

# Classical resummation and breakdown of strong-field QED

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QED perturbation theory has been conjectured to break down in sufficiently strong backgrounds, obstructing the analysis of strong-field physics. We show that the breakdown occurs even in classical electrodynamics, at lower field strengths than previously considered, and that it may be cured by resummation. As a consequence, an analogous resummation is required in QED. A detailed investigation shows, for a range of observables, that unitarity removes diagrams previously believed to be responsible for the breakdown of QED perturbation theory.

Examining the transition to classical physics can help us understand quantum theories, with topical examples being the classical post-Minkowskian expansion of general relativistic dynamics [1, 2], classical double copy [3, 4] and decoherence [5]. Whether classical or quantum, theories containing strong background fields are typically analysed using background field perturbation theory [6, 7], where the strong background is treated exactly (as a classical field), while particle scattering on the background is treated perturbatively. For quantum electrodynamics (QED) in strong fields, this amounts to employing the usual perturbative expansion in the fine structure constant,  $\alpha \ll 1$ , while fermion propagators are ‘dressed’ exactly by the background: this is also called the Furry expansion [8]. It is an essential tool for theory and experimental modelling, and underlies numerical particle-in-cell schemes used in astrophysics [9] and plasma physics [10].

However, the Ritus-Narozhny (RN) conjecture suggests that the Furry expansion breaks down for sufficiently strong fields [11, 12]; for constant fields, as originally formulated, one finds that the effective expansion parameter is not  $\alpha$ , but  $\alpha\chi^{2/3}$  [13], where  $\chi$  is proportional to the background field strength. The conjecture has been interpreted to hold for any background that can be approximated as “locally constant”, i.e. constant over typical “formation scales” [13, 14]. Due to the widespread use of the Furry, and related, expansions, it is crucial to understand their regime of applicability [15–19].

The RN conjecture applies at high field strength, and low energy [20, 21], suggesting it suits a classical analysis. Using this approach, we will show here explicitly that the breakdown of perturbation theory occurs already in the classical theory—and not just for constant fields. Crucially, this allows further progress than is possible in the quantum theory, as we can effectively resum the classical perturbative series to all orders. We will show that this resummation cures the unphysical behaviour associated with the breakdown of perturbation theory. As every term in the classical limit corresponds to some term in QED, our results have direct implication for the quantum theory: in particular, we find that perturbation theory breaks down for far lower intensities than predicted by the RN conjecture.

*Classical.* A strong background,  $f^{\mu\nu} = eF_{\text{ext}}^{\mu\nu}/m$ , for  $m$  and  $e$  the electron mass and charge respectively, is characterised by a dimensionless coupling  $\xi \sim f/\omega \gg 1$ , for  $\omega$  a typical frequency scale of the background. The classical equations of motions in such a background are

$$\ddot{x}^\mu = (f^{\mu\nu} + eF^{\mu\nu}/m)\dot{x}_\nu, \quad \partial_\mu F^{\mu\nu} = j^\nu, \quad (1)$$

in which  $F$  is the generated radiation field,  $x^\mu$  is the particle orbit and  $j^\nu$  its current. (Note that  $c = 1$  throughout.) The classical limit of the Furry expansion corresponds to treating  $f$ , therefore  $\xi$ , exactly, and  $e$  (made appropriately dimensionless) perturbatively. The zeroth order equations describe the Lorentz orbit in the background  $f$ , with no radiation. At higher orders, radiation and radiation back-reaction (‘RR’) appear [14, 22–27]. The assumption behind the Furry expansion is simply that these RR corrections, corresponding to higher powers of  $\alpha$  in QED, are subleading. We now give two examples which show this is not the case, leading to a breakdown of the perturbative expansion.

First, an electron in a rotating electric field  $\mathbf{E}(t) = E_0(0, \cos \omega t, \sin \omega t)$  can have a closed orbit, with energy  $m\gamma$  determined by  $\xi = eE_0/(m\omega)$  and  $\omega$  [28]. The Lorentz force prediction for the energy is  $\gamma^2 - 1 = \xi^2$ . Using Furry picture perturbation theory to calculate corrections to this, one finds it is not an expansion in some small parameter, but rather in powers of  $\epsilon_{\text{rad}}\xi^3$ , where  $\epsilon_{\text{rad}} := (2/3)(e^2/4\pi)(\omega/m)$ , with leading behaviour

$$\gamma^2 - 1 \sim \xi^2(1 - \epsilon_{\text{rad}}^2\xi^6 + \dots). \quad (2)$$

Hence, for sufficiently strong fields, the corrections become larger than the supposedly dominant Lorentz force contribution and the perturbative expansion breaks down, signalled in (2) by the unphysical result  $\gamma^2 - 1 < 0$ .

For our second example, consider an electron in an arbitrary plane wave (direction  $n_\mu$ , typical frequency  $\omega$ ,  $k_\mu := \omega n_\mu$ , phase  $\phi = k \cdot x$ ) with transverse electric field  $\mathbf{a}'(\phi)$ . According to the Lorentz force, i.e. zeroth order in perturbation theory, the lightfront energy component  $n \cdot p$  of the electron is conserved. The first perturbative correction to the final electron momentum  $p_{\text{out}}$  is

$$\frac{n \cdot p_{\text{out}}}{n \cdot p} = 1 - \frac{2}{3} \frac{e^2}{4\pi} \frac{k \cdot p}{m^4} \int d\phi |\mathbf{a}'(\phi)|^2 \equiv 1 - \Delta \quad (3)$$

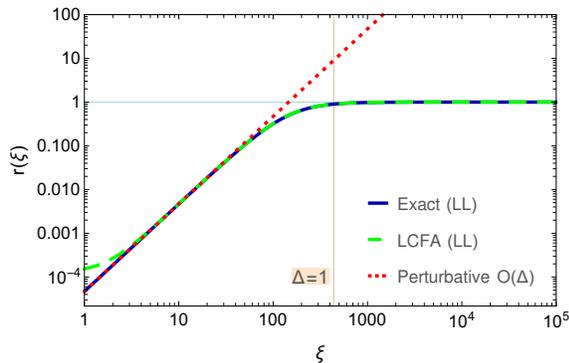


FIG. 1. Proportion,  $r(\xi)$ , of initial electron energy radiated in a 4-cycle circularly-polarised plane wave pulse  $\mathbf{a}(\phi) = m\xi \sin^2(\phi/8)[\cos \phi, \sin \phi]$  for  $\phi \in [0, 8\pi]$  and  $\mathbf{a}(\phi) = \mathbf{0}$  otherwise. The energy radiated is bounded by the initial electron energy in the LL result, but is unbounded for the Lorentz result. The LCFA discussed in the text is also shown: it characteristically over-predicts the radiated energy for  $\xi \sim \mathcal{O}(1)$ .

The effective expansion parameter is  $\Delta \propto \xi^2$ , which again may not be small; the expansion breaks down for  $\xi \gg 1$ , signalled here by the unphysical behaviour  $n \cdot p_{\text{out}} < 0$ .

In some cases it is possible to explicitly resum perturbative solutions to (1) [29]. A more general approach is to *effectively* resum the perturbative series by eliminating the electromagnetic variables from (1) to obtain the exact Lorentz-Abraham-Dirac (LAD) equation for the electron orbit [30]. For our first example, LAD implies that  $\gamma$  satisfies the equation  $\xi^2 = (\gamma^2 - 1)(1 + \epsilon_{\text{rad}}^2 \gamma^6)$  [28]. This recovers (2) if  $\epsilon_{\text{rad}}$  is treated perturbatively, but behaves as  $\gamma^2 - 1 \sim \xi^2 (\epsilon_{\text{rad}} \xi^3)^{-1/2}$  for  $\epsilon_{\text{rad}} \xi^3 \gg 1$ , i.e. resummation corrects the unphysical behaviour of perturbation theory. For plane waves, the solution to the LAD equation is not known, so we use the Landau-Lifshitz (LL) equation instead [31], which agrees exactly with LAD to low orders and is adequate classically [32, 33]. (What is important is that both LAD and LL equations provide all-order results.) The exact solution to LL yields [34]

$$\frac{n \cdot p_{\text{out}}}{n \cdot p} = \frac{1}{1 + \Delta} > 0, \quad (4)$$

so that, comparing to (3) [35], resummation again fixes the unphysical behaviour of perturbation theory.

Note that the coefficient of the integral in (3) is  $(2/3)\alpha\eta$ , with the QED energy invariant  $\eta := \hbar \mathbf{k} \cdot \mathbf{p}/m^2$ . This underlines that both perturbative and resummed results have their origin in QED [36]. To begin making connections to QED we first need to understand in more depth what changes when we go from low orders of perturbation theory to all-orders results. To do so we consider the energy-momentum  $K_\mu$  radiated by an electron in a plane wave. This is calculated by inserting the LL solution for the electron orbit into the fully relativistic Larmor formula [33]. We focus for simplicity on the lightfront momentum fraction  $r := n \cdot K/n \cdot p$ .

In Fig. 1, we show how the ratio  $r$  depends on the intensity of a pulsed plane wave. The leading,  $\mathcal{O}(\Delta)$ , result scales without bound as  $r \sim \Delta \sim \xi^2$ ; to this order in  $\Delta$  one has that  $r = 1 - n \cdot p_{\text{out}}/n \cdot p$ , hence reaching  $r > 1$  reflects the unphysical behaviour in (3). The behaviour of the all-orders, or resummed, result, is completely different: the total energy radiated is bounded by  $r \leq 1$ , as demonstrated by the plateau in Fig. 1. Thus the effect of resummation is very clear, and physically sensible, but to help understand it we analyse the *formation* of the emitted radiation as a function of phase  $\phi$ . Following the established procedure of expanding double phase integrals in the difference of two phases [37, 38], we develop a locally constant field approximation (LCFA) for our LL-corrected observables. Let  $\Delta(\phi)$  be defined as in (3) but with the integral extending only up to  $\phi$ , and define  $\mathcal{R} := 1 + \Delta(\phi)$ . Then we find the classically resummed, LCFA result [39]

$$\frac{dr}{d\phi} = -\frac{e^2 m^2}{4\pi k \cdot p} \int_0^\infty d\bar{s} \bar{s} \mathcal{R}^2 \left[ \text{Ai}_1(z) + \frac{2}{z} \text{Ai}'(z) \right], \quad (5)$$

where  $z = (\bar{s} \mathcal{R}^2 / \bar{\chi})^{2/3}$ ,  $\bar{\chi} = |\mathbf{a}'| k \cdot \mathbf{p}/m^3$ , and  $\bar{s} = n \cdot k_{\text{out}}/n \cdot p$  for  $k_{\text{out}}$  the radiation wavevector. Note the simple relation  $\bar{\chi} = \chi/\hbar$  relating the classical  $\bar{\chi}$  to the quantum parameter  $\chi$ . The LCFA is benchmarked against the exact result and found to agree excellently in the high-field limit in Fig. 1. If  $\Delta \rightarrow 0$  ( $\mathcal{R} \rightarrow 1$ ), (5) tends to the classical limit of the  $\mathcal{O}(\alpha)$  quantum result [14]. The  $\bar{s}$  integral in (5) can be written as an integral over a low-frequency region,  $\bar{s} \leq 1$ , plus a high-frequency region,  $\bar{s} > 1$ . The integral over the low-frequency region is exactly equal to the  $\mathcal{O}(\alpha)$  quantum result with recoil and spin set to zero. Hence, just like the QED result, it scales as  $\bar{\chi}^{2/3} \sim \xi^{2/3}$  in the high-field limit typical of the RN conjecture. However, the high-frequency ( $\bar{s} > 1$ ) contribution grows with a larger power,  $\sim \xi^2$ , and thus dominates the scaling of the classical rate (5) in agreement with previous expectations [22, 40].

In Fig. 2 we plot the local rate (5) for various  $\xi$ , and compare to the perturbative (Lorentzian) result without RR. Fig. 2a shows that the higher the pulse intensity, the earlier the majority of radiation is emitted and hence the quicker the electron is decelerated. Without RR, on the other hand, the rate of radiation is symmetric with the shape of the pulse: as much is emitted in the tail as in the rise. This is emphasised in Fig. 2b, in which we pick two phase points early and late in the pulse ( $\phi = 2\pi$  and  $\phi = 6\pi$  as also indicated in Fig. 2a), and illustrate how the rate of emission at those points changes as  $\xi$  is increased. The perturbative scaling,  $\sim \xi^2$  at small  $\xi$ , is corrected at large  $\xi$  to a scaling  $\sim \xi^{-2}$ . Clearly, resummation in  $\Delta$  has changed the large- $\xi$  behaviour.

The origin of these different behaviours can be traced back to the impact of RR corrections on the Airy argument  $z$  in (5): note that it is the behaviour of the analogous argument in QED results which determines the

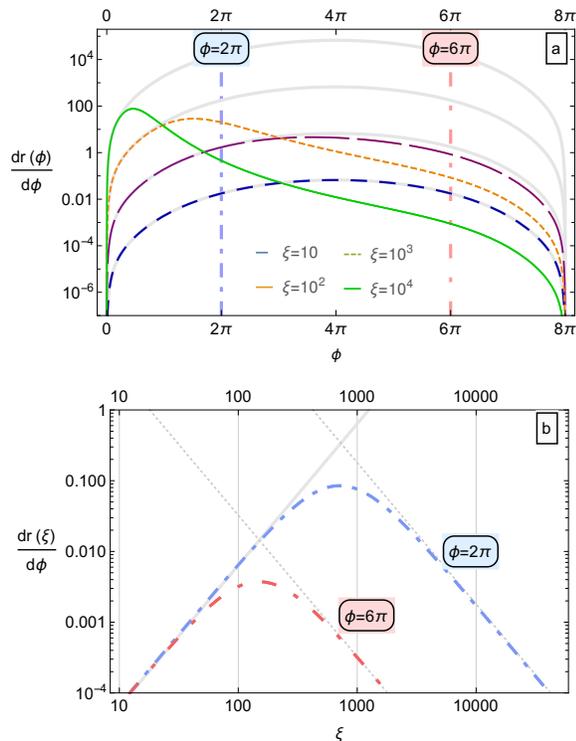


FIG. 2. a) Classically resummed LCFA rate of radiation (5) (coloured lines), compared to the  $\mathcal{O}(\Delta)$  result (faint gray lines). b) Intensity scaling of (5) at two fixed points in the pulse (on the rising and falling edge respectively). The same field and parameters were used as in Fig. 1.

large- $\xi$  asymptotic behaviour. Here we have,

$$\text{if } \Delta \ll 1: z \sim \left(\frac{\bar{s}}{\xi}\right)^{2/3}; \quad \text{if } \Delta \gg 1: z \sim (\bar{s}\xi^3)^{2/3}.$$

If RR corrections are neglected, large  $\xi$  yields *small* Airy arguments  $\sim (\bar{s}/\xi)^{2/3} \equiv (s/\xi)^{2/3}$ , which leads to the power-law  $\xi^2$ -dependence of the total emitted radiation associated with the breakdown of perturbation theory ( $r > 1$ , recall Fig. 1). With RR, though, large  $\xi$  yields a *large* Airy argument which *suppresses*  $dr/d\phi$  (leading to  $r \leq 1$ ). Hence, crucially, resummation reverses the asymptotic limit of the Airy functions compared to that expected from perturbation theory. We saw the physical consequence of this reversal above: the rate of radiation in the high- $\xi$  region is suppressed as  $\xi^{-2}$ ; hence even in high- $\xi$  pulses the radiation is mainly generated in the *small- $\xi$*  regions in the rising edge of the pulse, where the rate scales as  $\xi^2$ . The plateau in  $r$  at ever higher intensities is a consequence of a balance between ever-stronger decelerations over ever-shorter durations.

The above examples advance our understanding of the RN conjecture significantly: we have seen that the breakdown of perturbation theory in strong fields appears even in non-constant backgrounds, that it occurs classically, and that it can be resolved by classical resummation.

*Quantum.* When intensity increases, quantum effects can become relevant before large classical RR effects set in [32]. As we saw below (5), this can change the power of  $\xi$  or  $\chi$  in perturbative results, reducing the energy radiated compared to classical predictions [40], but it does *not* prevent perturbative breakdown, so that resummation is still required. Comparing scales reveals that resummation of classical effects becomes necessary, when  $\alpha\eta\xi^2 \sim 1$ , at far lower intensities  $\xi$  than required by the RN conjecture,  $\alpha(\eta\xi)^{2/3} \sim 1$  (neglecting pulse effects in both cases). This implies that the contributions which fix unphysical behaviour in the QED Furry expansion must include at least those which fix its classical limit. In this light we reconsider some of the observables above, but now in QED.

The final momentum of an electron scattering off a plane wave is given in QED by the expectation value of the momentum operator  $\hat{P}_\mu$  [41],

$$\langle \hat{P}_\mu \rangle := \sum_f \int d\mathbb{P}_f \left( \pi_\mu - \sigma_\mu + \lambda n_\mu \right), \quad (6)$$

in which  $d\mathbb{P}_f$  is the differential probability to obtain a final state  $f$ ,  $\pi_\mu$  is the Lorentz force momentum of the electron after traversing the wave [42],  $\sigma_\mu$  denotes the sum of momenta of any produced particles, and the coefficient  $\lambda$  is fixed by momentum conservation [43]. In the Furry expansion of (6), the  $\mathcal{O}(\alpha^0)$  contribution comes from elastic scattering ( $\sigma = \lambda = 0$ ) at tree level, yielding  $\langle \hat{P}_\mu \rangle = \pi_\mu$  as expected. Proceeding to higher orders in  $\alpha$ , the RN literature has focussed on self-energy corrections to elastic scattering, known to contain terms scaling like  $(\alpha\chi^{2/3})^n$  at  $n$ -loops in a constant crossed field [13]. These terms appear in (6) through the order  $\alpha^n$  coefficients of  $\pi_\mu$ . The breakdown of the Furry expansion is clear already to first order in  $\alpha$ . Including the one-loop electron self-energy, one finds [43]

$$\langle \hat{P}_\mu \rangle = \pi_\mu \left( 1 - 1.46 \frac{m}{n \cdot p} \alpha \chi^{2/3} + \dots \right), \quad (7)$$

with the correction dominating the leading term at large  $\chi$ . However, as with any inclusive observable, there are further contributions to (6). At  $\mathcal{O}(\alpha)$ , one-photon emission also contributes:

$$\langle \hat{P}_\mu \rangle = \pi_\mu + 2 \text{Im} \left[ \hat{P}_\mu \text{ (self-energy)} + \hat{P}_\mu \text{ (emission)} \right]$$

Strikingly, this contribution removes exactly the  $\alpha\chi^{2/3}$ -dependent self-energy terms from  $\langle \hat{P}_\mu \rangle$ . What remains has a perturbative expansion in  $\hbar$ , and recovers classical results as  $\hbar \rightarrow 0$ , including e.g. (4). (As shown in [43], this cancellation is *required* for the classical limit to exist at all, as otherwise  $\langle \hat{P}_\mu \rangle$  would contain terms of order  $1/\hbar$ .) Notably, we can argue that the cancellation of the self-energy terms must hold to *all* orders in  $\alpha$ . Observe

that the *total* coefficient of  $\pi_\mu$  in (7) is

$$\sum_f \int d\mathbb{P}_f = \mathbb{P}(e^- \rightarrow \text{anything}) = 1. \quad (8)$$

Unitarity therefore removes, to all orders in perturbation theory, the known loop contributions scaling with powers of  $\alpha\chi^{2/3}$ . The same argument holds for other variables such as the quantum variance in the momentum,  $\langle \hat{P} \rangle^2 - \langle \hat{P}^2 \rangle$ , and the *total* outgoing momentum, obtained by removing  $\sigma_\mu$  from (6).

This points to a previously unexplored mechanism by which parts of the Furry expansion are brought back under control. It also reinforces our findings for the classical theory: some of the QED terms previously identified as leading to perturbative breakdown actually drop out, at least for the observables considered above. Their RN scaling behaviour thus becomes irrelevant. The remaining terms, which contain the classical result (3), still need to be resummed. It is currently not known how to perform this resummation in QED, as even going beyond known  $\mathcal{O}(\alpha)$  results remains challenging. To see what is involved, we *sketch* the calculation of  $\mathcal{O}(\alpha^2)$  contributions to  $n \cdot \langle \hat{P} \rangle$  from two-photon emission. We take the LCFA expression of [44, Eq. (59)], and insert the photon lightfront momentum under the integral ( $q_1 + q_2$  in the notation of [44]), which turns the probability into the expectation value  $n \cdot \langle \sigma \rangle$ , see (6). We then examine the  $\hbar \rightarrow 0$  limit in order to compare with (4) expanded to second order in  $\Delta$ —note that in perturbation theory, both LAD and LL agree that the result is  $1 - \Delta + \Delta^2$ . Two-photon emission yields the expected classical term, albeit with a wrong coefficient,

$$\frac{n \cdot \langle \hat{P} \rangle}{n \cdot p} = 1 - \Delta + \frac{729}{512} \Delta^2 + \dots \quad (9)$$

as well as other classical terms, and terms of order  $1/\hbar$  (plus a tower of  $\hbar$  corrections.) By the same logic as employed at  $\mathcal{O}(\alpha)$ , the  $\mathcal{O}(\alpha^2)$  corrections to single photon emission [45], which we have not calculated here, and possibly to the LCFA, will be responsible for removing unwanted terms and transforming the  $\Delta^2$  coefficient in (9) from about 1.4 to unity. Similarly subtle cancellations also occur in [46, 47]. This also confirms that photon loops contribute to the classical limit [36, 48].

*Conclusions.* We have shown that the conjectured breakdown of perturbation theory in strong field QED should be expected to be a generic feature of background field perturbation theory. Furthermore, there is nothing intrinsically quantum about this result: the same problems arise classically. We have also shown that resummation (achieved through the use of exact, or all-orders, results), can fix the unphysical behaviour of perturbation theory, and that classical resummation becomes necessary at lower intensities than suggested by the RN conjecture. The implication for strong field QED is that

in order to obtain physically sensible results at high  $\chi$  (reached through high  $\xi$  at fixed  $\eta$  [20, 21]) one should resum *at least* all classical contributions at each loop order. The relevant diagrams include loops and photon emissions, cf. [49]. We have also seen, for several natural observables, that unitarity removes many previously considered diagrams scaling with powers of  $\alpha\chi^{2/3}$ . In the context of the RN conjecture, it may thus be misleading to look at only subsets of diagrams.

Let us finally comment on the impact of resummation on photonic observables [50]. Starting with a probe photon, a strong background field can cause helicity flip at one loop. In a constant crossed field this effect grows like  $\alpha\chi^{2/3}$  at one loop, and higher loop corrections are believed to scale with higher powers of the same [13]. It is a purely quantum effect, as is also seen by cutting the one-loop diagram to obtain the probability of pair production from a photon in a strong background. Our classical results are nevertheless relevant because higher order corrections contain photon loops, which (as above) include classical contributions—their imaginary parts describe real radiation emitted from the created pair [51], and we now know that such classical effects need to be resummed in the high-field limit. Furthermore, the one-loop effect itself exponentiates to a phase, yielding vacuum birefringence of the probe [52], so again resummation gives physically sensible results.

*The authors are supported by the EPSRC, grant EP/S010319/1 (AI, BK), and the Leverhulme Trust, grant RPG-2019-148 (TH, AI).*

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