^2y^3+dy^3xy^2+dy^3y^3+ \\ &\quad 3\,dx^2y^2-6\,dx\,dy\,y^2+6\,dy^2xy+9\,dy^2y^2-6\,dx\,y+6\,dy\,x+18\,dy\,y+6,- \\ &\quad (-16\,dx+64\,dy)(s+1)(s+2)(s+3)+\big(8\,dx^2y-24\,dx\,dy\,y+48\,dy^2y-84\,dx+336\,dy\big) \\ &\quad (s+1)(s+2)-\big(-3\,dx^3y^2+6\,dx^2dy\,y^2-9\,dx\,dy^2y^2+12\,dy^3y^2+22\,dx^2y-66\,dx\,dy\,y+ \\ &\quad 132\,dy^2y-78\,dx+312\,dy\big)(s+1)+y^3dx^4-dx^3dy\,y^3+dx^2dy^2y^3-dx\,dy^3y^3+dy^4y^3- \\ &\quad 3\,dx^3y^2+6\,dx^2dy\,y^2-9\,dx\,dy^2y^2+12\,dy^3y^2+6\,dx^2y-18\,dx\,dy\,y+36\,dy^2y-6\,dx+24\,dy > \end{aligned}$$

The $ann(f^s)$ is a left module generated by the elements of G_1 The next two steps are very long to write in full, but are computed as before. They yield the following expression for the Bernstein polynomial:

$$b_f(s) = (s+1)^2\,(s+\frac{3}{4})\,(s+\frac{1}{2})\,(s+\frac{3}{2})\,(s+\frac{5}{4}).$$

The operated D which is satisfies 4.1 can be found in the Appendix 6

Example 20. *In Example 18, we saw that the Bernstein polynomial for $f = (y^2 - x^3)$, is $b_f(s) = (s+1)\,(s+\frac{5}{6})\,(s+\frac{7}{6})$.*

Here, we will define two polynomials $f_1 = y\,f$ and $f_2 = x\,f$, and we will compute the Bernstein polynomial for each one by repeating all the steps in the previous examples. In this way we calculate the Bernstein polynomials:

$$\begin{aligned} b_{f_1}(s) &= (s+1)^2\,(s+\frac{5}{9})\,(s+\frac{7}{9})\,(s+\frac{8}{9})\,(s+\frac{10}{9})\,(s+\frac{11}{9})\,(s+\frac{13}{9}), \\ b_{f_2}(s) &= (s+1)^2\,(s+\frac{5}{8})\,(s+\frac{7}{8})\,(s+\frac{9}{8})\,(s+\frac{11}{8}). \end{aligned}$$

In this example, the two polynomials f_1 and f_2 have the same degree but there a huge difference between $b_{f_1}(s)$ and $b_{f_2}(s)$. We see that $b_{f_1}(s)$ if of degree 8 whilst $b_{f_2}(s)$ has only degree 6 and the roots of each are different except for -1. The operators relative to these polynomials are different as well (See the Appendix 2 and 3). The graphs of f_1 (See figure: 5.1) and f_2 (See figure:5.2) shows that the singularity of f_1 is more degenerate than the one for f_2: both curves comprise a cusp and a line passing through the cusp, but in the case of f_1 the line is also tangent to the cusp.

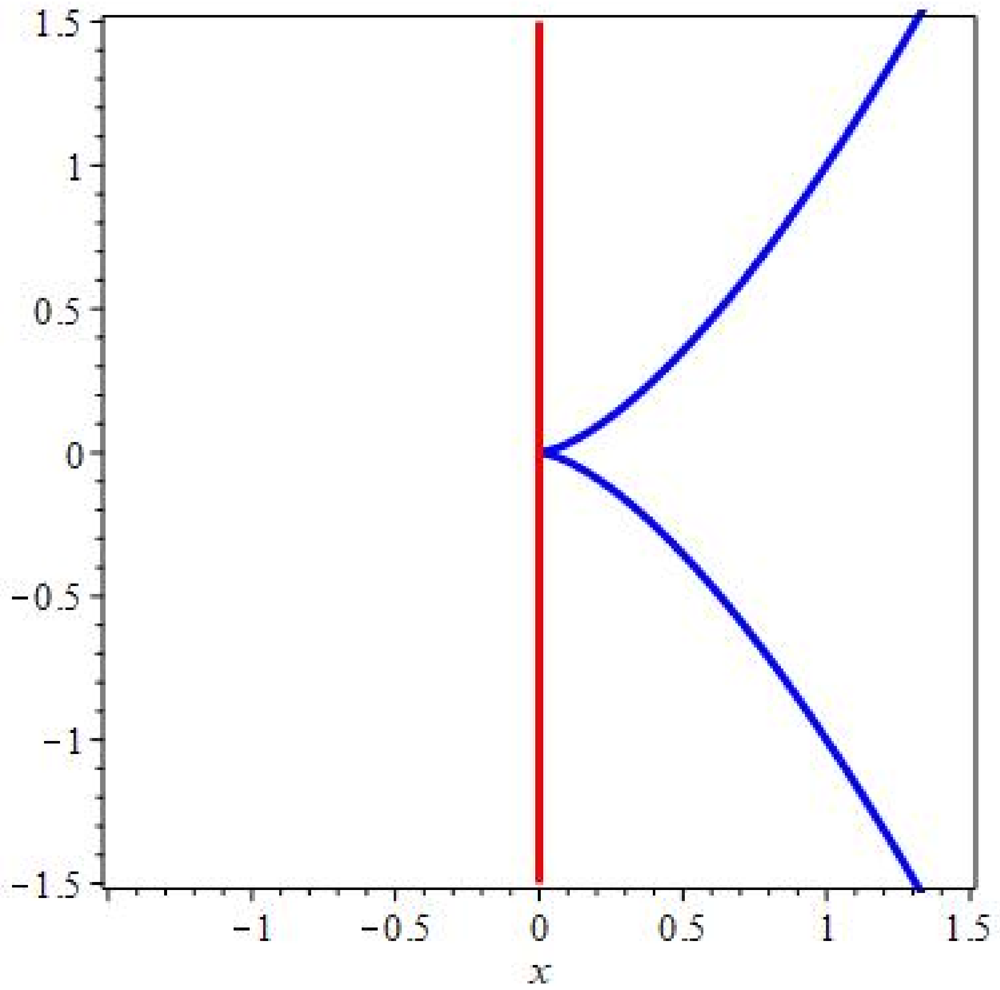


Figure 5.1: The graph of the function $\mathrm{f}_1(x,y) = y\,(y^2 - x^3)$

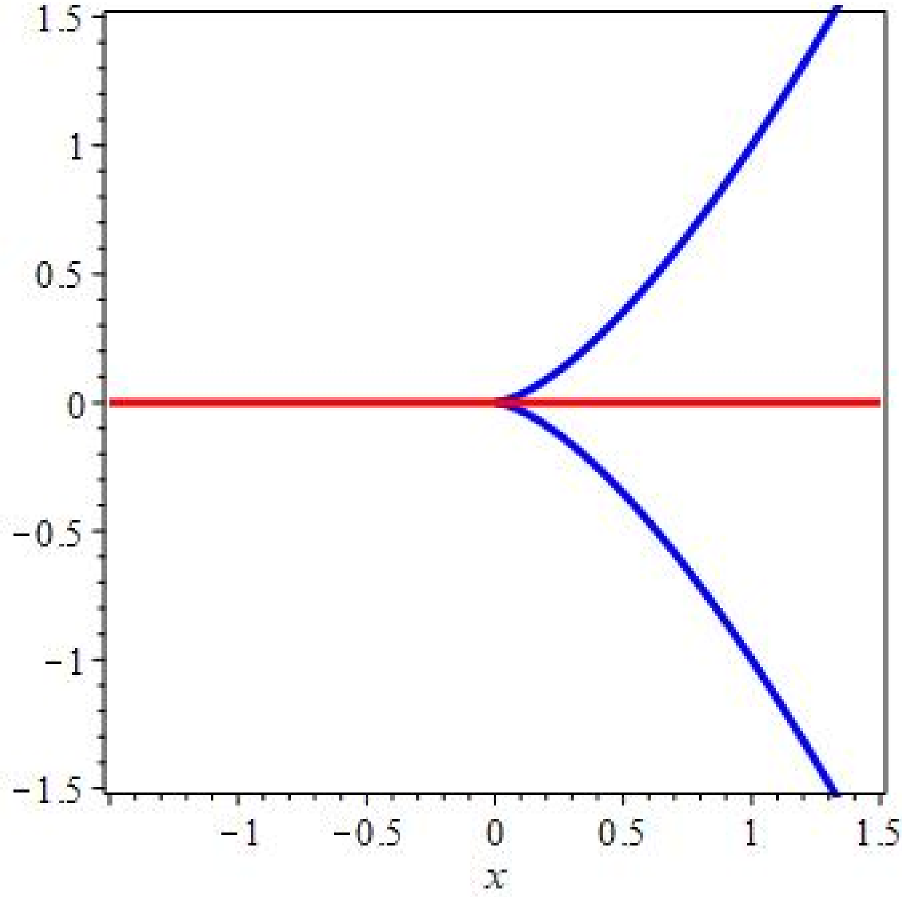


Figure 5.2: The graph of the function $f_2(x,y) = x(y^2 - x^3)$

Now, we will study four closely related polynomials, which are $g_1(x,y) = xy$, $g_2(x,y) = y(y-x^2)$, $g_3(x,y) = y(y-x^3)$ and $g_4(x,y) = y(y-x^4)$. We want between these four polynomial in case which polynomial have mostly closed to its tangent. The graphs of these polynomials show that the curve g_4 (See figure:5.6) is mostly closer to its tangent than others. Then g_3(See figure: 5.5) is closer to its tangent than g_2 (See figure:5.4) and g_2 is closer than g_1 (See figure:5.3). Now, we will compute Bernstein polynomial for each of these polynomials

$$
\begin{aligned}
b_{g_1}(s) &= (s+1), \\
b_{g_2}(s) &= (s+1)^2(s+\frac{3}{4})(s+\frac{5}{4}), \\
b_{g_3}(s) &= (s+1)^2(s+\frac{2}{3})(s+\frac{4}{3})(s+\frac{5}{6})(s+\frac{7}{6}), \\
b_{g_4}(s) &= (s+1)^2(s+\frac{5}{8})(s+\frac{7}{8})(s+\frac{9}{8})(s+\frac{11}{8}(s+\frac{3}{4})(s+\frac{5}{4}).
\end{aligned}
$$

When we compare between the roots of Bernstein polynomial of the above polynomial and the approaching between the curve of the each polynomial and its tangent, we see Bernstein polynomial for g_4 in degree 8 and for g_3 of degree 6 while in degree 4 and 2 for g_2 and g_1 respectively. The operators for g_3 and g_4 are found in Appendix 4 and 5 respectively.

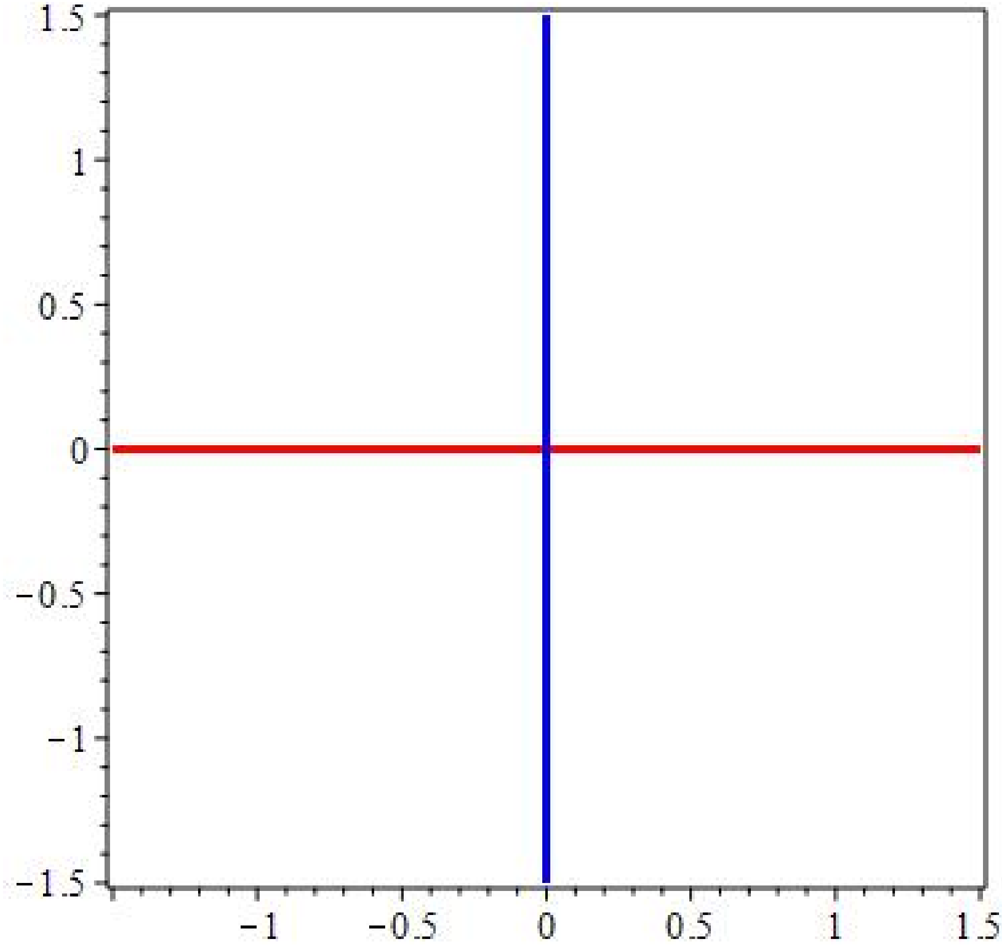


Figure 5.3: The graph of the function $g_1(x,y) = xy$

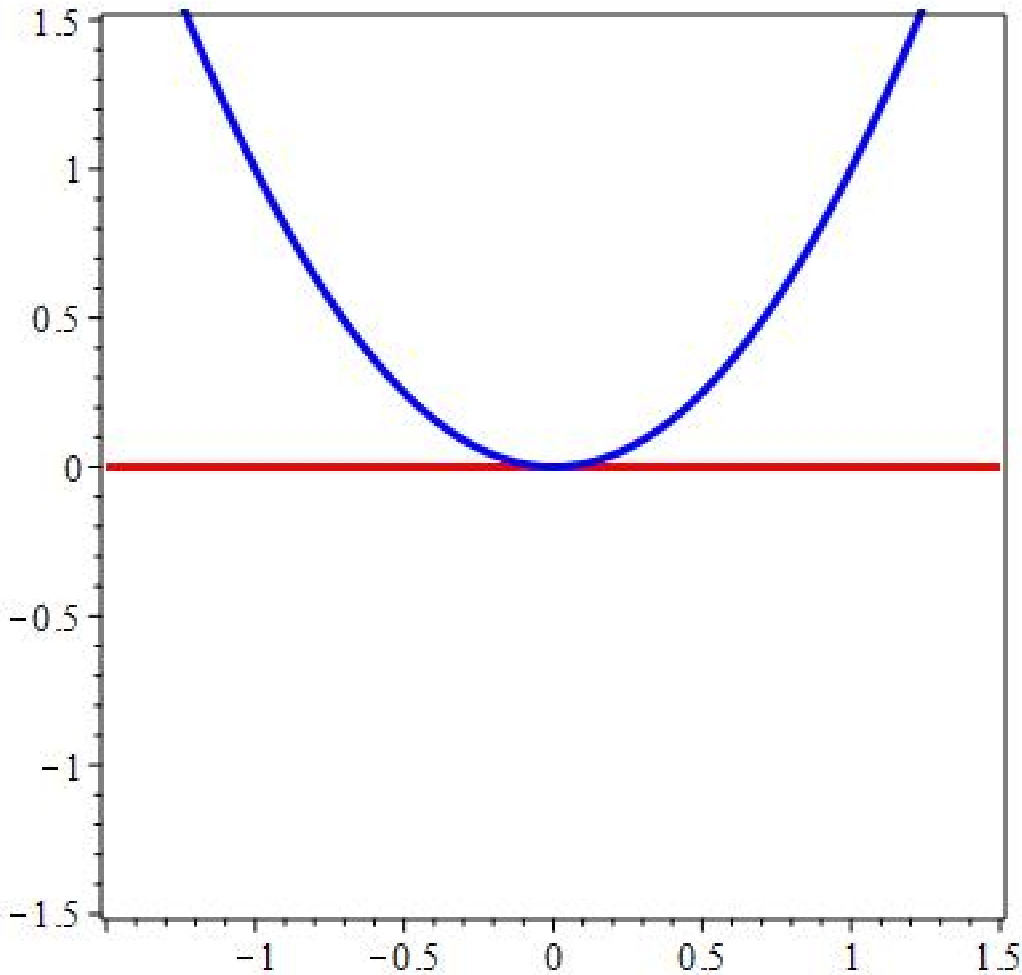


Figure 5.4: The graph of the function $g_2(x,y) = y(y-x^2)$

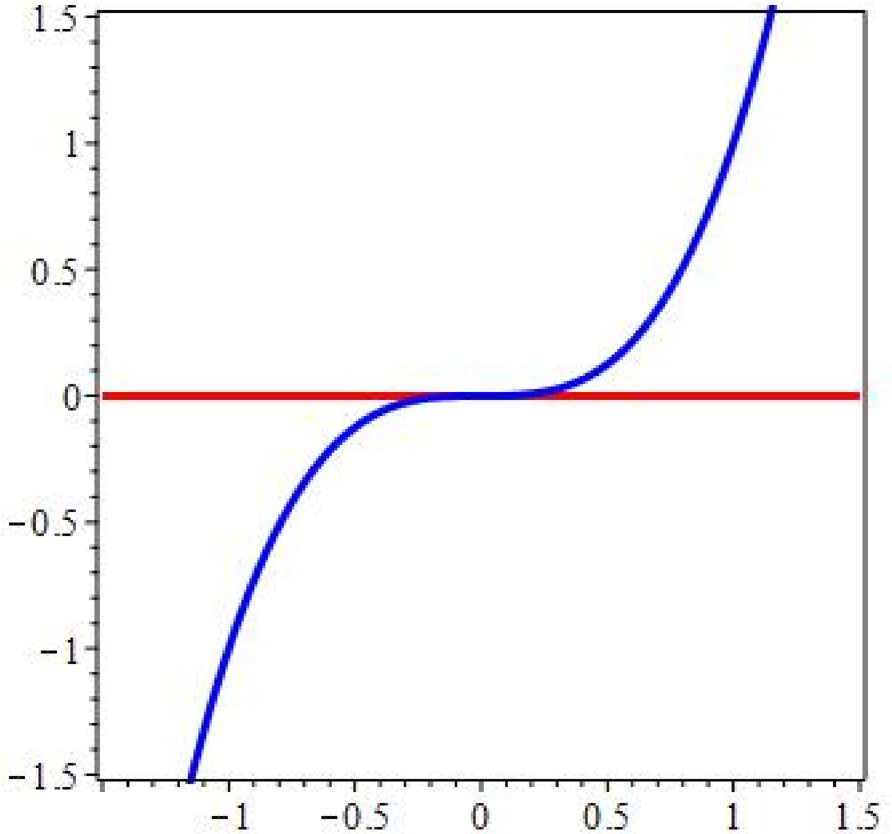


Figure 5.5: The graph of the function $g_3(x,y) = y(y-x^3)$

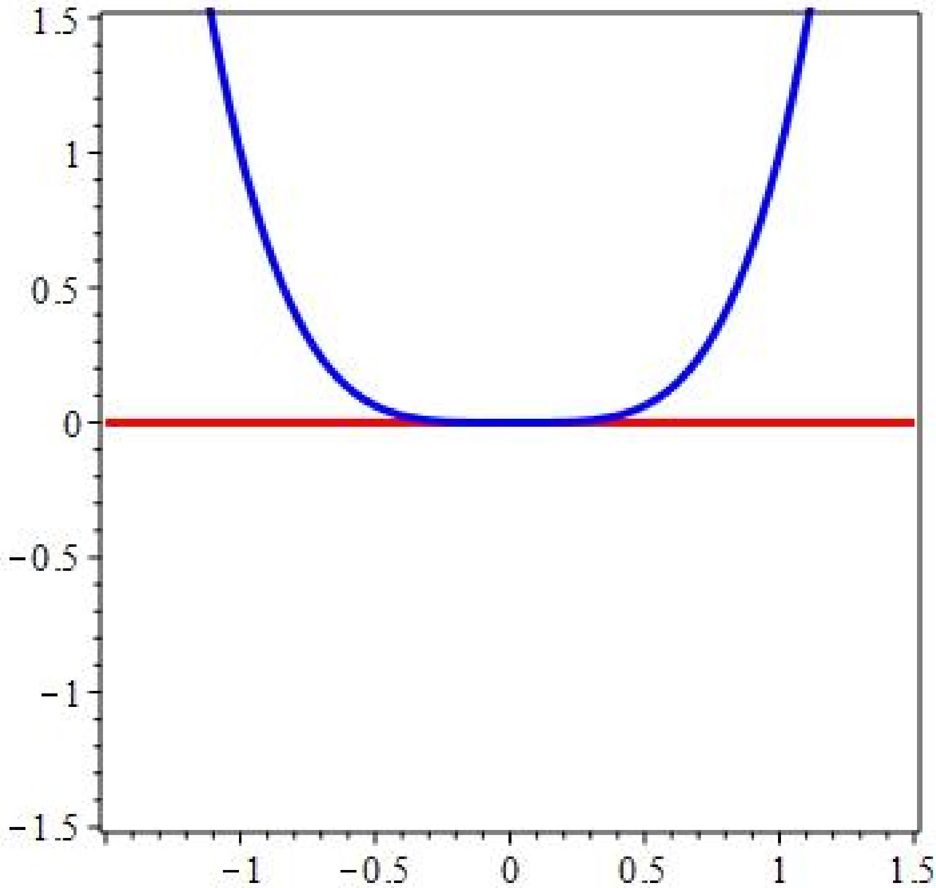


Figure 5.6: The graph of the function $g_4(x,y) = y(y-x^4)$

5.3.3 Three Dimensional Cases

For the case of three dimensional polynomial, there are a very complicated computations, so will give a simple example.

Example 21. *Let* $f = x^2 + y^2 + u^2$*, then* $I =< -\left(u^2 + x^2 + y^2\right) z + t, 2zx\,\partial_t + \partial_x, 2zy\partial_t + \partial_y, 2zu\partial_t + \partial u, -wz + 1 >$.

Repeating the methods above, we obtain

$$\left(\frac{\partial_x^2}{4} + \frac{\partial_y^2}{4} + \frac{\partial u^2}{4}\right)(x^2 + y^2 + u^2)^{s+1} = (s+1)\,(s + \frac{3}{2})\,(x^2 + y^2 + u^2)^s.$$

Chapter 6

D$-$Coh Polynomials for 2-Dimensional polynomials

6.1 Introduction

In this chapter we will give a method to calculate D-Coh polynomial,denoted by $\tilde{b}_f$, and we will study the relation between D-Coh polynomial $\tilde{b}_f$ and the Bernstein polynomial $b(f)$.

We will give two method to calculate $\tilde{b}_f$ for a homogeneous polynomial, a quasi-homogeneous polynomial and another polynomial which is neither homogeneous nor quasi-homogeneous.

6.2 Method

Let f be a polynomial with two independent variables, x and y, and $b_f(s)$ its Bernstein polynomial. The two operators d_1 and d_2 are defined as follows:

$$d_1 : \mathbb{C}[x,y] \to \Omega^1 \quad \text{defined by} \quad d_1(h) = f\, d(h) + (s+1)\, d(f)\, h\,,$$
$$d_2 : \Omega^1 \to \Omega^2 \quad \text{defined by} \quad d_2(\omega) = f\, d(\omega) + s\, d(f) \wedge \omega\,,$$

where $h \in \mathbb{C}[x,y]$, and Ω^1, Ω^2 are the spaces of one-forms and two-forms respectively, and s is a real number. One can easily check that $d_2 \circ d_1 = 0$ for every polynomial

$h \in \mathbb{C}[x,y]$. We consider the complex

$$\mathbb{C}[x,y] \xrightarrow{d_1} \Omega^1 \xrightarrow{d_2} \Omega^2,$$

and calculate the first cohomology group $H_s = ker(d_2)/im(d_1)$. We denoted by h_s for dimension of H_s. Our polynomial $\tilde{b}_f = \prod_{s'}(s-s')^{h_{s'}}$.
Our motivation to choose d_1 and d_2 is to find a relation between our polynomial $\tilde{b}_f$ and the Bernstein polynomial. For any polynomial $h \in \mathbb{C}[x,y]$, the normal derivative for $h\,f^{s+1}$ is

$$d(h f^{s+1}) = (s+1)\, f^s\, h\, d(f) + f^{s+1}\, d(h) = ((s+1)\, h\, d(f) + f\, d(h))\, f^s,$$

so we choose $d_1(h) = f\, d(h) + (s+1)\, d(f)\, h$. For any one $\omega \in \Omega_1$, the normal derivative for $\omega\, f^s$ is

$$d(\omega\, f^s) = f^s\, d(\omega) + s\, f^{s-1}\, d(f) \wedge \omega = (f\, d(\omega) + s\, d(f) \wedge \omega)\, f^{s-1},$$

so we choose $d_2(\omega) = f\, d(\omega) + s\, d(f) \wedge \omega$. By looking to the right hand side for the equation 4.1, $b(s)\, f^s$, we see its a Bernstein polynomial multiplied by f^s, for this reason we cared by the first cohomology group.
We will give many examples to explain how to calculate the cohomology group and the dimension of it as a vector space, and the calculation of our polynomial $\tilde{b}_f$.After these calculation, we will give our conjecture which is $\tilde{b}_f$ divisible by the Bernstein polynomial $b_f(s)$.

6.3 Motivation

The initial motivation to calculate the first cohomology group was to study the first integrals of vector fields with Darboux integrating factors Darboux (1878). We consider the following vector field

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y). \tag{6.1}$$

An invariant algebraic curve is a curve $f(x,y) = 0$ where the polynomial $f(x,y)$ satisfies $\frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = fL$, where L is called the polynomial cofactor Christopher & Llibre (2000). If the cofactor is a multiple of the divergence of the system, then there is an inverse integrating factor of the form f^{α} and the system 6.1 has a first integral ϕ given by Christopher & Llibre (2000)

$$\phi = \int \frac{Pdy - Qdx}{f^{\alpha}} \; or \tag{6.2}$$

$$d\phi = \frac{Pdy - Qdx}{f^{\alpha}}. \tag{6.3}$$

We are interested in cases where the function ϕ is of the form

$$\phi = \frac{h}{f^{\alpha}}, \tag{6.4}$$

where h is a polynomial, for more details see Cairó et al. (1999)and Christopher & Llibre (1999).

Our working hypothesis is that when $-\alpha$ is not a root of the Bernstein polynomial, $b_f(s)$, then there exists such an h.

We have not been able to prove this assertion, but the rest of the thesis is dedicated to demonstrating it for many examples.

When $-\alpha$ is a root of the Bernstein polynomial, then we expect the first integral ϕ to be more complex. For example, let f be any polynomial and let $\alpha = 1$ (corresponding to the root $s = -1$ which is always a factor of $b_f(s)$). In this case we have a first integral of the form $\phi = \ln f$ satisfying 6.2, where

$$\ln f = \int \frac{df}{f} .$$

So when $\alpha = 1$, the first integral is never of the form 6.4. Conversely, if we could establish our working hypothesis, then the existence of such a first integral for $\alpha = 1$ would show that $s = -1$ is always a root of $b_f(s)$ for any polynomial f.

For another example, let us take $f = y^2 - x^3$, we know that

$$b_f(s) = (s+1)\,(s+\frac{5}{6})\,(s+\frac{7}{6}) .$$

In this case, there exists a first integral of the form 6.4 whenever α is not equal to $1, 5/6$ and $7/6$, corresponding to the roots $s = -1, -5/6, -7/6$ of $b_f(s)$. In the case where $\alpha = 5/6$ we can find two polynomials P and Q which gives an inverse integrating factor $(y^2 - x^3)^{\frac{5}{6}}$ which does not integrate to the form 6.4. In more detail, if we let $u = \frac{y}{x^{\frac{3}{2}}}$, then

$$du = x^{-\frac{3}{2}}\,dy - \frac{3}{2} y x^{\frac{-5}{2}}\,dx = (x\,dy - \frac{3}{2} y\,dx)\,x^{\frac{-5}{2}} = (x\,dy - \frac{3}{2} y\,dx)\,(x^3)^{\frac{-5}{6}} .$$

By above, we can see that

$$\int \frac{(x\,dy - \frac{3}{2} y\,dx)}{(y^2 - x^3)^{\frac{5}{6}}} = \int \frac{du}{(u^2-1)^{\frac{5}{6}}} ,$$

which is given in terms of elliptic integrals. In the same way, for $\alpha = 7/6$ we have

$$\int \frac{(x^2\,dy - \frac{3}{2} y x\,dx)}{(y^2 - x^3)^{\frac{7}{6}}} = \int \frac{du}{(u^2-1)^{\frac{7}{6}}} .$$

The above integrals are very complicated in case of $-\alpha$ is one of the roots of Bernstein polynomial. While when $\alpha = \frac{1}{2}$ then then integral above will become

$$\int \frac{(x^4\,dy - \frac{3}{2}yx^3\,dx)}{(y^2-x^3)^{\frac{1}{2}}} = \int \frac{du}{(u^2-1)^{\frac{1}{2}}} = \ln(u+\sqrt{u^2-1})\,.$$

Our calculation of the first cohomology group as a vector space, we found for a specific polynomial f with specific value for S a set of generations in the form one form such as $p\,dy + q\,dx$. the relation between the cohomology group and the first integral is to find a nice form for $\int \frac{p\,dy+q\,dx}{f^s}$. If we looking to Example 25, we see the first cohomology when $s = -\frac{5}{6}$ is generated by $(2x\,dy + 3y\,dx) + Im(d_1)$ while when $s = -\frac{7}{6}$ is generated by $(2x^2\,dy + 3yx\,dx) + Im(d_1)$. For both cases we find $h \in \Omega_1$ with a nice integral $\int \frac{h}{f^\alpha}$. Consider another example, let $f = x^2 + y^2$ the Bernstein polynomial for f is $b_f(s) = (s+1)^2$ (see Example16) . So we can put $z = x + iy$, then

$$\int \frac{x\,dy + y\,dx}{x^2+y^2} = \int \frac{d(z\bar{z})}{z\bar{z}} = \int \frac{d\bar{z}}{\bar{z}} + \int \frac{dz}{z}$$

In the last example, consider $g = x^2 + y^2 + xy$, we set $w = \sqrt{x(x^2+y)}$, so $g = y^2 + w^2$ and we can find a nice integral by the same way as in the previous example.

6.4 Examples

6.4.1 Homogenous Polynomials

Example 22. *Let $f = x^3 + x^2\ y + x\ y^2 + x^3$, then the Bernstein polynomial for f is $b_f(s) = (s+1)^2\ (s+\frac{2}{3})\ (s+\frac{4}{3})$.*

We want to compute the first cohomology group for each root of the Bernstein polynomial: -1, $-\frac{2}{3}$ and $-\frac{4}{3}$. Let us compute the dimension of first cohomology group h_s when $s = -1$. In order to do that, as a first step we must calculate $ker(d_2)$ for some one form $\omega = p\,dy + q\,dx$, where p and q are general polynomials of degree up to 3.

$$\begin{aligned}\omega = \quad & (a_{10}y^3 + a_7x^3 + a_8x^2y + a_9xy^2 + a_4x^2 + a_5xy + a_6y^2 + a_2x + a_3y + a_1)\,dy + \\ & (b_{10}y^3 + b_7x^3 + b_8x^2y + b_9xy^2 + b_4x^2 + b_5xy + b_6y^2 + b_2x + b_3y + b_1)\,dx.\end{aligned}$$

Then, by applying d_2 to ω, we get

$d_2(\omega) = ((b_8 - b_7)x^5 + (-a_4 + b_5 - b_4)x^4 + (-a_8 - 2b_7 + 2b_9 + a_7)yx^4 + (-2a_9 - 3b_7 - b_8 + b_9 + 3b_{10} + 2a_7)x^3y^2 + (-2b_4 + 2b_6 - 2a_5)x^3y + (-2a_2 + b_3 - b_2)x^3 + (a_8 - a_9 - 2b_8 - 3a_{10} + 2b_{10} + 3a_7)x^2y^3 + (-3b_4 - b_5 + b_6 + a_4 - a_5 - 3a_6)x^2y^2 + (-2b_2 - a_2 - 3a_3)x^2y + (-b_1 - 3a_1)x^2 + (b_{10} + 2a_8 - b_9 - 2a_{10})xy^4 + (2a_4 - 2b_5 - 2a_6)xy^3 + (-b_3 - 3b_2 - 2a_3)xy^2 + (-2b_1 - 2a_1)xy + (a_9 - a_{10})y^5 + (-b_6 + a_5 - a_6)y^4 + (-2b_3 + a_2 - a_3)y^3 + (-3b_1 - a_1)y^2)\,dx\,dy.$

In order to calculate $ker(d_2)$ with respect to ω, the following homogenous linear system must be solved

$(b_8 - b_7) = (-a_4 + b_5 - b_4) = (-a_8 - 2b_7 + 2b_9 + a_7) = (-2a_9 - 3b_7 - b_8 + b_9 + 3b1_0 + 2a_7) = (-2b_4 + 2b_6 - 2a_5) = (-b_6 + a_5 - a_6) = (-2a_2 + b_3 - b_2) = (a_8 - a_9 - 2b_8 - 3a_{10} + 2b_{10} + 3a_7) = (-3b_4 - b_5 + b_6 + a_4 - a_5 - 3a_6) = (-2b_2 - a_2 - 3a_3) = (-b_1 - 3a_1) = (b_{10} + 2a_8 - b_9 - 2a_{10}) = (2a_4 - 2b_5 - 2a_6) = (-b_3 - 3b_2 - 2a_3) = (-2b_1 - 2a_1) = (a_9 - a_{10}) = (-2b_3 + a_2 - a_3) = (-3b_1 - a_1) = 0.$

One can easily to see that $ker(d_2)$ with respect to ω is a 5-dimensional vector space generated by

$Ker(d_2) =< (x^3 + x^2y + xy^2 + y^3)\,dy, (x^3 + x^2y + xy^2 + y^3)\,dx, (-x^2 - xy - y^2)\,dy + x^2\,dx, xy\,dx + x^2\,dy, y^2\,dx + xy\,dy > .$

Secondly, to calculate $Im(d_1)$, let us start with a polynomial $h = ax + by + c$, because

$d_1(h)$ is a one form with weight no more that 3 (like ω), then

$$d_1(h) = \left\langle \left(x^3+x^2y+xy^2+y^3\right) dy, \left(x^3+x^2y+xy^2+y^3\right) dx \right\rangle.$$

It can be seen that $Im(d_1)$ is 2-dimensional vector space generated by the above two elements. Finally, the generating set of the first cohomology $H_{-1} = \dfrac{ker(d_2)}{Im(d_1)}$ is

$$(\left(-x^2-xy-y^2\right) d_y + x^2 d_x) + Im(d_1), (xydx + x^2 d_y) + Im(d_1), (y^2 d_x + xyd_y) + Im(d_1).$$

Because the first two generators of $ker(d_2)$ can be written as a linearly combination of the generators of $Im(d_1)$, the first cohomology H_{-1} is generated by three element and hence $h_{-1} = 3$.

Now,we will repeat the same procedure for the second root of $b_f(s)$, which is $-\dfrac{2}{3}$, let us start with the same Ω, then

$$\begin{aligned} d_2(\Omega) = (&\frac{-4b_2+b_3-a_2-6a_3}{3}x^2y + \frac{a_2-4a_3-6b_2-b_3}{3}xy^2 + \frac{-6a_1-2b_1}{3}x^2 + \\ &\frac{3a_2-2a_3-3b_3}{3}y^3 + \frac{-3a_2-2b_2+3b_3}{3}x^3 + \frac{-2a_1-6b_1}{3}y^2 + \\ &\frac{-4a_{10}+6a_8+a_9+5b_{10}}{3}y^4x + \frac{4a_8-a_9-3b_8+2b_9-6a_{10}+7b_{10}+9a_7}{3}y^3x^2 + \\ &\frac{6a_4+a_5-4a_6-3b_5+2b_6}{3}y^3x + \\ &\frac{-3a_9+9b_{10}+2a_8+4b_9+7a_7-b_8-6b_7}{3}y^2x^3 + \frac{6b_9+5a_7+b_8-4b_7}{3}yx^4 + \\ &\frac{-a_5+4b_6+4a_4-b_5-6b_4-6a_6}{3}y^2x^2 + \frac{-3a_5+6b_6+2a_4+b_5-4b_4}{3}yx^3 + \\ &\frac{-2a_{10}+3a_9+3b_{10}}{3}y^5 + \frac{3a_5-2a_6}{3}y^4 + \frac{3a_7+3b_8-2b_7}{3}x^5 + \frac{-2b_4+3b_5}{3}x^4 + \\ &\frac{-4a_1-4b1}{3}xy)\,dxdy\,. \end{aligned}$$

In order to calculate $ker(d_2)$, we must solve the following linear system:

$$-4b_2+b_3-a_2-6a_3 = a_2-4a_3-6b_2-b_3 = -6a_1-2b_1 = -4a_1-4b1 = \\ 3a_2-2a_3-3b_3 = -3a_2-2b_2+3b_3 = -2a_1-6b_1 = -4a_{10}+6a_8+a_9+5b_{10} =$$

$4a_8-a_9-3b_8+2b_9-6a_{10}+7b_{10}+9a_7 = 6a_4+a_5-4a_6-3b_5+2b_6 = -3a_9+9b_{10}+2a_8+4b_9+7a_7-b_8-6b_7 = 6b_9+5a_7+b_8-4b_7 = -a_5+4b_6+4a_4-b_5-6b_4-6a_6 = -3a_5+6b_6+2a_4+b_5-4b_4 = -2a_{10}+3a_9+3b_{10} = 3a_5-2a_6 = 3a_7+3b_8-2b_7 = -2b_4+3b_5.$

One can easily to see that $ker(d_2)$ is a 4-dimensional vector space generated by

$Ker(d_2) =< (-x^3+xy^2-6y^3)\,dy+(-12x^3-7x^2y-6xy^2-5y^3)\,dx, (2x^3+x^2y+9y^3)\,dy+(15x^3+8x^2y+7xy^2+6y^3)\,dx, (x^2+2xy+3y^2)\,dy+(-3x^2-2xy-y^2)\,dx, x\,dy-x\,dy > .$

Let us calculate the $Im(h)$, where $h = ax+by+c$, then

$$\begin{aligned} d_1(h) &= ((b+\frac{a}{3})x^3+\frac{cx^2}{3}+(\frac{4b+2a}{3})yx^2+(\frac{5b}{3}+a)xy^2+\frac{2c}{3}xy+2by^3+cy^2)\,dy+ \\ &\quad (-2ax^3-cx^2-(\frac{5a}{3}+b)yx^2-(\frac{4a+2b}{3})xy^2-\frac{2c}{3}xy-(a+\frac{b}{3})y^3-\frac{c}{3}y^2)dx. \end{aligned}$$

From above, it can be seen that $Im(d_1)$ is a 3-dimensional vector space generated by

$$\begin{aligned} Im(d_1) &= < (\frac{1}{3}x^3+\frac{2}{3}x^2y+xy^2)\,dy+(-2x^3-\frac{5}{3}x^2y-\frac{4}{3}xy^2-y^3)dx, \\ &\quad (x^3+\frac{4}{3}x^2y+\frac{5}{3}xy^2+2y^3)dy+(-x^2y-\frac{2}{3}xy^2-\frac{1}{3}y^3)dx, \\ &\quad (\frac{1}{3}x^2+\frac{2}{3}xy+y^2)dy+(-x^2-\frac{2}{3}xy-\frac{1}{3}y^2)dx > . \end{aligned}$$

The first 3 generators of the generator set of $ker(d_2)$ can be written as linear combination of generators set of $Im(d_1)$, so the first cohomology group $H_{-\frac{2}{3}}$ is generated by one element which is $(x\,dy-y\,dx)+Im(d_2)$ and hence $h_{-\frac{2}{3}} = 1$

Now, let us calculate the dimension of first cohomology group for the last root of $b_f(s)$

which is $-\dfrac{4}{3}$. We start with the same ω as before, and find that

$$\begin{aligned} d_2(\omega) \quad = \quad & \Big(\frac{-8b_2-b_3-5a_2-12a_3}{3}x^2y + \frac{-5b_3-12b_2-8a_3-a_2}{3}xy^2 + \\ & \frac{-4b_1-12a_1}{3}x^2 + \frac{-9b_3-4a_3+3a_2}{3}y^3 + \frac{3b_3-4b_2-9a_2}{3}x^3 + \frac{-12b_1-4a_1}{3}y^2 + \\ & \frac{6a_8+b_{10}-6b_9-8a_{10}-a_9}{3}y^4x + \\ & \frac{-9a_9+9b_{10}-2a_8+2b_9+5a_7-5b_8-12b_7}{3}y^2x^3 + \\ & \frac{2a_8-5a_9-9b_8-2b_9-12a_{10}+5b_{10}+9a_7}{3}y^3x^2 + \\ & \frac{-6a_8+6b_9+a_7-b_8-8b_7}{3}yx^4 + \frac{-5a_5+2b_6+2a_4-5b_5-12b_4-12a_6}{3}y^2x^2 + \\ & \frac{6a_4-9b_5-8a_6-a_5-2b_6}{3}y^3x + \frac{-8b_4-b_5+6b_6-2a_4-9a_5}{3}yx^3 + \\ & \frac{-4a_{10}+3a_9-3b_{10}}{3}y^5 + \frac{-4a_6+3a_5-6b_6}{3}y^4 + \frac{3b_8-4b_7-3a_7}{3}x^5 + \\ & \frac{-4b_4-6a_4+3b_5}{3}x^4 + \frac{-8b_1-8a_1xy}{3}\Big)d_x\,d_y. \end{aligned}$$

By solving the system $d_2(\omega)=0$, we see $ker(d_2)$ with respect to ω is a 4-dimensional vector space generated by

$$\begin{aligned} ker(d_2) \quad = \quad & <xy^2\,dx + x^2y\,dy,\,(-x^2-2yx-3y^2)\,dy + (3x^2+2\,,yx+y^2)\,dx, x^2y\,dx + x^3\,dy, \\ & y^3\,dx + xy^2\,dy > . \end{aligned}$$

On the other hand, $Im(d_1)$ with respect to $h=ax+by+c$ is 3-dimensional vector space generated by

$$\begin{aligned} Im(d_1) \quad = \quad & < \left(-x^3-2x^2y-3xy^2\right)dy + \left(-x^2y-2xy^2-3y^3\right)dx, \left(3x^3+2x^2y+xy^2\right)dy \\ & + \left(3x^2y+2xy^2+y^3\right)dx, \left(-x^2-2yx-3y^2\right)dy + \left(3x^2+2yx+y^2\right)dx > . \end{aligned}$$

Again, the first 3 generators of the generator set of $ker(d_2)$ can be written as linear combination of generators set of $Im(d_1)$, so the first cohomology group $H_{-\frac{4}{3}}$ is generated by one element which is $(xy^2\,dy - y^3\,dx) + Im(d_2)$ and hence $h_{\frac{-4}{3}} = 1$.

Thus $\tilde{b}_f = (s+1)^3\,(s+\frac{2}{3})\,(s+\frac{4}{3})$ while $b_f(s) = (s+1)^2\,(s+\frac{2}{3})\,(s+\frac{4}{3})$.

Example 23. *Let $f = x^2+y^2$, the Bernstein polynomial for f is $b_f(s) = (s+1)^2$.*

Firstly we calculate the dimension of first cohomology group h_{-1}. We will start with $\omega = p\,dy + q\,dx$ as the same ω which we used in the previous example. By using the same strategy to calculate $ker(d_2)$ and $Im(d_1)$, we see that

$$ker(d_2) \;=\; <(x^2+y^2)\,d_y,(x^2+y^2)\,d_x,(-y)\,dy+x\,dx,(x)\,dy+y\,dx>,$$

and

$$Im(d_1) \;=\; <((x^2+y^2)\,d_y,(x^2+y^2)\,d_x> .$$

So the first cohomology group H_{-1} is generated by two elements which are $(y\ dy + x\ dx) + Im(d_1)$ and $(y\ dx - x\ dy) + Im(d_1)$, and hence $h_{-1} = 2$. Secondly, one can easily check that $Im(d_1) \subseteq ker(d_2)$ when $s \neq -1$. As a result, $\tilde{b}_f = (s+1)^2 = b_f$.

Example 24. *In this example, we will consider a particular case of a homogeneous polynomial of degree 4. Let $f = x^4+x^3y+x^2y^2+xy^3+y^4$. The Bernstein polynomial for f can be calculated as $b_f(s) = (s+1)^2\left(s+\frac{1}{2}\right)\left(s+\frac{3}{4}\right)\left(s+\frac{3}{2}\right)\left(s+\frac{5}{4}\right)$.*

To calculate the dimension of first cohomology group with repect to the roots of Bernstein polynomial, let we start with $\omega = p\,dy + q\,dx$, where p and q are a general polynomials of degree up to 6. When $s=-1$, one can check that $ker(d_2)$ is a 12-dimensional vector space generated by
$(\frac{-1}{3}x^6 - y^6 + x^5y + \frac{1}{3}xy^5)\,dy + (\frac{2}{3}x^5y - \frac{2}{3}y^6)\,dx, x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6)\,dy, (x^4y+x^3y^2+x^2y^3+xy^4+y^5)\,dy, x^4+x^3y+x^2y^2+xy^3+y^4)\,dy, (x^3+y^3)\,dy+$

$(-x^3-xy^2-y^3)\,dx, xy^2\,dx+x^2y\,dy, y^3\,dx+xy^2\,dy, -y^3\,dy+(x^3+x^2y+xy^2+y^3)\,dx, (-\frac{2}{3}x^6+\frac{2}{3}xy^5)\,dy+(xy^5+\frac{1}{3}x^5y-\frac{1}{3}y^6-x^6)\,dx, (-x^5+y^5)\,dy+(-x^5+y^5)\,dx, (x^5+x^4y+x^3y^2+x^2y^3+xy^4)\,dx, (x^4+x^3y+x^2y^2+xy^3+y^4)\,dx,$

whilst $Im(d_1)$ is an 8-dimensional vector space generated by

$(-x^4-x^3y-x^2y^2-xy^3-y^4)\,dx, (x^4+x^3y+x^2y^2+xy^3+y^4)\,dy, (-2x^5-2x^4y-2x^3y^2-2x^2y^3-2xy^4)\,dx, (x^5+x^4y+x^3y^2+x^2y^3+xy^4)\,dy+(-x^4y-x^3y^2-x^2y^3-xy^4-y^5)\,dx, (2x^4y+2x^3y^2+2x^2y^3+2xy^4+2y^5)\,dy+(-3x^6-3x^5y-3x^4y^2-3x^3y^3-3x^2y^4)\,dx, (x^6+x^5y+x^4y^2+x^3y^3+x^2y^4)\,dy+(-2x^5y-2x^4y^2-2x^3y^3-2x^2y^4-2xy^5)\,dx, (2x^5y+2x^4y^2+2x^3y^3+2x^2y^4+2xy^5)\,dy+(-x^4y^2-x^3y^3-x^2y^4-xy^5-y^6)\,dx, (3x^4y^2+3x^3y^3+3x^2y^4+3xy^5+3y^6)\,dy.$

When we look to the basis of $ker(d_2)$, there are 8 of its generators can be written as a linear combination of the basis of $Im(d_1)$, then H_{-1} is generated by

$(x^3dy+x^2ydx)+Im(d_1), (xy^2dx+x^2ydy)+Im(d_1), (y^3dx+xy^2dy)+Im(d_1), ((x^3+x^2y+xy^2+y^3)\,dy-x^3\,dx)+Im(d_1).$

Thus $h_{-1}=4$.

Let us repeat the same procedure with the second root of $b_f(s)$ which is $-\frac{3}{4}$. One can check that $ker(d_2)$ is also 12-dimensional vector space generated by

$xy\,dx+x^2\,dy, (x^2+xy)\,dy+(xy+y^2)\,dx, (2x^3+4x^2y+6xy^2+8y^3+4x^2)\,dy+(-8x^3-6x^2y-4xy^2-2y^3+4xy)dx, (-4x^6+11x^5y+9xy^5-16y^6+11x^2)dy+(12x^5y-5x^4y^2-7y^6+11xy)\,dx, (-1469x^6+825x^4y^2+880x^3y^3+3594xy^5+1120y^6+880x^2)\,dy+(-2992x^6+2163x^5y+240x^4y^2-55x^3y^3+2112xy^5-918y^6+880xy)\,dx, (7x^6+5x^2y^4-12xy^5+5x^2)\,dy+(16x^6-9x^5y-11xy^5+4y^6+5xy)\,dx, (-313x^5+150x^4y+165x^3y^2+180x^2y^3+768y^5+180x^2)\,dy+(-468x^5+275x^4y-30x^3y^2-15x^2y^3+288y^5+180xy)\,dx, (18x^4+21x^3y+24x^2y^2+27xy^3+$

$40y^4+21x^2)\,dy+(16x^4-6x^3y-3x^2y^2+3y^4+21xy)\,dx,(-1849x^6+165x^4y^2+3954xy^5+480y^6+2640x^2)\,dy+(528x^6+5943x^5y+3120x^4y^2+2805x^3y^3+2640x^2y^4+2112xy^5-1198y^6+2640xy)\,dx,(-179x^5+30x^4y+15x^3y^2+384y^5+180x^2)\,dy+(36x^5+385x^4y+210x^3y^2+195x^2y^3+180xy^4+144y^5+180xy)\,dx,(3x^4-3x^2y^2-6xy^3+16y^4+21x^2)\,dy+(40x^4+27x^3y+24x^2y^2+21xy^3+18y^4+21xy)\,dx,(7x^5+5xy^4-12y^5+5x^2)\,dy+(12x^5-5x^4y-7y^5+5xy)\,dx.$

Furthemore, $Im(d_1)$ is 9-dimensional vector space generated by

$(x^3+2x^2y+3xy^2+4y^3)\,dy+(-4x^3-3x^2y-2xy^2-y^3)\,dx,(x^4+2x^3y+3x^2y^2+4xy^3)\,dy+(-8x^4-7x^3y-6x^2y^2-5xy^3-4y^4)\,dx,(4x^4+5x^3y+6x^2y^2+7xy^3+8y^4)\,dy+(-4x^3y-3x^2y^2-2xy^3-y^4)\,dx,(x^5+2x^4y+3x^3y^2+4x^2y^3)\,dy+(-12x^5-11x^4y-10x^3y^2-9x^2y^3-8xy^4)\,dx,(4x^5+5x^4y+6x^3y^2+7x^2y^3+8xy^4)\,dy+(-8x^4y-7x^3y^2-6x^2y^3-5xy^4-4y^5)\,dx,(x^6+2x^5y+3x^4y^2+4x^3y^3+8x^4y+9x^3y^2+10x^2y^3+11xy^4+12y^5)\,dy+(-16x^6-15x^5y-14x^4y^2-13x^3y^3-12x^2y^4-4x^3y^2-3x^2y^3-2xy^4-y^5)\,dx,(4x^6+5x^5y+6x^4y^2+7x^3y^3+8x^2y^4)\,dy+(-12x^5y-11x^4y^2-10x^3y^3-9x^2y^4-8xy^5)\,dx,(8x^5y+9x^4y^2+10x^3y^3+11x^2y^4+12xy^5)\,dy+(-8x^4y^2-7x^3y^3-6x^2y^4-5xy^5-4y^6)\,dx,(12x^4y^2+13x^3y^3+14x^2y^4+15xy^5+16y^6)\,dy+(-4x^3y^3-3x^2y^4-2xy^5-y^6)\,dx.$

When we compare between the generating sets of $ker(d_2)$ and $Im(d_1)$,we see the last 10 generators of $ker(d_2)$ can be written as a linear combination of the generating set of $Im(d_2)$. As a result, $H_{-\frac{3}{4}}$ is generated by two elements and hence $h_{\frac{-3}{4}}=2$. For the another roots, we use the same procedure, giving

$H_{\frac{-5}{4}} =< \left(2x^3y+3x^2y^2\right)\,dy+\left(2x^2y^2+3xy^3\right)dx+Im(d_1)\,,\left(x^2y^2+2xy^3\right)\,dy+\left(xy^3+2y^4\right)\,dx+Im(d_1)>,$

$H_{-\frac{1}{2}} = <(xdy+ydx+)Im(d_1)>$.

Thus, $h_{-\frac{5}{4}} = 2$ and $h_{-\frac{1}{2}} = 1$. As a result, $\tilde{b}_f = (s+1)^4(s+\frac{3}{4})^2(s+\frac{3}{2})^2(s+\frac{5}{4})^2(s+\frac{1}{2})$, is divisible by $b_f(s) = (s+1)^2\left(s+\frac{1}{2}\right)\left(s+\frac{3}{4}\right)\left(s+\frac{3}{2}\right)\left(s+\frac{5}{4}\right)$.

In the previous examples, all the polynomial were homogenous and all generators of the kernel and image were also homogenous. Let us now see what happens for quasi-homogenous polynomial.

6.4.2 Quasi-Homogenous Polynomials

Definition 30. A polynomial $f \in \mathbb{K}[x_1,x_2,\ldots,x_r]$ is called quasi homogenous polynomial of weight $w = (w_1,w_2,\ldots,w_n)$, if

$f(r^{w_1}x_1, r^{w_2}x_2,\ldots,r^{w_n}x_n) = r^w f(x_1,x_2\ldots,x_n)$ for every $r \in \mathbb{K}$

Or f($x_1,x_2,\ldots,x_n$) is a quasi-homogeneous of weight w, if $f(y_1^{w_1},y_2^{w_2},\ldots,y_n^{w_n})$ is a homogenous with respect to $y_i's$ Steenbrink (1977)

For example, x^2+y^3 is a quasi-homogeneous of weight $(3,2)$ because $(x^2)^3+(y^3)^2$ is a homogeneous of degree 6.

Example 25. *Let $f := x^3+y^2$ which is quasi homogenous polynomial. The Bernstein polynomial of f is $b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})$, which has three simple roots.*

Let us calculate the first cohomology group for each root. For the first root -1, we start with $\omega = pdy+qdx$ where p and q are a homogenous polynomials with degree up to 6. Then

$d_2(\omega) = (b1 + b20x^3y^2 + b22xy^4 + b19x^4y + b16xy^5 + b26x^3y + b15x^2y^4 + b27x^2y^2 + b12x^5y + b8x^2y + b13x^4y^2 + b21x^2y^3 + b14x^3y^3 + b24x^4 + b25y^4 +$

$b17y^6 + b18x^5 + b23y^5 + b2x + b3y + b11x^6 + b10y^3 + b7x^3 + b4x^2 + b6y^2 + b28xy^3 + b5xy + b9xy^2)\,dx + (a1 + a7x^3 + a4x^2 + a6y^2 + a2x + a3y + a11x^6 + a17y^6 + a18x^5 + a23y^5 + a24x^4 + a25y^4 + a10y^3 + a8x^2y + a5xy + a9xy^2 + a12x^5y + a13x^4y^2 + a14x^3y^3 + a15x^2y^4 + a16xy^5 + a19x^4y + a20x^3y^2 + a21x^2y^3 + a22xy^4 + a26x^3y + a27x^2y^2 + a28xy^3)\,dy.$

Thus, $ker(d_2)$ is a 15-dimensional vector space generated by

$-3x^2dx + 2ydy, (x^3 + y^2)dy, (x^4 + xy^2)dy + (-x^3y - y^3)dx, (x^6 - y^4)dy + (-3x^5y - 3x^2y^3)dx, (y^3x^3 + y^5)dy, (x^3y^2 + y^4)dy, (x^3y + y^3)dy, (-x^5y - x^2y^3)dy + (y^2x^4 + xy^4)dx, (-3y^2x^4 - 3xy^4)dy + (y^3x^3 + y^5)dx, (-2x^4y - 2xy^3)dy + (x^3y^2 + y^4)dx, (-x^5 - x^2y^2)dy + (2x^4y + 2xy^3)dx, (x^5 + x^2y^2)dx, (x^4 + xy^2)dx, (x^3 + y^2)dx, (x^6 + x^3y^2)dx,$

whilst $Im(d_1)$ is a 13-dimensional vector space generated by

$(-x^3 - y^2)\,dx, (x^3 + y^2)\,dy, (-2x^4 - 2xy^2)\,dx, (x^4 + xy^2)\,dy + (-x^3y - y^3)\,dx, (2x^3y + 2y^3)\,dy + (-3x^5 - 3x^2y^2)\,dx, (x^5 + x^2y^2)\,dy + (-2x^4y - 2xy^3)\,dx, (2x^4y + 2xy^3)\,dy + (-x^3y^2 - y^4)\,dx, (3x^3y^2 + 3y^4)\,dy, (-4x^6 - 4x^3y^2)\,dx, (x^6 + x^3y^2)\,dy + (-3x^5y - 3x^2y^3)\,dx, (2x^5y + 2x^2y^3)\,dy + (-2y^2x^4 - 2xy^4)\,dx, (3y^2x^4 + 3xy^4)\,dy + (-y^3x^3 - y^5)\,dx, (4y^3x^3 + 4y^5)\,dy.$

One can easily to check that all of the generators of $ker(d_2)$ can be written as a linearly combination of the generators of $Im(d_1)$ except for $-3x^2dx + 2ydy$, so $h_{-1} = 1$.

For the second root $-\dfrac{5}{6}$, following the same steps, one can easily to see that the $ker(d_2)$ is a 16-dimensional vector space generated by

$< xydy + (\frac{-9}{2}x^3 - 3y^2)\,dx, (x^3 + \frac{4}{3}y^2)\,dy - \frac{1}{2}x^2ydx, x^2ydy + (-6xy^2 -$

$\frac{15}{2}x^4)\,dx,\ (xy^2+\frac{3}{4}x^4)\,dy+(-\frac{9x^3y}{8}-\frac{3}{4}y^3)\,dx,\ (x^3y^3+\frac{13y^5}{12})\,dy-\frac{1}{8}x^2y^4dx,\ (x^3y^2+\frac{10y^4}{9})\,dy-\frac{1}{6}x^2y^3dx,x^4y\,dy+(-\frac{27x^6}{2}-12x^3y^2)\,dx,(x^6-\frac{40y^4}{27})\,dy+(-\frac{25x^2y^3}{9}-\frac{7}{2}x^5y)\,dx,(x^3y+\frac{6}{5}y^3)\,dy+\frac{3}{10}x^5\,dx,y\,dy-\frac{3}{2}x^2\,dx,(-x^5y-\frac{7}{6}x^2y^3)\,dy+(xy^4+\frac{5}{4}x^4y^2)\,dx,(-3x^4y^2-\frac{10}{3}xy^4)\,dy+(y^5+\frac{3}{2}x^3y^3)\,dx,\frac{2}{3}x\,dy+y\,dx,(-\frac{2}{3}x^2y^2-\frac{1}{2}x^5)\,dy+(xy^3+\frac{5}{4}x^4y)dx,\frac{2}{15}y^3\,dy+(x^2y^2+\frac{6}{5}x^5)\,dx,-\frac{7}{3}xy^3\,dy+(y^4-27x^6-\frac{45x^3y^2}{2})\,dx>.$

However, 15 of these generators belong to the submodule $Im(d_1)$, generated by

$< -3x^2\,dx+2y\,dy,(-9x^3-6y^2)\,dx+2xy\,dy,(6x^3+8y^2)\,dy-3x^2y\,dx,(-15x^4-12xy^2)\,dx+2x^2y\,dy,(6x^4+8xy^2)\,dy+(-9x^3y-6y^3)\,dx,(14x^3y+14y^3)\,dy+(-21x^5-21x^2y^2)\,dx,(6x^5+8x^2y^2)\,dy+(-15x^4y-12xy^3)\,dx,(12x^4y+14xy^3)\,dy+(-9x^3y^2-6y^4)\,dx,(18x^3y^2+20y^4)\,dy-3x^2y^3\,dx,(-27x^6-24x^3y^2)\,dx+2x^4y\,dy,(6x^6+8x^3y^2)\,dy+(-21x^5y-18x^2y^3)\,dx,(12x^5y+14x^2y^3)\,dy+(-15x^4y^2-12xy^4)\,dx,(18x^4y^2+20xy^4)\,dy+(-9x^3y^3-6y^5)\,dx,(24x^3y^3+26y^5)\,dy-3x^2y^4\,dx>.$

So $H_{-\frac{5}{6}}$ is generated by $(2x\,dy+3y\,dx)+Im(d_1)$ and hence $h_{-\frac{5}{6}}=1$.

When we repeat the same procedure, we see $H_{-\frac{7}{6}}$ is generated by $(2x^2\,dy+3yx\,dx)+Im(d_1)$ and hence $h_{-\frac{7}{6}}=1$. As a result, $\tilde{b}_f=b_f(s)=(s+1)(s+\frac{5}{6})(s+\frac{7}{6})$

Example 26. *Let $f=x^2+y^4$, the Bernstein polynomial is $b_f(s)=(s+1)^2$. If we put $u=y^2$, then $f=x^2+u^2$, so by Example 23, we see H_{-1} is generated by two elements which are $(u\ du+x\ dx+Im(d_1))$ and $(u\ dx-x\ du)+Im(d_1)$. Thus $\tilde{b}_f=b_f(s)=(s+1)^2$.*

In the previous two examples, we see that $\tilde{b}_f=b_f$

Example 27. *Let $f=y^2-(x^2-1)^2$, the Bernstein polynomial is $b_f(s)=(s+1)^2$. Using the same method as before, we can consider $f=y^2-u^2$, then $\tilde{b}_f=b_f(s)=$*

$(s+1)^2$

6.4.3 Other Polynomials

Example 28. *Let $f = y(y^2 - x^3)$. We can calculate the Bernstein polynomial as $b_f(s) = (s+1)^2(s+\frac{5}{9})(s+\frac{7}{9})(s+\frac{8}{9})(s+\frac{10}{9})(s+\frac{11}{9})(s+\frac{13}{9})$. We will calculate the first cohomology group for each root as before.*

Let us start with first root, -1, and the same $\omega = p\,dy + q\,dx$, where p and q is a general polynomial with degree up to 6. The vector space $ker(d_2)$ with respect to ω is generated by

$ker(d_2) =< 3dxx^2y + 2dyy^2, (2x^3y^3 - 2y^5)dy, (2x^3y^2 - 2y^4)dy, (2x^3y - 2y^3)dy, (-2x^5y + 2x^2y^3)dy + (4x^4y^2 - 4xy^4)dx, dx(x^3y^3 - y^5) + dy(-2x^4y^2 + 2xy^4), (2x^4y - 2xy^3)dy + (-2x^3y^2 + 2y^4)dx, 3dxx^2y + 2dyx^3, (-2x^3y + 2y^3)dx, \left(-2x^4y + 2xy^3\right)dx >$,
whilst the image of d_1 is generated by

$Im(d_1) =< dx(x^3y - y^3), dy(-x^3y + y^3), dx(2x^4y - 2xy^3), dx(x^3y^2 - y^4) + dy(-x^4y + xy^3), dx(3x^5y - 3x^2y^3) + dy(-2x^3y^2 + 2y^4), dx(2x^4y^2 - 2xy^4) + dy(-x^5y + x^2y^3), dx(x^3y^3 - y^5) + dy(-2x^4y^2 + 2xy^4), dy(-3x^3y^3 + 3y^5), dx(4x^6y - 4x^3y^3), dx(3x^5y^2 - 3x^2y^4) + dy(-x^6y + x^3y^3), dx(2x^4y^3 - 2xy^5) + dy(-2x^5y^2 + 2x^2y^4), dx(x^3y^4 - y^6) + dy(-3x^4y^3 + 3xy^5), dy(-4x^3y^4 + 4y^6) >$.
One can easily check that
$H_{-1} =< (3x^2y\,dx + 2y^2\,dy) + Im(d_1), (3x^2y\,dx + 2x^3\,dy) + Im(d_1) >$.
Hence $h_{-1} = 2$.

For the next root, $-\frac{5}{9}$, we will start with the same ω and make the same calculation to see that:

$$ker(d_2) =< -3x^2y\,dx + \left(x^3-3y^2\right)dy, \left(-429x^5y+225x^2y^3\right)dx + \left(44x^6-180y^4\right)dy, -6x^2y^3\,dx + \left(11x^3y^2-15y^4\right)dy, -12x^2y^2\,dx + \left(13x^3y-21y^3\right)dy, \left(30x^4y^2-18xy^4\right)dx + \left(-13x^5y+21x^2y^3\right)dy, \left(-21x^3y^2+9y^4\right)dx + \left(13x^4y-21xy^3\right)dy, \left(21x^3y-9y^3\right)dx + \left(-4x^4+12xy^2\right)dy, \left(55x^4y-33xy^3\right)dx + \left(-\frac{22x^5}{3}+22x^2y^2\right)dy, 3y\,dx+2x\,dy >,$$

whilst

$$Im(d_1) =< -12x^2y\,dx + \left(4x^3-12y^2\right)dy, \left(-3x^3y-9y^3\right)dx + \left(4x^4-12xy^2\right)dy, -12x^2y^2\,dx + \left(-5x^3y-3y^3\right)dy, \left(6x^4y-18xy^3\right)dx + \left(4x^5-12x^2y^2\right)dy, \left(-3x^3y^2-9y^4\right)dx + \left(-5x^4y-3xy^3\right)dy, \left(15x^5y-39x^2y^3\right)dx + \left(4x^6-26x^3y^2+6y^4\right)dy, \left(6x^4y^2-18xy^4\right)dx + \left(-5x^5y-3x^2y^3\right)dy, \left(-3x^3y^3-9y^5\right)dx + \left(-14x^4y^2+6xy^4\right)dy, -12x^2y^4\,dx + \left(-23x^3y^3+15y^5\right)dy > .$$

One can easy to check that:

$$H_{-\frac{5}{9}} =< (3y\,dx+2x\,dy) + Im(d_1) >,$$

and hence $h_{-\frac{5}{9}} = 1$.

We will make the same calculation with the next root $-\dfrac{7}{9}$:

$$
\begin{aligned}
ker(d_2) \;=\; & -3x^2y\,dx + \left(x^3-3y^2\right)dy, \left(-165x^5y+126x^2y^3\right)dx + \left(10x^6-36y^4\right)dy, \\
& -3x^2y^3\,dx + \left(10x^3y^2-12y^4\right)dy, -6x^2y^2\,dx + \left(11x^3y-15y^3\right)dy, \\
& \left(24x^4y^2-18xy^4\right)dx + \left(-11x^5y+15x^2y^3\right)dy, \left(15x^3y^3-9y^5\right)dx + \\
& \left(-20x^4y^2+24xy^4\right)dy, \left(-15x^3y^2+9y^4\right)dx + \left(11x^4y-15xy^3\right)dy, \left(15x^3y-9y^3\right)d \\
& + \left(-2x^4+6xy^2\right)dy, \left(12x^4y-9xy^3\right)dx + \left(-x^5+3x^2y^2\right)dy, 3xy\,dx + 2x^2\,dy >,
\end{aligned}
$$

and

$$
\begin{aligned}
Im(d_2) \;=\; & < -18x^2y\,dx + \left(6x^3-18y^2\right)dy, \left(-9x^3y-9y^3\right)dx + \left(6x^4-18xy^2\right)dy, \\
& -18x^2y^2\,dx + \left(-3x^3y-9y^3\right)dy, -18xy^3\,dx + \left(6x^5-18x^2y^2\right)dy, \\
& \left(-9x^3y^2-9y^4\right)dx + \left(-3x^4y-9xy^3\right)dy, \left(9x^5y-45x^2y^3\right)dx + \left(6x^6-30x^3y^2\right)dy, \\
& -18xy^4\,dx + \left(-3x^5y-9x^2y^3\right)dy, \left(-9x^3y^3-9y^5\right)dx - 12x^4y^2\,dy, -18x^2y^4\,dx + \\
& \left(-21x^3y^3+9y^5\right)dy > .
\end{aligned}
$$

If we check the two two generator sets above for $ker(d_2)$ and $Im(d_1)$, we see that:

$$h_{-\frac{7}{9}} = < (3xy\,dx + 2x^2\,dy) + Im(d_2) >$$

and $h_{-\frac{7}{9}} = 1$.

For the other roots, we will give just the generator sets of the cohomology group and its dimensions.

After the appropriate calculations, we have:

$$
\begin{aligned}
H_{-\frac{8}{9}} &=< (3y^2\,dx + 2xy\,dy) + Im(d_1) >,\\
H_{-\frac{10}{9}} &=< (3xy^2 + 2x^2y\,dy) + Im(d_1) >,\\
H_{-\frac{11}{9}} &=< (3x^3y\,dx + 2x^4\,dy) + Im(d_1) >,\\
H_{-\frac{13}{9}} &=< (3x^4y\,dx + 2x^5\,dy) + Im(d_1) > .
\end{aligned}
$$

Then the dimension of cohomologies are $h_{-\frac{8}{9}} = h_{-\frac{10}{9}} = h_{-\frac{11}{9}} = h_{-\frac{13}{9}} = 1$. As a result, we see that

$$\tilde{b}_f = b_f(s) = (s+1)^2\,(s+\frac{5}{9})\,(s+\frac{7}{9})\,(s+\frac{8}{9})\,(s+\frac{10}{9})\,(s+\frac{11}{9})\,(s+\frac{13}{9}).$$

Thus, in this example $f = y(y^2 - x^3)$, we see that $\tilde{b}_f = b_f(s)$. However in the next example, we take $f = x(y^2 - x^3)$. The Bernstein polynomial has a different number of roots to $\tilde{b}_f(s)$.

Example 29. *Like the previous example, we will start with $s = -1$ and $\omega = p\,dx + q\,dy$ where p and q are general polynomials of degree up to 6. The vector spaces, $ker(d_2)$ and $Im(d_1)$, are given by*

$$
\begin{aligned}
ker(d_2) \;=\; & < 3y^2\,dx + 2xy\,dy,\ \left(-x^4 + xy^2\right) dy,\ \left(2x^5y - 2x^2y^3\right) dx + \left(-x^6 + x^3y^2\right) dy,\\
& \left(x^4y - xy^3\right) dx + \left(-x^5 + x^2y^2\right) dy,\ \left(-x^4y + xy^3\right) dy,\ \left(-x^4y^2 + xy^4\right) dy,\\
& \left(x^3 - y^2\right) dx,\ \left(-x^4 + xy^2\right) dx,\ \left(-x^5 + x^2y^2\right) dx,\ \left(x^4y^2 - xy^4\right) dx +\\
& \left(-2x^5y + 2x^2y^3\right) dy >,
\end{aligned}
$$

and

$$
\begin{aligned}
Im(d_1) \;=\; & < \left(x^4-xy^2\right)dx,\, \left(-x^4+xy^2\right)dy,\, \left(2x^5-2x^2y^2\right)dx,\, \left(x^4y-xy^3\right)dx+ \\
& \left(-x^5+x^2y^2\right)dy,\, \left(3x^6-3x^3y^2\right)dx+\left(-2x^4y+2xy^3\right)dy,\, \left(2x^5y-2x^2y^3\right)dx+ \\
& \left(-x^6+x^3y^2\right)dy,\, \left(x^4y^2-xy^4\right)dx+\left(-2x^5y+2x^2y^3\right)dy,\, \left(-3x^4y^2+3xy^4\right)dy > .
\end{aligned}
$$

By looking for the generating sets of $ker(d_2)$ and $Im(d_1)$, one can easily to check that:

$$H_{-1} =< 3y^2\,dx+2xy\,dy+Im(d_1), (y^2-3x^3)\,dx+xy\,dy+Im(d_1) >,$$

and the dimension of first cohomology group is $h_{-1}=2$.
For the next root,$s=-\dfrac{5}{8}$, and for the same ω, we see that

$ker(d_2) =< \left(-12x^3y+3y^3\right)dx + \left(8x^4-14xy^2\right)dy, \left(-4x^3+y^2\right)dx - 2xy\,dy, \left(-20x^4+11xy^2\right)dx - 6x^2y\,dy, \left(-12x^6+9x^3y^2\right)dx - 2x^4y\,dy, \left(-28x^5y+19x^2y^3\right)dx + \left(8x^6-14x^3y^2\right)dy, \left(-20x^4y^2+11xy^4\right)dx + \left(16x^5y-22x^2y^3\right)dy, \left(-4x^3y^3+y^5\right)dx + \left(8x^4y^2-10xy^4\right)dy, \left(-20x^4y+11xy^3\right)dx + \left(8x^5-14x^2y^2\right)dy, \left(-28x^5+19x^2y^2\right)dx - 6x^3y\,dy, 3y\,dx+2x\,dy >,$

and

$Im(d_1) =< \left(-16x^3+4y^2\right)dx - 8xy\,dy, \left(-8x^4-4xy^2\right)dx - 8x^2y\,dy, \left(-16x^3y+4y^3\right)dx - 8x^4\,dy, -12x^2y^2\,dx - 8x^3y\,dy, \left(-8x^4y-4xy^3\right)dx - 8x^5\,dy, \left(8x^6-36x^3y^2+4y^4\right)dx + \left(-24x^4y+8xy^3\right)dy, -8x^6\,dy - 12x^2y^3\,dx, \left(-8x^4y^2-4xy^4\right)dx + \left(-16x^5y+8x^2y^3\right)dy, \left(-16x^3y^3+4y^5\right)dx + \left(-24x^4y^2+16xy^4\right)dy > .$

So, $H_{\frac{-5}{8}} =< (3y\,dx+2x\,dy)+Im)d_1)$ and $h_{\frac{-5}{8}}=1$.
For the last three roots, we will give just the generators of the first cohomology group

and the dimension:

$$H_{-\frac{7}{8}} =< (3xy\,dx + 2x^2\,dy) + Im(d_1) >,$$
$$H_{-\frac{9}{8}} =< (3x^2y\,dx + x^3\,dy) + Im(d_1) >,$$
$$H_{-\frac{11}{8}} =< (3x^3y\,dx + x^4\,dy) + Im(d_1) >,$$

so $h_{-\frac{7}{8}} = h_{-\frac{9}{8}} = h_{-\frac{11}{8}} = 1$. As a result

$$\tilde{b}_f = b_f(s) = (s+1)^2\,(s+\frac{5}{8})\,(s+\frac{7}{8})\,(s+\frac{9}{8})\,(s+\frac{11}{8}). \tag{6.5}$$

As a result,$\tilde{b}_f$ and $b_f(s)$ are not always equal. We saw $\tilde{b}_f \neq b_f(s)$ in case $f = x^3 + x^2\,y + x\,y^2 + x^3$ while $b_g^{\sim} = b_g(s)$ in case of $g = x^2 + y^2$

Another way to compute the dimension of cohomology group h_s

For higher degree polynomials, f, the first cohomology group needs more extensive calculations. We will use the following way to calculate the dimension of the first cohomology group h_s, where s are the roots of $b_f(s)$.

Firstly, we start with $\omega = p\,dy + q\,dx$, where p and q are general polynomials up to degree j.

Secondly, we will consider $ker(d_2)$ as a n-dimensional vector space and $Im(d_1)$ as a m-dimensional vector space with respect to the bases

$x^n\,dy, x^{n-1}\,y\,dy, x^{n-2}\,y^2\,dy, \ldots, x\,dy, y\,dy, dy, x^n\,dx, x^{n-1}\,y\,dx, x^{n-2}\,y^2\,dx$

$, \ldots, x\,dx, y\,dx, dx.$

So we have n vectors as a bases of $ker(d_2)$ and m vectors as a bases of $Im(d_1)$

Thirdly, we make the new matrix M_s as follows

$$M_s = \left[\begin{array}{c:c} & kr_1 \\ I_n & kr_2 \\ & \vdots \\ & kr_n \\ \hdashline & Im_1 \\ O & Im_2 \\ & \vdots \\ & Im_m \end{array}\right]$$

Where I_n is the $n \times n$ identity matrix, 0 is the zero matrix, $kr_1, kr_2, \ldots, kr_n$ are the bases of $kr(d_2)$ as a vector space and $Im_1, Im_2, \ldots, Im_m$ are the bases of $Im(d_1)$ as a vector space as well. Since the $kr's$ is a bases for $ker(d_2)$ as a vector space so it should be a linearly independent and the same thing for $Im's$. For these reasons we make the matrix M_s in this way and if any vector from $kr's$ can be written as a linearly combinations of $Im's$ it will be a zero vector after we make row echelon form of M_s.
The matrix N_s is the row echelon form of M_s. If any one of the bases of $ker(d_2)$ can be written as a linear combination of the bases of $Im(d_1)$ it will become a zero vector in N_s, so

$$N_s = \left[\begin{array}{c|c} & k_1 \\ & k_2 \\ & \vdots \\ I_d & k_d \\ & 0 \\ & 0 \\ & \vdots \\ & 0 \\ \hline & Im_1 \\ O & Im_2 \\ & \vdots \\ & Im_m \end{array}\right]$$

where $d \leq n$ and $k's + Im(d_1)$ are the generator set of $kr(d_2)/Im(d_1)$ as a vector space. To make these generator set a bases, we put the vectors $k_1, k_2, \ldots, k_d$ in a matrix called K_s. If any vector from $k's$ can be written as a linear combinations from the others in K_s will become a zero after making Gaussian elimination of matrix K_s, so the bases of the first cohomology group with respect to s is the non zero vectors in Gaussian elimination of matrix K_s.And its number is h_s.

We will apply our method to two examples, the first is the case $f = x^2 + y^2$, we have just one root of the Bernstein polynomial $b_s = (s+1)^2$. From Example 23 one can see the bases of $ker(d_2)$ is $(x^2+y^2)d_y, (x^2+y^2)d_x, -yd_x + xd_y, xd_x + ydy$, so we can consider it as a 4-dimensional vector space with respect to the bases

$x^2dy, y^2dy, xdy, ydy, x^2dx, y^2dx, xdx, ydx$,

and the bases of $Im(d_1)$ is $(x^2+y^2)d_y, (x^2+y^2)d_x$, so we can cosider it as 2-dimensional vector space with respect to the same bases above. Then the bases of

$ker(d_2)$ can be written as

$$\{[1,1,0,0,0,0,0,0],[0,0,0,0,1,1,0,0]$$
$$[0,0,0,-1,0,0,1,0],[0,0,1,0,0,0,0,1]\},$$

And the bases of $Im(d_1)$ also can be written as

$$\{[1,1,0,0,0,0,0,0],[0,0,0,0,1,1,0,0]\}l.$$

The matrix M_{-1} is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

.

And N_{-1} is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

In N_{-1}, we see the first two rows are zero (if we ignore I_4), then kr_{-1} is

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

.

The Gaussian elimination of kr_{-1} still the same matrix, then $h_{-1} = 2$.

The second example is $f = x^3 + x^2\ y + xy^2 + x^3$. From the Example 22, we Know that $b_f(s) = (s+1)^2\ (s+\frac{2}{3})\ (s+\frac{4}{3})$ First, we take $s = -1$, since $ker(d_2)$ contain 5 independent elements, so we can consider is as a 5-dimensional vector space with respect to the bases , see 22

$x^3\,dy, x^2\,y\,dy, xy^2\,dy, y^3\,dy, x^2\,dy, xy\,d_y, y^2\,dy, x\,dy, y\,dy, dy,$

$x^3\,d_x, x^2\,y\,dx, xy^2\,dx, y^3\,dx, x^2\,dx, xy\,dx, y^2\,dx, x\,dx, y\,dx, dx.$

And $Im(d_1)$ as 2-dimensional vector space with respect to the same bases. Then the basis of $ker(d_2)$ is

$$\{[1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$
$$[0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0]$$
$$[0,0,0,0,1,1,1,0,0,0,0,0,0,0,-1,0,0,0,0,0]$$
$$[0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0]$$
$$[0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0]\},$$

and the basis of $Im(d_1)$ is

$$\{[1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$
$$[0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0]\}.$$

The matrix M_{-1} is

$$\begin{bmatrix}
1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0
\end{bmatrix}$$

.

And N_{-1} is

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0
\end{bmatrix}$$

In N_{-1}, we see the first and fifth rows are zero (if we ignore I_5), then kr_{-1} is

$$\begin{bmatrix}
0&0&0&0&1&1&1&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0
\end{bmatrix}.$$

The Gaussian elimination of kr_{-1} still the same matrix, then $h_{-1} = 3$.
For the second root of Bernstein polynomial $s = \dfrac{-2}{3}$. Since the Gröbner bases $ker(d_2)$ contains 4 elements, so we can consider it as a 4-dimensional vector space with respect to the bases
$x^3\,dy, x^2\,y\,dy, x\,y^2\,dy, y^3\,dy, x^2\,dy, x\,y\,d_y, y^2\,dy, x\,dy, y\,dy, dy, x^3\,d_x, x^2\,y\,dx,$
$x\,y^2\,dx, y^3\,dx, x^2\,dx, x\,y\,dx, y^2\,dx, x\,dx, y\,dx, dx,$
and $Im(d_1)$ as 3-dimensional vector space with respect to the same bases. Then the basis of $ker(d_2)$ is

$$\begin{aligned} \{[0,0,0,0,1,2,3,0,0,0,0,0,0,0,-3,-2,-1,0,0,0] \\ [-1,0,1,-6,0,0,0,0,0,0,-12,-7,-6,-5,0,0,0,0,0,0] \\ [2,1,0,9,0,0,0,0,0,0,15,8,7,6,0,0,0,0,0,0] \\ [0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0]\}, \end{aligned}$$

and the bases of $Im(d_1)$ are

$$\begin{aligned} \{[0,0,0,0,1,2,3,0,0,0,0,0,0,0,-3,-2,-1,0,0,0] \\ [3,4,5,2,0,0,0,0,0,0,0,0,-3,-2,-1,0,0,0,0,0,0] \\ [1,2,3,0,0,0,0,0,0,0,-6,-5,-4,-3,0,0,0,0,0,0]\}. \end{aligned}$$

Then the matrix $M_{\frac{-2}{3}}$ is

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & -7 & -6 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 1 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 15 & 8 & 7 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 4 & 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -5 & -4 & -3 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

and $N_{\frac{-2}{2}}$ is

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 7 & 6 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & -6 & -5 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & -1 & 0 & 0 & 0
\end{bmatrix}$$

One can easily to see that four rows from the first five rows from above matrix are zero if we ignore the identity matrix, so $h_{\frac{-2}{3}} = 1$

For the last root of Bernstein polynomial $s = \dfrac{-4}{3}$, since $ker(d_2)$ generated by 4 independent element , then one can consider it as a 4-dimensional vector space and $Im(d_1)$ as 3-dimensional vector space with respect to the same bases which we used in cases $s=-1$ and $s=\dfrac{-2}{3}$. Then the basis of $ker(d_2)$ is

$$\begin{aligned}\{&[0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0]\\&[1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0]\\&[0,0,0,0,-1,-2,-3,0,0,0,0,0,0,0,3,2,1,0,0,0]\\&[0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0]\},\end{aligned}$$

And the basis of $Im(d_1)$ is

$$\begin{aligned}\{&[0,0,0,0,-1,-2,-3,0,0,0,0,0,0,0,3,2,1,0,0,0]\\&[3,2,1,0,0,0,0,0,0,0,0,3,2,1,0,0,0,0,0,0]\\&[-1,-2,-3,0,0,0,0,0,0,0,0,-1,-2,-3,0,0,0,0,0,0]\}\end{aligned}$$

The Martix $M_{\frac{-4}{3}}$ is

$$\begin{bmatrix}
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&-1&-2&-3&0&0&0&0&0&0&0&3&2&1&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&-2&-3&0&0&0&0&0&0&0&3&2&1&0&0&0\\
0&0&0&0&3&2&1&0&0&0&0&0&0&0&0&3&2&1&0&0&0&0&0&0\\
0&0&0&0&-1&-2&-3&0&0&0&0&0&0&0&0&-1&-2&-3&0&0&0&0&0&0
\end{bmatrix}$$

and $N_{\frac{-4}{3}}$ is

$$\begin{bmatrix} 1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&-2&0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0 \\ 0&0&0&0&1&0&-1&0&0&0&0&0&0&0&0&1&0&-1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&2&0&0&0&0&0&0&0&0&0&1&2&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&2&3&0&0&0&0&0&0&0&-3&-2&-1&0&0&0 \end{bmatrix}$$

.

When we looks to the first 4 rows, we see 3 of them are not zero, but the Gaussian elimination of the matrix $kr_{\frac{-4}{3}}$ is

$$\begin{bmatrix} 0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

.

If we look to a bove matrix we see it have 1 non zero vector, that means $h_{\frac{-4}{3}} = 1$.

Our strategy to calculate the first cohomology group is to start with a $\omega = p\,d_x + q\,d_y$, where p and q are general polynomials up to certain small degree (m). Then we repeat the same steps with $\omega_1 = p_1\,d_x + q_1\,d_y$, where p_1 and q_1 are generals polynomial up to certain small degree (m_1) until we get a non-zero dimensional first cohomology group. In this step there is an important question. If we start with a higher weight ω, the dimension of the first cohomology group will change or will be the same?.

Our answer for this question, after calculating the dimension of first cohomology group for a higher weight ω and we saw that the result is still the same.

For example, we start in Example 22 with weight 1 in ω , we saw that the dimension of cohomology group is 0, and we got the same result with weight up to 2 in ω. But in weight up to 3, the dimension of the cohomology group have a non-zero dimension. We continue until up to weight 12 in ω and we saw that the result is still the same.

In Example 23, we found the non-zero dimensional cohomology group in ω of weight 1, but we succeed to consider until ω of weight 20, because this example have smaller degree than Example 22, and the result is still the same. In Example 25, we wrote the more complicated case which is when ω is up to weight 6, but we found the non- zero cohomology group when in ω is up to weight 2 in case $s=-1$ and $s=-\dfrac{7}{6}$ and when ω is of weight 1 in case $s=-\frac{5}{6}$.

Chapter 7

Conclusion and Future Work

In this thesis, we study the Bernstein polynomial and developed a method to calculate it. The previous authors founding many ways to calculate Bernstein polynomial without caring about the operator D which satisfy $D\, f^{s+1} = b_f(s)\, f^s$. In our method we gave a way how to calculate it. All references about the Bernstein polynomial gave a methods or algorithms without any example to explain to the reader how it works. We present many examples to explain how our method work. We start with very easy polynomials like $f = x^2$, and we continue with harder ones. Thinking geometrically like and in Example 20, we see that the polynomials which its curve is more closely to their tangent they do have a higher degree Bernstein polynomial. This motivate us to study some related polynomials and see the relation between the Bernstein polynomial and how much the closer is its curve with their tangent.

This work will give a way to us or to other authors to study the geometric relation for Bernstein polynomial.

In chapter six, we give our definition for the polynomials of the dimension of the first cohomology group. To calculate the Bernstien polynomial we start with f^{s+1} and we end with f^s. This give us the motivation to think on the derivative, and we define two operators

$$d_1 : \mathbb{C}[x,y] \to \Omega^1 \quad \text{defined by} \quad d_1(h) = f\, d(h) + (s+1)\, d(f)\, h\,,$$

$$d_2 : \Omega^1 \to \Omega^2 \quad \text{defined by} \quad d_2(\omega) = f\, d(\omega) + s\, d(f) \wedge \omega\,,$$

The first operator is the derivative of hf^{s+1}, and the second is the derivative of ωf^s, for some $h \in \mathbb{C}[x,y]$ and $\omega \in \Omega_1$. And we calculate the dimension of the first cohomology group for certain degree of ω.
After our calculation and many examples, we see that our polynomial $b_f^{\sim}$ is always divisible by the Bernstein polynomial.

7.1 Future work

1. In chapter five we present a way to calculate the operator D, but still we to more study more properties of this operator.

2. The geometric relation between the Bernstein polynomial and its graph needs deep study.

3. We calculate $b_f^{\sim}$ until certain degree ω. If we go to higher degree then it needs a complex calculation. We should try to prove it in general.

4. Our polynomial $b_f^{\sim}$ depends on the dimension of the first cohomology group. We will study the dimension of the higher cohomology group.

5. We study two dimensional polynomials and we calculate the first cohomology group. We will study three dimensional polynomials and its cohomology.

Appendix A

The Operator D relative to Bernstein polynomial

1. The operated D which is satisfy 4.1 when $f = (x-1)^2(x-2)^3$ is

$$D = -\frac{1952683\,dx^4}{1296} + \frac{889\,dx^6}{648} + \frac{982813\,dx^5}{1620} - \frac{9077172457\,s^2dx^4}{43200} - \frac{10961235421\,s^3dx^3}{3600} + \frac{188838533\,sdx^5}{16200} + \frac{3786139\,dx^5x^2}{2025} - \frac{8387593979\,s^2dx^3}{16200} - \frac{2115383\,dx^5x}{1200} - \frac{174569551\,sdx^4}{6480} - \frac{4996297\,dx^4x^2}{2700} - \frac{141902147\,sdx^3}{2700} - \frac{562\,x^3dx^3}{9} - \frac{5159\,dx^6x}{1296} + \frac{8155\,sdx^6}{432} + \frac{8267539693\,s^3dx^5}{16200} - \frac{211273972669\,s^4dx^4}{86400} + \frac{35581\,s^2dx^6}{324} + \frac{54816888547\,s^4dx^5}{32400} - \frac{313863473303\,s^5dx^4}{86400} + \frac{302659\,s^3dx^6}{864} - \frac{80320126559\,dx^3s^6}{1296} - \frac{34773860117\,dx^3s^5}{1080} - \frac{208034467\,s^3dx^4}{225} - \frac{24037615678\,s^4dx^3}{2025} + \frac{3254032787\,s^2dx^5}{32400} + \frac{100\,x^4dx^3}{9} - \frac{50\,dx^4x^5}{9} + \frac{2\,dx^5x^5}{27} + \frac{35\,dx^6x^4}{108} + \frac{1081\,dx^4x^4}{27} + \frac{58123\,dx^5x^4}{405} + \frac{12377313331\,s^5dx^5}{3240} - \frac{2157559279\,s^6dx^4}{1440} + \frac{858011\,s^4dx^6}{1296} - \frac{73108039045\,dx^3s^7}{864} + \frac{266333\,xdx^4}{90} + \frac{15220625\,s^{10}dx^5}{48} + \frac{106544375\,s^{11}dx^4}{96} - \frac{76103125\,s^{11}dx^3}{24} + \frac{10955\,s^6dx^6}{24} + \frac{8575\,s^7dx^6}{72} + \frac{1541288875\,s^{10}dx^4}{288} + \frac{126399875\,s^9dx^5}{72} + \frac{7493649475\,s^8dx^5}{1728} + \frac{37282519225\,s^9dx^4}{3456} - \frac{338830625\,s^{10}dx^3}{18} + \frac{16314678655\,s^7dx^5}{2592} + \frac{115955441885\,s^8dx^4}{10368} - \frac{43531839875\,s^9dx^3}{864} - \frac{1253\,dx^6x^3}{648} + \frac{1922627\,s^5dx^6}{2592} -$$

$$\frac{6944027\,dx^5x^3}{8100} + \frac{86161\,dx^4x^3}{270} - \frac{415019\,dx^3x^2}{405} + \frac{26269186955\,s^7dx^4}{5184} -$$
$$\frac{52018824925\,s^8dx^3}{648} + \frac{30820899349\,s^6dx^5}{5184} + \frac{5467\,dx^6x^2}{1296} +$$
$$\frac{1223063796463\,s^4xdx^4}{259200} + \frac{345411534311\,s^5xdx^4}{51840} + \frac{7939165505\,s^6xdx^4}{5184} +$$
$$\frac{3444101051\,sxdx^4}{64800} + \frac{16398547\,sdx^3x}{225} + \frac{205318204469\,s^3dx^5x^2}{129600} +$$
$$\frac{355\,s^2dx^5x^5}{72} + \frac{270625\,x^4dx^3s^2}{324} + \frac{4000\,x^4dx^3s}{27} + \frac{4874427785\,x^4s^5dx^5}{5184} -$$
$$\frac{14258946875\,s^9x^3dx^4}{3456} + \frac{34405\,s^2dx^6x^4}{1296} + \frac{13203125\,s^5dx^3x^4}{2592} +$$
$$\frac{962675\,s^5dx^6x^4}{5184} + \frac{3774964375\,s^8dx^5x^4}{3456} - \frac{2078125\,s^4dx^4x^5}{2592} + \frac{14425\,s^3dx^5x^5}{1296} +$$
$$\frac{4283507677\,x^4s^4dx^5}{10368} + \frac{5845\,sdx^6x^4}{1296} + \frac{1671875\,s^3dx^3x^4}{648} + \frac{148085\,s^3dx^6x^4}{1728} +$$
$$\frac{6109375\,s^4dx^3x^4}{1296} + \frac{424795\,s^4dx^6x^4}{2592} + \frac{8157920125\,s^7dx^5x^4}{5184} +$$
$$\frac{849585625\,s^6dx^5x^4}{576} - \frac{173125\,s^2dx^4x^5}{648} + \frac{82\,sdx^5x^5}{81} - \frac{1625\,sdx^4x^5}{27} -$$
$$\frac{134375\,s^3dx^4x^5}{216} - \frac{18663125\,s^5dx^3x^3}{648} - \frac{5626355\,s^5dx^6x^3}{5184} -$$
$$\frac{22009462375\,s^8dx^5x^3}{3456} + \frac{3727625\,s^4dx^4x^4}{648} + \frac{89017651\,s^3dx^5x^4}{720} -$$
$$\frac{51324820721\,s^4dx^4x^2}{17280} + \frac{59075501021\,s^5dx^3x}{1296} - \frac{571203125\,s^{10}x^3dx^4}{288} +$$
$$\frac{1953125\,s^6dx^3x^4}{648} + \frac{16625\,s^6dx^6x^4}{144} + \frac{64109375\,s^9dx^5x^4}{144} - \frac{78125\,s^5dx^4x^5}{144} +$$
$$\frac{5125\,s^4dx^5x^5}{432} - \frac{186383125\,s^9dx^5x^3}{72} + \frac{1668125\,s^5dx^4x^4}{432} -$$
$$\frac{171921875\,s^{10}x^2dx^3}{18} + \frac{1884743\,s^3dx^6x^2}{1728} + \frac{56993129515\,xs^6dx^3}{648} +$$
$$\frac{2090403653\,x^3s^4dx^4}{3456} - \frac{40763925775\,x^2s^5dx^3}{2592} - \frac{11851096993\,x^3s^3dx^5}{16200} -$$
$$\frac{40820047799\,x^2s^5dx^4}{10368} + \frac{857029925\,x^3s^5dx^4}{1152} - \frac{25268562191\,x^3s^4dx^5}{10368} -$$
$$\frac{39619398925\,x^2s^6dx^3}{1296} + \frac{4375\,s^7dx^6x^4}{144} - \frac{38828125\,s^{11}x^3dx^4}{96} +$$
$$\frac{1953125\,s^7dx^3x^4}{2592} + \frac{7765625\,s^{10}dx^5x^4}{96} - \frac{390625\,s^6dx^4x^5}{2592} + \frac{3125\,s^5dx^5x^5}{648} -$$
$$\frac{45040625\,s^{10}dx^5x^3}{96} + \frac{171875\,s^6dx^4x^4}{162} - \frac{38828125\,s^{11}x^2dx^3}{24} + \frac{151081\,sdx^6x^2}{2592} -$$
$$\frac{142219\,sdx^6x}{2592} - \frac{192678315499\,s^3dx^5x}{129600} - \frac{414379\,s^2dx^6x}{1296} -$$
$$\frac{1278690133189\,s^4dx^5x}{259200} - \frac{1765253\,s^3dx^6x}{1728} - \frac{9283745561\,x^3s^2dx^5}{64800} +$$

$$\frac{3078765359\,x^3s^3dx^4}{12960} - \frac{67022477\,x^3sdx^5}{4050} + \frac{688198387\,x^3s^2dx^4}{12960} -$$
$$\frac{7418738863\,x^2s^4dx^3}{1296} + \frac{3459109\,x^3sdx^4}{540} - \frac{1558135177\,x^2s^3dx^3}{1080} -$$
$$\frac{775358173\,x^2s^2dx^3}{3240} - 37988257\,x^2sdx^3 1620 - \frac{17216630573\,s^2x^2dx^4}{64800} +$$
$$\frac{3851103863\,xs^3dx^3}{900} + \frac{2333379017\,sdx^5x^2}{64800} + \frac{9795370223\,s^3xdx^4}{5400} -$$
$$\frac{37884697351\,s^2dx^5x}{129600} - \frac{273728377\,sdx^4x^2}{8100} + \frac{1953470677\,s^2dx^3x}{2700} +$$
$$\frac{53731374769\,s^2xdx^4}{129600} - \frac{1098322601\,sdx^5x}{32400} + \frac{882623\,s^2dx^6x^2}{2592} - \frac{34741\,sdx^6x^3}{1296} +$$
$$\frac{96404304475\,s^6dx^5x^2}{5184} + \frac{670523\,s^4dx^6x^2}{324} - \frac{4077090395\,s^7xdx^4}{324} -$$
$$\frac{90032124205\,s^6dx^5x}{5184} - \frac{25375\,s^7dx^6x^3}{144} - \frac{96775\,s^6dx^6x^3}{144} - \frac{1053025\,s^3dx^3x^3}{72} -$$
$$\frac{872893\,s^3dx^6x^3}{1728} - \frac{5769625\,s^4dx^3x^3}{216} - \frac{415541\,s^4dx^6x^3}{432} -$$
$$\frac{23849983625\,s^7dx^5x^3}{2592} - \frac{3736611335\,s^6dx^5x^3}{432} + \frac{156460\,s^2dx^4x^4}{81} +$$
$$\frac{3001361\,sdx^5x^4}{1080} + \frac{140939\,sdx^4x^4}{324} + \frac{5815075\,s^3dx^4x^4}{1296} + \frac{312814469\,s^2dx^5x^4}{12960} -$$
$$\frac{26057246875\,x^2dx^3s^8}{648} - \frac{766045\,x^3dx^3s^2}{162} - \frac{15053\,x^3dx^3s}{18} -$$
$$\frac{47604521875\,s^8x^3dx^4}{10368} - \frac{2292932125\,s^7x^3dx^4}{864} - \frac{21952071875\,s^9x^2dx^3}{864} +$$
$$\frac{151640640107\,s^4dx^5x^2}{28800} + \frac{4020989\,s^5dx^6x^2}{1728} - \frac{11244541\,s^5dx^6x}{5184} +$$
$$\frac{34326581861\,s^5dx^5x^2}{2880} - \frac{48164254727\,s^5dx^5x}{4320} - \frac{1670585\,s^4dx^6x}{864} -$$
$$\frac{2734375\,x^3dx^3s^7}{648} - \frac{1375000\,x^3dx^3s^6}{81} - \frac{25070921587\,s^3dx^4x^2}{21600} +$$
$$\frac{8061182357\,s^2dx^5x^2}{25920} + \frac{36184959193\,s^4dx^3x}{2160} + \frac{31994375\,s^{10}dx^5x^2}{32} +$$
$$\frac{796578125\,s^{10}dx^4x^2}{96} - \frac{89149375\,s^{10}dx^5x}{96} + \frac{54359375\,s^{11}dx^4x^2}{32} -$$
$$\frac{76103125\,s^{11}xdx^4}{32} + \frac{54359375\,s^{11}dx^3x}{12} - \frac{192395\,s^6dx^6x}{144} + \frac{206815\,s^6dx^6x^2}{144} -$$
$$\frac{50225\,s^7dx^6x}{144} - \frac{369665625\,s^{10}xdx^4}{32} - \frac{740075875\,s^9dx^5x}{144} + \frac{11054625\,s^9dx^5x^2}{2} +$$
$$\frac{18025\,s^7dx^6x^2}{48} - \frac{895908779\,x^3s^5dx^5}{162} - \frac{724763525\,x^3s^6dx^4}{2592} -$$
$$\frac{6816963625\,x^2s^7dx^3}{162} + \frac{15699065225\,s^8dx^5x^2}{1152} + \frac{6573366875\,s^9x^2dx^4}{384} -$$
$$\frac{43849963025\,s^8dx^5x}{3456} - \frac{27166529875\,s^9xdx^4}{1152} + \frac{967421875\,s^{10}xdx^3}{36} +$$

$$\frac{1420955345\,s^7dx^5x^2}{72}+\frac{64554053875\,s^8x^2dx^4}{3456}-\frac{95395984145\,s^7dx^5x}{5184}-\frac{28948297175\,s^8xdx^4}{1152}+\frac{31040625625\,s^9xdx^3}{432}+\frac{52540054475\,s^7x^2dx^4}{5184}+\frac{148172142125\,s^8xdx^3}{1296}+\frac{420424675\,s^6x^2dx^4}{5184}+\frac{77965400675\,s^7xdx^3}{648}-\frac{407323\,s^2dx^6x^3}{2592}+\frac{6710144\,dx^3x}{2025}-\frac{651757\,dx^3}{270}$$

2. The operated D which is satisfy 4.1 when $f=x\,(y^2-x^3)$ is

$$D = \frac{3\,x^2dx^3dy^2s}{1024}+\frac{27\,x^2dy^4s^2}{256}+\frac{3\,x^2dx^3dy^2}{1024}+\frac{27\,sdy^4x^2}{128}+\frac{3\,xdx^2dy^2s^2}{256}+\frac{ydx^4dy\,s}{512}+\frac{9\,ydx\,dy^3s^2}{128}+\frac{1701\,x^2dy^4}{16384}+\frac{93\,dy^2sdx^2x}{2048}+\frac{ydx^4dy}{512}+\frac{9\,sdy^3dx\,y}{64}-\frac{dx^4s^2}{256}+\frac{3\,dx\,dy^2s^3}{32}+\frac{69\,xdx^2dy^2}{2048}+\frac{567\,ydx\,dy^3}{8192}-\frac{sdx^4}{128}+\frac{69\,s^2dy^2dx}{128}-\frac{dx^4}{256}+\frac{1689\,dy^2sdx}{2048}+\frac{3081\,dx\,dy^2}{8192}$$

3. The operated D which is satisfy 4.1 when $f=y\,(y^2-x^3)$ is

$$D = -\frac{7\,dy^5y^2}{729}-\frac{80\,dy^4y}{6561}-\frac{74812\,dx^3dy}{1594323}-\frac{9440\,dy\,sdx^3}{59049}-\frac{2\,s^3y^2dy^5}{243}+1/27\,s^4dy^4y-\frac{19\,s^2y^2dy^5}{729}+\frac{26\,s^3dy^4y}{243}-\frac{80\,s^3dy\,dx^3}{729}-\frac{344\,dy^3x^2dx^2}{177147}-\frac{10936\,dy\,xdx^4}{1594323}-\frac{640\,dy^3dx^3y^2}{531441}-\frac{32\,s^2ydx^6}{59049}-\frac{64\,sydx^6}{59049}-\frac{16\,dy\,s^4dx^3}{729}-\frac{20\,sy^2dy^5}{729}+\frac{595\,s^2dy^4y}{6561}-\frac{11852\,s^2dy\,dx^3}{59049}+\frac{56\,sdy^4y}{6561}-\frac{8\,dy\,s^3xdx^4}{2187}-\frac{272\,dy\,s^2xdx^4}{19683}-\frac{8\,dy^3s^2dx^3y^2}{6561}-\frac{16\,dy^3sdx^3y^2}{6561}+\frac{71\,dx\,yxdy^4}{59049}-\frac{79\,s^2dy^3dx\,x}{19683}+\frac{2680\,sdy^3dx\,x}{177147}-\frac{2\,s^3dy^3dx\,x}{81}+\frac{8\,dy^2s^3ydx^3}{2187}+\frac{68\,s^2dy^2dx^3y}{6561}-\frac{s^4dy^3dx\,x}{81}+\frac{1712\,sdy^2dx^3y}{177147}+\frac{14\,dy^4x^2ydx^2}{19683}-\frac{2\,dy^3s^3x^2dx^2}{729}-\frac{50\,dy^3s^2x^2dx^2}{6561}-\frac{1208\,dy^3sx^2dx^2}{177147}-\frac{14\,dy^5dx\,y^2x}{6561}-\frac{872\,dy^2xydx^4}{1594323}-\frac{3016\,dy\,sxdx^4}{177147}+\frac{776\,dy^3s}{2187}+\frac{172\,dy^3}{2187}+\frac{163\,s^2dx\,yxdy^4}{6561}+\frac{890\,sdx\,yxdy^4}{59049}+\frac{4\,dy^4s^2x^2ydx^2}{6561}+\frac{26\,sdy^4x^2ydx^2}{19683}-\frac{4\,s^2dy^5dx\,y^2x}{2187}-\frac{26\,sdy^5dx\,y^2x}{6561}-\frac{8\,dy^2s^2xydx^4}{19683}-\frac{56\,dy^2sxydx^4}{59049}+\frac{8\,s^3dy^4dx\,yx}{729}-\frac{2560\,ydx^6}{4782969}+\frac{1279\,dy^3s^2}{2187}+1/9\,s^4dy^3+\frac{34\,dy^3s^3}{81}+\frac{524\,dx^3dy^2y}{177147}+\frac{1204\,dy^3dx\,x}{177147}$$

4. The operated D which is satisfy 4.1 when $f=y\,(y-x^3)$ is

$$d=-\frac{23\,dx^3dy}{972}-1/24\,dy\,sdx^3-\frac{13\,dy\,xdx^4}{7776}-\frac{s^2dy\,dx^3}{54}+\frac{71\,sdy^2}{72}+\frac{sdy^2dx^3y}{432}-\frac{11\,s^2xdy^2dx}{144}+1/24\,dy^3dx\,xy+1/4\,dy^2s^3+\frac{dy^2xydx^4}{2592}-\frac{dy\,sxdx^4}{1296}+\frac{13\,dy^2}{36}+1/12\,sdy^3dx\,xy-1/24\,s^3dy^2dx\,x-\frac{s^2dy^2x^2dx^2}{72}-1/36\,dy^2sx^2dx^2+\frac{dy^2xdx}{108}+1/24\,s^2dy^3dx\,yx-\frac{dx^6}{11664}+\frac{7\,ydy^3s^2}{16}+\frac{71\,ydy^3s}{144}+\frac{dx^3dy^2y}{216}+\frac{13\,ydy^3}{72}-\frac{11\,dy^2sdx\,x}{432}-\frac{dy^2dx^2x^2}{72}+\frac{7\,s^2dy^2}{8}+1/8\,s^3ydy^3-\frac{dy\,dx^5x^2}{7776}$$

5. The operated D which is satisfy 4.1 when $f=y(y-x^4)$ is

$$D=\frac{285\,dy^2ydx^4}{524288}-\frac{1267\,dy\,sdx^4}{262144}-\frac{151\,dy\,s^2dx^4}{32768}-\frac{3\,dy\,s^3dx^4}{2048}+\frac{28821\,sdy^2}{16384}+\frac{9\,s^2xdy^2dx}{128}+\frac{1935\,dy^3dx\,xy}{32768}+1/4\,dy^2s^5+\frac{11\,dy^2s^4}{8}+\frac{385\,dy^2s^3}{128}+\frac{6165\,dy^2}{16384}+\frac{6863\,sdy^3dx\,xy}{32768}-\frac{885\,dy\,dx^4}{524288}-\frac{217\,s^3dy^2dx\,x}{2048}-\frac{43\,s^2dy^2x^2dx^2}{1024}-\frac{3389\,dy^2sx^2dx^2}{131072}+\frac{1845\,dy^2xdx}{32768}+\frac{35\,s^2dy^3dx\,yx}{128}+\frac{209\,ydy^3s^2}{128}+\frac{28821\,ydy^3s}{32768}+\frac{6165\,ydy^3}{32768}-\frac{3\,s^3dy^2x^3dx^3}{2048}-\frac{9\,s^2dy^2x^3dx^3}{2048}-\frac{579\,dy^2sx^3dx^3}{131072}+\frac{dy^2yx^2dx^6}{524288}+\frac{195\,dy^3yx^2dx^2}{32768}-\frac{dy\,sdx^6x^2}{262144}-\frac{15\,s^4dx\,dy^2x}{128}-1/32\,s^5dy^2dx\,x-\frac{s^4dy^2x^2dx^2}{128}-\frac{61\,s^3dy^2x^2dx^2}{2048}+\frac{23\,dy^2yxdx^5}{524288}-\frac{dy\,sdx^5x}{32768}+\frac{45\,dy^2sydx^4}{65536}+\frac{dy^2s^2ydx^4}{4096}+\frac{4805\,dy^2sdx\,x}{32768}-\frac{765\,dy^2dx^2x^2}{131072}+\frac{209\,s^2dy^2}{64}+\frac{dy^2syxdx^5}{65536}+1/32\,s^4dy^3ydx\,x+\frac{79\,s^3dy^3ydx\,x}{512}+\frac{3\,s^3dy^3yx^2dx^2}{512}+\frac{9\,s^2dy^3yx^2dx^2}{512}+\frac{579\,dy^3syx^2dx^2}{32768}-\frac{dx^8}{4194304}-\frac{195\,dy^2x^3dx^3}{131072}-\frac{25\,dy\,dx^6x^2}{2097152}-\frac{dy\,dx^5x}{16384}+1/8\,s^5dy^3y-\frac{dy\,x^3dx^7}{2097152}$$

6. The operated D which is satisfy 4.1 when $f=x^4+x^3y+x^2y^2+xy^3+y^4$ is

$$d=\frac{1167\,dx^3dy}{320000}-\frac{12831\,dx^2dy^2}{640000}-\frac{1601\,dx\,dy^3}{320000}+\frac{15767\,dx^5y}{768000}-\frac{650761\,dy^5y}{7680000}-\frac{140243\,dy^5x}{5120000}-\frac{97273\,dy^6y^2}{960000}+\frac{191\,dy^4}{10240}+\frac{573\,dx^4}{40000}+\frac{5202503\,sdy^7x^2y^2dx}{61440000}-\frac{2699\,dy^3y^4xdx^6}{512000}+\frac{123853\,dy^4y^4xdx^5}{5120000}-\frac{140973\,dy^5y^4xdx^4}{5120000}+\frac{96721\,dy^6y^4xdx^3}{5120000}-\frac{200263\,dy^7y^4xdx^2}{15360000}-\frac{16529\,sdy^8xy^3}{480000}+\frac{251569\,s^2dy^7xy^2}{3840000}-\frac{830671\,sy^2dy^8x^2}{30720000}+\frac{1316171\,ys^2dy^7x^2}{15360000}-\frac{130477\,ysdy^8x^3}{30720000}-\frac{26479\,ys^3dy^6x}{320000}+\frac{73601\,dy^8y^2dx\,x^3}{20480000}+\frac{23819\,dy^8y^3dx\,x^2}{12288000}-\frac{232729\,dy^3s^2y^3dx^4}{768000}+\frac{768721\,s^2dy^4y^3dx^3}{1536000}-$$

$$\frac{75341\,s^2dy^5y^3dx^2}{384000}+\frac{180841\,s^2dy^6y^3dx}{1536000}+\frac{7\,dy\,s^3y^2dx^5}{320}-\frac{11297\,dy^2s^3y^2dx^4}{38400}+$$
$$\frac{16069\,dy^3s^3y^2dx^3}{25600}+\frac{48859\,s^3dy^4y^2dx^2}{384000}+\frac{177\,dy^7ydx^3x^5}{5120000}+\frac{3591\,dy^3sdx^5x^4}{2560000}-$$
$$\frac{1771\,dy^2sdx^6x^4}{2560000}+\frac{33\,dy^3y^2dx^7x^4}{2560000}-\frac{243\,dy^4y^2dx^6x^4}{5120000}+\frac{131\,dy^5y^2dx^5x^4}{2560000}+$$
$$\frac{23\,dy^2ydx^7x^4}{320000}-\frac{77\,dy^3ydx^6x^4}{160000}+\frac{4759\,dy^4ydx^5x^4}{5120000}+\frac{11\,dy\,sdx^7x^4}{320000}+$$
$$\frac{33\,dy^2s^2dx^6x^4}{640000}-\frac{1167\,dy^3s^2dx^5x^4}{1280000}+\frac{247\,s^2dy^4dx^4x^4}{64000}-\frac{1504107\,sdy^7xy^2}{20480000}-$$
$$\frac{3834857\,s^2dy^6xy}{15360000}+\frac{21733\,sydy^7x^2}{3840000}+\frac{3443\,s^2dy^6dx^2x^4}{320000}-\frac{3477\,s^2dy^7dx\,x^4}{1280000}-$$
$$\frac{89\,s^2dy^5dx^3x^4}{10000}+\frac{11\,s^2dy^3y^4dx^5}{2560}-\frac{537\,s^2dy^4y^4dx^4}{25600}+\frac{153\,s^2dy^5y^4dx^3}{5120}-$$
$$\frac{59\,s^2dy^6y^4dx^2}{5120}+\frac{4589\,s^2dy^7dx\,y^4}{384000}+\frac{11\,dy^2s^3y^3dx^5}{1920}-\frac{217\,sdy^7y^5dx^2}{64000}-$$
$$\frac{289\,dy\,s^2dx^6x^3}{320000}-\frac{297\,dy^2s^3dx^5x^3}{640000}-\frac{245257\,s^3dy^5y^2dx}{384000}-\frac{19\,dy\,s^4ydx^4}{480}+$$
$$\frac{187\,dy^2s^4dx^3y}{2400}+\frac{3773\,s^4dy^3ydx^2}{16000}-\frac{1639\,s^4dy^4ydx}{4800}+\frac{33\,dy^2s^4dx^4x^2}{32000}-$$
$$\frac{1049\,dy^3s^4x^2dx^3}{80000}+\frac{85237\,sdy^4}{1280000}+\frac{843\,sdx^4}{64000}+\frac{3\,y^2dx^6}{1600}+\frac{9\,dy^9x^5}{5120000}+\frac{77\,dx^7x^3}{320000}+$$
$$\frac{45733\,s^2dy^4}{640000}+\frac{11\,s^2dx^4}{3200}+\frac{899\,dy^9y^5}{512000}+\frac{5237\,dy^8x^4}{10240000}+\frac{7163\,dy^7x^3}{2560000}+\frac{1347\,s^3dy^4}{80000}+$$
$$\frac{481031\,dy^7y^3}{10240000}-\frac{6899\,x^2dx^6}{2560000}-\frac{95279\,dy^6x^2}{5120000}+\frac{17293\,dx^5x}{2560000}+\frac{236827\,dy^8y^4}{10240000}+$$
$$\frac{199871\,dy^8x^2y^2}{15360000}-\frac{879\,dy^4dx^4x^4}{2560000}+\frac{sdx^7x^3}{6400}+\frac{4001\,dy^8dx\,y^5}{3072000}-\frac{137449\,s^2dy^7y^3}{768000}+$$
$$\frac{26669\,s^3dy^6y^2}{64000}+\frac{6059\,ys^4dy^5}{48000}+\frac{13069\,sdy^8y^4}{1536000}+\frac{3\,dy\,y^3dx^6}{1600}+$$
$$\frac{127751\,dy^2y^3dx^5}{1536000}-\frac{233\,dy^3s^4xydx^3}{9600}+\frac{10241\,s^4dy^4xydx^2}{120000}+\frac{1473\,s^4dy^5xydx}{32000}+$$
$$\frac{12157\,sdy^6ydx^3x^4}{5120000}+\frac{21479\,sdy^8ydx\,x^4}{7680000}-\frac{33\,dy^2s^2y^3dx^6x}{12800}+\frac{83\,s^2dy^3y^3dx^5x}{4000}-$$
$$\frac{1873\,s^2dy^4xy^3dx^4}{38400}+\frac{7591\,s^2dy^5xy^3dx^3}{192000}-\frac{219\,s^2dy^6xy^3dx^2}{16000}+\frac{15503\,s^2dy^7xy^3dx}{960000}+$$
$$\frac{121\,dy^2s^3xy^2dx^5}{7680}-\frac{6281\,dy^3s^3y^2dx^4x}{128000}-\frac{12503\,s^3dy^4y^2dx^3x}{1920000}-\frac{33\,dy^2s^2ydx^6x^3}{80000}+$$
$$\frac{33941\,sdy^8dx\,y^4x}{3840000}-\frac{13037\,sdy^3y^2dx^6x^3}{7680000}+\frac{7553\,sdy^4y^2dx^5x^3}{1536000}-$$
$$\frac{76159\,sdy^7ydx^2x^4}{15360000}-\frac{3169\,dy\,syx^2dx^6}{96000}+\frac{1813651\,dy^2sydx^5x^2}{7680000}-$$
$$\frac{2349009\,sdy^3ydx^4x^2}{5120000}+\frac{9040463\,sdy^4dx^3yx^2}{61440000}+\frac{643573\,sdy^5yx^2dx^2}{5120000}-$$
$$\frac{164609\,sdy^6yx^2dx}{2457600}-\frac{55\,dy\,sxy^2dx^6}{1536}+\frac{69567\,s^2dy^5xy^2dx^2}{640000}-$$

$$\frac{4866727\,s^2dy^6y^2xdx}{15360000}+\frac{2\,dy\,s^3xydx^5}{75}-\frac{56179\,dy^2s^3xydx^4}{384000}-\frac{25661\,dy^3s^3xydx^3}{768000}+$$
$$\frac{2790487\,s^3dy^4xydx^2}{3840000}-\frac{1367831\,s^3dy^5xydx}{3840000}-\frac{421\,sdy^3x^2y^3dx^6}{128000}+$$
$$\frac{77153\,sdy^4x^2y^3dx^5}{7680000}-\frac{46081\,sdy^5x^2y^3dx^4}{3840000}+\frac{8057\,sdy^6x^2y^3dx^3}{960000}-$$
$$\frac{19753\,sdy^7x^2y^3dx^2}{1920000}-\frac{413\,dy^3s^2ydx^5x^3}{2560000}-\frac{33\,sdy^3y^4xdx^6}{25600}+\frac{71\,sdy^4y^4xdx^5}{9600}-$$
$$\frac{49759\,sdy^5y^4xdx^4}{3840000}+\frac{9997\,sdy^6y^4xdx^3}{1280000}-\frac{30401\,sdy^7y^4xdx^2}{3840000}+$$
$$\frac{140581\,sdy^8y^2dx\,x^3}{15360000}+\frac{101843\,sdy^8y^3dx\,x^2}{7680000}-\frac{33\,dy^2s^4xydx^4}{6400}+\frac{3\,sy^2dx^6}{1280}-$$
$$\frac{22341\,dy^3dx^4x^3}{640000}+\frac{2483\,dy^4y^5dx^5}{1536000}-\frac{4787\,dy^5y^5dx^4}{1024000}+\frac{2989\,dy^6y^5dx^3}{1024000}+$$
$$\frac{15121\,dy^9y^4x}{2560000}-\frac{228601\,dy\,dx^4x}{1280000}+\frac{1857571\,dy^2dx^3x}{4096000}-\frac{6467\,xydx^6}{768000}+\frac{s^3dx^5x}{320}+$$
$$\frac{97\,s^2dx^5x}{6400}+\frac{2759\,dy^3dx^5x^4}{1280000}+\frac{17\,dy\,dx^7x^4}{256000}-\frac{1147\,dy^2dx^6x^4}{1280000}+\frac{21\,dy^{10}yx^5}{2560000}-$$
$$\frac{21\,sdy^9x^5}{640000}+\frac{793\,dy^6dx^3x^5}{2560000}+\frac{1049\,dy^8dx\,x^5}{5120000}-\frac{17\,dy^5dx^4x^5}{512000}-\frac{2929\,dy^7dx^2x^5}{5120000}+$$
$$\frac{48047\,dy^9y^3x^2}{10240000}+\frac{5457\,dy\,s^2dx^3}{320000}-\frac{2131\,dy^2s^2dx^2}{320000}-\frac{11421\,dy^3s^2dx}{160000}+$$
$$\frac{4056641\,ysdy^5}{30720000}+\frac{983\,sydx^5}{15360}+\frac{647\,dy\,sdx^3}{20000}-\frac{31893\,dy^2sdx^2}{640000}-\frac{11351\,sdy^3dx}{320000}-$$
$$\frac{217\,sx^2dx^6}{51200}+\frac{43853\,dy\,dx^5x^2}{640000}-\frac{4638879\,dy^2dx^4x^2}{20480000}+\frac{231523\,dy^3x^2dx^3}{1280000}-$$
$$\frac{5109\,s^4dy^6x^2}{160000}+\frac{1097\,s^3dy^7x^3}{160000}-\frac{3\,xy^2dx^7}{6400}+\frac{dy^2y^4dx^6}{3200}+\frac{11113\,dy^3y^4dx^5}{512000}-$$
$$\frac{13799\,dy^4y^4dx^4}{160000}+\frac{618679\,dy^5y^4dx^3}{10240000}-\frac{442309\,dy^6y^4dx^2}{10240000}+\frac{629837\,dy^7dx\,y^4}{30720000}-$$
$$\frac{2907\,dy^5dx^3x^4}{1024000}-\frac{3713\,dy^7dx\,x^4}{5120000}+\frac{53897\,dy^6dx^2x^4}{20480000}+\frac{593117\,dy^2dx^5x^3}{20480000}+$$
$$\frac{97\,s^2ydx^5}{1920}+\frac{1499\,dy^9yx^4}{10240000}+\frac{823\,sdy^8x^4}{5120000}+\frac{21379\,dy^9y^2x^3}{15360000}+\frac{59\,s^3dy\,dx^3}{32000}+$$
$$\frac{1029\,s^3dy^2dx^2}{160000}-\frac{827\,s^3dy^3dx}{32000}-\frac{1922629\,dy^3y^3dx^4}{3840000}+\frac{1788083\,dy^4y^3dx^3}{3840000}-$$
$$\frac{7923193\,dy^5y^3dx^2}{30720000}+\frac{77513\,dy^6y^3dx}{480000}-\frac{15162589\,sdy^6y^2}{30720000}+\frac{1549079\,s^2dy^5y}{1536000}+$$
$$\frac{108637\,dy^4x^3dx^3}{20480000}+\frac{27841\,dy^5x^3dx^2}{5120000}-\frac{16799\,dy^6dx\,x^3}{20480000}-\frac{43519\,s^2dy^6x^2}{640000}+$$
$$\frac{3577\,dy^8x^3y}{960000}+\frac{74521\,dx^5dy\,y^2}{768000}-\frac{4002439\,dx^4dy^2y^2}{3840000}+\frac{46981757\,dx^3dy^3y^2}{30720000}-$$
$$\frac{5788361\,dx^2dy^4y^2}{7680000}+\frac{12430541\,dx\,dy^5y^2}{30720000}-\frac{482327\,dy^6xy}{5120000}-\frac{222347\,dx^4dy\,y}{384000}+$$
$$\frac{45734579\,dx^3dy^2y}{30720000}-\frac{89921\,dx^2dy^3y}{128000}+\frac{2234097\,dx\,dy^4y}{10240000}-\frac{127\,sdy^3ydx^6x^4}{512000}+$$

$$
\begin{aligned}
&\frac{3869\, s dy^4 y dx^5 x^4}{5120000} + \frac{285687\, dy^3 s y dx^5 x^3}{10240000} - \frac{14557\, y s dy^5 dx^4 x^4}{15360000} + \\
&\frac{75031\, s^2 dy^4 x^3 y dx^4}{7680000} - \frac{258809\, s^2 dy^5 x^3 y dx^3}{7680000} + \frac{331531\, s^2 dy^6 x^3 y dx^2}{7680000} - \\
&\frac{12167\, s^2 dy^7 y dx\, x^3}{1280000} - \frac{209\, dy^2 s^2 x^2 y^2 dx^6}{76800} + \frac{29627\, s^2 dy^3 y^2 dx^5 x^2}{1920000} - \\
&\frac{59037\, s^2 dy^4 x^2 y^2 dx^4}{2560000} - \frac{10931\, s^2 dy^5 x^2 y^2 dx^3}{2560000} + \frac{163399\, s^2 dy^6 x^2 y^2 dx^2}{7680000} + \\
&\frac{17407\, s^2 dy^7 x^2 y^2 dx}{7680000} - \frac{85229\, s dy^5 x^3 y^2 dx^4}{15360000} + \frac{34951\, s dy^6 x^3 y^2 dx^3}{5120000} - \\
&\frac{180799\, s dy^7 x^3 y^2 dx^2}{15360000} + \frac{273943\, s^3 dy^5 x y^2 dx^2}{1920000} - \frac{77249\, s^3 dy^6 y^2 x dx}{384000} - \\
&\frac{229\, dy\, s^2 y x^2 dx^6}{38400} + \frac{517081\, dy^2 s^2 y dx^5 x^2}{7680000} - \frac{1863091\, dy^3 s^2 y dx^4 x^2}{15360000} - \\
&\frac{133021\, s^2 dy^4 dx^3 y x^2}{1024000} + \frac{277951\, s^2 dy^5 y x^2 dx^2}{768000} - \frac{1106509\, s^2 dy^6 y x^2 dx}{3072000} + \\
&\frac{135037\, dy^2 s x y^2 dx^5}{307200} - \frac{4072379\, s dy^3 y^2 dx^4 x}{3072000} + \frac{2746197\, s dy^4 y^2 dx^3 x}{2560000} - \\
&\frac{22950769\, s dy^5 x y^2 dx^2}{61440000} + \frac{3342569\, s dy^6 y^2 x dx}{15360000} + \frac{16297\, dy\, s^2 x y dx^5}{76800} - \\
&\frac{314003\, dy^2 s^2 x y dx^4}{307200} + \frac{14362949\, dy^3 s^2 x y dx^3}{15360000} + \frac{2214741\, s^2 dy^4 x y dx^2}{2560000} - \\
&\frac{10609903\, s^2 dy^5 x y dx}{15360000} - \frac{223819\, s dy^4 x^3 y dx^4}{12288000} - \frac{359369\, s dy^5 x^3 y dx^3}{15360000} + \\
&\frac{2225401\, s dy^6 x^3 y dx^2}{61440000} + \frac{26827\, s dy^7 y dx\, x^3}{3840000} + \frac{7\, dy\, s x^2 y^2 dx^7}{6400} - \frac{39113\, dy^2 s x^2 y^2 dx^6}{1536000} + \\
&\frac{549747\, s dy^3 y^2 dx^5 x^2}{5120000} - \frac{943229\, s dy^4 x^2 y^2 dx^4}{6144000} + \frac{207949\, s dy^5 x^2 y^2 dx^3}{2457600} - \\
&\frac{834783\, s dy^3 dx^4 x^3}{20480000} - \frac{dy^2 dx^7 x y^4}{12800} - \frac{3\, dy\, dx^7 x y^3}{6400} + \frac{5129\, s dy^9 y^4 x}{1920000} - \\
&\frac{20747\, s^2 dy^8 x y^3}{480000} - \frac{31\, dy^6 y^2 dx^4 x^4}{320000} + \frac{633\, dy^7 y^2 dx^3 x^4}{2560000} - \frac{893\, dy^8 y^2 dx^2 x^4}{7680000} - \\
&\frac{2611\, dy^9 y^2 dx\, x^4}{15360000} + \frac{11\, dy^3 y^3 dx^7 x^3}{160000} - \frac{791\, dy^4 y^3 dx^6 x^3}{2560000} + \frac{1391\, dy^5 y^3 dx^5 x^3}{3072000} - \\
&\frac{dy^6 y^3 dx^4 x^3}{2500} + \frac{5417\, dy^7 y^3 dx^3 x^3}{7680000} - \frac{299\, dy^8 y^3 dx^2 x^3}{480000} - \frac{2279\, dy^9 y^3 dx\, x^3}{5120000} - \\
&\frac{143\, dy^4 y^5 dx^6 x}{384000} + \frac{307\, dy^5 y^5 dx^5 x}{256000} - \frac{7\, dy^6 y^5 dx^4 x}{6400} + \frac{119\, dy^7 y^5 dx^3 x}{128000} - \\
&\frac{107\, dy^8 y^5 dx^2 x}{192000} - \frac{69\, dy^9 dx\, y^5 x}{256000} + \frac{62851\, s dy^6 dx^2 x^4}{5120000} - \frac{6109\, s dy^7 dx\, x^4}{2560000} + \\
&\frac{653\, dy^6 y dx^3 x^4}{245760} + \frac{24161\, dy^8 y dx\, x^4}{15360000} - \frac{55991\, s dy^5 dx^3 x^4}{5120000} + \frac{187\, dy^2 s^4 dx^3 x}{8000} + \\
&\frac{11319\, s^4 dy^3 x dx^2}{160000} - \frac{1639\, s^4 dy^4 x dx}{16000} + \frac{dy^2 s y^4 dx^6}{2560} + \frac{1473\, s dy^3 y^4 dx^5}{51200} - \\
&\frac{30481\, s dy^4 y^4 dx^4}{256000} + \frac{99267\, s dy^5 y^4 dx^3}{1024000} - \frac{15653\, s dy^6 y^4 dx^2}{256000} + \frac{7229\, s dy^7 dx\, y^4}{204800} +
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1981\,dy^2s^2y^3dx^5}{38400}-\frac{19\,dy^8ydx^2x^5}{5120000}-\frac{59\,dy^9ydx\,x^5}{2560000}-\frac{11\,sdy^5dx^4x^5}{640000}+\\
&\frac{69\,sdy^6dx^3x^5}{256000}-\frac{13\,sdy^7dx^2x^5}{20000}+\frac{391\,sdy^8dx\,x^5}{1280000}-\frac{33\,dy^6ydx^4x^5}{2560000}-\frac{587\,dy^8dx\,y^4x}{5120000}-\\
&\frac{6641\,dy\,s^3xdx^4}{64000}+\frac{13619\,dy^2s^3dx^3x}{51200}+\frac{286123\,dy^3s^3xdx^2}{1280000}-\frac{23639\,s^3dy^4xdx}{51200}+\\
&\frac{3\,dy\,sy^3dx^6}{1280}+\frac{845\,dy^2sy^3dx^5}{6144}-\frac{212807\,sdy^3y^3dx^4}{256000}+\frac{2941549\,sdy^4y^3dx^3}{3072000}-\\
&\frac{1000309\,sdy^5y^3dx^2}{2048000}+\frac{1696029\,sdy^6y^3dx}{5120000}+\frac{307\,dy\,s^2y^2dx^5}{2400}+\frac{3401\,s^4dy^4x^2dx^2}{160000}+\\
&\frac{3717\,s^4dy^5x^2dx}{160000}-\frac{391\,dy^3s^3y^3dx^4}{12800}+\frac{7949\,s^3dy^4y^3dx^3}{192000}+\frac{851\,s^3dy^5y^3dx^2}{192000}-\\
&\frac{1403\,s^3dy^6y^3dx}{38400}-\frac{11\,dy^2s^4y^2dx^4}{384}+\frac{311\,dy^3s^4y^2dx^3}{4800}+\frac{1159\,s^4dy^4y^2dx^2}{24000}-\\
&\frac{67\,s^4dy^5y^2dx}{640}+\frac{11\,dy^3y^4dx^7x^2}{128000}-\frac{2359\,dy^4y^4dx^6x^2}{3840000}+\frac{2497\,dy^5y^4dx^5x^2}{1920000}-\\
&\frac{17\,dy^6y^4dx^4x^2}{16000}+\frac{2233\,dy^7y^4dx^3x^2}{1920000}-\frac{697\,dy^8y^4dx^2x^2}{640000}-\frac{2033\,dy^9y^4dx\,x^2}{3840000}+\\
&\frac{395993\,dy\,ydx^5x}{1536000}-\frac{32817947\,dy^2ydx^4x}{30720000}+\frac{21793649\,dy^3ydx^3x}{20480000}-\\
&\frac{3843871\,dy^5x^2ydx^2}{61440000}+\frac{1080499\,dy^6ydx\,x^2}{30720000}+\frac{14007\,sdy^4dx^2x^2}{128000}-\\
&\frac{935829\,sdy^5dx\,x^2}{20480000}+\frac{2925113\,dy^2y^2dx^5x}{7680000}-\frac{28180973\,dy^3xy^2dx^4}{30720000}+\\
&\frac{10539301\,dy^4xy^2dx^3}{15360000}-\frac{21257509\,dy^5xy^2dx^2}{61440000}+\frac{107551\,dy^6xy^2dx}{614400}-\\
&\frac{24129071\,sydy^6x}{61440000}-\frac{273263\,dy\,sydx^4}{192000}+\frac{1213213\,dy^2sydx^3}{307200}-\\
&\frac{41971453\,sdy^3ydx^2}{30720000}-\frac{678239\,sdy^4ydx}{2560000}-\frac{552379\,dy\,sdx^4x}{1280000}+\\
&\frac{6067917\,dy^2sdx^3x}{5120000}-\frac{418043\,s^2dy\,dx^4x}{1280000}+\frac{4877813\,s^2dy^2dx^3x}{5120000}-\\
&\frac{12563\,s^2dy^3xdx^2}{1280000}-\frac{2734541\,s^2dy^4dx\,x}{5120000}-\frac{3779089\,dy^4yxdx^2}{10240000}+\\
&\frac{2781631\,dy^5yxdx}{12288000}-\frac{1633501\,sdy^3xdx^2}{4096000}-\frac{444821\,sdy^4xdx}{5120000}-\\
&\frac{118379\,dy^2s^2y^2dx^4}{96000}+\frac{20816647\,dy^3s^2y^2dx^3}{7680000}-\frac{1471433\,s^2dy^4y^2dx^2}{1920000}-\\
&\frac{200913\,s^2dy^5y^2dx}{512000}-\frac{6641\,dy\,s^3ydx^4}{19200}+\frac{341231\,dy^2s^3dx^3y}{384000}+\\
&\frac{1419779\,dy^3s^3ydx^2}{1920000}-\frac{118519\,s^3dy^4ydx}{76800}-\frac{430537\,dy^5x^3y^2dx^4}{61440000}+\\
&\frac{170401\,dy^6x^3y^2dx^3}{20480000}-\frac{780223\,dy^7x^3y^2dx^2}{61440000}-\frac{53527\,dy^4x^3ydx^4}{2048000}-\\
&\frac{440307\,sdy^6x^2y^2dx^2}{10240000}-\frac{19\,dy\,s^4xdx^4}{1600}+\frac{87659\,dy^5x^3ydx^3}{20480000}-\frac{65069\,dy^6x^3ydx^2}{30720000}+
\end{aligned}
$$

$$\frac{196141\,dy^7ydx\,x^3}{61440000}-\frac{272093\,sdy^4x^3dx^3}{10240000}+\frac{1190841\,sdy^5x^3dx^2}{20480000}-\frac{7867\,sdy^6dx\,x^3}{256000}-$$
$$\frac{253021\,dy^2x^2y^2dx^6}{7680000}+\frac{486073\,dy^3y^2dx^5x^2}{3840000}-\frac{1533067\,dy^4x^2y^2dx^4}{10240000}+$$
$$\frac{5469713\,dy^5x^2y^2dx^3}{61440000}-\frac{1982111\,dy^6x^2y^2dx^2}{30720000}+\frac{1839539\,dy^7x^2y^2dx}{61440000}-$$
$$\frac{235283\,dy\,yx^2dx^6}{7680000}+\frac{1561333\,dy^2ydx^5x^2}{7680000}-\frac{7097337\,dy^3ydx^4x^2}{20480000}+$$
$$\frac{5582843\,dy^4dx^3yx^2}{30720000}+\frac{66749\,dy\,s^2dx^5x^2}{1280000}+\frac{131431\,dy\,sdx^5x^2}{1280000}-$$
$$\frac{282439\,dy^7ydx^2x^4}{61440000}-\frac{1958469\,dy^2sdx^4x^2}{5120000}-\frac{1001083\,dy^2s^2dx^4x^2}{5120000}+$$
$$\frac{5318841\,sdy^3x^2dx^3}{20480000}+\frac{46007\,dy^3s^2x^2dx^3}{1280000}+\frac{1687161\,s^2dy^4x^2dx^2}{5120000}-$$
$$\frac{891469\,s^2dy^5x^2dx}{5120000}+\frac{302177\,dy^3y^3dx^5x}{1920000}-\frac{3797561\,dy^4xy^3dx^4}{15360000}+$$
$$\frac{9964349\,dy^5xy^3dx^3}{61440000}-\frac{6578893\,dy^6xy^3dx^2}{61440000}+\frac{892523\,dy^7xy^3dx}{20480000}-\frac{3\,s^2xydx^6}{640}-$$
$$\frac{1031\,sxydx^6}{76800}+\frac{22823\,dy^3s^2dx^4x^3}{5120000}-\frac{321873\,s^2dy^4x^3dx^3}{5120000}+\frac{478691\,s^2dy^5x^3dx^2}{5120000}-$$
$$\frac{390977\,s^2dy^6dx\,x^3}{5120000}+\frac{179\,dy^2x^2y^3dx^7}{192000}-\frac{16337\,dy^3x^2y^3dx^6}{1920000}+$$
$$\frac{234347\,dy^4x^2y^3dx^5}{10240000}-\frac{428881\,dy^5x^2y^3dx^4}{20480000}+\frac{185413\,dy^6x^2y^3dx^3}{12288000}-$$
$$\frac{65899\,dy^7x^2y^3dx^2}{4096000}+\frac{79\,dy\,x^2y^2dx^7}{38400}+\frac{syx^2dx^7}{1920}+\frac{193\,dy\,s^3dx^5x^2}{32000}-$$
$$\frac{1393\,dy^2s^3dx^4x^2}{80000}-\frac{85141\,dy^3s^3x^2dx^3}{1280000}+\frac{264391\,s^3dy^4x^2dx^2}{1280000}-$$
$$\frac{31603\,s^3dy^5x^2dx}{640000}+\frac{1889\,sdy^7x^3}{409600}+\frac{17957\,s^3dy^5x}{64000}+\frac{570501\,dy^8xy^3}{20480000}-$$
$$\frac{594391\,s^2dy^6y^2}{7680000}+\frac{452651\,s^3dy^5y}{480000}+\frac{127\,yx^2dx^7}{192000}+\frac{861373\,sdy^5x}{20480000}-$$
$$\frac{3441\,dy^3xdx^2}{16000}+\frac{1547819\,s^2dy^5x}{5120000}+\frac{102281\,dy^7x^2y}{5120000}-\frac{371671\,sdy^6x^2}{5120000}+$$
$$\frac{1013029\,dy^7xy^2}{20480000}+\frac{278579\,dy^4dx\,x}{4096000}+\frac{1001\,sdx^5x}{51200}-\frac{33883\,dy^4dx^2x^2}{819200}+$$
$$\frac{149623\,dy^5dx\,x^2}{5120000}-\frac{79779\,s^3dy^6x^2}{1280000}+\frac{4793\,s^2dy^7x^3}{320000}+\frac{6059\,s^4dy^5x}{160000}-$$
$$\frac{2989\,dy^7y^5dx^2}{1024000}-\frac{16283\,dy\,dx^6x^3}{2560000}+\frac{21\,dy^{10}y^5x}{128000}+\frac{49\,dy^{10}y^2x^4}{640000}+\frac{651\,dy^{10}y^3x^3}{2560000}+$$
$$\frac{7\,dy^{10}y^4x^2}{20000}-\frac{9\,s^2x^2dx^6}{6400}+\frac{s^3ydx^5}{96}+\frac{2227\,s^3dy^7y^3}{48000}+\frac{131\,s^4dy^6y^2}{6000}-$$
$$\frac{4061\,s^2dy^8y^4}{192000}+\frac{221\,s^2dy^8x^4}{640000}+\frac{131\,sdy^9y^5}{64000}-\frac{1404549\,sdy^7y^3}{10240000}+$$
$$\frac{11\,dy^2sy^2dx^7x^3}{76800}+\frac{11\,dy^2sy^3dx^7x^2}{38400}+\frac{11\,dy^2sydx^7x^4}{640000}+\frac{17\,dy\,sydx^7x^3}{38400}+$$

$$
\begin{aligned}
&\frac{341\,dy^2 s^3 y dx^5 x^2}{128000} + \frac{11437\,dy^3 s^3 y dx^4 x^2}{960000} - \frac{10531\,s^3 dy^4 dx^3 y x^2}{128000} + \\
&\frac{103459\,s^3 dy^5 y x^2 dx^2}{640000} - \frac{35759\,s^3 dy^6 y x^2 dx}{192000} - \frac{153\,dy^2 s y^3 dx^6 x}{10240} + \\
&\frac{55553\,s dy^3 y^3 dx^5 x}{512000} - \frac{1823429\,s dy^4 x y^3 dx^4}{7680000} + \frac{2550739\,s dy^5 x y^3 dx^3}{15360000} - \\
&\frac{840757\,s dy^6 x y^3 dx^2}{7680000} + \frac{389641\,s dy^7 x y^3 dx}{3072000} - \frac{63\,dy\,s^2 x y^2 dx^6}{6400} + \\
&\frac{44151\,dy^2 s^2 x y^2 dx^5}{256000} - \frac{697287\,dy^3 s^2 y^2 dx^4 x}{1280000} + \frac{2128071\,s^2 dy^4 y^2 dx^3 x}{5120000} - \\
&\frac{65971\,dy^2 s y dx^6 x^3}{7680000} + \frac{311813\,s dy\,y dx^5 x}{768000} - \frac{4893843\,s dy^2 y dx^4 x}{2560000} + \\
&\frac{20456709\,s dy^3 y dx^3 x}{10240000} - \frac{8904581\,s dy^4 y x dx^2}{61440000} + \frac{1268389\,s dy^5 y x dx}{30720000} + \\
&\frac{89599\,dy^2 s dx^5 x^3}{2560000} - \frac{113123\,dy^3 y^2 dx^6 x^3}{30720000} - \frac{332963\,dy^2 y dx^6 x^3}{30720000} - \frac{847\,dy\,s dx^6 x^3}{128000} + \\
&\frac{7769\,dy^2 s^2 dx^5 x^3}{1280000} + \frac{96221\,dy^4 y^2 dx^5 x^3}{12288000} + \frac{1940249\,dy^3 y dx^5 x^3}{61440000} + \frac{46279\,s^3 dy^7 x y^2}{320000} - \\
&\frac{64363\,s^2 y^2 dy^8 x^2}{3840000} + \frac{23833\,y s^3 dy^7 x^2}{384000} - \frac{67\,y s^2 dy^8 x^3}{120000} - \frac{7991\,y s^4 dy^6 x}{80000} - \\
&\frac{151\,s dy^9 y^3 x^2}{256000} - \frac{201\,s dy^9 y x^4}{512000} - \frac{10031\,s dy^9 y^2 x^3}{7680000} - \frac{6907\,dy^2 x y^3 dx^6}{307200} - \\
&\frac{23719\,dy\,y^2 x dx^6}{768000} + \frac{217\,s dy\,y^2 dx^5}{1024} - \frac{812783\,s dy^2 y^2 dx^4}{384000} + \frac{28850497\,s dy^3 y^2 dx^3}{7680000} - \\
&\frac{17099989\,s dy^4 y^2 dx^2}{10240000} + \frac{1529973\,s dy^5 y^2 dx}{2560000} - \frac{27847\,s^2 dy\,y dx^4}{25600} + \\
&\frac{1632111\,s^2 dy^2 y dx^3}{512000} - \frac{145807\,s^2 dy^3 y dx^2}{1920000} - \frac{13501969\,s^2 dy^4 y dx}{7680000} + \\
&\frac{587\,s dy^8 dx\,y^5}{384000} + \frac{143\,s dy^4 y^5 dx^5}{76800} - \frac{87\,s dy^5 y^5 dx^4}{16000} + \frac{217\,s dy^6 y^5 dx^3}{64000} + \\
&\frac{2293\,dy^3 s^3 dx^4 x^3}{320000} - \frac{7353\,s^3 dy^4 x^3 dx^3}{320000} + \frac{1431\,s^3 dy^5 x^3 dx^2}{40000} - \frac{24803\,s^3 dy^6 dx\,x^3}{640000} + \\
&\frac{14881\,s dy^4 dx^4 x^4}{5120000} - \frac{7939\,dy^5 y dx^4 x^4}{7680000} + \frac{2021\,dy^2 y^2 dx^7 x^3}{3840000} + \frac{1691\,dy\,y dx^7 x^3}{1920000}
\end{aligned}
$$

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