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AN INVESTIGATION ABOUT HIGH SCHOOL MATHEMATICS TEACHERS' BELIEFS ABOUT TEACHING GEOMETRY

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AN INVESTIGATION ABOUT HIGH SCHOOL MATHEMATICS TEACHERS' BELIEFS ABOUT TEACHING GEOMETRY

by

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ABSTRACT

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An Investigation about High School Mathematics Teachers' Beliefs about the Teaching of Geometry

There continues to exist a dilemma about how, why and when geometry should be taught. The aim of this study was to examine high school mathematics teachers' beliefs about geometry and its teaching with respect to its role in the curriculum, the uses of manipulatives and dynamic geometry software in the classroom, and the role of proofs. In this study *belief* is taken as subjective knowledge (Furinghetti and Pehkonen, 2002). Data were collected from 520 teachers using questionnaires that included both statements that required responses on a Likert scale and open-ended questions. Also an intervention case study was conducted with one teacher. A three factor solution emerged from the analysis that revealed *a disposition towards activities, a disposition towards an appreciation of geometry and its applications* and *a disposition towards abstraction*. These results enabled classification of teachers into one of eight groups depending on whether their scores were positive or negative on the three factors. Knowing the teacher typology allows for appropriate professional development activities to be undertaken. This was done in the case study where techniques for scaffolding proofs were used as an intervention for a teacher who had a positive disposition towards activities and appreciation of geometry and its applications but a negative disposition towards abstraction. The intervention enabled the teacher successfully to teach her students how to understand and construct proofs.

The open-ended responses on the questionnaire were analysed to obtain a better understanding of the teachers' beliefs. Four themes, the *formal, intuitive, utilitarian* and the *mathematical,* emerged from the analysis, which support the modal arguments given by Gonzalez and Herbst (2006). The findings reveal a disconnect between some high school teachers' beliefs about why geometry is important to study and the current position of the Standards Movement; and between whether geometry should be taught as part of an integrated curriculum or as a one-year course.

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I have undertaken this study to make a contribution to the field of mathematics education which will eventually benefit pupil learning.

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Brenda Strassfeld

April, 2008

AUTHOR'S DECLARATION

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award.
A programme of advanced study was undertaken which included semi-annual seminars at The Centre for Teaching Mathematics at the University of Plymouth. Relevant conferences were regularly attended at which work was presented.

Publications:

Strassfeld, B. and Graham, E. (2006), High School Teachers' Beliefs: The Use of Manipulatives, in Fraser-Abder, P. and Wallace, R. J. (eds), *Pedagogical Issuers in Science, Mathematics and Technology Education*, Volume 2, 6-23.

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Activities, Appreciation, Abstraction: Secondary School Mathematics Teachers' Beliefs about Teaching and Learning Geometry

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High School Mathematics Teachers' Beliefs: Activities, Applications, and Abstractions

April 2005 -- National Council of Teachers of Mathematics Research Pre-session Poster Presentation,.Anaheim
An Investigation of High School Mathematics Teachers' Beliefs: Teaching/Learning Geometry

May 2003 -- Sharing Our Success in Science and Math Teaching: Exceeding the standards in Math and Science, New York University, New York
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CHAPTER 1 – INTRODUCTION: GEOMETRY IN SCHOOL

1.1 INTRODUCTION

Since ancient times the importance of learning geometry has been recognised as a key element of a well grounded education. Plato claimed that geometry was *"the essential in the training of philosophers."* (Heath, 1981, p. 284) Geometry was studied at Oxford as early as the end of thirteenth century (Slaught, 1912). In the seventeenth century pupils studied Euclid at Cambridge. Imagine how prominent geometry was before Newton! Slaught claimed, *"In the universities of Great Britain, Euclid met with no competition."* (p. 21) In fact, *"During the second half of the eighteenth century England had come to be the only country where Euclid was practically the only geometry text used."* (p. 23) Euclid began to be studied on a large scale in English secondary schools in the middle of the nineteenth century.

Similarly, because admission examinations to American universities in the nineteenth century required a knowledge of geometry, American high schools began offering courses in Euclidean geometry at this time (Herbst, 2002).

In 1892, The Committee of Ten (Eliot, 1893) was appointed by the National Education Association to determine the purpose of High Schools in the United States. They recommended that pupils in grades 5-8 (ages 10-13) study concrete geometry for at least one hour a week. Concrete geometry involved measuring, constructing, estimating and designing – applying geometry to the pupils' immediate environment. The hope was that if pupils had been successfully taught concrete geometry in these middle school grades the pupils would be able to master both formal plane and solid geometry in their high schools. They proposed that pupils should study formal or demonstrative geometry for an average of two and one-half periods per week in grades 10 and 11. They suggested, *"As soon as the student has acquired the art of rigorous demonstration, his work should cease to be merely receptive. He should begin to devise* *constructions and demonstrations for himself.*" (Eliot, 1893, p. 129) Pupils have to first learn how to construct proofs then they can work like mathematicians making models and their own conjectures.

In 1908, The American Federation of Teachers of Mathematical Sciences together with The National Education Association established a Committee of Fifteen to *"study and report upon the question of a geometry syllabus."* (Slaught, 1912, p.3) The committee was composed of eight representatives from secondary schools and seven representatives from universities. Over 5000 copies of the report were distributed prior to its presentation at The National Education Association meeting in 1912. The committee recommended a balance of informal and formal work in geometry, with the informal work starting in elementary grades. At the high school level, they suggested that geometry should be taught in the tenth grade and they included 100 theorems that should be formally proved in the course. They recommended that these theorems be followed by concrete questions and applications. They proposed inclusion of analytic geometry in order to foster the connection between algebra and geometry. In its time this report was widely accepted (Herbst, 2002; Gonzalez and Herbst, 2006). The geometry items appearing on college entrance examinations were limited to the theorems that the committee listed in its report.

Over the next one hundred years, the recommendations by organisations and committees that came after the Committee of Ten and the Committee of Fifteen, such as The National Council of Teachers of Mathematics and Mathematical Association, supported a stronger emphasis on informal geometry in the lower grades. This was never fully realised because of the importance elementary school teachers attach to the teaching of number and operations. This resulted in less geometry being taught in high school. There was also a demand for less rigour and an integration of topics such as geometry and algebra in high school (Gonzales and Herbst, 2006). Many states in the

United States started to offer integrated curricular in mathematics so that pupils would be exposed to different domains of mathematics. The geometry topics that would remain in the curriculum were an issue for continued debate. Proof and mathematical structure were deemphasised in the integrated courses.

At the beginning of the twenty first century, the *Principles and Standards for School Mathematics* (NCTM, 2000) has included proof as an actual standard and recommended it to *"be a part of the mathematics education of all students"* (Knuth, 2002b, p. 62) from pre-kindergarten through high school.

Following the course of geometry is like riding a roller coaster with its many ups and downs. I knew that I wanted to be a part of that ride in the twenty first century.

1.2 AN OVERVIEW OF THE STUDY

The study of geometry is something that fascinates many people. Personally, this has been true since I first started studying it in the tenth grade. The world around us is full of examples of geometry. The mobile on the baby's crib, the toddler's shape sorter, the pre-school child's jigsaw puzzle are encounters one has with geometry at an early age with some of these encounters even coming before one's earliest experiences with number. A group of pre-service graduate students were recently given an assignment which required them to silently make a poster about geometry, working in small groups. The students within each group were not allowed to verbally communicate with one another. One group drew the sun, mountains, trees and water. They entitled their poster "Geometry is Everywhere," which endorses my own views about this subject.

I am fascinated with geometry and intrigued by the teaching and learning of it. Students are intrigued by shapes when they are young. So what happens in their school experiences that make many students dislike learning geometry? My own geometry teacher would only accept a proof if it was written exactly how she wanted it, leaving little room for any creativity and causing much anxiety in my fellow students. In the course of my career as a mathematics teacher educator I have been intrigued by how geometry is taught by teachers and learned by students. I have tried to explore how geometry can be taught most effectively and how students' minds can be challenged to make the learning of geometry interesting and intellectually stimulating. Are geometry teachers today like my teacher or are they like Socrates in Plato's *Meno* (The dialogue known as the *geometry experiment* can be found in Appendix J)? Freudenthal (1971) claimed that the earliest lesson in the history of education *"is a lesson of geometry, the Socratic lesson Meno's slave was taught on doubling the square. Socrates taught the slave not the solution of the problem nor solving the problem, but finding the solution by trial and error. He did not teach a ready made solution but the way of reinventing the solution."* (p. 414)

This method of teaching was considered *"the first and most celebrated"* method of discovery teaching. (Cooney, Davis, and Henderson, 1975, pp. 136-7) Fernandez (1994) analysed the excerpt noting both positive and negative aspects of this teacher student dialogue. Socrates used a visual to help the slave boy, but it became rather cluttered. There are fifty questions asked of the slave boy of which thirty-six required only a yes or no response.

> To the modern reader, the dominance of "yes/no" questions conveys an image of mathematics discourse in which ideas flow primarily from teacher to student; the teacher is the sole possessor of knowledge; and answers are either right or wrong. Such images perpetrate the unrealistic expectation that "the teacher is always right" which in turn undermines the student's knowledge, inhibits the student's thinking, and minimizes the students' role in classroom discourse. (Fernandez, 1994, p.45)

On a positive note, the question and answer format of the dialogue is more desirable than a classroom where the teacher *"states mathematical propositions and directs his students to memorise them"* (p.45) which was my high school geometry experience.

Socrates' believed that anyone is capable of discovering mathematical ideas. How many high school mathematics teachers have that belief today? How do they teach geometry to their students? Is there a balance between the concrete, such as the use of manipulatives in the classroom, and the abstract, such as doing proofs? I am curious about the answers to these questions. More specifically, I decided to investigate high school teachers' beliefs about geometry because this is a topic which has always fascinated me and has been a focus of my professional work in teacher development. My goal was to try to understand the variety of beliefs held by high school teachers about the nature of geometry as a subject in its own right; beliefs about the role of geometry in the curriculum; beliefs about the use of manipulatives and dynamic geometry software packages in the classroom; and beliefs about doing proofs, teaching proofs, and learning proofs,

Over 30 years ago a survey was conducted with almost 1000 high school mathematics teachers from the United States (Gearhart, 1975). The goal of the survey was to find out what the teachers thought should be included in a high school geometry course. No questions were asked about the role of manipulatives which at that time were already available although dynamic geometry had yet to be invented.

Priorities in School Mathematics was another survey, which was supported by the National Science Foundation, and designed by the National Council of Teachers of Mathematics in 1977 to collect information from mathematics teachers who were subscribers to the Mathematics Teacher or the Arithmetic Teacher Journals (NCTM), mathematics supervisors, junior college mathematics teachers, principals, presidents of school boards or presidents of parent teacher organisations on their beliefs and reactions to the possible changes to the mathematics curriculum of the 1980s (Lindquist, 1984; Suydam, 1981, 1985). Although a total of 10,000 people was surveyed with different surveys for each population, the average response rate was only 29%. There was a companion survey distributed in Canada, which was also interested in all areas of mathematics (Worth, Cathcart, Kieren, Worth, and Forth, 1981). The results of the United States surveys served as a database that shaped recommendations for curriculum changes suggested in An Agenda for Action (NCTM, 1980).

At about the same time, Mathematics Counts (Cockcroft, 1982) included a broad range of learning outcomes and recommendations for teaching in England and Wales (Ernest, 1991). Similar documents were published in the United States (NCTM, 1989, 1991) based on the earlier recommendations from An Agenda for Action (NCTM, 1980).

A large amount of curriculum reform has taken place in the last two decades. In the United States the National Council of Teachers of Mathematics published the Professional Standards for School Mathematics (PSSM, 2000) and in the United Kingdom, Department for Education and Employment, DfEE published the Revised National Curriculum (1999) with the curriculum recommendations for what and how geometry should be taught. The report of the working group of The Royal Society and Joint Mathematical Council (2001) claimed that the most important issue for 11-16 geometry,

> ...is to ensure that teachers have the knowledge, understanding, skills, and resources to teach geometry in a way which genuinely captures pupils' interest and imagination, while developing their thinking and reasoning skills, their power of visualisation, their ability to apply and model, and their understanding. (p. viii)

Taking into account the current curriculum reform, I believe it is now the time to conduct an investigation into the current beliefs of high school teachers about the nature and teaching of geometry.

This investigation into teachers' beliefs about geometry and its teaching and learning should be useful to teacher educators and school administrators as they develop curricula for the future.

The research questions addressed are:

- What are high school mathematics teachers' beliefs about the role of geometry in the curriculum?
- What are high school teachers' beliefs about the use of manipulatives and dynamic geometry software packages?
- What are high school teachers' beliefs about the role of proof?

I surveyed high school mathematics teachers' beliefs about geometry using previous surveys (Gearhart, 1975) as an initial source for questionnaire statements. I have included a review of the literature, in Chapter 2, that provided me with a background about belief research and previous research about teaching and learning geometry. In chapter 3, I discuss the research methodology used in this study including information about questionnaire design and other methods of effective data collection and analysis. I describe the results of my pilot questionnaire including reasons for its revision in Chapter 4. The quantitative results of the descriptive data are presented in Chapter 5 and the qualitative results are presented in Chapter 7. Chapter 6 contains the factor analysis of the data and describes the three factor solution that leads to the eight typologies of teachers. Conclusions and implications for further study are presented in Chapter 8.

CHAPTER 2 - REVIEW OF THE LITERATURE

2.1 INTRODUCTION

> *"Not only has interest in the affective domain been long standing, but it is an area to which considerable research continues to be directed."* (Leder and Grootenboer, 2005, p. 1)

This investigation concerns secondary school mathematics teachers' beliefs about the teaching and learning of geometry. In the first part of this chapter I will discuss the current literature about the definition and classification of beliefs, the affective domain in general, belief research in mathematics education including teachers' beliefs about mathematics and mathematics education, problem solving and school reform, the difference between knowledge and beliefs, and domain specific beliefs. I conclude this section of the chapter with the definition of beliefs on which this study is based.

In the second part of this chapter I will discuss current research on educational aspects of geometry including research on geometric thought, geometry's role in the curriculum, the role of manipulatives in the classroom, the role of dynamic geometry software, and the role of proof.

2.2 BELIEFS

Researchers in various disciplines have used different definitions and classifications of beliefs. (Abelson, 1979; Goldin, 2002; Green, 1971; Hart, 1989; McLeod, 1989a; Pajares, 1992; Presmeg, 2002; Rokeach, 1972; Scheffler, 1965; Schoenfeld, 1986; Thompson, 1984, 1992; Törner, 2002). In the section below some of their conclusions are described.

2.2.1 The Definition and Classification of Beliefs

The difficulty of defining beliefs has been documented in several research studies.

Scheffler (1965) stated "*It will, in any case, never be reasonable to take belief simply as a matter of verbal response: belief is rather a 'theoretical' state characterizing, in subtle ways, the orientation of the person in the world.*" (p. 89-90)

Rokeach (1972), a social psychologist, on the other hand, accepts a person's verbal response as a belief. He defined beliefs as "*any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase, 'I believe that...'*" (p.113). He identified three types of beliefs:

1. Descriptive or existential beliefs which describes the object of belief as true or false.
2. Evaluative beliefs which describes the object of belief as good or bad.
3. Prescriptive or exhortatory beliefs which advocate a certain course of action or a certain state of existence as desirable or undesirable. All beliefs are 'predispositions to actions'.

He defined an attitude as "*a relatively enduring organization of beliefs around an object or situation predisposing one to respond in some preferential manner*" (p. 112).

Pajares (1992) stated that:

> ...defining beliefs is at best a game of player's choice. They travel in disguise and often under alias-attitudes, values, judgments, axioms, opinions, ideology, perceptions, conceptions, conceptual systems, preconceptions, dispositions, implicit theories, explicit theories, personal theories, internal mental processes, action strategies, rules of practice, practical principles, perspectives, repertories of understanding, and social strategies, to name but a few that can be found in the literature. (p. 309)

Leder and Forgasz (2002) included a table of selected definitions of beliefs from the field of psychology during the years 1970-1997. They concluded that: "*...given the common elements evident among many of the definitions, much useful work can be done without full and rigid agreement about the precise definition adopted.*" (p. 96) Table 2.1 provides definitions of beliefs from a subset of the psychologists included in Leder and Forgasz (2002) whose definitions have been used in papers mentioned in this thesis.

Authors	Key elements of the definition
Rokeach (1972)	"A belief is any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase, 'I believe that...'" (p. 113) An attitude is defined simply as an organization of interrelated beliefs around common object, with certain aspects of the object being at the focus of attention for some persons, and other aspects for other people (p. 116)...Each belief within an attitude organization is conceived to have three components: a *cognitive* component [it represents a person's knowledge], an *affective* component [the belief can arouse affect], and a *behavioral* (sic) component [leads to some action, when suitably activated]. (p. 113)
Fishbein & Ajzen (1975)	"Whereas attitude refers to a person's favorable (sic) or unfavorable (sic) evaluation of an object, belief represents the information he has about the object. Specifically a belief links an object to some attribute.... The object of a belief may be a person, a group of people, an institution, a behavior (sic), a policy, an event, etc., and the associated attribute may be any object, trait, property, quality, characteristic, outcome, or event." (p. 12)

Table 2.1 *Selected Definitions of Beliefs* (Leder and Forgasz, 2002, pp. 96-97)

Furinghetti and Pehkonen (2002) found that it is inappropriate to expect that one single definition of beliefs will be suitable for all the fields of application. They included nine characterisations of beliefs that they had gathered from research literature (1987-1998) in a questionnaire they designed. This questionnaire was sent to 22 mathematics educators who had carried out research in the field of beliefs. Their characterisations of beliefs included in the questionnaire are listed in Table 2.2.

Characterisation #1 (Hart, 1989, p.44)	"we use the word belief to reflect certain types of judgments about a set of objects"
Characterisation #2 (Lester et al., 1989, p. 77)	"beliefs constitute the individual's subjective knowledge about self, mathematics, problem solving, and the topics with in problem statements"
Characterisation #3 (Lloyd & Wilson, 1998, p. 249)	"we use the word conceptions to refer to a person's general mental structures that encompass knowledge, beliefs, understandings, preferences, and views"
Characterisation #4 (Nespor, 1987, p. 321)	"Belief systems often include affective feelings and evaluations, vivid memories of personal experiences, and assumptions about the existence of entities and alternative worlds, all of which are simply not open to outside evaluation or critical examination in the same sense that the components of knowledge systems are"
Characterisation #5 (Ponte, 1994, p. 169)	"Beliefs and conceptions are regarded as part of knowledge. Beliefs are the incontrovertible personal 'truths' held by everyone, deriving from experience or from fantasy, with a strong affective and evaluative component."
Characterisation #6 (Pehkonen, 1998, p.44)	"we understand beliefs as one's stable subjective knowledge (which also includes his feelings) of a certain object or concern to which tenable grounds may not always be found in objective considerations"
Characterisation #7 (Schoenfeld, 1992, p. 358)	"beliefs – to be interpreted as an individual's under-standings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behaviour"
Characterisation #8 (Thompson, 1992, p. 132)	"A teacher's conceptions of the nature of mathematics may be viewed as that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics."
Characterisation #9 (Torner & Grigutsch, 1994, p.213)	"Attitude is a stable, long-lasting, learned predisposition to respond to certain things in a certain way. The concept has a conative (action) aspect."

Table 2.2 ***Characterisation of Beliefs*** (Furinghetti and Pehkonen,2002, p.47)

The respondents, in the Furinghetti and Pehkonen (2002) study, were asked whether they agreed or disagreed with the characterisations not knowing whom the authors of the statements were, and to state their reasons for agreement or disagreement, possible improvements, and personal characterisations. Eighteen mathematics educators responded to the questionnaire, but no clear pattern was observed. The answers were most unified in relation to Ponte's characterisation (#5 above) where 15 of the 18 respondents disagreed with the statement. The relationship between knowledge and beliefs was one of the major points of disagreement. The characterisations that most respondents agreed with (11 of the 18) were those of Schoenfeld #7 and Thompson #8.

Furinghetti and Pehkonen have not exhausted the list of characterisations of beliefs and I have included earlier definitions and characterisations of beliefs (Green, 1971; Rokeach, 1972; Scheffler, 1965), definitions and characterisations from the same period but not included in Furinghetti and Pehkonen's list (Bar-Tal, 1990; Pajades, 1992), and later definitions and characterisations (Goldin, 2002; Lester, 2002; Yackel and Rasmussen, 2002).

Green (1971) proposed a multidimensional perspective of the structure of beliefs. Green stated:

> We may, therefore, identify three dimensions of belief systems. First there is the quasi-logical relation between beliefs. They are primary or derivative. Secondly, there are relations between beliefs having to do with their spatial order or their psychological strength. They are central or peripheral. But there is a third dimension. Beliefs are held in clusters, as it were, more or less in isolation from other clusters and protected from any relationship with other sets of beliefs. Each of these characteristics of belief systems has to do not with the content of our beliefs, but with the way we hold them. (pp. 47 – 48)

Researchers in mathematics education have used Green's perspective to analyse teachers' beliefs (Cooney, Shealy, and Arvold, 1998).

Goldin (2002) defined beliefs as '*multiply-encoded cognitive/affective configurations, to which the holder attributes some type of truth value (empirical truth, validity, logical truth, or religious truth)*'. He asserted that:

> The stability of beliefs in individuals has much to do with the interaction of belief structures not only with affect (feelings) but with meta-affect (feelings about feelings)-that thru their psychological interplay, meta-affect and belief structures sustain each other… Affect stabilizes beliefs and beliefs establish meta-affective contexts. (p.59)

Goldin (2002) defined a belief structure as a set of mutually consistent, mutually supportive, or mutually reinforcing beliefs in individuals. An extensive belief structure that is culturally or socially shared is a belief system.

Currently many researchers have also looked at beliefs from a sociological perspective. (Yackel and Rasmussen, 2002; Lerman, 2002)

Yackel and Rasmussen (2002) claimed that beliefs are cognitive in nature - a person's understanding of things. Beliefs are the *'psychological correlates of norms'* and evolve together as a dynamic system. They defined social norms *"as taken-as-shared beliefs that constitute a basis for communication and make possible the smooth flow of classroom interaction."* (p. 316) Beliefs are an individual's understanding of *'normative expectancies'*.

Tsamir and Tirosh (2002) focused on intuitive beliefs that are *"particular, immediate forms of cognition that refer to statements and decisions that exceed the observable facts"* (p. 331). They stated Fischbein's (1987) characteristics of intuitive beliefs:

- Self evidence-person perceives them as being true and in need of no further justification.
- Intrinsic certainty- they are associated with a feeling of certitude or intrinsic conviction.
- Perseverance-they are robust.
- Coerciveness- the individual tends to reject alternative interpretations, those that would contradict his or her intuitions.
- Extrapolativeness-intuitive- intuitive cognitions have the capacity to extrapolate beyond an empirical support.
- Globality- intuitive beliefs are accepted as structured, meaningful, unitary representations, as opposed to logically acquired cognitions which are sequential and analytical.

Students sometimes have intuitive beliefs about mathematical ideas that are not compatible with *"formal mathematical definitions and theorems."* (p. 341) For example the mathematical definition of the word *'similar'* differs from what pupils' intuitive or everyday conception of similar objects.

Bar-Tal (1990) stated that the study of beliefs can be classified into four areas:

1. acquisition and change of beliefs
2. structure of beliefs
3. effects of beliefs
4. content of beliefs

In studies investigating teachers' beliefs the researcher is interested in one or more of the above areas. Before continuing to describe studies investigating teachers' beliefs I think it is appropriate to discuss the affective domain in which researchers claim beliefs are included (Hart, 1989; McLeod, 1989a, 1989b, 1992; Goldin, 2002).

2.2.2. The Affective Domain

McLeod (1989b) analysed the affective domain, describing affect in terms of beliefs, attitudes, and emotions. He discussed the central role of affect in problem solving; the importance of the social context in the study of affective factors in mathematics learning; the need to integrate research on cognition and affect; and methodical issues and their implications for further research on affective factors in the teaching and learning of mathematics. McLeod proposed a theoretical framework for investigating the affective factors that help or hinder performance in mathematical problem solving. The framework included the following factors:

1. Magnitude and direction of the emotion.
2. Duration of the emotion.
3. Level of awareness of the emotion.
4. Level of control of the emotion.

We need to know the ways in which these factors interact with different types of cognitive processes, the different types of instructional environments, and the differing beliefs that students hold.

Hart (1989) described the different meanings psychologists and mathematics educators ascribe to the words attitude, affect, affective domain, belief system, emotion, and anxiety. She summarised some of the consistencies and inconsistencies among the meanings as follows:

> Belief - Certain types of judgments about a set of concepts.
>
> Attitude - Emotional reaction to the object, behaviour toward the object, beliefs about the object.
>
> Emotion – Hot, gut-level reaction.
>
> Affect – Synonymous with emotion. (p.44)

Hart (1989) suggested that one reason researchers from various disciplines and within the same discipline have had difficulty communicating effectively with one another is the lack of common usage of terms such as attitude and belief. Her view about miscommunications resulting from the lack of common definitions for *beliefs* and *attitudes* is not shared by Leder and Forgasz (2002) who believe that although the definitions of terms differ there is enough commonality for researchers to understand each other. Hart's view was expressed five years before Padjares' (1992) all inclusive description of beliefs.

In the Concise Dictionary of Psychology attitude is defined as *"a stable, long-lasting, learned predisposition to respond to certain things in a certain way. The concept has a cognitive (belief) aspect, an affective (feeling) aspect, and a conative (intention) aspect".* (Statt, 1998, p.10) This definition is in agreement with Hart's (1989) definition of *attitude*.

McLeod (1989a) stated two ways that attitudes about mathematics develop.

> First, attitudes may result from the automatizing (sic) of a repeated emotional reaction to mathematics; for example, if a student has repeated negative experiences with geometric proofs, the emotional impact will usually lessen in intensity over time. Eventually, the emotional reaction to geometric proof will become more automatic, there will be less physiological arousal, and the response will become a stable one that can probably be measured through the use of a questionnaire. Another second source of attitudes is the assignment of an already existing attitude to a new but related task. A student who has a negative attitude toward geometric proof may attach the same attitude to proofs in algebra. (p.249)

Goldin (2002) characterised beliefs, emotions, attitudes, and values, ethics and morals as sub-domains of affective representations.

- Beliefs- internal representations to which the holder attributes truth or validity. Beliefs are usually stable, highly cognitive and may be highly structured.
- Emotions- rapidly changing states of feeling, mild to intense, which are usually local or embedded in a context.
- Attitudes- moderately stable predispositions toward ways of feeling in situations. They involve a balance of affect and cognition.
- Values, ethics, and morals- deeply held preferences, possibly characterised as "personal truths". They are stable, highly affective as well as cognitive, and may be highly structured.

Each individual has these self-guiding road signs. However, the individual is the product of his socio-cultural environment. Thus, he can be expected to share its belief systems, communally shared emotions, accepted attitudes, values, ethics and morals.

Leder and Grootenboer (2005) cited Grootenboer's (2003) model of conceptions of the affective domain in Figure 2.1.

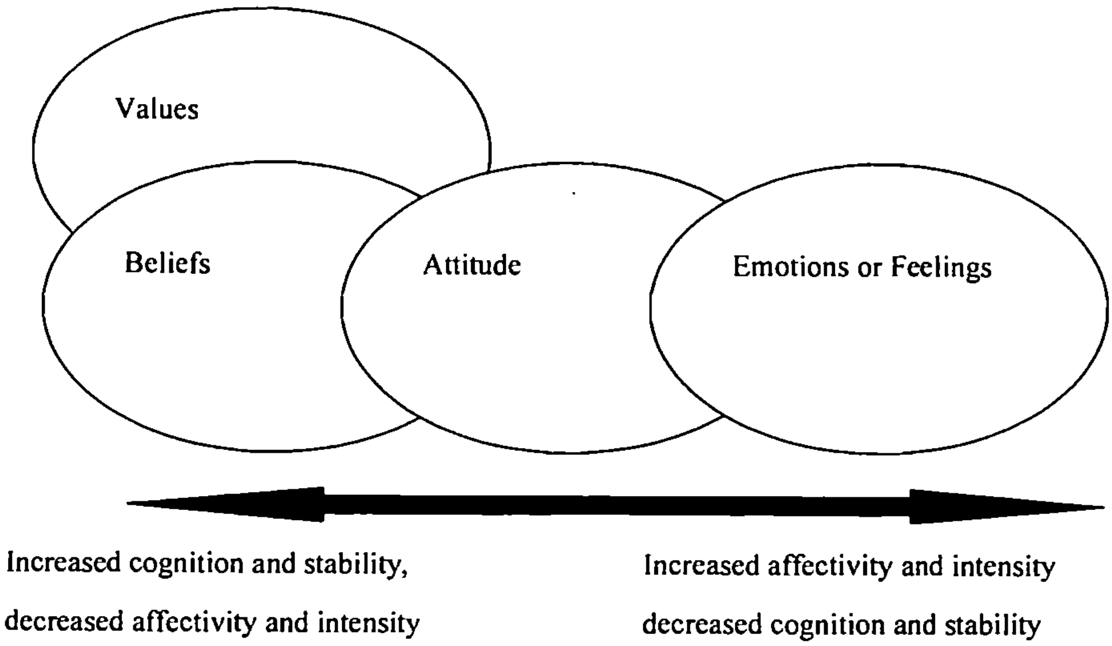


Figure 2.1 ***Conceptions of the affective domain*** **(p. 2)**

In the last two decades there have been studies of both teachers' and students' beliefs about mathematics, problems solving, and school reform. In the sections below we will describe in detail some of those studies involving teachers' beliefs.

2.2.3 Beliefs Research in Mathematics Education

McCleod (1992) stated *"although affect is a central concern of students and teachers, research on affect in mathematics education continues to reside on the periphery of the field."* (p. 575) Concerning teachers' beliefs specifically, Schoenfeld (1992) claimed that there is a "*moderate but growing literature*". Now more than a decade later, research into the affective domain is no longer on the periphery and the research on teachers' beliefs has become fairly extensive. In fact in November 1999 there was an international meeting about mathematics related beliefs. Many of the presentations at the conference became chapters in *Beliefs: A Hidden Variable in Mathematics Education* edited by Leder, Pehkonen and Törner (2002).

Next some of the early studies on teachers' beliefs about mathematics and its teaching and learning are described. In the last section of this chapter mention is also made of recent belief studies.

Thompson's (1984) work was one of the first studies of beliefs within the field of mathematics education. She investigated three junior high school mathematics teachers and their conceptions of mathematics and mathematics teaching. Her intent was to identify what constituted the teachers' beliefs. In particular, Thompson sought to discover how a teacher's professed beliefs, views, and preferences about mathematics and mathematics teaching are reflected in their instructional practices. The common focus of research studies prior to her work was predominantly on the behaviour of the teacher rather than the teacher's thoughts. Thompson argued that there is reason to believe that a relationship exists between one's conception of mathematics and one's teaching of mathematics, but "*very little is known about the role that teachers' conceptions of the subject matter and its teaching might play in the genesis and evolution of instructional practices characteristic of their teaching.*" (p. 105) Thompson warned that "*failure to recognize the role that the teachers' conceptions might play in shaping their behaviour is likely to result in misguided efforts to improve the quality of mathematics instruction in schools.*" (p. 106).

Thompson used the method of case studies to report for each teacher on their conceptions of mathematics, mathematics teaching, and their criteria for judging effectiveness of instruction. She found that, for the most part, teachers' preferences and views of mathematics were reflected in their teaching practices. Although all three teachers believed that mathematics was relevant to daily life and served as an important tool for solving problems, none of them incorporated applications into their lessons. As reasons for not teaching applications, the participants cited a lack of interest in the application, a lack of familiarity with the application, and deficiencies in the students' mathematical backgrounds. Also, the differing views of mathematics, what constitutes mathematical understanding, and the purpose or benefit of lesson planning held by each teacher had an impact on their views about teaching. The most striking inconsistencies that she found concerning teachers' beliefs about teaching were encouragement of student participation; use of a wide variety of instructional approaches; and realisation of their goals within the context of mathematics education. The reasons given for these inconsistencies were adherence to lesson plans; reduction of potential discipline problems; general dissatisfaction with teaching; reliance on the textbook; lack of familiarity with alternative explanations; and following the path of least resistance. Thompson concluded that:

> Teachers' beliefs, views, and preferences about mathematics and its teaching, regardless of whether they are consciously or unconsciously held, play a significant, albeit subtle, role in shaping the teachers' characteristic patterns of instructional behaviour. In particular, the observed consistency between the teachers' professed conceptions of mathematics and the manner in which they typically presented the content strongly suggests that the teachers' views, beliefs, and preferences about mathematics do influence their instructional practice. Teachers possess conceptions that are general and not specific to the teaching of mathematics. They also have conceptions about their students and the social and emotional make-up of their class. These perceptions appear to play a significant role in affecting instructional decisions and behaviour. For some teachers these conceptions are likely to take precedence over other views and beliefs specific to the teaching of mathematics. (p. 125)

Thompson (1992) stated *"it is important that researchers that are interested in examining teachers' beliefs make explicit to themselves and to others the perspectives from which they are approaching their work. This is particularly important because of the interpretative nature of most studies in this line of research."* (p. 137) Thompson gave a historic overview of the study of beliefs in the 20th century. She also discussed some of the philosophical distinctions between knowledge and beliefs. She included the research that had been done along with theoretical models, methodology, and findings. Some of the papers cited were case studies of a few teachers. Her conclusions included the implications of her work and recommendations for future study. The main conclusions are listed below.

1. Belief systems are dynamic.
2. Relationships between beliefs and practice are dialectic, not simply cause and effect.
3. There is a need to explore whether and how teachers' belief and/or knowledge (conceptions) relate to their experience.
4. There is a need to study the extent to which teachers' and students' conceptions interact during instruction.

She agreed with Scheffler (1965), that verbal expression alone is not evidence of belief. Thompson (1992) suggested that:

> ...researchers interested in studying teachers' beliefs should give careful consideration to the concept from both a philosophical as well as a psychological perspective. Philosophical works can be helpful in clarifying the nature of beliefs. Psychological studies may prove useful in interpreting the nature and the relationship between beliefs and behaviour as well as understanding the function and structure of beliefs. (p.129)

2.2.3.1 Teachers' Beliefs about Mathematics

Ernest (1989) claimed that the bases for teachers' practices are their beliefs and views of the nature of mathematics. He identified three philosophical views of mathematics:

- Problem solving view
- Instrumentalist view
- Platonist view

Teachers having a problem solving view of mathematics believe that mathematics is a *"dynamic, continually expanding field of human creation and invention, a cultural product."* (p. 250) Mathematics is a process of inquiry. It is not a finished product. Such a view is considered 'fallibilistic' since its results are open to correction and revision. Teachers having a instrumentalist view of mathematics believe that *"mathematics is an accumulation of unrelated facts, rules, and skills to be used for some external end."* (p.250). Teachers, with a Platonist view of mathematics, believe that mathematics is *"a static but unified body of knowledge."* (p.250) They believe *"mathematics is discovered, not created."* (p.250) Such a view is considered 'absolutist' since mathematics is seen as unquestionable and certain.

Ernest associated these three views of mathematics with three teachers' roles:

- Facilitator – Confident problem posing ad problem solving is the intended outcome
- Instructor – Skills mastery with correct performance is the intended outcome
- Explainer – Conceptual understanding with unified knowledge is the intended outcome (p. 251)

Ernest (1991) claimed that the disparity between teachers' espoused beliefs and actual practices were due to *"the constraints and opportunities provided by the social context of teaching."* (p. 290) Social context includes the mandated curriculum along with its textbooks, assessments, and expectations of other people such as administrators, parents and students.

Skott (2002) similarly concluded that motives behind teachers' classroom practices are not necessarily dependent on the teachers' espoused beliefs but surface in the course of complex classroom interactions.

> ...to acknowledge the simultaneous existence of multiple, possibly conflicting, actual and virtual communities of a teacher's practice. Each of these may play a role when different objects of the teachers' activity emerge

> in the course of the classroom interaction. From this perspective, the focus of classroom research on teachers' beliefs is not to state congruence or conflict between beliefs and practices, but to disentangle the ways in which – from the teachers' perspective – the multiple communities interact and frame the emergence of different objects of his or her activity. (p. 4-216)

Greer, Verschaffel, and De Corte (2002) stated *"assessment is a major agent of belief shaping. Assessment impacts instruction because it transmits powerful signals conveying the goals of instruction, what counts as competence in mathematics, and what forms of mathematical performance are valued."* (p. 287)

In my own experience, I find it quite amazing that teachers can spend several months teaching students how to prove theorems, when the state test may contain only a six mark question on proof. These teachers believed that teaching proof is integral to the study of geometry and emphasised it regardless of the policies of the state test setters.

Cooney, Shealy, and Arvold (1998) suggested that *"teachers' beliefs about mathematics and how to teach mathematics are influenced in significant ways by their experiences with mathematics and schooling long before they enter the formal world of mathematics education."* (p. 306) As such, Cooney and his colleagues examined the belief structures of four pre-service secondary mathematics teachers as they completed the last two years of their teacher preparation coursework including student teaching. The beliefs data collected through surveys, classroom observations, written assignments, and interviews was analysed using Green's (1971) multidimensional perspective of the structure of beliefs.

The analysis of the data revealed that each of the four teachers wanted approval for what he or she believed was the role of a good mathematics teacher. For example, one of the teachers believed that the purpose of teaching mathematics was to prepare students to enter the world of work. Over the course of time he began to see how the use of technology (which he did not value at the beginning of this study) could facilitate his goals for teaching mathematics. He also valued the thoughts, opinions, and suggestions of his classmates — many of which he held as peripheral beliefs that he later assimilated into his repertoire of centrally held beliefs.

Cooney and his colleagues also reported that "*a teacher's movement from conceptualizing knowledge as something emanating from external beings toward conceptualizing knowledge as something emanating from interrelationships between self and others is an important consideration in conceptualizing teachers' professional development.*" (p. 329)

Stigler and Hiebert (1999) analysed videos of eighth grade mathematics lessons from Germany, Japan, and the U.S. that were part of the Third International Mathematics and Science Study (TIMSS). They viewed teaching as a cultural activity where the "*script for teaching*" (p.87) rests on a set of core beliefs about the nature of mathematics, about how students learn and about the role of the teacher in the classroom. They claimed that teaching has to be "*understood in relation to the cultural beliefs*" (p.88) which surround it. Even though teachers reported having implemented reform measures into their teaching, the videos showed little evidence of this. "*We learned that teaching is not a simple skill but rather a complex cultural activity that is highly determined by beliefs and habits that work outside the realm of consciousness.*" (p. 103)

Aguirre and Speer (2000) explored the relationship between two secondary school mathematics teachers' beliefs and goals using video and interviews. They analysed how beliefs influenced the decisions teachers made during classroom interaction. "*By investigating the influential role beliefs play in the teaching process we can obtain a better understanding of the teaching we see in the classroom.*" (p. 354)

They claimed that:

1. Beliefs played a central role in shaping the moment to moment practice of teaching.

2. Beliefs are most likely to become apparent during a shift in teachers' goals.

They defined beliefs as conceptions, personal ideologies, world views and values that shape practice and orient knowledge.

Raymond's (1997) study has been included in this review of the literature because her model of the relationships between beliefs and practices was the first model to include teachers' prior school experiences. Their prior experiences can have an important influence on teachers and should inform teacher education programs. Raymond (1997) investigated the relationship between six beginning elementary teachers' professed beliefs about mathematics and its instruction, and the teachers' actual teaching practices. She defined beliefs as *"personal judgments about mathematics formulated from experiences in mathematics, learning mathematics, and teaching mathematics."* (p. 552) Data collection lasted approximately 10 months and consisted of six hour-long interviews, five classroom observations, lessons plans, a concept map activity, and a questionnaire about mathematics beliefs. Raymond used the work of Ernest (1989) as a means for analysing and discussing teachers' beliefs about the nature of mathematics and its teaching and learning. She began her data analysis by categorising her data as beliefs, teaching practices, and influences on beliefs and practices and the degree of inconsistency between them. She further subdivided the beliefs category into beliefs about the nature of mathematics and the nature of its teaching and learning. The teaching practice category was subdivided into *"tasks, discourse, environment, and evaluation."* (p.555) Influences on beliefs and practice were subdivided into *"social teaching norms, immediate classroom situation, prior school experiences, and other* *influences.* " (p. 556) The data regarding beliefs about mathematics content, teaching, and learning were categorised as *"traditional, primarily traditional, an even mix of traditional and non-traditional, primarily non-traditional, and non-traditional.* " (p. 556).

Raymond (1997) reported on one fourth grade teacher whose beliefs about teaching and learning were *"most inconsistent with her practice.* " (p. 553) but agreed with the other participants about their *"primary influences on beliefs and practice.* " (p. 553) She found that the teacher in this case believed that her teacher preparation program had minimal impact on her instructional practices and moderate impact on her beliefs. Raymond pointed out that *"the primary goal of mathematics teacher preparation should be to stimulate the examination and development of beliefs about mathematics and mathematics pedagogy because teacher education programs are likely to have more influence on beliefs than on specific practices.* " (p. 572)

Collier (1972) conducted a study intended to measure prospective elementary school teachers' beliefs about mathematics and mathematics instruction. Of interest is that although this is a paper about teachers' beliefs the author does not bother defining beliefs. The items on the questionnaires were clearly statements that have appeared on questionnaires in other belief studies. The participants were categorised by their academic records and were then placed into one of four groups: Group I — no prior enrolment in college mathematics courses; Group II — completion of one mathematics course; Group III — completion of two mathematics courses; Group IV — completion of two mathematics courses and a pedagogy course. The participants responded to a list of 80 questions by rating them on a six-point scale, where 1 represented "strongly disagree" and a 6 represented "strongly agree." The items themselves were designed to measure a formal-informal dimension of teachers' beliefs about mathematics and mathematics instruction. After using quantitative methods such as a two-way ANOVA and individual t-tests to analyse the data, Collier concluded that prospective teachers *"enter elementary teacher education programs with neutral beliefs; they do not view mathematics as formal or informal."* (p. 159) The teachers' beliefs about mathematics instruction are also neutral. After two college mathematics courses their beliefs about mathematics and mathematics instruction remain neutral, but the views of mathematical high achievers become somewhat informal. Upon completion of two college mathematics courses and a pedagogy course, students have *"a slightly informal view of mathematics with high achievers having a more informal view than low achievers"* and a *"moderately informal belief about mathematics instruction."* (p. 159) The importance of this early study is that Collier concluded that beliefs formed from prior experiences may be difficult to change and that *"most of the students tested had not been exposed to courses which had formation of beliefs as specific course objectives."* (p. 159) Although this study was conducted twenty-five years before Raymond's study (1997) the implications for teacher preparation programs suggested by Raymond seemed already relevant from Collier (1972).

2.2.3.2 Beliefs about Problem Solving

Anderson, White, and Sullivan (2005) presented a model that identified teachers' problem-solving beliefs and practices. They investigated factors that may impact these practices. They included six models that had previously been used to investigate teachers' beliefs and practices as is shown in Table 2.3.

Researcher	Summary of Models and Key Factors
Romberg (1984)	The model includes teachers' beliefs and mathematics content as determining factors in teachers' plans, classroom actions, and student performance.
Guskey (1986, 2002)	A linear model of teacher change that proposes a sequence of events from professional development to new practices in the classrooms with a change in teachers' beliefs and attitudes if there is a change in student learning outcomes.
Fennema, Carpenter, and Peterson (1989)	A model for curriculum development that connects teaching and learning and includes teachers' knowledge, beliefs, and decisions as influencing factors on instruction and students' learning.
Flexer, Cumbo, Borko, Mayfield, and Marion (1994)	This model of teachers' belief systems includes beliefs about children's learning and appropriate mathematics content with beliefs about instruction and assessment as factors influencing practice.
Ernest (1991)	A model of espoused and enacted beliefs recognising the influence of teachers' conceptions of knowledge and mathematics, their views about mathematics teaching and learning, and acknowledges the constraints and opportunities of the classroom and school setting.
Raymond (1997)	A model of the relationships between teachers' beliefs and practices that recognises the influence of a range of new factors including teacher education programs, experiences, teachers' and students' lives outside of school, and teachers' personality traits. Key factors that account for inconsistencies are social teaching norms and the immediate classroom situation.

Table 2.3 ***Six Models used to Investigate Teachers' Beliefs and Practices*** **(p.16)**

Anderson, White, and Sullivan (2005) proposed a new model as shown in Figure 2.2 below incorporating beliefs, knowledge about mathematics and how children learn, practices, and the social context of teaching which includes experiences and constraints. In order to deal with disparities the model includes professed beliefs as a subset of beliefs and reported practice as a subset of practices.

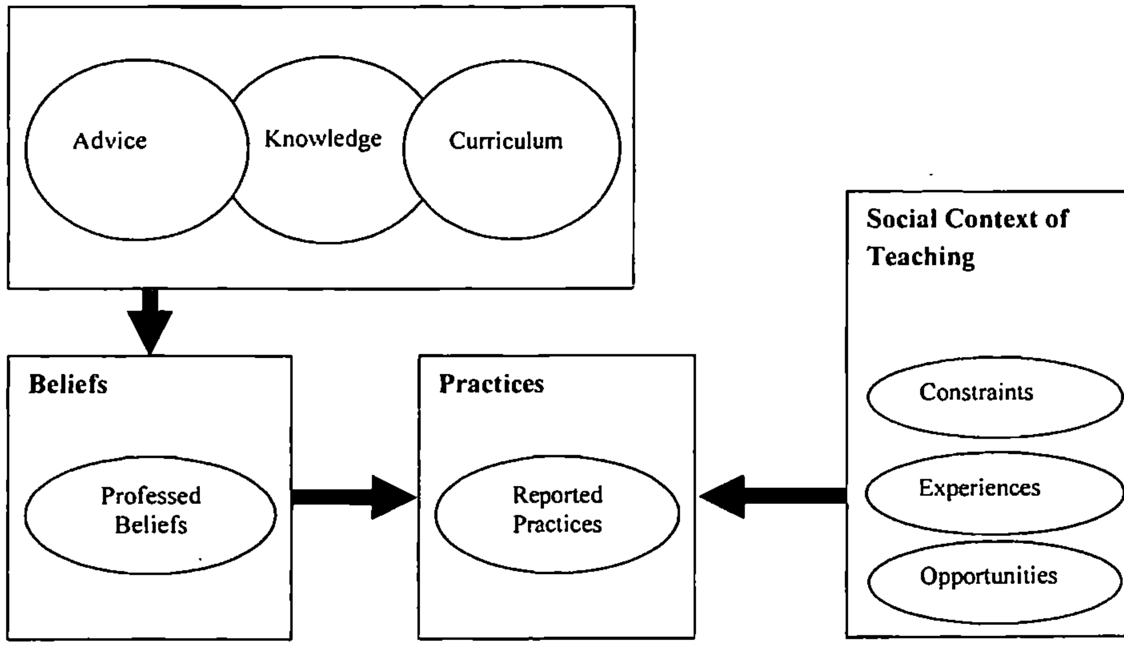


Figure 2.2 ***A model of the factors that impact on teachers' problem-solving beliefs and practices*** **(Anderson, White, and Sullivan, 2005, p.18)**

Anderson et al. (2005) used the model in Figure 2.2 in a study of 162 primary school teachers' problem-solving beliefs and practices to guide both instrument design and data analysis. A survey consisting of both Likert scales and open-ended was used to gather data. The first two sets of survey items contained statements made by two imaginary teachers about problem solving. One teacher had what would be considered a *traditional* teaching approach with a view of problem solving as being an *end* and the second teacher had a *contemporary* teaching approach with a view of problem solving as being a *means.* Another survey item listed 20 statements related to teaching approaches. Respondents had to rate the frequency of their use of these approaches as *hardly ever*, *sometimes*, *often*, and *almost always*.

Semi-structured interviews were conducted with a sample of nine teachers. They represented the range of problem solving beliefs and practices and they taught in a variety of school contexts. A subset of two teachers who were interviewed was chosen to be observed teaching 'problem solving lessons.' The results of the analysis of the data provided evidence for a revision of the model of the relationships between beliefs and practices in Figure 2.2. Knowledge which included not only knowledge about curricula but also knowledge about the students' individual needs and the teachers' own experiences as learners of mathematics was a major factor impacting on teachers' practice. The constraints on implementing a problem solving approach outweighed the opportunities that supported the implementation.

Anderson et al. revised their model, as shown in Figure 2.3, making it cyclic to acknowledge the influence of social context on knowledge and beliefs. They included beliefs as subjective knowledge (Furinghetti and Pehkonen, 2002).

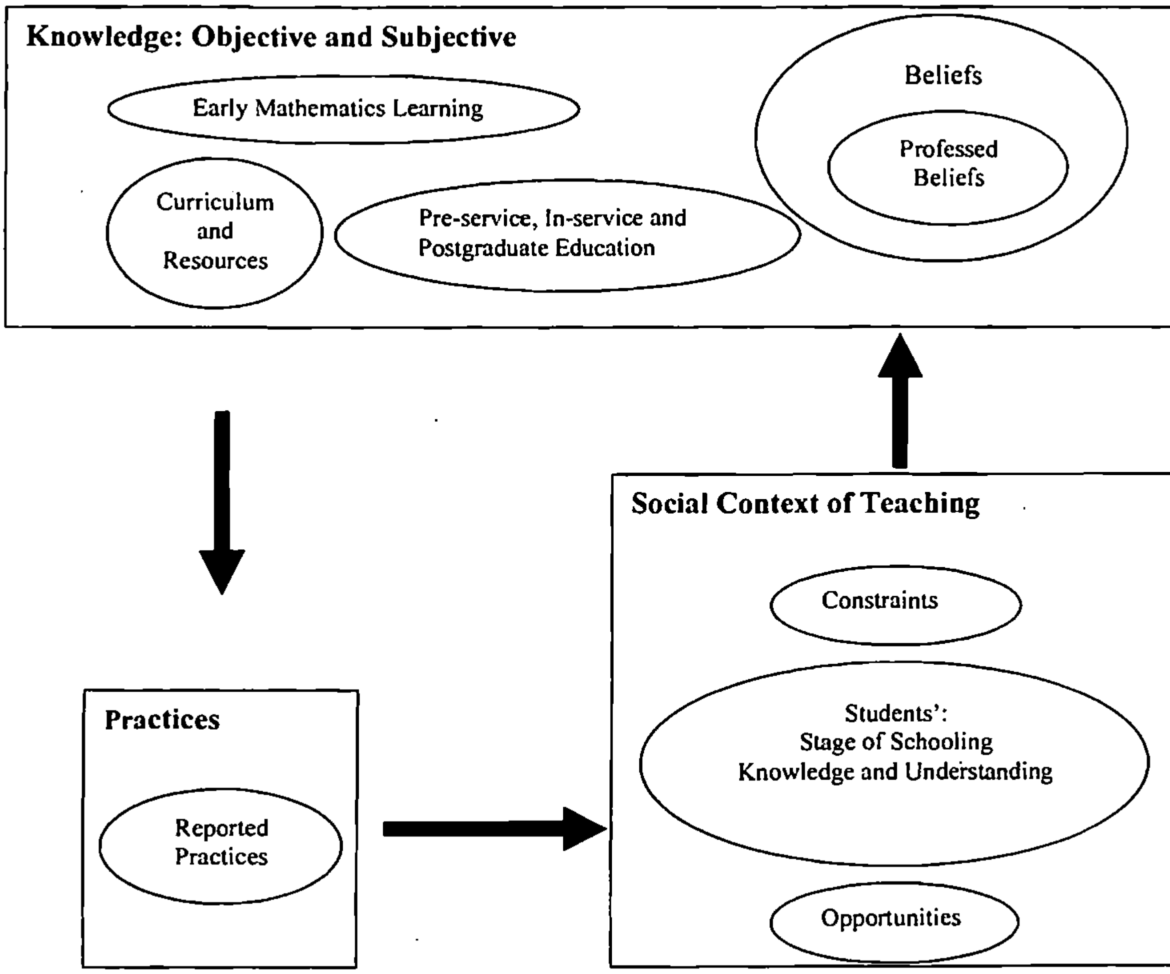


Figure 2.3 ***A revised model of the factors that impact on teachers' reported beliefs and practices*** (Anderson, White, and Sullivan, 2005, p. 34)

The model in Figure 2.3 includes all the factors that I believe impact on teaching practices. Although my study is investigating teachers' beliefs I consider this model to be the theoretical framework for my study.

Pehkonen and Törner (1996) said that one "*meaning of beliefs lies in their inertia force for change: Experienced teachers believe to know through their long-term practice, what kind of mathematics teaching is (in their eyes) good.*" (p. 101) They stated that beliefs have a component in both the cognitive and affective domain. "*Beliefs are situated in the 'twilight zone' between the cognitive and affective domain.*" (p. 101) Teachers' beliefs are essential since teachers play central roles in organising the learning environment in their classrooms. Pehkonen and Törner defined an individual's mathematical beliefs as:

> ...The compound of his subjective (experience-based) implicit knowledge (and feelings) concerning mathematics and its teaching/learning. Conceptions could be understood as conscious beliefs, and thus differ from so-called primitive beliefs which are often unconscious. We think that in the case of conceptions, the cognitive component will be stressed, whereas the affective component is emphasized in primitive beliefs. (p.102)

2.2.3.3 Differentiating Beliefs from Knowledge

While the purpose of this study is to investigate teachers' beliefs regarding the teaching and learning of geometry, I must have a means for separating teachers' beliefs about mathematics from teachers' knowledge of mathematics. I will try to make an attempt here even though Pajares (1992) claimed "*distinguishing knowledge from belief is a daunting undertaking.*" (p.309)

Plato defined knowledge as "*justified true belief*". (McDowell, 1987, p.94, 201d) Objective knowledge is accepted by the community and subjective knowledge does not need to be evaluated.

Similarly Thompson (1992) claimed,

> From a traditional epistemological perspective, a characteristic of knowledge is general agreement about procedures for evaluating and judging its validity; knowledge must meet criteria involving canons of evidence. Beliefs, on the other hand, are often held or justified for reasons that do not meet those criteria, and, thus, are characterized (sic) by a lack of agreement over how they are evaluated or judged. (p. 130)

Bar-Tal (1990) viewed beliefs as units of knowledge. He posited, "*Beliefs constitute the totality of an individual's knowledge, including what people consider as facts, opinions, hypotheses, as well as faith.*" (p.12) This definition of beliefs differs from those of other social psychologists who view beliefs as subjective knowledge.

Knowledge, according to Bar Tal, "*encompasses all the beliefs accumulated through our own experience, thinking, or as a result of contact with other individuals or their products.*" (p.5)

I have found Scheffler's (1965) definition of knowledge most helpful, because it is presented in a propositional format.

> This definition sets three conditions for *knowing that*, and we shall refer to these as the *belief* condition, the *evidence* condition, and the *truth* condition.
>
> *X* knows that *Q* if and only if
>
> (i) *X* believes that *Q*
>
> (ii) *X* has adequate evidence that *Q* and
>
> (iii) *Q*. (p.21)

This definition of '*knowing*' is more widely accepted than the above definition of Bar Tal (1990).

Nespor (1987) provided a conceptualisation of beliefs consisting of six structural features based upon the work of Abelson (1979) who had proposed seven features that differentiate belief systems from knowledge systems. Nespor's six features are existential presumption, alternativity, affective and evaluative loading, episodic structure, non-consensuality, and unboundedness. Abelson had included an additional feature that beliefs could be held with varying degrees of certitude (variable credences). The features important to my investigation are the *existential presumption* and *affective and evaluative loading.*

The *existential presumption* considers that the individual believer has assumptions or beliefs about existence or non-existence of an entity. Pajares (1992) referred to them as "*incontrovertible, personal truths*". (p.309) "*These entities are usually central organizing categories in the belief system, and as such, they may play an unusual role* *which is not typically to be found in the concepts of straight knowledge systems.*" (p. 357) Nespor's study included two mathematics teachers who had strong beliefs about students' 'ability', 'maturity', and 'laziness'. One of the teachers believed that attaining proficiency in mathematics was only realisable through drill and practice, and that a lack thereof was a sign of the student's laziness resulting in a failure to complete assignments. The second teacher believed that success in learning mathematics was dependent upon a student's maturity. This teacher emphasised that the students' communication with one another was essential in achieving the goal of mastering mathematics. Nespor's analysis of the data led him to assert that 'ability', 'maturity' and 'laziness' "*were not simply descriptive terms, they were labels for entities thought to be embodied by the students.*" (p. 318) In Nespor's view, "*the reification of transitory, ambiguous, conditional, or abstract characteristics into stable, well-defined, absolute, and concrete entities is important because entities tend to be seen as immutable — as beyond the teacher's control and influence.*" (p. 318)

Abelson (1979) stated, "*Belief systems rely heavily on evaluative and affective components.*" (p. 358) Nespor (1987) found that belief systems are frequently connected to affective and evaluative components such as feelings, moods, and personal evaluations. These components are grounded in personal preferences, and they tend to act by themselves apart from other cognitive processes in contrast to systems of knowledge. The analysis of the data by Nespor led him to believe that "*a less obvious arena in which affect is important is that of teachers' conceptions of subject matter. The values placed on course content by the teachers in the TBS study often influenced how they taught the content.*" (p. 319) There were four history teachers in his study, three of them believed that teaching history effectively included engaging students in meaningful activities such as analysing history as an inter-related corpus of knowledge instead of a string of separate events. They believed it was important to teach students practical skill such as organising a notebook or outlining a chapter. They de-emphasised rote memorising of dates or the reciting sections of historically important documents. Nespor further found that these history teachers did not spend much time teaching material that would be taught a second time or that would not be focused on in later grades. These findings indicate that affective and evaluative components directly impact a teacher's decisions about lesson planning. Nespor claimed "*Affect and evaluation can thus be important regulators of the amount of energy teachers will put into activities and how they will expend energy on an activity.*" (p.320)

Goldin (2002) characterised knowledge as beliefs that are true, correct or valid. Lester (2002) suggested that to make sharp distinctions between beliefs and knowledge is "*unhelpful and probably wrongly headed*". Instead he thought of beliefs as a special form of knowledge - namely personal, internal knowledge in contrast with external knowledge - knowledge from some community consensus of practice. This internal knowledge directs a person's actions.

Törner (2002) stated that the question of the distinction between knowledge and beliefs is academic. "*However, for many individual persons no sharp borderline is drawn between knowledge and beliefs.*" (p. 82)

2.2.3.4 Domain Specific Beliefs

The studies described above investigated teachers' beliefs about mathematics, mathematics teaching and mathematics learning in general or with respect to problem solving or school reform. This research examines teachers' beliefs about teaching and learning geometry. In this section I consider how beliefs about geometry relate to beliefs about mathematics in general.

Törner (2002) called the beliefs about mathematics in general '*global beliefs*', beliefs about every mathematical term or procedure, he called '*subject matter beliefs*' and the beliefs about an area of mathematics such as geometry '*domain specific beliefs*.' (p. 86-87) A question that appears in both my questionnaire and in my pilot interviews is whether the teachers teach geometry in ways that are different from their teaching of other topics in mathematics. Törner (2002) asked the following open research question:

> What mental structures link global beliefs, domain-specific beliefs and subject matter beliefs? Do the sum of the beliefs from the individual fields of mathematics constitute beliefs about mathematics as a whole, or do general attitudes tend to imprint subjective perceptions more in the individual domains? (p. 87)

Since different fields of mathematics have different characteristics, are global beliefs stronger than domain specific or subject matter beliefs? Is the belief structure a top down or bottom up influence structure as shown in Table 2.4?

Top-down influence	↓	Global Beliefs	↑	Bottom-up influence
		Domain-specific Beliefs		
		Subject-matter beliefs		

Table 2.4 ***Different Belief structures according to Törner*** (p. 87)

Törner (2002) conducted a qualitative study with six graduate pre-service upper secondary school teachers. The six participants were asked to write three essays, each of two to four pages, on their experiences with calculus lessons. The essay themes were: "*Calculus and me – how I experienced Calculus at school and university*", "*How I would have liked to have learned calculus*", and "*How I would like to teach calculus*". (p.88) He concluded from an analysis of the data *"that domain-specific beliefs must be considered in terms of global views of mathematics."* (p.90) Global beliefs address *"a more structural-axiomatic organization of mathematics."* (p.90) Since mathematics is often taught in modules, most high school and university courses do not usually *"induce a pluralistic world view of mathematics"*. (p.90) Törner concluded that more research is needed in the area of domain specific beliefs.

Aguirre (in press), provided additional evidence of domain-specific beliefs. She found that high school teachers, when faced with implementing district-mandated mathematics reform initiatives, expressed different views about the domains of geometry, algebra and probability linked to the level of abstraction within the domain and their perception of the usefulness of the domain for the future.

2.2.4 Measuring Beliefs

Lester (2002) stated that a fundamental problem facing belief researchers is that *"much of this research may rest on a shaky logical foundation. Specifically, a basic assumption is that beliefs influence peoples' - both students' and teachers' - thinking and action. However, it is also often assumed that beliefs lie hidden and so can be studied only by inferring them from how people think and act."* (p. 346) He suggested two ways to solve this problem:

1. Insist that studies of beliefs involve very careful conceptual and methodological analyses.
2. Develop research methods to uncover beliefs directly rather than infer them from teachers' actions. (p.346)

Leder and Forgasz (2002) claimed that *"the advantages and disadvantages of the techniques used to measure attitudes and beliefs continue to be debated in the literature."* (p.98) They summarised various methods for measuring beliefs. These are:

1. Likert Scales which are summated rating scales
2. Projective techniques
3. Checklists/inventories
4. Physiological measures
5. Repertory grid techniques
6. Interviews-an orally administered questionnaire. The 'structured' interview consists of a predetermined list of specific questions to be asked. One advantage of an unstructured interview is that it can uncover views not

anticipated in advance. The semi-structured interview is a combination of these approaches and is used by many researchers.

7. Observations (pp. 98-99)

They summarised ten recent beliefs studies in mathematics education according to the theme of the study, the beliefs to be measured and key instrument used to measure the beliefs. In six of the ten studies questionnaires were used with three of the six using Likert scales. One of the studies used open-ended questions. These are the kinds of instruments used to collect data in this research.

2.2.5 Definition of Beliefs Adopted for My Study

For this study the characterisation of knowledge and beliefs suggested by Furinghetti and Pehkonen (2002) has been adopted. They consider two types of knowledge: objective and subjective. Objective knowledge has to be true whether proved by experiment and/or socially accepted; subjective knowledge is knowledge constructed by an individual. Therefore belief is taken as subjective knowledge.

2.2.6 Recent Belief Studies

There are some recent studies that have been published since this research began. These studies investigated the connection between beliefs and practices of secondary mathematics teachers (Barkatsas and Malone 2005; Beswick, 2005; Karaagac and Threlfall, 2004), pre-service and in-service teachers' beliefs about reform (Aguirre, in press; Cady, Meier and Lubinski, 2006; Gooya, 2007; Webb and Webb, 2006), and teachers' beliefs about problem solving (Anderson, White, and Sullivan, 2005). They took place in Australia (Anderson, White, and Sullivan, 2005; Beswick, 2005, 2007), Cyprus (Charalambous, Philippou, and Kyriakides, 2002), England (Watson and DeGeest, 2005), Greece (Barkatsas and Malone, 2005), Iran (Gooya, 2007), South Africa (Webb and Webb, 2006), Turkey (Karaagac and Threlfall, 2004) and the United

States (Aguirre, in press; Cady, Meier and Lubinski, 2006; Langrall, Alagic and Rayl, 2004).

In many of these studies the researchers stated that beliefs can be defined in many ways and then they proceeded to give the definition of beliefs that they adopted for their studies and continued from there. They found no need to belabour the point. As a researcher reading their studies I can understand their perspectives, as there is no longer the ambiguity described by Hart (1989). For example Barkatsas and Malone (2005) investigated Greek secondary mathematics teachers' beliefs about mathematics and its teaching and learning using McLeod's (1992) characterisation of beliefs and Raymond's (1997) model of relationships between teachers' mathematical beliefs and practice. They employed principal component analysis with varimax rotation, extracting a five component solution which they called orientations: *socio-constructivist, dynamic problem driven, static – transmission, mechanist – transmission,* and *cooperating orientation.*

Beswick (2005) who also studied secondary mathematics teachers' beliefs about mathematics and its teaching and learning used Ajzen and Fishbein's definition of beliefs (Leder and Forgasz, 2002) which is anything a person thinks of as true. Her study took place in Australia. Charalambous, Philippou, and Kyriakides (2002) studied 229 Cypriot teachers' beliefs about the nature of mathematics in order to examine the efficiency of Ernest's three dimensional model (1989, 1991). Teachers responded to a questionnaire containing both Likert items and open-ended questions. Factor analysis, a data reduction technique, was used to identify underlying factors that could account for the large number of significant correlations between responses. Five factors were extracted that represented combinations of Ernest's three dimensional model. Four 'relatively homogeneous' groups of teachers were identified through further analysis.

Charalambous et al. (2002) found the domain of beliefs to be complex (Raymond, 1997). This included the suggestion that teachers' beliefs about the nature of mathematics might influence their beliefs about teaching and learning mathematics. They found inconsistencies between teachers' beliefs and reported practices, but found that previous reported practice influenced beliefs (Raymond, 1997; Anderson et al, 2005; Thompson, 1992).

Of the studies mentioned above only two of them examined the beliefs of mathematics teachers about geometry (Gooya, 2007; Langrall, Alagic and Rayl, 2004). The teachers in both studies were involved in professional development initiatives. Langrall et al. (2004) investigated the epistemological and geometry-related beliefs of 88 middle school mathematics teachers from a Midwestern city in the United States participating in a two year professional development project with the implementation of a standards based curriculum (NCTM, 2000) as its goal. Gooya (2007) studied Iranian secondary school teachers' beliefs about curricula changes in the context of professional development surrounding the use of new geometry textbooks incorporating reform ideologies.

Langrall et al. (2004) claimed:

> Geometry has not been generally taught in the middle school at the level of complexity now called for to help students link mathematical concepts, as recommended in the NCTM connection standard:
>
> > Instructional programs from pre-kindergarten through grade 12 should enable all students to:
> >
> > - Recognize and use connections among mathematical ideas;
> > - Understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
> > - Recognise and apply mathematics in context outside of mathematics. (NCTM, 2000, p.64) (p.2)

Langrall et al. hypothesised

1. If a number of teachers in the current study view knowledge as structurally *simple*, a similar percentage would be inclined to view geometry as more about the *step-by-step* application of memorised rules rather than as a way of thinking involving a set of integrated *concepts*.

2. If a number of teachers viewed learning as a relatively *quick* process, a similar percentage would be disinclined to view geometry as *time-consuming*.

3. If a number of teachers viewed knowledge as structurally *simple* and learning as a relatively *quick* process, a similar percentage would be disinclined to view geometry as a subject involving *word problems*, as these would typically involve *effort* to deliberate about geometry's *use* within real world settings.

The 88 participants completed a 102 item questionnaire that the researchers adapted from already existing belief survey instruments but substituted the word *geometry* instead of the word *mathematics* in the original.

The results of the survey supported hypothesis #1 with 39% of the respondents *somewhat agreeing* or *strongly agreeing* that knowledge is simple and 38% of the respondents *somewhat agreeing* or *strongly agreeing* that doing geometry involves rule based step-by-step procedures. Alternatively 44% of the respondents *somewhat disagreed* or *strongly disagreed* that knowledge is simple and a similar percentage (45.4%) *somewhat agreed* or *strongly agreed* that geometry is conceptual. The success of any professional development intervention depends on appropriate accommodations for teachers with such diverse beliefs.

Similarly, survey results supported hypothesis #2. Teachers for the most part believed that geometry is difficult and/or that more sophisticated teaching methods are needed to develop understanding. The results of the survey did not support hypothesis #3. Almost half of the respondents *somewhat disagreed* or *strongly disagreed* with the statement that geometry is useful. Langrall et al. account for this, *"Believing that word problems are not part of geometry suggests that teachers' prior experiences with learning geometry may have excluded much emphasis on word problems or that* *teachers do not see many real world applications for geometry.* " (p. 11)

They concluded that professional development for middle school mathematics teachers has to address the issue of "*how to spot and explore the quantitative/spatial aspects of everyday life (word problems make geometry real and relevant and require effort to unpack).* " (p. 12) Their findings suggested that professional development for teachers with mixed epistemological beliefs should focus on: "*developing connected conceptual understanding, increasing insight into students' thinking, and relating geometry to the real world.* " (p. 12)

Unlike in Langrall et al. (2004) where the participants met for monthly four hour sessions, in Gooya (2007) there were 100 hours of professional development for 480 teachers over a 10 day period. Although there was some reform of mathematics education in 1992 in Iran, there was no change to the main geometry textbook which contained only deductive reasoning and very few real world applications. Finally new geometry textbooks with reform oriented approaches were written and 480 teachers took part in a nationwide professional development program with the goal of successful implementation of these textbooks. There were 130 participants who were experienced teachers with traditional views about teaching and learning geometry. Included in this group of teachers were 30 teachers who considered themselves solely as geometry teachers. The remaining 350 participants did not have much experience teaching geometry and did not have 'firm beliefs' about teaching and learning geometry. In this paper the author does not explain how she measured their geometric beliefs at the start of the professional development program. This study was qualitative and involved analysis of reflective writings, open-ended questionnaire, video-taped group and whole class discussions, oral communications and teacher notes.

Gooya (2007) identified three categories of teachers that emerged from her analysis of the data. The more conservative of the experienced teachers held fast to their beliefs that the changes to the geometry texts were unnecessary and useless. Gooya called this category of teachers '*traditionalists.*' She identified the teachers who were willing to try new approaches but had some reservation since these approaches were not in accord with their beliefs as '*incrementalists.*' The '*innovators*' were the teachers that embraced the curriculum changes. Their beliefs were aligned with those of the curriculum developers. The findings showed that in-service professional development can help in the implementation of reform ideologies.

This was not the case with pre-service elementary school teachers in the United States. Cady, Meier and Lubinski (2006) conducted a longitudinal study of elementary school teachers as they transitioned from pre-service to experienced teachers. This paper reported on two of the participants in the Cognitively Guided Instruction Project that had taken place while they were doing their field experience in pre-service education. The goal of the project was to provide experiences and discussions to *"challenge pre-service teachers' traditional beliefs about mathematics teaching and learning and to provide alternative models for teaching mathematics. The objective was that teachers fully implement mathematics education reform practices in their classrooms as novice teachers."* (p. 3) Two questionnaires both using Likert scales were among the instruments used to collect data. The goal of the study was to find out whether these two teachers were able sustain the beliefs and practices promoted by the CGI Project. Although the teachers experienced the same pre-service education and taught at similar schools their beliefs as experienced teachers were different. One reason for the difference was due to the different professional development programs with which they were involved. My interest in this study revolved around the lengthy questionnaires to which participants responded and the issues that were responsible for the changes in the teachers' beliefs.

As I complete this section of the review of the literature on beliefs I want to reiterate that for this study I have adopted the characterisation of knowledge and beliefs suggested by Furinghetti and Pehkonen (2002), where belief is taken as subjective knowledge. I have seen in the literature the way that beliefs are measured and the key instruments used to measure them. Several studies used questionnaires with Likert scales together with open-ended questions. This is the kind of instrument that I decided to use to collect data. I have also found few studies about high school teachers' domain specific beliefs about geometry and its teaching and learning.

2.3 GEOMETRY

"*Let no one ignorant of geometry enter my doors*" is the inscription carved over the entrance to Plato's (492-348 B.C.) academy. (O'Daffer, 1980) Over 2400 years later we can still ask the basic question: "*What is geometry*?" for example Allendorfer (1969) stated, *"In geometry . . . there is not even agreement as to what the subject is about."* (p.165)

The Oxford English Dictionary (second edition, 1989) defines geometry as the science that investigates properties and relations of magnitude in space, as lines, surfaces, and solids. In the etymological sense, geometry is the art of measuring ground.

O'Daffer (1980) defined geometry as "*the study of space and spatial relations*". Mason (1989) defined geometry as *"dynamics of the mind; what is 'seen'; incidence properties invariant under isometries and similarities."* (p. 36). He says that the real importance of geometry to him is *"as a domain in which the fact that there are necessary and inescapable facts can be experienced, developed, manipulated to* *produce new facts and for those that wish, organized into a deductive scheme.* " (p.43) The National Council of Teachers of Mathematics (NCTM) said: "*Geometry offers a means of describing, analyzing, and understanding the world and seeing beauty in its structure*," (NCTM, 2000, p.308). Similarly, these examples illustrate the diversity of thinking about the very definition of geometry, however there are many suggestions that geometry is an important topic to study.

The National Research Council said: "*Geometry is a vibrant and exciting part of mathematics and a key to understanding our world*" (Leitzel, 1991).

O'Daffer (1980) stated

> ...geometric form and structure have always permeated the universe and that humans have been immersed in a geometric environment from the very beginning. As early inhabitants observed the world around them, they began to abstract geometric ideas and draw pictures to represent them. Later it became useful to name them, to define them more accurately to enhance communication, and to study the more complex relationship between these abstracted ideas. Finally, these refined ideas were reapplied to the real world in simple as well as sophisticated situations. (p.91)

O'Daffer suggested that geometry could be studied in three ways:

1. With a focus on its origins in nature and imitations in human-made objects.
2. As a logical, organized body of knowledge like Euclid did.
3. As a formal, axiomatic structure as Einstein did. (p. 91)

When Einstein referenced non-Euclidean geometry in a lecture he gave in 1921 he said, "*To this interpretation of geometry I attach great importance, for should I have not been acquainted with it, I would never have been able to develop the theory of relativity.*" (O'Daffer, p. 91)

2.3.1 Models of Geometric Development

Bell, Costello, and Küchemann (1983) described the three stages of teaching and learning geometry that were identified in a 1923 report of the Mathematical Association.

> *Stage A* is a 'preliminary experimental stage', based on practical situations, and on drawing and measuring.
>
> *Stage B* is the deductive stage where the student 'learns to prove theorems and riders and to write out proofs'.
>
> *Stage C* is the systematising stage where the aim is 'to arrange the theorems in a logical sequence depending on a comparatively small number of axioms.' (p. 222)

These stages have some similarity to the levels of geometric thinking proposed by Dina and Pierre van Hiele, two Dutch teachers, who in the 1950s were concerned with the difficulties their students encountered when learning geometry. The van Hieles believed that they were teaching on one level while their students were thinking on a different level. They claimed that if the teacher and student were reasoning on two different levels, then they would not be able to understand one another. This observation ultimately led the van Hieles to describe five levels of geometric thinking.

The van Hiele Model of the development of geometric thought has been used as a framework for interpreting students' understanding of geometric ideas and for developing teaching programs for geometry (Crowley, 1987; Fuys, Geddes, and Tischler, 1988; Hoffer, 1981; Mayberry 1983; Shaughnessy and Burger, 1985; Usiskin, 1982; van Hiele, 1999; van Hiele-Geldof, 1984/1958; Wirszup, 1976).

Wirszup (1976) investigated the levels in the Soviet Union in the 1960s and introduced them in the United States in 1974. He claimed that the majority of the high school students were at an earlier level of development of geometric thinking than the course they were taking demanded. There were several research projects related to the van Hiele levels conducted in the United States during the 1980s. Usiskin (1982) tested Wirszup's claim in the United States. The purpose of his study was "*to test the ability of the van Hiele theory to describe and predict the performance of students in secondary school geometry*" (p.8). Mayberry (1983) studied pre-service elementary teachers. Shaughnessy and Burger (1985) studied K-12 students. Fuys, Geddes, and Tischler (1988) studied grades 6 and 9 students and teachers and textbooks from grades K-8.

The Van Hieles numbered their levels 0-4 (Fuys, Geddes, and Tischler, 1988; Jones, 1998). Many of the subsequent researchers renumbered the levels 1-5. (Burger and Shaughnessy, 1986; Hoffer, 1981; Usiskin, 1982)

<u>LEVEL 0:</u> Learners can identify and name geometric figures based on their physical appearance. (Visual/ Recognition level)

<u>LEVEL 1:</u> Learners can analyse figures in terms of their components and discover properties of a figure experimentally. They cannot formally define figures and cannot recognise relationships between figures. Properties are not yet logically ordered. (Descriptive/analysis level)

<u>LEVEL 2:</u> Learners can use informal deduction to see relationships between different figures, since properties are logically ordered at this level. Learners can form meaningful definitions and can use them to justify relationships. They can follow a proof but have trouble beginning a proof or writing it if it is different from what they previously experienced, because the role of axioms, theorems, and their converses is not fully understood. (Informal deduction/ordering level)

<u>LEVEL 3:</u> The student can prove theorems and establish relationships between the theorems. (Formal deduction level)

<u>LEVEL 4:</u> The student can establish theorems in different ways and analyse and compare them. Geometry is seen in the abstract. A theory can be developed without any concrete interpretation. (Abstract/rigour)

Besides the five levels of geometric thought the van Hiele Model includes several properties that characterise the levels of thinking. The first property is that the levels are sequential. A student must proceed through the five levels in order. A student must succeed in the previous level in order to proceed to the next level. The second property is called advancement. Students can succeed from level to level based on how they transform their information. They should understand the concepts and not just memorise the skills involved by rote. The third property is intrinsic and extrinsic. Crowley (1987) explained, "*The inherent objects at one level become the objects of study at the next level.*" (p. 4) The fourth property is linguistics. Students should progress from level to level and begin to make connections between figures by using comparative language associated with the respective level. The last property is mismatch. If a student is at one level and the teacher is instructing at another level, the student cannot successfully understand the information. (Crowley, 1987)

Usiskin (1982) carried out research on the van Heile Model by testing 2699 students enrolled in a one-year geometry course from 13 schools throughout the United States. His project, the Cognitive Development and Achievement in Secondary School Geometry (CDASSG), developed a twenty-five question multiple-choice test called the van Hiele Geometry Test that has been used in a range of research settings. Usiskin and Senk (1990) found that the test answered two questions: (1) Is the theory *descriptive*, in the sense that a unique level can be assigned to each student?; and if so (2) is the theory *predictive*, in the sense that the students' van Hiele level can be used to predict his or her performance in a traditional tenth-grade geometry course? They found that a student's van Hiele level is a good predictor of their ability to write proofs. Seventy percent of the students tested were at van Hiele levels 0 and 1 before taking the geometry course. They found that only those students entering at level two had a good chance of understanding and producing proofs. Usiskin (1982) concluded:

1. In the form given by the van Hieles, level 5 (abstract level) either does not exist or is not testable. All other levels are testable. (p. 79)
2. Over two-thirds and perhaps as many as nine-tenths of students respond to test items in ways which make it easy to assign them to a van Hiele level. (p. 80)
3. Arbitrary decisions regarding the number of correct responses necessary to attain a level can affect the level assigned to many students. (p. 80)
4. Considering those students at a given van Hiele level in the autumn, there is a great variability in the change in van Hiele level from autumn to spring. (p.81)
5. Van Hiele level is a very good predictor of concurrent performance on a multiple-choice test of standard geometry content. Van Hiele level is also a good predictor of concurrent performance on a proof test. (p. 82)
6. In geometry classes that have studied proof, the van Hiele levels of most students toward the end of the school year are too low to afford a high likelihood of success in geometry proof. (p. 83)
7. In geometry classes that study proof, the autumn van Hiele levels of over half the students are too low to afford even a 2 in 5 chance of success at proof. (p. 84)
8. Using van Hiele levels as the criterion, almost half of geometry students are placed in a course in which their chances of being successful at proof are only 50-50. (p.85)

9. Many students leave the geometry course not versed in basic terminology and ideas of geometry. (p.87)
10. The ability to learn geometry, from facts through proof, is equal between the sexes. (p. 88)

There were researchers who had psychometric concerns about the CDASSG Van Hiele Geometry Test. Wilson (1990) reanalysed the data from Usiskin's investigation through the use of a probabilistic model. He gave a more detailed meaning to the concept of testability. He threw some doubt on the Van Hiele Geometry Test and suggested ways to improve that instrument. Crowley (1990) provided an alternative analysis of the reliability associated with the Van Hiele Geometry Test. She suggested that if the instrument is to be used to assign a van Hiele level to students, it needed to have more reliability studies conducted on it. "*By providing a starting point for assessing levels, the Van Hiele Geometry Test has made a valuable contribution to research on van Hiele levels.*" (p 240) Usiskin and Senk (1990) agreed that the Van Hiele Geometry Test needed improvement, but none of the studies that used the instrument had *"found performance in high school geometry significantly different from that of students in our study."* (p. 244)

Mayberry (1983) investigated the van Hiele levels of 19 undergraduate pre-service elementary school teachers, specifically testing the hierarchical nature of the levels. Although the study was limited it did confirm the Usiskin (1982) results that 70% of the response patterns of the students who had taken high school geometry were not at the proper level to understand the deductive nature of geometry.

Shaughnessy and Burger (1985) analysed the thoughts of over seventy primary and secondary school students about geometric concepts through activity-based interviews. They found that *"most students in high school geometry have a lot of difficulty with* *deduction and proof"* because *"they don't understand the role or meaning of an axiomatic system."* (p.419) The students resort to memory to get through their geometry course. Their findings corroborated Usiskin's (1982) results, which revealed that although traditional high school geometry courses, at the time of their study, were taught at van Hiele level 3, many of the students were reasoning at level 1. As a result of this discrepancy, students were left with negative attitudes towards geometry and did not appreciate the need for proof. Shaughnessy and Burger concluded that there was a need to include informal geometry before formal geometry in the high school curriculum.

The Brooklyn College Project (Fuys, Geddes and Tischler, 1988) had four main objectives:

> 1. To develop and document a working model of the van Hiele levels based on several sources that the Project had translated from Dutch to English.
>
> 2. To characterise the thinking in geometry of sixth and ninth graders in terms of levels—in particular, at what levels are students?; do they show potential for progress within a level or to a higher level?; and what difficulties do they encounter?.
>
> 3. To determine if teachers of grades 6 and 9 can be trained to identify van Hiele levels of geometry thinking of students and of geometry curriculum materials.
>
> 4. To analyse current geometry curriculum as evidenced by American text series (grades K-8) in light of the van Hiele model. (p.1)

The results of this study supported the hierarchical nature of the first three levels. *"The results indicated that pre-service and in-service teachers can learn to identify van Hiele levels of thinking in student responses and in text materials."* (p. 154) The teachers who participated in the study gained insights into geometry, by working through the prepared modules.

It is important for teachers to introduce worthwhile tasks which enable students who are functioning on different van Hiele levels to approach the task from their particular level (Crowley, 1987; Fuys, Geddes and Tischler, 1988; Mayberry, 1983; Shaughnessy and Burger, 1985). Teachers should be able to recognise that students may be unable to perform higher level tasks unless they have made the transfer to that level. Appropriate teaching is necessary in order to encourage that transfer.

Van Hiele (1999) answered the question about how students develop geometric thinking:

> I believe that development is more dependent on instruction than on age or biological maturation and that types of instructional experiences can foster, or impede development. ...instruction intended to foster development from one level to the next should include sequences of activities, beginning with an exploratory phase, gradually building concepts and related language, and culminating in summary activities that help students integrate what they have learned into what they already know. (p. 311)

Malloy (1999) worked with middle school students, who were on different van Hiele levels. She engaged them in an activity about perimeters. The students each planned strategies at their own levels of thought. Their *objects of thought* were dependent on their van Hiele levels. Malloy used guiding questions for a group discussion which helped to extend student thinking to the next level.

There have been challenges to the Van Hiele Theory by several researchers. (Gutiérrez, Jaime and Fortuny, 1991; Pegg 1997a, b; Pegg and Currie, 1998)

Gutiérrez, Jaime and Fortuny (1991) challenged the levels in the van Hiele Theory. They theorised that the van Hiele geometric thought levels are not discrete and they presented an additional method to evaluate and identify those learners who are in transition between levels. Although they looked specifically at three-dimensional geometry, their method could be used for any topic where the van Hiele levels can be applied. They concluded that a student could operate on two consecutive reasoning levels at the same time. This does not mean that the van Hiele levels are not hierarchical, as the van Hieles claimed. Rather, since the human reasoning process does not behave in a linear manner the van Hiele models would have to be adapted to reflect this thinking process. Another conclusion they reached was that students showed a better acquisition of van Hiele level 2 than level 1. This was true in the Brooklyn College study too. (Fuys, Geddes and Tischler, 1988) They felt this could be due to a fault in their test, their methods of evaluation, or the teaching methods used in the classroom. Since thirty-three out of the fifty participants in this study were ages 21-22, I suspect that previous classroom experiences or lack of them could account for this finding.

Pegg (1997) suggested a modification of the second van Hiele level (level 1 according to the van Hieles) by splitting it into two parts A and B, where A is the part of the level where *"figures are identified in terms of a single property"* and B is the part of the level where *"figures are identified in terms of properties which are seen as independent of one another."* (p. 391)

Pegg and Currie (1998) found that *"the current level descriptors are narrow and easily generalisable to a range of question types common in school geometry."* (p. 335) They have tried to broaden the descriptors to allow for more inclusive criteria while at the same time maintaining consistency with the original ideas of the van Hieles. They made use of the Structure of Observed Learning Outcome (SOLO) Taxonomy (Biggs and Collis, 1982), which like the van Hiele theory was designed to facilitate learning activities.

> The van Hiele levels are a series of signposts of cognitive growth reached through a teaching/learning process as opposed to some biological maturation. SOLO, however is particularly applicable to judging the quality of instructional

> dependent tasks. It is concerned with evaluating the quality of students' responses to various stimulus items. (p. 3-337)

In the van Hiele theory the levels describe people, but in the Solo Taxonomy the levels describe students' responses. The three Solo levels of responses are associated with van Hiele levels two and three and can be used to broaden the descriptors at these levels.

Having reviewed the research literature on the ways students learn geometry attention now turns to the research literature focusing on the teaching of geometry.

2.3.2 Teaching Geometry

Jones (2000) claimed,

> Teaching geometry effectively involves, among other things, appreciating the history and cultural context of geometry, knowing how to recognise interesting geometrical problems and theorems, understanding the many and varied uses to which geometry is put, and incorporating all these things into the practice of teaching in the classroom. (p. 109)

A significant additional factor in teaching geometry effectively is an appreciation of how students receive and process the material that is being taught.

One component of a teachers' professional knowledge is to have an understanding of how students think about and conceptualize the mathematics that they teach. As Davis (1986) pointedly observed, *"A teacher who is not concerned with how the students think will not succeed in 'teaching' much mathematics."* (p. 274)

Davis asked whether tenth-grade geometry teachers actually teach mathematics or simply take their students through a set of procedures. His answer is that in most cases it appears that they do not. To illustrate what he means, he presents the following sequence of events:

> (1) The teacher assigns the task of proving a certain theorem, the proof to be handed in the next day; (2) on the next day, some students (usually only a few)

> come to class with correct proofs and pass them in. Clearly, the teacher did not teach *these* how to make the proof; they worked it out at home, either by themselves, or with parental help; (3) But most students come to class the next day without a proof and report that they were unable to make one. What does the teacher do for these students? Typically, the teacher *shows* them a proof. But this does not answer the question. The real question was: *How does someone who, initially, does not know how to make a proof, go about the task of analyzing the problem so that they ultimately ARE able to make a proof?* Typically the teacher does not attempt to deal with the question. (Davis, 1986, p. 274)

Moise (1975) and Schoenfeld (1986) suggested that when geometry is taught properly students have the opportunity to do real mathematics in precisely the same way that research mathematicians do. This is one of the reasons that Chazan and Yerushalmy (1997), Gonzalez and Herbst (2006), Herbst (2002) gave for keeping Euclidean geometry in the secondary school curriculum. In this Euclidean environment, students can, ideally, experience the deductive development of an axiomatic system. Chazan and Yerushalmy (1997) questioned whether Euclidean geometry should be replaced or modified in the secondary school curriculum. They believed that there should revisions to the traditional course and that these revisions should be of the kind that could be supported by dynamic geometry software.

Mason (1989) suggested **why** we teach geometry:

> ...to strengthen and help organise sense of space; to educate awareness that there are certain geometrical facts; to gain direct contact with the world of mathematics accessed through the mind. (p. 36)

Mason (1989) also suggested **how** we teach geometry:

> By encouraging and supporting pupils in *working on* rather than *working through* mathematical tasks; by bringing attention to the power of mental imagery, and extensions of the mental on paper and electronically. (Mason, 1989, p. 36)

According to Freudenthal (1973) the teaching of geometry had not been successful worldwide because the deductivity "*was imposed on the learner.*" (p.402) In an earlier paper Freudenthal (1971) stated, *People today believe that geometry failed because it was not deductive enough; to my opinion it failed because its deductivity could not be* *reinvented by the learner but only imposed.*" (p. 418)

In many classrooms geometry is taught at the recall level. Fuys, Geddes and Tischler (1988) suggested reasons why teaching only for recall or rote learning should be avoided:

> First, such teaching prevents students from engaging in appropriate thinking about geometry topics. For example, students are simply not learning much geometry if they memorize relationships such as 'all squares are rectangles' and 'area of a rectangle is base times height,' without trying to explain them, at least intuitively. Second, students tend to forget or confuse memorized information and are often unable to apply it, especially in non-routine situations. Third,...conveys the meta-cognitive message that learning geometry is just a matter of memorization. This in turn, prevents students from even wondering if properties are true, and if so, why. (p. 156)

This suggestion is in agreement with Skemp's (1976) work, which distinguished between instrumental and relational understanding. He described instrumental understanding as '*rules without reason*' whereas relational understanding is "*knowing both what to do and why.*" (p.2) Skemp would call teaching for recall 'teaching instrumental mathematics.'

In a similar vein, Moise (1975) suggested two major hazards in the teaching of mathematics:

> 1. It is much easier to drill students in a repertory (sic) of routines than to teach them the real meaning of the things they are asked to do. In courses taught under pressure - and most of them are - the temptation to settle for the repertory (sic) is almost irresistible. In fact, the art of yielding to this temptation is highly developed.
>
> 2. Even if we do our best to "teach for understanding," the fact remains that most of the ideas that we teach lead to processes for solving problems; and unless we do something drastic to prevent it, *the process tends to replace the problem in the mind of the student.* In practice, people remember not the ideas that are explained to them but the ideas that they *use*: and what students really learn, in a mathematics course, is whatever they use in doing their homework. (p.473)

This suggests that if students were empowered to construct meanings for themselves they would be more likely to learn mathematics.

Moise (1975) stated that geometry is a course where the problem of developing meaning for the existence of mathematical objects does not arise. "*The ideas of point, line, plane, circle, sphere, angle, congruence, and so on are immediate abstractions from common observation and experience.*" (p.475) He feels that the intuitive nature of geometric concepts is helpful in exacting definitions.

> Nearly every geometric definition can be-and commonly is- preceded by a picture that conveys an intuitive idea. The definition can be checked against the pictures, with a view to finding out whether the definition really describes the idea that it is supposed to describe. (Moise, 1975, p. 475)

The use of definitions has a special place in mathematical discourse. Students can understand geometrical definitions and cite them "*in their homework papers in the same way in which a highly trained mathematician would.*" (p. 476)

Lim and Moore (2002) examined the effects of teaching high school geometry on students needing remedial tutoring by using non-goal specific problems rather than using worked examples. They found that the participants in the non-goal specific group showed greater improvements in test scores. They solved problems faster, were more efficient, and made fewer errors. This study provided evidence that the manner of presentation of instructional content in a geometry class affects student learning.
The fact that effective instruction affects student learning had been recognised by The Mathematical Association of America in its document *A Call for Change* (Leitzel, 1991), which described the collegiate mathematical experiences that prepare the "*ideal*" mathematics teacher. The recommendations for high school teachers were a need for thorough understanding of geometry from synthetic, transformational, and algebraic perspectives and not limited to the plane, but including higher dimensions. This need

arose from the increasing variety of geometric applications in the world, including imaging techniques and knot theory (Usiskin, 1980, 2007).

Ten years later, as the geometry content of the curriculum changed the MAA recognised the changes and made further recommendations.

The Mathematics Association of America (2001) stated that

> ...high school geometry was once a year-long course of synthetic Euclidean plane geometry that emphasized logic and formal proof. Recently, many high school texts and teachers have adopted a mixture of formal and informal approaches to geometric content, de-emphasizing axiomatic developments of the subject and increasing attention to visualization and problem solving. Many schools use computer software to help students do geometric experiments---investigations of geometric objects that give rise to conjectures that can be addressed by formal proof. Some curricula approach Euclidean geometry by focusing primarily on transformations, coordinates, or vectors; and new applications of geometry to robotics and computer graphics. These approaches illustrate how mathematics is used in the workplace in ways that are accessible and interesting to high school students. (p. 41)

The Mathematics Association of America (2001) recommended that to be well-prepared to teach the geometry in high school, mathematics teachers need:

- Mastery of core concepts and principles of Euclidean geometry in the plane and space.
- Understanding and facility with a variety of methods and associated concepts and representations, including transformations, coordinates, and vectors.
- Understanding of trigonometry from a geometric perspective and skill in using trigonometry to solve problems.
- Knowledge of some significant geometry topics and applications such as tiling, fractals, computer graphics, robotics, and visualisation.
- Ability to use dynamic drawing tools to conduct geometric investigations emphasising visualisation, pattern recognition, conjecturing, and proof.
- Understanding of the nature of axiomatic reasoning and the role that it has played in the development of mathematics, and facility with proof. (Conference Board of the Mathematical Sciences [CBMS], p. 41)

Before CBMS (2001), Grover and Conner (2000) surveyed over 100 universities across the United States about their undergraduate geometry for pre-service high school mathematics teachers. They found that 40% of the courses emphasised Euclidean geometry or a mixture of Euclidean and non-Euclidean geometries, 20% emphasised analytic and projective geometries, and 23% took a survey approach that gave students a short introduction to several geometries. The teaching approach in 63% of these courses was straight lecture. There was group work in the remainder of the courses, but only three quarters of the instruction in these classes could be considered standards based. After analysing their data, they did not find a 'typical' geometry course.

At the same time, in the UK, The Royal Society (2001) published, *Teaching and Learning Geometry 11-19*, which recommended that *"the most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources which are disseminated effectively and coherently to, and by, teachers."* (p. 19)

2.3.2.1 Geometry in the School Curriculum

The first of my research questions concerning high school teachers' beliefs about teaching geometry is: What is the role of geometry in the curriculum?

Discussions and studies about the role of geometry in the school curriculum have been on going for many years. Reeve (1930), in the Fifth Yearbook of the National Council of Teachers of Mathematics, claimed:

> In the tenth grade the pupil is plunged headlong into the study of formal geometry without any previous preparation in or experience with informal geometry as a background. The next problem is, therefore, to consider the importance of beginning the study of geometry earlier and spreading it over a longer period of time. (p. 6)

In the UK, the Mathematical Association (1959) published the book Mathematics in Secondary Modern School, in which they reported:

> For the modern school pupil the value of experimental work in geometry will depend to no small extent upon choice of practical themes. The risk of work in geometry becoming desultory or time wasting is greatly reduced if back ground experiences are in themselves worthwhile and interesting. (Mathematical Association, 1959, p. 124)

The Royal Society (2001) recommended that geometry become a significant part of the curriculum. They suggested a minimum of three hours a week devoted to teaching mathematics with 25-30% of the time being devoted to geometry. They recommended a name change from *shape, space, and measure* back to geometry and that the word *numeracy* in documents should be replaced with the word *mathematics*. Numeracy connotes arithmetic whereas mathematics can be any of its domains.

The recommendations from the professional organisations in both the USA and the UK advocated geometry being taught informally in the early grades.

2.3.2.1.1 Teaching Geometry in Elementary School

Before the National Council of Teachers of Mathematics introduced their Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) there was little emphasis (O'Daffer, 1980) on geometry in elementary school classrooms. Kline (1973) and Shaughnessy and Burger (1985) were among the researchers that recommended introducing informal geometry in elementary school.

O'Daffer (1980) suggested that many elementary school teachers omitted teaching geometry because they

> ...are often operating on premises established when they studied geometry in high school. They seem to feel that geometry is a rigorous, proof-oriented type of subject that would be uninteresting and difficult for them and the children in their classes. This emphasis on the deductive process is sometimes even reinforced in pre-service courses for elementary teachers and has often led teachers to a narrow view that has limited their ability either to view geometry creatively or to enjoy geometric activities. (p. 93)

The NCTM (1989) suggested

> In grades K-4, the mathematics curriculum should contain two- and three-dimensional geometry so that students can-

- describe, model, draw, and classify shapes;
- investigate and predict the results of combining, subdividing, and changing shapes;
- develop spatial sense (An intuitive feel for one's surroundings and the objects in them. p.49);
- relate geometric ideas to number and measurement ideas;
- recognise and appreciate geometry in their world. (p. 48)

Agreeing with the NCTM Standards (1989), Ball (1992) advocated putting a greater emphasis on geometry in elementary schools. Davis (1994) suggested using various manipulatives, drawings, and the computer program, LOGO, when teaching geometry to elementary school pupils, instead of emphasising technical terms as was proposed by The School Mathematics Study Group. Davis claimed,

> Now, children are clearly interested in geometry, but using natural language in a precise and constrained way to describe abstract mathematical entities is not high on their lists, and probably does not contribute much to increasing their ability to visualise complex geometric arrangements. (p. 4)

2.3.2.1.2 Geometry in the Secondary School Curriculum

The National Council of Teachers of Mathematics has said, "*Geometry is a natural place for the development of student reasoning and justification skills.*" (NCTM, 2000) Herbst (2002) stated that when universities made geometry a requirement for admission in the 1840's American high schools started to offer geometry courses. Students then had to master the "*Euclidean body of knowledge as developed by a text.*" (p. 288)

Kline (1973) said that with the teaching of Euclidean geometry the traditional curriculum "*becomes deductive.*" (p.6) He claimed that this "*shift from mechanical algebra to deductive geometry bothers most students*" (p. 6) since they have not yet learned what "proof" is and must master this concept and its requirements in addition to learning the subject matter of geometry. Whether proof is taught or not Kline believed that "*the traditional method of teaching results in far too much of one kind of learning-*

- memorisation. " (p.7) Kline suggested that an intuitive approach to learning geometry should be taken and he insisted that *"This recommendation may appear to be treason to mathematics, but it is loyalty to pedagogy.* " (p.157)

Kline included the use of pictures, reasoning by analogy, and induction as part of the intuitive approach. There is a place for deductive proof to be introduced and taught after the student has understood a result and appreciates that the argument for it is plausible. He said, *"In no case should one start with the deductive approach, even after students have come to know what this means. The deductive proof is the final step.* " (p.162) However, Kline felt it was important to keep synthetic geometry in which Euclidean geometry is the base, in the curriculum, since geometry *"furnishes the pictorial interpretation of much analytic work.* " (p.154)

Hoffer (1981), like Kline, believed that students needed *"to explore more with pictures and manipulative devices"* (p. 11) before starting to work with deductive proofs. He stated that in a good geometry course *"it is important to provide students with adequate experiences to develop both sides of the brain.* " (p. 11): the left hemisphere dealing with logic and analytic function and the right hemisphere dealing with spatial functions. He stated five basic geometric skills that high school students should learn before spending time doing formal proofs. These are: visual skills, verbal skills, drawing skills, logical skills and applied skills. He gave examples of what each of the skills look like at each van Hiele level.

On the other hand, Moise (1975) believed in maintaining the traditional Euclidean geometry course. His reason was:

> ...the search for coherence and order, over and above the search for new facts, is a vital part of scientific thought. I believe that systematic geometry is by far the most elementary example of this, and the fact that it was historically the first example is not an accident. I think it is the only example that young students are likely to understand. Note that here the whole is greater than the sum of its

parts. If the facts of elementary geometry were taught piecemeal, as digressions in other course, with no regard to the way in which they fit together, then the educational effect would be quite different. (p. 477)

At the same time, Bell, Costello, and Küchemann (1983) in the UK agreed with Kline (1973) on the issue of current geometry courses not being traditional Euclidean courses. Bell et al. argued that transformation geometry, which had widespread acceptance as a topic in the 1960's and 1970's in the UK, could be used to develop an understanding of geometry. They claimed, *"The appropriateness of transformation geometry depends on the objectives which it is intended to achieve."*(p. 154) Transformation geometry can enhance students' understanding of geometrical relationships but most secondary school students were challenged by the study of transformations.

Cox (1985) agreed with Hoffer (1981), Shaughnessy and Burger (1985) and Kline (1973) that at least the first semester of the traditional plane geometry course should be devoted to informal geometry without formal proofs. He stated that it was unreasonable to expect students to write proofs about concepts they don't understand and to use skills that they have not fully developed. He suggested the need to offer a variety of curriculum options in order to increase students' geometric literacy. *"If we do not wish to relegate our students to almost certain failure and frustration in geometry, we must begin instruction at their level of competence and understanding."* (Cox, 1985, p. 405) His solution was to divide students into two groups. Students continuing onto college would take a geometry course that included some work with proofs while the students not contemplating attending college would take a year-long informal geometry course.

Kilpatrick (1985), was in agreement with Moise (1975) who said that *"It is geometric intuition that rescues calculus courses from meaningless formalism"* (p. 475), and believed that the study of geometry develops the mathematical intuition that students will need in order to study mathematics in a more abstract form. But Kilpatrick (1985) said that students needed to go *"beyond the formal development of Euclidean geometry"* (pp. 27-28) and *"study properties of geometric figures under various transformations"* (p.28) in a similar way to Bell (1983). Kilpatrick (1985) claimed there is too little emphasis on three-dimensional geometry. Students who wish to pursue careers in engineering, architecture, graphic design, chemistry, biology, physics, and medicine *"need well developed spatial abilities"* (p. 28) in both two and three dimensions.

Niven (1987), in answering the question of whether geometry could 'survive in the secondary curriculum', proposed nine recommendations in order to make geometry a more 'attractive subject'. The first is to teach geometry in the same way that algebra and calculus are taught - without emphasising rigour too much. He suggested that geometry should not be taught as a general introduction to the axiomatic structure of mathematics, but should be taught by simply introducing basic ideas and concepts that will intrigue students and make them want to study geometry further. By putting too much emphasis on rigour and theorems and definitions, he felt that students become bored or even worse scared by geometry and therefore do not want to learn it. For example if one was to teach geometric proofs by just doing proofs day after day, the students would probably become bored and many would not even understand why they were doing them. Some researchers have suggested that a much more student centred and interesting method must be used to introduce proofs to change students' attitudes. For example, Niven (1987) suggested introducing geometry through algebra or by using various pictures and other less rigorous methods.

The second of Niven's suggestions for making geometry more appealing is to get to the "heart" of it as quickly as possible. One of the most famous theorems is Pythagoras', but it doesn't appear until quite late in most textbooks. Since this theorem is the foundation of much work in mathematics, he felt that it should be introduced much sooner. The reasoning behind a statement like this is that if the topic was taught earlier or more time and emphasis were put on it instead of just rushing through it in order to complete the curriculum, as is often the case, then the students would have a better understanding of both the theorem and geometry itself.

The next suggestion for making geometry a more attractive subject is to *"use the techniques of algebra and analytical geometry as well as the classical Euclidean methods."* (p. 40) The various methods of doing geometry should be taught to students since some are not only useful, but also interesting. Moreover, students should be exposed to a greater variety of ways in which geometrical ideas can be taught and explained.

The fourth recommendation is — *"use diagrams in all explanations, especially proofs."* (p.41) Too often students are confused by what is being taught, where something as simple as a picture would help clarify the misunderstanding. As geometry is a visual subject, very often a topic would be easier to understand if an appropriate diagram was used as well as the usual words. One thing that must be pointed out however, is that the diagram must be accurate and all cases must be discussed.

The fifth recommendation is to relate geometry to the real world. Putting geometry in the context of a real world application makes the subject come alive for the students and makes then realise that there are reasons it should be learned since it is used in the "real world."

The sixth suggestion is to eliminate the wordiness in geometry. Too many times things are proved, said or explained with more words than necessary and it is these extra words that confuse the students. Even when too many words are not used, sometimes very complicated words are used, more than is really necessary.

Postponing or omitting the proofs of very difficult theorems is Niven's seventh suggestion. Too often, we show our students how to prove something that is far too complicated for them to understand and they simply get lost and frustrated, which does little for their perception of the subject. Niven believed *"We should abandon all proofs and offer cookbook courses"* (p.44) where students are given a method for working out an example.

The eighth suggestion refers to textbooks and the fact that they do not contain enough problems of intermediate difficulty to challenge the students. Most textbooks contain very simple problems, which is fine, provided that there are only a few and that there are more challenging problems to test and develop the students' knowledge. Moreover, if we expect our students to pass difficult standardised tests, they need to be exposed to problems of an appropriate level of difficulty, which is something that most textbooks currently in use lack.

The final recommendation for improving the geometry curriculum is to explain to the students the trisection of an angle problem, as well as to show them that it is possible given certain situations. For example, if marks are allowed on a straightedge and a compass is used, trisection becomes very easy. However, with an unmarked straightedge as the Greeks used, trisection is impossible. He made this recommendation because he stated, *"Many students will come away from their geometry course persuaded that it is impossible to trisect an angle. Some of these students may even become 'trisection nuts' and 'solve' the problem that has confounded mathematicians for centuries!"* (p.45)

Woodward (1990) advocated that the high school geometry course should contain a laboratory component, in which the students can engage in explorations in order to collect and record data and form conclusions. He based his argument for this on the results of the National Assessment of Educational Progress' fourth mathematics assessment where fewer than fifty percent of the eleventh grade students who had taken geometry could apply the Pythagorean theorem in a routine problem.

Hansen (1998) stated that although he didn't think that children should be formally taught Euclid's postulates, he believed that in order for teachers to teach with a proper perspective they themselves should know the postulates.

Rowlands and Carson (2006) claimed that *"geometry provides an ideal venue for inducting students into proof and the formalism of mathematics and to encourage them to think as mathematicians."* (p.72) They proposed the inclusion of seventeen '*primary events*' from geometry's developmental history (Carson and Rowlands, 2006; Rowlands and Carson, 2006) into the existing geometry curriculum in secondary school.

Gonzalez and Herbst (2004) investigated the development of different perspectives for teaching and learning geometry in high school through an analysis of a set of articles from the *Mathematics Teacher* journal between 1908 and 2002 and other documents from that era. They identified two major trends.

> One trend frames high school geometry within the structure of deductive reasoning and expects students to get acquainted with proofs and formal mathematics. The other trend stresses on the connections between mathematics and the real world. In this second trend, the formality of proofs becomes less important while the opportunity to study relationships between geometry and other branches of mathematics or even other disciplines is more relevant. (p.1)

Gonzalez and Herbst (2006) discussed four "modal discourses" that proposed new definitions for the high school geometry course during the twentieth century. By 'modal discourses' they meant, *"not necessarily ideologies explicitly promulgated by individuals but central tendencies around which opinions of various individuals could converge."* (p.13) They claimed, *"Authors rarely subscribed to a unique, well defined, modal argument. Still, their writings permit to isolate those four modal arguments as ideal types of justification for the study of geometry."* (p. 22)

The four modal discourse or arguments (Gonzalez and Herbst, 2006) that give geometry its reason for being in the curriculum:

1. Mathematical argument- justified the study of geometry as an opportunity to experience the work of doing mathematics, including using proof to understand geometric concepts.
2. Formal argument- defined the study of geometry as a case of logical reasoning.
3. Utilitarian argument- stated that geometry would provide tools for the future work of non-mathematical studies.
4. Intuitive argument- aligned the geometry course with opportunities to learn a language that would allow students to model the world. (p. 13)

A summary of these arguments can be found in Table 2.5.

	Formal argument	*Utilitarian argument*	*Mathematical argument*	*Intuitive argument*
What is geometry	Geometry is a case of logical reasoning.	Geometry is a tool for dealing with applications in other fields.	Geometry is a conceptual domain that permits students to experience the work of mathematicians.	Geometry provides a language for our experiences with the real world.
Views about mathematical activity	Transferring formal geometry reasoning to logical abilities.	Studying concepts and problems that apply to work settings.	Applying deductive reasoning through the study of geometric concepts.	Modelling problems using geometric ideas while reasoning intuitively.
Expectations about students	All students require logical reasoning to be good citizens and to participate in a democracy.	All students will be part of the workforce in the future.	All students can simulate the work of mathematicians.	All students could develop skills but their abilities vary
Characteristics of problems in the geometry curriculum	Applying logical thinking to mathematical and real-life situations.	Relating geometric concepts and formulas to model real-world objects or to solve problems emerging in job situations.	Making conjectures and proving theorems deductively.	Exploring intuitively geometric ideas towards formality and integrating algebra and geometry.
The place of proofs	Proofs as a method of thinking and as an opportunity to practice deductive reasoning detached from geometric concepts.	Proofs not as important as problems that apply geometry to future jobs.	Proofs as original problems providing opportunities to experience the activity of mathematicians.	Proofs following informal appreciation of geometric concepts; blurring differences between definitions, postulates and theorems.

Table 2.5 ***Elements within the four modal arguments defining the geometry course*** (Gonzalez and Herbst, 2006, p. 23)

A major change of the 20th century was to expose students to other geometrical approaches such as coordinate or transformational approaches to Euclidean geometry and even an introduction to non-Euclidean geometry rather than what is considered the synthetic approach that continued to be the dominant geometry in the classroom. There had been little pockets of consensus when defining the nature of school geometry and practices have changed very little throughout the years. However the teaching of geometry had to accommodate different interests in order to survive.

The new vision set in *Principles and Standards for School Mathematics* (NCTM, 2000) tried to resolve many of the discourses of the 20th century. Gonzalez and Herbst (2006) suggested that a lack of awareness of the underlying assumptions behind the discourses might result in lack of coherence of what is expected from the high school geometry course.

2.3.2.2 Using Manipulatives

The second question I would like this study to consider is: *What is the role of manipulatives in the high school classroom?*

Successful use of manipulatives requires the teachers believing in their effectiveness. They have to believe that the manipulative is not just a "toy". They also have to understand the connections between the concrete manipulative and abstract mathematics and how manipulatives can help their students to make these connections. The National Council of Teachers of Mathematics has encouraged the use of concrete manipulatives at all grade levels since 1940. What exactly falls into the category of a manipulative?

Kline (1973) suggested that a mathematics laboratory should be incorporated into the mathematics classroom to strengthen the intuitive approach to teaching. Although he did not use the word manipulative at the time, he did say that the laboratory should contain *"apparatus of various sorts which could be used to demonstrate physical* *happenings from which mathematical results can be inferred."* (p.40) He specifically mentioned Cuisenaire rods and geoboards.

Prevost (1985) suggested a manipulative approach to topics in junior high school. He complained that teachers used *"too few devices that allow the students to 'do geometry' rather to 'watch geometry'."* (p.412)

Fuys, Geddes, and Tischler (1988) reported that the teachers who participated in the Brooklyn College Project, which advocated the use of manipulatives, were unanimous in their endorsement *"of the hands-on visual concrete approach to developing geometric concepts for students in grades 6-9."* (p.155)

Mason (1989) said that ever since the first published educational reports there has been discussion about the role of and need for *"practical equipment in the classroom."* (p.38) Cockcroft (1982) suggested that mathematics teaching at all levels should provide *"opportunities of investigational work"* (p.71) which includes the use of 'practical work'. The National Curriculum (Department for Education and Employment, 1999) states that pupils in key stage 1 should *"observe, handle, and describe common 2-D and 3-D shapes"* (p. 19) and also *"create 2-D and 3-D shapes."* (p. 19) Similarly at key stage 2 pupils should *"make and draw with increasing accuracy 2-D and 3-D shapes."* (p.25)

Thomas (1992) defined a **manipulative** as any object used by children to model some process or their thinking about some concept.

Spikell (1993) defined **manipulatives** as physical, real world objects that can be used to teach mathematical ideas, concepts, principles, and skills to students. He stated that manipulatives were once regarded as supplementary resource materials in the classroom, but today they are viewed as important instructional aids in school mathematics programs. He claimed that as manipulatives have become more available, their effective use in instruction may have decreased, because teachers have inadequate initial preparation and follow up support in the use of manipulatives. The early adopters of manipulatives in the classroom benefited from the relationship they had with the developers of the manipulatives movement of the 1960s and 1970s. They were caught up in the excitement of new ideas.

> They **believed** that manipulatives were a powerful teaching aid and did not have to be convinced of their potential value. Moreover, they had the requisite interest, motivation, and skill to discover for themselves, with minimal help, how to incorporate manipulatives in their instructional programs. In short, they required minimal formal preparation to use manipulatives. (Spikell, 1993, p. 219)

Spikell suggested that in order to use manipulatives properly, teachers must understand three things: the content embodied in the manipulative; specific activities with the manipulative that can be used to teach the content; and the effective pedagogy for teaching the content with the manipulative. He wrote the book *Teaching Mathematics with Manipulatives* (Spikell, 1993), which provides a frame of reference for teachers to enable them to teach effectively when working with manipulatives.

Moyer (2001) similarly defined **manipulatives** as physical objects designed to represent abstract mathematical ideas explicitly and concretely. Students "manipulate" these physical objects that *"have both visual and tactile appeal"* (p.176) and allow for hands-on experiences. She claimed that manipulatives became popular because researchers' beliefs about how children learn changed. They believed that for students' learning to be permanent, students must understand what they are learning. *"The impact of theories and research connecting students' actions on physical objects with mathematical learning has had an important influence on the emergence and use of manipulatives in the K-8 classrooms."* (p.176)

Moyer studied how and why ten middle school teachers used manipulatives in their classrooms. The teachers found them fun to use but not really necessary for teaching and learning mathematics. They used them for enrichment, for playing games, and problem solving. The decision of when to use the manipulatives did not necessarily depend on the concept being taught, but rather on the amount of time remaining during a class period, the day of the week (Fridays were most often manipulative days), or the behaviour of the class (good behaviour was rewarded with manipulative use). The teachers believed that when using manipulatives the class was doing fun mathematics, but real mathematics was reserved for paper and pencil, textbooks, and teacher lectures. Using manipulatives in the classroom is beneficial if the students can eventually link their actions with these manipulatives to abstract concepts. The teachers' role is to create environments that allow for this to happen. Moyer suggests:

> It is the mediation by students and teachers in shared and meaningful practices that determines the utility of the manipulatives. Therefore, the physicality of concrete manipulatives does not carry the meaning of the mathematical ideas behind them. Students must reflect on their actions to build meaning. (p. 177)

Leitzel (1991) stated that recent research on the learning of geometry (Kline, 1973; Mason, 1989) required concrete experiences with geometric figures and relationships to occur prior to a formal axiomatic study of geometry. He believed that these experiences should involve active participation, experimentation, and the use of different kinds of materials and models.

> For the middle school mathematics teachers such concrete experiences are important not only in the development of their own geometric understanding but also in the enhancement of their knowledge of the stages through which geometric understanding evolves. (Leitzel, p.19)

Ball (1992) believes manipulatives are motivational, but she also believes that there is no magic involved with using manipulatives. Although they provide a kinaesthetic experience that can enhance perception and thinking, they do not themselves carry meaning or insight. She argued, "*The manipulative itself cannot on its own carry the intended meanings and uses.*" (p. 18)

Ball (1992) felt that there is no need for any further debate about the purpose of using manipulatives and their role in helping students learn mathematics. She stated, "*Manipulatives, and the underlying notion that understanding comes through the fingertip, have become part of the educational dogma: using them helps students, not using them hinders students.*" (p. 17) A problem that Ball cited was that when using manipulatives there is room for multiple interpretations and confusion. She claimed

> One of the reasons that we as adults may overstate the power of concrete representation to deliver accurate mathematical messages is that we are "seeing" concepts that we already understand. That is, we who already have the conventional mathematical understandings can 'see' correct ideas in the material representations but for children who do not have the same mathematical understandings that we have, other things can be reasonably 'seen'. (Ball, 1992, p. 17)

Viadero (2007) reported on studies that showed that use of manipulatives does not guarantee success in learning. She cited Uttal, a psychology professor who said, "*The critical question for researchers now is to find out how and when manipulatives should be used.*" In Uttal's recent study, as reported by Viadero (2007), "*the researchers found that children taught to do two-digit subtraction by the traditional written method performed just as well as children who used a commercially available set of manipulatives made up of individual blocks that could be interlocked to form units of 10.*" (p. 12) The lessons involving the manipulatives were time consuming, taking three times longer than the traditional lessons. These students when using the manipulatives had difficulty "*transferring their knowledge to paper-and-pencil representations.*" (p. 12)

Viadero (2007) also cited Clements, a professor of learning and instruction, who suggested, "*In some cases, teachers might also find that "virtual" manipulatives on a computer screen could be more effective than the real thing.*" (p.13) In a study that he conducted with Sarama in the 1990s, they found that a group of middle school students using only a textbook to learn geometric transformations concepts were outscored by two other groups: one group using the Logo computer software program and the other group using manipulatives together with pencil and paper. What is of even more interest is that the computer using group performed better than the other groups on a test given three weeks later. The retention of the computer group was better because the those lessons "*required students to be more explicit about their learning.*" (p.13) Students had to type in commands to manipulate shapes on the screen "*instead of mindlessly rotating or taking apart a block.*" (p.13)

Teachers have to learn how to use manipulatives effectively in order to help their students make the appropriate connections. Roberts (2007) cautioned, "*Be careful how you use manipulatives and models in your classroom; they may be hazardous to mathematical learning.*" (p. 9) When using materials that were not rigid, her students were determined to reshape the manipulatives so that their erroneous conjectures were realised.

Secondary school mathematics teachers should have the ability to see underlying connections and themes. They should think about manipulatives as one of several useful pedagogical tools. They should have the ability to create activities whether they are using manipulatives, dynamic geometry, or doing proofs that uncover central habits of mind such as going from a particular to the general.

2.3.2.3 Dynamic Geometry

A third question I hope to answer through the results of this study is: *What do high* *school teachers believe is the role of dynamic geometry software packages in the teaching and learning of geometry?*

There are several well known dynamic geometry software programs used throughout the world to enhance student learning in geometry. These include *Cabri* (LaBorde and Bellemain, 1994), *Cinderella* (Richter-Gebert and Kortenkamp, 1999), *Geometer's Sketchpad* (Jackiw,1995), and its precursor, *Geometric Supposer* (Schwartz and Yerushalmy, 1985).

Chazan (1990) described ways teachers could use the software The Geometric Supposers (1985) to address the process standards presented in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). Teachers could create an environment for inquiry through the use of dynamic geometry. Students can generate conjectures and verify them and/or generalise them.

> One of the most important innovative aspects of this approach is that students are not trying to prove statements that they know are true (by virtue of being in the textbook) and that they know have been proved year after year in geometry classes. Some of the statements that they try to prove may not be true. Others that are true may not be present in their textbooks and may even be unfamiliar to their teachers. (Chazan, 1990, p.630)

Sibley (1998) believed that these geometry software packages provide a valuable way for students to build their intuition and prepare them for proof.

Hansen (1998) also believed that while computer graphics can enhance the teaching and learning of most topics in geometry, he didn't see the need to abandon classical geometry. He said, *"You do not have to make use of new topics in order to make use of these new tools."* (p.9) Students can be given engaging tasks in Euclidean geometry.

Olive (2000) said:

> At the secondary level dynamic geometry can (and should) completely transform the teaching and learning of mathematics. Dynamic geometry turns mathematics into a laboratory science rather than the game of mental gymnastics, dominated by computation and symbolic manipulation, that it has become in many of our secondary schools.
> http://jwilson.coe.uga.edu/olive/Portugal/Portugal_paper.html

However, he warned that

> While there have been many personal accounts of the powerful learning that can take place when students of all ages work with dynamic geometry technology (my own included), there have been very few well designed research projects to study the effects on learning in such environments.
> http://jwilson.coe.uga.edu/olive/Portugal/Portugal_paper.html

A great deal of additional research on dynamic geometry software has taken place in the last ten to fifteen years, since these tools have become available.

A summary of some of this research was presented by Jones (2002). He stated, *"Over the last two decades, dynamic geometry software has become one of the most widely used pieces of software in schools and colleges all over the world."* (p. 18) Jones concluded:

> A variety of research shows that interacting with dynamic geometry software can help students to explore, conjecture, construct and explain geometric relationships. It can even provide them with the basis from which to build deductive proofs. Overall, this research has found that discussions and group work in the classroom are important components. (p.19)

Three of Jones' conclusions from studying the research on dynamic geometry software are:

1. Dynamic geometry software used inappropriately makes no significant difference (and might make things worse).
2. Dynamic geometry software integrated intelligently with curriculum and pedagogy produces measurable learning gains.
3. What matters is *how* dynamic geometry software is used. (p.20)

In other words, dynamic geometry software is a tool that can improve mathematical understanding, but only if the teacher knows how to use it as an appropriate part of instruction. Just as the mathematics is not 'magically' in the manipulatives, it is not 'magically' in the dynamic geometry software. The software itself cannot guide the student from *"perceptive to theoretical thinking."*

Accascina and Rogora (2006) agree with Jones (2002) in that dynamic geometry software is useful for teaching and learning geometry. They claimed that Cabri3D helped students create good concept images of three dimensional objects.

Jiang (2002) found The Geometer's Sketchpad (GSP) to be an excellent teaching and learning tool for pre-service secondary school teachers. In his work, using GSP enhanced the pre-service teachers' reasoning and proving skills. *"Sketchpad explorations can not only encourage students to make conjectures, they can foster insight for constructing proofs."* (p.722)

The documents Principles and Standards in School Mathematics (NCTM, 2000) in the United States and the National Curriculum (DfEE, 1999) in the United Kingdom recommend the use of dynamic software packages even in elementary/primary school.

> Geometry has always been a rich arena in which students can discover patterns and formulate conjectures. The use of dynamic geometry software enables students to examine many cases, thus extending their ability to formulate and explore conjectures. (NCTM, 2000, p. 309)

Some teachers that oppose the use of dynamic geometry software argue that students may think that their investigations are proof enough. De Villiers (1999) has suggested a solution,

> When students have already thoroughly investigated a geometric conjecture through continuous variation with dynamic software like Sketchpad, they have little need for further conviction. Therefore verification serves as little or no motivation for doing a proof. However, I have found it relatively easy to solicit

> further curiosity by asking students *why* they think a particular result is true; that is, to challenge them to try and *explain* it. (p. 8)

Other 'pitfalls' with dynamic geometry software were cited by Scher (2003) and more recently by de Villiers (2007). Scher (2003) claimed that certain misconceptions, associated with the accuracy of measurement, arise when students work in dynamic environments that usually don't exist in a static geometry environment. He also claimed that students with limited background cannot distinguish between the inherent properties of the figures they are working with and behaviour that is specific to the dynamic geometry software. For example, dragging different vertices of a figure might result in a change of dimensions or just a movement across the screen. In either case, the geometry of the figure remains unchanged. The students may think that how the figure moves is as important as its properties.

Ruthven (2006) interviewed teachers from eleven high school mathematics departments. He found that the dynamic geometry software was the main form of technology used in six of the participating schools. He observed lessons conducted by three of the teachers and had post lesson debriefings. These three teachers were chosen for the observations because of the different pedagogical approaches they reported in their interviews. He found that teachers differed in the degree that they allowed their students to use the dynamic geometry software for themselves. One teacher found the dynamic geometry software to be difficult to operate. She provided her students with prepared figures so that they would only have to do a limited amount of construction for themselves. De Villiers (2007) considers this a good idea because construction "*requires a solid understanding of necessary and sufficient conditions*" (p.49) which means that the students would be have to be operating at van Hiele level 2. When students are exploring prepared shapes they are operating on a lower van Hiele level. The explorations could actually be the catalyst that moves them to the next level.

Another teacher in Ruthven's (2006) study felt the time involved was not worthwhile since the examinations did not require deep investigations. He also felt it would take students a long time to master the program. De Villiers (2007) suggested that students can explore geometrical problems with dynamic geometry with minimal exposure to the software. He claimed that teachers should *"expose students to the specific skills necessary for a particular learning context."* (p. 49)

Ruthven also found that the reasons the teachers used dynamic geometry software were to find a more efficient generation of data than was possible by hand. He concluded that this finding is in line with the emphasis the curriculum places on arithmetic computation. Nevertheless, the use of dynamic geometry software can hardly be considered successful if as de Villiers declared, teachers use it as a *"glorified blackboard without really changing the curriculum of activities or teaching style."* (p.48)

2.3.2.4 Proof in Geometry

The fourth question I want this study to answer is: *What do high school teachers believe about the role of proof in the high school classroom?*

The National Council of Teachers of Mathematics in its most recent standards document (NCTM, 2000) has increased the emphasis placed on proof in grades pre-kindergarten through twelve over their previous standards document (NCTM, 1989).

> Students should see the power of deductive proof in establishing the validity of general results from given conditions. The focus should be on producing logical arguments and presenting them effectively with careful explanation of the reasoning, rather than on the form of proof used (e.g., paragraph proof or two-column proof). A particular challenge for high school teachers is to integrate technology in their teaching as a way of encouraging students to explore ideas and develop conjectures while continuing to help them understand the need for proofs or counterexamples of conjectures. (NCTM, 2000, p.309)

One should first ask, *"What is proof?"*

Mason (1989) suggests that proof is convincing others about the truth of a statement. Knuth, Choppin, Slaughter, and Sutherland (2002) explored the development of middle school students' competencies in justifying and proving. They elaborated on and utilised the six level framework proposed by Waring (2000) (which is similar to the van Hiele levels):

> *Level 0:* Students are ignorant of the need for, or existence of, proof.
>
> *Level 1:* Students are aware of the notion of proof, but consider checking a few cases as sufficient.
>
> *Level 2:* Students are aware that checking a few cases is not sufficient, but are satisfied that either i) checking extreme cases or random cases is proof, or ii) use of a generic example forms a proof for a class of objects.
>
> *Level 3:* Students are aware of a need for a general argument, but are unable to produce such arguments themselves. However, they are likely to understand the generation of such arguments. This also includes the ability to follow a short chain of deductive reasoning. (Knuth et al. (2002) extended this level to include students' understanding of various concepts such as definitions and necessary and sufficient conditions that are prerequisites to understanding and producing deductive arguments).
>
> *Level 4:* Students are aware of the need for a general argument, are able to understand the generation of such an argument, and are able to produce such arguments themselves in a limited number of familiar contexts.
>
> *Level 5:* Students are aware of the need for a general argument, are able to understand the generation of such an argument are able to produce such arguments themselves in a variety of contexts both familiar and unfamiliar. (Knuth et al., 2002 p.1694).

Formal proof is meaningless for students who are thinking at Van Hiele levels 0 or 1. They do not doubt the validity of their empirical observations. Why justify the obvious? (Jiang, 2002) Proof oriented geometry courses require thinking at least at van Hiele level two. Students who are not operating at a high enough level may become frustrated when learning about proofs. Kline (1973) claimed,

> If the teacher proves a theorem of mathematics, the student will still be struggling to understand the theorem, its proof, and its meaning. While undergoing such struggles the student is not likely to be impressed with the intellectual content and what the human mind has accomplished. In him the theorem and proof produce bewilderment and confusion. (p. 10)

Knuth et al. (2002) stated that teachers' own understanding of proof and its place in school mathematics may be enhanced by engaging in discussions focusing on students' competencies in doing proofs. I would add that teachers should be aware of the van Hiele levels of their geometry students. The middle school students in this study were operating between levels zero and two.

Hanna (1995) claimed that although the main role of proof in mathematics is *"justification and verification,"* its main role in mathematics education is *"explanation."* She believed that proof should be part of the curriculum because it promotes mathematical understanding. Formal proof was emphasised by the "new math" of the 1960s. Kline (1973) claimed that students just memorised definitions and proofs because the level of the material was beyond them. The movements that followed, such as 'back to basics', and later, 'problem solving', shifted the emphasis away from proof. Curriculum decisions and misinterpretations of learning theories such as constructivism, have both contributed to a decline of proof in geometry. If the teacher's role is interpreted to be 'a guide on the side' then the teacher cannot *"take an active part in helping students understand why proof is needed and when it is valid."* (Hanna, 1995, p. 45)

Hoyles (1997) also feels that the effect of curriculum can cause a detrimental effect in students' approaches to proof. She surveyed 2500 secondary school pupils in the UK with questions about proof in algebra and geometry. She found that the responses to the geometry questions were quite poor. She attributed this to the *"almost complete disappearance of geometrical reasoning"* (p. 14) from the National Curriculum. This curriculum is organised into four attainment targets (Department for Education and Employment Education, 1995) with the third being 'shape, space, and measures.' Proof, separated from the content areas, is found in the first target called 'using and applying mathematics.' Each attainment target was divided into hierarchical levels. In terms of proof this meant that it became "*official that proof is very hard and only for the most able.*" (p. 9) "*Proof requires the coordination of a range of competences—identifying assumptions, organising logical arguments—each of which, individually, is by no means trivial.*" (p. 7) The National Curriculum (DfEE, 1999) has since been revised to include geometric reasoning in key stages 3 and 4 as part of the shape, space, and measurement attainment target. At key stage 3 reasoning includes the ability to:

- Distinguish between practical demonstration, proof, conventions, facts, definitions, and derived properties
- Explain and justify inferences and deductions using mathematical reasoning
- Show step by step deduction in solving a geometry problem (p.36)

Geometrical reasoning at key stages three and four includes the ability to understand a proof of the sum of the angles of a triangle and a proof of the exterior angle theorem. The breadth of study at these key stages includes "*activities that develop short chains of deductive reasoning and concepts of proof in algebra and geometry.*" (p. 38) Only students showing 'exceptional performance' beyond the eight level descriptors are expected to "*use the conditions for congruent triangles in formal geometric proofs (for example, to prove the base angles of an isosceles triangle are equal).*" (p. 92) Kline (1973) agreed that the concept of proof is fundamental in mathematics. In their geometry courses students have the opportunity to learn one of the great features of the subject.

> But since the final deductive proof of a theorem is usually the result of a lot of guessing and experimenting and often depends on an ingenious scheme which permits proving the theorem in the proper logical sequence, the proof is not necessarily a natural one, that is, one which would suggest itself readily to the adolescent mind. Moreover, the deductive argument gives no insight into the difficulties that were overcome in the original creation of the proof. Hence the

> student cannot see the rationale and he does the same thing in geometry that he does in algebra. He memorizes the proof. (p. 6)

Kline also believed that the students have to discover the need for rigour rather than having it imposed on them. They have to experience the passage from what they regard as obvious to the not-so-obvious and then move on to find the need for a proof themselves.

Hoffer (1981) suggested that if formal proofs are introduced at too early a stage in a geometry course there may be many students who have not reached the level of geometric thinking required for the proof. Therefore he suggested that the informal development of concepts and vocabulary should occur in the first semester and that deductive proof becomes the focus of the second semester.

Schoenfeld (1988) conducted a series of studies exploring students' understanding of geometry. He found that even though high school students took a full year high school geometry course, which focused on proving theorems about geometric objects, they experimented when trying to do geometric construction problems. They did not use their "proof-related knowledge" in this context. He conjectured that by experiencing a certain type of mathematics instruction students may come to believe:

1. The processes of formal mathematics (e.g., "proof") have little or nothing to do with discovery or invention. Corollary: Students fail to use information from formal mathematics when they are in "problem-solving" mode.

2. Students who understand the subject matter can solve assigned mathematics problems in five minutes or less. Corollary: Students stop working on a problem after just a few minutes as they believe that if they haven't solved it it's because they don't understand the material (and therefore give up in frustration and will not solve it).

3. Only geniuses are capable of discovering, creating, or really understanding mathematics. Corollary: Mathematics is studied

passively, with students accepting what is passed down "from above" without expectation that they can make sense of it for themselves.

4. One succeeds in school by performing the tasks, to the letter, as described by the teacher. Corollary: Learning is an incidental by-product to "getting the work done." (p. 151)

Schoenfeld observed one class, at least once per week for an entire year. The class was well run, had no discipline problems, and scored in the top 15% on the state test. The primary goal of instruction was for the students to do well on the state test. The students memorised the required proofs and how to produce the required constructions. Results of the study showed that students believed that proof had nothing to do with construction. He stated:

> Proof had always served as confirmation of information that someone (usually the teacher or mathematicians at large) already knew to be true; they provided the "justification" for constructions. But ask these students to discover a construction, and they do not see that any proof arguments are relevant at all. For these students, a construction is right when it "works." They are in "discovery mode," and proofs have never helped them to discover. Confronted with a construction problem they make their best guess, and they test it by trying it out and seeing if their attempt meets their empirical standards. Such behaviour was learned, alas as an unintended by-product of their instruction. (p.157)

Students take their "cues" from their teachers. Classroom experience affects students' beliefs about mathematics. Teachers need to examine their own beliefs about proofs in order to understand how they may influence their students. McCrone, Martin, Dindyal and Wallace (2002) studied the relationship between the ability of students to construct proofs and their teachers' '*pedagogical choices*'. They defined pedagogical choices "*to include the choice of mathematical tasks, the ways the teacher allocated time for activities, the instructional strategies (direct instruction, cooperative learning, investigations), and the teachers' expectations about student ability that may be reflected in the choices.*" (p. 1702) They studied four teachers in geometry classes which were based entirely on proofs. Teachers in their study did not use manipulatives at all and made infrequent use of technology. McCrone et al. also explored "*possible connections between the social environment in the classroom and the students' ability to construct proofs.*" (p. 1707) The classroom social environment can be thought of as social and sociomathematical norms (Cobb and Yackel, 1996) such as the expectation that "*all mathematics problems can be can be solved in a relatively short period of time.*" (p. 1708)

McCrone et al. (2002) and Senk (1985) found that students have difficulties with constructing proofs, especially when no helpful suggestions are provided. If teachers strongly convey the idea that proofs are necessary to fully understand and appreciate the fundamental geometrical principles being taught, students may become more interested and involved in learning about and doing proofs. Otherwise, doing proofs becomes a dry, rote classroom drill. As earlier researchers have reported, doing formal proofs should come after students have made some sense of the underlying geometrical and mathematical ideas through hands-on explorations (Battista and Clements 1995; Freudenthal, 1971; Hoffer, 1981; Kline, 1973).

According to De Villiers (1999),

> Traditionally the function of proof has been seen almost exclusively as being to verify the correctness of mathematical statements. The idea is that proof is used mainly to remove either personal doubt or the doubt of skeptics, an idea that has one-sidedly dominated teaching practices and most discussion and research on the teaching of proof. (p. 1)

Olive (2000) explained De Villiers theories about the role of proof:

> De Villiers (1999) expanded the role and function of proof beyond that of verification. If students see proof only as a means of verifying something that is "obviously" true then they will have little incentive to generate any kind of logical proof once they have verified through their own experimentation that

> something is always so. De Villiers (1999) suggests that there are at least five other roles that proof can play in the practice of mathematics: explanation, discovery, systematization, communication, and intellectual challenge. He points out that the conviction that something is true most often comes *before* a formal proof has been obtained. It is this conviction that propels mathematicians to seek a logical *explanation* in the form of a formal proof. Having convinced themselves that something must be true through many examples and counter examples, they want to know *why* it must be true. (p. 11)

Taking into consideration the roles of proof suggested by De Villiers, Knuth (2002a) studied 16 in-service secondary school mathematics teachers' conceptions of proof. His use of the word conception included both subject matter knowledge and beliefs. The teachers recognised various roles of proof in mathematics. However, he found they did not view proof as a tool for learning mathematics. *"An informed conception of proof-one that reflects the essence of proving in mathematical practice –must include a consideration of proof in each of these roles"*:

1. To verify that a statement is true.
2. To explain why a statement is true.
3. To communicate mathematical knowledge.
4. To describe or create new mathematics.
5. To systematise statements into an axiomatic system. (p. 381)

Although the teachers recognised these roles of proof, Knuth believed that

> ...perhaps if teachers were to pay explicit attention to these roles during their instruction, they would provide classroom experiences with proof that would enable students to go beyond the limited conceptions of proof that students traditionally developed. For example, one might question whether high school geometry students are able to view the proofs that they construct in class as interrelated- that is, whether these students are cognizant (sic) of the particular axiomatic system (typically Euclidean geometry) that provides the structure for their work. Teachers holding a view of proof as a means of systematizing might be more likely to provide opportunities for students to reflect on their work through this particular lens. (p. 399)

The teachers did not mention the role of proof in promoting understanding. Teachers view proof as a topic of study rather than as a tool for communicating and studying mathematics (Knuth, 2002a,b). Their previous experiences with proofs, when they themselves were students, focused on the final product and this experience now influences their own approach to teaching proof. In current high school classes many students spend time verifying statements that are either intuitively obvious or that they know have been proven before. Instructional practices of this nature may serve to limit the teachers' conceptions of proof. Some teachers believed that a *"proof is a fallible construct"* (p. 401) and many teachers needed to test a proof with empirical evidence to reach a stronger level of conviction regarding the truth of a proof's conclusion. Knuth concluded that *"teachers need, 'as students', to experience proof as a meaningful tool for studying and learning mathematics."* (p. 403) He suggested that

> ...future research needs to explore more fully the conceptions of proof that teachers must have as they help students learn to reason mathematically. What do teachers need to know about proof and how do they draw on and use this knowledge in the act of teaching? What conceptions of proof are necessary in selecting and designing tasks to present to students? Which are essential for making sense of and changing one's practice to more closely reflect reform recommendations about proof?" (p. 404)

Another purpose of Knuth's study was to examine whether secondary school mathematics teachers were prepared to include proofs and proving in their instruction as was recommended by the NCTM standards. His findings suggested that *"the successful enactment of such practices might be difficult for teachers."* (p. 83) Many of the teachers in his study viewed proof as an appropriate goal for only a minority of students.

Even though learning proof has not had much success in the high school geometry class (Senk, 1985), Wu (1996) claimed the high school Euclidean geometry is the most suitable course for learning to work with mathematical proof because the proofs can be supported by visuals, are for the most part relatively short, and require only a few concepts.

McClure (2000) also found that Euclidean geometry was the best course for teaching proof, but his focus was on university students. He found that it is a not uncommon that university *"students have great difficulty in making the transition from their early mathematical training to an environment in which proof is emphasized."* (p.44) In order to alleviate this problem, many mathematics departments have created *'bridge courses'* for the students' first encounter with formal proof. These courses focus either on set theory, elements of analysis or linear algebra. McClure (2000) argues that the only satisfactory way to respond to the students' difficulties is to begin with Euclidean geometry. Most students *"enjoy finding clever solutions to challenging problems, but have no natural appetite for technical aspects of mathematics."* (p.44) Therefore McClure suggests, *"Euclidean geometry is a very favorable (sic) place to begin a student's serious mathematical training because it involves familiar objects that can be thought about both visually and verbally and the statements it makes about these objects are readily intelligible."* (p.45) Jones and Rodd (2001) claimed that if proof continues to reside in high school geometry then the *"challenge is to develop teaching methods which do not turn pupils off or get them to solely learn by rote (as appears to be the case in the past)."* (p.98)

Herbst (2002) suggested that one of the main reasons for including proof in the curriculum is to have students experience what is involved in the work of mathematicians. He studied the history of the two-column proof in school geometry. In the late nineteenth century there were concerns about the school's responsibility for students' intellectual activity. This was at about the same time that the custom of the two-column proof was developed. This development was *"A viable way for instruction to meet the demand that every student should be able to do proofs."* (p.285)

Herbst (2002) traced back the mandate that students should learn the 'art' of proving in the high school geometry course to The Committee of Ten report issued in 1893. At that time the report recognised that students were memorising the proofs in the geometry texts. The committee recommended instructional change. In the textbooks of the early 1840s, proofs were written in paragraph form. Neither general descriptions of proofs nor methods of proving were included in the texts. The main goal of instruction was to train the reasoning faculties of students through reading and reproducing a text.

The computer has made possible new ways of justification. Computers can validate very long proofs such as the four-colour theorem (Appel and Haken, 1977). Hanna (1995) stated that mathematicians debate whether mathematical truths can *"be established by computer graphics and other forms of experimentation."* (p. 44) She believed that these debates confirm the central role of proof in mathematics. *"There has never been a single set of universally accepted criteria for validity of a mathematical proof. Yet mathematicians have been united in their insistence on the importance of proof."* (p.44)

Students could use proof as a way of creating new results when using dynamic geometry. They generate conjectures and try to verify their truth by producing deductive proofs (Knuth, 2002 a, b).

One's epistemological beliefs about mathematics in general will undoubtedly influence one's beliefs about the role of proof in the high school classroom (Hanna, 1995; Knuth, 2002 a, b). Those teachers having a problem solving view of mathematics (Ernest, 1991) may look askance at the "Euclidean programme" which Lakatos (1976) believed presented mathematics as "authoritative", "infallible" and "irrefutable." Similarly if teachers have a 'Platonistic view' of mathematics or 'absolutist' philosophy (Ernest, 1989, 1991) will they teach using manipulatives or dynamic geometry software packages?

2.4 HAS ANYTHING REALLY CHANGED?

The use of dynamic geometry software packages seems to have enhanced the teaching and learning of geometry. But to what extent has this happened? How widespread is its use? Do teachers believe this? These are questions I would like to investigate.

Similarly, we have seen arguments for and against relying on manipulatives to teach mathematics. What are today's high school teachers' beliefs about their use?

What about the topic of geometry in general? In 1987, Usiskin stated two major problems concerning school geometry: one being poor student performance, stemming from the fact that *"there is no standard curriculum for elementary school geometry that is comparable to the curriculum that exists for arithmetic."* (p.18) The result is that when students get to the higher grades, they either opt not to take geometry, or of those that do, most do not fare that well or get very far. The reason being that they do not have an adequate enough background to study more intricate topics.

Usiskin made four suggestions on how to remedy the performance dilemma:

- Specify an elementary school curriculum by grade
- Do not use algebra as a requirement for studying geometry
- Require a certain amount of comprehension in geometry from all students
- Require all teachers who teach some level of mathematics to take geometry in college

Now twenty years later, the idea of having an elementary school geometry curriculum has not been fully realised, but there are the geometry content standards for all grades (NCTM, 1989, 2000).

Students who do poorly in their algebra class may not be given the opportunity to continue on to study geometry. Some students may be visual learners and therefore would do much better in geometry than in algebra. Every student should be given a chance to study geometry; despite how good or bad they did in prior mathematics courses.

If students are required to have a significant amount of comprehension in geometry, then students as well as teachers would put more time and effort into geometry, therefore students' achievement would be much higher. Do teachers believe that all students should learn geometry?

Mathematics teachers should be taking geometry courses in college to ensure their own competence in the subject. Far too often, many mathematics teachers (whether elementary or secondary) have not seen geometry since high school (if even then) and subsequently have poor subject knowledge and are not well prepared to teach it. This results in the teachers either skipping or rushing through certain parts of geometry and what they do teach may not even be correct or may be misunderstood by the teacher and often as a consequence, misunderstood by their students. How confident are the geometry teachers in our schools today?
These questions must be adequately addressed to ensure that our students are given the best opportunities to learn mathematics.

In the next chapter I will discuss the methodologies employed in this study to investigate the above issues. In chapter 4 I discuss the pilot study. In chapters 5 and 6 the reader will find the results of the quantitative analysis. Chapter 7 contains the qualitative analysis of the data. Chapter 8 contains the conclusions of this study along with implications for further research.

CHAPTER 3 – RESEARCH METHODOLOGY

3.1 INTRODUCTION

In chapter one the overall research question was introduced: What are high school mathematics teachers' beliefs about the teaching and learning of geometry? In order to make this study operational this general research question was further subdivided into more specific, concrete questions (Cohen, Manion, and Morrison, 2000). These questions are:

1. What are teachers' beliefs about the nature of geometry as a subject and its role in the curriculum?

2. What are teachers' beliefs about the use of manipulatives in the classroom?

3. What are teachers' beliefs about the use of dynamic geometry software packages in the classroom?

4. What are teachers' beliefs about doing proofs, teaching proofs, and students learning proofs?

A review of the literature has convinced me of the need for a combination of research methodologies in order to answer the above questions. For this study a mixture of both quantitative and qualitative methodologies was used because I believed I could gain some understanding of teachers' beliefs using quantitative analysis of a questionnaire (Leder and Forgasz, 2002), but in order to gain a deeper understanding and explain the results of the quantitative research it was necessary to use some form of qualitative methodologies (Ely, Anzul, Friedman, Garner and McCormack-Steinmetz, 1991; Merriam, 1998). This chapter contains a short summary of the differences between quantitative and qualitative research followed by a description of the methodologies employed in this study.

3.2 QUALITATIVE AND QUANTITATIVE RESEARCH

Qualitative and quantitative research methods are both used extensively in educational research. Both are legitimate forms of inquiry and the methodology selected for a particular research task should depend on the questions being asked.

Quantitative methods were initially developed in the biological, physical and behavioural sciences. Borg and Gall (1989) suggested that these methods are best exemplified by the research of experimental psychologists. Other names for this methodology are conventional, traditional, or positivist. The quantitative method in educational research is based on the scientific method and involves experimentation and mathematical analysis of the data in order to validate theory.

The second paradigm, qualitative research, has slowly gained acceptance in the last 40 years. It was originally developed by sociologists and anthropologists. It is also known as ethnographic, post-positivistic or naturalistic research. The qualitative method in educational research involves the interpretation of subjective meanings that individuals place upon their actions. Borg and Gall (1989) posited "*Qualitative research is much more difficult to do well than quantitative research because the data collected are usually subjective and the main measurement tool for collecting the data is the investigator himself*" (p. 380). Some of the differences between quantitative and qualitative studies are shown in Table 3.1.

QUANTITATIVE	QUALITATIVE
Hypotheses are stated in advance. The investigator selects variables and makes predictions. The task is to verify or refute. It takes a deductive approach.	The investigator chooses an issue to study. Hypotheses emerge from exploratory studies. It takes an inductive approach.
The sample size is usually large so that statistical methods can be applied.	Usually, relatively small sample sizes are used.
The investigator gathers data through instruments such as questionnaires, tests etc.	The investigator is the principal instrument for data collection.
The investigator assumes an unbiased stance.	The investigator is aware of his/her own biases and strives to capture the subjective reality of the participants.
Knowledge gained is objective and quantifiable.	Knowledge gained is about understanding the meaning of the experience.
It assumes that reality is stable, observable, and measurable.	Multiple realities are constructed socially by individuals.

Table 3.1 *Some Differences Between Quantitative and Qualitative Studies*

Borg and Gall (1989) discussed studies that successfully used a combination of quantitative and qualitative methodology. Once the quantitative data had provided the basic research evidence, the qualitative data rounded out the picture providing examples and deeper insights. This is what happened with my study of high school mathematics teachers' beliefs about the teaching and learning of geometry. I had teachers respond to a questionnaire that served as the instrument to get their personal information and surveyed their beliefs. I followed this up with pilot interviews and a case study to gain deeper insight.

3.2.1 Quantitative Methodology

Leder and Forgasz (2002) summarised various methods for measuring beliefs. They identified Likert Scales, open response questionnaires, interviews, and observations as possible ways of gaining information about beliefs.

Oppenheim (1966) stated "*A questionnaire is not just a list of questions or a form to be filled out. It is essentially a scientific instrument for measurement and for collection of particular kinds of data*" (p. 2). Questionnaires can be used to collect both quantitative and qualitative data. He emphasised the importance of pilot work in that it could help with reduction of non-response rates and the ordering and actual wording of the questions. Di Martino (2004) promoted the use of questionnaires on a large scale because they are easy to administer.

3.2.1.1 Questionnaire design

Oppenheim (1966) talked about two types of survey design – descriptive and analytic. The purpose of the descriptive survey is to count a representative sample of the population. It then makes inference about the whole population. It answers the question of 'how many' in the population have a certain characteristic. Public opinion polls and census are examples of descriptive surveys. The descriptive survey does show relationships between variables. The analytic survey explores the relationships between variables and is designed to answer the 'why' questions. Analytic surveys are also known as relational surveys. They are more concerned with prediction rather than description.

Questionnaires have certain limitations. The beliefs that the researcher considers important are selected *a priori*. One way to avoid this is for the researcher to select items that come from various sources creating an item pool. In this way the researcher limits 'influencing' the questionnaire statements.

Questionnaires can contain both open and closed questions. Oppenheim (1966) defined a closed question as "*one in which the respondent is offered a choice of alternative replies*" (p. 40). Open questions are not followed by any choice.

Cohen et al (2000) stated that open-ended questions allow respondents to answer questions in their own voice. Oppenheim (1966) suggested that "*The chief advantage of the open question is the freedom it gives to the respondent*" (p. 41).

I examined teachers' beliefs and wanted to find a way to group together teachers with similar beliefs. Cohen et al (2000) discussed Bennett's research conducted in 1976 about the relationship between teaching styles and pupil progress. Bennett employed a factor analysis technique known as principal component analysis followed by varimax rotation. This technique allowed Bennett to reduce the 28 variables in his original questionnaire to 19 variables. Bennett then went on to develop multi-dimensional typologies of teacher behaviour through the use of factor analysis. Oppenheim (1966) also suggested using factor analysis on questionnaire data in order to find factors that the questionnaire items have in common. After a review of the literature to find a current, accepted method of factor analysis, consultation with colleagues about the benefit of the different techniques of factor analysis, and actually trying several of the techniques, I chose to use principal component analysis with varimax rotation.

3.2.1.2 Factor Analysis

Background

Cureton and D'Agostino (1983) stated "*Factor analysis is partly a mathematical science and partly an art*" (p. xix). It is a science in that there are specific procedures and calculations that must be done to the data to get results. But it becomes an art in the way the results are interpreted.

In 1901, Karl Pearson was the first person to make known an explicit procedure for a factor analysis. Charles Spearman was responsible for the further development of factor analysis in 1904. Psychological and educational test scores were the first variables studies that used factor analysis. Thurstone later coined the term factor analysis in his classic work written in 1947 (Harman, 1976).

Factor analysis is used as a general term to refer to an entire family of data reduction techniques that look for "clumps" or groups among the inter-correlations of a set of variables. The techniques analyse the correlations between variables, but do not address causal relationships. When interpreting the results of factor analysis we are trying to find the underlying processes that have created the correlations among the variables. Factor analysis became more popular when computers enabled researchers to apply them for large data sets. Factor analysis (FA) has been applied to the behavioural and social sciences as well as to medicine, biology, chemistry and geology.

> "Traditionally, factor analysis has been used to explore the possible underlying structure in a set of interrelated variables without imposing any preconceived structure on the outcome. At its crudest, no thought might be given to the selection of variables; rather, the data, because they happen to be numerous as with a questionnaire or attitude scale items, are submitted for analysis in a 'let's see what happens' spirit. However it is unusual to find social scientists starting research in such an empty-headed way. In most instances, the analysis is preceded by a hunch as to the factors which may emerge..." Child (1990, p. 6).

Factor analysis attempts to produce a smaller number of linear combinations of the original variables that accounts for most of the variability in the pattern of correlations. Comrey (1973) suggested that a researcher would use factor analysis to get an idea about the underlying constructs that might explain the inter-correlations among a large collection of variables. Factor analysis can help researchers "*...gain a better understanding of the complex and poorly defined interrelationship among a large number of imprecisely measured variables*" (Comrey, 1973, p.1).

The Mathematics Behind Factor Analysis

What is a factor?

A factor is a group of variables that have a common characteristic that can be determined using correlations. One can think of factors as constructs that are postulated in order to find an explanation for the inter-correlations amongst variables. Another term for factors is latent variables, because they are not observed, counted or measured directly (Cureton and D'Agostino, 1983).

Matrix Interpretation

Mathematically speaking, the goal of factor analysis is to define a set of axes in p space, where p is the number of variables, which better describes the space than the set of vectors arranged within it and then to interpret what the axes, factors or components, represent. These axes are the eigenvectors. Correlation coefficients are the cosines of the angles between the axes. Loading of a variable on a factor or component is the cosine of the angle between the variable vector and the eigenvector (axis). This is the correlation between a variable and a component. A more detailed mathematical interpretation of factor analysis can be found in Appendix G.

Goals in the use of Factor Analysis

I have used factor analysis to statistically analyse the 48 Likert type statements in the questionnaire. Factor analysis is a statistical technique that can be used to reduce a large number of independent variables to a smaller, more coherent set of variables. (Child, 1990; Comrey,1973; Cureton and D'Agostino, 1983; Dunteman,1989; Harman,1976; Jackson, 1991;Tabachnick and Fidell, 2001). Factor analysis seeks to make order out of chaos (Child, 1990). Harman (1976) stated "*A satisfactory solution will yield factors which convey all the essential information of the original set of variables. Thus, the chief aim is to attain scientific parsimony or economy of description,*" (p. 4). The analysis produces a small set of factors that summarise the relationships among the larger set of variables.

3.2.1.3 Avoiding Errors in Research

Oppenheim (1966) stated "*All research is involved in the never-ending fight against error*" (p. 20). He listed possible sources of error:

- Faults in the questionnaire design
- Unreliability and lack of validity of various techniques used
- Sampling errors
- Errors due to non-response
- Faulty interpretation of results
- Bias errors due to:
    - Questionnaire design and questionnaire wording

- Respondents' misunderstanding or unreliability
- Coding responses

Bias in a questionnaire statement:

Questionnaire statements may be misunderstood by some respondents because of the following problems:

- The statement may be too vague or abstract
- The statement may be a leading statement
- The statement may ask for information the respondent does not have or has forgotten

Bias can also be introduced due to types of non-response to the questionnaire, when the returns are no longer representative of the population from which they were selected.

Bias can also be due to non-response to individual items on the questionnaire.

I tried to avoid questionnaire bias through the piloting process. Ambiguous questions were removed.

3.2.1.4 Reliability

Reliability of a questionnaire refers to its consistency. Will we get the same results if we administer the questionnaire again? Oppenheim (1966) suggested that on attitude questionnaires we should not rely on single questions, but rather on a set of questions or an attitude scale. He also suggested using Cronbach's alpha to measure reliability. Cronbach's alpha is not a statistical test. It is a coefficient of consistency. Its formula is:

$\alpha = \frac{N - \bar{r}}{(1 + (N - 1)\bar{r})}$ where N is equal to the number of items and $\bar{r}$ is the average inter-item correlation among the items. A high alpha indicates that the items are measuring the same underlying construct.

A single question may not be enough to reflect one's beliefs. Hence I included several different statements about manipulatives, dynamic geometry and proofs on my questionnaire which will discussed in chapters 4 and 5.

3.2.1.5 Validity

Internal validity of questionnaire tells us whether the questionnaire item is really measuring what it is supposed to be measuring. Oppenheim (1966) claimed that the main difficulty with attitude statements is lack of criteria. We can't necessarily predict behaviour from beliefs. Conversely, we cannot infer beliefs from behaviour with any validity. Teachers may do certain things in their classes that do not necessarily reflect their actual beliefs (Thompson, 1992).

Cohen et al (2000) suggested "*In quantitative data validity might be improved through careful sampling, appropriate instrumentation and appropriate statistical treatments of the data*" (p. 105). In this study I tried to adhere to this by being careful with my sampling, by piloting and revising the questionnaire, and by carefully analysing the data.

"*External validity is concerned with the extent to which the findings of one study can be applied to other situations*," (Merriam, 1998, p. 207). She said that a researcher can strengthen external validity by using standard sampling procedure. The respondents were from many different cities throughout the United States, Canada, Australia, and England. I tried to adhere to the procedure to the best of my ability.

3.2.2 Qualitative Methodology

3.2.2.1 Case Studies

Cohen et al (2000) stated "...*case studies investigate and report the complex dynamic and unfolding interactions of events, human relationships and other factors in a unique instance*" (p.181). In a case study the researcher is interested in an instance of a 'bounded system'.

When Merriam (1998) wrote her first book about case studies in 1988 she defined case study in terms of its end product. "*A qualitative case study is an intensive, holistic, description and analysis of a single instance, phenomenon, or social unit,*" (Merriam, 1988, p. 21; Merriam, 1998, p. 27). In the second edition she concluded that bounding the object of study is the most defining characteristic of this type of research. She further stated that if the object of study is not bounded then it is not a case.

Merriam (1998) suggested, "*A case study design is employed to gain an in-depth understanding of the situation and meaning for those involved*" (p. 19). She discussed three data collection techniques that can be used in case studies-observations, interviews, and analysing documents.

Lancy (1993) stated "*The case study whether it is used alone or as part of a large-scale quantitative study is the method of choice for studying interventions or innovations*" (p.140). He quoted Yin's (1984) applications of case studies.

Two purposes of these case study applications are:

1. To explain causal links that are too complex for a survey

2. To richly describe the real-life context in which an intervention has occurred

A case study of a particular teacher was conducted after a factor analysis was performed on the data. I wanted to see how useful the results of the factor analysis were. This case study might be considered an instrumental case study since it was undertaken in order to gain insight into numeric results. In trying to understand the teacher studied, I hoped to gain an understanding that could improve practice.

Merriam (1998) identified three special features of case studies: descriptive, particularistic, and heuristic. The descriptive case study provides a rich narrative account as an end product, the particularistic case study focuses on a specific event or phenomenon, and the heuristic develops categories in order to examine initial assumptions.

Merriam (1998) also described case studies by their intent (See Table 3.2).

TYPE OF CASE STUDY	INTENT
Descriptive case studies	Presents a detailed account of the phenomenon under study.
Interpretive case studies	Although this type of case contains rich descriptions, the descriptions are used to support, illustrate or challenge theoretical assumptions held prior to data gathering.
Evaluative case studies	Involves description, explanation, and judgement.

Table 3.2 *Types of Case Studies*

I have summarised some of the strengths and weaknesses of case studies that Cohen et al (2000) listed from Nisbet and Watt (1984):

Strengths of case studies:

- The results are more easily understandable by a wide audience because they are usually written in everyday language
- They are strong on reality
- They catch features that may be lost in large-scale data
- They can be undertaken by a one researcher rather than an entire team
- They can build in uncontrolled variables

Weaknesses of case studies:

- They are not open to cross checking which means they could be subjective
- There may be observer bias involved
- They may not be generalisable

There are also problems of reliability and validity in case study research. Each case may be unique in some way that would make it inconsistent with other cases. The bases of qualitative studies "*...include the uniqueness and idiosyncrasy of situations, such that the study cannot be replicated – that is their strength rather than their weakness*" (Cohen et al, 2000, p. 119).

The believability and usefulness of qualitative research is captured by the idea of trustworthiness. Trustworthiness is an alternative to the ideas of reliability and validity found in quantitative research. Ely et al. (1991) stated that trustworthiness is more than a set of procedures. It is a "*personal belief system that shapes the procedures in process.*"

(p.93) The researcher is fully involved in a qualitative study. The researcher is the instrument. Ely et al. (1991) characterise trustworthiness by the following elements:

- The process of the research is carried out fairly
- The products represent as closely as possible the experiences of the people who are studied
- Ethical principles ground

  (1) How data are collected and analysed

  (2) How one's own assumptions and conclusions are checked

  (3) How participants are involved

  (4) How results are communicated (p. 93)

3.3 THE ENACTED WORK PLAN

Design of the questionnaire

A questionnaire was designed by collecting statements that would lead towards answers for the research questions that were stated at the beginning of this chapter. Scale items and open-ended response statements were included thus combining both research methods described above. Oppenheim (1966) suggested collecting a pool of items from the literature. He further suggested using a Likert scale where respondents place themselves on a continuum from "*strongly agree*" to "*agree*", "*uncertain*", "*disagree*", and "*strongly disagree*". Likert scales provide more precise information about the respondents' degree of disagreement or agreement. Further details about the questionnaire design can be found in chapters 4 and 5.

Pilot the questionnaire

Oppenheim (1992) said "*Questionnaires do not emerge fully fledged; they need to be created or adapted, fashioned and developed to maturity after many abortive test flights*" (p.47).

The questionnaire was piloted to check for clear directions to respondents, ambiguous statements, and sequencing of statements. The questionnaire was then revised based on the feedback from the pilot study.

Distribute the revised questionnaires

The revised questionnaires were then distributed. I wanted teachers to respond to a questionnaire that would serve as the instrument to obtain details of their personal information and survey their beliefs. I wanted to follow this up with a case study to gain deeper insight into the respondents' beliefs.

Conduct pilot interviews

Merriam (1998) suggested that "*interviewing is necessary when we cannot observe behaviour, feelings, or how people interpret the world around them,*" (p.72). I wanted to conduct pilot interviews to see if the questions I asked would give me the information I needed to answer my research questions. I interviewed two teachers from the United States and one teacher from the United Kingdom. The pilot interview questions can be found in Appendix D and a transcribed interview can be found in Appendix E. The interview data was not an essential part of the analysis since the results from the factor analysis as reported in chapter 6 were so rich.

Analysis of the data using quantitative and qualitative methods

Analyses of the data were performed using both quantitative and qualitative methods. Chi-square analysis was used on the personal data and factor analysis on the scale items. Factor analysis is a data reduction technique that is used to make sense out of the data by analysing the correlations between variables. These quantitative methods found relationships between the variables under study. The quantitative analysis of the descriptive data can be found in chapter 5 and the results of the factor analysis of the data in chapter 6.

Coding was used to analyse the open response items on the questionnaire. In qualitative analysis *coding* is a process that creates and assigns categories or themes for the data. The analysis of the qualitative data can be found in chapter 7.

Conduct a case study

Finally, a case study was conducted in order to reconfirm the results from the above analyses. Cohen et al (2000) state that the researcher does not always have to adhere to the criteria of representativeness in case study research. It could very well be that an event will occur infrequently, but may be crucial to the understanding of the case.

There are two types of observations in case study research - participant observation and non-participant observation. The participant observer engages in the activities she sets out to observe. The non-participant observer observes like 'a fly on the wall'.

Cohen et al (2000) said that the most typical method of observation is the unstructured ethnographic account of teachers' work in the natural surroundings of their classrooms. I took the role of non-participatory observer.

3.4 CONCLUSIONS

Undertaking a mixed methods study although an arduous task because of the amount of data collected was extremely rewarding because using quantitative methods first yielded results that were then corroborated and enhanced by the qualitative methods as can be seen in Chapters 5, 6, and 7.

CHAPTER 4 - PILOT STUDY

4.1 INTRODUCTION

This chapter discusses the pilot study designed to explore the following research questions first posed in chapter 1:

- What are high school mathematics teachers' beliefs about the role of geometry in the curriculum?
- What are high school teachers' beliefs about the use of manipulatives and dynamic geometry software packages?
- What are high school teachers' beliefs about the role of proof?

The chapter describes the process of questionnaire design and administration which is followed by results and discussion. The chapter concludes with the implications of the pilot study for subsequent questionnaire redesign and use on a larger scale.

4.2 QUESTIONNAIRE DESIGN

A questionnaire is a useful instrument for collecting data about beliefs (Cohen et al, 2000; Leder and Forgasz, 2002; Oppenheim, 1966).

In order to make the research question operational I decided to investigate themes that I thought could answer the question.

Initially the pilot study centred on the following themes:

1. The role of geometry in the high school curriculum
2. The use of manipulatives in geometry
3. The use of dynamic geometry systems
4. The role of proof
5. Affective factors in teaching and learning geometry

The questionnaire was designed with the purpose of investigating teachers' beliefs about teaching and learning geometry and included questions relating to the above themes. Raymond (1997) used a questionnaire as one of the instruments to survey elementary school teachers about the nature of mathematics, the learning of mathematics, and the teaching of mathematics. I adapted these themes from Raymond's work and created more specific statements about the respondents' beliefs regarding the nature of geometry, the learning of geometry and the teaching of geometry rather than her more general statements about mathematics. For example:

Statement 24. My students enjoy doing geometric proofs.

Statement 31. I enjoy doing geometric proofs.

Statement 58. I enjoy teaching my students how to do geometric proofs.

I designed some statements that would satisfy my curiosity about whether teachers believed that all students should study geometry. Other statements on the questionnaire were adapted from the questionnaire that The National Council of Teachers of Mathematics (NCTM) used to survey high school geometry teachers (Gearhart, 1975). Gearhart's survey contained 57 statements. The statements that I adapted from his questionnaire can be found in Table 4.1.

Gearhart's statements	My pilot statements
1. The average college prep student regards it as unnecessary to prove theorems he regards as obvious.	45. It is unnecessary for students to prove theorems they regard as obvious
3. I enjoy teaching geometry to average college prep students.	1. I enjoy teaching geometry
5. The geometry course is valuable to high school mathematics students.	2. Geometry is valuable for HS students
7.Learning to write proofs is important for high school mathematics students.	5. Learning to construct proofs is important for HS students
8. Developing students' spatial perception is a primary objective of the geometry course.	6. Developing spatial sense is a primary objective of teaching geometry
10. The approach to geometry should be more concrete, using models, etc.	33. It is important to use hands-on activities to explore geometric ideas
14. The average college prep student finds the geometry course boring.	7. Students find geometry boring

Table 4.1 *Adaptation of Gearhart's Questionnaire to Pilot Questionnaire*

Further questions were adapted from a questionnaire about graphing calculator usage (Fleener, 1995). For example, I adapted my statement 21: *Using manipulatives in the teaching of geometry is motivational* from Fleener's statement 3: *Calculators are motivational.*

The particular language of some of the statements was adapted from an analysis of different discourses about geometry studies that were found in articles published in the National Council of Teachers of Mathematics' *Mathematics Teacher* journal between 1908 and 2002 (Gonzalez and Herbst, 2004). An example of one such statement is: *Students can experience the activity of mathematicians through their work in geometry class.* This statement concerned the expectations of students in the 'mathematical argument' for why geometry should be included in the high school curriculum.

I originally used a four point Likert scale which ranged from 1 representing *strongly disagree* up to 4 representing *strongly agree.*

A team of researchers and educators from the United States and England who reviewed the first draft of the questionnaire suggested using a five point Likert scale that added an undecided option into the choices. Questions that they thought were unclear or ambiguous were either discarded or rewritten.

The first part of the revised pilot questionnaire which can be found in Appendix A contained a five-point Likert scale with 59 items, where

5 corresponded to *strongly agree,*

4 corresponded to *agree*

3 corresponded to *undecided*

2 corresponded to *disagree*

1 corresponded to *strongly disagree.*

The second part of the questionnaire was designed to gain data about the respondents' background and experience and asked for factual data such as gender, undergraduate major (first degree), pre-service training, number of years of teaching experience, size and location of their schools.

The third section of the questionnaire consisted of an open-ended written response to the statement: *Geometry is an important/not important topic for high school students to study because....*

As a follow up, two pilot interviews were conducted--one in the US and one in the UK. The questions used in the pilot interviews can be found in Appendix C and the transcribed interview can be found in Appendix D.

I checked reliability of the questionnaire by using the Cronbach's reliability test. I created new variables when I changed any statement on the questionnaire that was negatively worded to a positive statement. The Cronbach's α was 0.848 for the 59 variables. This implies that the questionnaire was very reliable.

4.3 RESULTS

The revised pilot questionnaire (See Appendix A) was distributed to 44 high school mathematics teachers. Some were sent directly to local urban high schools. The other respondents were either teachers attending a graduate course at a local college or contacts from outside New York State, who responded either by email or through the regular post. In total there were 40 respondents yielding a 91% response rate.

The frequency tables of responses can be found in Appendix B.

4.3.1 Analysis of the Respondents' Personal Data

While the experience of the teachers ranged to beyond 25 years, there was a substantial proportion (57.5%) of relatively new teachers, with less than 5 years experience. The details are provided in Table 4.2. This does reflect the national profile of the United States, where there are many teachers with up to five years teaching experience (National Center for Education Statistics, 2006). Table 4.3 shows that at least 85% of the teachers had a mathematics or mathematics education undergraduate major. In addition, 45% of the teachers also had graduate degrees. The majority of the respondents taught in urban high schools, which reflected the way in which the questionnaire were distributed, but there were responses from all of the main classifications of schools, as shown in Table 4.4. The sample included equal numbers of male and female respondents, as shown in Table 4.5.

Number of years teaching	Frequency	Percent
1-5	23	57.5
6-10	3	7.5
11-15	3	7.5
16-20	6	15
21-25	1	2.5
> 25	3	7.5
No Response	1	2.5

Table 4.2 ***Teaching Experience of Pilot questionnaire respondents***

Undergraduate major (first degree)	Frequency	Percent
Mathematics related	34	85
Other	5	12.5
No Response	1	2.5

Table 4.3 ***Undergraduate information about respondents***

Location of school	Frequency	Percent
Urban	28	70
Suburban	9	22.5
Rural	1	2.5
No Response	2	5
Total	38	100

Table 4.4 ***Respondents' school location***

Gender	Frequency	Percent
Male	20	50
Female	20	50

Table 4.5 ***Respondents' Gender***

Table 4.6 shows that the majority of the teachers had used manipulatives in their teaching, but in contrast the majority had not used dynamic geometry packages, as shown in table 4.7.

I have used manipulatives to teach geometrical concepts	Frequency	Percent
No	11	27.5
Yes	29	72.5
Total	40	100

Table 4.6 ***Respondents use of manipulatives***

I have used a dynamic geometry software package with my students	Frequency	Percent
No	25	62.5
Yes	14	35
Total	39	97.5

Table 4.7 ***Respondents use of Dynamic Software***

Table 4.8 shows that relatively only a small proportion of the teachers have taught year-long geometry courses. From Table 4.9, it can be seen that the vast majority of teachers are delivering courses where geometry is an integrated topic. Thus it is clear that geometry is not being taught as a substantial topic in its own right, which reflects the fact that in some states there has been a move towards integrated courses. This illustrates that the state in which the respondent teaches may be a factor in determining their views about the teaching of geometry and that in a large scale survey teachers from a variety of states should be included.

I have taught geometry as a one-year course	Frequency	Percent
No	29	72.5
Yes	11	27.5
Total	40	100

Table 4.8 ***Respondents teaching a year-long geometry course***

I have taught geometry as a topic in an integrated curriculum	Frequency	Percent
No	2	5
Yes	38	95
Total	40	100

Table 4.9 ***Respondents teaching an integrated curriculum***

Although small, the sample seems to be moderately representative of teachers in the United States, except with respect to the length of course and integrated teaching of geometry, which has been highly influenced by the policy in New York State. It is interesting to note the difference between the level of use of manipulatives and dynamic geometry packages.

4.3.2 Analysis of the Responses to the Likert Statements

For analysis purposes the responses for *strongly agree* and *agree* were grouped together into a single response- *agree*. Similarly, *strongly disagree* and *disagree* were grouped together into a single response-*disagree*. The percentages of responses to statements from the questionnaire can be found in Table 4.10. In Table 4.10 the following notation is used:

A: Agree

U: Uncertain

D: Disagree

The percentage responses to all the questionnaire statements can be found in Appendix B.

Pilot Questionnaire Statements	A	U	D
1. I enjoy teaching geometry.	95%	0%	5%
2. Geometry is valuable for HS students	87.5%	10%	2.5%
3. I refer to theorems when teaching geometry	5%	2.5%	92.5%
4. Most HS students find geometry difficult	52.5%	27.5%	20%
5. Learning to construct proofs is important for HS students	72.5%	17.5%	7.5%
6. Developing spatial sense is a primary objective of teaching geometry	60%	22.5%	15%
7. Students find geometry boring	17.5%	27.5%	55%
8. The greatest value of geometry is the exposure it gives students to the deductive method	62.5%	35%	2.5%
9. I prove geometrical results so that my students can apply them to solve problems	60%	17.5%	22.5%
10. Geometry should be included in the curriculum for all students	85%	7.5%	7.5%
11. There are some things in geometry like proofs that are best memorised	12.5%	25%	62.5%
12. Dynamic geometry enables students to enjoy learning geometry	77.5%	22.5%	0%
13.Geometry should be a full, one-year course	60%	32.5%	7.5%
14. Geometry class is a good environment in which to develop the principles of proof	87.5%	10%	0%
15. High school geometry should not contain proof.	10%	15%	72.5%
16.Geometric ideas should be embedded in the curriculum in all grades	87.5%	12.5%	0%
17. Visuals should be an integral part of the geometry curriculum	100%	0%	0%
18. Students should learn to do geometric constructions	82.5%	10%	7.5%
19. HS students should be able to write two column proofs	67.5%	15%	17.5%
20. Geometry is a way of seeing structure in the world	77.5%	17.5%	5%
21. Using manipulatives in the teaching of geometry is motivational	95%	5%	0%
22. Geometry should only be taught to very able students	7.5%	15%	77.5%
23. Students can explore mathematics as mathematicians might	77.5%	22.5%	0%
24. My students enjoy doing proofs	10%	47.5%	40%
25. I lack the confidence to teach HS geometry	0%	5%	95%
26. Geometry has many real world applications	90%	5%	5%
27. Students should be taught how to produce valid mathematical arguments	90%	2.5%	7.5%
28. Manipulatives help students to grasp the basic ideas of geometry	87.5%	7.5%	5%
29. Geometry offers a means of describing, analyzing, and understanding the world	82.5%	17.5%	0%
30. All students should have familiarity with dynamic geometry	52.5%	27.5%	15%
31. I enjoy doing mathematical proofs	82.5%	15%	2.5%

Table 4.10 (a) ***Percentages of responses to statements on the pilot questionnaire***

Pilot Questionnaire Statements	A	U	D
32. HS students should experience other geometries besides Euclidean	57.5%	30%	12.5%
33. It is important to use hands-on activities to explore geometric ideas	80%	20%	0%
34. Proofs done in HS should be short	22.5%	40%	37.5%
35. It is beneficial to use manipulatives as an integral part of my geometry lessons	82.5%	15%	2.5%
36. Students find it difficult to use dynamic geometry packages	5%	62.5%	27.5%
37. Critiquing arguments is an important aspect of proving	70%	22.5%	7.5%
38. The use of manipulatives makes learning geometry fun	85%	10%	5%
39. More interesting geometrical problems can be explored with dynamic geometry than without it	55%	37.5%	2.5%
40. Geometry is an exercise in memorisation	7.5%	22.5%	70%
41. Algebraic skills should be strengthened in geometry	82.5%	10%	5%
42. HS geometry should be initially hands-on with proofs coming later in the course	52.5%	42.5%	5%
43. I am familiar enough with dynamic geometry to use it in my teaching	50%	17.5%	32.5%
44. Students should discover theorems in geometry	85%	12.5%	2.5%
45. It is unnecessary for students to prove theorems they regard as obvious	7.5%	17.5%	75%
46. Geometry is where students can validate conjectures using deductions	90%	10%	0%
47. More time should be spent on analytic geometry and other topics in geometry rather than on proving	40%	42.5%	17.5%
48. It is more important for students to apply theorems learned rather than explore geometric properties	10%	52.5%	35%
49. Proofs written in paragraph form are acceptable	65%	12.5%	20%
50. A main goal of geometry is to teach students how to reason	75%	17.5%	5%
51. If a student makes a conjecture about a geometrical idea that is not in the curriculum, the teacher should allow the class time to prove or disprove the conjecture	70%	25%	5%
52. Dynamic geometry can take the place of rigorous proofs.	20%	47.5%	27.5%
53. I am confident about teaching geometry	92.5%	7.5%	0%
54. I apply many theorems without proving them	37.5%	22.5%	40%
55. Geometry appeals to my visual, aesthetic, and intuitive senses	85%	15%	0%
56. Students should be made aware of the historical background of geometry	85%	12.5%	2.5%
57. Studying geometry leads to a positive attitude about mathematics	55%	35%	10%
58. I enjoy teaching geometrical proofs	82.5%	12.5%	5%
59. When teaching geometry, connections to real world applications should be made	90%	7.5%	2.5%

Table 4.10 (b) ***Percentages of responses to statements on the pilot questionnaire***

An initial observation from the results was that there was a tendency for quite a number of teachers to give uncertain responses. There were 24 statements for which more than

20% of the respondents were uncertain and 10 statements where over one-third of the respondents were uncertain. There is also a lack of consensus on 22 statements which represents 37.3% of the entire questionnaire. In a study that used a similar approach, Fleener (1995) defined consensus statements when over 70% of the responses were in the same category, which could be agreement or disagreement. The impact of this is considered again later in this chapter along with other revisions that were made before the main study.

4.3.3 Observations from the data

Although the sample was small, it was possible to make some interesting observations. The majority of these observations arise from the identification of inconsistencies in the data. These are now described in some detail.

Proof

The responses to statements 15 and 58, which can be found in Table 4.10, indicate that the majority of the teachers feel that high school geometry courses should contain proof and that they enjoy teaching geometrical proof. The response to statement 24 shows that very few of the teachers believe that their students enjoy doing geometrical proof. The question of what happens in the teaching of proof is clearly an area for further investigation, as the teachers enjoy doing and teaching proofs but their enjoyment is not instilled in their students.

Dynamic Geometry

As described above, some statements attracted a large number of uncertain responses. This was particularly true for the statements concerning the use of dynamic geometry systems and could be explained by the fact that many of the teachers said that they did not use dynamic geometry systems and did not have the experience to give positive or negative responses with any conviction. However, there are some quite contradictory responses to the statements about dynamic geometry systems. For example, while only

35% of teachers say that they use a dynamic geometry system, 50% of the teachers feel that are confident enough to use such a system and 77.5% of the teachers felt that dynamic geometry systems enhance students' learning of geometry. This raises an interesting question about why the usage of dynamic geometry systems is so low when many teachers feel that it could benefit their pupils.

Manipulatives

In section 4.3.1, the difference between the use of manipulatives and dynamic geometry was noted. This is another area worthy of investigation.

Curriculum Structure

From the response to statement 13, it can be seen that over 60% of the teachers agreed that geometry should be a one-year course, but only 27.5% had taught geometry in this way. Further it should be noted that only 5% of the teachers had not taught geometry as a topic in an integrated curriculum. The question that arises is why do so many teachers believe that geometry should be a year-long course, when they have so little experience of teaching it in that way? Perhaps there is a degree of dissatisfaction with the integrated course. Moise (1975) stated that geometry loses its structure and coherence when it is taught as part of an integrated course.

4.3.4 Further Exploration of the data

To explore further the issues raised in the previous section, a chi-squared analysis was applied to the data, to try to answer the following questions:

1) Is there a relationship between use of manipulatives and use of dynamic geometry?
2) Is there any relationship between gender and manipulative use?
3) Is there any relationship between gender and dynamic geometry use?
4) Is there any relationship between teaching experience and manipulative use?

5) Is there any relationship between teaching experience and dynamic geometry use?
6) Is there any relationship between taking methods courses and manipulative use?
7) Is there any relationship between taking methods courses and dynamic geometry use?
8) Is there any relationship between taking a geometry course as an undergraduate and manipulative use?
9) Is there any relationship between taking a geometry course as an undergraduate and dynamic geometry use?
10) Is there any relationship between undergraduate major (first degree) and manipulative use?
11) Is there any relationship between whether geometry is taught as a one-year course or part of an integrated curriculum and dynamic geometry use?
12) Is there any relationship between having a graduate degree and manipulative use?
13) Is there any relationship between having a graduate degree and dynamic geometry use?
14) Is there any relationship between whether geometry is taught as a one year course or part of an integrated curriculum and manipulative use?
15) Is there any relationship between undergraduate major and dynamic geometry use?
16) Is there any relationship between location of school (urban, suburban, or rural) and manipulative use?
17) Is there any relationship between location of school and dynamic geometry use?
18) Is there any relationship between size of school and manipulative use?
19) Is there any relationship between size of school and dynamic geometry use?

One has to be careful when applying the Chi-squared statistic to data when the expected frequency is less than five in any cell or if the value of any cell is less than one (Conover, 1999). Any implication reached should be further investigated with a larger sample.

The only statistically significant relationship ($p < 0.05$) was found for question 1 as shown in Table 4.11. There were some relationships for questions 8, 11, and 15 as shown in Tables 4.12, 4.13, 4.14 but none of these relationships were at a statistically significant level.

	I have used a dynamic geometry software package with my students		
I have used manipulatives to teach geometric concepts	No	Yes	Total
No	10 (7)	1 (4)	11
Yes	15 (18)	13 (10)	28
Total	25	14	39
Chi-squared = 4.78 $p = 0.0287$ Expected frequencies in brackets.			

Table 4.11 *Relationship between manipulatives use and dynamic geometry use*

The results in Table 4.11 show that teachers who use manipulatives also use dynamic geometry software more than would be expected. It suggests the possibility that a teacher who uses one type of mathematical tool would try using other tools too.

Looking at Table 4.12 there is a relationship between using manipulatives and taking an undergraduate geometry course. It appears that those who have a graduate degree are more likely to use manipulatives.

	I have used manipulatives to teach geometric concepts		
I have taken geometry courses as an undergraduate	No	Yes	Total
No	4 (2)	3 (5)	7
Yes	7 (9)	26 (24)	33
Total	11	29	40
Chi-squared = 3.74 $p = 0.053$ Expected frequencies in brackets			

Table 4.12 ***Relationship between manipulative use and undergraduate geometry courses***

Similarly in Table 4.13 there is a relationship between dynamic geometry software use and having taught geometry as a topic in an integrated curriculum. Although not significant, this result reflects the current situation in many high schools, since 95% of the teachers have taught an integrated course, almost anyone using dynamic geometry would fall into this category. I intend to further investigate these relationships with a larger sample (Chapter 5).

	I have used a dynamic geometry software package with my students		
I have taught geometry as a topic in an integrated curriculum	No	Yes	Total
No	0 (1)	2 (1)	2
Yes	25 (24)	12 (13)	37
Total	25	14	39
Chi-squared = 3.76 $p = 0.052$ Expected frequencies in brackets			

Table 4.13 ***Relationship between dynamic geometry use and teaching geometry as a topic in an integrated curriculum***

In Table 4.14 there is a relationship between dynamic geometry software use and one's undergraduate major that is not significant. It is probable that teachers having a mathematics related undergraduate major are more likely to be using dynamic geometry software than teachers who did not major in mathematics related fields. These results however may not reflect the availability of the software at the respondents' schools. I intend to further investigate these relationships with a larger sample (Chapter 5).

	I have used a dynamic geometry software package with my students		
Undergraduate major (first degree)	No	Yes	Total
Mathematics related	20 (22)	13 (11)	33
Other	5 (3)	0 (2)	5
Total	25	13	38
Chi-squared = 2.99 $p = 0.084$ Expected frequencies in brackets			

Table 4.14 ***Relationship between undergraduate major and dynamic geometry use***

4.3.5 Crosstabulations between Likert statements and Personal Data

Due to the number of uncertain responses to questionnaire statements about the use of dynamic geometry software packages I wanted to see if I could identify any relationships between those statements and some of the personal data that might give me insights for further investigation (Chapter 5).

For analysis purposes the responses for *strongly agree* and *agree* were grouped together into a single response- *agree*. Similarly, *strongly disagree* and *disagree* were grouped together into a single response-*disagree*.

Statistically significant results occurred when Chi-squared statistical tests were applied to some of the crosstabulations between the Likert statements and the personal data. As stated above, one has to be careful about any implications made if any cells have expected frequencies less than five or if the value of any cell is less than one. I intend to further investigate these relationships with a larger sample (Chapter 5).

Some of the findings seemed obvious, for example there is a significant relationship between teachers who have used dynamic geometry software with their students and teachers' belief that dynamic geometry enables students to enjoy learning geometry as shown in Table 4.15. Teachers who use dynamic geometry software believe that dynamic geometry enables students to enjoy learning geometry significantly more than teachers who do not use dynamic geometry software.

	I have used dynamic geometry software with my students		
Dynamic geometry enables students to enjoy learning geometry	No	Yes	Total
Undecided	9 (6)	0 (3)	9
Agree	16 (19)	14 (11)	30
Total	25	14	39
Chi-squared = 6.55 $p = 0.01$ Expected frequency in brackets			

Table 4.15 ***Crosstabulations between Statement 12 and use of dynamic geometry software***

I thought that I would find a similar relationship between the statement *I have used manipulatives to teach geometrical concepts* and the statement: *Dynamic geometry enables students to enjoy learning geometry*. There was a statistically significant relationship between the two statements as shown in Table 4.16. Teachers who use manipulatives believe that dynamic geometry enables students to enjoy learning geometry significantly more than teachers who do not use manipulatives. I intend to further investigate these relationships with a larger sample (Chapter 5).

	I have used manipulatives to teach geometric concepts		
Dynamic geometry enables students to enjoy learning geometry	No	Yes	Total
Undecided	5 (2)	4 (7)	9
Agree	6 (9)	25 (22)	31
Total	11	29	40
Chi-squared = 4.58 $p = 0.032$ Expected frequencies in brackets			

Table 4.16 ***Crosstabs between Statement 12 and use of manipulatives***

This pattern of relationships repeated itself and there were Likert statements that had significant relationships with both statements *I have used manipulatives to teach geometrical concepts* and *I have used a dynamic geometry software package with my students* as shown in Tables 4.17 and 4.18.

There was a statistically significant relationship between the two statements *I have used manipulatives to teach geometrical concepts* and statement 33: *It is important to use hands-on activities to explore geometric ideas* as shown in Table 4.17. Significantly more teachers than expected who use manipulatives agree with the statement. I intend to further investigate this relationship with a larger sample (Chapter 5).

	I have used manipulatives to teach geometric concepts		
It is important to use hands-on activities to explore geometric ideas	No	Yes	Total
Undecided	6 (2)	2 (6)	8
Agree	5 (9)	27 (23)	32
Total	11	29	40
Chi-squared = 11.32 $p = 7.68 \times 10^{-4}$ Expected frequencies in brackets			

Table 4.17 ***Crosstabs between Statement 33 and use of manipulatives***

There was a statistically significant relationship between the two statements *I have used a dynamic geometry software package with my students* and statement 33: *It is important to use hands-on activities to explore geometric ideas* as shown in Table 4.18. Significantly more teachers than expected that use dynamic geometry software packages with their students agree with the statement. I intend to further investigate this relationship with a larger sample (Chapter 5).

	I have used dynamic geometry software with my students		
It is important to use hands-on activities to explore geometric ideas	No	Yes	Total
Undecided	8 (5)	0 (3)	8
Agree	17 (20)	14 (11)	31
Total	25	14	39
Chi-squared = 5.64 $p = 0.018$ Expected frequencies in brackets			

Table 4.18 ***Crosstabs between Statement 33 and use of dynamic geometry software***

There were Likert statements that had a significant relationship with either the statement: *I have used manipulatives to teach geometrical concepts* or the statement: *I have used a dynamic geometry software package with my students* but not with both statements as shown in Tables 4.19 and 4.20; 4.21 and 4.22.

There was **not** a statistically significant relationship between the two statements *I have used manipulatives to teach geometrical concepts* and statement 30: *All students should have familiarity with dynamic geometry* as shown in Table 4.19. Teachers' belief about whether students should have familiarity with dynamic geometry software is independent of whether the teacher uses manipulatives. I intend to further investigate this relationship with a larger sample (Chapter 5).

	I have used manipulatives to teach geometric concepts		
All students should have familiarity with dynamic geometry	No	Yes	Total
Disagree	1 (2)	5 (4)	6
Undecided	6 (3)	5 (8)	11
Agree	4 (6)	17 (15)	21
Total	11	27	38
Chi-squared = 4.95 $p = 0.084$ Expected frequencies in brackets			

Table 4.19 ***Crosstabs between Statement 30 and use of manipulatives***

There was a statistically significant relationship between the two statements *I have used a dynamic geometry software package with my students* and statement 30: *All students should have familiarity with dynamic geometry* as shown in Table 4.20. Significantly more teachers than expected that use dynamic geometry software with their students agree with the statement. By using the software teachers are giving their students familiarity with it. I intend to further investigate this relationship with a larger sample (Chapter 5).

	I have used dynamic geometry software with my students		
All students should have familiarity with dynamic geometry	No	Yes	Total
Disagree	6 (4)	0 (2)	6
Undecided	10 (7)	1 (4)	11
Agree	7 (12)	13 (8)	20
Total	23	14	37
Chi-squared = 13.79 $p = 0.001$ Expected frequencies in brackets			

Table 4.20 ***Crosstabs between Statement 30 and use of dynamic geometry software***

There was a statistically significant relationship between the two statements *I have used manipulatives to teach geometrical concepts* and statement 39: *More interesting geometrical problems can be explored with dynamic geometry than without it* as shown in Table 4.21. Significantly more teachers than expected who use manipulatives agree with the statement. I intend to further investigate this relationship with a larger sample (Chapter 5).

	I have used manipulatives to teach geometric concepts		
More interesting geometrical problems can be explored with dynamic geometry than without it	No	Yes	Total
Disagree	0 (0)	1 (1)	1
Undecided	8 (4)	7 (11)	15
Agree	3 (6)	19 (16)	22
Total	11	27	38
Chi-squared = 7.25 $p = 0.027$ Expected frequencies in brackets			

Table 4.21 *Crosstabs between Statement 39 and use of manipulatives*

There was **not** a statistically significant relationship between the two statements *I have used a dynamic geometry software package with my students* and statement 39: *More interesting geometrical problems can be explored with dynamic geometry than without it* as shown in Table 4.22. Teachers' belief about whether more interesting geometry problems can be explored with dynamic geometry is independent of their use of the software. I intend to further investigate this relationship with a larger sample (Chapter 5).

	I have used dynamic geometry software with my students		
More interesting geometrical problems can be explored with dynamic geometry than without it	No	Yes	Total
Disagree	0 (1)	1 (0)	1
Undecided	12 (9)	3 (6)	15
Agree	11 (13)	10 (8)	21
Total	23	14	37
Chi-squared = 4.53 $p = 0.104$ Expected frequencies in brackets			

Table 4.22 *Crosstabs between Statement 39 and use of dynamic geometry software*

4.4 REVISION AND REFINEMENT OF THE PILOT QUESTIONNAIRE

As reported in section 4.3.2, a large percentage of respondents chose the undecided option on the five-point Likert scale which produced the results shown in Table 4.10.

Because of the small sample size and all the undecided responses in the above tables as shown in Tables 4.15-4.22 I was hindered in trying to make inferences from the data. As a consequence, I decided to use a six-point Likert scale for future questionnaires in an effort to force respondents "off the fence" so to speak, by eliminating the undecided choice as a possibility. I could have reverted back to a four point Likert scale but I wanted a more "continuous" scale so I added two more options: *disagree slightly more than agree* and *agree slightly more than disagree*.

I also found the need to eliminate any ambiguous questions. As an example, I came to see that statement 55: *Geometry appeals to my visual, aesthetic and intuitive senses* is ambiguous since the researcher could not guess which of the three or even if all of the three *senses* were being referred to in any individual response.

Some respondents found the questionnaire too long. The final version of the questionnaire contained 48 Likert type statements. I also added two open response questions to the one that already existed in the pilot questionnaire. The open-ended response questions were placed before the request for personal data. In other words, I exchanged the second and third parts of the questionnaire. I did this because I thought the questionnaire would be less laborious with the personal data at the end of it rather than in the middle of it. I created a version for American teachers that can be found in Appendix B and a version for United Kingdom teachers that can be found in Appendix C. I created the UK version to avoid such misunderstandings over terminology as the fact that 'high school' in the US is referred to as 'secondary school' in the UK. Also what is referred to as an 'undergraduate major' in the US is thought of as a 'first degree' in the UK.

I decided to add questions about membership of professional organisations and/or attendance at professional meetings to the personal data section of the questionnaire. I was curious to find out whether there were any significant relationships between belonging to a professional organisation or attending professional meetings and use of manipulatives and/or dynamic geometry software.

4.5 CONCLUSIONS FROM THE PILOTING PROCESS

4.5.1 The Process

Oppenheim (1992) stated that every aspect of a questionnaire has to be explored beforehand to make sure it works as intended. Piloting a questionnaire helped me to refine or eliminate ambiguous questions, determine an appropriate scale, and adjust open-ended response questions in order to gain a better understanding of respondents' beliefs. I have also learned that I need a large enough sample from a variety of schools to get less biased results. For example the 20 teachers from the same school had little access to manipulatives and almost no access to any dynamic geometry software. I was able to take the opportunity to ask for additional personal data in the revised questionnaire that might help me to identify significant relationships that could impact on my study.

4.5.2 Areas for Investigation

Many of the issues raised in this pilot study are worthy of further study since the sample size was small and the implications from the chi-squared analyses have to be taken cautiously. For example is there a statistically significant relationship between the use of manipulatives and the use of dynamic geometry software packages when looking at a large sample of high school mathematics teachers? Do respondents think of dynamic geometry packages as sophisticated manipulatives? Already the results seem to indicate that this is not the case since we have statements that are statistically significant with respect to one and not the other as shown in Tables 4.19 and 4.20; 4.21 and 4.22.

Does the relationship found in Table 4.12 between the use of manipulatives and taking an undergraduate geometry course become statistically significant as the sample size increases?

Similarly do the relationships found in Tables 4.13 and 4.14 between dynamic geometry software use and having taught geometry as a topic in an integrated curriculum or having an undergraduate mathematics related major (first degree) become statistically significant as the sample size increases?

Do any of the relationships that were statistically significant in this pilot study stay statistically significant as the sample size increases?

These questions along with the questions about the other relationships found in section 4.3.4 will be further explored when the revised questionnaire is analysed in chapter 5.

CHAPTER 5 - DESCRIPTIVE ANALYSIS OF THE DATA

5.1 INTRODUCTION

This chapter contains an analysis of the descriptive data for the revised questionnaire which was distributed in the 2004-2005 school year. (The frequency of responses tables to the 48 Likert type statements and 15 personal data questions can be found in Appendix F). I was looking to answer the questions that were raised in Chapter 4 Sections 4.3.3, 4.3.4 and 4.6.3 and other questions that arose from the data such as whether there are statistically significant gender differences with respect to teachers' beliefs about teaching or learning geometry. This was done by looking for statistically significant relationships between variables such as the gender of the respondents and their responses to statements on the questionnaire that would help me better understand high school mathematics teachers' beliefs about the teaching and learning of geometry. A further analysis of the data using factor analysis is discussed in Chapter 6.

5.2 THE SAMPLE

The questionnaire contained 48 Likert type statements, three open ended response questions and a number of personal data statements. It was distributed to high school mathematics teachers from the United States, Australia, and Canada. A slightly different version of the questionnaire was used in England because I tried to avoid misunderstandings over terminology. These versions can be found in Appendices B and C. There were fewer than 20 responses in total from outside the United States making it impossible to compare the results from different countries in this study. DiMartino, (2004) and Leder and Forgasz (2002) have written that questionnaires are easy to administer. I took that as a fact when I decided to use a questionnaire to collect data about teachers' beliefs. But I found, as have many other researchers, that while questionnaires may be easy to *administer*, but they are not necessarily easy to get back.

I received 520 responses out of 750 questionnaires that were distributed, a decent return rate, but I had made considerable repeated efforts to obtain an even better one.

My sample consisted of an almost equal number of males and females although a few respondents did not specify their gender, as shown in Table 5.1. There were several significant gender differences that will be discussed later in this chapter.

Gender	Frequency	Percent
Male	240	46.2
Female	268	51.5
No Response	12	2.3
Total	520	100

Table 5.1 ***Respondents' Gender***

The teaching experience of the respondents ranged from 1 to 49 years. However as shown in Table 5.2, almost one-third of the respondents were relatively inexperienced teachers who had taught for five years or less. Regarding this result one may ask whether newer teachers were more willing to respond to the questionnaire or whether the turn over rate is such that there is a large percentage of new teachers in many schools (National Center for Education Statistics, 2006).

Number of years teaching	Frequency	Percent
1-5	172	33.1
6-10	76	14.6
11-15	59	11.3
16-20	66	12.7
21-25	38	7.3
26-30	31	6.0
> 30	57	11.0
No Response	21	4.0
Total	520	100

Table 5.2 ***Teaching Experience of Respondent***

The majority of the respondents had a mathematics or mathematics education undergraduate major/first degree as shown in Table 5.3. Mathematics related majors included a major in statistics, computers and engineering.

Undergraduate major/first degree	**Frequency**	**Percent**
Mathematics related	336	64.6
Other	163	31.3
No Response	21	4.0
Total	520	100

Table 5.3 ***Undergraduate information about respondents***

A large percentage, 73.8%, of respondents had graduate degrees as shown in Table 5.4. In many states in the United States teachers need to obtain a graduate degree to teach beyond 5 years.

I have a graduate degree	**Frequency**	**Percent**
No	120	23.1
Yes	384	73.8
No Response	16	3.1
Total	520	100

Table 5.4 ***Graduate information about respondents***

The majority of the respondents taught in inner city high schools as shown in Table 5.5. The types of schools in which respondents taught led to statistically significant different results that are examined later in this chapter. The sample is reasonably representative of the population.

Location of school	**Frequency**	**Percent**
Inner City	321	61.7
Suburban	103	19.8
Rural	31	6.0
Other	31	6.0
No Response	34	6.5
Total	520	100

Table 5.5 ***Respondents' school location***

5.3 COLLATING THE DATA

The responses to the 48 Likert statements were numerically coded from 1-6 with 1 being *strongly disagree* and 6 being *strongly agree*. The package SPSS was used to find the frequencies for the descriptive data, which are presented in Appendix D and to calculate the crosstabulations between many of the variables. A chi-squared analysis was performed on the crosstabulations to determine whether the variables were independent. For the analysis, I grouped the responses *strongly disagree*, *moderately disagree*, and *disagree slightly more than agree* into a single response --*disagree*. Similarly, I grouped *strongly agree*, *moderately agree*, and *agree slightly more than disagree* into a single response – *agree*. One has to be careful when applying the Chi-squared statistic to data when the expected frequency is less than five in any cell, so grouping the data in this way allowed for a consistent way of dealing with this issue. Otherwise more than one-third of the contingency tables contained cells with expected frequency less than five. There was no such grouping necessary for the factor analysis (Chapter 6).

The Chi Square Test was used to test for statistical significance (Conover, 1999). The key calculations used are shown below.

The expected value of each cell is calculated using the formula:

$$Expected\ Value = \frac{Row\ Total\ \times Column\ Total}{Sample\ Size}$$

For example, when rolling a fair six sided die twenty-four times the expected value for each of the possible outcomes would be four.

The Chi Square statistic is a calculated using the formula:

$$Chi\ Square\ Statistic = \sum \frac{(O-E)^2}{E}$$

O is the observed frequency of responses and E is the expected frequency of responses. It is big if the observed frequency is not similar to the expected frequency.

The percentages of respondents that agreed (**A**), disagreed (**D**) or did not respond (**NR**) can be found in Table 5.6. I refer to these results again when needed later in the chapter.

Questionnaire Statement	A	D	NR
1. I enjoy teaching geometry.	94.6%	4.8%	0.6%
2. Geometry is valuable for HS students.	99%	1%	
3. Most HS students find geometry difficult.	91%	7.8%	1.2%
4. Learning to construct proofs is important for HS students.	86.7%	12.7%	0.6%
5. Developing spatial sense is a primary objective of teaching geometry.	90.7%	8.5%	0.8%
6. Geometry should be included in the curriculum for **all** students.	92.9%	6.3%	0.8%
7. There are some things in geometry like proofs that are best memorised.	34.5%*	64.8%	0.8%
8. Dynamic geometry enables students to enjoy learning geometry.	88.1%	3.4%	8.5%
9. Geometry should occupy a significant place in the curriculum.	92.9%	6.4%	0.8%
10. High school geometry should not contain proof.	23.1%	76.9%	3.7%
11. Visuals such as diagrams and sketches should not be an integral part of the geometry curriculum.	7.9%	91.9%	0.2%
12. Students should learn how to do geometric constructions with straight edge and compass.	84.6%	14.4%	1%
13. HS students should be able to write rigorous proofs in geometry.	62.3%*	37.4%	0.2%
14. Using manipulatives in the teaching of geometry is motivational.	94.7%	4.7%	0.8%
15. Geometry should only be taught to very able students.	14.4%	85.4%	0.2%
16. My students enjoy doing geometric proofs.	33.4%*	60.6%	6%
17. I lack the confidence to teach HS geometry.	5.2%	93.8%	1%
18. Geometry has many real world applications.	95.6%	3.1%	1.3%
19. Manipulatives help students to grasp the basic ideas of geometry.	95.5%	2.5%	2%
20. All students should have familiarity with dynamic geometry.	79.6%	15.4%	5%
21. I enjoy doing geometric proofs.	88%	11.2%	0.8%
22. HS students should experience other geometries besides Euclidean (e.g. transformational, Non Euclidean).	80.3%	18.2%	1.5%
23. It is important to use hands-on activities to explore geometric ideas.	94.4%	3.7%	1.9%
24. It is beneficial to use manipulatives as a component of my geometry lessons.	89.8%	7.5%	2.7%

Table 5.6 (a) ***Percentages of respondents' responses to statements on the questionnaire*** **(statements 1 to 24)**
*no consensus

Questionnaire Statement	A	D	NR
25. Students find it difficult to use dynamic geometry packages.	32.6%*	51.6%	15.8%
26. The use of manipulatives makes learning geometry fun	93.3%	3.8%	2.9%
27. More interesting geometrical problems can be explored with dynamic geometry than without it.	79.5%	8.8%	11.7%
28. Geometry is an exercise in memorisation.	16.9%	81.6%	1.2%
29. Initially, HS geometry should be hands-on with proofs coming later in the course.	75.5%	23%	1.3%
30. I am familiar enough with dynamic geometry to use it in my teaching.	57.7%*	39.9%	2.5%
31. HS students should discover theorems in geometry.	88.9%	10.8%	0.4%
32. It is unnecessary for students to prove theorems they regard as obvious.	32.9%*	66.4%	0.8%
33. Geometry is where students can validate conjectures using deductions.	94%	3.9%	2.1%
34. More time should be spent on analytic geometry and other topics in geometry rather than on proving.	67.5%*	31.1%	1.3%
35. Proofs written in paragraph form are acceptable.	87.9%	10.4%	1.7%
36. A main goal of geometry is to teach students how to reason.	92.9%	6.6%	0.6%
37. If a student makes a conjecture about a geometrical idea that is not in the curriculum, the teacher should allow the class time to prove or disprove the conjecture.	93.2%	5.4%	1.2%
38. Dynamic geometry can take the place of rigorous proofs.	41.5%*	50%	8.5%
39. I am confident about teaching geometry.	95.6%	3.6%	0.8%
40. Students should be made aware of the historical background of geometry.	92.9%	6.9%	0.2%
41. Studying geometry leads to a positive attitude about mathematics.	82.5%	16%	1.5%
42. When teaching geometry, connections to real world applications should be made.	98%	2%	
43. Students can experience the activity of mathematicians through their work in geometry class.	92.2%	7.7%	2.1%
44. I enjoy teaching my students how to do geometric proofs.	78.8%	17.5%	3.7%
45. Geometry enables ideas from other area of mathematics to be pictured.	92.9%	3.9%	3.3%
46. The main goal of geometry is to illustrate the order and coherence of a mathematical system.	75.4%	22.8%	1.7%
47. Applying geometrical concepts and thinking will help students in their future occupations or professions.	91.3%	7.3%	1.3%
48. I enjoy proving theorems for my students.	80.2%	17%	2.9%

Table 5.6 (b) ***Percentages of respondents' responses to statements on the questionnaire*** **(statements 25 to 48)**

*no consensus

5.3.1 Consensus

Fleener (1995) defined consensus statements when over 70% of the responses were in the same category, which could be agreement or disagreement. The consensus statements are indicated in Table 5.6. Consensus was found on all but eight statements. This is an improvement over the pilot study where there were 22 statements with no consensus. There was no consensus for:

Statement 7: *Some things like proofs are best memorised.*

Statement 13: *High school students should be able to write rigorous proofs in geometry.*

Statement 16: *My students enjoy doing geometric proofs.*

Statement 25: *Students find dynamic geometry difficult.*

Statement 30: *I am familiar enough with dynamic geometry to use it confidently in my teaching.*

Statement 32: *It is unnecessary for students to prove theorems that they regard as obvious.*

Statement 34: *More time should be spent on analytic geometry and other topics rather than on proving.*

Statement 38: *Dynamic geometry can take the place of rigorous proof.*

It is interesting to note that the majority of the statements on which there was no consensus involved proof in geometry. The question of whether proofs should be included in the high school geometry curriculum has been debated for many years (Batista and Clements, 1995; Gearhart, 1975; Gonzalez and Herbst, 2004; Hanna, 1995; Hoffer, 1981; Hoyles, 1997; Kline, 1973; Knuth, McCrone, 2002; Senk, 1985; Schoenfeld, 1988).

5.3.2 Reliability

In order to test the reliability of the questionnaire, I recoded any of the 48 Likert type statements that were negatively worded so that all statements were positively directed.

There were 359 valid cases (69%) where every statement had been rated. Cronbach's alpha test was performed using listwise deletion based on all the variables in the procedure for the valid cases. Listwise deletion means that if a respondent left out even one response to any of the 48 statements, then their questionnaire was not included in the analysis. The Cronbach's alpha measured 0.852 which indicates high reliability.

5.4 FINDINGS

Although crosstabulations were performed between almost all variables I have only included the tables where there were statistically significant results with $p < 0.05$.

5.4.1 Findings about the use of manipulatives

There have been mixed messages about the use of manipulatives (Ball, 1992; Fuys, Geddes, Tischler, 1988; Howard, Perry and Tracey, 1997; Kline, 1973; Mason 1989, Moyer, 2001;National Council of Teachers of Mathematics, 1989,1991,2000; Spikell, 1993; Thomas, 1992) and I wanted to find out what this particular sample of respondents believed and practiced.

Out of the 506 responses to the statement in the personal data section: *I have used manipulatives to teach geometrical concepts*, 80.2% responded *yes* and 19.8% responded *no* as shown in Table 5.7.

I have used manipulatives to teach geometrical concepts	Frequency	Percent
No	100	19.2
Yes	406	78.1
No Response	14	2.7
Total	520	100

Table 5.7 *Respondents' use of manipulatives*

Four out of the 48 Likert type statements on the questionnaire were about manipulatives and two others were about using a hands–on approach when teaching geometry as shown in Table 5.8.

Statements about Manipulatives	A	D
14. Using manipulatives in the teaching of geometry is motivational.	94.7%	4.7%
19. Manipulatives help students to grasp the basic ideas of geometry.	95.5%	2.5%
24. I think it is beneficial to use manipulatives such as mirrors as a component of my geometry lessons.	89.8%	7.5%
26. The use of manipulatives makes the learning geometry fun.	93.3%	3.8%
23. It is important to use hands-on activities to explore geometric ideas.	94.4%	3.7%
29. Initially, high school geometry should be hands-on with proofs coming later in the course	75.5%	23%

Table 5.8 ***Statements about Manipulatives on the Geometry Beliefs Questionnaire***

There was consensus for all statements, but not as strong for statement 29. I was curious about why it was the case that fewer teachers agreed that initially, high school geometry should be hands-on with proofs coming later in the course.

I wanted to find out if there were any significant differences in the responses to the statements about manipulatives between users and nonusers of manipulatives. In order to determine whether there were any relationships between these variables I used the Chi-squared statistic. I crosstabulated each of the six Likert statements listed in Table 5.8 with the statement from the personal data section: *I have used manipulatives to teach geometric concepts*. Each of the Tables 5.9-5.14 contains the observed frequencies and their totals. The expected frequencies for each cell, rounded to the nearest whole number, are in brackets.

$$Expected\ Value = \frac{Row\ Total\ \times Column\ Total}{Sample\ Size}$$

I found statistically significant results for each of the statements except for statement 29.

Respondents who have used manipulatives agreed significantly more than expected to statement 14 *using manipulatives is motivational*. The expected value for respondents who use manipulatives to be in agreement with statement 14 is 386 [(405 x 478)/502]. The fact that 395 respondents who use manipulatives agreed with statement 14 is statistically significant ($p = 7.03 \times 10^{-7}$). Looking at the results from another perspective, it was expected that 5 [(97x24)/502] respondents who do not use manipulatives would disagree with statement 14, but in actuality 14 disagreed as shown in Table 5.9. This seems to imply that teachers do not use manipulatives because they do not believe that they are motivational. Those teachers who have tried using manipulatives have found them to be motivational, while those who have not don't know this. This idea runs through the cross tabulations between the statement: *I have used manipulatives...* and the other statements about manipulatives on the questionnaire except for statement 29.

	I have used manipulatives to teach geometric concepts		
Using manipulatives is motivational	No	Yes	Total
Disagree	14 (5)	10 (19)	24
Agree	83 (92)	395 (386)	478
Total	97	405	502
Chi-squared = 24.61 ($p = 7.03 \times 10^{-7}$) Expected frequencies in brackets			

Table 5.9 ***Crosstabulation between statement 14 and manipulatives use***

I obtained similar results with statements 19, 24, 26, and 23 (see Tables 5.10, 5.11, 5.12, and 5.13). The results were similar because respondents who have used manipulatives agreed significantly more than expected with these statements.

There were 400 respondents who used manipulatives and believed that manipulatives help students to grasp basic ideas. We would only have expected 392 respondents to have this belief. Although it is only a difference of eight persons it is a statistically significant difference. Looking at it from another perspective, we would only expect 2 respondents who do not use manipulatives to disagree with statement 19. Actually there were 10 respondents who disagreed as shown in Table 5.10. This seems to imply that teachers do not use manipulatives because they believe they are not helpful to students. One has to be careful about making implications about the results when applying the Chi-squared statistic to data when the expected frequency is less than five in any cell.

	I have used manipulatives to teach geometric concepts		
Manipulatives help students grasp basic ideas	No	Yes	Total
Disagree	10 (2)	3 (11)	13
Agree	84 (92)	400 (392)	484
Total	94	403	497
Chi-squared = 29.29 ($p = 6.23 \times 10^{-8}$) Expected frequencies in brackets			

Table 5.10 ***Crosstabulation between statement 19 and manipulatives use***

Similarly, there were 387 respondents who used manipulatives and believed that it is beneficial to use manipulatives in their lessons. We would only have expected 369 respondents to have this belief. Looking at it from another perspective, we would expect 7 respondents who do not use manipulatives to disagree with statement 24. Actually there were 25 respondents that disagreed as shown in Table 5.11. This seems to imply that teachers do not use manipulatives because they believe that manipulatives are not beneficial to their lessons. Respondents who believe they are beneficial but do not use them may not have them readily available.

	I have used manipulatives to teach geometric concepts		
Beneficial to use manipulatives in my lessons	No	Yes	Total
Disagree	25 (7)	14 (32)	39
Agree	66 (84)	387 (369)	453
Total	91	401	492
Chi-squared = 58.44 ($p = 2.09 \times 10^{-14}$) Expected frequencies in brackets			

Table 5.11 ***Crosstabulation between statement 24 and manipulatives use***

Likewise there were 395 respondents who used manipulatives that believed that manipulatives make geometry learning fun. We would have only expected 385 respondents to have this belief. Although it is only a difference of ten persons it is a statistically significant difference. Looking at it from another perspective, we would expect 3 respondents who do not use manipulatives to disagree with statement 26. Actually there were 13 respondents who disagreed as shown in Table 5.12. This seems to imply that teachers do not use manipulatives because they believe that manipulatives do not make learning geometry fun.

	I have used manipulatives to teach geometric concepts		
Manipulatives makes learning geometry fun	No	Yes	Total
Disagree	13 (3)	6 (16)	19
Agree	77 (87)	395 (385)	472
Total	90	401	491
Chi-squared = 33.13 $(p = 8.63 \times 10^{-19})$ Expected frequencies in brackets			

Table 5.12 ***Crosstabulation between statement 26 and manipulatives use***

There were 396 respondents who used manipulatives that believed that it is important to use hands-on activities. We would only have expected 386 respondents to have this belief. Although it is only a difference of ten persons it is a statistically significant difference. Looking at it from another perspective, we would expect 4 respondents who do not use manipulatives to disagree with statement 23. Actually there were 14 respondents who disagreed as shown in Table 5.13. This seems to imply that teachers do not use manipulatives because they believe that it is not important to do hands-on activities. This result reveals a stronger statistically significant relationship than the relationship found between these variables in Table 4.17 of the pilot study.

	I have used manipulatives to teach geometric concepts		
It is important to use hands-on activities	No	Yes	Total
Disagree	14 (4)	5 (15)	19
Agree	81 (91)	396 (386)	477
Total	95	401	496
Chi-squared = 37.94 ($p = 7.29 \times 10^{-10}$) Expected frequencies in brackets			

Table 5.13 ***Crosstabulation between statement 23 and manipulatives use***

It is also of note to consider why teachers who believe the statements that manipulative use is motivational and fun or that it is important to do hands-on activities don't actually use them?

The use of manipulatives and teachers' beliefs about whether a geometry course should be initially hands-on with proof coming later (statement 29) are independent of each other, in other words, there is no significant relationship between the statements as shown in Table 5.14.

When factor analysis was performed on the 48 variables (statements) of the questionnaire, all the statements relating to manipulatives except for statement 29 loaded onto the same factor together with statements about dynamic geometry. I named this factor "activities" (see Chapter 6). Statement 29 loaded negatively onto the factor I named "abstraction". This could mean that respondents who have a disposition towards doing proofs are not in favour of having students engaged in hands-on activities in their classes.

	I have used manipulatives to teach geometric concepts		
Geometry should be hands–on with proofs coming later	No	Yes	Total
Disagree	24 (23)	93 (94)	117
Agree	72 (73)	310 (309)	382
Total	96	403	499
Chi-squared = 0.1597 ($p = 0.6894$) Expected frequencies in brackets			

Table 5.14 *Crosstabulation between statement 29 and manipulatives use*

5.4.1.1 Manipulatives and Gender

Is there any relationship between gender and manipulative use? Statistically significant results were found when the chi-squared statistic was applied to responses to the statements *I have used manipulatives to teach geometrical concepts* and the respondents' gender as shown in Table 5.15. For this particular sample I have found that female high school teachers use manipulatives significantly more than the male teachers. Further study is needed to see if this is true in general and if so, why?

	I have used manipulatives to teach geometric concepts		
Gender	No	Yes	Total
Female	35 (52)	230 (213)	265
Male	63 (46)	176 (193)	239
Total	98	406	504
Chi squared = 13.8779 ($p = 1.9507 \times 10^{-4}$) Expected frequencies in brackets			

Table 5.15 *Crosstabulation between gender and manipulatives use*

5.4.1.2 Manipulatives and Professional Organisations

Statistically significant results were found when the chi-squared test was applied to the statement *I am a member of NCTM etc.* and the statement *I have used manipulatives to teach geometrical concepts* as shown in Table 5.16. The expected frequency for members of professional organisations to use manipulatives is 183, but the responses show that 200 of these members use manipulatives.

	I have used manipulatives to teach geometric concepts		
I am a member of NCTM, ATM etc.	No	Yes	Total
No	71 (54)	204 (221)	275
Yes	28 (45)	200 (183)	228
Total	99	404	503
Chi squared = 14.45 $(p = 1.44 \times 10^{-4})$ Expected frequencies in brackets			

Table 5.16 ***Crosstabulation between membership in professional organisations and manipulatives use***

Similarly statistically significant results were found when the chi-squared test was applied to the statements *I have attended at least 2 NCTM national meetings* and *I have used manipulatives to teach geometrical concepts* as shown in Table 5.17. More respondents who attend professional meetings use manipulatives than was expected.

	I have used manipulatives to teach geometric concepts		
I have attended at least 2 NCTM meetings	No	Yes	Total
No	83 (68)	257 (272)	340
Yes	16 (31)	140 (125)	156
Total	99	397	496
Chi squared = 13.41 $(p = 2.49 \times 10^{-4})$ Expected frequencies in brackets			

Table 5.17 ***Crosstabulation between attendance at professional meetings and manipulatives use***

It is interesting that membership of professional organisations and attendance at professional conferences is significant with respect to manipulative use. Does membership of a professional organisation and/or attendance at conferences provide more awareness of manipulatives and their uses or do teachers who use manipulatives join professional organisations and attend professional meetings more often than teachers who don't use manipulatives? Do teachers who believe in using manipulatives join organisations and/or attend meetings to learn more about their profession?

5.4.1.3 Manipulatives and Dynamic Geometry Software

Statistically significant results were found when the chi-squared test was applied to the statements *I have used dynamic geometry software with my students* and *I have used manipulatives to teach geometrical concepts* as shown in Table 5.18. More teachers than expected who use dynamic geometry software also use manipulatives. It could be implied that respondents may have considered dynamic geometry software packages as sophisticated manipulatives. These results show a stronger relationship then the results when the same variables were cross tabulated in the pilot study as shown in Table 4.11.

	I have used manipulatives to teach geometric concepts		
I have used dynamic geometry software with my students	No	Yes	Total
No	81 (61)	226 (246)	307
Yes	19 (39)	179 (159)	198
Total	100	405	505
Chi squared $=21.36 \quad (p=3.8\times10^{-6})$ Expected frequencies in brackets			

Table 5.18 *Cross tabulation between use of dynamic geometry and manipulatives*

5.4.1.4 Manipulatives and University Degrees

I conducted a statistical test to determine if there was any relationship between use of manipulatives and the type of undergraduate (first degree) or graduate degree the respondents had. When considering the teachers' undergraduate major I divided majors into five groups: business majors (including majors in accounting, finance, marketing, and economics); education (including all education majors except for mathematics education); mathematics (including pure and applied mathematics, mathematics education, actuarial science, statistic majors, and computer science); science (including all science content areas), and other majors, a category that included history, art, psychology etc (see Appendix D for frequencies of undergraduate majors and graduate degrees). This grouping of majors did not produce any statistically significant results.

I used a similar grouping for graduate degrees, adding one further group for respondents without graduate degrees and a second further group for respondents with unspecified graduate degrees. I did not find any statistical significance when working with this grouping.

When I instead used two categories for the undergraduate major: one called-mathematics related undergraduate major (first degree) which included mathematics education, statistics, and computers and the second for any other undergraduate major I found a statistically significant result with respect to use of manipulatives as shown in Table 5.19. We would expect to find 270 respondents from this sample who have mathematics related undergraduate majors (first degrees) to use manipulatives. Actually, 279 reported that they use manipulatives which is a statistically significant difference with $p = 0.0344$.

	I have used manipulatives to teach geometric concepts		
Undergraduate major (first degree)	No	Yes	Total
Mathematics	55 (64)	279 (270)	334
Other	40 (31)	124 (133)	164
Total	95	403	498
Chi-squared = 4.47 ($p = 0.0344$) Expected frequencies in brackets			

Table 5.19 ***Crosstabulation between undergraduate major and manipulatives use***

Similarly I found a statistically significant result when comparing whether respondents had a graduate degree with manipulative use as shown in Table 5.20. There were 316 teachers who have some type of graduate degree and that reported using manipulatives. This was significantly more than the 305 expected respondents with $p = 0.0038$. This result may imply that teachers who attended graduate school might have taken courses that introduced them to manipulatives use that they then incorporated into their practice.

	I have used manipulatives to teach geometric concepts		
I have a graduate degree	No	Yes	Total
No	35 (24)	85 (96)	120
Yes	65 (76)	316 (305)	381
Total	100	401	501
Chi-squared = 8.37 ($p = 0.0038$) Expected frequencies in brackets			

Table 5.20 *Crosstabulation between having a graduate degree and manipulatives use*

5.4.1.5 Manipulatives and Teaching Experience

When I compared the number of years of teaching experience with manipulatives use the use of manipulatives was independent of the teaching experience of the respondents. In other words, there were no statistically significant results. This really surprised me. It was contrary to what I had anticipated.

No matter how the respondents' years of experience were grouped, $p > 0.05$. I thought that respondents with fewer years experience would have been exposed to manipulatives use in their teacher preparation courses. Actually132 new teachers used manipulatives although the expected number of teachers was 139 as shown in Table 5.21.

	I have used manipulatives to teach geometric concepts		
Number of years of teaching experience	No	Yes	Total
0-5	40 (33)	132 (139)	172
6-10	9 (14)	67 (62)	76
11-15	10 (11)	49 (48)	59
16-20	11 (12)	54 (53)	65
21-25	8 (7)	28 (29)	36
26-30	4 (6)	27 (25)	31
Over 30	13 (11)	46 (48)	59
Total	95	403	498
Chi-squared = 6.379 ($p = 0.382$) Expected frequencies in brackets			

Table 5.21 *Crosstabulation between number of years teaching and manipulatives use*

5.4.1.6 Manipulatives and School size

I also wanted to investigate whether the use of manipulatives was linked to the size of the respondents' school. When I applied the chi-squared test to the variables *school size* and *manipulatives use I* did not find a statistically significant relationship between these variables as shown in Table 5.22. The use of manipulatives was independent with respect to school size. This really surprised me. I would have thought that smaller schools might be more likely to have manipulatives available for their teachers to use. Actually 98 respondents who teach in schools with fewer than 1000 students use manipulatives whereas it was expected that 91 respondents would use manipulatives. This was not a statistically significant difference.

	I have used manipulatives to teach geometric concepts		
Number of students in my school	No	Yes	Total
≤ 1000	16 (23)	98 (91)	114
1001-2000	23 (24)	99 (98)	122
2001-3000	20 (21)	87 (86)	107
3001-4000	19 (14)	52 (57)	71
Over 4000	15 (11)	40 (44)	55
Total	93	376	469
Chi-squared = 6.63 (p = 0.1568) Expected frequencies in brackets			

Table 5.22 ***Crosstabulation between number of students in school and manipulatives use***

5.4.1.7 Manipulatives and Type of School

There was statistical significance when comparing the type of school with manipulatives use. Manipulatives are used more than expected in suburban and rural high schools and less than expected in inner city and other types of high schools such as private schools as shown in Table 5.23. According to the data we would expect 254 respondents from inner city schools to use manipulatives, but only 248 reported using manipulatives. We would also expect 82 teachers from suburban schools and 25 teachers from rural school to use manipulatives. There were 90 teachers from suburban schools who reported using manipulatives and 27 from rural schools. This result is significant with $p = 0.025$. We can perhaps say that there are fewer materials available to inner city teachers, which results in the reduced use of manipulatives. The other category included private schools and schools for the gifted where perhaps a more traditional approach is taken when teaching mathematics.

	I have used manipulatives to teach geometric concepts		
Type of high school	No	Yes	Total
Inner city	69 (63)	248 (254)	317
Suburban	12 (20)	90 (82)	109
Rural	4 (6)	27 (25)	31
Other	10 (6)	20 (24)	30
Total	95	385	480
Chi-squared = 9.31 $p = 0.025$ Expected frequencies in brackets			

Table 5.23 ***Crosstabulation between location of school and manipulatives use***

At this point in my analysis I knew there were statistically significant differences between manipulatives use and gender and manipulatives use and type of school. I wanted to know whether male teachers in the suburbs or in rural schools used manipulatives significantly more than urban male teachers. I used log-linear modelling looking at the main effects of manipulatives use, gender, and type of school and their interactions. I did not find any other statistically significant relationships than those I found using chi-squared analysis.

5.4.1.8 Manipulatives and Length of Course

I investigated whether there is a relationship between the way geometry is taught, for instance as part of course or as a year-long course, and the use of manipulatives. I found that when geometry is taught as a one-year course there is a statistically significant relationship as shown in Table 5.24. More respondents (273) than was expected (258) used manipulatives when teaching geometry as a full year course.

Perhaps when more time is devoted to a subject a greater range of approaches can be used to teach the subject?

	I have used manipulatives to teach geometric concepts		
I have taught geometry as a full year course	No	Yes	Total
No	51 (36)	131 (146)	182
Yes	48 (63)	273 (258)	321
Total	99	404	503
Chi-squared = 12.55 ($p = 3.965 \times 10^{-4}$) Expected frequencies in brackets			

Table 5.24 ***Crosstabulation between geometry as a full year course and use of manipulatives***

There was no statistically significant difference with respect to the use of manipulatives when respondents taught geometry as a topic in an integrated curriculum as shown in Table 5.25.

	I have used manipulatives to teach geometric concepts		
I have taught geometry as a topic in an integrated curriculum	No	Yes	Total
No	25 (22)	86 (89)	111
Yes	75 (78)	320 (317)	395
Total	100	406	506
Chi-squared = 0.68 ($p = 0.409$) Expected frequencies in brackets			

Table 5.25 ***Crosstabulation between geometry as a topic in an integrated curriculum and use of manipulatives***

The fact that there was a statistical significance when geometry is taught for a full year and no statistical significance when geometry is part of integrated curriculum may reflect the fact that when taught as part of an integrated curriculum geometry might be considered as a context for algebra and not as a subject in its own right. Teachers use the geometry context to practice algebraic skills.

5.4.1.9 Manipulatives and Undergraduate Courses

I wanted to know what the effects were of having taken an undergraduate geometry course or courses in teaching methods/pedagogy on the use of manipulatives. When I applied a chi-squared test to these variables I found there was statistical significance between taking teaching methods/pedagogy courses and use of manipulatives as shown in Table 5.26.

We would expect 339 respondents who have taken teaching methods /pedagogy courses to use manipulatives. There were 348 respondents who reported using manipulatives. Looking at this from another perspective, we would expect 16 respondents who have not taken teaching methods/pedagogy not to use manipulatives, but in actuality there were 25 respondents who did not use manipulatives. The implication of these results for teacher preparation is extremely important. The pedagogy course may indeed have an influence on whether teachers use manipulatives in their classrooms. This influence could be either positive or negative depending on how good the implementation is.

	I have used manipulatives to teach geometric concepts		
I have taken mathematics methods courses	**No**	**Yes**	**Total**
No	25 (16)	56 (65)	81
Yes	73 (82)	348 (339)	421
Total	98	404	502
Chi-squared = 7.9087 $p = 0.0049$ Expected frequencies in brackets			

Table 5.26 ***Crosstabulation between taking mathematics methods courses and use of manipulatives***

I did not find any statistically significant relationship between taking an undergraduate geometry course and use of manipulatives as shown in Table 5.27. The undergraduate geometry courses that the respondents took may have had little relationship with the geometry that the respondents teach. This was similar to the findings in the pilot study as shown in Table 4.12.

	I have used manipulatives to teach geometric concepts		
I have taken an undergraduate geometry course	No	Yes	Total
No	37 (30)	114 (121)	151
Yes	62 (69)	290 (283)	352
Total	99	404	503
Chi-squared = 3.17 ($p = 0.075$) Expected frequencies in brackets			

Table 5.27 ***Crosstabulation between taking undergraduate geometry courses and use of manipulatives***

Sixteen of the forty-eight Likert type statements were eliminated when factor analysis, a data reduction technique was performed on the questionnaire data (Chapter 6). These 16 statements did not correlate highly with the other 32 statements. I decided to look separately at these sixteen variables and investigate their relationships with some of the personal data variables such as the use of manipulatives and the use of dynamic geometry software.

5.4.1.10 Manipulatives and Spatial Sense

I found statistical significance when I crosstabbed use of manipulatives with statement 5: *developing students' spatial sense is a primary objective of teaching geometry* as shown in Table 5.28. We would expect 368 respondents who use manipulatives to believe that developing a student's spatial sense is a primary goal of geometry. In actuality, 377 respondents who use manipulatives believe the statement with $p = 1.65 \times 10^{-4}$. This implies that teachers who use manipulatives believe that they help to develop spatial awareness.

	I have used manipulatives to teach geometric concepts		
Developing students' spatial sense is a primary objective of geometry	No	Yes	Total
Disagree	18 (9)	25 (34)	43
Agree	82 (91)	377 (368)	459
Total	100	402	502
Chi-squared = 14.19 ($p = 1.65 \times 10^{-4}$) Expected frequencies in brackets.			

Table 5.28 *Crosstabulation between statement 5 and manipulatives use*

5.4.1.11 Manipulatives and Type of Student

We would expect 14 teachers who do not use manipulatives to agree with statement 15: *Geometry should only be taught to very able students* as shown in Table 5.29. In actuality 23 teachers who do not use manipulatives agreed with this statement. This is a significant difference with p = 0.0052. Perhaps if these teachers used manipulatives their beliefs about who should take a geometry course would change. It can be implied that if respondents didn't teach able students who might be abstract thinkers but instead taught more needy students they would recognise a need to use manipulatives to make the geometry more concrete.

	I have used manipulatives to teach geometric concepts		
Geometry should only be taught to very able students	No	Yes	Total
Disagree	77 (86)	356 (347)	433
Agree	23 (14)	49 (58)	72
Total	100	405	505
Chi-squared = 7.80 ($p = 0.0052$) Expected frequencies in brackets			

Table 5.29 *Crosstabulation between statement 15 and manipulatives use*

5.4.1.12 Manipulatives and Beliefs about Dynamic Geometry

More teachers (230) than expected (221) who use manipulatives disagree with statement 25: *Students find dynamic geometry difficult* as shown in Table 5.30. From another perspective, we would expect 26 teachers who did not use manipulatives to agree with this statement. In actuality 35 teachers who did not use manipulatives agreed. Do non-users of manipulatives believe that there are certain manipulatives that confuse the students rather than aid them in their understanding of geometry? This question was not asked directly in the questionnaire but can be implied from Tables 5.10 and 5.11.

	I have used manipulatives to teach geometric concepts		
Students find dynamic geometry difficult	No	Yes	Total
Disagree	34 (43)	230 (221)	264
Agree	35 (26)	126 (135)	161
Total	69	356	425
Chi-squared = 5.77 (p = 0.016) Expected frequencies in brackets			

Table 5.30 ***Crosstabulation between statement 25 and manipulatives use***

There is a statistically significant relationship between familiarity with dynamic geometry software and use of manipulatives as shown in Table 5.31. The number of teachers who used manipulatives and agreed with statement 30: *I am familiar enough with dynamic geometry to use it* (260) was statistically significantly greater than expected (234). There isn't any way of determining whether these teachers equate dynamic geometry software with manipulatives. They may think of dynamic geometry software packages as tools or as sophisticated manipulatives.

	I have used manipulatives to teach geometric concepts		
I am familiar enough with dynamic geometry to use it	No	Yes	Total
Disagree	66 (40)	138 (164)	204
Agree	30 (56)	260 (234)	290
Total	96	398	494
Chi-squared = 37.05 ($p = 1.15 \times 10^{-9}$) Expected frequencies in brackets.			

Table 5.31 ***Crosstabulation between statement 30 and manipulatives use***

Just as I found in the pilot study, there is a statistically significant relationship between use of manipulatives and teachers' belief in statement 8: *Dynamic geometry software packages such as Geometer's Sketchpad or Cabri enable students to enjoy learning geometry*. We would expect 368 respondents that use manipulatives to agree with this statement. Actually, 372 respondents agreed which is significant with $p = 0.013$. Looking at it from another perspective, teachers who don't use manipulatives (7), disagree with statement 8 more than expected (3) as shown in Table 5.32.

	I have used manipulatives to teach geometric concepts		
Dynamic geometry enables students to enjoy learning geometry	No	Yes	Total
Disagree	7 (3)	11 (15)	18
Agree	73 (77)	372 (368)	445
Total	80	383	463
Chi-squared = 6.12 $p = 0.013$ Expected frequencies in brackets.			

Table 5.32 ***Crosstabulation between statement 8 and manipulatives use***

As stated previously there is a statistically significant relationship between use of manipulatives and use of dynamic geometry as shown in Table 5.18. There is also a strong relationship between the use of manipulatives and statement 20: *All high school students should have used dynamic geometry* as shown in Table 5.33. We would expect 325 of the respondents who use manipulatives to agree with this statement. Actually, 341 of the respondents agreed, which is significant with $p = 6.62 \times 10^{-7}$. In the pilot study there was **not** a significant relationship between these two variables as shown in Table 4.19. The wording of the statement was changed from: *All students should have familiarity with dynamic geometry* to *All high school students should have used dynamic geometry.* I don't believe that the change in wording accounted for the statistically significant relationship that resulted in the study. I believe it is the result of the increased sample size with many more teachers who are familiar with dynamic geometry software. We could conclude that manipulative users believe that high school students should use dynamic geometry software.

	I have used manipulatives to teach geometric concepts		
All HS students should have used dynamic geometry	No	Yes	Total
Disagree	31 (15)	48 (64)	79
Agree	61 (77)	341 (325)	402
Total	92	389	481
Chi-squared = 24.72 ($p = 6.62 \times 10^{-7}$) Expected frequencies in brackets.			

Table 5.33 ***Crosstabulation between statement 20 and manipulatives use***

5.4.1.13 Manipulatives and Confident Teachers

There is a statistically significant relationship between teachers who have confidence in teaching geometry and the use of manipulatives. Significantly more teachers (394) than expected (390) who used manipulatives agreed with statement 39: *I am confident about my teaching of geometry* as shown in Table 5.34. Although this is a small difference it is statistically significant with $p = 0.011$.

	I have used manipulatives to teach geometric concepts		
I am confident about my teaching of geometry	No	Yes	Total
Disagree	8 (4)	11 (15)	19
Agree	90 (94)	394 (390)	484
Total	98	405	503
Chi-squared = 6.4 ($p = 0.011$) Expected frequencies in brackets			

Table 5.34 *Crosstabulation between statement 39 and manipulatives use*

5.4.1.14 Manipulatives and Attitude

There is a statistically significant relationship between use of manipulatives and teachers' belief in statement 41: *Studying geometry leads to a positive attitude about mathematics* as shown in Table 5.35. We would expect 336 respondents that use manipulatives to agree with this statement. In actuality, 347 respondents agreed which is statistically significant with $p = 5.796 \times 10^{-4}$.

	I have used manipulatives to teach geometric concepts		
Studying geometry leads to a positive attitude about mathematics	No	Yes	Total
Disagree	27 (16)	54 (65)	81
Agree	70 (81)	347 (336)	417
Total	97	401	498
Chi-squared = 11.84 ($p = 5.796 \times 10^{-4}$) Expected frequencies in brackets.			

Table 5.35 *Crosstabulation between statement 41 and manipulatives use*

5.4.1.15 Manipulatives and Applications

There is a statistically significant relationship between the use of manipulatives and teachers' belief in statement 47: *Applying geometrical concepts and thinking will help students in their future occupations* as shown in Table 5.36. We would expect 371 of the respondents who use manipulatives to agree with this statement. Actually, 379 respondents agreed which is significant with $p = 8.81 \times 10^{-4}$.

From another perspective there were 15 respondents who do not use manipulatives who disagreed with this statement. It was expected that only 7 respondents would disagree with this statement. There is an implication here that teachers who do not use manipulatives do not necessarily believe that their students will ever need geometry later in their lives.

	I have used manipulatives to teach geometric concepts		
Applying geometrical concepts and thinking will help students in their future occupations	No	Yes	Total
Disagree	15 (7)	22 (30)	37
Agree	83 (91)	379 (371)	462
Total	98	401	499
Chi-squared = 11.06 ($p = 8.81 \times 10^{-4}$) Expected frequencies in brackets			

Table 5.36 ***Crosstabulation between statement 47 and manipulatives use***

There were statistically significant relationships between the use of geometry and teachers' beliefs about its real world applications. We would expect 389 of the respondents who use manipulatives to agree with statement 18: *Geometry has many real world applications*. Actually, 394 of the respondents agreed which is significant with $p = 0.0016$ as shown in Table 5.37.

	I have used manipulatives to teach geometric concepts		
Geometry has many real world applications	No	Yes	Total
Disagree	8 (3)	8 (13)	16
Agree	89 (94)	394 (389)	483
Total	97	402	499
Chi-squared = 9.86 ($p = 0.0016$) Expected frequencies in brackets.			

Table 5.37 ***Crosstabulation between statement 18 and manipulatives use***

Similarly there is a statistically significant relationship between the use of geometry and teachers' belief about statement 42: *When teaching geometry connections to the real world should be made* as shown in Table 5.38. We would expect 398 of the respondents that use manipulatives to agree with this statement. Actually, 401 of the respondents agreed which is significant with $p = 0.015$.

	I have used manipulatives to teach geometric concepts		
When teaching geometry connections to the real world should be made	**No**	**Yes**	**Total**
Disagree	5 (2)	5 (8)	10
Agree	95 (98)	401 (398)	496
Total	100	406	506
Chi-squared = 5.88 ($p = 0.015$) Expected frequencies in brackets.			

Table 5.38 ***Crosstabulation between statement 42 and manipulatives use***

The relationship between teachers who use manipulatives and the belief that geometry has many real world applications is stronger than the relationship between teachers who use manipulatives and the belief that when teaching geometry the real world connections should be made. Are connections to real applications being made by teachers whenever possible?

5.4.1.16 Manipulatives and Geometry in the Curriculum

There is a statistically significant relationship between the use of manipulatives and teachers' belief in statement 9: *Geometry should occupy a significant place in the curriculum* as shown in Table 5.39. We would expect 377 respondents who use manipulatives to agree with this statement. Actually, 385 respondents agreed which is statistically significant with $p = 1.4 \times 10^{-4}$. From a different perspective, we would expect 7 respondents who did not use manipulatives to disagree with the statement. Actually there were 15 teachers who did not use manipulatives and don't agree with statement 9. Perhaps through the use of manipulatives teachers can come to understand the role that geometry plays in the curriculum.

	I have used manipulatives to teach geometric concepts		
Geometry should occupy a significant place in the curriculum	No	Yes	Total
Disagree	15 (7)	18 (26)	33
Agree	85 (93)	385 (377)	470
Total	98	403	503
Chi-squared = 14.5 ($p = 1.4\times10^{-4}$) Expected frequencies in brackets.			

Table 5.39 ***Crosstabulation between statement 9 and manipulatives use***

There is a statistically significant relationship between the use of manipulatives and teachers' belief in statement 22: *High School students should experience other geometries besides Euclidean* as shown in Table 5.40. We would expect 325 of the respondents who use manipulatives to agree with this statement. Actually, 332 of the respondents agreed which is significant with $p = 0.031$.

Teachers who use manipulatives might use them to investigate properties in other geometries. For example, they might use spheres to investigate spherical geometry. Teachers who do not use manipulatives do not have the means to make other geometries accessible to most high school students.

	I have used manipulatives to teach geometric concepts		
HS students should experience other geometries besides Euclidean	No	Yes	Total
Disagree	26 (19)	69 (76)	95
Agree	71 (78)	332 (325)	403
Total	97	401	498
Chi-squared = 4.66 ($p = 0.031$) Expected frequencies in brackets			

Table 5.40 ***Crosstabulation between statement 22 and manipulatives use***

There is a statistically significant relationship between the use of manipulatives and teachers' belief in statement 43: *Students can experience the activities of mathematicians through their work in geometry class* as shown in Table 5.41. We would expect 366 of the respondents who use manipulatives to agree with this statement. Actually, 373 of the respondents agreed which is significant with $p = 0.0029$. When students do investigations using manipulatives they are exploring the various conjectures they have made. They are able to verify which conjectures might be true and which are false, similar to mathematicians trying to verify their conjectures.

	I have used manipulatives to teach geometric concepts		
Students can experience the activities of mathematicians through their work in geometry class	No	Yes	Total
Disagree	15 (8)	25 (32)	40
Agree	82 (89)	373 (366)	455
Total	97	398	495
Chi-squared = 8.85 ($p = 0.0029$) Expected frequencies in brackets.			

Table 5.41 ***Crosstabulation between statement 43 and manipulatives use***

There is a statistically significant relationship between the use of manipulatives and teachers' belief in statement 45: *Geometry enables ideas from other areas to be pictured* as shown in Table 5.42. We would expect 380 of the respondents who use manipulatives to agree with this statement. Actually, 385 respondents agreed which is significant with $p = 0.0025$. Teachers who use manipulatives can demonstrate concepts or have their students investigate concepts from other areas of mathematics such as algebra and calculus.

	I have used manipulatives to teach geometric concepts		
Geometry enables ideas from other areas to be pictured	No	Yes	Total
Disagree	9 (4)	11 (16)	20
Agree	84 (89)	385 (380)	469
Total	93	396	489
Chi-squared = 9.14 ($p = 0.0025$) Expected frequencies in brackets.			

Table 5.42 ***Crosstabulation between statement 45 and manipulatives use***

5.4.1.17 Conclusions about Manipulatives

In this section I have discussed many of the relationships between the use of manipulatives and other variables. Some of the relationships were not as surprising as others such as the statistically significant relationship between the use of manipulatives and the belief that manipulatives make learning geometry fun.

Other relationships were surprising such as the relationships between manipulative use and school size or manipulatives' use and teaching experience. I was amazed to find that manipulatives are used or not used with the same frequency no matter what the school size is and no matter how long the teacher has been teaching.

There are findings that impact teacher education such as a need for undergraduate pedagogy courses where future teachers can become familiar with manipulatives. There is also a need to make explicit for future teachers the relationships between undergraduate geometry courses they take and the high school geometry they eventually will teach.

Since there is a statistically significant relationship between having a graduate degree and use of manipulatives it makes sense for school policy makers to require that all their teachers obtain graduate degrees if they want to encourage the use of manipulatives.

The finding that urban teachers use manipulatives significantly less than their suburban counterparts should encourage advocates to try to obtain funds for schools to purchase manipulatives or to provide professional development to instruct teachers in the best ways to use manipulatives since research has shown that use of manipulatives can improve students' understanding of mathematics (Fuys, Geddes, and Tischler, 1988; Mason, 1989; Moyer, 2001).

5.4.2 Findings About The Use Of Dynamic Geometry

Vagn Lundsgaard Hansen (1998) and Jiang (2002) believed that dynamic geometry can enhance the teaching and learning of most topics in geometry. Do the questionnaire respondents agree? There were 507 responses to the personal data statement: *I have used dynamic geometry software with my students.* 39% of these respondents have used dynamic geometry software and 61% have not as shown in Table 5.43.

I have used a dynamic geometry software package with my students	**Frequency**	**Percent**
No	309	59.4
Yes	198	38.1
No Response	13	2.5
Total	520	100

Table 5.43 ***Respondents' use of Dynamic Software***

The 48 Likert statements on the questionnaire included six statements about dynamic geometry software as shown in Table 5.44.

Questionnaire Statements	**A**	**D**
8. Dynamic geometry software packages enable students to enjoy learning geometry	88.1%	3.4%
20. Ideally, all high school students should have used dynamic geometry software	79.6%	15.4%
25. Students find it difficult to use dynamic geometry software	32.6%	51.6%
27. More interesting geometrical problems can be explored with dynamic geometry than without it	79.5%	8.8%
30. I am familiar enough with dynamic geometry to use it	57.7%	39.9%
38. Dynamic geometry can take the place of rigorous proofs	41.5%	50.0%

Table 5.44 ***Statements about dynamic geometry software on the Geometry Beliefs Questionnaire***

Just as I did with the responses about the use of manipulatives, for analysis purposes I grouped responses *strongly disagree*, *moderately disagree*, and *disagree slightly more than agree* into a single response-*disagree*. Similarly, I grouped *strongly agree*, *moderately agree*, and *agree slightly more than disagree* into a single response- *agree*.

As shown in Table 5.6 there was consensus on three of the statements: statement 8: *Dynamic geometry software packages enable students to enjoy learning geometry*; statement 20: *Ideally, all high school students should have used dynamic geometry software*; and statement 27: *More interesting geometrical problems can be explored with dynamic geometry than without it.*

As shown in Table 5.6 there was no consensus on the remaining three statements: statement 25: *Students find it difficult to use dynamic geometry* software; statement 30: *I am familiar enough with dynamic geometry to use it*; and statement 38: *Dynamic geometry can take the place of rigorous proofs.*

The lack of consensus on half of the statements about dynamic geometry made me curious as to why this was so.

In order to determine whether there were any relationships between the variables I used the Chi-squared statistical test. I crosstabbulated the six Likert statements as shown in Table 5.44 with the statement from the personal data section: *I have used dynamic geometry software with my students*. Each of the Tables 5.45-5.50 contains the observed frequencies and their totals. The expected frequencies for each cell, rounded to the nearest whole number, are in brackets. I found statistically significant results for each of the statements except for statements 8 and 27.

There was no statistically significant relationship between teachers who have used dynamic software with their students and teachers' belief about statement 8: *Dynamic* *geometry software packages enable students to enjoy learning geometry* as shown in Table 5.45. The belief that dynamic geometry software packages enable students to enjoy learning geometry is independent of whether respondents use dynamic geometry software with their students or not. Teachers may not be using the software because it is unavailable to them at their schools. This is possibly an equity issue where the wealthier schools buy software licenses but the poorer schools do not have the funds necessary for a site license.

	I have used dynamic geometry software with my students		
Dynamic geometry software packages enable students to enjoy learning geometry	No	Yes	Total
Disagree	14 (10)	4 (7)	18
Agree	252 (256)	194 (190)	446
Total	266	198	464
Chi-squared = 3.20 ($p = 0.07$) Expected frequencies in brackets.			

Table 5.45 ***Crosstabulation between statement 8 and dynamic geometry use***

There is a statistically significant relationship between the use dynamic geometry software with students and statement 20: *Ideally, all high school students should have used dynamic geometry software* as shown in Table 5.46. More teachers who use dynamic geometry software with their students believe statement 20 than would be expected. There were 223 respondents who have not used dynamic geometry software with their students but believe that students should use this software. This was less than the expected number of 238 respondents. Teachers who use dynamic geometry software believe that students should use it. There were 62 respondents that do not use dynamic geometry with their students and that do not believe their students should use dynamic geometry software. I think this may reflect lack of knowledge of the software by some of the respondents.

	I have used dynamic geometry software with my students		
Students should use dynamic geometry	No	Yes	Total
Disagree	62 (47)	17 (32)	79
Agree	223 (238)	180 (165)	403
Total	285	197	482
Chi-squared = 14.64 ($p = 1.299 \times 10^{-4}$) Expected frequencies in brackets.			

Table 5.46 ***Crosstabulation between statement 20 and dynamic geometry use***

There is a statistically significant relationship between the use of dynamic geometry software and statement 25: *Students find it difficult to use dynamic geometry software* as shown in Table 5.47. More teachers who had not used dynamic geometry agreed with statement 25 than would be expected. Could these teachers be projecting their own reasons for not using dynamic geometry onto their students? Why do they believe that students find dynamic geometry difficult to use? Is this belief pervasive among teachers?

	I have used dynamic geometry software with my students		
Students find dynamic geometry difficult to use	No	Yes	Total
Disagree	120 (143)	145 (122)	265
Agree	111 (88)	51 (74)	162
Total	231	196	427
Chi-squared = 21.86 ($p = 2.93 \times 10^{-6}$) Expected frequencies in brackets.			

Table 5.47 ***Crosstabulation between statement 25 and dynamic geometry use***

I found that there was **not** a statistically significant relationship between use of dynamic geometry software and statement 27: *More interesting geometrical problems can be explored with dynamic geometry than without it* as shown in Table 5.48. The result in the pilot study was similar as shown in Table 4.22. This result is surprising in that I would have expected that teachers who use dynamic geometry software would find significantly more interesting problems to explore with the software. What type of investigations are teachers doing with dynamic geometry software?

	I have used dynamic geometry software with my students		
More interesting problems can be explored with dynamic geometry than without it	No	Yes	Total
Disagree	29 (26)	17 (20)	46
Agree	223 (226)	179 (176)	402
Total	252	196	448
Chi-squared = .96 ($p = 0.33$) Expected frequencies in brackets.			

Table 5.48 ***Crosstabulation between statement 27 and dynamic geometry use***

When the chi-squared test was applied to the statements *I am familiar with dynamic geometry* and *I have used dynamic geometry software with my students* I found a statistically significant relationship as shown in Table 5.49. There were 109 teachers who were familiar with dynamic geometry but who have not used it with their students. This could be the result of unavailability of dynamic geometry software licenses in many high schools. There were 16 respondents who used dynamic geometry software with their students without being familiar enough with it- that could make using dynamic geometry software not enjoyable for students. The very big difference between observed and expected frequencies is the reason for the large chi-square value and very small p value.

	I have used dynamic geometry software with my students		
I am familiar enough with dynamic geometry to use it	No	Yes	Total
Disagree	188 (122)	16 (82)	204
Agree	109 (175)	182 (116)	291
Total	297	198	495
Chi-squared = 149.51 $(p = 2.22 \times 10^{-34})$ Expected frequencies in brackets.			

Table 5.49 ***Crosstabulation between statement 30 and dynamic geometry use***

There is a statistically significant relationship between the use of dynamic geometry software and statement 38: *Dynamic geometry can take the place of rigorous proofs* as shown in Table 5.50. More teachers than expected, who use dynamic geometry software, believe dynamic geometry can take the place of rigorous proof. In what ways do teachers believe dynamic geometry can take the place of rigorous proof?

	I have used dynamic geometry software with my students		
Dynamic geometry can take the place of rigorous proof	No	Yes	Total
Disagree	162 (148)	95 (109)	257
Agree	106 (120)	102 (88)	208
Total	268	197	465
Chi-squared = 6.86 $(p = 0.0088)$ Expected frequencies in brackets.			

Table 5.50 ***Crosstabulation between statement 38 and dynamic geometry use***

5.4.2.1 Dynamic Geometry and Gender

Unlike with the use of manipulatives there was no statistically significant difference between gender and use of dynamic geometry as shown in Table 5.51. This result may answer an earlier question as to whether teachers think of dynamic geometry software as a type of manipulative. This result suggests that the answer is no since there are statistically significant gender differences with the use of manipulatives but not with the use of dynamic geometry.

	I have used dynamic geometry software with my students		
Gender	No	Yes	Total
Female	160 (163)	107 (104)	267
Male	148 (140)	90 (93)	238
Total	308	197	505
Chi-squared = .27 ($p = 0.60$) Expected frequencies in brackets.			

Table 5.51 ***Crosstabulation between gender and dynamic geometry use***

5.4.2.2 Dynamic Geometry and Professional Organisations

Similar to the findings in Table 5.16 about the use of manipulatives, there is a statistically significant relationship between membership of professional organisations and the use of dynamic geometry as shown in Table 5.52. The expected frequency for members in professional organisations to use dynamic geometry is 89, but the actual number is 124. Is it the professional organisation promoting dynamic geometry usage or is it that teachers who are more likely to use dynamic geometry become members of professional organisations?

	I have used dynamic geometry software with my students		
I am a member of NCTM, ATM etc.	No	Yes	Total
No	203 (168)	71 (106)	274
Yes	105 (140)	124 (89)	229
Total	308	195	503
Chi-squared = 41.896 ($p = 9.62 \times 10^{-11}$) Expected frequencies in brackets.			

Table 5.52 ***Crosstabulation between membership of professional organisation and dynamic geometry use***

There is also a statistically significant relationship between attendance at professional meetings and use of dynamic geometry as shown in Table 5.53.

	I have used dynamic geometry software with my students		
I have attended at least 2 NCTM/ATM meetings	No	Yes	Total
No	245 (211)	95 (129)	340
Yes	63 (97)	94 (60)	157
Total	308	189	497
Chi-squared = 46.469 ($p = 9.31 \times 10^{-12}$) Expected frequencies in brackets.			

Table 5.53 ***Crosstabulation between attendance at professional meetings and dynamic geometry use***

Do teachers who attend profession meetings have more access to dynamic geometry software because they come from wealthier schools that have computer laboratories? Or does attendance at professional meetings encourage teachers to use dynamic geometry?

5.4.2.3 Dynamic Geometry and University Degrees

There was no statistical significance between respondents' use of dynamic geometry software and the respondents' undergraduate major/first degree as shown in Table 5.54 or whether a respondent had a graduate degree as shown in Table 5.55. Again this result bears evidence to the fact that respondents were not thinking of dynamic geometry software as a manipulative, since there were statistically significant results when the use of manipulatives were crosstabulated with university degrees as shown in Tables 5.19 and 5.20.

	I have used dynamic geometry software with my students		
Undergraduate major (first degree)	No	Yes	Total
Mathematics	202 (206)	134 (130)	336
Other	103 (99)	58 (62)	161
Total	305	192	497
Chi-squared = .68 ($p = 0.4087$) Expected frequency in brackets			

Table 5.54 ***Crosstabulation between undergraduate major and use of dynamic geometry***

Whether a respondent used or did not use dynamic geometry software with their students was independent of whether or not they had a graduate degree as shown in Table 5.55.

	I have used dynamic geometry software with my students		
I have a graduate degree	No	Yes	Total
No	78 (74)	42 (46)	120
Yes	231 (235)	151 (147)	382
Total	309	193	502
Chi-squared = .7914 (p = 0.3737) Expected frequencies in brackets.			

Table 5.55 ***Crosstabulation between having a graduate degree and use of dynamic geometry***

5.4.2.4 Dynamic Geometry and Teaching Experience

There was a statistically significant relationship between respondents' use of dynamic geometry and their teaching experience. Teachers with between 11-15 and 26-30 years experience used dynamic geometry significantly more than expected. New teachers, with five or fewer years of teaching experience, used dynamic geometry significantly less than expected as shown in Table 5.56. This is a surprising result. One would expect a new teacher coming out of a teacher preparation program to use dynamic geometry software with their students. There may be several factors at play here: new teachers may be getting jobs in needy schools where there are no updated computer laboratories or maybe the teacher preparation courses are not successful in promoting the use of dynamic geometry. If the reason is the latter than this result informs teacher preparation programs about what might be happening with their most recent graduates.

	I have used dynamic geometry software with my students		
Number of years teaching	No	Yes	Total
0-5	114 (104)	57 (67)	171
6-10	48 (46)	28 (30)	76
11-15	25 (36)	34 (23)	59
16-20	41 (40)	25 (26)	66
21-25	26 (23)	11 (14)	37
26-30	13 (19)	18 (12)	31
Over 30	36 (35)	21 (22)	97
Total	303	194	497
Chi-squared = 17.275 ($p = 0.0083$) Expected frequencies in brackets.			

Table 5.56 ***Crosstabulation between number of years teaching and use of dynamic geometry***

5.4.2.5 Dynamic Geometry and Type of School

There was a statistically significant relationship between respondents' use of dynamic geometry and the type of school in which they taught. Respondents who taught in suburban and rural schools used dynamic geometry with their students more than expected and inner city respondents used dynamic geometry significantly less than expected as shown in Table 5.57. As stated previously suburban schools may have updated computer laboratories and access to software packages whereas inner city schools may be overcrowded and lack funding for software licenses.

	I have used dynamic geometry software with my students		
Type of high school	No	Yes	Total
Inner city	238 (196)	79 (121)	317
Suburban	36 (63)	66 (39)	102
Rural	11 (19)	20 (12)	31
Private	10 (14)	12 (8)	22
Other	3 (6)	6 (3)	9
Total	298	183	481
Chi-squared = 68.81 ($p = 4.05 \times 10^{-14}$) Expected frequencies in brackets.			

Table 5.57 ***Crosstabulation between location of school and use of dynamic geometry***

5.4.2.6 Dynamic Geometry and School Size

I wanted to investigate whether school size had an effect on the use of dynamic geometry. When I crosstabbulated the number of students in school with respondents' use of dynamic geometry software I obtained statistically significant results as shown in Table 5.58. There is a relationship between school size and use of dynamic geometry software. In smaller schools (≤ 2000 students) more teachers used dynamic geometry than expected. In schools having between two and three thousand students the number of teachers using dynamic geometry was about what was expected. In larger schools (> 3000 students) a significantly smaller number of teachers used dynamic geometry than was expected. This could be due to the fact that schools with large numbers of students lack the space for computer laboratories or that there might be problems with classroom management. Again this result differs from an earlier finding reported in Table 5.22 that there was no statistical significance between school size and the use of manipulatives.

	I have used dynamic geometry software with my students		
Number of students in my school	No	Yes	Total
≤ 1000	56 (70)	58 (44)	114
1001-2000	63 (75)	59 (47)	122
2001-3000	67 (66)	41 (42)	108
3001-4000	53 (44)	18 (27)	71
Over 4000	49 (34)	6 (21)	55
Total	288	182	470
Chi-squared = 35.18 ($p = 4.266 \times 10^{-7}$) Expected frequencies in brackets.			

Table 5.58 ***Crosstabulation between number of students in respondents' school and the use of dynamic geometry***

5.4.2.7 Dynamic Geometry and Length of Course

I investigated whether there is a relationship between the way geometry is taught, for instance as part of course or as a year-long course, and the use of dynamic geometry software. I found that when geometry is taught either as a one-year course or as part of an integrated course there are statistically significant relationships. More respondents than were expected used dynamic geometry software when teaching geometry as a full year course as shown in Table 5.59 but fewer respondents than expected used dynamic geometry when teaching geometry as a topic in an integrated course as shown in Table 5.60. When geometry is taught as a topic in an integrated curriculum there is less time to incorporate dynamic geometry software. New York State is about to reinstate geometry as a full year course instead of as part of an integrated curriculum as it has been for over twenty years. They are also starting to provide site licenses with the goal that eventually every high school will have one.

	I have used dynamic geometry software with my students		
I have taught geometry as a full year course	No	Yes	Total
No	129 (111)	52 (70)	181
Yes	180 (198)	143 (125)	323
Total	309	197	406
Chi-squared = 11.81 $(p = 5.88 \times 10^{-4})$ Expected frequencies in brackets.			

Table 5.59 ***Crosstabulation between geometry as a full year course and the use of dynamic geometry software***

There was no significant relationship between these variables in the pilot study as shown in Table 4.13.

	I have used dynamic geometry software with my students		
Geometry taught as a topic in an integrated curriculum	No	Yes	Total
No	52 (68)	60 (44)	112
Yes	256 (240)	138 (154)	394
Total	308	198	506
Chi-squared = 12.59 ($p = 3.87 \times 10^{-4}$) Expected frequencies in brackets.			

Table 5.60 ***Crosstabulation between geometry as a topic in an integrated course and the use of dynamic software***

5.4.2.8 Dynamic Geometry and Undergraduate Courses

I wanted to know whether having taken an undergraduate geometry course or courses in mathematical methods/pedagogy had an impact on the use of dynamic geometry. When I crosstabulated these variables I found there was statistical significance between those taking methods courses and the use of dynamic geometry as shown in Table 5.61, but there was no statistically significant relationship between taking an undergraduate geometry course and the use of dynamic geometry as shown in Table 5.62.

	I have used dynamic geometry software with my students		
I have taken mathematics methods courses	No	Yes	Total
No	64 (49)	17 (32)	81
Yes	243 (258)	179 (164)	422
Total	307	196	503
Chi-squared = 13.12 ($p = 2.92 \times 10^{-4}$) Expected frequencies in brackets			

Table 5.61 ***Crosstabulation between taking mathematics methods/pedagogy courses and the use of dynamic software***

	I have used dynamic geometry software with my students		
I have taken an undergraduate geometry course	No	Yes	Total
No	100 (92)	51 (59)	151
Yes	207 (215)	146 (138)	353
Total	307	197	504
Chi-squared = 2.56 (p = 0.1099) Expected frequencies in brackets.			

Table 5.62 ***Crosstabulation between taking an undergraduate geometry course and the use of dynamic software***

Is dynamic geometry software used or even mentioned in undergraduate geometry courses that are not tied to pedagogy courses? This result can also inform teacher preparation programs that university content course may not make the necessary technological connections.

When factor analysis was performed on the questionnaire (Chapter 6) sixteen of the forty-eight Likert type statements were eliminated since they did not load strongly on the factors extracted. I decided to look separately at these sixteen variables and investigate their relationships to the use of dynamic geometry software.

5.4.2.9 Dynamic Geometry and Enjoyment of Teaching Geometry

When I crosstabulated statement 1: *I enjoyed teaching geometry* with the use of manipulatives there was no statistically significant relationship. Similarly with statements 7: *There are some things in geometry, like proofs that are best memorised* and statement 28: *geometry is an exercise in memorisation.* There was a statistically significant relationship between each of these three statements and respondents' use of dynamic geometry with their students.

Respondents who use dynamic geometry, enjoy teaching geometry significantly more than those that do not use it as shown in Table 5.63.

	I have used dynamic geometry software with my students		
I enjoy teaching geometry	No	Yes	Total
Disagree	21 (15)	4 (10)	25
Agree	285 (291)	194 (188)	479
Total	306	198	504
Chi-squared = 5.98 ($p = 0.014$) Expected frequencies in brackets.			

Table 5.63 ***Crosstabulation between statement 1 and the use of dynamic geometry software***

5.4.2.10 Dynamic Geometry and Memorisation

Fewer respondents than expected that used dynamic geometry with their students believed that some things in geometry are best memorised as shown in Table 5.64.

	I have used dynamic geometry software with my students		
Some thing in geometry like proofs are best memorised	No	Yes	Total
Disagree	184 (202)	147 (129)	331
Agree	123 (105)	50 (68)	173
Total	307	197	504
Chi-squared = 11.48 ($p = 7.04 \times 10^{-4}$) Expected frequencies in brackets.			

Table 5.64 ***Crosstabulation between statement 7 and the use of dynamic geometry software***

Fewer respondents than expected that used dynamic geometry with their students believed that geometry is an exercise in memorisation as shown in Table 5.65.

	I have used dynamic geometry software with my students		
Geometry is an exercise in memorisation	No	Yes	Total
Disagree	240 (253)	177 (164)	417
Agree	64 (51)	20 (33)	84
Total	304	197	501
Chi-squared = 10.18 ($p = 0.0014$) Expected frequencies in brackets.			

Table 5.65 ***Crosstabulation between statement 28 and the use of dynamic geometry software***

The results in Tables 5.64 and 5.65 give testimony to the belief that when students use dynamic geometry software there is less reliance on memorisation and more reliance on understanding.

5.4.2.11 Conclusions about Dynamic Geometry Software

In this section I have discussed many of the relationships between the use of dynamic geometry and other variables.

There are findings that impact teacher education such as a need for undergraduate pedagogy courses where future teachers can become familiar with dynamic geometry software. There is also a need to make explicit for future teachers the relationships between undergraduate geometry courses and high school geometry.

In this section we have answered the question of whether the statistically significant relationship between the use of dynamic geometry software and the use of manipulatives is due to the fact that teachers consider dynamic geometry to be a sophisticated manipulative? We have seen that the responses to some dynamic geometry questions have been significantly different from responses to manipulative questions. Teachers do not believe that dynamic geometry software packages are sophisticated manipulatives.

5.4.3 Findings About Respondents' Beliefs Regarding Proofs

The questionnaire contained 17 statements about proof: 14 explicit statements and 3 implicit statements as shown in Table 5.66.

Questionnaire Statements	A	D
4. Learning to construct proofs is important for High School students	86.7%	4.8%
7. There are some things in geometry like proofs that are best memorised	34.5%	64.8%
10. High school geometry should not contain proof	23.1%	76.9%
13. HS students should be able to write rigorous proofs in geometry	62.3%	37.4%
16. My students enjoy doing geometric proofs	33.4%	60.6%
21. I enjoy doing geometric proofs	88%	11.2%
29. Initially, HS geometry should be hands-on with proofs coming later in the course	75.5%	23%
31. HS students should discover theorems in geometry	88.9%	10.8%
32. It is unnecessary for students to prove theorems they regard as obvious	32.9%	66.4%
33. Geometry is where students can validate conjectures using deductions	94%	3.9%
34. More time should be spent on analytic geometry and other topics in geometry rather than on proving	67.5%	31.1%
35. Proofs written in paragraph form are acceptable	87.9%	10.4%
36. A main goal of geometry is to teach students how to reason	92.9%	6.6%
37. If a student makes a conjecture about a geometrical idea that is not in the curriculum, the teacher should allow the class time to prove or disprove the conjecture	93.2%	5.4%
38. Dynamic geometry can take the place of rigorous proofs	41.5%	50%
44. I enjoy teaching my students how to do geometric proofs	78.8%	17.5%
48. I enjoy proving theorems for my students	80.2%	17%

Table 5.66 ***Statements about proof on the Geometry Beliefs Questionnaire***

Just as I did with the responses about both the use of manipulatives and dynamic geometry, for analysis purposes I grouped responses *strongly disagree*, *moderately disagree*, and *disagree slightly more than agree* into a single response - *disagree*. Similarly, I grouped *strongly agree*, *moderately agree*, and *agree slightly more than disagree* into a single response - *agree*.

In the pilot study 72.5% of the respondents disagreed with the statement that high school geometry should not contain proofs. In this study 76.9% of the respondents disagreed (See Table 5.66, statement 15). Also, in the pilot study 82.5% of the respondents enjoyed teaching geometrical proofs. In this study 78.8% of the respondents enjoy teaching geometrical proofs (See Table 5.66, statement 44). In the pilot 82.5% of the respondents enjoyed doing mathematical proofs. In this study I changed the statement to be more specific: *I enjoy doing geometrical proofs* (See Table 5.66, Statement 21). This yielded 88% agreement among respondents. In this study I added statement: *I enjoy proving theorems for my students.* The reason for adding this statement was to try to distinguish between respondents' enjoyment of doing proofs for themselves and for their students and for having to teach their students how to do proofs. There was 80.2% agreement with this statement. The fact that there were different responses to these statements leads me to conclude that the respondents recognised the differences in the statements and responded accordingly.

I cross tabulated the 17 Likert type statements about proof with the respondents' personal data information: The impact of their gender, their teaching experience, the type of school in which they teach, their undergraduate major/first degree, whether they had a graduate degree, whether they took an undergraduate geometry course, whether they took mathematics methods/pedagogy courses, whether they used manipulatives, whether they used dynamic geometry, whether they taught geometry as a full year course and whether they taught geometry as an integrated course on their attitudes was investigated. I have reported the statistically significant results in Tables 5.67-5.76.

There were two statistically significant results for statement 4: *Learning to construct proofs is important for high school students.* Significantly more respondents than expected who have taught geometry as a year-long course agree with this statement as shown in Table 5.67. Similarly significantly more respondents that had mathematics related undergraduate/first degree agreed with statement 4 as shown in Table 5.68.

	I have taught geometry as a full year course		
Learning to construct proofs is important for HS students	No	Yes	Total
Disagree	30 (22)	31 (39)	61
Agree	150 (158)	291 (283)	441
Total	180	322	502
Chi-squared = 5.36 (p = 0.021) Expected frequencies in brackets.			

Table 5.67 *Crosstabulation between statement 4 and teaching geometry as a full year course*

	Undergraduate major/First Degree		
Learning to construct proofs is important for HS students	Mathematics related	Other	Total
Disagree	33 (41)	28 (20)	61
Agree	303 (295)	132 (140)	435
Total	336	160	496
Chi-squared = 5.92 (p = 0.015) Expected frequencies in brackets.			

Table 5.68 *Crosstabulation between statement 4 and respondents' undergraduate major*

When I crosstabulated statement 7: *There are some things in geometry like proofs that are best memorised*, I found three statistically significant relationships. More respondents than expected that use dynamic geometry with their students disagreed with statement 7 as shown in Table 5.69. If one believes heavily in memorisation one is less likely to use dynamic geometry software as shown in Table 5.65. More respondents than expected that are members of professional organisations and that have attended professional meetings disagreed with statement 7 as shown in Tables 5.70 and 5.71.

	I have used dynamic geometry software with my students		
There are some things in geometry like proofs that are best memorised	No	Yes	Total
Disagree	184 (202)	147 (129)	331
Agree	123 (105)	50 (68)	173
Total	307	197	504
Chi-squared = 11.48 ($p = 7.04 \times 10^{-4}$) Expected frequencies in brackets.			

Table 5.69 ***Crosstabulation between statement 7 and use of dynamic geometry software***

	I am a member of NCTM, ATM (etc.)		
There are some things in geometry like proofs that are best memorised	No	Yes	Total
Disagree	165 (178)	164 (151)	329
Agree	106 (93)	65 (78)	171
Total	271	229	500
Chi-squared = 6.35 ($p = 0.012$) Expected frequencies in brackets.			

Table 5.70 ***Crosstabulation between statement 7 and membership of professional organisation***

	I have attended at least 2 NCTM meetings		
There are some things in geometry like proofs that are best memorised	No	Yes	Total
Disagree	208 (221)	116 (103)	324
Agree	129 (116)	41 (54)	170
Total	337	157	494
Chi-squared = 7.022 ($p = 0.008$) Expected frequencies in brackets.			

Table 5.71 ***Crosstabulation between statement 7 and attendance at professional meetings***

I found no statistically significant relations between statement 10: *High school geometry should not contain proof* and the respondents' personal data.

I found one rather unusual statistically significant relationship between statement 13: *High School students should be able to write rigorous proofs in geometry* and respondents that have taught geometry as a topic in an integrated curriculum. More respondents than expected who have taught geometry as part of integrated curriculum believe statement 13. There wasn't a statistically significant relationship between this statement and respondents who have taught geometry as a full year course. I have no explanation for this. I would have expected the opposite since when geometry is just a topic in a curriculum I would assume there would be less time for rigorous proof.

I found a statistically significant relationship between statement 16: *My students enjoy doing geometric proofs* and membership of professional organisations. More respondents than expected that are members of professional organisations believe that their students enjoy doing geometric proofs. Does this imply that members of professional organisations have 'more tricks of the trade' so to speak to make learning how to do geometric proofs enjoyable?

	I am a member of NCTM, ATM (etc.)		
My students enjoy doing geometric proofs	No	Yes	Total
Disagree	176 (165)	129 (140)	305
Agree	81 (92)	88 (77)	169
Total	257	217	474
Chi-squared = 4.19 ($p = 0.04$) Expected frequencies in brackets.			

Table 5.72 ***Crosstabulation between statement 16 and membership of professional organisation***

Significantly more respondents than expected who have taught geometry as a year-long course agree with statement 21: *I enjoy doing geometric proofs* as shown in Table 5.73. Respondents who teach geometry as a year-long course might prefer teaching geometry to teaching other mathematics topics: one of the reasons they might enjoy teaching geometry is because they enjoy doing geometric proofs.

	I have taught geometry as a full year course		
I enjoy doing geometric proofs	No	Yes	Total
Disagree	28 (20)	28 (36)	56
Agree	153 (161)	292 (284)	445
Total	181	320	505
Chi-squared = 5.26 ($p = 0.022$) Expected frequencies in brackets.			

Table 5.73 ***Crosstabulation between statement 21 and teaching geometry as a full year course***

Significantly more respondents than expected who have taught geometry as a year-long course agree with statement 44: *I enjoy teaching my students how to do geometric proofs* as shown in Table 5.74. There is more time in a year long course to actually teach students how to do geometric proofs which may explain this finding.

	I have taught geometry as a full year course		
I enjoy teaching my students how to do geometric proofs	No	Yes	Total
Disagree	44 (31)	45 (58)	89
Agree	126 (139)	271 (258)	397
Total	170	316	486
Chi-squared = 10.01 ($p = 0.0016$) Expected frequencies in brackets.			

Table 5.74 ***Crosstabulation between statement 44 and teaching geometry as a full year course***

I found statistically significant relationships between statement 48: *I enjoy proving theorems for my students* and respondents who have taught geometry as a year-long course and who have membership of professional organisations as shown in Tables 5.75 and 5.76. When teachers have a year to teach geometry they have time prove theorems. They don't have to rush. They can show their students interesting proofs and the students have the opportunity to reflect on the concepts.

	I have taught geometry as a full year course		
I enjoy proving theorems for my students	No	Yes	Total
Disagree	42 (29)	42 (55)	84
Agree	130 (143)	276 (263)	406
Total	172	318	490
Chi-squared =9.88 ($p = 0.0017$) Expected frequencies in brackets.			

Table 5.75 ***Crosstabulation between statement 48 and teaching geometry as a full year course***

There is a statistically significant relationship between membership of professional organisations and statement 48: *I enjoy proving theorems for my students* as shown in Table 5.76. This may be the case because professional organisations provide their members with journals and other professional development materials that may contain interesting theorems to prove for their students. Members of professional organisations may have a more problem solving approach to teaching geometry and prefer their students to prove their own theorems rather than proving theorems for their students.

	I am a member of NCTM, ATM (etc.)		
I enjoy proving theorems for my students	No	Yes	Total
Disagree	37 (46)	48 (39)	85
Agree	226 (217)	178 (187)	404
Total	263	226	489
Chi-squared = 4.35 ($p = 0.037$) Expected frequencies in brackets.			

Table 5.76 ***Crosstabulation between statement 48 and membership of professional organisation***

5.4.4 Findings About Membership Of Professional Organisations And Attendance At Professional Meetings

These findings were not part of the research question for my dissertation but provide important information that needs further investigation. Of the 520 questionnaire respondents, 229 acknowledged their membership of a professional organisation such as the National Council of Teachers of Mathematics (NCTM) or the Association of Teachers of Mathematics (ATM) as shown in Table 5.77, and 157 of the respondents have attended two or more professional meetings as shown in Table 5.78.

I belong to a professional organization	Frequency	Percent
No	275	52.9
Yes	229	44.0
No Response	16	3.1
Total	520	100

Table 5.77 ***Respondents' membership of professional organisation***

I have attended at least 2 professional meetings	Frequency	Percent
No	341	65.6
Yes	157	30.2
No Response	22	4.2
Total	520	100

Table 5.78 ***Respondents' attendance at professional meetings***

There is a statistically significant relationship between the type of high school in which this sample of teachers is employed and their membership of professional organisations as shown in Table 5.79 and their attendance at professional meetings as shown in Table 5.80. Significantly more teachers from suburban and other high schools such as private schools are members of professional organisations and attend professional meetings than do teachers from inner city high schools.

	I am a member of NCTM, ATM (etc.)		
I teach in:	No	Yes	Total
Inner city HS	206 (174)	111 (143)	317
Suburban HS	38 (56)	64 (46)	102
Other	18 (32)	41 (27)	59
Total	262	216	478
Chi-squared = 40.01 ($p = 2.054 \times 10^{-9}$) Expected frequencies in brackets.			

Table 5.79 ***Crosstabulation between the type of high school in which employed and membership of professional organisations***

	I have attended at least 2 NCTM meetings		
I teach in:	No	Yes	Total
Inner city HS	254 (219)	61 (96)	315
Suburban HS	53 (69)	47 (31)	100
Other	21 (40)	36 (17)	57
Total	328	144	472
Chi-squared = 59.94 ($p = 9.652 \times 10^{-14}$) Expected frequencies in brackets.			

Table 5.80 ***Crosstabulation between type of high school in which employed and attendance at professional meetings***

Some interesting findings related to gender can be identified from these data. Significantly more of the females in this sample are members of professional organisations than are the males as shown in Table 5.81 and also the females attend more professional meetings than their male counterparts as shown in Table 5.82.

	I am a member of NCTM, ATM etc.		
Gender	No	Yes	Total
Female	129 (144)	135 (120)	264
Male	144 (129)	94 (109)	238
Total	273	229	502
Chi-squared = 6.84 ($p = 0.0089$) Expected frequencies in brackets.			

Table 5.81 ***Crosstabulation between gender and membership of professional organisations***

	I have attended at least 2 NCTM meetings		
Gender	No	Yes	Total
Female	161 (178)	99 (82)	260
Male	178 (161)	58 (75)	236
Total	339	157	496
Chi-squared = 10.42 ($p = 0.0012$) Expected frequencies in brackets.			

Table 5.82 ***Crosstabulation between gender and attendance at professional meetings***

There is a statistically significant relationship between membership of professional organisations and both the respondents' undergraduate majors (first degrees) and the area of the respondents' graduate degree. Those respondents with mathematics related majors belonged to professional organisations in significantly higher numbers than those with other majors as shown in Table 5.83. In the case of graduate degrees, those respondents without a graduate degree attended significantly fewer professional meetings than those with a graduate degree as shown in Table 5.84.

	I am a member of NCTM, ATM etc.		
Undergraduate major/first degree	No	Yes	Total
Business	27 (19)	17 (25)	44
Education	10 (9)	11 (12)	21
Mathematics	58 (93)	157 (122)	215
Science	53 (32)	21 (42)	74
Other	20 (16)	16 (20)	36
Total	168	222	390
Chi-squared = 55.78 ($p = 2.229 \times 10^{-11}$) Expected frequencies in brackets.			

Table 5.83 ***Crosstabulation between undergraduate major/first degree and membership of professional organisations***

	I am a member of NCTM, ATM etc.		
Graduate degree:	No	Yes	Total
Business	9 (6)	2 (5)	11
Education	37 (43)	41 (35)	78
Mathematics	108 (121)	111 (98)	219
Science	15 (9)	2 (8)	17
Other	13 (14)	12 (11)	25
Yes	18 (17)	12 (13)	30
No degree	75 (66)	44 (53)	119
Total	275	225	499
Chi-squared = 18.94 ($p = 0.0042$) Expected frequencies in brackets.			

Table 5.84 ***Crosstabulation between respondents' graduate degree and membership of professional organisations***

There are statistically significant relationships between both undergraduate major/first degree, area of graduate degree and attendance at professional meetings. Mathematics and mathematics education majors are more likely to attend professional meetings than respondents holding business related, education related, science related, or other majors as shown in Table 5.85. In the case of graduate degrees, those respondents without a graduate degree attended significantly fewer professional meetings as shown in Table 5.86.

	I have attended at least 2 NCTM meetings		
Undergraduate major /first degree	No	Yes	Total
Business	34 (30)	10 (14)	44
Education	13 (14)	8 (7)	21
Mathematics	206 (222)	119 (103)	325
Science	48 (39)	9 (18)	57
Other	32 (28)	9 (13)	41
Total	333	155	488
Chi-squared = 14.11 ($p = 0.007$) Expected frequencies in brackets.			

Table 5.85 ***Crosstabulation between undergraduate major/first degree and attendance at professional meetings***

	I have attended at least 2 NCTM meetings		
Graduate degree:	No	Yes	Total
Business	11 (9)	2 (4)	13
Education	47 (54)	32 (25)	79
Mathematics	135 (147)	79 (67)	214
Science	14 (12)	3 (5)	17
Other	20 (16)	3 (7)	23
Yes	22 (21)	9 (10)	31
No grad degree	91 (81)	27 (37)	118
Total	340	155	495
Chi-squared = 16.78 ($p = 0.01$) Expected frequencies in brackets.			

Table 5.86 ***Crosstabulation between respondents' graduate degree and attendance at professional meetings***

I found statistically significant relationships between respondents' teaching experience and membership of professional organisations as shown in Table 5.87 and between respondents' teaching experience and their attendance at professional meetings as shown in Table 5.88. The number of respondents with fewer than 10 years of teaching experience who were members of professional organisations was significantly less than expected while the number of respondents with more than 10 years of experience was significantly more than expected. I found similar results for attendance at professional meetings.

	I am a member of NCTM, ATM etc.		
Number of years teaching	No	Yes	Total
0-5	112 (92)	58 (78)	170
6-10	44 (41)	32 (35)	76
11-15	29 (32)	30 (27)	59
16-20	31 (35)	34 (30)	65
21-25	18 (20)	19 (17)	37
26-30	12 (17)	19 (14)	31
Over 30	22 (31)	35 (26)	57
Total	268	227	495
Chi-squared = 20.52 ($p = 0.0022$) Expected frequencies in brackets.			

Table 5.87 ***Crosstabulation between number of years teaching and membership of a professional organisation***

	I have attended at least 2 NCTM meetings		
Number of years teaching	No	Yes	Total
0-5	154 (115)	15 (54)	169
6-10	60 (52)	16 (24)	76
11-15	32 (40)	26 (18)	58
16-20	37 (44)	28 (21)	65
21-25	21 (25)	16 (12)	37
26-30	10 (20)	20 (10)	30
Over 30	20 (37)	34 (17)	54
Total	334	155	489
Chi-squared = 96.74 $(p = 1.198 \times 10^{-18})$ Expected frequencies in brackets.			

Table 5.88 ***Crosstabulation between number of years teaching and attendance at professional meetings***

I wanted to know what the effects of having taken an undergraduate geometry course or courses in mathematical methods/pedagogy were on membership of professional organisations and on attendance at professional meetings. When I crosstabulated these variables I found there was a statistical significance between taking methods courses and membership of professional organisations as shown in Table 5.89 and between taking methods courses and attendance at professional meetings as shown in Table 5.90. I did not find any statistically significant relationships between taking an undergraduate geometry course and membership of professional organisations as shown in Table 5.91 or between taking an undergraduate geometry course and attendance at professional meetings as shown in Table 5.92.

	I am a member of NCTM, ATM etc.		
I have taken mathematics methods courses	No	Yes	Total
No	58 (44)	23 (37)	81
Yes	215 (229)	205 (191)	420
Total	273	228	501
Chi-squared = 11.41 $(p = 7.298 \times 10^{-4})$ Expected frequencies in brackets.			

Table 5.89 ***Crosstabulation between taking methods courses and membership of professional organisations***

	I have attended at least 2 NCTM meetings		
I have taken mathematics methods courses	No	Yes	Total
No	67 (55)	13 (25)	80
Yes	272 (284)	143 (131)	415
Total	339	156	495
Chi-squared = 10.30 ($p = 0.0013$) Expected frequencies in brackets.			

Table 5.90 ***Crosstabulation between taking methods courses and attendance at professional meetings***

	I am a member of NCTM, ATM etc.		
I have taken an undergraduate geometry course	No	Yes	Total
No	90 (82)	60 (68)	150
Yes	183 (191)	168 (160)	351
Total	273	228	501
Chi-squared = 2.62 ($p = 0.1055$) Expected frequencies in brackets.			

Table 5.91 ***Crosstabulation between taking undergraduate geometry courses and membership of professional organisations***

	I have attended at least 2 NCTM meetings		
I have taken an undergraduate geometry course	No	Yes	Total
No	108 (102)	42 (48)	150
Yes	230 (236)	115 (109)	345
Total	338	157	495
Chi-squared = 1.373 ($p = 0.24$) Expected frequencies in brackets.			

Table 5.92 ***Crosstabulation between taking undergraduate geometry courses and attendance at professional meetings***

Could the size of the school have an effect on whether a respondent is a member of a professional organisation or attends professional meetings? I found that schools with fewer than 2000 students have significantly more teachers who belong to professional organisations as shown in Table 5.93 and attend professional meetings as shown in Table 5.94 than schools with more than 2000 students.

	I am a member of NCTM, ATM etc.		
Number of students in my school	No	Yes	Total
≤ 1000	55 (62)	60 (53)	115
1001-2000	49 (65)	71 (55)	120
2001-3000	65 (58)	42 (49)	107
3001-4000	43 (39)	28 (32)	71
Over 4000	42 (30)	13 (25)	55
Total	254	214	468
Chi-squared = 24.41 ($p = 6.6 \times 10^{-5}$) Expected frequencies in brackets.			

Table 5.93 ***Crosstabulation between school size and membership of professional organisations***

These results may be due to personalisation in smaller schools equates to professionalism. There are fewer mathematics teachers in small schools. Teachers get to know each other better than in large schools. They may come together to plan and share ideas. This is what I call professionalism.

	I have attended at least 2 NCTM meetings		
Number of students in my school	No	Yes	Total
≤ 1000	73 (78)	39 (34)	112
1001-2000	66 (82)	52 (36)	118
2001-3000	76 (75)	32 (33)	108
3001-4000	61 (49)	9 (21)	70
Over 4000	45 (38)	9 (16)	54
Total	321	141	462
Chi-squared = 26.42 ($p = 2.6 \times 10^{-5}$) Expected frequencies in brackets.			

Table 5.94 ***Crosstabulation between school size and attendance at professional meetings***

5.5 CONCLUSIONS

In this chapter I have answered the questions originally raised in the pilot study that can be found in section 4.3.4.

My analysis of the data has identified a number of statistically significant relationships between aspects of the teaching and learning of geometry and the attitudes and attributes of the teachers.

With regard to gender I found that female high school teachers use manipulatives significantly more than their male counterparts. There were no statistically significant gender differences with respect to the 48 statements on the questionnaire or with the use of dynamic geometry software. As a by-product of this study we found that significantly more females than males are members of professional organisations and attend professional meetings. It is important for teacher educators and administrators to encourage male high school teachers to use manipulatives to promote student understanding. If teacher educators and administrators want to promote gender equality and professionalism they should encourage more male teachers to join professional organisations and attend professional meetings.

The findings show that there is a statistically significant relationship between the use of manipulatives and both membership of professional organisations and attendance at professional meetings. Even though we cannot assume a causal relationship we can ask whether being a member of a professional organisation and/or attending professional meetings affects a teacher's beliefs about using manipulatives or whether a teacher who believes in using manipulatives joins professional organisations or attends professional meetings and so gains insights into how best to use manipulatives. Similar results were found for the use of dynamic geometry.

There is a statistically significant relationship between the use of manipulatives and a teacher's belief that it is important to use hands-on activities when teaching geometry, that using manipulatives is motivational, that manipulatives help students grasp basic ideas, that it is beneficial to use manipulatives in their lessons and that manipulatives make learning geometry fun. What is troubling is that there are teachers who have these beliefs but do not use manipulatives. These teachers may not have access to manipulatives or may feel that they do not have the time to use manipulatives because of the amount of material they have to cover.

It was interesting to find that teachers' experience and school size do not matter significantly with regard to the use of manipulatives but that school type does. All three of these variables are significant with regard to use of dynamic software. Suburban school districts have the money to supply their teachers with both manipulatives and dynamic software systems. Teachers in smaller high schools may find it easier to take their students to a computer laboratory to work with dynamic geometry. The type of school is also significant with respect to membership of professional organisations and attendance at professional meetings. Money may be a large factor, personalisation in smaller schools equating to professionalism may be another.

I would have assumed that newer teachers that have fewer than 5 years of experience would have been exposed to dynamic geometry in their own training courses and would be more likely to use it than teachers who have been teaching for many years. This was not the case. I could only assume that these less experienced teachers might be teaching in inner city schools where they have less access to dynamic geometry software. I cross tabulated number of years of experience and type of school and found that there were 115 respondents that have fewer than 5 years teaching experience teaching in inner city high schools. Lack of resources in these schools could be a big factor, which explains why the newest teachers have not used dynamic geometry as much as expected.

Teachers who have graduate degrees use manipulatives significantly more than teachers who do not have graduate degrees. There is no similar significant finding for the use of dynamic geometry. This finding raises a question about the emphasis that graduate teacher education programs place on the use of dynamic geometry software.

We have found time and again significant relationships between the use of manipulatives and positive beliefs about the use of dynamic geometry (see Tables 5.13, 5.26, 5.25, 5.27, and 5.33).

There is significant use of manipulatives and dynamic geometry by teachers who teach year-long geometry courses. There is no significant use of manipulatives by teachers in an integrated course. This may influence those policy makers who favour use of manipulatives to reconsider how geometry is taught.

Suburban high school teachers used manipulatives and dynamic geometry with their students significantly more than inner city teachers. They also were members of professional organisations and attended professional meetings significantly more often than teachers from other schools. This could very well be a monetary issue. In this study I found fewer new teachers in the suburban high school that may mean a higher rate of retention in the suburbs.

Professed high school mathematics teachers' beliefs about manipulatives and dynamic geometry may not be enacted in practice due to the social context of their teaching situation. There may not be manipulatives or dynamic geometry available at the schools where they teach or the administration or colleagues discourage their use.

One must be careful about generalising findings from one sample to the entire population of high school teachers. In future chapters I will look at the data from alternative perspectives. (See chapters 6, 7, and 8).

CHAPTER 6 - FACTOR ANALYSIS

6.1 INTRODUCTION

In Chapter 3 I described the development of factor analysis, its mathematical interpretation and the goals of the use of factor analysis in this project. There are several techniques associated with factor analysis and I have applied a number of them to my data. Despite the differences in these techniques, the results produced were similar and I finally chose to use the factor analysis technique that is known as principal component analysis with varimax rotation. This chapter will discuss the steps taken when doing a factor analysis and the results of that factor analysis on my data.

6.2 STEPS IN FACTOR ANALYSIS

6.2.1 Selecting and Measuring a Set of Variables

The first step in factor analysis is selecting and measuring a set of variables. In this study the variables are the 48 statements from the questionnaire. I used SPSS to generate a correlation matrix for the variables. It checked the suitability of the data through two tests: Bartlett's Test of Sphericity, which hypothesises that the correlations in a correlation matrix are zero and the Kaiser-Meyer-Olkin (KMO) test measure of sampling adequacy, which is the ratio of the sum of the squared correlations to the sum of the squared correlations plus sum of squared partial correlations. If the partial correlations are small then the value of the KMO approaches 1. Good factor analysis requires the KMO test to produce a value greater than or equal to 0.6 and for the results of the Bartlett test not to be significant. The results for my data are shown in Table 6.1, and as they satisfy the criteria it was possible to proceed confidently with the factor analysis.

Kaiser-Meyer-Olkin Measure of Sampling Adequacy.		.877
Bartlett's Test of Sphericity	Approx. Chi-Square	5081.323
	df	496
	Sig.	.000

Table 6.1 ***KMO and Bartlett's Test***

The factor analysis extracted a set of factors from the correlation matrix. SPSS allows the researcher to rotate the factors to increase interpretability. The job of the researcher is to interpret the results.

Two main issues determining the suitability of the data for factor analysis are the sample size - the larger the better - and the strength of the relationship among the variables.

The problem with a small sample size is that the factors obtained from small data sets do not generalise as well as those from a larger sample and that the correlation coefficients among the variables are less reliable in small samples. Comrey (1973), Stevens (1992), Tabachnick and Fidell (1996) said you have to have at least 300 cases for factor analysis to be an appropriate method. Child (1990) said the overall sample size is not as important as the ratio of subjects to items. Harman (1976) recommended a 10 to 1 ratio meaning 10 cases for each item to be factor analysed, while Stevens in an earlier edition of his book suggested a 5 to 1 ratio.

The strength of the inter-correlations among the items can be determined by looking in the correlation matrix for coefficients greater than 0.3. If few of these are found then a factor analysis should not be performed. (The terms in the matrix lie in the range $-1 < r < 1$). Tabachnick and Fidell (2001) recommended that a researcher should not use factor analysis if upon inspection of the inter-correlation matrix there are no correlations in excess of 0.3.

6.2.2 Factor Extraction

The researcher must determine the number of factors that best describe the underlying relationship among the variables. The researcher would like to work with as few factors as possible but still needs to explain as much of the variance in the original data set as possible. Kaiser's criterion and Catell's scree test (Child, 1990) are two techniques that can help a researcher decide the number of factors to keep as shown in Figure 6.1. SPSS uses several approaches to identify the number of underlying factors that include principal components, maximum likelihood factoring, and principal factors. I tried all three techniques as shown in Table 6.2.

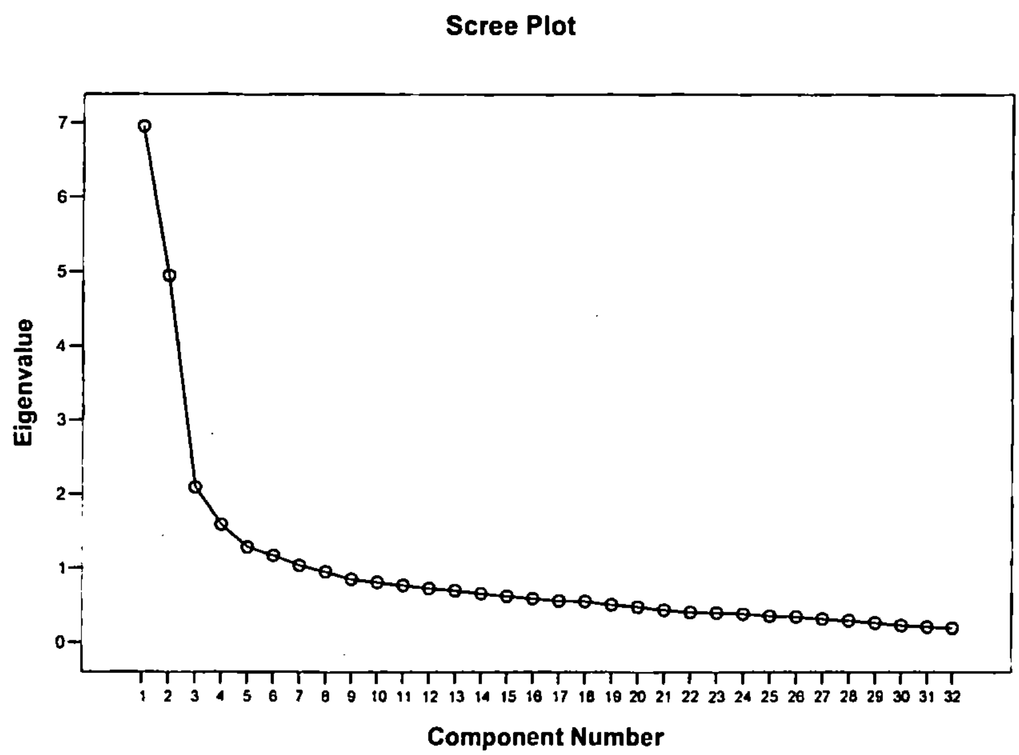


Figure 6.1 *Scree Plot*

I used Kaiser's criterion or the eigenvalue rule that states that only factors with eigenvalues greater than or equal to 1.0 are kept for further investigation. The eigenvalue of a factor represents the amount of total variance explained by that factor.

A shortcoming of this technique is that too many factors may be kept. I used this criterion since it was the SPSS default setting. At first I also used the default setting that deleted listwise missing values. Using Principal Component Analysis (PCA), a technique of factor analysis, 13 components were extracted that explained 62.667% of the variance as shown in Table 6.2. When using the same analysis but with pairwise deletion of missing values 12 factors were extracted accounting for 59.991% of the variance. When I examined the loading I found there were only 2 or 3 loadings on factors 7-13. I then decided to examine the scree test and choose fewer factors.

The Catell's scree test involves plotting each of the eigenvalues of the factors and looking for a point where the shape of the curve changes direction and becomes horizontal. All the factors above the break in the plot, or the elbow, are kept because these factors contribute most to the explanation of the variance in the data set. There was a break after the first 3 factors and a second break after the fifth factor. When I first ran PCA on my data the number of respondents was still small, but I was still able to make some sense of my data when five factors were extracted. As the number of respondents increased I was able to make more sense of my data when three factors were extracted.

A loading or saturation is a correlation between the factor and the variable. Stevens (1992) suggested that a general variable should share at least 15% of its variance with the factor that it will help name. This means using loadings with absolute value of about 0.40 or greater for interpretation purposes since $(0.4)^2 = 0.16$.

I eventually discarded the variables that did not load onto any factor with the above criteria and performed factor analysis again.

Data snooping is encouraged when doing factor analysis. Data snooping is accomplished by trying various techniques of extraction, varying the number of factors and the rotational methods with each run. "*Analysis terminates when the researcher* *decides on the preferred solution*" (Tabachnick and Fidell, 2001, p.609). To better interpret the results I tried various methods of rotation as shown in Table 6.2. Originally, PCA gave me a unique solution. The results were no longer unique under rotation. Once rotation is applied to PCA the technique is considered a factor analysis.

6.2.3 Factor rotation and interpretation

SPSS shows you which variables to clump together to create a factor. It does not label or interpret each of the factors. There are two main approaches to rotation: orthogonal and oblique. Orthogonal rotation results in uncorrelated factor solutions. The varimax method is the more commonly used technique for orthogonal rotation.

Oblique rotation results in correlated factor solutions. Direct oblimin is the more commonly used technique for oblique rotation. For this technique, the researcher must assume that the underlying constructs of orthogonal rotations are independent. These solutions are easier to interpret than those resulting from oblique rotation. Many researchers conduct both rotations and then report the one that is easier to interpret. (Hoping that each variable loads strongly on only one factor, and each factor represents by a number of strongly loading variables). Varimax rotation simplifies the columns of the factor loading by maximising the variance of the squared loadings. By loading high, for the most part, on one factor and low on the other factors, rotations result in a simplification of the initial solution where variables might have moderate loading across a number of factors.

Extraction method	Rotation method	Number of variables	Number of components	Total variance Explained (%)
Principal Components Analysis (PCA) (listwise)	Varimax	48	13*	62.667
PCA (pairwise)	Varimax	48	12*	59.991
PCA (listwise)	Varimax	48	5	41.662
PCA (pairwise)	Varimax	48	5	41.575
PCA (listwise)	Varimax	48	4	37.728
PCA (pairwise)	Varimax	48	4	37.442
PCA (recoded)**(list)	Varimax	48	12*	60.854
PCA (recoded)**(pair)	Varimax	48	11*	57.819
PCA (listwise)	Varimax	48	3	33.204
PCA (pairwise)	Varimax	48	3	32.913
PCA (listwise)	Varimax	37	3	39.269
PCA (pairwise)	Varimax	42	3	36.539
PCA (pairwise)	Varimax	39	3	38.196
PCA (pairwise)	Varimax	36	3	40.435
PCA (listwise)	Varimax	35	3	41.029
PCA (listwise)	Varimax	34	3	41.982
PCA (listwise)***	Varimax	32	3	43.782
PCA (pairwise)	Varimax	34	3	42.448
PCA (pairwise)	Varimax	33	3	42.978
PCA (pairwise)	Varimax	32	3	43.661
PCA (pairwise)	Varimax	31	3	44.279
PCA (recoded)	Varimax	43	3	35.931
PCA (recoded)	Varimax	42	3	36.471
PCA (recoded)	Varimax	41	3	37.013
PCA (listwise)	Oblimin	48	3	33.204
PCA (pairwise)	Oblimin	31	3	44.279
Maximum likelihood (listwise)	Varimax	48	3	28.956
Maximum likelihood (pairwise)	Varimax	48	3	28.705
Maximum likelihood (pairwise)	Varimax	33	3	38.164
Maximum likelihood (pairwise)	Oblimin	31	3	38.627
Maximum likelihood (pairwise)	Varimax	27	3	42.466
Principal axis factoring (pairwise)	Varimax	32	3	38.294
Principal axis factoring (listwise)	Varimax	32	3	38.333
PCA (mean)****	Varimax	48	12	59.103
PCA (mean)****	Varimax	48	3	32.261
PCA (mean)****	Varimax	38	3	37.818
PCA (mean)****	Varimax	35	3	40.533
PCA (mean)****	Varimax	34	3	41.545
PCA (mean)****	Varimax	33	3	42.380
PCA (mean)****	Varimax	31	3	43.983
PCA (mean)****	Varimax	29	3	45.494

Table 6.2 ***Results of various extraction methods using SPSS***

*(SPSS default-Eigenvalues > 1)

**3.5 replaced missing entries for dynamic geometry statements

*** Method chosen for analysis of data

****Missing values replaced with the mean

The method I chose was principal component analysis with orthogonal (varimax) rotation. The other methods explained less of the variance or contained variables that loaded on to more than one factor. I tried using a maximum likelihood factor analysis but this assumes that the original variables follow a multivariate normal distribution whereas PCA requires no distributional assumptions. I have excluded cases (respondents) listwise. This means that those cases (respondents) that have missing values for any of the variables were excluded from the analysis. When using pairwise exclusion we exclude cases (respondents) with missing values for either or both of the pair of variables in computing the statistic. The last entry in Table 6.2 explained 45.494% of the variance. I did not want to use this method because the variable statement *geometry should initially be hands-on with proof coming later* had a loading that I found interesting and I did not want to have to omit it simply to explain a little more of the variance. This statement loaded negatively on factor three, which surprised me as I thought it might load positively on factor one where the other variable dealing with a hands-on statement loaded. This has to be further investigated.

The first three factors extracted from every rotation that I tried were the three factors found in the rotated component matrix (See Table 6.3). The only changes were the order in which they occurred and that more variables loaded on each of the factors as I decreased their number. For instance, for the default extraction of factors with eigenvalues having absolute value greater than 1, variables about manipulatives loaded on the first factor and variables about dynamic geometry loaded on a later factor. As the number of factors decreased, more of these variables loaded on the same factor as shown in Table 6.3.

Questionnaire Statements	Factor		
	1	2	3
26. The use of manipulatives makes learning geometry fun.	.781		
23. It is important to use hands-on activities to explore geometric ideas.	.771		
8. Dynamic geometry enables students to enjoy learning geometry.	.729		
24. I think it is beneficial to use manipulatives as a component of my geometry lessons.	.726		
14. Using manipulatives in the teaching of geometry is motivational.	.714		
20. All HS students should have used dynamic software.	.708		
19. Manipulatives help students grasp the basic ideas in geometry.	.696		
27. More interesting geometrical problems can be explored with dynamic geometry than w/o.	.644		
30. I am familiar enough with dynamic geometry to use it confidently in my teaching.	.421		
31. HS students should discover theorems in geometry.	.407		
41. Studying geometry leads to a positive attitude about mathematics.		.681	
45. Geometry enables ideas from other areas to be pictured.		.651	
43. Students can experience the activities of mathematicians through their work in geometry class.		.650	
47. Applying geometrical concepts and thinking will help students in their future occupations or professions.		.647	
9. Geometry should occupy a significant place in the curriculum.		.625	
2. Learning geometry is valuable for HS students.		.580	
6. Geometry for all students.		.576	
42. When teaching geometry connection to real world applications should be made.		.554	
18. Geometry has many real world applications.		.534	
46. Main goal of geometry is to illustrate the order and coherence of a mathematical system.		.476	
40. Students should be made aware of the historical background of geometry.		.468	
4. Learning to construct proofs is important.			.725
10. High school geometry should not contain proofs.			-.710
13. HS students should be able to write rigorous proofs in geometry.			.680
44. I enjoy teaching geometric proofs.			.679
21. I enjoy doing do geometric proofs.			.630
48. I enjoy proving theorems for my students.			.624
34. More time should be spent on analytic geometry and other topics rather than on proving.			-.610
38. Using dynamic geometry can take the place of rigorous proof.			-.512
16. My students enjoy doing geometric proofs.			.477
32. It is unnecessary for students to prove theorems that they regard as obvious.			-.428
29. HS geometry should initially be hands on with proof coming later in the course.			-.410

Table 6.3: ***Rotated Component Matrix Extraction Method: Principal Component Analysis. Rotation Method: Varimax with Kaiser Normalisation. N=386***

6.2.4 Missing Data

Jackson (1991) suggested methods for dealing with missing data. SPSS allows us to replace each incidence of a missing value with the average of all available data in the sample for that particular variable. We then can obtain the correlation matrix for this adjusted set of data. SPSS also allows us to obtain each correlation coefficient in the matrix on the basis of all data vectors in the data set for which neither value is missing for that particular pair. I ran the data with listwise deletion of variables, pairwise deletion of variables, and the mean in place of the missing value. The results were very similar so I chose the method that gave me the most interpretable results.

6.2.5 Reliability

I tested each of the three rotated factors for reliability. Reliability tells us about the stability of the position of the loading when measured at different times and in different ways. I had to change the direction of the 4 variables that loaded negatively onto factor 3. The Cronbach's alphas for the three factors were 0.805, 0.827, and 0.802 respectively. This tells us that the statements loading on each of the factors separately are reliable. I could use 3 short questionnaires in place of my questionnaire and obtain similar results.

6.2.6 Orthogonality

I checked whether the 3 factors were orthogonal to each other by taking dot products. The values produced were -0.12 between factors 1 and 2, -0.08 between factors 2 and 3, and 0.05 between factors 1 and 3. As these values were all close to zero, I was confident enough to use an orthogonal (varimax) rotation rather than an oblique (oblimin) rotation of the extracted factors.

6.3 THREE FACTOR SOLUTION

I ran the data with listwise deletion of variables, pairwise deletion of variables, and the mean in place of missing values. The first three factors extracted from every rotation that I tried were the same as the factors identified in Table 6.3. I identified the three factor solution as "The Triple A: Activities, Applications & Appreciation, and Abstractions". We can interpret the factors in terms of teachers' dispositions:

Factor 1- A disposition towards doing activities

Factor 2- A disposition towards appreciation of geometry and its applications

Factor 3- A disposition towards abstraction.

6.3.1 Factor Scores

I saved the factor scores as variables. These scores allow me to identify each respondent's disposition. If a respondent scores high on all 3 factors we can probably conclude that (s)he is involved with doing geometric activities, appreciating geometry and its applications and doing proofs. Table 6.4 lists all eight groups to which a respondent could belong depending on combinations of factor scores in terms of whether they are positive or negative. Every respondent that completed the entire Likert part of the questionnaire fitted into one of the eight groups.

Group	Factor 1	Factor 2	Factor 3	Number of respondents
1	Positive	Positive	Positive	65 (16.8%)
2	Positive	Positive	Negative	64 (16.6%)
3	Positive	Negative	Positive	41 (10.6%)
4	Positive	Negative	Negative	42 (10.9%)
5	Negative	Positive	Positive	59 (15.3%)
6	Negative	Positive	Negative	36 (9.4%)
7	Negative	Negative	Positive	42 (10.9%)
8	Negative	Negative	Negative	37 (9.6%)

Table 6.4: ***Factor score profiles***

I wanted to explore whether there was a relationship between groups and gender, membership of professional organisations, attendance at professional meetings, undergraduate major (first degree), having a graduate degree, years of teaching experience, type of school, taking geometry courses and taking methods courses. I performed chi-squared analysis on the cross tabulations of these variables. The results are found in Table 6.5.

Gender was found to be independent with respect to the eight groups. Gender was significant when relating it specifically to manipulative use (Chapter 5). Female high school mathematics teachers use manipulatives significantly more than male mathematics teachers.

Significantly more NCTM members were in groups one and two than expected. This seems to indicate that they are more positive about teaching geometry and also may be more inclined to work with manipulatives, use dynamic geometry and emphasise applications than teachers who are non-members (Chapter 5).

Other significant relationships were between the groups and whether teachers had a graduate degree, took geometry courses, methods courses and whether geometry was taught as a full year course. Significantly more teachers who have graduate degrees are in group one and significantly less are in group eight. Similarly with teachers who have taken geometry and methods courses and who have taught geometry for a full year.

Relationship between Groups and:	Chi-squared =	$p =$	Significant
Gender	3.189	0.867	No
NCTM Member (or member of another organisation)	19.98	0.0056	Yes
Attend professional meetings	8.22	0.31	No
Undergraduate major	6.07	0.531	No
Graduate degree	43.7	2.44×10^{-7}	Yes
Years of teaching experience	56.44	0.067	No
Type of school	27.89	0.143	No
Took geometry course(s) as an undergraduate	15.22	0.03	Yes
Took methods course(s)	19.77	0.006	Yes
Taught geometry as a 1 year course	25.2	6.98×10^{-4}	Yes
Taught geometry as a topic in an integrated curriculum	2.378	0.94	No

Table 6.5: ***The relationship between the eight groups and other covariates***

There were no significant relationships between the groups and attendance at professional meetings, undergraduate major, experience, type of school and whether geometry is taught as a topic in an integrated curriculum.

6.4 CONCLUSION

The analysis revealed a three-component model of teacher dispositions that at first seemed to correspond to three philosophies of mathematics that occur in mathematics teaching (Ernest, 1989). These are the instrumentalist, Platonist, and problem solving view of mathematics. The first factor that I extracted which I call *a disposition towards activities* corresponds to the problem solving view of mathematics. The second extracted factor called *an appreciation of geometry and its applications* includes teachers who believe that geometry has real world applications. This factor could be said to loosely correspond to the instrumentalist view of mathematics but my factor implies a positive disposition whereas the instrumentalists' view appears to be totally utilitarian. The third factor that I extracted called *a disposition towards abstraction* corresponds to the Platonist view of mathematics.

The respondents were further divided into eight groups depending on their factor scores. Significant relationships were found between these groups and other covariates. The dispositions also relate loosely to the four arguments identified by Gonzales and Herbst (2004) that defined the geometry course in the twentieth century American mathematics curricula. These four arguments are:

- The intuitive argument where geometry is explored informally
- The mathematical argument where the emphasis is on making conjectures and proving theorems deductively
- The utilitarian argument where the emphasis is on geometric application
- The formal argument where geometry is a case of logical reasoning

The disposition towards activities supports the intuitive argument for the existence of a geometry course in the secondary school curriculum. The disposition towards an appreciation of geometry and its applications supports the utilitarian argument. Finally the disposition towards abstraction supports the mathematical and formal arguments. It is possible to use qualitative analysis to try to further understand these relationships and this is discussed in Chapter 7.

CHAPTER 7 - ANALYSIS OF THE QUALITATIVE DATA

7.1 INTRODUCTION

> *"One does not set out to do qualitative research; one sets out to advance the knowledge or understanding of some portion of the field of mathematics education and then searches for the most effective ways of achieving this goal."* (Pirie, 1998, p. 21)

The first part of this chapter contains the analysis of the free responses to the open ended questions in the questionnaire. Qualitative methods were utilised (Ely, Anzul, Friedman, Garner, McCormack and Steinmetz, 1991; Cohen, Manion and Morrison, 2000; Merriam, 1998). Cohen et al. (2000) believed that "*It is the open-ended responses that might contain the 'gems' of information that otherwise might have not been caught in the questionnaire.*" (p. 255) Milne (2007) stated, "*The power of qualitative analysis is that it narrows our vision but also supports us to go deeper whereas quantitative research gives us a broad-based view of the field of research in which we are interested.*" (Personal communication)

As I read the responses to the open ended questions, I coded every phrase, fragmented sentence or word into meaningful units to help identify initial themes or categories. I debriefed with colleagues to check my codes. For example, respondent #54 wrote, "*It develops mathematical reasoning, real world applications of mathematics, and is the foundation for a lot of advanced* (sic) *math*" in response to question 49a: *Is geometry an important topic for high school to study*? I coded *it develops mathematical reasoning* as *reasoning*; *real world applications of mathematics* as *applications*, and *is the foundation for a lot of advanced math* as *connections*. This process and the analysis of its results can be found in section 7.2 below.

The second part of the chapter contains a discussion and an analysis of a follow up questionnaire containing five open-ended questions. This was sent to a sample of the original respondents in order to triangulate the results from the factor analysis reported in Chapter 6.

The last part of this chapter describes the case of Rose, a high school mathematics teacher who was at the end of her third year of teaching. During the previous year, she had been one of the respondents to both the questionnaire and its follow up. Her scores on the three extracted factors (See chapter 6) were positive on factor 1: a disposition towards activities, positive on factor 2: a disposition towards appreciation of geometry and its applications, and negative on factor 3: a disposition towards abstraction. These scores placed her in my Group 2 (positive, positive, negative).

Based on Rose's factor scores, our conversations and my observations in her class I provided what might be considered an intervention that enabled Rose to teach her students how to do proofs. The intervention is described below, along with her responses to a further follow-up questionnaire. This chapter concludes with an analysis and the findings of this case study.

7.2 OPEN ENDED QUESTIONS

Merriam (1998) discusses several approaches used to analyse qualitative data. She claims that educational researchers use category schemes to classify the data. These category schemes can be pre-existing or they may arise from the data itself. The method that I used to analyse the open ended questions is called content analysis. "*The process involves the simultaneous coding of raw data and the construction of categories that capture relevant characteristics of the document's content*" (Merriam, 1998, p. 160).

Any form of communication, especially written, can be analysed using this technique (Borg and Gall, 1989; Cohen et al, 2000; Merriam, 1998). Borg and Gall (1989) listed the principal objectives of content analysis:

- Produce descriptive information
- Cross validate research findings
- Test theories and hypotheses and explore relations

I wanted to get a better understanding of whether teachers believe it is important for high students to study geometry, whether teachers believe that their students think studying geometry is important and finally whether teachers teach geometry differently from other mathematics content.

Therefore the questionnaire used in this study contained 3 open ended questions:

49a. Is geometry an important topic for high school students to study?

YES NO Please explain.

b. Do you think that students consider studying geometry in high school important?

YES NO Please explain.

50. In what ways do you think that teaching geometry differs from teaching other mathematics content such as algebra?

Since Gonzalez and Herbst (2006) identified four major themes, which they refer to as arguments or discourses, around the importance of geometry in the high school curriculum, I used their themes as a framework against which I analysed the responses to the open-ended questions using content analysis. I hoped that a content analysis of the responses to these questions would yield descriptive information that would help me gain a better understanding of the teachers' beliefs and would hopefully strengthen my findings from the quantitative data. I also wanted to see if my findings supported previous research concerning teachers' beliefs about mathematics in general (Aguirre, in press; Ernest, 1989)

7.2.1 Question 49a: Is Geometry an Important Topic for High School Students to Study? Yes No Please Explain

Gonzalez and Herbst (2006) identified four themes or arguments that represented reasons for geometry to be part of the United States high school mathematics curriculum in the twentieth century. These arguments emerged from an analysis of articles from the National Council of Teachers of Mathematics journal -- *Mathematics Teacher*. They admittedly limited the generisability of their research because they only studied papers from an American journal, but the place of geometry in the secondary curriculum is an international issue (Jones, 2001).

The four arguments were listed in Chapter 2 and at the end of Chapter 6 and are:

- The intuitive argument
- The mathematical argument
- The utilitarian argument
- The formal argument

The three factors extracted through principal component analysis with varimax rotation as discussed in Chapter 6 support the four arguments of Gonzalez and Herbst (2004, 2006) in that the disposition towards activities supports the intuitive argument for the existence of a geometry course in the secondary school curriculum. The disposition towards appreciation of geometry and its applications supports the utilitarian argument.

Finally the disposition towards abstraction supports the formal argument. It could be argued that the mathematical argument is also supported by the disposition towards abstraction because the characteristics of its problems in the geometry curriculum are *"making conjectures and proving theorems deductively."* (Gonzalez and Herbst, 2006, p. 23) But the place of proof in the mathematical argument *"as original problems providing opportunities to experience the activity of mathematicians"* differs from the place of proof in the formal argument which is as a *" method of thinking and as an opportunity to practice deductive reasoning detached from geometric concepts."* The formal argument for the justification of the geometry course started in the 19th century when educators thought that the reasoning skills learned through a geometry course could be transferred to other areas (Gonzalez and Herbst, 2006). I believe a disposition towards abstractions coupled with either of the other two dispositions would support the mathematical argument.

7.2.1.1 The Positive Responses

Almost all (94.4%) respondents agreed that geometry is an important topic for high school students to study. There was a total of 520 respondents to the questionnaire. The responses to question 49a are shown in Table 7.1.

Question number	**Responses**				
	Yes	No	Yes and No	Missing response	Total
49a	491	3	3	23	520

Table 7.1 ***Responses to Question 49a***

The respondents' explanations give a deeper insight into why high school teachers believe it is important to study geometry. Ten themes emerged from the coding and analysis of the responses and can be found in Table 7.2.

Theme	Frequency of response
Reasoning and thinking	178
Real world applications	96
Problem solving	16
Visualisation	37
Spatial sense	33
Connections to algebra and other areas of mathematics	46
Proof	21
Beauty and structure	23
Curriculum and tests	10
Communication	2

Table 7.2 ***Themes emerging from the analysis of question 49a***

The question now is whether some of these categories can be combined? Can they be related to the Gonzales and Herbst's (2006) four arguments as discussed above and shown in Table 7.3?

Data Coding

To give an example of the coding process for my data: Respondent #9 wrote in response to question 49a, "*Geometric proofs encourage students to reason. The reasoning skills subsequently developed can be applied to any occupation that requires rigorous thinking.*" I placed this response into 3 categories: reasoning and thinking, proofs and real world applications.

Respondent #12 wrote *"Geometric proof leads to clear thinking."*

Respondent #266 wrote *"Proofs make students use their reasoning skills."*

After reviewing the responses that had *proof* as one of their themes, I found that most of them also had *thinking and reasoning* as another theme. I combined the two themes that seem to relate to formal argument suggested by Gonzales and Herbst (2006). The formal argument they put forth says that geometry is a case of logical reasoning. *"The value of studying geometry was located in becoming skilled at building arguments using the same reasoning used in the geometry course. Proofs were not important because of the leverage they gave to understand particular mathematical concepts but as students' opportunities to learn, practice and apply deduction."* (p. 13) Therefore the 'new' theme was called *formal reason for studying geometry*. More than 34% of the respondents were in this category. An example of a response in this category is:

"Geometry teaches students to use deductive reasoning and logic which will definitely help them in many academic and real life situations." (#389)

The themes of *real world applications* and *connections to algebra and other areas of mathematics* can be combined into a theme called *utilitarian reason for studying geometry* (Gonzalez and Herbst, 2006). They claimed that in the utilitarian argument *"decisions as to what the geometry course should include are made according to the relevance of the topics in applying geometrical concepts or geometric thinking to the students' future occupations or professions"* (p. 16).

In the theme of *problem solving* the following responses were found:

"Helps them to become successful problem solvers." (#247)

"Helps to develop problem solving skills." (#201)

"A good problem solving tool." (#152)

These responses and others that were similar seem to indicate that the problem solving theme can be included in the *utilitarian* theme.

In the mathematics argument proposed by Gonzalez and Herbst (2006), as summarised in Table 7.3, proof is classified as more than an exercise in logic.

	Formal argument	*Utilitarian argument*	*Mathematical argument*	*Intuitive argument*
What is geometry?	Geometry is a case of logical reasoning.	Geometry is a tool for dealing with applications in other fields.	Geometry is a conceptual domain that permits students to experience the work of mathematicians.	Geometry provides a language for our experiences with the real world.
Views about mathematical activity	Transferring formal geometry reasoning to logical abilities.	Studying concepts and problems that apply to work settings.	Applying deductive reasoning through the study of geometric concepts.	Modelling problems using geometric ideas while reasoning
Expectations about students	All students require logical reasoning to be good citizens and to participate in a democracy.	All students will be part of the workforce in the future.	All students can simulate the work of mathematicians.	All students could develop skills but their abilities vary
Characteristics of problems in the geometry curriculum	Applying logical thinking to mathematical and real-life situations.	Relating geometric concepts and formulas to model real-world objects or to solve problems emerging in job situations.	Making conjectures and proving theorems deductively.	Exploring intuitively geometric ideas towards formality and integrating algebra and geometry.
The place of proof	Proof as a method of thinking and as an opportunity to practice deductive reasoning detached from geometric concepts.	Proof not as important as problems that apply geometry to future jobs.	Proof as original problems providing opportunities to experience the activity of mathematicians.	Proof following informal appreciation of geometric concepts; blurring differences between definitions, postulates and theorems.

Table7.3 ***Elements within the four modal arguments defining the geometry course*** (Gonzalez and Herbst (2006), p. 23)

According to Gonzalez and Herbst (2006) a major goal of the geometry course is to have students experience the activities of mathematicians. "*One common notion among proponents of the mathematical argument, regardless of the way in which the avowed goals were achieved, is that the study of geometry remained within the realm of mathematical activity and focused on knowing geometry*" (p. 18). The value of geometry is in its structure as a mathematical system (Moise, 1973).

"*Geometry helps students with structure and organization. It incorporates the main ideas of math: communications, connections, problem solving, and logical reasoning so beautifully.*" (#171)

"It's beautiful." (#44)

"Deepens the understanding of mathematics in the world." (#182)

The beauty and structure of geometry relates to the mathematical argument from the perspective of Moise (1975) so I combined them into a theme called *the mathematical reason for studying geometry*. No respondent stated that students in a geometry class can experience mathematics in the same way that mathematicians do.

In Principles and Standards in School Mathematics (PSSM) (NCTM, 2000) one of the geometry standards for Pre K-12 is to "*Use visualization, spatial reasoning, and geometric modeling to solve problems.*" (p. 308) Some responses included in the *visualization* theme were:

"Geometry enhances visualization." (#17)

"It allows students to develop their visual learning." (#93)

Some responses in the *spatial* category were:

"*Helps students develop spatial sense.*" (#207)

"*It is also important for students to understand the properties in the world around them.*" (#198)

Respondent #321 wrote, *"It is one of the few areas of mathematics that lends itself to visualization and spatial concepts."*

Similarly, respondent #235 wrote, *"Spatial visualization – facts about geometric shapes are important for students to be exposed to."*

Since PSSM (2000) grouped visualisation and spatial reasoning together I decided to make them one category. I looked again to Gonzalez and Herbst (2006) to find a relationship between this category and their intuitive argument as summarised in Table 7.3. *"The core idea sustaining proponents of the intuitive argument is the principle that geometry provides lenses to understand, to experience and model the physical world."* (p. 20) I renamed the new theme *the intuitive reasons for studying geometry*. I included the *communication theme in the intuitive reasons* for studying geometry since Gonzalez and Herbst (2006) claimed that "*Geometry provides a language for our experiences in the real world*" (p. 23) as the essence of what geometry is according to the intuitive argument.

Finally, the theme of curriculum and testing contained the following statements:

"*Geometry should be studied at least at some level.*" (#191)

"*As part of the curriculum it is determined to be important.*" (#142)

"*It is required for SAT testing.*" (#48)

"*Geometry is part of a basic mathematics education and all students should have some experience with it.*" (#174)

These statements can be included in the *utilitarian reasons for studying geometry.* The summary of the new themes that emerged from question 49a can be found in Table 7.4. When the categories are collapsed there is a loss of subtlety of meanings.

Original categories	New Themes
Reasoning and thinking; Proof	Formal reason for studying geometry
Real world applications; Problem solving; Connections to algebra and other mathematics; Curriculum and tests	Utilitarian reason for studying geometry
Beauty and structure	Mathematical reason for studying geometry
Visualisation; Spatial; Communication	Intuitive reason for studying geometry

Table 7.4 ***High school geometry teachers' reasons for including geometry in the curriculum***

Implications of the data

More than 34% of the respondents gave the *formal reason* for studying geometry. Gonzalez and Herbst (2006) believed that although at the beginning of the twentieth century the argument for a geometry course was that "*...geometry would carry the burden of developing students' capacities for deductive reasoning unlike any other* *subject in high school"* (p. 8), they reported that by the end of the twentieth century there were other expectations for the teaching and learning of geometry. The expectations for geometry students expressed in The Principles and Standards for School Mathematics (NCTM, 2000, p. 308) include:

- Analyse characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
- Use visualisation, spatial reasoning, and geometric (sic) modelling to solve problems
- Use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture.

In the current standards, the formal argument no longer plays the role that it did for hundreds of years. Gonzalez and Herbst (2006) claimed:

> A notable change in the rhetoric of the Standards movement is that in spite of the value put on students' learning of geometry, the formal argument plays no role in the justification of the study of geometry within the rhetoric of the Standards movement. There are not justifications of learning geometry based upon the idea that students could apply geometric reasoning to other domains in their lives. The Reasoning and Proof Standard embeds the justifications for proof at all levels. (p.24)

Students should not be thinking and reasoning only in their geometry class. The PSSM (NCTM, 2000) stated that *"Students should develop an appreciation of mathematical justification in the study of all mathematical content."* (p. 342) Gonzalez and Herbst (2006) therefore concluded, *"Geometry does not carry the burden of teaching reasoning skills to high school students. Rather, students' development of capacity for logical deductions (in mathematics) should lead students to have a deeper understanding of geometric notions."* (p. 25) They include evidence from the PSSM that supports a combination of their other three arguments. My findings suggest that the formal argument is still popular. Geometry teachers believe that the geometry course is where students learn thinking and reasoning skills that they can use in other domains. Gonzales and Herbst (2006) recognise this as a possible tension between policy makers and teachers.

> But those developments in the justification for the geometry course are the discussion at the level of opinion leaders and policy makers. Actual schools, parents, teachers, and students might well continue to hold geometry instruction accountable to procure the stakes identified by the formal argument. Our research suggests that at a minimum, instructional policy that seeks to promote the vision of the NCTM Principles and Standards will have to contend with those expectations and find a serious way to talk to stake holders about the kind of transfer that is reasonable to expect from school studies. (p. 28)

The standards movement promotes students learning how to think and reason throughout their school years and in all mathematics courses. Therefore they have broadened the expectations in the geometry class to include a wider range of geometric ideas and topics. Teachers who hold fast to the *formal reason for teaching geometry* may not emphasise these other areas, thus creating a tension.

7.2.1.2 The Negative Responses to Question 49a: Is Geometry an Important Topic for High School Students to Study?

Although almost every respondent answered *yes* to the above question, there were three respondents who answered *no*. Two of these respondents wrote the statements below.

Respondent #98 wrote, "*It is better to focus on the foundations of numeration.*"

Respondent #153 wrote, "*Not for every student in it. Should not be a graduation requirement. For able students, such as students who will take calculus it is critical and really teaches them to think mathematically.*"

What is interesting about these two respondents is that their factor scores placed them both in Group 8 (Chapter 6). This means that they had negative scores on all three factors that were extracted through factor analysis. They have negative dispositions towards activities, towards an appreciation of geometry and its applications, and towards abstractions. Respondent #98 answered, "*No, they are kids*" in response to question 49b: Do you think that students consider studying geometry in high school important? This may reflect the respondent's negative attitude about his students.

Respondent #153 responded, "*yes and no- regular students-no, advanced students generally realise how important it is,*" to question 49b. This respondent believes that not all students have the ability to learn geometry.

The third **no** response was from respondent #128 who wrote, "*For those students who detest mathematics- geometry is useless torture.*" This respondent's factor scores placed him in Group 7 where members have high negative scores on the first two factors. This respondent had a moderately high positive score on factor three. He has a disposition towards abstraction and away from activities and appreciation and applications of geometry.

Implications

One could assume that respondents #98 and #153 have a negative disposition towards the teaching and learning of geometry since they had negative factor scores on all three factors. It appears that these teachers have a deficit view of students, believing that most students lack the ability to learn geometry. Respondent #128 is in group 7 and said that those students who hate mathematics find it a torture to study geometry. There were other respondents who believed that because geometry is different from other areas of mathematics some students actually like it better. Furthermore, in Aguirre's (in press) study, teachers believed that students having difficulties in algebra should be studying geometry instead. If teachers spend a lot of time teaching geometrical abstractions to students who do not like mathematics, then the students may come to see studying geometry as '*torture*'

7.2.1.3 Mixed Responses

There were three respondents that answered **yes and no** to question 49a. Only respondent #205 gave an explanation: *computer graphics-yes; otherwise-no*. This respondent is in Group 4 with a positive factor score on the first factor- a disposition towards activities. The open ended response gives us more detail of what this respondent's disposition towards activities might mean. This respondent believes that it is important for students to study computer graphics.

The four themes that emerged from the responses to question 49a supported the four modal arguments identified by Gonzalez and Herbst (2006). It could be argued that some of the responses that were originally in the category labelled *connections* could be considered part of the *mathematical argument* for the geometry course as proposed by Fehr (1972), but the main point here is that I did not find any different themes. Earlier research (Suydam, 1985) claimed that respondents to the *Priorities in School Mathematics* survey (NCTM, 1981) believed that geometry is taught in order to *develop logical thinking abilities* (p. 481) which corresponds to the *formal reason for studying geometry* as shown in Table 7.4. The other reasons for studying geometry were to *"develop spatial intuitions about the real world"* (p. 481), to *"impart the knowledge needed to study more mathematics"* (p. 481) and to *"teach the reading and interpretation of mathematical arguments."* (p. 481) These reasons correspond to the *intuitive, utilitarian,* and *mathematical reasons* found in Table 7.4.

7.2.2 Question 49b-Do you think that students consider studying geometry in high school important? Yes No Explain

The responses to this question are shown in Table 7.5. There were comments from some of the 82 respondents who did not initially answer yes or no.

Question number	Responses				
	Yes	No	Yes and No	Missing response	Total
49b	146	261	31	82	520

Table 7.5 *Responses to Question 49b*

As I coded the data I found the categories listed in Table 7.6. Below I include examples of responses in each of the categories.

Categories	Frequencies
Geometry is not relevant	103
Geometry is relevant	55
Logical thinking	21
Proofs	6
No proofs	23
Geometry is Difficult	22
Geometry is boring	3
Geometry is memorisation	6
Test driven and requirement	27
Dislikes Geometry	13
Discovery and enjoyment	15
Values Geometry	7
Relation to other mathematics	9
Geometry is too easy	1
Mathematical Maturity	6
Dependent on the teacher and curriculum	25
Negative attitude about students	43

Table 7.6 ***Categories Emerging from Question 49b***

7.2.2.1 Categories of responses to question 49b

There were both yes and no responses to question 49b with the same reason given. For example, some respondents answered yes and gave an explanation that suggested geometry was relevant to the students' lives whereas some respondents answered **no** and explained how irrelevant geometry was to the students' lives. The pedagogical implications will be discussed later in this chapter.

7.2.2.1.1 Geometry is not relevant

The following responses indicated that teachers believe that their students do not think it is important to study geometry because it is not relevant to their lives:

"Students don't see the connection between geometry and real world applications." (#7)

"They don't see enough connection to the real world." (#55)

"They find no applications to their life 'don't need it' and 'will never use it'. They see no immediate need for geometry." (#113)

"They constantly ask where they will use this. They do not seem to believe that they won't be able to pay others to do the painting, carpeting, real world math for them." (#50)

"They don't see the relation to real world logic and its application outside of certain professions (engineer, architect)." (#29)

"They find it hard to relate it to their current lives." (#302)

"Although students can see how geometry connects to real life, they feel it is not related to their lives." (#304)

"Where there is no application of concept except through memorization and proofs – they don't see!" (#370)

"The question is answered differently by different students, however many students do not see the connection of geometry to life outside the classroom." (#381)

"The curriculum does not connect it to their lives." (#375)

"Students do not seem to understand the full benefit of geometry as it applies to all areas of life." (#385)

"It's not always the most fun and this may cause them to ask why the topic is relevant." (#338)

The responses are divided between connections to real world applications and applications that personally affect the students' everyday lives. Teachers believe students don't see either of these types of connections.

7.2.2.1.2 Geometry is Relevant

The number of teachers who believed students find geometry relevant is slightly more than half the number of teachers who believed that students find geometry irrelevant.

Here are some examples:

"They enjoy it. They see, feel the reality around them." (#277)

"They can relate it to real life experiences, problems (art and architecture etc.)." (#54)

"The ones that can connect geometry to their environment do." (#289)

"Maybe not at first but eventually they can relate it to their environment-making connections." (#301)

"Geometry is a branch of math that students can easily relate to the real world." (#125)

Of note is how much teachers referred to students as "they" rather than saying "I believe….."?

Pedagogical implications

Depending on what teachers believe geometry is about will affect whether they make few connections between the 'real' world and geometry. Even if the curriculum does not include applications is it possible for teachers to go beyond the curriculum in order to connect geometry to the students' lives? (Cockcroft, 1994)

7.2.2.1.3 Logical Thinking

There were 21 responses to question 49b that belong to the *logical thinking* category, some having a **yes** response and some having a **no** response. Responses in this category explicitly refer to 'logic', thinking and/or 'reasoning'.

"Yes-think logically." (#168)

"Yes-systematic development of the reasoning process." (#133)

"As I say to students, 'If you can't prove that the base angles of an isosceles triangle are congruent, then how are you going to convince an employer to hire you or a client to purchase your goods or services?'" (#292)

"Many will <u>never</u> get this experience again and it helps them with other thinking." (#353)

"Students do not like geometry and often fail to reason in a sequence." (#386)

"It means they have to think logically and they don't want to do that." (#27)

"They don't know or haven't been taught how to think for themselves, reason, and justify opinions." (#457)

"No, they don't understand that they are training their reasoning skills." (#240)

Most of the responses in this category relate to the *formal reason for studying geometry*. (Gonzalez and Herbst, 2006) Respondent #27 seems to speak negatively about students' attitudes towards geometry. I will address negatives comments below.

7.2.2.1.4 Proofs

Currently geometry in secondary school is either taught as a topic in an integrated curriculum or as separate course for a year. In the year long course there is an emphasis on proof. Another category with both yes and no responses is about whether students like or dislike doing proofs.

"They hate proofs and do not see why proving things and the ability to prove things is important." (#215)

A yes response was followed by the comment: *"Except proofs."* (#218)
"They don't see the meaning of proofs." (#435)

"They think that proofs are futile and useless. It's just a class they need to pass to graduate." (#196)

"Some students find the course unnecessarily confining in terms of the structure and process of writing proofs." (#187)

"Geometry – yes, proofs – no. They don't realize the importance of learning how to reason."

"No-Most do not enjoy proofs." (#160)

Positive response with respect to proofs:

"Yes-By studying geometry the students begin to reason out with proofs." (#185)

"Yes-it depends on their level of abstract thinking. College bound students usually understand the importance of proofs." (#236)

"Yes-but they find proofs difficult and equate geometry to proofs. They don't realise the geometry they learn around the proofs." (#227)

A response that can be coded as not relevant and also fits into the proof category:

"There is no real life application to studying proofs. Euclid is dead and let him remain buried."

This respondent (#183) responded yes to question 49a but added: "*But not proofs.*"

Finally this respondent answered question 50: *"Proofs are too rigorous for students."*

Most of the negative responses focused on the belief that students find that doing proofs is a useless exercise. Other negative responses were that students find proofs difficult. Some of the positive responses were in reference to 'able' students doing geometry proofs.

Pedagogical implications

Teachers who are required to teach students how to do proofs must find more engaging methods of teaching proof that will appeal to students. Serra (2003, 2007) uses flow charts for doing proofs. I use a tactile method that will be described below. The PSSM (NCTM, 2000) recommends that students learn to justify and explain their answers in early grades. If students start these processes at an early age they should be able to continue using them in the upper grades.

7.2.2.1.5 Geometry is Difficult

There were respondents who claimed that geometry is difficult for some students. This is contrary to findings from Aguirre (in press) where teachers believed geometry was easier for students because it is less abstract. If the geometry course in Aguirre's study was more rigorous her teachers might have believed differently. The teachers were using reform curricula that were based on hands on explorations and did not stress proof.

"It is too rigorous for them." (#17)

"Many lower level students dislike geometry, but many dislike math in general. It is too rigorous for them." (#191)

"The students who lack in reading and writing skills struggle. Experts at applied math may struggle with proofs." (#249)

One respondent said that students find geometry easy:

"No-they find it way too easy and develop bad habits as a result." (#52)

"I think the diminishing role of proofs makes it less challenging." (#474)

As in the responses about proof there were responses that addressed students' abilities. A careful look at these responses suggested a recoding of the categories into general themes that I describe below.

7.2.2.1.6 Dependent on the teacher and curriculum

There were respondents who said that the answer to question 49b depends either on the geometry curriculum and/or on how geometry is taught. It is the teacher that makes the difference with respect to whether or not students believe geometry is important for them to learn. For example, respondent #30 answered *"Yes – But teacher's job is to help them see importance."* Or respondent #204 who stated, *"No-because it is not taught in a way that allows students to see the connections to real life."* I strongly believe this to be at the heart of the issue. This respondent's answer to question 50 describes how teaching geometry differs from teaching other topics with the following response: *"Geometry offers opportunities in every lesson for hands-on exploration, dynamic discovery and/or manipulative extension to enrich student understanding and allow them to 'construct' their own meaning."* This teacher has found a way to help his students understand the importance of geometry.

Similarly respondent #274 wrote, *"If it's approached in a creative fashion the students will understand its importance."*

Another respondent described his role in the geometry classroom. *"Yes, because I explain to them and show them the benefits of their increased ability to think logically, understand their surroundings, and argue conclusively, my students know geometry is important"* (#22).

This respondent's answer to question 50, which asks how teaching geometry is different from teaching other mathematics, was:

"A student who works hard can excel in algebra without understanding what he/she is doing. In a <u>proper geometry</u> class, a student develops understanding and doesn't really excel unless he/she can demonstrate understanding and use reason. Students learn to problem solve by analysing material in geometry class. There is a level of geometry that can be taught that is like algebra in that students can memorize their way through it, but that is not an appropriate class for the average to the above average student. Geometry is like calculus in that some thinking has to be done to really understand."

"It is taught too algorithmically and not connected enough to real life." (#388)

"No-Students feel it has no real life applications. Teachers must show them that there are." (#91)

A teacher can teach a geometry course that is thought to be successful because of student achievement on assessments, but may be unsuccessful if the students develop misconceptions about the nature of geometry. (Schoenfeld, 1988) An example of this is that students may not see the connections between proofs and constructions unless they are made explicit by the teacher.

Implications for pedagogy

Students can't necessarily make the connection between geometry and the real world or between geometry and other topics in mathematics. That has to be the role of the teacher. As respondent #206 stated, "*They don't always see the connection, unless you point it out.*" The teacher does not have to tell the students what the connections are, but should find ways to guide the students to discover these connections for themselves. Even if the curriculum used for the course does not make connections between topics that would enhance students' understanding, then it is still necessary for the teacher to take some extra time to make these connections. It may be well worth the extra time.

"They don't see the connections at first until I give them a real world problem." (#37)

Respondent #37 addresses the issue by trying to find a way to make geometry more relevant for his students. Teachers have to address issues of relevance in all subject areas.

There are textbooks, for example Serra (2007), which contain many different real world applications of geometric concepts that could be used to motivate pupils in their lessons. It is the job of the teacher to help students make connections as reported below.

Respondent #247 wrote, *"Depends on teacher and curriculum. If teacher can show real life connections and all part (a)* (here respondent is referring to his response to question 49a which was: *Real world connections show students its importance...) students will appreciate the importance."*

"The curriculum does not connect it to their lives." (#375)

"Students are frequently overwhelmed with proofs or they are not shown any applications." (#360)

"The course is often presented axiomatically to those who are not ready and devoid of practical applications." (#413)

There were respondents who believe that when students discover the geometry themselves they realise its importance.

"Yes-in our school it is made a 7 month long topic that they discover on their own." (#207)

Respondent #466 claimed, *"As educators we fail to instil a love of learning and are pressed for time which leaves us with little time to do activities based learning."*

"Many teachers don't understand geometry well enough to teach it well." (#124)

"Inferior status of geometry in the school curriculum stems from a lack of familiarity of educators with the nature of geometry and with advances that have taken place in its development." (#118)

"Yes-The student follows the lead of the teacher. If the teacher is excited by geometry the students usually follow suit. It is important not to require too much rigor. Students should be able to explain (with reasons) a conclusion." (#59)

The above respondents believe that the geometry course does not succeed by itself, its success rests with the teachers and their presentation of the material.

7.2.2.1.7 Geometry is Boring

There were respondents who believed that students find geometry boring.

"It seems to be boring to them." (#205)

Pedagogical implications

Teachers need to make an effort to make their geometry course more exciting for their students.

7.2.2.1.8 Geometry requires Memorising

One of the reasons students find geometry boring is because they have to memorise many definitions and theorems. One of the respondents gave the following reason:

"No-Many lack the prerequisites for appreciating its importance. The curriculum is not aligned with many students' level of thought development, so they memorise and dislike geometry." (#203)

If teachers themselves view geometry as difficult, boring and requiring memorising it will impact on their students' learning of the subject. (Beswick, 2007)

7.2.2.1.9 Geometry is a Required Course

College bound students usually need to take three years of mathematics in high school.

"Yes-My students are college bound honor (sic) *students who want to succeed academically. This does not mean that they understand how geometry will benefit them, just that they know it is an important part of academic success."* (#51)

"Yes-They need the class in order to graduate." (#490)

"Yes-Most students understand that geometry is an integral part of high school math and know that it is important to graduate." (#503)

"No -Not for geometry as a subject. Most study it as a requirement for college admission." (#359)

"No-They just see it as another requirement." (#496)

"It is part of the state mandated test." (#468)

Students who think it is important to study geometry only because it is a requirement should be made aware of others reasons for studying geometry.

7.2.2.1.10 Mathematically Mature

There were respondents that felt that students were not mature enough in their mathematical development to really understand geometry.

"No-Most often do not see the relevance at this point in their education. I don't think they are truly mature enough for the broad content of geometry." (#502)

"No-many lack the prerequisites for appreciating its importance. The curriculum is not aligned with many students' level of thought development, so they memorise and dislike geometry." (#203)

"No-Most students do not like geometry because the material (theory) is taught well before they have the mathematical maturity to comprehend it." (#96)

Pedagogical implications

Although teachers talk about 'mathematical maturity', not one of them explicitly mentioned the van Hiele levels of geometric thought, although respondent #203 did mention levels of thought development. Teacher education courses and professional development activities should include discussions about models of geometric thought. If teachers are made aware of these models, it might help them in structuring their lessons to make geometry more accessible to their students.

7.2.2.1.11 Values Geometry

Some teachers believed that the students value geometry but still do not see its importance while other teachers believe students do not value geometry.

"No-I have discussed this with students. They see the value of geometry but don't consider it important." (#28)

"No-until students begin to see the value in what they are learning, they do not consider it important." (#374)

"No-students are not exposed to geometry long enough to appreciate its value and importance. Geometry is not taught as one solid course; so students don't see its beauty and magnitude." (#389)

Although respondent #389 believes that students do not value geometry, the response suggests that the respondent herself does value geometry. This may not be the case as below I provide an analysis of her responses to the open-ended section of the questionnaire together with the factor analysis results from chapter 6.

There was a positive response from respondent # 167: *"Yes-I think more students who are interested in advancing see the value in the ways that geometry is linked to many real life situations."*

Finally, there was a yes and no response from respondent #128: "Yes *and no-depends on what they want to do after high school. If they want to further their education they will probably value it otherwise no."*

Both respondents #167 and #128 believe the 'advancing' student believes in the value of geometry.

Pedagogical implications

How can the average student find value in the geometry course? Teachers should believe that geometry is valuable to all students (DfEE, 1999; NCTM, 2000). The responses do not support that belief.

7.2.2.1.12 Geometry in Relation to Other Domains of Mathematics

There were both yes and no reasons that geometry is important to students based on geometry's relationship or lack of relationship to other areas of mathematics.

Positive Responses

There was a variety of positive reasons that teachers believe students have about studying geometry in relation to other mathematics such as algebra. These included the belief that geometry is more relevant than algebra and that geometric explorations have helped students understand algebra better.

"Yes-Many students understand the importance of supplementing their knowledge of algebra with geometric facts and procedures." (#42)

"Yes-Most students with a good aptitude for mathematics like geometry." (#208)

"Yes-it seems more relevant to their day to day activities than say, algebra." (#213)

"Yes-Students have discussed how exploring geometry has made clearer their understanding of algebra." (#94)

Negative Responses

For the most part teachers believe that students do not think that geometry is related to other mathematics. The students do not see the connections and do not think of geometry as being 'real' mathematics.

"They don't see how geometry is related to math (arithmetic, algebra, etc.)." (#12)

"No-Students do not think geometry is important. They don't see how geometry is considered mathematics." (#252)

"No-Geometry is so different from their other classes and they don't apply what they learned until precalc at which point they can't remember what they learned during geometry." (#225)

"No-For those that place any value in mathematics, they consider algebraic competence the most important thing." (#223)

"No-Students do not consider geometry real math because it's not dealing with numbers...." (#111)

"No-They think arithmetic skills are the only important ones." (#184)

"No-By the time students begin studying geometry, they have already developed a fear of math." (#166)

"Some students cannot see the connection because they are clouded by the idea that they hate math." (#163)

Pedagogical implications

Geometry can't remain ambiguous. Students need to understand why geometry is mathematics and how it relates to other branches of mathematics.

7.2.2.1.13 Enjoyment and Discovery

There were teachers who believe their students enjoy studying geometry.

"I'm not sure they think about its importance. I find some students enjoy geometry more than other branches of mathematics (the future lawyers?) While others find geometry more challenging." (#230)

"Yes and no-When students have to study mathematics they tend to back away from all forms of mathematics. After teaching them and allowing them to discover geometrical concepts they begin to like it." (#220)

Pedagogical implications

Teacher education programmes and professional development should include ideas on how to make geometry more meaningful and at the same time more enjoyable for students.

7.2.2.1.14 Negative attitudes

There were forty-three responses that I would categorise as having a *negative attitude about students*. These responses seem to refer to high school students' attitudes about learning in general.

"Students in HS don't consider anything mathematical or in most cases educationally important." (#246)

"I don't think they consider studying any subject area important anymore. Students have a completely different agenda from prior times." (#169)

"They are teenagers. The only thing really important to them is not their math class." (#416)

"Most students don't see any math as important. They don't see most classes as important." (#46)

"Students don't see the importance of learning." (#27)

"Most students don't even think that HS is important." (#33)

"Do they think anything in HS is important?" (#8)

"Are you kidding?" (#65)

"I think most students feel that most of what they study in HS is a waste of time." (#164)

"Most don't care." (#132)

"Lazy." (#117)

"It is hard to get them to think that anything is important." (#314)

"Students do not want to study any math in HS. They figure if they can add, subtract, multiply and divide that's enough." (#159)

"Your 'general' HS student thinks very little is important to study. Only the high-powered student with definite goals will appreciate the value in studying geometry." (#152)

Only respondent #152 referred to students' ability in relationship to studying. Beswick (2007) claimed that in the absence of the belief that *"all students can learn, one can only wonder what purpose teachers would see their work having."* (p. 114) In section 7.2.2.1.5 above I reported on the category that teachers believe students find geometry difficult. Respondent #191 claimed:

"Many lower level students dislike geometry, but many dislike math in general. It is too rigorous for them."

Watson and DeGeest (2005) researched how teachers were able to successfully teach secondary mathematics to 250 low achieving students. They found it was not the methods of instruction or the materials used, but rather the teachers' belief in the *"worth of all students."* (p.226)

Implications for pedagogy

The undercurrent of negativism in the above responses represents about 8% of the respondents. It is still enough to make one wonder about how high school teachers feel about or what they expect from the average student. Even if we look at the above responses from a lens that suggests that the teachers are responding out of frustration or cynicism Davis (2007, 1955) wrote, "*The attitudes of the teacher are communicated in a subtle, nonverbal way to the student, and give rise to certain definite attitudes in the student which have a decisive influence on his problem solving ability*" (p. 524). What message do the above responses send to students? Do teachers hold these beliefs and do nothing about them? Davis (2007, 1955) suggested, "*By building self-confidence, by pointing out the positive paths to achievement, by encouraging faith in the possibility of success, we do the optimal job of teaching*" (p. 524).

Chou (2007) suggests that teachers' beliefs about their students shape their expectations about student learning and eventually affects student learning. In his study of six teachers' perspectives of indigenous students that took place in urban schools in Taiwan he found factors that could either obstruct or encourage the students' success. One of the teachers he interviewed said,

> I realize these children are not slow. Many teachers think Indigenous students are incompetent at academic subjects. Many Indigenous students just give up when teachers show this attitude. We just have to understand them – to work with them better....Sometimes teachers adjust the curriculum by suggesting a lowering of expectations, such as not giving Indigenous students academically demanding assignments. There is a fine line between wanting to adjust the curriculum to meet the student's capacity and actually challenging the student. Chou (2007, p. 22)

If we remove the word 'Indigenous' from the above quote it could be coming from any of the respondents who expressed negative attitudes about high school students. Teachers who send negative messages about their capabilities to their students can have a negative impact on student learning.

Teachers who may have deficit views of students might lead teachers to decide not to include a topic rather than trying to develop strategies for learning that can be incorporated into the curriculum.

Teachers have to try to motivate their students to learn. Some respondents suggested using technology and discussing applications of geometry as possible 'hooks' as in the next two responses to question 49b:

"Students rarely take anything seriously, although I found students interested in geometry. I always have to hold on to this interest with technology." (#287)

"Students today need to see real world applications or they don't care about learning it." (#156)

There are implications for teacher education and professional development suggested by the responses reported above. As secondary programmes prepare teachers to teach mathematics they might provide courses to help teachers understand the culture of contemporary high school students. Similarly professional development must address teachers' current beliefs about their students.

7.2.2.2 Themes for the responses to question 49b

After examining the categories and conferring with colleagues I decided on four themes: teachers' beliefs about students' attitudes and abilities, beliefs about the nature of geometry, beliefs about teaching geometry, beliefs about geometry's relationship to other mathematics as shown in Table 7.7. When the categories are collapsed there is a loss of subtlety of meanings. An example of this is with the category of *enjoyment and discovery* where enjoyment is an attitude but discovery could be considered either the nature of geometry or a pedagogical strategy for teaching geometry.

Original Categories	New Themes
Geometry is relevant; geometry is not relevant; geometry is boring; geometry is difficult; enjoy geometry; geometry is easy; dislikes geometry; mathematical maturity; values geometry negative attitudes	Students' attitudes and abilities
Proofs; no proofs; logical thinking; memorisation; discovery	The nature of geometry
Dependent on teacher and curriculum; test driven and required	Teaching geometry
Geometry's relationship to other mathematics	Geometry's relationship to other mathematics

Table 7.7 ***Themes emerging from responses to question 49b***

The nature of teachers beliefs about students' attitudes and abilities encompasses such things as: students are not interested, they are immature, they think geometry is irrelevant, they are not intelligent enough (as "college bound" students are). An example: *"Yes-it is difficult for most of my 'regular' geometry students but they feel like they need it to learn to reason logically and as a foundation to higher math* (sic) *and to college."* (#26) In response to question 49a this respondent stated that she taught geometry with the van Hiele levels in mind.

The nature of teachers' beliefs about the subject is that its focus is proof, it involves discovery, and it requires memorisation. The nature of how the course is taught includes the role of the teacher and curriculum, the fact that it is required and tested. The theme of relationships to other mathematics includes beliefs about its connections to and comparisons of geometry to other fields of mathematics.

The original category of discovery and enjoyment was split - discovery could be considered about the nature of geometry or as a pedagogical strategy for teaching geometry whereas enjoyment is an attitude about geometry. Although some responses might have to be moved from their original category to best fit into one of the themes, the themes exhausted all the responses to question 49b.

7.2.3 Question 50-In what ways do you think that teaching geometry differs from teaching other mathematics content such as algebra?

Some of the responses to question 50 seemed very rich and contained a lot of detail. Given the rich detail and the groups that were identified in chapter 6 using my factor analysis, I wanted to know if I could predict the group to which a respondent belonged from their written comments. The richer the response the more likely it was that I could correctly predict the group. For example a creative respondent (#118) wrote:

> *It must be remembered that Euclid wrote for persons preparing for the study of philosophy. So the way you teach geometry should be (since we study trigonometry, analytic geometry, geometry) giving this problem: 'A ship sails the ocean. It left Boston with cargo of wool. It grosses 200 tons. It is bound for Le Havre. There are 12 passengers on board. The wind is blowing E-N-East. The clock points to a quarter past three in the afternoon. It is the Month of May. How old is the captain?' Gustave Flaubert.*

This respondent belongs to group 5, scoring negative, positive, positive on factors 1, 2, and 3 respectively. His response to question 49a was *"Its axiomatic method was considered the best introduction to deductive reasoning. (Formal method was stressed for effective educational purposes)."* His response to question 49b was *"Inferior status of geometry in the school curriculum stems from a lack of familiarity on the part of educators with the nature of geometry and with advances that have taken place with its development."* The respondent appreciates classical Euclidean geometry. By analysing his responses to the three open questions I predicted him to be positive on factors 2 and 3. Nothing was mentioned about manipulatives or activities, so I predicted that he was negative on factor 1, which places him in group 5, which agrees with the results of the factor analysis.

Similarly respondent #128 said, *"It is much more rigorous and tight. Teachers better* *know the material.*" This respondent belongs to group 7 because he scored a negative, negative, positive on factors 1, 2, and 3 respectively. In section 7.2.1.2 above we noted this teacher's negative response to question 49a: "*For those students who detest mathematics- geometry is useless torture.*" His response to question 49b was yes and no, "*Depends on what they want to do after high school. If they want to further their education they will probably value it. Otherwise – no.*" This respondent had a moderately high positive score on factor three. He has a disposition towards abstraction and away from activities and appreciation and applications of geometry. It is possible that this respondent teaches a very abstract geometry course appropriate for able students. The categories that emerged from the initial coding of the responses to question 50 can be found in Table 7.8.

Categories	Frequencies
Geometry is more visual	139
Geometry is more hands-on	81
Geometry is more spatial	32
More applications in geometry	60
Geometry and reasoning	74
Geometry equated to proof	48
Abstraction	50
No differences	14
Discovery, enjoyment, and creativity	32
More difficult/easy to teach	12
Geometry as a mathematical system	12
Geometry and Memorisation	18
Geometry is less algorithmic	6
Geometry involves more reading, writing, and vocabulary	26
Others	6

Table 7.8 *Categories Emerging from Question 50*

7.2.3.1 Geometry is more visual

There were 139 respondents who claimed they could use more visuals when teaching geometry than when teaching algebra. It was not clear from some of those respondents whether they meant using manipulatives and/or diagrams. There were other respondents who mentioned hands-on and manipulatives specifically.

"Geometry is so visual, that I find it an easier branch of mathematics to teach." (#360)

"It is easier to use visuals." (#131)

"More visuals are needed in geometry than in algebra." (#125)

"Need to visualize the objects, much more critical thinking." (#123)

"It is a more visual approach to learning, which algebra does not offer in all situations." (#253)

"More visual. Greater ability to use hands-on manipulatives and models." (#122)

"Very visual." (#117)

"More visual. We can use more manipulatives to motivate the students." (#127)

"It is more visual which can potentially draw a larger audience if taught correctly." (#268)

Respondent #268 believes that geometry could be popular but is dependent on the way it is taught. This is similar to what some respondents said in response to question 49b above.

Similarly, respondent #273 stated, *"It can be argued that geometry demands more spatial visualisation than algebra does. Not all mathematics teachers are equipped to teach high school geometry well."*

The success of the geometry course lies with the teacher who must be both knowledgeable and teach geometry in a way that engages the students.

7.2.3.2 Geometry is more hands-on

There were 81 respondents who claimed that geometry is more hands-on than algebra. The teacher can have students discover geometric properties through explorations.

"Involves hands-on approach; a conclusion can be discovered rather than taught." (#121)

"Geometry is more tactile- it involves visualization and manipulation much more than algebra." (#119)

"Necessitates more opportunities for discovery." (#248)

"It requires manipulatives and a visualisation unlike others which require following directions." This response was from respondent #98 who answered **no** to question 49a above and who answered "No *they are kids*" to question 49b. This respondent has factor scores that place him in group 8 with negative scores on all 3 factors. The implication here is that although he knows what teaching geometry may entail he doesn't believe it is that important for high school students to learn geometry. The results of the analysis would inform his managers that perhaps this teacher should teach content that he believes is important for students to learn.

7.2.3.3 Geometry equated with proof

There were 48 respondents, who believe that the fact that proof is included in the geometry course makes the geometry course different from other high school courses. A subset of theses respondents believed the geometry course only contains Euclidean proofs.

"Teaching geometry is very different than teaching algebra. Words need to be defined and memorized. Theorems need to be stated and proven. Students need to be taught how to mark diagrams based on given information in the proof, otherwise proving a theorem is not possible." (#252)

"Geometry is not straight forward. Students have to discuss how the theorems work and when to apply them-this is difficult for students." (#251)

"Part of geometry involves learning for the sake of learning. Also a lot of algebra in high school involves 1 step; proof writing involves multi-steps." (#249)

"First verbal math course. Students have difficulty organizing themselves so proofs are difficult. There are few immediate rewards to geometry. Most kids need immediate rewards." (# 245)

"The concept of proof makes it much more difficult than algebra. The practice of proofs makes it a torturous (sic) *chore for weaker students, one that they all too often give up on."* (#215)

These responses shed some light on the questionnaire result where only 33.4% of the respondents believed that their students enjoy doing geometry proofs (Table 5.6a).

7.2.3.4 Geometry and Reasoning

There were 74 respondents who believe that the difference between teaching geometry and teaching other mathematical domains is that more reasoning is involved.

"Geometry involves more thinking and reasoning than algebra." (#324)

"In algebra you can teach by a lot of practice. In geometry which requires practice utilizing reasoning skills." (#431)

"One of the most important topics in life because they are taught how to reason."(#457)

Reasoning is a processing standard for all mathematics in all grades (DfEE, 1999; NCTM, 1989, 2000). Are students reasoning in any of their mathematics classes before they take a class in geometry? If they do not take a geometry course does that mean they never learn how to reason?

7.2.3.5 Geometry as a Mathematical System

Only 12 respondents believe that in geometry classes students learn about the structure of mathematical systems.

"Development of a mathematical system. Undefined terms, defined terms, theorems..." (#293)

"I think that geometry is the first time where they start to study an axiomatic structure in mathematics." (#474)

Two questions come to mind while reading these responses. Can mathematical structure be discussed in an algebra course? In curricula where geometry is a topic that is studied for a number of weeks, do teachers spend any time teaching about mathematical structure?

I believe that in a year-long course there is an opportunity to focus on structure.

7.2.3.6 Geometry and Memorising

There were eighteen respondents that claimed that teaching geometry differs from teaching other mathematics because of the amount of memorising involved. What is of interest is that some respondents believe that geometry requires less memorising:

"It requires more spatial understanding and less memorization (sic) *of steps."* (#313)

while other respondents believe geometry requires more memorising:

"More memorisation is required (definitions, previous theorems, etc.) and attention to detail is required..." (p. 51)

If teachers have either a Platonistic or instrumentalist view of geometry (Ernest, 1989, 1991) and believe that geometry is a list of postulates, definitions, and theorems to be memorised, whether or not they are part of a mathematical structure, then this belief will have a strong influence on their beliefs about teaching geometry (Ernest, 1989, 1991; Raymond, 1997).

7.2.3.7 Geometry is Less Algorithmic

Similarly, if teachers have a problem solving view of geometry (Ernest, 1989, 1991) it may influence them to believe that geometry is less algorithmic to teach.

"Every problem is different-no algorithms." (#20)

"Algebra, trigonometry and statistics contain a more strict algorithmic approach. Geometry allows for a more free-association and individual approach." (#14)

These responses give a glimpse into how some respondents approach the teaching of algebra.

7.2.3.8 More Applications in Geometry

Teachers can find more real world applications for geometry than for algebra. For example the teacher can have students measure the height of a building indirectly using properties of similar triangles. There were 60 respondents that believe that geometric applications and connections distinguish geometry from other branches of mathematics.

"More real world applications in everyday life." (#377)

"It is more practical." (#120)

"Geometry is easier to relate to the world. Shapes are less abstract than other mathematical models." (#126)

There are textbooks that contain interesting real world applications of geometry, which teachers could use to enhance their knowledge of suitable applications. (Serra, 2007)

7.2.3.9 Spatial Reasoning

There were 32 respondents who believe that geometry involves more spatial reasoning than other topics in mathematics.

"Geometry requires a degree of spatial reasoning and spatial intelligence that makes it difficult for some students, and then some students are better at geometry for the same reason." This reply was from respondent #153 who answered **no** to question 49a.

His response to question 49b was *"Yes and no; regular students-no, advanced students generally realize how important it is".*

This respondent had 3 negative factor scores placing him in group 8. From his responses to the three opened questions it is clear that he believes that geometry should not be taught to all students.

Although teachers' beliefs about mathematics in general have been researched extensively there has been little research on what Törner (2002) called domain specific beliefs (Aguirre, in press). Domain specific beliefs are associated with a specific field of mathematics unlike subject specific beliefs which are associated with a topic (certain topics such as functions have been extensively researched).

Aguirre (in press) studied teachers in an urban United States high school. She found that the teachers expressed different views about the domains of geometry, algebra, and probability with respect to the implementation of reform curricula. The teachers in the study believed that geometry was more '*concrete*', '*visual*' and '*tangible*' to students than algebra was. *"The paper demonstrates how teachers distinguished among these domains along at least two dimensions: the role of abstraction in the domain and the role of the domain's utility for future career and educational pathways."* (Aguirre, in press)

The teachers in her study believed that algebra is more abstract than geometry and therefore less accessible to students. One of the teachers described the abstraction of algebra as a stumbling block for students, but geometry was 'okay' because it is 'concrete and you can manipulate things'. Some of the respondents in my study have the same belief.

7.2.3.10 Abstraction

There were respondents who believe geometry is less abstract (or more concrete) than other mathematical domains and there were respondents who believe that geometry is a more abstract mathematical domain.

Geometry is less abstract than algebra

The teachers who responded that geometry is less abstract wrote about geometry as being concrete, visual and lends itself to using manipulatives to grasp geometric ideas.

"Geometry is viewed as concrete whereas algebra is abstract." (#250)

"It is much easier and makes things more obvious using manipulatives. It is a lot easier for most students to understand because it's less abstract and more visual than other topics like algebra." (# 246)

Geometry is more abstract

The respondents who believe geometry is more abstract suggest that students have to be 'visualisers' in order to represent the abstractions. Some respondents believe that the abstract nature of geometry is in the areas of proof and construction.

"Geometry is more abstract and students have to be more visual to do geometry." (#243)

"More abstract when talk about proofs and construction; more difficult symbolism and terminology, ~ etc." (#367)

Since some responses state that geometry is more abstract than algebra and some that geometry is less abstract it is important to have a definition of 'abstraction'. Steen (1990) characterised abstractions in several ways which include symbols, equivalence, logic, similarity and recursion. He claimed that there are abstractions in all domains of mathematics.

The seventeen teachers in Aguirre's (in press) study about domain specific beliefs and mathematics reform only focused on symbol manipulation in the domain of algebra. They did not find geometry abstract.

> *Both Sara and Joscelyn described geometry as a domain that all students could learn and algebra a domain only some students could learn. They believed that students experience difficulties when required to formalise or codify mathematical relationships into symbolic notation. For these two teachers abstraction is an important dimension distinguishing algebra from geometry.* (p. 23)

It appears that the geometry taught at BVHS (the high school in Aguirre's study) was very concrete. There is no mention of proof and the geometry seems very intuitive. The teachers described geometry as a domain that all students could learn because of the decreased role of abstraction and its increased utility. They suggested that not all students should study algebra. It depended on the students' career choice. These teachers were opposed to increasing the school's mathematics requirement.

Respondent #414 is in agreement with the teachers in Aguirre's study. She stated, *"I think that geometry can be much more intuitive than algebra and that the role of the math teacher is to develop that intuition."* But there are teachers in my study who disagree with the teachers in Aguirre's study. Respondent #198 writes, *"I think that the art of reasoning is emphasised much more in geometry than any other classes. I also* *think that students become very frustrated in geometry. The ease of most problems and difficulties of most proofs make the subject an obstacle to teach."* This teacher finds geometry problems not dealing with proofs easy to teach thus agreeing with Aguirre's teachers. It is when proof is a part of the curriculum that geometry becomes more difficult to teach. It is apparent that many teachers are unfamiliar with the recommendations of the PSSM that suggest proof and reasoning be woven throughout the grades and in every mathematics content area (NCTM, 2000; Stylianides, 2007). Along these lines respondent #199 writes, *"I don't believe they should be different. I'd love to see more algebraic proof and less geometric proof."*

Noguera and Wing (2006), when trying to close the achievement gap at Berkeley High School, encountered similar opposition to that expressed by Aguirre's teachers. Noguera and Wing had suggested increasing the school mathematics requirement from two years to four years. At Berkeley there were two tracks of geometry for the students: regular geometry and honours geometry. Ninth graders entering the school had four possible mathematics placements. The two lowest tracked placements did not lead to geometry in the tenth grade. With only a two year mathematics requirement it was possible for some students never to encounter geometry in high school. This also seems to be the case in some parts of Canada. Respondent #389 claimed,

> *"Algebra is being taught for many years allowing students to absorb the ideas of one level and build on more abstract ideas on the next level. In Canada, geometry is being taught in grade 12 as part of a course "Geometry and Discrete Math." Since not all university programs require the credit for this course, only a small percentage of all students take it, mostly those who apply for engineering and architecture programs."*

7.2.3.11 Geometry is Easier/More difficult to Teach

There were 12 respondents who claimed that geometry was either easier or harder to teach.

"There is a tangible element to it that can make it both more and less difficult." (#310)

"Algebra is easier for most students to understand." (#325)

Respondent #349 claimed that it takes more work to teach geometry.

"In some ways it takes the most work on the part of the teacher, since drawing skills, technology skills, knowledge of applications, necessary tools and manipulatives are so important to making the subject meaningful."

The teacher should not be walking into a geometry class without being well prepared.

7.2.3.12 Geometry is more Reading, Writing and Vocabulary

There were 26 respondents who believed that there is more reading, writing, and vocabulary in geometry classes.

"1)Geometry has more vocabulary and reading. 2) Geometry is less sequential. 3) Geometry favors (sic) *visual learners. 4) Geometry is better supported by interactive software. 5) Geometry is of greater value to students going into the trades."* (#28)

This respondent used the word 'values' in answer to all three open-ended questions. I would predict that this respondent is in either groups 1 or 2, because I can't determine his belief about proof from his responses. He is in group 2 with a high negative factor score on disposition towards abstractions.

"There is much more language involved." (#16)

7.2.3.13 No Difference

There were 14 respondents who believe there is no difference between teaching geometry and other mathematics courses.

"Good teaching regardless of subject is independent of topic." (#514)

"It doesn't." (#339)

"Some parts are the same-the proofs are what is different because you have to justify what you write." (#376)

"There shouldn't be a difference since they supplement each other." (#476)

It is possible that respondent #514 has global beliefs of teaching mathematics that are not domain specific (Törner, 2002)

7.2.3.14 Enjoyment, Discovery and Creativity

There were 32 respondents who believe geometry is a course where students can engage in discovery lessons and apply their creativity more readily than in other mathematical domains. Three of the thirty-two respondents stated that geometry is fun.

"There is room for a lot of discovery and plenty of real life problems." (#40)

"I find it more fun! There are more hands on activities to incorporate. It is more concrete and visual. The students can hold the solids and see the drawings and theorems fit together." (#335)

"Geometry requires more creative ability, also it is crucial that difficult terminologies be explained properly." (#274)

"More ways to discover concepts. Makes them own their own learning." (#510)

Each of the first three statements can belong to other categories above. Respondent #510 answered the question in relation to learning geometry.

There were also 6 responses in the 'other' category. Respondent #52 stated, *"There is nothing that compares to the opportunity to make mistakes in a long algebra problem."* I did not have a category for this response, but it could possibly fit into a general theme.

7.2.4 Teachers' Positive Attitudes

A group of respondents believed that a teacher's positive attitude towards geometry can have a positive influence on student learning.

"Students recognize algebra as 'math'. They frequently do not understand the connections between geometry and other areas of mathematics. It is my job to make the connection. The fact that I love geometry and really believe it is a valuable study for students usually helps a lot." (#62)

After the initial coding I consulted with colleagues and identified themes that emerged from the data as shown in Table 7.9. The positive attitude of respondent #62 might fit into the enjoyment category, but is easily included in a teaching geometry theme. The two themes that emerged were the *nature of geometry* theme and the *teaching and learning geometry* theme. When the categories are collapsed there is a loss of subtlety of meanings. An example of this is with the category of *enjoyment and discovery* where enjoyment is an attitude but discovery could be considered either the nature of geometry or a pedagogical strategy for teaching geometry.

Original Categories	New Themes
Geometry is visual; hand-on; spatial; reasoning; focus on proofs; memorisation; geometry as a mathematical system; abstraction; less algorithmic; discovery	Nature of geometry
Easy to teach; difficult to teach; attitude; geometry involves more reading, writing, and vocabulary; enjoyment, and creativity; no difference; applications	Teaching and learning geometry

Table 7.9 ***Themes emerging from responses to question 50***

7.2.5 Concluding Remarks about the Analysis of the Open ended Responses

Four themes, the *formal, intuitive, utilitarian* and the *mathematical,* emerged in the analysis of question 49a as shown in Table 7.1 above about the reasons for studying geometry, which support the modal arguments given by Gonzalez and Herbst (2006). There were only three teachers who believed it is not important to study geometry.

The Standards Movement (NCTM, 2000) supports the utilitarian, intuitive and mathematical reasons for studying geometry. The formal reason for studying geometry is no longer as powerful as it was in the early part of the twentieth century. Teachers in this study still believe in the importance of the formal argument. This could impact what is stressed in the geometry classroom.

The analysis of question 49b raised an interesting issue. What happens to student learning when the teachers have negative attitudes about the students?

"Good teaching is good teaching regardless of the content. Instruction needs to be delivered in a way that is meaningful and motivates the student. Once a student is interested the learning process becomes easier." (#96)

Teachers' beliefs about what is important shape their practice. Ball and Cohen (1996) claimed, *"...Teachers are influenced by what they think of their students, about what students bring to instruction, students' probable ideas about the content at hand and about the trajectory of their learning that content"* (p. 7).

Four themes emerged from the data for question 49b. They are *students' attitudes and abilities*, the *nature of geometry, teaching geometry* and *geometry's relationship to other mathematics*. There were two themes that emerged from the data for question 50: the *nature of geometry* and *teaching and learning geometry*. Teachers' epistemological beliefs about geometry and teachers' beliefs about teaching and learning geometry are themes for both questions 49b and 50. Are these beliefs synchronous? The beliefs about the nature of geometry include Platonist, the instrumentalist and the problem solving (Ernest, 1989, 1991). Within the theme of teaching mathematics we find dispositions towards abstractions, dispositions towards problem solving and dispositions toward an appreciation of geometry and its applications which were the factors extracted in chapter 6.

If the beliefs about the nature of geometry are not synchronous with the beliefs about its teaching and learning the result may be ineffective teaching.

Teaching geometry perhaps requires teachers to exercise different skills to those needed when teaching algebra. For example, respondent #349 summed it up nicely, *"In some ways it takes the most work on the part of the teacher, since drawing skills, technology* *skills, knowledge of applications, necessary tools and manipulatives are so important to making the subject meaningful.*" This respondent belongs to group 2 (see chapter 6) and does not have a positive disposition towards abstraction. His emphasis is on activities and applications and he hopes to give his students a meaningful geometry experience. Respondent #278, a member of group 5, claimed that, "*To teach geometry a teacher needs <u>patience</u> and needs to be very ready to answer questions-often the questions seem like they are from left field but may not be!*" Patience is needed when teaching proofs to students. Respondent #356 suggests that geometry involves more planning than other mathematical domains:

> *The teaching of geometry involves helping students to think in more detailed terms than in other subjects. Developing a student's ability to reason and break down their thought processes in order to develop a proof or to recognise the application of properties involves more planning on the part of the teacher.*

Respondent #356 belongs to group 8 and believes in the importance of teaching geometry (response to question 49a), but believes students entering geometry lack the pre-knowledge to appreciate its value (response to question 49b) so that it is therefore more difficult to teach them geometry. He has low negative scores on factors 1 and 2, and a slightly higher negative on factor 3. His responses suggest that he has reflected on these issues and is not positive about teaching geometry.

Respondent #281 believes that geometry is harder to teach and requires more preparation.

"*Geometry is more abstract to explain sometimes. Geometry needs a lot of preparation the day before. More work is needed than algebra.*"

Finally, respondent #514 claims that, "*Good teaching regardless of subject is independent of topic.*" This respondent has global beliefs about teaching mathematics, but there are places where a mathematics teacher is called a geometry teacher if that is the only domain of mathematics he or she teaches (Gooya, 2007). There are best practices for teaching all aspects of mathematics but each domain requires different strengths. Respondent #479 states that, "*...All teachers can teach algebra 1, but not all teachers can teach geometry....*" Similarly, respondent #204 claims, "*In geometry students need to think which is difficult to teach. Algebra concepts are more drill oriented.*"

Respondent #389 is an anomaly. Her response to question 49a was, "*Yes-geometry teaches students to use deductive reasoning and logic which will definitely help them in many academic and real life situations.*" Her response to question 49b was, "*No-students are not exposed to geometry long enough to appreciate its value and importance. Geometry is not taught as one solid course; so students don't see its beauty and magnitude.*" Her response to question 50 was, "*Algebra is being taught for many years allowing students to absorb the ideas of one level and build on more abstract ideas on the next level. In Canada, geometry is being taught in grade 12 as part of a course "Geometry and Discrete Math." Since not all university programs require the credit for this course, only a small percentage of all students take it, mostly those who apply for engineering and architecture programs.*" From an analysis of these three responses I could only predict a positive factor score on factor 2 and possibly a positive factor score on factor 3. She has given the formal argument for including geometry in the curriculum, but lives in a country where geometry is a part of a course that most students do not take. When I checked her factor scores they were all low negatives placing her in group 8. This surprised me and made me realise that I would need more information from respondents in order to accurately predict the groups to which they belong. A short follow up questionnaire would be an appropriate instrument to give me the data I would need for a more accurate and detailed analysis.

7.3 THE FOLLOW-UP QUESTIONNAIRE

I decided to create a follow up questionnaire, that can be found in Appendix H, for the following reasons:

- As a data source to provide triangulation of the results from the factor analysis described in Chapter 6

- To gain richer responses about some issues asked about in the original questionnaire

I emailed the follow up questionnaire to a sample of the respondents, who had identified that they were willing to be involved with further aspects of the research on the original questionnaire.

7.3.1 Triangulation of Results

Through an analysis of the follow up questionnaire I wanted to see if it was possible to identify to which of the eight groups (Chapter 6) a respondent belonged. The importance of being able to do this is that instead of having to get teachers to answer a lengthy questionnaire we could get an accurate analysis of their beliefs using the short open ended questionnaire. In this section I have included responses to the follow-up questionnaire from members of several of the eight groups.

The directions for the follow up questionnaire were: Please answer the questions to the best of your ability.

An example of a respondent whose group was easily identifiable follows. Respondent #40 wrote:

1. What do you most love about geometry and why?

I like that you can visualize and have practical applications

2. What is your most memorable experience or experiences as a student in a geometry class?

Students being able to see logic of reasoning after proof

3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

   Manipulatives-shapes to see what figures look like

4. Is there any topic or topics that are in the current curriculum that you believe should be eliminated? Please explain.

   Spend less time on formal proofs

5. Do you include real world applications in your geometry course? What are these and why are they included?

   Yes, area and volume

This respondent's suggestion to spend less time on formal proofs led me to conclude that she had a negative factor score on component three – a disposition towards abstraction. Although the responses are not very descriptive, I thought she might have positive factor scores on factors one and two – a disposition towards activities since she uses manipulatives and a disposition towards appreciation of geometry and its applications since she claims she loves applications. I would venture to say that this respondent is in Group 2. Checking the original results shows that she would indeed be placed in this group. Such results are important because they can inform assistant principals as to the appropriate geometry courses for teachers to teach and the types of professional development needed to benefit the teacher and their students.

Another respondent (#397) answered the follow up questionnaire:

1. What do you most love about geometry and why?

I love the fact that you are building a system of mathematics from the bottom up and you can really see the structure of mathematical systems and how changing one definition can change the entire system.

2. What is your most memorable experience or experiences as a student in a geometry class?

I remember proofs and liking the structure of them.

3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

   I use the Geometer's Sketchpad computer program for constructions and discovery learning. I also use a variety of hand-held manipulatives such as folding paper, Miras, solids, string so that students can see for themselves the rules evolving and why certain things are true.

4. Is there any topic or topics that are in the current curriculum that you believe should be eliminated? Please explain.

   I can't think of any specifics that need to be eliminated – some topics are more useful than others for future math courses (right triangles etc.) but I believe we have a good balance of topics.

5. Do you include real world applications in your geometry course? What are these and why are they included?

   We talk about architecture, engineering, art – golden ratio...

I concluded that this respondent definitely had a positive score on factor three – a disposition towards abstraction. He uses different manipulatives and Geometer's Sketchpad which would give him a positive score on factor one – a disposition towards activities. He talks about interesting applications which would have me believe he had a positive score on factor two – a disposition towards an appreciation and applications of geometry. I was wrong! His score on factor 2 was a low negative, which placed him in Group 3 and not in Group 1. If he misread a single statement on the questionnaire it could have thrown his factor score off or it could mean that although he includes real world applications his belief is that it may not be necessary. Also he may not believe that geometry is for all students. I concluded that I had to be careful about relying on the short questionnaire alone when trying to profile respondents.

Respondent #51 was a second respondent whose factor score placed her in group 3.

1. What do you most love about geometry and why?

I was a very strong math student in high school and did algebra, trigonometry and calculus independently. The only class I had to go to was geometry because it didn't come naturally to me. I have to say the best thing about geometry is that it helped me develop mathematical skills that I didn't have including a spatial sense. The other thing I love about geometry is all its connections to art.

2. What is your most memorable experience or experiences as a student in a geometry class?

As I said I was a very strong math student and all the other mathematics came very easily to me. Geometry was different and I had a lot to learn. I remember being aware that it was harder for me, but at the time I attributed it to the fact that it wasn't real mathematics. (I don't think that now)

3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

I like to use lots of different colored markers and chalk. This enables students to see the different parts of the whole. It also enables students to see overlapping parts as two separate parts. I found that students who can't visualize well find this very helpful. I also like to use Geometer's Sketchpad - this speeds up the drawing process and many students enjoy the computer much more than trying to draw it themselves. Another advantage is that students can try many related cases easily which is good for exploring theorems.

4. Is there any topic or topics that are in the current geometry curriculum that you believe should be eliminated? Please explain why.

I taught honors (sic) *and I thought everything should be included. If someone taught slower kids and couldn't cover everything that would be a reason to exclude areas of geometry.*

5. Do you include real world applications in your geometry course? What are these and why are they included?

The only real world applications that I included were those that the book included. I admit that this isn't one of my strengths. I believe that this is something I would have changed if I had continued teaching high school geometry. I think that applications are *important for students who feel math is very theoretical and irrelevant. As a student myself I loved math for its theory and it took me a while to see that isn't what attracts most students.*

When respondents claimed geometry is more visual or that they used manipulatives it was sometimes difficult to determine exactly what they meant. In the above response the teacher uses coloured chalk and markers to mark off overlapping pieces. I can surmise that she does this when doing proofs with her class giving her a positive score on factor 3. She uses Geometer's Sketchpad but does not mention manipulatives which gives her a low positive on factor 1. She herself admits in response to question 5 that she is not strong on applications, which gives her a low negative on factor 2. So her positive, negative, positive scores place her in group 3. In this case, I was able to determine whether the scores were high or low based on her responses.

Respondent #13 had some similar responses but was not in the same group as respondent #51.

1. What do you most love about geometry and why?

   The unexpected simplicity that arrives from a seemingly complex situation. It makes me feel that there really is order in the universe.

2. What is your most memorable experience or experiences as a student in a geometry class?

   Being asked to solve challenging problems by Mr. Slavin at Lincoln H.S.

3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

   Colored (sic) *chalk to highlight specific parts of the diagram, making it easier for students to focus on one part and then another.*

   Geometer's Sketchpad is helpful too, but I haven't used it as much as I might have liked.

4. Is there any topic or topics that are in the current geometry curriculum that you believe should be eliminated? Please explain why.

Too much has already been eliminated.

5. Do you include real world applications in your geometry course? What are these and why are they included?

Rarely. Probably not as much as I should, and probably because I find them uninteresting.

Respondent #13 is seems to have a positive disposition towards abstraction. He does not mention manipulatives but says that Geometer's Sketchpad is helpful but does not use it too much. He could have a low positive or low negative score on factor 1. His score on the first factor was negative. I had to return to his responses to the original questionnaire to find out whether he used manipulatives or not. Finally, he rarely uses real world applications which might give him a negative score on factor 2, but he has a strong appreciation of geometry which I found in his response to question 49a: "*It is stimulating and thought provoking.*" He has a low positive score on factor 2. His negative, positive, high positive puts him in group 5. To get the best 'picture' of a respondent the researcher has to take all the data into account.

Respondent #44 gave exuberant responses.

1. What do you most love about geometry and why?

    It's beauty, the way it makes sense. The way it gets you to think about things. For instance if you are working on a more traditional algebraic question and you try to visualize it geometrically it indubitably becomes more interesting and easier, at least for me, to understand.

    I love Euclid, the way the proofs build on each other, how nice it is to work within a system.

    I love that it connects to art and architecture, that it fills our world with beauty.

2. What is your most memorable experience or experiences as a student in a geometry class?

I studied at St. John's College where we spent three fourths of the year studying Euclid's elements. I just remember getting to that last proof in book 12 that explains why there are only 5 possible Euclidean solids and I remember thinking, wow, that is so beautiful and it just makes so much sense.

3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

I have to admit I'm a big fan of Geometer's Sketchpad, though usually I just want to go old school and make a ton of constructions. I love thinking about what is and isn't possible with a straightedge and a compass.

4. Is there any topic or topics that are in the current geometry curriculum that you believe should be eliminated? Please explain why.

I guess it depends on what the purpose of geometry is. Unfortunately many of my favorite parts of geometry are also the parts I think should be eliminated because they are not that useful. At the same time if those areas could be what would most greatly engage students, then they should be left in. I guess I have not seen enough people who are able to engage the students around these areas, so unless I'm teaching (or someone with the same passion and interests) probably leaving it be would be better.

5. Do you include real world applications in your geometry course? What are these and why are they included?

Of course. Geometry is so fundamental to everything we do. How animals work and live, how we as humans live, what we do to our world. Everything around the golden rectangle, fractals, tessellations and transformations are practical and beautiful. Geometric ways of understanding the other math we do is essential as well and I think that often ties to the practical and real world. Why do bees make hexagonal honeycomb? Why are buildings so dependent on rectilinear shapes? All these interesting civil engineering applications, building bridges etc...

Respondent #44 is passionate about geometry. He had positive scores, although not especially high, on all three factors placing him in group 1.

Respondent #225 loves proofs as an adult but hated geometry as a student.

1. What do you most love about geometry and why?

As an adult, I love proofs; they force me to sit down and truly understand why something works. As a teacher, I loved the chapter on parallelograms because I *felt my students were capable of understanding them and that they were still challenging.*

2. What is your most memorable experience or experiences as a student in a geometry class?

I honestly remember absolutely hating geometry as a student. It was the worst that I ever did grade wise in any course.

3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

Because most of my students were very technologically inclined, Geometer Sketchpad was great, however, I had a difficult time monitoring all of my 25 students in a lab at the same time. I never found the perfect solution to this problem.

4. Is there any topic or topics that are in the current geometry curriculum that you believe should be eliminated? Please explain why.

I think maybe the section on logic was unnecessary only because they never really use it. The level of the proofs was so basic that they didn't need logic to prove them and thus it was not relevant to any other section in the course.

5. Do you include real world applications in your geometry course? What are these and why are they included?

I used real world applications in the course I taught in NYC because the book was all applied. When I changed school districts, I would have loved to have brought some of that into the course, however, there was no time There were school-wide exams at midterm and final (for which the questions were decided by the department chair) and if I added something else in, I lost time and put my students at a disadvantage for these tests. Unfortunately, there was very little time to supplement or deviate at all from the curriculum.

This respondent does not discuss applications with her students. She is doing proofs but hated it as a student. Her early experiences with proof had a negative influence on her and she has a negative disposition towards abstraction. She uses Geometer's Sketchpad but has classroom management issues. Her factor scores are positive, negative, negative which places her in group 7.

Respondent #240 had interesting responses.

1. What do you most love about geometry and why?

This question assumes I love geometry (I'm an algebraist myself). I guess its diagram-y goodness is useful for solving certain kinds of problems. Celtic knots are cool too. I have to respect geometry since the once happily algebraic group theory almost always winds up being somehow geometric. (Solving a 15 puzzle, for example.)

2. What is your most memorable experience or experiences as a student in a geometry class?

As a student, my clearest memories are my graduate work. I think you're asking about my high school experience. For that, my math team coach gave me a copy of an article in Mathematics Teacher that dealt with infinite area sums in a pentagon. All I remember was reading that over and over again until it made sense.

3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

Coloured chalk. It makes it easier to highlight the areas I'm talking about. I've been known to do the usual "string around a can lid" to teach circumference formulas, and I cut up the plate to show the formula for the area of a circle. Slightly lame, but it makes for better classroom entertainment than "here's 50 problems, do them." Graph paper is also nice for teaching area concepts...to people who should have learned them in the 6th grade

4. Is there any topic or topics that are in the current geometry curriculum that you believe should be eliminated? Please explain why.

The curriculum wastes a ton of time on circle, area, and volume concepts that should have been mastered in junior high.

5. Do you include real world applications in your geometry course? What are these and why are they included?

> *Its tough to find real-world examples of geometry suitable for the topic the state wants us to spend that DAY on. There was that newspaper article about the man arrested because he was dealing drugs within 1000 feet of a school zone. The court upheld that the distance is calculated using the Pythagorean theorem, not by adding the two straight-line distances one would have to walk in Manhattan to get from the arrest site to the school. (It worked out that if you added the legs it was more than 1000 feet, but the hypotenuse was under 1000 feet so he got extra jail time.) The article suggested the school system hire him when he gets out.*

Firstly, respondent #240 tolerates geometry but it is not his passion. There is an undercurrent of negativism in the responses to questions 3, 4 and 5. The respondent is 'putting down' his students' ability in questions 3and 4. He is being critical of the curriculum in questions 4 and 5. He has negative scores on all 3 factors which places him in group 8. The school management could decide whether someone who has three negative factor scores should teach geometry.

Most of the respondents to the follow up questionnaire loved proofs because of its puzzle-like nature. One respondent wrote:

"I always loved the proofs because it was like unravelling a mystery or like doing a puzzle."

The analysis of the follow up questionnaire can inform department chairpersons which teachers might be best suited to teach geometry.

7.4 THE CASE OF ROSE

One of the questions that I wanted to follow up after the preliminary questionnaire was: What happens in a class where a teacher is required to teach geometric proof but has scored negatively on factor 3: a disposition towards abstraction? Could something be done to help a teacher overcome a negative disposition towards abstraction?

The rest of this chapter describes the case of Rose, a high school mathematics teacher who was at the end of her third year of teaching. During the previous year, she had been one of the respondents to the questionnaire. Her scores on the three extracted factors (See chapter 6) were positive on factor 1: a disposition towards activities, positive on factor 2: a disposition towards appreciation of geometry and its applications, and negative on factor 3: a disposition towards abstraction. These scores placed her in Group 2 (positive, positive, negative).

I observed Rose's class several times. I also had the results of the factor analysis and my intention was to find a way to make her comfortable teaching students about proof. The intervention is described below, along with her responses to a further follow-up questionnaire. When she taught with the movable cards containing statements and reasons which were part of the intervention she felt more at ease.

7.4.1 The Study

In my position as a mathematics specialist-consultant, I was carrying out professional development in Rose's school and the principal suggested that I observe Rose's class in which she was about to start teaching geometric proof. This provided an opportunity for me to delve further into these questions, using Rose as an 'opportunistic sample'. It was in the position of observer participant that I was present in and observed Rose's class seven times taking extensive field notes.

I met with Rose each morning to discuss her lesson plan for the day and after each of these classes to conduct a debriefing with her. During these sessions I made several suggestions, such as always listing all six corresponding parts of congruent triangles on the board when referring to them and marking them on the diagrams. She implemented these suggestions and others described below in her class almost immediately. I also gave Rose a copy of the new five question open response questionnaire which she completed. The hope was that responses to this last questionnaire would give further insight into teachers' beliefs that were not captured in the original. The questionnaire can be found in Appendix H and Rose's responses can be found in section 7.4.4. This study was presented to Rose so that she could concur or refute any inferences made.

7.4.2 Rose In Her Second Year Of Teaching

Rose who has an undergraduate degree in mathematics education had taught ninth and tenth grade mathematics in a small urban high school for two years. She was in her late twenties, when I started to work with her. She is enthusiastic in the classroom and she exhibits good classroom management skills.

When completing the original questionnaire, she had included her email address and telephone number so that I could contact her for further questioning. When the factor analysis was performed on all the Likert data (See chapter 6), her factor scores on the first two factors: **a disposition towards activities** and **a disposition towards appreciation of geometry and its applications** were both low positive. Her factor score on factor 3: **a disposition towards abstraction** was a high negative. I therefore went back to look at her actual responses to a number of statements on the questionnaire.

She responded *disagree slightly more than agree* to the following statements:

4. Learning to construct proofs is important for high school students.

6. Geometry should be included in the curriculum for all students.

13. High school students should be able to write rigorous proofs in geometry.

This indicated to me that Rose was concerned about teaching average or below average students how to do proofs.

Rose responded *agree slightly more than disagree* to these statements:

1. I enjoy teaching geometry.

2. Learning geometry is valuable for high school students.

9. Geometry should occupy a significant place in the curriculum.

10. High school geometry should not contain proofs.

21. I enjoy doing geometric proofs.

These responses appear to show that Rose believed that geometry is worth learning and that she did enjoy teaching geometry as long as she did not have to teach students how to do proof. She herself likes doing proofs.

Rose responded *strongly disagree* to the statements:

16. My students enjoy doing geometric proofs.

44. I enjoy teaching my students how to do geometric proofs.

Rose responded *moderately agree* to the statement:

48. I enjoy proving theorems for my students.

These responses and the conversations that I had with her led me to conclude that she was uncomfortable about teaching students how to do proofs. The fact that she enjoyed proving theorems for students and doing proofs gave me a glimmer of hope that she might reconsider teaching proofs if she was armed with the appropriate tools and therefore more confident.

7.4.3 Rose In Her Third Year Of Teaching

By her third year of teaching, Rose had a desire to teach mathematics to upper grade students and so she sought and accepted a position at another small urban high school whose students were supposedly "more academic" than at Rose's first school. I was doing short-term professional development at the school where I worked with three of the four mathematics teachers. The principal asked me to work with both Rose and another teacher who were both starting a unit on proof in geometry. Although I observed both teachers and suggested similar interventions this study focuses on Rose because she was an identified respondent to my questionnaire.

Ball, Bass, and Hill (2004) studied teaching and suggested eight types of problem-solving that teachers do in their 'work of teaching'. These are

1. *Design mathematically accurate explanations that are comprehensible and useful for students*
2. *Use mathematically appropriate and comprehensible definitions*
3. *Represent ideas carefully, mapping between a physical or graphical model, the symbolic notation and the operation or process*
4. *Interpret and make mathematical and pedagogical judgements about students' questions, solutions, problems and insights*
5. *Be able to respond productively to students' mathematical questions and curiosities*
6. *Make judgements about the mathematical quality of instructional materials and modify as necessary*
7. *Be able to pose good questions and problems that are productive for students' learning*
8. *Assess students' mathematics learning and take next steps*

Kazima and Adler (2006) condensed the eight aspects of problem-solving into six in their study of the teaching of probability:

1. *Definitions*
2. *Explanations*
3. *Representations*

4. *Working with students' ideas*
5. *Restructuring tasks*
6. *Questioning*

Teaching students how to prove theorems involves all of the above. Rose exhibited these problem solving skills in her teaching of other aspects of geometry. Could Rose incorporate these skills when teaching proof? I believed I could share a method of teaching students how to do proofs that would be appealing to Rose. The method is described below.

7.4.3.1 Congruent Triangles

The students in the class were learning how to prove geometrical results. They were mostly tenth graders who had already learned definitions and properties of triangles and quadrilaterals in the ninth grade or the beginning of the tenth grade.

Rose used the concept of congruent triangles as a vehicle for introducing students to proving conjectures.

The following is a snapshot of the type of questions Rose asked on the first day of the unit:

"What makes triangles congruent?" Students respond that the triangles have to be exactly the same. Rose then drew a picture of two triangles on the board. How can I show that triangle *ABC* is congruent to triangle *DEF* (See Figure 7.1) based on the information given?

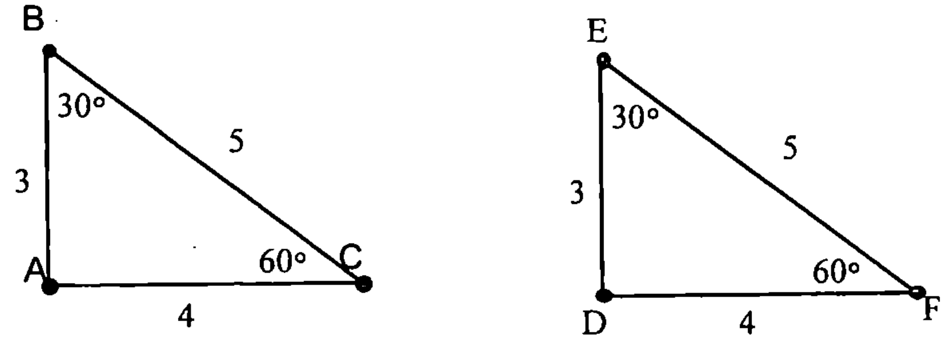


Figure 7.1 ***Rose's example of congruent triangles***

The students recognised that the triangles were congruent from the given information. No student noticed that these triangles couldn't really exist because in a 30-60 triangle the length of the side opposite the 30 degree angle is equal to half the length of the hypotenuse. Bills, Dreyfus, Mason, Tsamir, Watson, and Zaslavsky (2006) asserted that when selecting instructional examples the teacher should take into account 'learners' preconceptions and prior experience'. Zaslavsky and Zodik (in press) studied what considerations went into teachers' choices of examples. They found there was a tension between the desire to construct real-life examples and mathematical accuracy. A random choice of example could lead to an impossibility. When Rose and I discussed her example she was surprised at what she had done. She expressed a desire to be more careful about her choice of examples in the future.

Another question that Rose posed was whether the information given was sufficient to prove triangles congruent: She drew the diagram shown in Figure 7.2 and asked students, "In the square ABCD, is ΔABC congruent to ΔADC?"

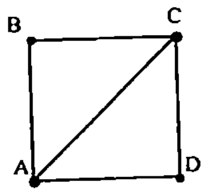
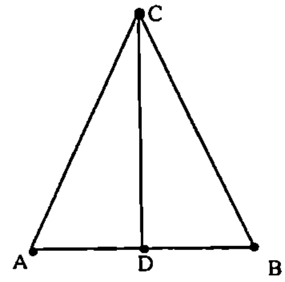


Figure 7.2 ***Rose's second example***

Her students had to remember the properties of a square in order to answer this question. They knew the sides were all congruent. Rose wanted the students to focus on SSS congruent to SSS. Rose had the students rely heavily on the visual aspects of the problem. I suggested that she have the students investigate the other congruence relationships. I loaned her Michael Serra's book *Discovering Geometry: an Investigative Approach* (2003). She prepared a hands-on lesson for the investigation: Is ASA a Congruence Shortcut? Rose gave each group of students a work sheet with a line segment and two angles drawn on it and asked them to construct a triangle. The students used scissors and tape to cut out the segment and angles and paste them together to form a triangle. The worksheet can be found in Appendix I. She had the groups compare their results. Rose placed the results up on the bulletin board.

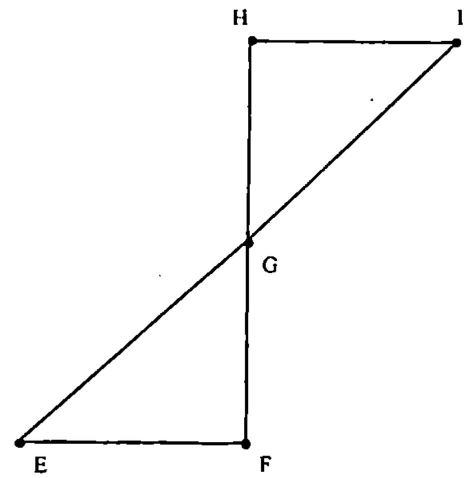
Rose kept telling me that she was anxious about having the students do actual proofs. I gave her three worksheets from a set of worksheets I had received from Sandra Gundlach, a teacher, who had presented them at a conference. The first one had six statements to prove along with a diagram for each (See Figure 7.3). The next two sheets had mixed up answers to each of the proofs from the first sheet. I brought in envelopes with the given, the "to prove", and the diagram for each of the six proofs taped onto the outside and the cut up statements and reasons inside.

1. Given: D is the midpoint of $\overline{AB}$
$\angle ADC \cong \angle BDC$
Prove $\angle A \cong \angle B$



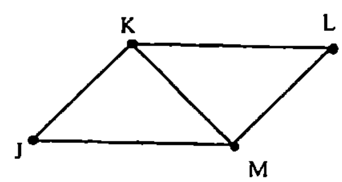
2. Given: $\overrightarrow{CD}$ bisects $\angle ACB$
$\angle ADC \cong \angle BDC$
Prove $\overline{CA} \cong \overline{CB}$

3. Given: G is the midpoint of $\overline{EI}$
$\angle E \cong \angle I$
Prove $\angle H \cong \angle F$



4. Given: $\overline{HI} \parallel \overline{EF}$
$\overline{HI} \cong \overline{EF}$
Prove: G is the midpoint of $\overline{EI}$

5. Given: $\overline{KL} \cong \overline{JM}$
$\overline{JK} \cong \overline{LM}$
Prove: $\angle J \cong \angle L$



6. Given: $\overline{JK} \parallel \overline{LM}$
$\overline{JK} \cong \overline{LM}$
Prove: $\overline{KL} \cong \overline{JM}$

Figure 7.3 ***Sheet 1: Proving Triangles Congruent***

Rose took proof #1 (See Figure 7.4) and enlarged the cut up statements and reasons. She taped them to the blackboard, wrote the given and to prove statements, and drew the accompanying diagram. Some of the students had difficulty with how to use the definition of midpoint. Rose used coloured chalk effectively to illustrate. My suggestion was to use Geometer's Sketchpad to demonstrate angle bisectors in proof #2, but the technician was not available to bring a laptop to Rose's classroom. (I mention this here to make the point that even if a teacher wants to use technology it is not always readily available.) Students complained that one angle looked bigger than rather than equal to the other angle. (Sometimes such arguments are productive but in this case time was wasted).

D is the midpoint of $\overline{AB}$	$\overline{AD} \cong \overline{DB}$	$\angle A \cong \angle B$	Given
$\angle ADC \cong \angle BDC$	$\overline{CD} \cong \overline{CD}$	$\Delta ACD \cong \Delta BCD$	Given
Reflexive	CPCTC	Def of midpoint	SAS

Figure 7.4 ***Proof #1 Mixed up answers***

Rose used a metaphor of identical twins to help the students understand that corresponding parts of congruent triangles are congruent. "If the twins are identical, what can you say about their eye colour, their height etc.?" The students responded, "They are the same." "So if the triangles are congruent by SSS, SAS or ASA, what can you say about the other parts of the triangles?" The students were able to understand this concept. In the United States some teachers abbreviate the statement *corresponding parts of congruent triangles are congruent* – CPCTC. Unfortunately many students use the abbreviation but fail to remember what it represents.

Some students struggled with the logical sequencing of the steps. In proof #4 they placed the statement G is a midpoint of EI in the middle of the proof. One student, Gary said, "You have to look at cause and effect." This was a useful insight.

Eventually Rose used the same format for proofs that she found in the text. She assessed how the students were doing by giving them a quiz where all the statements and reasons were written in mixed-up order on the page and the students had to put the proof together correctly. She was pleased with the results.

7.4.4 Rose's Response to the Five Question Follow-Up

Rose's responses to the follow up questionnaire discussed in 7.3 above were:

1. What do you most love about geometry and why?

I love geometry proofs. I feel they help students think logically. A proof is like a jigsaw puzzle where everything must fit and when it is complete it's a nice accomplishment. Proofs make students realize that nothing in geometry can be taken for granted there always has to be a reason.

Rose's response indicates a positive experience with proofs, but I knew from conversations with her that she was worried about teaching proofs. Her next response gave me a glimpse into why she was anxious about teaching students how to do proofs.

2. What is your most memorable experience or experiences as a student in a geometry class?

My teacher explained the topics very thoroughly. However eliminated geometry proofs from the curriculum. I feel this turned me off from proofs for quite some time.

3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

I use coloured chalk to outline certain things so the students can see it more clearly. I have also used string and rulers so students can measure angles and they can see the *relationship between angles such as alternate interior angles and a linear pair. I have used sketchpad in the past. The visual is very important in geometry since once students see the relationship visually they can apply it to any problem.*

This question was included in this short questionnaire because responses to the open ended questions that talked about geometry being more visual than algebra did not reveal enough information on how teachers used visuals in their geometry classes.

4. Is there any topic or topics that are in the current geometry curriculum that you believe should be eliminated? Please explain why.

I believe constructions should be eliminated from the curriculum, time does not allow for it.

This question was included to try to find out what teachers do not value in geometry. The way the curriculum is arranged in Rose's state, geometry is part of integrated courses. Constructions are taught in the first course and proofs are taught in the second course. There is no context for the unit on construction. It is left to the last lessons of the course. Rose cannot do justice to the topic and therefore wanted to see it eliminated.

5. Do you include real world applications in your geometry course? What are these and why are they included?

Geometry is a topic in mathematics that lends itself to real world application. I tell my students geometry is something that is used in every field in the working world. Construction works as well as carpentry works need to know geometry. Individuals who work in advertising need to think about space when they make up an advertisement. Police officers need to use geometry when they are on a chase or when a shooting occurs. This year I took my students outside in the courtyard and we went around looking at the building and trying to find quadrilaterals and explain their properties and purpose by looking at them as well as their purpose in the building.

I was able to understand Rose better from her responses to this short open-ended questionnaire. Her own experiences with proof in high school (Raymond, 1997) influenced her belief that it would not be easy to teach students how to prove. Rose did not understand the relationship between constructions and proof (Schoenfeld, 1988) and felt that teaching constructions should be eliminated from the curriculum. The curriculum emphasises the procedure for constructions. Since Rose's high school teacher did not teach proof to the class she may have had the students working aimlessly at constructions which is what Rose did in her own class and felt it was a waste of time. In Rose's responses, the *formal, intuitive,* and *utilitarian reasons* for studying geometry can be found.

7.4.5 Case Study Conclusions

Rose's factor scores on the questionnaire placed her in group 2. Rose had a fear of teaching students how to do proofs. From her response to question 2 above we find that because Rose's teacher did not teach her how to do proofs when she herself took a geometry class, she was reluctant to now teach her own students how to do proofs. From another perspective, Rose left her first high school teaching job in order to teach at a school with more academic students. Not all of her students at the second high school were as academic as she expected. She might have believed that many of them were not capable of doing proofs. I created an intervention by showing her an approach to teaching proofs that fitted well to her disposition to work in a hands-on manner and use manipulatives since she had a positive score on factor 1. She used the intervention successfully in her class and has now requested to teach two sections of this course in the coming year. She has also taken an intermediate level training course in Geometer's Sketchpad during the summer in order to become more adept at using it in her class when she is teaching geometry (Cinco and Eyshinskiy, 2006).

In this one case, by looking at the factor scores I was able to find an appropriate intervention for the teacher. Can one look at the factor scores of other respondents and introduce them to interventions that would help them in their teaching of geometry? We can't generalise Rose's success to others since Rose was already implementing most of the aspects of problem solving in her work as a teacher (Ball, Bass, and Hill, 2004; Kazima and Adler, 2006).

Rose believed that she has a professional responsibility to continue learning and perfecting her craft. Beswick (2007) refers to this belief as "*commitment to seeking out 'second voices' and is related to a propensity to reflect on one's practice with a view to continual improvement*" (p. 115). She attributed the notion of "second voices" to Lerman (1997). Rose was willing to incorporate suggestions made to help improve her practice. Teachers who are unwilling to listen to "second voices" may not be able practice their espoused beliefs.

7.4.6 Follow-Up: Rose In Her Fourth Year Of Teaching

During Rose's fourth year of teaching I observed her class at the beginning and towards the end of her unit on proof. She again used investigations to verify conjectures about when triangles are congruent (Serra, 2007). She displayed the results of these investigations on the classroom walls. She also used the cut out statements and reasons that I had shown her the previous year. She increased the number of proofs that her students did using this method. Her questioning had improved. She had the students planning out their proofs. She asked, "Why does this belong here? Why can't it be placed earlier in the proof?"

On examinations she included matching up statements and reasons instead of cutting them out. She then had the students put the matched up pairs into a formal proof. Her examination and a homework problem done by a student can be found in Appendix J. Some of her students were finally able to complete proofs on their own. There was another geometry teacher in the school who successfully used the intervention. She was more confident than Rose in her teaching of geometry and once I showed her the movable cards, she created card sets for many proofs and had the students work in groups and present their solutions when they completed the proofs.

7.5 CONCLUSIONS

The findings seem to show a possible disconnect between some high school teachers' beliefs about why it is important to study geometry and the current position of the Standards movement. The PSSM (NCTM, 2000) was released in 2000. Afterwards 49 of the 50 states in the United States adopted Standards based on PSSM. The Standards provide reasons for including geometry in the curriculum that mirror the mathematics, utilitarian and intuitive arguments for its inclusion (Gonzalez and Herbst, 2006). Many teachers are using the formal argument. For example respondent #496 in response to question 50 that asks about the difference between teaching geometry and other areas of mathematics claims, *"...in most of their math classes they are always asked to give a correct answer, but in geometry they are asked why is that the answer and can you prove it."* If teachers are waiting for the geometry class, which not every student takes, to ask *"Why?"* and *"Can you prove it?"* then mathematics education is facing a major challenge. Teacher education programs and professional development interventions which encourage teachers to challenge students early in their mathematics courses should be developed.

Respondent #154 expresses her view on what a geometry class might look like if 'taught correctly':

> *I feel that there is so much more "exploring" and concluding and allowance for different ways of doing things and seeing things. I also strongly believe that if taught "correctly" it would be nearly impossible for a student to do well just from 'memorization* (sic)*." I think this is the reason so many students find it so frustrating. They are used to "memorizing" and "doing" problems that are similar to ones in class where they get "an answer." Geometry requires much more thinking. I love it!*

Respondent #96 suggests: *"Good teaching is good teaching regardless of the content. Instruction needs to be delivered in a way that is meaningful and motivates the students. Once a student is interested, the learning process becomes easier."*

Further implications

Some of the responses to the open questions above have claimed that students have difficulty with proof and that some teachers find geometry difficult to teach because of this.

In the case study we find Rose who moved schools but still seemed to have a deficit view of the students she taught. What would have happened to the students in Rose's class if I did not encourage her to use proofs with her geometry class? It seemed to me that she needed to be encouraged to believe in her own ability to teach proofs and in the ability of her students to do proofs. How many other mathematics teachers are there out there not using proofs with their students because they believe that proofs are too difficult?

Are other teachers encouraged to use proofs with their students even if the teachers think their students are not capable? It would seem to me that there might be implications for mathematics curriculum. If the standards suggest students should learn proof and curriculum designers are including proof in their textbooks and programs what can be done for this disconnect?

CHAPTER 8 – CONCLUSIONS

This investigation into teachers' beliefs about geometry and their approaches to its teaching and learning, using both quantitative and qualitative methods of analysis, has tried to answer the following research questions posed in Chapter 1:

- What are high school mathematics teachers' beliefs about the role of geometry in the curriculum?
- What are high school teachers' beliefs about the use of manipulatives and dynamic geometry software packages?
- What are high school teachers' beliefs about the role of proof in geometry?

As a result of the analysis I have found eight typologies for high school geometry teachers. I also have important results that illuminate the findings around my original research questions. These are findings that inform teachers, teacher trainers and curriculum planners and will be discussed in this chapter.

8.1 TYPOLOGIES OF GEOMETRY TEACHERS

SPSS was used to perform factor analysis (Chapter 6) on the data from the 48 Likert statements on the questionnaire. A three component solution was extracted using principal component analysis with varimax rotation. The three components are:

- A disposition towards activities
- A disposition towards an appreciation of geometry and its applications
- A disposition towards abstraction

The respondents' scores on the three factors extracted allowed for the creation of eight typologies of teachers as shown in Table 8.1. These characteristics lead me to believe that a teacher belonging to Group 1, with positive scores on all 3 factors, is probably best suited to teaching geometry.

Group	Factor 1	Factor 2	Factor 3	Number of respondents
1	Positive	Positive	Positive	65 (16.8%)
2	Positive	Positive	Negative	64 (16.6%)
3	Positive	Negative	Positive	41 (10.6%)
4	Positive	Negative	Negative	42 (10.9%)
5	Negative	Positive	Positive	59 (15.3%)
6	Negative	Positive	Negative	36 (9.4%)
7	Negative	Negative	Positive	42 (10.9%)
8	Negative	Negative	Negative	37 (9.6%)

Table 8.1: *Factor score profiles*

Several statistically significant results were found when the chi-squared test was applied to the crosstabulations between the groups to which respondents belonged and their personal data. Perhaps of most importance is the significant relationship found between teaching geometry as a one year course and results from factor analysis of the data. More teachers than expected who were in Group 1 taught geometry as a year long course. In other words teachers who teach a one year course in geometry are more likely to be in group 1.

There were also significant results with respect to taking undergraduate geometry courses, methods courses (pedagogy), having a graduate degree and membership of professional organisations. Teachers' content knowledge and pedagogical content knowledge are related to the group to which the teacher belongs in that significantly more teachers than expected who took undergraduate geometry courses, or who took methods (pedagogy) courses, or who had a graduate degree belonged to Group1. Similarly, significantly more teachers than expected who are members of professional organisations belong to Group 1. Although not statistically significant the group that contained the most teachers with a mathematics education major (first degree) was Group 1. Interestingly, the groups that contained the most mathematics majors were Groups 2 and 5. Teachers in Group 5 had a negative, positive, positive on factors 1, 2, and 3, respectively. Just over 45% of the teachers with a mathematics degree do not have a positive disposition towards activities. The group that contained the teachers who use manipulatives most was Group 2. The group that contained the most teachers who don't use manipulatives or dynamic geometry software was Group 5. Groups 1 and 2 contained the largest number of teachers using dynamic geometry software. Groups 1 and 2 contained the largest number of female teachers while the largest number of male teachers belonged to Groups 1 and 5.

8.2 IMPLICATIONS FOR SUCCESSFUL PROFESSIONAL DEVELOPMENT

Knowing the typologies to which teachers belong could allow for very specific and prescriptive professional development programmes to be developed to meet specific needs and goals.

At about the same time that I started investigating high school mathematics teachers' beliefs about teaching and learning geometry another study examining 88 middle school mathematics teachers' epistemological and geometry-related beliefs was carried out (Langrall, Alagic and Rayl, 2004).

These researchers found that the majority of teachers believed that geometry is difficult and/or that more sophisticated teaching methods are needed to develop understanding. Also, almost half of the respondents *somewhat disagreed* or *strongly disagreed* with the statement that geometry is useful. Langrall et al. account for this, saying that:

"Believing that word problems are not part of geometry suggests that teachers' prior experiences with learning geometry may have excluded much emphasis on word problems or that teachers do not see many real world applications for geometry." (p. 11) Their study included teachers who did not learn geometry in high school, so they

had very limited prior experiences with geometry. This result is significantly different from the teachers in my study where over 95% of the teachers, as shown in Table 5.6a, believed that geometry has many real world applications. The high school teachers in my study had stronger geometry backgrounds than the middle school teachers in Langrall's study with 68.3% of my respondents having taken an undergraduate geometry course.

The content knowledge of teachers is an important factor when making connections between geometry and the real world as well as between geometry and other mathematics. The success of any professional development depends on appropriate provisions for teachers with such diverse beliefs.

For teachers, in my study, belonging to groups 3, 4, 7 and 8 who have a negative score on factor 2: *A disposition towards appreciation of geometry and its applications*, an appropriate professional development would focus on making them more aware of the applications of geometry and on developing an appreciation of the subject. In practice this might include providing books and reading materials that contain real world applications of geometry and examples e.g. in nature, news clips, architecture magazines, building instructions etc. The teachers could also be guided to find concepts that could be embedded in the given situation in a similar way to that suggested by Langrall et al. for the middle school teachers. Teachers can then begin to see how geometric topics are involved in everyday life and can develop an appreciation of geometry.

Langrall et al. proposed grouping teachers according to three teacher characteristics in professional development settings. The characteristics are conceptual level, content knowledge and job commitment. Langrall et al. identified eight profiles depending on the teacher's 'level' on the three characteristics. These eight profiles function in a similar manner to my eight groups with their high conceptual level, high knowledge, and high job commitment being most receptive to professional development. Similarly, my positive disposition towards activities, positive disposition towards appreciation of geometry and its applications and positive disposition towards abstractions could be used to predict the best candidates for teaching high school geometry.

I believe it would be counterproductive to form a group during a professional development workshop with only teachers from Group 8 because with respect to geometry, these teachers do not seem to have any strengths on which to build. My study identifies the needs which professional development for geometry teachers might address, while Langrall et al. identifies the teachers who might respond. Putting them together creates a powerful approach for professional development.

As was shown in Table 6.3, there are ten items loading on factor 1 and eleven on each of factors 2 and 3. The first loading on factor 1 suggests a strong belief about manipulatives: they make geometry learning fun. Of interest, is an equally strong loading on factor 3: that learning to construct proofs in geometry is important. Although only loadings greater than 0.4 were reported in Table 6.3, each item loaded onto all three factors to some degree either positively or negatively. In chapter 6, it was suggested that the three extracted factors may correspond to Ernest's (1989) three views of mathematics. A disposition towards activities corresponds nicely to his problem solving view and a disposition towards abstractions corresponds to his Platonic view. A disposition towards appreciation of geometry and its applications did not quite fit Ernest's instrumentalist view of mathematics. Ernest was studying global beliefs about mathematics and my study investigated the domain specific beliefs about geometry (Törner, 2002; Aguirre, in press). The disposition towards appreciation of geometry and its applications includes the beliefs that studying geometry leads to a positive attitude about mathematics, that geometry should occupy a significant place in the curriculum and that geometry is for all students. I believe that factor 2 is the lynchpin for identifying teachers who may be able to bring out the better of the two extremes. In other words a disposition towards abstraction can be associated with a traditional view of teaching and a disposition towards activities can be associated with a 'constructivist' view of teaching which is more student centred (Ernest, 1989). Teachers in groups 1-4 have a positive disposition towards activities, teachers in groups 1, 3, 5 and 7 have a positive disposition towards abstractions, and teachers in groups 1, 2, 5, and 6 have a positive disposition towards appreciation and applications of geometry as shown in Table 8.1. Therefore teachers in groups 1, 2 and 5 may have beliefs that make them the most viable candidates for teaching geometry, depending on the goals of the curriculum.

Building on teachers' positive strengths can be used to improve their dispositions that are negative. This result can be seen in the case study where I provided professional development and scaffolding for teaching proof. Rose, who was in Group 2 with a negative disposition towards proof, was enabled to successfully teach her students how to do proofs. (Chapter 7)

8.3 EUCLIDEAN ZEALOTS

Before performing factor analysis on the data, I used descriptive analysis to find percentages of agreement or disagreement with the Likert statements and to analyse the respondents' personal data (Chapter 5). Chi squared tests were applied to the cross tabulations between Likert statements and personal data (Chapter 5).

The quantitative analysis of statement 29, which can be found in chapter 5: *Initially high school geometry should be hands-on with proofs coming later in the course* yielded an important result. Although there were five other statements on the questionnaire referring to the use of manipulatives or hands-on activities, statement 29 had the lowest consensus among the respondents (75.5% as shown in Table 5.6b). Also, when the chi-squared test was applied to the cross-tabulation between statement

29 and the statement *I have used manipulatives to teach geometrical concepts* these statements were found to be independent of each other with $p = 0.6894$ as shown in Table 5.14. The other five statements had a statistically significant relationship with the statement *I have used manipulatives to teach geometrical concepts* as shown in Tables 5.9-5.13. Finally when the factor analysis was performed on the data, as shown in Table 8.2, statement 29 loaded negatively on factor three - *a disposition towards abstractions* whereas the other five statements loaded positively on factor 1-*a disposition towards activities.*

The negative loading on factor 3 indicates that there are respondents who are absolutists or Platonists (Ernest, 1989) who believe that proof should always be taught as it has been taught and who believe that introducing it via hands-on activities detracts from proof or cheapens it in some way. They believe that proofs are the primary focus of high school geometry and must be thought of as such, not as difficult to understand distractions of a curriculum focused on hands-on activities. Respondent #474 who is in Group 7 stated, *"I think that the diminishing role of proof makes geometry less challenging."*

There may be teachers who were in group 7, as shown in Table 8.1, having negative factor scores on factors 1 and 2 and a positive score on factor 3, who believe that geometry should not be taught to all students. These teachers may skip the applications part of the textbook because, as geometry purists, they believe geometry should be taught for its own sake with applications being of little or no importance to them. These teachers would be unlikely candidates for professional development according to Gooya (2007) and Langrall et al (2004) unless they are wisely placed together with teachers in Groups 1 and 2.

8.4 BELIEFS ABOUT GEOMETRY IN THE HIGH SCHOOL CURRICULUM

Over 90% of the respondents believed that geometry should occupy a significant place in the curriculum and over 60% of the respondents have taught geometry as a one year course.

This last result introduces a very important dilemma associated with the design of the mathematics curriculum in schools. Should geometry be integrated into the mathematics curriculum as it is in the United Kingdom or is it more appropriately taught as a one-year stand alone course called 'Geometry' as it is in many states of the United States (Schoenfeld, 1994)?

Gonzales and Herbst (2006) stated that at the beginning of the twentieth century *"The high school geometry course with its promise of training in mathematical reasoning was the beacon of non-integration"* (p. 5). Eventually some states in the United States did fuse the study of various mathematical disciplines into integrated courses.

Fehr (1972/2006) was concerned about the isolation of geometry in the high school curriculum. He believed that the rest of the world taught geometry as part of an integrated curriculum.

> Of all the developed countries of the world, the only country that retains a year sequence of a modified study of Euclid's synthetic geometry is the United States. We must immediately give serious consideration to presenting our high school students with a mathematics education that will not leave them anachronistic when they enter the university or enter the life of adult society. (p. 379)

On the other hand, Moise (1975) was against teaching geometry as part of an integrated curriculum because it would lose its structure and coherence. For over twenty years New York State high schools were teaching geometry as part of an integrated course that included a mix of many topics. Moise was correct, geometry lost its structure. It became a vehicle for practicing algebraic manipulation. From September 2008 New York State will once again be offering a year long geometry course. The benefits of integration did not outweigh the need for structure and coherence in the geometry curriculum. One reason for the change back to a one year course is to have a common understanding among the school districts throughout the state and the entire country of what the New York State high school curriculum actually is.

The intent of the proposed courses is that it go beyond the teaching of skills and procedures. There will be a focus on conceptual understanding rather than on memorisation and rote learning. The New York State Standards Committee's recommendation for the geometry course is that it be taught with a problem solving approach so that students will gain a deep understanding of geometric concepts (Brosnan and McSweeney, 2005). There will not be a return to the memorisation of "statement reason" geometric proof. They stated,

> While we believe students need to understand the essence of mathematical reasoning and proof, and that we need to be able to apply this knowledge to situations which are new to them, we do not believe they must formally prove every geometric relationship. For many students, such a course would be dull and boring, and certainly would not accomplish the goals of this committee. We envision students: exploring geometric relationships, discussing with classmates what relationships can be deduced from the knowledge given, working with physical models of plane figures and solids and using available software in their explorations. (p. 3)

My research supports geometry being taught as a one year course in high school. There were many statistically significant results regarding teaching geometry as a one year course whereas there were few statistically significant results with respect to teaching geometry as part of an integrated curriculum. I would consider these results to be statistically significant in a negative direction. More respondents than expected used manipulatives and dynamic geometry software when teaching geometry as a one year course, but fewer respondents than expected used dynamic geometry when teaching geometry as a topic in an integrated curriculum. Similar results were found with respect to applications. Of great importance is the fact that teachers who have taught geometry as part of an integrated curriculum agree significantly less than expected that geometry has many real world applications as shown in Table 8.2. As stated in the conclusions to chapter 5, geometry as a topic in an integrated curriculum seems to lack the mathematical rigour that is historically associated with this area of mathematics.

	I have taught geometry as a topic in an integrated curriculum		
Geometry has many real world applications	No	Yes	Total
Disagree	0 (3)	16 (13)	16
Agree	109 (106)	375 (378)	484
Total	109	391	500
Chi-squared = 4.6 ($p = 0.0318$) Expected frequencies in brackets			

Table 8.2 ***Crosstabulation between statement 18 and teaching geometry as a topic in an integrated curriculum***

As a result of this finding, I cross tabulated other questionnaire items with the personal statements: *I have taught geometry as a one year course* and *I have taught geometry as a topic in an integrated curriculum*. Significantly more respondents than expected who have taught geometry as a one year course agree that *geometry enables ideas from other area of mathematics to be pictured, applying geometrical concepts and thinking will help students in their future occupations or professions* and that students *should be made aware of the historical background of geometry* as shown in Tables 8.3, 8.4, and 8.5 respectively.

	I Have Taught Geometry as a One Year Course		
Geometry enables ideas from other area of mathematics to be pictured	No	Yes	Total
Disagree	12 (7)	8 (13)	20
Agree	160 (165)	308 (303)	468
Total	172	316	488
Chi-squared = 5.60 (p = 0.018) Expected frequencies in brackets			

Table 8.3 ***Crosstabulation between statement 45 and teaching geometry as a one year course***

When teaching geometry for a full year, teachers have the time to make connections to other areas of mathematics, to explore and discuss interesting applications and find ways to help students understand the relevance of geometry to their lives, which many respondents believe makes geometry important for students to learn.

	I Have Taught Geometry as a One Year Course		
Applying geometrical concepts and thinking will help students in their future occupations or professions	No	Yes	Total
Disagree	20 (13)	16 (23)	36
Agree	158 (165)	305 (298)	463
Total	178	321	499
Chi-squared = 6.69 (p = 0.0097) Expected frequencies in brackets			

Table 8.4 ***Crosstabulation between statement 47 and teaching geometry as a one year course***

When teaching geometry for a full year teachers have time to explore the historical background of geometry. They may want to teach geometry using the 17 suggested modules of Carson and Rowlands (2006).

	I Have Taught Geometry as a One Year Course		
Students should be made aware of the historical background of geometry	No	Yes	Total
Disagree	24 (13)	11 (22)	35
Agree	158 (169)	311 (300)	469
Total	182	322	504
Chi-squared = 17.18 ($p = 3.40 \times 10^{-5}$) Expected frequencies in brackets			

Table 8.5 ***Crosstabulation between statement 40 and teaching geometry as a one year course***

Furthermore, more respondents than expected who have taught geometry as a full year course enjoy doing geometric proofs, enjoy teaching students how to do geometric proofs and enjoy proving theorems for their students as shown in Tables 5.73, 5.74 and 5.75 respectively.

8.5 TEACHERS' NEGATIVITY

There were a number of responses to question 49b: *Do you think that students consider studying geometry in high school important*, that seemed to suggest a negative attitude about students. This existential presumption can be a real problem as researchers have suggested it is immutable (Abelson, 1979; Nespor, 1987; Parjares, 1992 and Rokeach, 1972). For example, a teacher with this belief characterises students as 'lazy' and will absolve themselves from trying to help the student. Some of the negative comments can be found in Chapter 7.

8.6 BELIEFS ABOUT MANIPULATIVES

Some teachers believe that manipulatives help students to grasp the basic ideas of geometry, that using manipulatives in the teaching of geometry is motivational, that the use of manipulatives makes learning geometry fun, that it is beneficial to use manipulatives as a component of their geometry lessons and that it is important to use hands-on activities to explore geometric ideas. Statistically significant relationships involving the use of manipulatives can be found in Table 8.6.

I have used manipulatives to teach geometric concepts is significant to:	*P*
Gender	$p = 1.9507 \times 10^{-4}$
Membership of professional organisations	$p = 1.44 \times 10^{-4}$
Attendance at professional meetings	$p = 2.49 \times 10^{-4}$
Use of dynamic geometry software	$p = 3.8 \times 10^{-6}$
Undergraduate major (first degree)	$p = 0.0344$
Graduate degree	$p = 0.0038$
Type of high school	$p = 0.025$
Taught geometry as a full year course	$p = 3.965 \times 10^{-4}$
Took mathematics methods (pedagogy) course	$p = 0.0049$
Belief that developing students' spatial sense is a primary objective of geometry	$p = 1.65 \times 10^{-4}$
Belief that geometry should only be taught to able students	$p = 0.0052$
Belief that students find geometry difficult	$p = 0.016$
Familiar enough with dynamic geometry to use it	$p = 1.15 \times 10^{-9}$
Belief that dynamic geometry enables students to enjoy learning	$p = 0.013$
Belief that all high school students should use dynamic geometry	$p = 6.62 \times 10^{-7}$
Confidence in teaching geometry	$p = 0.011$
Belief that studying geometry leads to a positive attitude about mathematics	$p = 5.796 \times 10^{-4}$
Belief that applying geometrical concepts and thinking will help students in their future occupations	$p = 8.81 \times 10^{-4}$
Belief that geometry has many real world applications	$p = 0.0016$
Belief that when teaching geometry connections to the real world should be made	$p = 0.015$
Belief that geometry should occupy a significant place in the curriculum	$p = 1.4 \times 10^{-4}$
Belief that HS students should experience other geometries besides Euclidean	$p = 0.031$
Belief that students can experience the activities of mathematicians through their work in geometry class	$p = 0.0029$
Belief that geometry enables ideas from other areas to be pictured	$p = 0.0025$

Table 8.6 ***Statistically significant relationships for using manipulatives***

Although there was a statistically significant relationship between the use of manipulatives and teaching geometry as a full-year course, there was not a statistical relationship between the use of manipulatives and teaching geometry as part of an integrated curriculum. More teachers than expected used manipulatives when they taught geometry as a full year course. As discussed above when geometry is just a topic in an integrated curriculum there is less coherence and structure and less time to use manipulatives.

8.7 BELIEFS ABOUT DYNAMIC GEOMETRY SOFTWARE

Statistically significant relationships involving the use of dynamic geometry systems can be found in Table 8.7.

I have used dynamic geometry software with my students is significant to:	***P***
Membership of professional organisations	$p = 9.62 \times 10^{-11}$
Attendance at professional meetings	$p = 9.31 \times 10^{-12}$
Use of manipulatives	$p = 3.8 \times 10^{-6}$
Type of high school	$p = 4.05 \times 10^{-14}$
Number of years teaching	$p = 0.0083$
Number of students in school	$p = 4.266 \times 10^{-7}$
Taught geometry as a full year course	$p = 5.88 \times 10^{-4}$
Taught geometry as a topic in am integrated curriculum	$p = 3.87 \times 10^{-4}$
Took mathematics methods (pedagogy) course	$p = 2.92 \times 10^{-4}$
Belief that dynamic geometry can take the place of rigorous proof	$p = 0.0088$
Belief that students find dynamic geometry difficult to use	$p = 2.93 \times 10^{-6}$
Familiar enough with dynamic geometry to use it	$p = 2.22 \times 10^{-34}$
Belief that all high school students should use dynamic geometry	$p = 1.299 \times 10^{-4}$
Enjoy teaching geometry	$p = 0.014$
Belief that some things in geometry like proofs are best memorised	$p = 7.04 \times 10^{-4}$
Belief that geometry is an exercise in memorisation	$p = 0.0014$

Table 8.7 ***Statistically significant relationships for using dynamic geometry software***

The fact that teachers use dynamic geometry software and manipulatives significantly more when geometry is a full year course is an advocate in itself for geometry to be taught as a full year course. There was no significant relationship between the use of manipulatives and teaching geometry as part of an integrated curriculum, but there was a statistically significant relationship in the negative direction between the use of dynamic geometry software and teaching geometry as part of an integrated curriculum. Teachers use dynamic geometry software significantly less when teaching geometry as part of an integrated curriculum.

8.8 BELIEFS ABOUT DOING PROOFS IN GEOMETRY

As shown in Table 8.8, there are statistically significant relationships between *geometry being taught as a full year course* and *teachers enjoy doing geometric proofs*, *enjoy teaching their students how to do geometric proofs* and *enjoy proving theorems for their students*.

Learning to construct proofs is important for HS students is significant to:	***P***
Undergraduate major (first degree)	$p = 0.015$
Taught geometry as a full year course	$p = 0.021$
Some things in geometry like proofs are best memorised is significant to:	***P***
I have used dynamic geometry software with my students	$p = 7.04 \times 10^{-4}$
Membership of professional organisations	$p = 0.012$
Attendance at professional meetings	$p = 0.008$
My students enjoy doing geometric proofs is significant to:	***P***
Membership of professional organisations	$p = 0.04$
I enjoy doing geometric proofs is significant to:	***P***
Taught geometry as a full year course	$p = 0.022$
I enjoy teaching my students how to do geometric proofs is significant to:	***P***
Taught geometry as a full year course	$p = 0.0016$
I enjoy proving theorems for my students is significant to:	***P***
Membership of professional organisations	$p = 0.037$
Taught geometry as a full year course	$p = 0.0017$

Table 8.8 *Statistically significant relationships for proof statements*

One can conclude from the above results that if policy makers and administrators want teachers to use manipulatives and dynamic geometry software then geometry should be taught as a full year course. Similarly, if administrators want teachers to 'enjoy' teaching students about proof in geometry it should be done as part of a full year course and not as part of an integrated curriculum.

Another conclusion to be made concerning proof was obtained from the responses to the follow up questionnaire. There were teachers who believed that constructions should be eliminated from the curriculum. When geometry is taught in bits and pieces as part of an integrated curriculum then the topic of constructions could 'show up' anywhere at anytime and be totally disconnected from proof. Teaching constructions as a procedure to be memorised has a negative effect on students' views of what mathematics is (Schoenfeld, 1988). When constructions are taught with an understanding in connection to their proof then students can see the beauty of mathematics. If teachers are not making these connections because they are not being made in the textbook then that is a curriculum issue, but if teachers are not making these connections because they themselves never learned them then it becomes an issue for teacher education and professional development programmes.

8.9 BELIEFS ABOUT THE ROLE OF APPLICATIONS IN GEOMETRY

I included statements relating to applications and connections on both my questionnaire and its follow up because I believe that the role of applications in geometry is an important issue. Factor 2, which was extracted through factor analysis, is called an appreciation of geometry and its applications. Teachers in groups 1, 2, 5 and 6 all have a positive disposition towards applications. Some respondents said they don't spend enough time on geometric applications. About half of the middle school teachers in Langrall et al. (2004) did not believe that geometry is useful. I went back to 4 statements from the questionnaire that dealt with applications or connections.

18. *Geometry has many real world applications.*

42. *When teaching geometry, connections to real world applications should be made.*

45. *Geometry enables ideas from other area of mathematics to be pictured.*

47. *Applying geometrical concepts and thinking will help students in their future occupations or professions.*

I crosstabulated these statements with the personal data and found several statistically significant results. Those teachers who use manipulatives agreed significantly more than expected that geometry has many real world applications compared with those teachers who do not use manipulatives, as shown in Table 8.9. One has to be careful when applying the Chi-squared statistic to data when the expected frequency is less than five in any cell (Conover, 1999).

$$Expected\ Value = \frac{Row\ Total\ \times Column\ Total}{Sample\ Size}$$

and

$$Chi\ Square\ Statistic = \sum \frac{(O-E)^2}{E}$$

	I have used manipulatives to teach geometric concepts		
Geometry has many real world applications	No	Yes	Total
Disagree	8 (3)	8 (13)	16
Agree	89 (94)	394 (389)	483
Total	97	402	499
Chi-squared = 9.86 ($p = 0.0017$) Expected frequencies in brackets			

Table 8.9 ***Crosstabulation between statement 18 and manipulatives use***

Similarly, teachers who use manipulatives agree significantly more than expected that when teaching geometry connections to real world applications should be made, as shown in Table 8.10. The use of manipulatives facilitates making connections since models of real world problems can be made using manipulatives. Again, One has to be careful when applying the Chi-squared statistic to data when the expected frequency is less than five in any cell (Conover, 1999).

	I have used manipulatives to teach geometric concepts		
When teaching geometry, connections to real world applications should be made	No	Yes	Total
Disagree	5 (2)	5 (8)	10
Agree	95 (98)	401 (398)	496
Total	100	406	506
Chi-squared = 5.88 (p = 0.015) Expected frequencies in brackets			

Table 8.10 ***Crosstabulation between statement 42 and manipulatives use***

Teachers who have taken undergraduate geometry courses and methods courses agree significantly more than expected that, when teaching geometry, connections to real world applications should be made, as shown in Tables 8.11 and 8.12. One has to be careful when applying the Chi-squared statistic to data when the expected frequency is less than five in any cell (Conover, 1999). Small differences between the observed and the expected values led to statistically significant results. This was not the case with statement 18. The belief that geometry has many real world applications is independent of whether or not teachers took undergraduate geometry or methods course. Many teachers are familiar with basic applications of geometry, such as perimeter and area problems, from their own secondary school experiences.

	I have taken an undergraduate geometry course		
When teaching geometry, connections to real world applications should be made	No	Yes	Total
Disagree	6 (3)	4 (7)	10
Agree	145 (148)	351 (348)	496
Total	151	355	506
Chi-squared = 4.43 (p = 0.035) Expected frequencies in brackets			

Table 8.11 ***Crosstabulation between statement 42 and taking undergraduate geometry courses***

	I have taken mathematics methods (pedagogy) courses		
When teaching geometry, connections to real world applications should be made	No	Yes	Total
Disagree	4 (2)	6 (8)	10
Agree	77 (79)	418 (416)	495
Total	81	424	505
Chi-squared = 4.35 (p = 0.037) Expected frequencies in brackets			

Table 8.12 ***Crosstabulation between statement 42 and taking mathematics methods courses***

Of great importance is the fact that teachers who have taught geometry as part of an integrated curriculum agree significantly less than expected that geometry has many real world applications, as shown in Table 8.2. As was stated above, geometry as a topic in an integrated curriculum lacks the mathematical rigour that is historically associated with this area of mathematics.

I have shown that teachers believe that teaching geometry as a one year course is important. I have also shown that taking geometry content courses, pedagogy courses, and having a graduate degree has an effect and makes a difference for geometry teachers. Similarly, being a member of a professional organisation has an effect and makes a difference for mathematics teachers in their beliefs about the use of manipulatives and dynamic geometry software.

8.10 QUESTIONS FOR FURTHER STUDY

- The analysis of the data has shown that teaching geometry as a one year course allows for a balance between the inductive use of manipulatives and dynamic geometry software and the deductive use of proof. If high schools begin to include one year courses in geometry for their students how difficult is it for the teacher to teach the one year course if they were not taught that way themselves?

- I have investigated qualitatively teachers' beliefs about how teaching geometry differs from teaching other domains of mathematics. What would happen if factor analysis was used to analyse a questionnaire about the beliefs of algebra or calculus teachers? Would there be a similar three factor solution for other mathematical domains?

- I found that teachers have a disposition towards abstraction. How does this disposition manifest itself when teaching algebra, for example?

- One of the conclusions of this study is that knowing the group to which a teacher belongs would be helpful in the professional development of the teacher. I have shown through the case study of the teacher, Rose, how an intervention can be used effectively for a teacher in Group 2. The question remaining is whether using the information about the groups would be useful in large scale professional development. A comparison study where some teachers are grouped according to the results of their responses on the questionnaire while other teachers are randomly grouped could be conducted.

Also, it would be useful to explore what the common actual practices are for teachers belonging to the same group.

8.11 POSSIBLE METHODOLOGICAL IMPROVEMENTS

In retrospect several changes could have been instituted to improve the methodology used in this study. The pilot study could have been conducted on a larger sample of teachers so that a pilot factor analysis could be made. This analysis could have provided the information on the number of factors to extract and therefore might have eliminated the need for all the trial analyses in Table 6.2.

The Likert scales used on the questionnaire could have contained an odd number of values. There should have been an undecided choice. Trying to force respondents off the fence led to missing values. When the factor analysis was run with the missing values replaced by the mean there was little change in the results.

Many of the missing values were on statements pertaining to dynamic geometry so one must be careful about making inferences about dynamic geometry from the study.

In order to check for educational significance when there was statistical significance, tests for effect size should been made. There were several tables in both chapters 5 and 8 where there was little difference between the observed and expected frequencies but there was statistical significance.

More careful categorisation of qualitative data could have been made so as not to lose subtleties of meanings.

8.12 SOME FURTHER THOUGHTS AND CONCLUSIONS

Many teachers in my study believe that the geometry course is where the students learn how to reason. Respondent #28 claimed, *"It is one of the few courses that teach how to produce logical support or argument."* Similarly, respondent #236 stated, *"No other topic allows for intensive training in logical thinking."* Are students reasoning in other mathematics classes? What happens if a student does not take a geometry course in high school?

The PSSM (2000) advocates that students learn how to reason throughout their school years. It states:

> Mathematics programs should give students in grades K-12 opportunities
>
> - To make and investigate mathematics conjecture
> - To develop and evaluate mathematical arguments and proofs
> - To select various types of reasoning methods and proofs (NCTM, 2000, p. 342)

Senk, Thompson, and Johnson (2007) examined United States textbooks used in algebra I and II courses to find the types and extents of reasoning and proof there were for the topics of exponents and logarithms. They examined 3503 exercises and found only 6.8% of them contained *proof related reasoning*, with most of these exercises involving specific instances and not general properties. They found very few opportunities for students to make conjectures or evaluate arguments.

Proof is not happening in algebra I and minimally in algebra II. If it doesn't occur in geometry when will high school students be exposed to it? If students have to wait for geometry to prove their ideas--and many don't even take geometry-- then the mathematics education community has to address this issue.

I have created a picture of what a representative sample of teachers of geometry believes.

- They believe geometry should be a year long course and not subsumed into an integrated course
- They believe as summarised in Tables 8.6 that manipulatives should be used and are motivational
- They believe as summarised in Table 8.7 that all students should use dynamic geometry software

- They believe that geometry forces students to give clear reasons and arguments--proofs--for their thinking, in a manner that they don't believe other courses demand.

With some refinement, such as reducing it to the 32 items with loading greater than 0.4 and with a Likert scale that contains an *undecided* option which would hopefully eliminate incomplete questionnaires, I believe the questionnaire can function as a tool with which to characterise teachers' belief systems so that specific and prescriptive interventions and pre-service/in-service courses can be designed and developed and needs addressed.

The typologies indicate who is most likely to have the characteristics most hoped for in a geometry teacher: a combination of positive scores on at least two of the three factors is desirable.

I sincerely hope that the findings from this investigation will have a positive impact upon the future of the teaching of geometry by influencing policy makers, administrators, mathematics professors, teacher educators and in-service professional developers.

APPENDICES

APPENDIX A - PILOT QUESTIONNAIRE

Dear Fellow Educator,
Thank you in advance for completing the following questionnaire. This is part of a research project concerning the teaching and learning of geometry. If you have any questions, please contact me at bs49@nyu.edu.
Brenda Strassfeld

Please read each statement and check the appropriate response:

	Strongly Agree	Agree	Undecided	Disagree	Strongly Disagree
1. I enjoy teaching geometry.					
2. Learning geometry is valuable for high school students.					
3. I do not refer to theorems when teaching geometry.					
4. I think most high school students find geometry difficult.					
5. Learning to construct proofs is important for high school students.					
6. Developing students' spatial sense is a primary objective of teaching geometry.					
7. Students find geometry boring.					
8. The greatest value of geometry is the exposure it gives students to the deductive method.					
9. I prove geometrical results so that my students can apply them to solve problems.					
10. Geometry should be included in the curriculum for all students.					
11. There are some things in geometry, like proofs that are best memorized.					
12. Dynamic geometry software packages such as Geometer's Sketchpad or Cabri Geometry enable students to enjoy learning geometry.					
13. Geometry should be a full, one-year course.					
14. Geometry is a good environment in which to develop the principles of proof.					
15. High school geometry should not contain proof.					
16. Geometric ideas should be embedded in the curriculum in all grades.					
17. Visuals such as diagrams and sketches should be an integral part of the geometry curriculum.					

18. Students should learn how to do geometric constructions.					
19. High school students should be able to write two column proofs in geometry.					
20. Geometry is a way of seeing structure in the world.					
21. Using manipulatives in the teaching of geometry is motivational.					
22. Geometry should only be taught to very able students.					
23. Geometry is a course where students can explore mathematics as mathematicians might.					
24. My students enjoy doing geometric proofs.					
25. I lack the confidence to teach high school geometry.					
26. Geometry has many real world applications.					
27. Students should be taught how to produce valid mathematical arguments.					
28. Manipulatives help students to grasp the basic ideas of geometry.					
29. Geometry offers a means of describing, analyzing, and understanding the world.					
30. All students should have familiarity with Geometer's Sketchpad (or a similar dynamic geometry software package).					
31. I enjoy doing geometric proofs.					
32. High school students should experience other geometries besides Euclidean geometry (e.g. transformational, non Euclidean).					
33. It is important to use hands-on activities to explore geometric ideas.					
34. Proofs done in high school geometry lessons should be short.					
35. I think it is beneficial to use manipulatives as an integral part of my geometry lessons.					
36. Students find it difficult to use dynamic geometry software packages such as Geometer's Sketchpad or Cabri Geometry.					
37. Critiquing arguments is an important aspect of proving.					
38. The use of manipulatives makes learning geometry fun.					
39. More interesting geometrical problems can be explored with a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry than without it.					

40. Geometry is an exercise in memorization.					
41. Algebraic skills should be strengthened in geometry.					
42. High school geometry should be initially hands-on with proofs coming later in the course.					
43. I am familiar enough with a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry to use it confidently in my teaching.					
44. Students should discover theorems in geometry.					
45. It is unnecessary for students to prove theorems that they regard as obvious.					
46. Geometry is where students can validate conjectures using deduction.					
47. More time should be spent on analytic geometry and other topics in geometry rather than on proving.					
48. It is more important for students to apply theorems learned rather than explore geometric properties.					
49. Proofs written in paragraph form are acceptable.					
50. A main goal of geometry is to teach students how to reason.					
51. If a student makes a conjecture about a geometrical idea that is not in the curriculum, the teacher should allow the class time to prove or disprove the conjecture.					
52. Using a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry to demonstrate geometric properties and relationships can take the place of rigorous proofs.					
53. I am confident about teaching geometry.					
54. In class, I apply many theorems without proving them.					
55. Geometry appeals to my visual, aesthetic and intuitive senses.					
56. Students should be made aware of the historical background of geometry.					
57. Studying geometry leads to a positive attitude about mathematics.					
58. I enjoy teaching geometrical proofs.					
59. When teaching geometry, connections to real world applications should be made.					

60. Undergraduate major______________

61. Number of years teaching mathematics____ Grade levels______________

Circle the appropriate response:

62. Gender: M F

63. I have taken mathematics methods courses (i.e. courses on how to teach various aspects of mathematics): Yes No

64. I have taken geometry courses as an undergraduate: Yes No

65. I have a graduate degree: Yes No If yes, in what area?__________________

66. I have taught geometry as a 1 year course: Yes No

67. I have taught geometry as a topic in an integrated curriculum Yes No

68. I have used manipulatives to teach geometrical concepts: Yes No

69. I have used a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry with my students: Yes No

70. I teach in: an urban high school a suburban high school a rural high school

71. The number of students in my high school is approximately________.

Please answer the next question in the space provided. If you need more space please use the back of this sheet.

72. Geometry is an important/not an important topic for high school students to study because:

If you are willing to answer a few more questions based on your responses please include your name and phone number so that we can set up a convenient time for a short interview:

Name:

Phone number:

APPENDIX B - USA QUESTIONNAIRE DISTRIBUTED SEPTEMBER 2004 – JULY 2005

Dear Fellow Educator,

Thank you in advance for completing the following questionnaire. This is part of a research project concerning the teaching and learning of geometry. When I refer to *manipulatives* I mean tactile objects that students can use such as tiles and plastic mirrors. If you have any questions, please contact me at bs49@nyu.edu.

Brenda Strassfeld

Please read each statement and check the appropriate response:

	Strongly Agree	Moderately Agree	Agree Slightly more than Disagree	Disagree Slightly more than Agree	Moderately Disagree	Strongly Disagree
1. I enjoy teaching geometry.						
2. Learning geometry is valuable for high school students.						
3. I think many high school students find geometry difficult.						
4. Learning to construct proofs is important for high school students.						
5. Developing students' spatial sense is a primary objective of teaching geometry.						
6. Geometry should be included in the curriculum for all students.						
7. There are some things in geometry, like proofs that are best memorized.						
8. Dynamic geometry software packages such as Geometer's Sketchpad or Cabri Geometry enable students to enjoy learning geometry.						
9. Geometry should occupy a significant place in the curriculum.						
10. High school geometry should not contain proofs.						
11. Visuals such as diagrams and sketches should not be an integral part of the geometry curriculum.						
12. Students should learn how to do geometric constructions with straight edge and compass.						

	Strongly Agree	Moderately Agree	Agree Slightly more than Disagree	Disagree Slightly more than Agree	Moderately Disagree	Strongly Disagree
13. High school students should be able to write rigorous proofs in geometry.						
14. Using manipulatives in the teaching of geometry is motivational.						
15. Geometry should only be taught to very able students.						
16. My students enjoy doing geometric proofs						
17. I lack the confidence to teach geometry in high school.						
18. Geometry has many real world applications.						
19. Manipulatives help students to grasp the basic ideas of geometry.						
20. Ideally, all high school students should have used Geometer's Sketchpad (or a similar dynamic geometry software package).						
21. I enjoy doing geometric proofs.						
22. High school students should experience other geometries besides Euclidean geometry (e.g. transformational, non Euclidean).						
23. It is important to use hands-on activities to explore geometric ideas.						
24. I think it is beneficial to use manipulatives such as mirrors as a component of my geometry lessons.						
25. Students find it difficult to use dynamic geometry software packages such as Geometer's Sketchpad or Cabri Geometry.						
26. The use of manipulatives makes learning geometry fun.						

	Strongly Agree	Moderately Agree	Agree Slightly more than disagree	Disagree Slightly more than agree	Moderately Disagree	Strongly Disagree
27. More interesting geometrical problems can be explored with a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry than without it.						
28. Geometry is an exercise in memorization.						
29. Initially, high school geometry should be hands-on with proofs coming later in the course.						
30. I am familiar enough with a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry to use it confidently in my teaching.						
31. High school students should discover theorems in geometry.						
32. It is unnecessary for students to prove theorems that they regard as obvious.						
33. Geometry is one topic where students can validate conjectures using deduction.						
34. More time should be spent on analytic geometry and other topics in geometry rather than on proving.						
35. Proofs written in words are acceptable.						
36. A main goal of geometry is to teach students how to reason.						
37. If a student makes a conjecture about a geometrical idea that is not in the curriculum, the teacher should allow time to prove/disprove the conjecture.						

	Strongly Agree	Moderately Agree	Agree Slightly more than disagree	Disagree Slightly more than agree	Moderately Disagree	Strongly Disagree
38. Using a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry to demonstrate geometric properties and relationships can take the place of having students do rigorous proofs.						
39. I am confident about my teaching of geometry.						
40. Students should be made aware of the historical background of geometry.						
41. Studying geometry leads to a positive attitude towards mathematics.						
42. When teaching geometry, connections to real world applications such as art should be made.						
43. Students can experience the activity of mathematicians through their work in geometry class.						
44. I enjoy teaching my students how to do geometric proofs.						
45. Geometry enables ideas from other areas of mathematics to be pictured.						
46. The main goal of geometry is to illustrate the order and coherence of a mathematical system.						
47. Applying geometrical concepts and thinking will help students in their future occupations or professions.						
48. I enjoy proving theorems for my students.						

Please answer the next questions in the spaces provided. If you need more space please use the back of the questionnaire.

49a. Is geometry an important topic for high school students to study? YES NO Please explain. b. Do students consider studying geometry in high school important? YES NO Please explain.
50. In what ways do you think that teaching geometry differs from teaching other mathematics content such as algebra?

Personal Data:

Undergraduate major _______________

Number of years teaching mathematics____ Grade levels_______________

Circle the appropriate response:

Gender: M F

I have taken mathematics methods courses (i.e. courses on how to teach various aspects of mathematics): Yes No

I have taken geometry courses as an undergraduate: Yes No

I have a graduate degree: Yes No If yes, in what area?_______________

I have taught geometry as a 1 year course: Yes No

I have taught geometry as a topic in an integrated curriculum Yes No

I have used manipulatives to teach geometrical concepts: Yes No

I have used a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry with my students: Yes No

I am a member of the National Council of Teachers of Mathematics: Yes No

I have attended national or regional meetings of NCTM at least 2 times: Yes No

I teach in: an inner city high school a suburban high school a private high school a rural high school

The total number of students in my high school is approximately________.

If you are willing to answer a few more questions based on your responses please include your name and phone number so that we can set up a convenient time for a short interview:

Name:

Phone number:

Email:

APPENDIX C - UK VERSION OF QUESTIONNAIRE

Dear Fellow Educator,
Thank you in advance for completing the following questionnaire. This is part of a research project concerning the teaching and learning of geometry. When I refer to *manipulatives* I mean tactile objects that students can use such as tiles and plastic mirrors. If you have any questions, please contact me at bs49@nyu.edu.
Brenda Strassfeld
Please read each statement and tick the appropriate response:

	Strongly Agree	Moderately Agree	Agree Slightly more than Disagree	Disagree Slightly more than Agree	Moderately Disagree	Strongly Disagree
1. I enjoy teaching geometry.						
2. Learning geometry is valuable for secondary school students.						
3. I think many secondary school students find geometry difficult.						
4. Learning to construct proofs is important for secondary school students.						
5. Developing students' spatial sense is a primary objective of teaching geometry.						
6. Geometry should be included in the curriculum for all students.						
7. There are some things in geometry, like proofs that are best memorised.						
8. Dynamic geometry software packages such as Geometer's Sketchpad or Cabri Geometry enable students to enjoy learning geometry.						
9. Geometry should occupy a significant place in the curriculum.						
10. Secondary school geometry should not contain proofs.						
11. Visuals such as diagrams and sketches should not be an integral part of the geometry curriculum.						
12. Students should learn how to do geometric constructions with straight edge and compass.						
13. Secondary school students should be able to write rigorous proofs in geometry.						

	Strongly Agree	Moderately Agree	Agree Slightly more than Disagree	Disagree Slightly more than Agree	Moderately Disagree	Strongly Disagree
14. Using manipulatives in the teaching of geometry is motivational.						
15. Geometry should only be taught to very able students.						
16. My students enjoy doing geometric proofs.						
17. I lack the confidence to teach geometry in secondary school.						
18. Geometry has many real world applications.						
19. Manipulatives help students to grasp the basic ideas of geometry.						
20. Ideally, all secondary students should have used Geometer's Sketchpad (or a similar dynamic geometry software package).						
21. I enjoy doing geometric proofs.						
22. Secondary school students should experience other geometries besides Euclidean geometry (e.g. transformational, non Euclidean).						
23. It is important to use hands-on activities to explore geometric ideas.						
24. I think it is beneficial to use manipulatives such as mirrors as a component of my geometry lessons.						
25. Students find it difficult to use dynamic geometry software packages such as Geometer's Sketchpad or Cabri Geometry.						
26. The use of manipulatives makes learning geometry fun.						

	Strongly Agree	Moderately Agree	Agree Slightly more than Disagree	Disagree Slightly more than Agree	Moderately Disagree	Strongly Disagree
27. More interesting geometrical problems can be explored with a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry than without it.						
28. Geometry is an exercise in memorisation.						
29. Initially, secondary school geometry should be hands-on with proofs coming later in the course.						
30. I am familiar enough with a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry to use it confidently in my teaching.						
31. Secondary students should discover theorems in geometry.						
32. It is unnecessary for students to prove theorems that they regard as obvious.						
33. Geometry is one topic where students can validate conjectures using deduction.						
34. More time should be spent on analytic geometry and other topics in geometry rather than on proving.						
35. Proofs written in words are acceptable.						
36. A main goal of geometry is to teach students how to reason.						
37. If a student makes a conjecture about a geometrical idea that is not in the curriculum, the teacher should allow time to prove/ disprove the conjecture.						

	Strongly Agree	Moderately Agree	Agree Slightly more than Disagree	Disagree Slightly more than Agree	Moderately Disagree	Strongly Disagree
38. Using a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry to demonstrate geometric properties and relationships can take the place of having students do rigorous proofs.						
39. I am confident about my teaching of geometry.						
40. Students should be made aware of the historical background of geometry.						
41. Studying geometry leads to a positive attitude towards mathematics.						
42. When teaching geometry connections to real world applications such as art should be made.						
43. Students can experience the activity of mathematicians through their work in geometry class.						
44. I enjoy teaching my students how to do geometric proofs.						
45. Geometry enables ideas from other areas of mathematics to be pictured.						
46. The main goal of geometry is to illustrate the order and coherence of a mathematical system.						
47. Applying geometrical concepts and thinking will help students in their future occupations or professions.						
48. I enjoy proving theorems in geometry for my students.						

Please answer the next questions in the spaces provided. If you need more space please use the back of the questionnaire.

49a. Is geometry an important topic for secondary school students to study? YES NO Please explain.   b. Do you think that students consider studying geometry in secondary school important?
50. In what ways do you think that teaching geometry differs from teaching other mathematics content such as algebra?

Personal Data:

First degree _______________

Number of years teaching mathematics____ Key stages_______________

Circle the appropriate response:

Gender: M F

I have taken mathematics methods courses (i.e. courses on how to teach various aspects of mathematics): Yes No

I have taken geometry courses as an undergraduate: Yes No

I have a post-graduate degree: Yes No If yes, in what area?_______________

I have taught geometry as a 1 year course: Yes No

I have taught geometry as a topic in an integrated curriculum Yes No

I have used manipulatives to teach geometrical concepts: Yes No

I have used a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry with my students: Yes No

I am a member of the: ATM MA Other

I have attended at least 2 meetings of the above: Yes No

I teach in: an inner city comprehensive secondary school a selective school
a rural comprehensive secondary school an independent secondary school
The number of students in my secondary school is approximately________.

If you are willing to answer a few more questions based on your responses please include your name and phone number so that we can set up a convenient time for a short interview:

Name:

Phone number:

Email:

APPENDIX D – PILOT INTERVIEW QUESTIONS

1. What mathematics content area do you enjoy teaching the most?

2. Describe one of your favourite lessons in that area.

If the answer to question 1 was not geometry **ASK**

3. **How do you feel about teaching geometry?**
4. **Do you feel confident teaching geometry?**

5. **What is geometry?**

6. Describe a recent geometry lesson that went well.

7. Describe a recent geometry lesson that went badly.

8. **Do you think that geometry should be included in the curriculum for all students? Why or why not?**

9. What role should proof play in secondary mathematics? In geometry?

10. **If manipulatives have not been mentioned ask: Do you incorporate the use of manipulatives in your classes? Why and in what ways or why not?**

11. **If technology has not been mentioned ask: Do you incorporate technology into your lessons? Why and in what ways or why not?**

12. Do you teach every content area in a similar way?

13. In what ways do you assess your students?

14. Does the format of your assessments differ across content areas? In what ways?

APPENDIX E – TRANSCRIBED PILOT INTERVIEW

I-What mathematics content area do you enjoy teaching the most?

T- I like teaching algebra.

I- Describe one of your favourite lessons in algebra.

T-I like it when you are introducing the younger children year age 11-12 to algebra for the first time and they are first getting to know the idea of an algebraic variable and you get to talk about putting numbers into boxes and labelling the boxes. I think they suppose they are doing difficult maths when they start doing algebra and they really enjoy that.

I- How do you feel about teaching geometry?

T- I quite enjoy it. I enjoy geometry myself so I am quite enthusiastic about it but I am aware that not all students enjoy doing geometry.

I- Why?

T- I think they like it when it's involving drawing things and using rulers and compasses they enjoy constructions but I think when it gets on to proof they find that very difficult and all but the very brightest ones seem to dislike that.

I- Do you feel confident teaching geometry?

T- Yes I feel confident myself. Yes.

I-Can you describe a recent geometry lesson that you did that went very well.

T-I suppose I always try to think of good ways of proving circle theorems so what I tend to do is particularly with students who are not particularly able is to first of all get them to see what is going on with circle theorems by drawing circles and measuring the angles and doing a few examples and finding out what the rules are and then going back over with the rigorous proofs and actually proving it.. I think they are happier with proofs once they actually have seen that the rules work.

I- Do you have a lesson in geometry lesson that did not go well?

T-Let me think. Let me think. I think probably again on the proofs with the younger students trying to prove for example that the angles of a triangle add up to 180. They all seem quite happy to accept they do but when you are actually trying to get out the idea that you are going to prove it I think with 11-12 year olds they don't see why you should have to prove it if you can see. They can see, they claim they see it works.

I-I am wondering what the proof would look like? They would write the reasons for each step?

T-Yes.

I-You use the parallel postulate to prove it?

T-Yes that has come into the curriculum fairly recently. Up until 3 or 4 years ago we would not have done that. They would simply have to be aware of the fact. Yes, we would construct parallel lines and they would prove it by writing out step by step.

I-At what age is that requirement?

T-That would be age 12.

I-Wow that's real young. **{Judgmental!!}**

T-Yes, yes and proofs of alternate interior angle rules and those angle rules that all comes at age 12.

I-And stuff about quadrilaterals and parallelograms they do that?

T-They would do that going on into the next year at about age 13.

I-And circle proofs that you were talking about?

T-Oh that would be later that would be age 15.

I-And they are proving inscribed angle measured by one-half the arc? What's the kinds of stuff..?

T-That would be hmmm that would be I'm just trying to think how I did it. There would be two or three different ways you could do that aren't there?

I-Yeah, well yeah which that's the theorem that you are talking about inscribed angles?

T-I've got to be honest I can't remember exactly how I did it.

I-Well you have to draw a line I think to get a central angle.

T-Yes you have to draw.., put in the chord and then put in...

I-It's usually, I guess the proofs that require drawing an actual line that probably gives grief to many students.

T-Yes, yes. The one that I always, that in fact that I don't like to teach is that I have to go back to look at is the second theorem

I-Which is?

T-You've got the chord and the tangent and the angle inside the segment is the same as the angle the chord makes with the tangent.

I-Oh.

T- That's difficult to prove.

I-Not half of it?

T-The one that I'm thinking of they are equal. In my book it is called the alternative segment theorem. There are probably other names for it. But I don't like teaching it because I can't remember the proof.

I-And do you think it is a good idea to have **geometry in the curriculum for all students?**

T-Not for all students, no. I think the more able students, the students that enjoy maths, that enjoy geometry I think it is important. But I think very weak students who have difficulties with number work need to concentrate on number.

{Should ask: Can you say more?}

I-I see the next question is: What role should proof play in secondary mathematics? Do you do proofs in algebra also, or in number theory?

T-Yes, proof is very much more in the curriculum now. It has been brought in the recent revisions. I don't think that it should necessarily be there at all levels I think at the higher levels.

I-At age 12 they are already doing proofs?

T-Yes, yes.

I-And in geometry they are starting at age 12?

T-Yes. They are doing algebraic proofs as well. They are expected at age 14 when they do that SATS test.

I-What kind of algebraic proofs?

T- Something like proving that if you add two even numbers together you get an even.

I-So in this country they get to see proof in more than one context?

T-Yes, yes.

I-Do you use any of those little, well we call them manipulatives, concrete materials like tiles and mirrors and stuff?

T-Yes, yes. Certainly on work on reflections we would use mirrors. It depends on the ability of the students. If they are very able they can go straight into doing it without mirrors. But with weaker you would certainly have to use mirrors there.

I-Any other kinds of materials you use?

T-We tend to use,,,,, cubes when we do three dimensional work building three dimensional shapes –little cubes that lock together.

I-Right. So what are you using that for volume?

T-Yes, yes. Getting them to make cuboids.

I-At what age is that going to happen?

T-We are doing that with 11 and 12 year olds.

I-They are not proving stuff?

T-No although I actually heard quite an interesting lecture last year about how you can use things like that to do algebraic proofs so that building for example triangle numbers or square numbers actually building them up little cubes although you are seeing them as squares rather than as cubes and using them to build up the idea of algebraic...

I-And what about technology- do use Cabri?

T-I don't. I'd like to. I really want to find out how to use it. We haven't got it on the school system.

I-So there is no technology with geometry at all?

T-No, I wish we did have.

I-But they do the constructions with compasses and stuff like that? At age 12 they are also doing compass constructions?

T- Yes they have been learning to. Yes.

I-That would be like construct the perpendicular bisector?

T-Yes. They would already have done constructing triangles with given lengths of sides.

I-And when you teach algebra and geometry and other content area it is basically the same way? Or when you teach geometry is there anything that you do that is different? In terms of planning lessons?**(GIVE HER TIME-WHY ADD THIS ABOUT LESSON PLANS?!)**

T-I think it takes longer probably to plan geometry lessons because I haven't got dynamic software available to me in the classroom at the moment. And because I'm not confident at that as well. I do Most of the drawings have to be done on the white board. And so it is the case of drawing what you want as accurately as you on the whiteboard which I find very difficult. That's why I would like to be able to use dynamic geometry software. Because that would take away all the...

I-So when you are teaching geometry visuals play an important role? The diagrams

T-Yes, I try to produce worksheets that's why it takes longer to plan because rather than me doing all the drawing on the board I like to make sure that I've got good quality worksheets for them to work with.

I-So do they copy, your students or do they just work on the worksheets?

T-A bit of each. Yes, a bit of each.

I-How do you assess your students? Let's say specifically in geometry.

T-Well everything is done in modules. In my particular school everything is done in very short modules and at the end of each module... homework is part of the assessment and a short test at a time and so geometry test would be something like construct something using ruler and compass and we would mark it in terms of accuracy and...

I-Would there be proof on it also?

T-Hmmm, yes, I'm just trying to think in terms of things like alternate angles.

I-I'm wondering would they just give them a figure and say this angle is 30 degrees, what is the other angle?

T-That would be likely to be on the test, yes. What they would be asked to do is give a reason for example so they would be asked what is the angle labelled x and why? Give a reason. So one mark would be for getting the angle and the other mark would be for explaining that it was to do with alternate angle so there is a bit of an idea of explaining.

I-So they do push the explaining! So I guess the students realise it is important to learn to explain their answers. Do you do that in other content areas also?

T-Yes.

APPENDIX F - FREQUENCIES FOR QUESTIONNAIRE VARIABLES

Statement 1: I enjoy teaching geometry

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	5	1.0	1.0	1.0
	2.00	9	1.7	1.7	2.7
	3.00	11	2.1	2.1	4.8
	4.00	30	5.8	5.8	10.6
	5.00	139	26.7	26.9	37.5
	6.00	323	62.1	62.5	100.0
	Total	517	99.4	100.0	
Missing	System	3	.6		
Total		520	100.0		

Statement 2: Learning geometry is valuable for high school students

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	2.00	1	.2	.2	.2
	3.00	4	.8	.8	1.0
	4.00	33	6.3	6.3	7.3
	5.00	99	19.0	19.0	26.3
	6.00	383	73.7	73.7	100.0
	Total	520	100.0	100.0	

Statement 3: I think many secondary school students find geometry difficult.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	9	1.7	1.8	1.8
	2.00	9	1.7	1.8	3.5
	3.00	23	4.4	4.5	8.0
	4.00	108	20.8	21.0	29.0
	5.00	187	36.0	36.4	65.4
	6.00	178	34.2	34.6	100.0
	Total	514	98.8	100.0	
Missing	System	6	1.2		
Total		520	100.0		

Statement 4: Learning to construct proofs is important for secondary school students.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	8	1.5	1.5	1.5
	2.00	15	2.9	2.9	4.4
	3.00	43	8.3	8.3	12.8
	4.00	111	21.3	21.5	34.2
	5.00	174	33.5	33.7	67.9
	6.00	166	31.9	32.1	100.0
	Total	517	99.4	100.0	
Missing	System	3	.6		
Total		520	100.0		

Statement 5: Developing students' spatial sense is a primary objective of teaching geometry.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	4	.8	.8	.8
	2.00	13	2.5	2.5	3.3
	3.00	27	5.2	5.2	8.5
	4.00	101	19.4	19.6	28.1
	5.00	186	35.8	36.0	64.1
	6.00	185	35.6	35.9	100.0
	Total	516	99.2	100.0	
Missing	System	4	.8		
Total		520	100.0		

Statement 6: Geometry should be included in the curriculum for all students.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	5	1.0	1.0	1.0
	2.00	9	1.7	1.7	2.7
	3.00	19	3.7	3.7	6.4
	4.00	53	10.2	10.3	16.7
	5.00	109	21.0	21.1	37.8
	6.00	321	61.7	62.2	100.0
	Total	516	99.2	100.0	
Missing	System	4	.8		
Total		520	100.0		

Statement 7: There are some things in geometry, like proofs that are best memorized.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	130	25.0	25.2	25.2
	2.00	100	19.2	19.4	44.6
	3.00	107	20.6	20.7	65.3
	4.00	96	18.5	18.6	83.9
	5.00	55	10.6	10.7	94.6
	6.00	28	5.4	5.4	100.0
	Total	516	99.2	100.0	
Missing	System	4	.8		
Total		520	100.0		

Statement 8: Dynamic geometry software packages such as Geometer's Sketchpad or Cabri Geometry enable students to enjoy learning geometry.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	1	.2	.2	.2
	2.00	10	1.9	2.1	2.3
	3.00	7	1.3	1.5	3.8
	4.00	108	20.8	22.7	26.5
	5.00	173	33.3	36.3	62.8
	6.00	177	34.0	37.2	100.0
	Total	476	91.5	100.0	
Missing	System	44	8.5		
Total		520	100.0		

Statement 9: Geometry should occupy a significant place in the curriculum.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	4	.8	.8	.8
	2.00	4	.8	.8	1.6
	3.00	25	4.8	4.8	6.4
	4.00	79	15.2	15.3	21.7
	5.00	193	37.1	37.4	59.1
	6.00	211	40.6	40.9	100.0
	Total	516	99.2	100.0	
Missing	System	4	.8		
Total		520	100.0		

Statement 10: High / Secondary school geometry should not contain proofs.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	193	37.1	37.1	37.1
	2.00	115	22.1	22.1	59.2
	3.00	92	17.7	17.7	76.9
	4.00	66	12.7	12.7	89.6
	5.00	35	6.7	6.7	96.3
	6.00	19	3.7	3.7	100.0
	Total	520	100.0	100.0	

Statement 11: Visuals such as diagrams and sketches should not be an integral part of the geometry curriculum.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	382	73.5	73.6	73.6
	2.00	62	11.9	11.9	85.5
	3.00	34	6.5	6.6	92.1
	4.00	5	1.0	1.0	93.1
	5.00	10	1.9	1.9	95.0
	6.00	26	5.0	5.0	100.0
	Total	519	99.8	100.0	
Missing	System	1	.2		
Total		520	100.0		

Statement 12: Students should learn how to do geometric constructions with straight edge and compass.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	7	1.3	1.4	1.4
	2.00	31	6.0	6.0	7.4
	3.00	37	7.1	7.2	14.6
	4.00	105	20.2	20.4	35.0
	5.00	163	31.3	31.7	66.6
	6.00	172	33.1	33.4	100.0
	Total	515	99.0	100.0	
Missing	System	5	1.0		
Total		520	100.0		

Statement 13: High / Secondary school students should be able to write rigorous proofs in geometry.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	46	8.8	8.9	8.9
	2.00	62	11.9	11.9	20.8
	3.00	87	16.7	16.8	37.6
	4.00	152	29.2	29.3	66.9
	5.00	108	20.8	20.8	87.7
	6.00	64	12.3	12.3	100.0
	Total	519	99.8	100.0	
Missing	System	1	.2		
Total		520	100.0		

Statement 14: Using manipulatives in the teaching of geometry is motivational.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	3	.6	.6	.6
	2.00	6	1.2	1.2	1.7
	3.00	15	2.9	2.9	4.7
	4.00	81	15.6	15.7	20.3
	5.00	158	30.4	30.6	51.0
	6.00	253	48.7	49.0	100.0
	Total	516	99.2	100.0	
Missing	System	4	.8		
Total		520	100.0		

Statement 15: Geometry should only be taught to very able students.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	286	55.0	55.1	55.1
	2.00	95	18.3	18.3	73.4
	3.00	63	12.1	12.1	85.5
	4.00	36	6.9	6.9	92.5
	5.00	24	4.6	4.6	97.1
	6.00	15	2.9	2.9	100.0
	Total	519	99.8	100.0	
Missing	System	1	.2		
Total		520	100.0		

Statement 16: My students enjoy doing geometric proofs.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	90	17.3	18.4	18.4
	2.00	108	20.8	22.1	40.5
	3.00	117	22.5	23.9	64.4
	4.00	100	19.2	20.4	84.9
	5.00	58	11.2	11.9	96.7
	6.00	16	3.1	3.3	100.0
	Total	489	94.0	100.0	
Missing	System	31	6.0		
Total		520	100.0		

Statement 17: I lack the confidence to teach geometry in secondary school.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	404	77.7	78.4	78.4
	2.00	68	13.1	13.2	91.7
	3.00	16	3.1	3.1	94.8
	4.00	14	2.7	2.7	97.5
	5.00	8	1.5	1.6	99.0
	6.00	5	1.0	1.0	100.0
	Total	515	99.0	100.0	
Missing	System	5	1.0		
Total		520	100.0		

Statement 18: Geometry has many real world applications.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	1	.2	.2	.2
	2.00	5	1.0	1.0	1.2
	3.00	10	1.9	1.9	3.1
	4.00	31	6.0	6.0	9.2
	5.00	100	19.2	19.5	28.7
	6.00	366	70.4	71.3	100.0
	Total	513	98.7	100.0	
Missing	System	7	1.3		
Total		520	100.0		

Statement 19: Manipulatives help students to grasp the basic ideas of geometry.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	1	.2	.2	.2
	2.00	4	.8	.8	1.0
	3.00	8	1.5	1.6	2.5
	4.00	74	14.2	14.5	17.1
	5.00	154	29.6	30.2	47.3
	6.00	269	51.7	52.7	100.0
	Total	510	98.1	100.0	
Missing	System	10	1.9		
Total		520	100.0		

Statement 20: Ideally, all high / secondary students should have used Geometer's Sketchpad (or a similar dynamic geometry software package).

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	12	2.3	2.4	2.4
	2.00	31	6.0	6.3	8.7
	3.00	37	7.1	7.5	16.2
	4.00	116	22.3	23.5	39.7
	5.00	145	27.9	29.4	69.0
	6.00	153	29.4	31.0	100.0
	Total	494	95.0	100.0	
Missing	System	26	5.0		
Total		520	100.0		

Statement 21: I enjoy doing geometric proofs.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	15	2.9	2.9	2.9
	2.00	13	2.5	2.5	5.4
	3.00	31	6.0	6.0	11.4
	4.00	58	11.2	11.2	22.7
	5.00	149	28.7	28.9	51.6
	6.00	250	48.1	48.4	100.0
	Total	516	99.2	100.0	
Missing	System	4	.8		
Total		520	100.0		

Statement 22: High / Secondary school students should experience other geometries besides Euclidean geometry (e.g. transformational, non Euclidean).

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	17	3.3	3.3	3.3
	2.00	32	6.2	6.3	9.6
	3.00	45	8.7	8.8	18.4
	4.00	135	26.0	26.4	44.7
	5.00	142	27.3	27.7	72.5
	6.00	141	27.1	27.5	100.0
	Total	512	98.5	100.0	
Missing	System	8	1.5		
Total		520	100.0		

Statement 23: It is important to use hands-on activities to explore geometric ideas.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	4	.8	.8	.8
	2.00	5	1.0	1.0	1.8
	3.00	10	1.9	2.0	3.7
	4.00	61	11.7	12.0	15.7
	5.00	147	28.3	28.8	44.5
	6.00	283	54.4	55.5	100.0
	Total	510	98.1	100.0	
Missing	System	10	1.9		
Total		520	100.0		

Statement 24: I think it is beneficial to use manipulatives such as mirrors as a component of my geometry lessons.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	7	1.3	1.4	1.4
	2.00	17	3.3	3.4	4.7
	3.00	15	2.9	3.0	7.7
	4.00	94	18.1	18.6	26.3
	5.00	150	28.8	29.6	55.9
	6.00	223	42.9	44.1	100.0
	Total	506	97.3	100.0	
Missing	System	14	2.7		
Total		520	100.0		

Statement 25: Students find it difficult to use dynamic geometry software packages such as Geometer's Sketchpad or Cabri Geometry.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	59	11.3	13.5	13.5
	2.00	98	18.8	22.4	35.8
	3.00	112	21.5	25.6	61.4
	4.00	107	20.6	24.4	85.8
	5.00	44	8.5	10.0	95.9
	6.00	18	3.5	4.1	100.0
	Total	438	84.2	100.0	
Missing	System	82	15.8		
Total		520	100.0		

Statement 26: The use of manipulatives makes learning geometry fun.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	3	.6	.6	.6
	2.00	9	1.7	1.8	2.4
	3.00	8	1.5	1.6	4.0
	4.00	91	17.5	18.0	22.0
	5.00	167	32.1	33.1	55.0
	6.00	227	43.7	45.0	100.0
	Total	505	97.1	100.0	
Missing	System	15	2.9		
Total		520	100.0		

Statement 27: More interesting geometrical problems can be explored with a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry than without it.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	7	1.3	1.5	1.5
	2.00	11	2.1	2.4	3.9
	3.00	28	5.4	6.1	10.0
	4.00	97	18.7	21.1	31.2
	5.00	155	29.8	33.8	64.9
	6.00	161	31.0	35.1	100.0
	Total	459	88.3	100.0	
Missing	System	61	11.7		
Total		520	100.0		

Statement 28: Geometry is an exercise in memorization.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	239	46.0	46.5	46.5
	2.00	111	21.3	21.6	68.1
	3.00	76	14.6	14.8	82.9
	4.00	45	8.7	8.8	91.6
	5.00	36	6.9	7.0	98.6
	6.00	7	1.3	1.4	100.0
	Total	514	98.8	100.0	
Missing	System	6	1.2		
Total		520	100.0		

Statement 29: Initially, secondary school geometry should be hands-on with proofs coming later in the course.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	26	5.0	5.1	5.1
	2.00	34	6.5	6.6	11.7
	3.00	60	11.5	11.7	23.4
	4.00	102	19.6	19.9	43.3
	5.00	165	31.7	32.2	75.4
	6.00	126	24.2	24.6	100.0
	Total	513	98.7	100.0	
Missing	System	7	1.3		
Total		520	100.0		

Statement 30: I am familiar enough with a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry to use it confidently in my teaching.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	93	17.9	18.3	18.3
	2.00	70	13.5	13.8	32.1
	3.00	44	8.5	8.7	40.8
	4.00	64	12.3	12.6	53.5
	5.00	108	20.8	21.3	74.8
	6.00	128	24.6	25.2	100.0
	Total	507	97.5	100.0	
Missing	System	13	2.5		
Total		520	100.0		

Statement 31: High / Secondary students should discover theorems in geometry.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	9	1.7	1.7	1.7
	2.00	18	3.5	3.5	5.2
	3.00	29	5.6	5.6	10.8
	4.00	109	21.0	21.0	31.9
	5.00	191	36.7	36.9	68.7
	6.00	162	31.2	31.3	100.0
	Total	518	99.6	100.0	
Missing	System	2	.4		
Total		520	100.0		

Statement 32: It is unnecessary for students to prove theorems that they regard as obvious.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	110	21.2	21.3	21.3
	2.00	115	22.1	22.3	43.6
	3.00	120	23.1	23.3	66.9
	4.00	83	16.0	16.1	82.9
	5.00	66	12.7	12.8	95.7
	6.00	22	4.2	4.3	100.0
	Total	516	99:2	100.0	
Missing	System	4	.8		
Total		520	100.0		

Statement 33: Geometry is one topic where students can validate conjectures using deduction.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	3	.6	.6	.6
	2.00	3	.6	.6	1.2
	3.00	14	2.7	2.8	3.9
	4.00	86	16.5	16.9	20.8
	5.00	221	42.5	43.4	64.2
	6.00	182	35.0	35.8	100.0
	Total	509	97.9	100.0	
Missing	System	11	2.1		
Total		520	100.0		

Statement 34: More time should be spent on analytic geometry and other topics in geometry rather than on proving.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	20	3.8	3.9	3.9
	2.00	42	8.1	8.2	12.1
	3.00	100	19.2	19.5	31.6
	4.00	122	23.5	23.8	55.4
	5.00	151	29.0	29.4	84.8
	6.00	78	15.0	15.2	100.0
	Total	513	98.7	100.0	
Missing	System	7	1.3		
Total		520	100.0		

Statement 35: Proofs written in words are acceptable.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	3	.6	.6	.6
	2.00	16	3.1	3.1	3.7
	3.00	35	6.7	6.8	10.6
	4.00	73	14.0	14.3	24.9
	5.00	198	38.1	38.7	63.6
	6.00	186	35.8	36.4	100.0
	Total	511	98.3	100.0	
Missing	System	9	1.7		
Total		520	100.0		

Statement 36: A main goal of geometry is to teach students how to reason.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	5	1.0	1.0	1.0
	2.00	10	1.9	1.9	2.9
	3.00	19	3.7	3.7	6.6
	4.00	69	13.3	13.3	19.9
	5.00	168	32.3	32.5	52.4
	6.00	246	47.3	47.6	100.0
	Total	517	99.4	100.0	
Missing	System	3	.6		
Total		520	100.0		

Statement 37: If a student makes a conjecture about a geometrical idea that is not in the curriculum, the teacher should allow time to prove/ disprove the conjecture.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	3	.6	.6	.6
	2.00	8	1.5	1.6	2.1
	3.00	17	3.3	3.3	5.4
	4.00	105	20.2	20.4	25.9
	5.00	202	38.8	39.3	65.2
	6.00	178	34.2	34.6	99.8
	56.00	1	.2	.2	100.0
	Total	514	98.8	100.0	
Missing	System	6	1.2		
Total		520	100.0		

Statement 38: Using a dynamic geometry software package such as Geometer's Sketchpad or Cabri Geometry to demonstrate geometric properties and relationships can take the place of having students do rigorous proofs.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	83	16.0	17.4	17.4
	2.00	77	14.8	16.2	33.6
	3.00	100	19.2	21.0	54.6
	4.00	84	16.2	17.6	72.3
	5.00	98	18.8	20.6	92.9
	6.00	34	6.5	7.1	100.0
	Total	476	91.5	100.0	
Missing	System	44	8.5		
Total		520	100.0		

Statement 39: I am confident about my teaching of geometry.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	4	.8	.8	.8
	2.00	7	1.3	1.4	2.1
	3.00	8	1.5	1.6	3.7
	4.00	32	6.2	6.2	9.9
	5.00	118	22.7	22.9	32.8
	6.00	347	66.7	67.2	100.0
	Total	516	99.2	100.0	
Missing	System	4	.8		
Total		520	100.0		

Statement 40: Students should be made aware of the historical background of geometry.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	6	1.2	1.2	1.2
	2.00	10	1.9	1.9	3.1
	3.00	20	3.8	3.9	6.9
	4.00	146	28.1	28.1	35.1
	5.00	182	35.0	35.1	70.1
	6.00	155	29.8	29.9	100.0
	Total	519	99.8	100.0	
Missing	System	1	.2		
Total		520	100.0		

Statement 41: Studying geometry leads to a positive attitude towards mathematics.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	6	1.2	1.2	1.2
	2.00	31	6.0	6.1	7.2
	3.00	46	8.8	9.0	16.2
	4.00	151	29.0	29.5	45.7
	5.00	157	30.2	30.7	76.4
	6.00	121	23.3	23.6	100.0
	Total	512	98.5	100.0	
Missing	System	8	1.5		
Total		520	100.0		

Statement 42: When teaching geometry connections to real world applications such as art should be made.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	2.00	5	1.0	1.0	1.0
	3.00	5	1.0	1.0	1.9
	4.00	55	10.6	10.6	12.5
	5.00	169	32.5	32.5	45.0
	6.00	286	55.0	55.0	100.0
	Total	520	100.0	100.0	

Statement 43: Students can experience the activity of mathematicians through their work in geometry class.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	1	.2	.2	.2
	2.00	11	2.1	2.2	2.4
	3.00	28	5.4	5.5	7.9
	4.00	93	17.9	18.3	26.1
	5.00	205	39.4	40.3	66.4
	6.00	171	32.9	33.6	100.0
	Total	509	97.9	100.0	
Missing	System	11	2.1		
Total		520	100.0		

Statement 44: I enjoy teaching my students how to do geometric proofs.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	14	2.7	2.8	2.8
	2.00	23	4.4	4.6	7.4
	3.00	54	10.4	10.8	18.2
	4.00	87	16.7	17.4	35.5
	5.00	156	30.0	31.1	66.7
	6.00	167	32.1	33.3	100.0
	Total	501	96.3	100.0	
Missing	System	19	3.7		
Total		520	100.0		

Statement 45: Geometry enables ideas from other areas of mathematics to be pictured.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	2	.4	.4	.4
	2.00	4	.8	.8	1.2
	3.00	14	2.7	2.8	4.0
	4.00	52	10.0	10.3	14.3
	5.00	214	41.2	42.5	56.9
	6.00	217	41.7	43.1	100.0
	Total	503	96.7	100.0	
Missing	System	17	3.3		
Total		520	100.0		

Statement 46: The main goal of geometry is to illustrate the order and coherence of a mathematical system.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	10	1.9	2.0	2.0
	2.00	35	6.7	6.8	8.8
	3.00	74	14.2	14.5	23.3
	4.00	155	29.8	30.3	53.6
	5.00	161	31.0	31.5	85.1
	6.00	76	14.6	14.9	100.0
	Total	511	98.3	100.0	
Missing	System	9	1.7		
Total		520	100.0		

Statement 47: Applying geometrical concepts and thinking will help students in their future occupations or professions.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	1	.2	.2	.2
	2.00	10	1.9	1.9	2.1
	3.00	27	5.2	5.3	7.4
	4.00	88	16.9	17.2	24.6
	5.00	185	35.6	36.1	60.6
	6.00	202	38.8	39.4	100.0
	Total	513	98.7	100.0	
Missing	System	7	1.3		
Total		520	100.0		

Statement 48: I enjoy proving theorems in geometry for my students.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	1.00	17	3.3	3.4	3.4
	2.00	30	5.8	5.9	9.3
	3.00	41	7.9	8.1	17.4
	4.00	107	20.6	21.2	38.6
	5.00	154	29.6	30.5	69.1
	6.00	156	30.0	30.9	100.0
	Total	505	97.1	100.0	
Missing	System	15	2.9		
Total		520	100.0		

Undergraduate major (first degree) of Respondents

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		17	4.9	4.9	4.9
	accounting	6	1.7	1.7	6.7
	acct/fin	1	.3	.3	7.0
	Applied math	1	.3	.3	7.2
	Applied math/sta	2	.6	.6	7.8
	arts	1	.3	.3	8.1
	BA	1	.3	.3	8.4
	bio	1	.3	.3	8.7
	Bio-chem	1	.3	.3	9.0
	broad/journ	1	.3	.3	9.3
	busadmin	3	.9	.9	10.1
	business	8	2.3	2.3	12.5
	Chinese studies	1	.3	.3	12.8
	comm arts	1	.3	.3	13.0
	comp/engin	1	.3	.3	13.3
	comp/psy	1	.3	.3	13.6
	computer	7	2.0	2.0	15.7
	eastasianst	1	.3	.3	15.9
	eco/stat	1	.3	.3	16.2
	economics	9	2.6	2.6	18.8
	education	7	2.0	2.0	20.9
	elemeduc	3	.9	.9	21.7
	eng/appphysi	1	.3	.3	22.0
	engineering	24	7.0	7.0	29.0
	english	4	1.2	1.2	30.1
	finance/busa	1	.3	.3	30.4
	french	1	.3	.3	30.7
	generalstudi	1	.3	.3	31.0
	history	2	.6	.6	31.6
	informatics	1	.3	.3	31.9
	liberalarts	1	.3	.3	32.2
	math	151	43.8	43.8	75.9
	math&	18	5.2	5.2	81.2
	mathed	43	12.5	12.5	93.6
	mis	1	.3	.3	93.9
	mmss	1	.3	.3	94.2
	operres/eng	1	.3	.3	94.5
	physics	4	1.2	1.2	95.7
	premed	1	.3	.3	95.9
	psychology	6	1.7	1.7	97.7
	psych/econ	1	.3	.3	98.0
	Russianlang	1	.3	.3	98.3
	scied	1	.3	.3	98.6
	science	1	.3	.3	98.8
	seceduc	1	.3	.3	99.1
	sis	1	.3	.3	99.4
	statistics	1	.3	.3	99.7
	textilemana	1	.3	.3	100.0
	Total	345	100.0	100.0	

Teaching Experience of Respondents (years)

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	.00	1	.2	.2	.2
	.50	4	.8	.8	1.0
	1.00	25	4.8	5.0	6.0
	1.50	4	.8	.8	6.8
	2.00	29	5.6	5.8	12.6
	2.50	3	.6	.6	13.2
	3.00	40	7.7	8.0	21.2
	4.00	25	4.8	5.0	26.3
	5.00	41	7.9	8.2	34.5
	6.00	14	2.7	2.8	37.3
	6.50	1	.2	.2	37.5
	7.00	9	1.7	1.8	39.3
	7.50	1	.2	.2	39.5
	8.00	15	2.9	3.0	42.5
	9.00	7	1.3	1.4	43.9
	10.00	29	5.6	5.8	49.7
	11.00	9	1.7	1.8	51.5
	12.00	14	2.7	2.8	54.3
	13.00	9	1.7	1.8	56.1
	14.00	8	1.5	1.6	57.7
	15.00	19	3.7	3.8	61.5
	16.00	13	2.5	2.6	64.1
	17.00	12	2.3	2.4	66.5
	18.00	10	1.9	2.0	68.5
	19.00	5	1.0	1.0	69.5
	20.00	26	5.0	5.2	74.7
	21.00	7	1.3	1.4	76.2
	22.00	7	1.3	1.4	77.6
	23.00	9	1.7	1.8	79.4
	24.00	3	.6	.6	80.0
	25.00	12	2.3	2.4	82.4
	26.00	5	1.0	1.0	83.4
	27.00	8	1.5	1.6	85.0
	28.00	3	.6	.6	85.6
	29.00	1	.2	.2	85.8
	30.00	14	2.7	2.8	88.6
	31.00	6	1.2	1.2	89.8
	32.00	6	1.2	1.2	91.0
	33.00	12	2.3	2.4	93.4
	34.00	3	.6	.6	94.0
	35.00	17	3.3	3.4	97.4
	36.00	4	.8	.8	98.2
	37.00	3	.6	.6	98.8
	38.00	2	.4	.4	99.2
	40.00	1	.2	.2	99.4
	41.00	1	.2	.2	99.6
	43.00	1	.2	.2	99.8
	49.00	1	.2	.2	100.0
	Total	499	96.0	100.0	
Missing	System	21	4.0		
Total		520	100.0		

Response to Question: I have taken mathematics methods courses (i.e. courses on how to teach various aspects of mathematics).

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		15	2.9	2.9	2.9
	N	81	15.6	15.6	18.5
	Y	424	81.5	81.5	100.0
	Total	520	100.0	100.0	

Response to Question: I have taken geometry courses as an undergraduate.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		14	2.7	2.7	2.7
	N	151	29.0	29.0	31.7
	Y	355	68.3	68.3	100.0
	Total	520	100.0	100.0	

Postgraduate Degrees of Respondents

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		16	3.1	3.1	3.1
	accounting	1	.2	.2	3.3
	actuarial	1	.2	.2	3.5
	admin	2	.4	.4	3.8
	appliedmat	1	.2	.2	4.0
	appliedmath	1	.2	.2	4.2
	biometry	1	.2	.2	4.4
	business	2	.4	.4	4.8
	businessadmi	1	.2	.2	5.0
	businessed	1	.2	.2	5.2
	cancerresear	1	.2	.2	5.4
	chemistry	3	.6	.6	6.0
	cis	1	.2	.2	6.2
	clinicalpsyc	1	.2	.2	6.3
	communicatio	1	.2	.2	6.5
	comped	1	.2	.2	6.7
	compscied	1	.2	.2	6.9
	counseling	1	.2	.2	7.1
	curr&instru	1	.2	.2	7.3
	curr&instruc	2	.4	.4	7.7
	curr/ed	1	.2	.2	7.9
	curriculum	3	.6	.6	8.5
	divinity	1	.2	.2	8.7
	economics	4	.8	.8	9.4
	edadmin	6	1.2	1.2	10.6
	edleadershi	1	.2	.2	10.8
	Edpsych	1	.2	.2	11.0
	educ	2	.4	.4	11.3
	educ/mba	1	.2	.2	11.5
	education	28	5.4	5.4	16.9
	educcomputn	1	.2	.2	17.1
	eductech	3	.6	.6	17.7
	elect/commun	1	.2	.2	17.9
	eng/phy/stat	1	.2	.2	18.1
	engineering	6	1.2	1.2	19.2
	english	1	.2	.2	19.4
	finance	2	.4	.4	19.8
	finearts	1	.2	.2	20.0
	humanorgproc	1	.2	.2	20.2
	instrtech	1	.2	.2	20.4
	instructions	2	.4	.4	20.8
	instructtech	2	.4	.4	21.2
	insttech	1	.2	.2	21.3
	journalism	1	.2	.2	21.5
	law	2	.4	.4	21.9
	liberalstudi	1	.2	.2	22.1
	librarysci	1	.2	.2	22.3
	lit	1	.2	.2	22.5
	machinelear	1	.2	.2	22.7
	mat	1	.2	.2	22.9
	MAT	1	.2	.2	23.1
	MAT/MBA	1	.2	.2	23.3
	math	64	12.3	12.3	35.6
	math&	5	1.0	1.0	36.5
	mathed	141	27.1	27.1	63.7
	mathed&	2	.4	.4	64.0
	MBA	1	.2	.2	64.2
	MBA/manag	1	.2	.2	64.4
	medicine	1	.2	.2	64.6
	MPA	1	.2	.2	64.8
	multiculed	1	.2	.2	65.0
	N	120	23.1	23.1	88.1
	philo	1	.2	.2	88.3
	philosophy	1	.2	.2	88.5
	physics	3	.6	.6	89.0
	psych	1	.2	.2	89.2
	psychstat	1	.2	2	89.4
	qualityassur	1	.2	.2	89.6
	reading/comp	1	.2	.2	89.8
	russianlang	1	.2	.2	90.0
	schcounseln	1	.2	.2	90.2
	schooladmin	1	.2	.2	90.4
	scied	1	.2	.2	90.6
	sciencoed	1	.2	.2	90.8
	seceduc	1	.2	.2	91.0
	sis	1	.2	.2	91.2
	societes&cu	1	.2	.2	91.3
	speced/fina	1	.2	.2	91.5
	speceduc	7	1.3	1.3	92.9
	statistics	2	.4	.4	93.3
	teachlead	1	.2	.2	93.5
	urbanplan	1	.2	.2	93.7
	urbanpolicy	1	.2	.2	93.8
	Y	32	6.2	6.2	100.0
	Total	520	100.0	100.0	

Response to Question: I have taught geometry as a 1 year course.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		15	2.9	2.9	2.9
	N	182	35.0	35.0	37.9
	Y	323	62.1	62.1	100.0
	Total	520	100.0	100.0	

Response to Question: I have taught geometry as a topic in an integrated curriculum.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		13	2.5	2.5	2.5
	N	112	21.5	21.5	24.0
	Y	395	76.0	76.0	100.0
	Total	520	100.0	100.0	

Response to Question: I have used manipulatives to teach geometrical concepts

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		14	2.7	2.7	2.7
	N	100	19.2	19.2	21.9
	Y	406	78.1	78.1	100.0
	Total	520	100.0	100.0	

Response to Question: I have used dynamic geometry software with my students

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		13	2.5	2.5	2.5
	N	309	59.4	59.4	61.9
	Y	198	38.1	38.1	100.0
	Total	520	100.0	100.0	

Response to Question: I am a member of the: NCTM / ATM / MA / Other

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		16	3.1	3.1	3.1
	N	275	52.9	52.9	56.0
	Y	229	44.0	44.0	100.0
	Total	520	100.0	100.0	

Response to Question: I have attended at least 2 meetings of the above.

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		22	4.2	4.2	4.2
	N	341	65.6	65.6	69.8
	Y	157	30.2	30.2	100.0
	Total	520	100.0	100.0	

Gender of the Respondents

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		12	2.3	2.3	2.3
	F	268	51.5	51.5	53.8
	M	240	46.2	46.2	100.0
	Total	520	100.0	100.0	

Types of School in which the Respondents Teach

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		34	6.5	6.5	6.5
	gifted	1	.2	.2	6.7
	independent	3	.6	.6	7.3
	inner	321	61.7	61.7	69.0
	private	22	4.2	4.2	73.3
	rural	31	6.0	6.0	79.2
	selective	4	.8	.8	80.0
	specialty	1	.2	.2	80.2
	suburban	103	19.8	19.8	100.0
	Total	520	100.0	100.0	

Size of Respondents' Schools

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	39.00	1	.2	.2	.2
	100.00	2	.4	.4	.6
	106.00	1	.2	.2	.8
	140.00	1	.2	.2	1.1
	150.00	1	.2	.2	1.3
	170.00	1	.2	.2	1.5
	173.00	1	.2	.2	1.7
	182.00	1	.2	.2	1.9
	200.00	9	1.7	1.9	3.6
	220.00	2	.4	.4	4.2
	225.00	1	.2	.2	4.4
	250.00	3	.6	.6	5.1
	300.00	4	.8	.8	5.9
	340.00	1	.2	.2	6.1
	350.00	2	.4	.4	6.5
	368.00	1	.2	.2	6.8
	380.00	1	.2	.2	7.0
	400.00	9	1.7	1.9	8.9
	405.00	1	.2	.2	9.1
	415.00	1	.2	.2	9.3
	420.00	2	.4	.4	9.7
	425.00	1	.2	.2	9.9
	430.00	1	.2	.2	10.1
	450.00	6	1.2	1.3	11.4
	500.00	10	1.9	2.1	13.5
	550.00	1	.2	.2	13.7
	600.00	5	1.0	1.1	14.8
	650.00	3	.6	.6	15.4
	700.00	6	1.2	1.3	16.7
	750.00	2	.4	.4	17.1
	800.00	7	1.3	1.5	18.6
	832.00	1	.2	.2	18.8
	834.00	1	.2	.2	19.0
	839.00	1	.2	.2	19.2
	850.00	6	1.2	1.3	20.5
	900.00	3	.6	.6	21.1
	950.00	2	.4	.4	21.5
	1000.00	13	2.5	2.7	24.3
	1100.00	8	1.5	1.7	25.9
	1150.00	2	.4	.4	26.4
	1200.00	20	3.8	4.2	30.6
	1250.00	2	.4	.4	31.0
	1300.00	15	2.9	3.2	34.2
	1350.00	1	.2	.2	34.4
	1400.00	15	2.9	3.2	37.6
	1450.00	1	.2	.2	37.8
	1500.00	20	3.8	4.2	42.0
	1550.00	2	.4	.4	42.4
	1600.00	12	2.3	2.5	44.9
	1680.00	1	.2	.2	45.1
	1700.00	2	.4	.4	45.6
	1750.00	1	.2	.2	45.8
	1800.00	5	1.0	1.1	46.8
	1850.00	1	.2	.2	47.0
	1900.00	1	.2	.2	47.3
	2000.00	14	2.7	3.0	50.2
	2100.00	3	.6	.6	50.8
	2200.00	4	.8	.8	51.7
	2300.00	14	2.7	3.0	54.6
	2400.00	5	1.0	1.1	55.7
	2500.00	27	5.2	5.7	61.4
	2600.00	3	.6	.6	62.0
	2700.00	7	1.3	1.5	63.5
	2800.00	8	1.5	1.7	65.2
	2900.00	2	.4	.4	65.6
	3000.00	37	7.1	7.8	73.4
	3100.00	1	.2	.2	73.6
	3200.00	4	.8	.8	74.5
	3400.00	3	.6	.6	75.1
	3500.00	19	3.7	4.0	79.1
	3600.00	6	1.2	1.3	80.4
	3800.00	2	.4	.4	80.8
	4000.00	36	6.9	7.6	88.4
	4100.00	1	.2	.2	88.6
	4200.00	6	1.2	1.3	89.9
	4300.00	3	.6	.6	90.5
	4400.00	2	.4	.4	90.9
	4500.00	19	3.7	4.0	94.9
	4600.00	2	.4	.4	95.4
	4700.00	2	.4	.4	95.8
	4900.00	1	.2	.2	96.0
	5000.00	17	3.3	3.6	99.6
	5200.00	1	.2	.2	99.8
	5500.00	1	.2	.2	100.0
	Total	474	91.2	100.0	

Grade Levels Taught by the Respondents

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid		37	7.1	7.1	7.1
	1-10	1	.2	.2	7.3
	1-12	1	.2	.2	7.5
	1-grad	1	.2	.2	7.7
	10-11	1	.2	.2	7.9
	10-12	8	1.5	1.5	9.4
	10-grad	1	.2	.2	9.6
	10	3	.6	.6	10.2
	10,11	4	.8	.8	11.0
	11-12	1	.2	.2	11.2
	11-16	1	.2	.2	11.3
	11,12	3	.6	.6	11.9
	12	1	.2	.2	12.1
	2-grad	1	.2	.2	12.3
	3-11	1	.2	.2	12.5
	4-12	1	.2	.2	12.7
	5-11	1	.2	.2	12.9
	5-12	4	.8	.8	13.7
	6-10	2	.4	.4	14.0
	6-11	1	.2	.2	14.2
	6-12	16	3.1	3.1	17.3
	6-7	1	.2	.2	17.5
	6,7,9-11	1	.2	.2	17.7
	6,7,9-12	1	.2	.2	17.9
	7-10	2	.4	.4	18.3
	7-11	5	1.0	1.0	19.2
	7-12	44	8.5	8.5	27.7
	7-9	4	.8	.8	28.5
	7-college	1	.2	.2	28.7
	7-grad	1	.2	.2	28.8
	7	1	.2	.2	29.0
	7,9,10	2	.4	.4	29.4
	8-10	3	.6	.6	30.0
	8-11	3	.6	.6	30.6
	8-12	10	1.9	1.9	32.5
	8-16	1	.2	.2	32.7
	8-college	1	.2	.2	32.9
	9-10	11	2.1	2.1	35.0
	9-11	29	5.6	5.6	40.6
	9-12	276	53.1	53.1	93.7
	9-13	3	.6	.6	94.2
	9-college	1	.2	.2	94.4
	9-grad	1	.2	.2	94.6
	9	8	1.5	1.5	96.2
	9,10	13	2.5	2.5	98.7
	9,11	2	.4	.4	99.0
	9,11,12	1	.2	.2	99.2
	college	1	.2	.2	99.4
	k-12	1	.2	.2	99.6
	N-9	1	.2	.2	99.8
	prek-grad	1	.2	.2	100.0
	Total	520	100.0	100.0	

APPENDIX G – MATHEMATICAL INTERPRETATIONS OF FACTOR ANALYSIS

Matrix Interpretation

Mathematically speaking, the goal of factor analysis is to define a set of axes in p space, where p is the number of variables, which better describes the space than the set of vectors arranged within it and then to interpret what the axes, factors or components, represent. These axes are the eigenvectors. Correlation coefficients are the cosines between the angles. Loading of a variable on a factor or component is the cosine of the angle between the variable vector and the eigenvector (axis). This is the correlation between a variable and a component.

Definition. Let A be an nxn matrix. A scalar λ is called an *eigenvalue* of A (also referred to in the literature as characteristic root, latent root, principal value, or singular value) if there is a non-zero vector $v \neq 0$ called an *eigenvector* (also referred to in the literature as characteristic vector or latent vector) such that $Av=\lambda v$. (Matrix A stretches the eigenvector by v an amount specified by λ. $(A-\lambda I)\ v=0$ is a homogenous linear system of equations. This system has a nonzero solution if the coefficient matrix $A-\lambda I$ is singular. A scalar λ is an eigenvalue of the matrix A iff λ is a solution to the characteristic equation $\det(A-\lambda I)=0$.The sum of the eigenvalues of a matrix= its trace (sum of the diagonal entries). The product of its eigenvalues = its determinant. The eigenspace is the subspace spanned by $\ker(A-\lambda I)$.

Algebraic Interpretation

The first principal component, y_1, is a linear combination of $x_p x_2 \ldots x_p$ ($y_1 =$ $a_{11}x_1+a_{12}x_2+\ldots+a_{1p}x_p$ such that the variance of y_1 is maximized given the constraint that the sum of the squared weights is equal to 1 ($\Sigma a_{1i}^2 = 1$). The x_i's are random variables and can be either standard scores or deviation from the mean scores. If the variance of y_1 is maximized then so is the sum of the squared correlations of y_1 with the original $x_1 x_2 \ldots x_p$ variables. The second principal component is $y_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p$. The variable y_2 has the next largest sum of squared correlation with the original variables and is uncorrelated with y_1. Also $\Sigma a_{2i}^2 = 1$. In PCA the weights $(a_1, a_2, \ldots, a_p)$ are mathematically determined to maximize the sum of squared correlations of the principal components with the original variables.

Factor analysis using principal component analysis:

$x_1 = a$

The researcher is interested in reducing the number of variables from p to a smaller set of k derived variables that retain most of the information in the original p variables. The k derived variables if considered as independent variables will maximize the prediction of the original p variables. Principal components are the k derived variables that maximize the variance accounted for in the original variables. (In our case: $p = 48$, $k = 3$).

APPENDIX H – FOLLOW UP QUESTIONNAIRE

Please answer the questions to the best of your ability.

1. What do you most love about geometry and why?

2. What is your most memorable experience or experiences as a student in a geometry class?

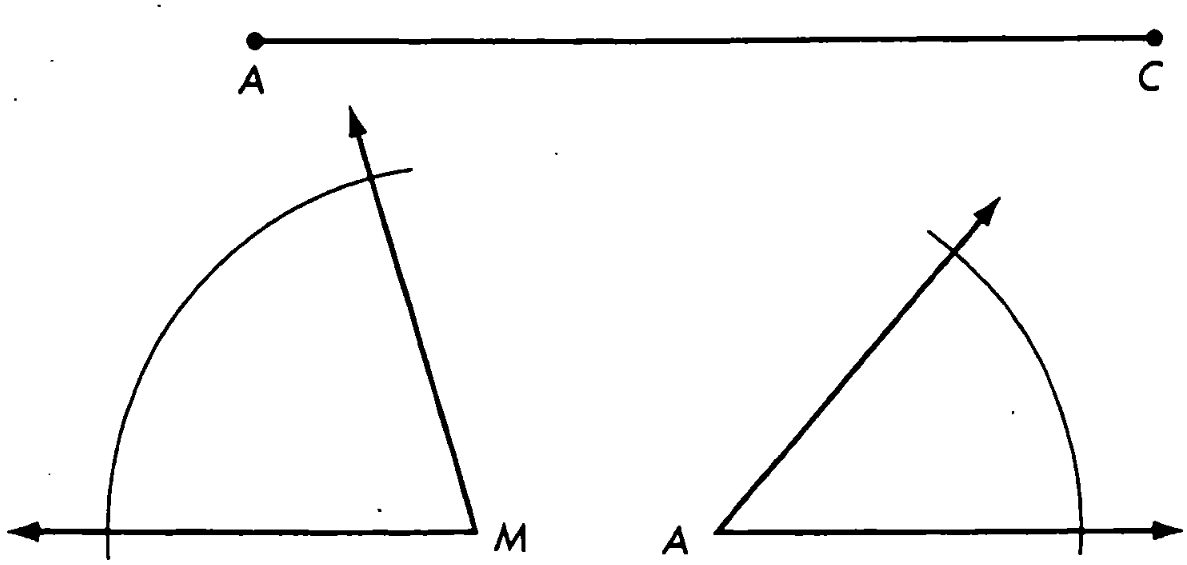
3. What do you use when teaching geometry to enable your students to explore the visual aspects of the subject? Please include your reasons for these.

4. Is there any topic or topics that are in the current geometry curriculum that you believe should be eliminated? Please explain why.

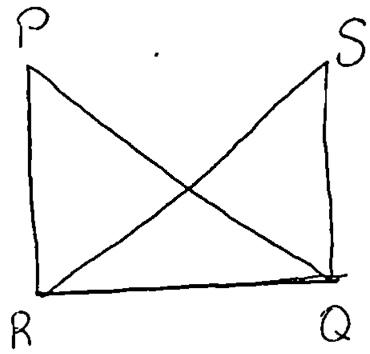
5. Do you include real world applications in your geometry course? What are these and why are they included?

APPENDIX I - WORK SHEET FOR ROSE'S CLASS

SAA or
AAS



APPENDIX J - ROSE'S EXAM AND STUDENT HOMEWORK SOLUTION



1) Given: $\overline{PR} \cong \overline{SQ}$

$\overline{PR} \perp \overline{RQ}$

$\overline{SQ} \perp \overline{RQ}$

Prove: $\overline{PQ} \cong \overline{SR}$

(Need to put mixed up answers into a formal proof)

Mixed up answers

$\overline{RQ} \cong \overline{RQ}$	$\overline{RR} \cong \overline{SQ}$	$\angle PRQ$ and $\angle SQR$ ~~form~~ are right angles	SAS
Given	$\angle PRQ \cong \angle SQR$	Given	Right angles are congruent
Reflexive property	$\overline{PR} \perp \overline{RQ}$	$\overline{PQ} \cong \overline{SR}$	$\overline{SQ} \perp \overline{RQ}$
Given	Def. of $\perp$ lines	$\Delta PRQ \cong \Delta SQR$	CPCTC

Rose's Student's Homework

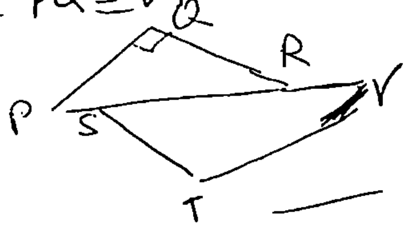
Given $\overline{PSRV}$ Prove $\overline{PQ} \cong \overline{VT}$

$\overline{PS} \cong \overline{VR}$

$\overline{RQ} \parallel \overline{ST}$

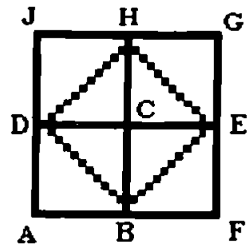
$PQ \perp QR$

$VT \perp \overline{TS}$



S	R
$\overline{PSRV}$	Given
$\overline{PS} \cong \overline{VR}$	Given
$\overline{SR} \cong \overline{SR}$	Reflexive
$\overline{PS} + \overline{SR} \cong \overline{SR} + \overline{RV}$	add. postulate
$\overline{PR} \cong \overline{SV}$	add. postulate (S)
$PQ \perp QR$ & $VT \perp TS$	Given
$\angle Q$ & $\angle T$ are right $\angle$'s	Def of $\perp$ lines
$\angle Q \cong \angle T$	right $\angle$'s are $\cong$ (A)
$\overline{RQ} \parallel \overline{ST}$	Given
$\angle QRP \cong \angle VST$	alt. interior $\angle$ theorem (A)
$\triangle QPR \cong \triangle TVS$	SAA $\cong$ SAA
$\overline{PQ} \cong \overline{VT}$	cpctc

APPENDIX K – PLATO'S MENO: THE GEOMETRY EXPERIMENT



SOCRATES: Tell me, boy, is not this our square of four feet? (ABCD.) You understand?
BOY: Yes.
SOCRATES: Now we can add another equal to it like this? (BCEF.)
BOY: Yes.
SOCRATES: And a third here, equal to each of the others? (CEGH.)
BOY: Yes.
SOCRATES: And then we can fill in this one in the corner? (DCHJ.)
BOY: Yes.
SOCRATES: Then here we have four equal squares?
BOY: Yes.
SOCRATES: And how many times the size of the first square is the whole?
BOY: Four times.
SOCRATES: And we want one double the size. You remember?
BOY: Yes.
SOCRATES: Now does this line going from corner to corner cut each of these squares in half?
BOY: Yes.
SOCRATES: And these are four equal lines enclosing this area? (BEHD.)
BOY: They are.
SOCRATES: Now think. How big is this area?
BOY: I don't understand.
SOCRATES: Here are four squares. Has not each line cut off the inner half of each of them?
BOY: Yes.
SOCRATES: And how many such halves are there in this figure? (BEHD.)
BOY: Four.
SOCRATES: And how many in this one? (ABCD.)
BOY: Two.
SOCRATES: And what is the relation of four to two?
BOY: Double.
SOCRATES: How big is this figure then?

BOY: Eight feet.
SOCRATES: On what base?
BOY: This one.
SOCRATES: The line which goes from corner to corner of the square of four feet?
BOY: Yes.
SOCRATES: The technical name for it is 'diagonal'; so if we use that name, it is your personal opinion that the square on the diagonal of the original square is double its area.
BOY: That is so, Socrates.
SOCRATES: What do you think, Meno? Has he answered with any opinions that were not his own?
MENO: No, they were all his.
SOCRATES: Yet he did not know, as we agreed a few minutes ago.
MENO: True.
SOCRATES: But these opinions were somewhere in him, were they not?
MENO: Yes.
SOCRATES: So a man who does not know has in himself true opinions on a subject without having knowledge.
MENO: It would appear so.
SOCRATES: At present these opinions, being newly aroused, have a dream-like quality. But if the same questions are put to him on many occasions and in different ways, you can see that in the end he will have a knowledge on the subject as accurate as anybody's.
MENO: Probably.
SOCRATES: This knowledge will not come from teaching but from questioning. He will recover it for himself.
MENO: Yes.
SOCRATES: And the spontaneous recovery of knowledge that is in him is recollection, isn't it?
MENO: Yes.
SOCRATES: Either then he has at some time acquired the knowledge which he now has, or he has always possessed it. If he always possessed it, he must always have known; if on the other hand he acquired it at some previous time, it cannot have been in this life, unless somebody has taught him geometry. He will behave in the same way with all geometrical knowledge, and every other subject. Has anyone taught him all these? You ought to know, especially as he has been brought up in your household.
MENO: Yes, I know that no one ever taught him.
SOCRATES: And has he these opinions, or hasn't he?
MENO: It seems we can't deny it.
SOCRATES: Then if he did not acquire them in this life, isn't it immediately clear that he possessed and had learned them during some other period?
MENO: It seems so.
SOCRATES: When he was not in human shape?
MENO: Yes.
SOCRATES: If then there are going to exist in him, both while he is and while he is not a man, true opinions which can be aroused by questioning and turned into knowledge, may we say that his soul has been for ever in a state of knowledge? Clearly he always either is or is not a man.
MENO: Clearly.

SOCRATES: And if the truth about reality is always in our soul, the soul must be immortal, and one must take courage and try to discover-that is, to recollect what one doesn't happen to know, or (more correctly) remember, at the moment.
MENO: Somehow or other I believe you are right.
SOCRATES: I think I am. I shouldn't like to take my oath on the whole story, but one thing I am ready to fight for as long as I can, in word and act: that is, that we shall be better, braver and more active men if we believe it right to look for what we don't know than if we believe there is no point in looking because what we don't know we can never discover.
MENO: There too I am sure you are right.

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HIGH SCHOOL MATHEMATICS TEACHERS' BELIEFS: THE USE OF MANIPULATIVES

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Abstract

This paper shares some of the preliminary results of the research that is being conducted for the dissertation High School Mathematics Teachers' Beliefs about the Teaching and Learning of Geometry to be submitted to the Department of Mathematics and Statistics of the University of Plymouth, UK. We will focus on high school teachers' beliefs about the use of manipulatives in their geometry classes. We have found that in a sample of 520 questionnaire respondents there were statistically significant differences in manipulative use with respect to gender, membership in professional organizations, attendance at professional conferences, undergraduate degree, having a graduate degree, school location, and undergraduate teacher preparation.

Introduction

The following research questions emerged from a pilot questionnaire distributed to high school mathematics teachers during the 2003-2004 school year.

- What are high school mathematics teachers' beliefs about the role of geometry in the high school curriculum?
- What are high school mathematics teachers' beliefs about the role manipulatives play in the geometry classroom?
- What are high school mathematics teachers' beliefs about the use of dynamic geometry software in high school?
- What are high school mathematics teachers' beliefs about the role of proof in high school geometry?

With the above questions in mind, a revised beliefs questionnaire about teaching and learning geometry containing 48 Likert type statements, three open ended response questions and a number of personal data statements was distributed to high school mathematics teachers from four countries: the United States, Australia, Canada and England during the 2004-2005 school year. There were 520 respondents: 268 females (52.8%), 240 males (47.2%), and 12 teachers that did not specify their gender. This paper reports on some of the findings concerning teachers' beliefs about the role of manipulatives in the classroom.

Manipulatives in the Literature

Successful use of manipulatives requires the teachers buying into them. They have to believe that the manipulative is not just a "toy". They have to understand the connections between the concrete manipulative and the abstract mathematics. The National Council of Teachers of Mathematics has encouraged the use of concrete manipulatives at all grade levels since 1940. Before going further it is worth considering exactly what falls into the category of a manipulative.

Kline (1973) suggested that a mathematics laboratory should be incorporated in the mathematics classroom to strengthen the intuitive approach to teaching. Although he did not use the word manipulative at the time, he did say that the laboratory should contain "apparatus of various sorts which could be used to demonstrate physical happenings from which mathematical results can be inferred." He mentioned Cuisenaire rods and geoboards.

Fuys, Geddes, & Tischler (1988) reporeted that the teachers who participated in the Brooklyn College Project were unanimous in their endorsement "of the hands-on visual concrete approach to developing geometric concepts for students in grades 6-9" (p. 155). Mason (1989) said that ever since the first published educational reports there has been discussion about the role and need for "practical equipment in the classroom" (p. 38). Thomas (1992) defined a manipulative object as any object used by children to model some process or their thinking about some concept. Spikell (1933) defined manipulatives as physical, real world objects that can be used to teach mathematical ideas, concepts, principles, and skills to student. He stated that manipulatives were once regarded as supplementary resource materials in the classroom, but today they are viewed as important instructional aids in school mathematics programs. He claimed that as manipulatives have become more available, their effective use in instruction may have decreased. He said that this is because teachers have inadequate initial preparation and follow up support in the use of manipulatives. The early adopters of manipulatives in the classroom benefited from the relationship they had with the developers of the manipulative movement of the 1960s and 1970s. The were caught up in the excitement of new ideas.

> "They believed that manipulatives were a powerful teaching aid and did not have to be convinced of their potential value. Moreover, they had the requisite interest, motivation, and skill to discover for themselves, with minimal help, how to incorporate manipulative in their instructional programs. In short, the required minimal formal preparation t use manipulatives." (p. 219).

Spikell suggested that in order to use manipulatives properly, teachers must understand three things: the content

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embodied in the manipulative; specific activities with the manipulative that can be used to teach the content; and the effective pedagogy for teaching the content with the manipulative. He wrote the book Teaching Mathematics with Manipulatives that provides a frame of reference model for teachers to use when working with manipulatives. Ball (1992) stated that there is no magic involved with using manipulatives. They do not themselves carry meaning or insight. They provide a kinesthetic experience that can enhance perception and thinking.

Moyer (2001) defined manipulatives as physical objects designed to represent abstract mathematical ideas explicitly and concretely. Students "manipulate" these physical objects that "have both visual and tactile appeal" (p. 176) and allows for hands-on experiences. She claimed that manipulatives became popular because researchers' beliefs about how children learn changed. In order for their learning to be permanent, students must understand what they are learning. Moyer studied how and why ten middle school teachers used manipulatives in their classrooms. The teachers found them fun to use but not necessary for teaching and learning mathematics. They used them for enrichment, for playing games, and problem solving. The decision of when t use the manipulatives did not necessarily depend on the concept being taught, but rather on the amount of time remaining during a class period, the day of the week (Fridays were most often manipulative days), or the behavior of the class (good behavior was rewarded with manipulative use). Teachers believed that when suing manipulatives, the class was doing fun mathematics, but real mathematics was reserved for paper and pencil, textbooks, and teacher lecture. Using manipulatives in the classroom is beneficial if the students can eventually link their actions with manipulatives to abstract concepts. The teacher's role is to create environments that allow for this. Moyer postulated,

> It is the mediation by students and teachers in shared and meaningful practices that determines the utility of the manipulatives. Therefore, the physicality of concrete manipulatives does not carry the meaning of the mathematical ideas behind them. Students must reflect on their actions to build meaning (p.177).

Leitzel (1991) stated that recent research into the learning of geometry (Kline, 1973; Mason, 1989) claimed the need for concrete experiences with geometric figures and relationships to occur prior to a formal axiomatic study of geometry. These experiences should involve active participation, experimentation and the use of different kinds of materials and models. "For the middle school mathematics teachers, such concrete experiences are important not only in the development of their own geometric understanding but also in the enhancement of their knowledge of the stages through which geometric understanding evolves." (a Call for Change, p 19). The Rand Report (2003) suggested that secondary school mathematics teachers need to think deeply about simple things. They need to have the ability to see underlying connections and themes. They should have the ability to create activities whether they are using manipulatives or dynamic geometry or doing proofs that uncover central habits of mind such as going from a particular to the general.

In a study of 939 Australian teachers of which 336 taught in secondary school, Howard, Perry and Tracey (1997) reported that only 15 secondary teachers used manipulatives regularly. Their study suggested that secondary teachers need to develop a greater awareness of the ways in which manipulatives can be used to support student learning.

Craine (2004) surveyed mathematics department chairpersons in 158 secondary schools in Connecticut about contemporary high school geometry courses. There were no questions about use of manipulatives. In order to create classroom such as those suggested by NCTM (2000) and The Rand Report (2003), teachers' beliefs about the role of manipulatives should be examined.

The literature suggest that when manipulatives are used, if at all, it is not considered as an essential component of the lesson.

Methodology

The part of the study reported in this paper uses quantitative methods. To obtain data, a questionnaire was used. Some of the questions on the questionnaire were adapted from the questionnaire that The National Council of Teachers of Mathematics (NCTM) used to survey high school geometry teachers (Gearhardt, 1975). Other questions were adapted from a questionnaire about graphing calculator usage (Fleener, 1995). Responses to the Likert type statements were numerically coded from 1-6 with 1 being strongly disagree and 6 being strongly agree. SPSS was used to look at the frequencies of the descriptive data and cross-tabs between variables. Chi-squared analysis was performed on the crosstabs. Factor analysis was performed on the 48 Likert type statements (results not reported here).

Findings

Of the 506 responses to the statement in the personal data sections: .I have used manipulatives to teach geometrical concepts, 406 teachers (80.2%) responded yes and 100 (19.8%) responded no. Four out of the 48 Likert type statements on the questionnaire were about manipulatives and two others were about using a hands-on approach when teaching geometry (Table 1). For analysis purposes we grouped responses strongly disagree, moderately disagree and disagree slightly more than agree into a single response-disagree. Similarly, we grouped strongly agree, moderately agree, and agree slightly more than disagree into a single response - agree. We found that 95.3% agreed with statement 19, 92.3% agreed with statement 24, 96.o% agreed with statement 26, and 96.3% agreed with statement 23. There was quite a drop in the percentage of respondents that agreed with statement 29. Only 76.6% agreed with this statement. We were curious as to why this was so.

Table 1

Statements about Manipulatives on the Geometry Beliefs Questionnaire

	Strongly Agree	Moderately Agree	Agree slightly more than disagree	Disagree slightly more than agree	Moderately disagree	Strongly disagree
14. Using manipulatives in the teaching of geometry is [illegible].						
19. Manipulatives help students to grasp the basic ideas of geometry.						
24. I think it is beneficial to use manipulatives such as mirrors as a component of my geometry lessons.						
26. The use of manipulatives makes the learning geometry fun.						
23. It is important to use hands-on activities to explore geometric ideas.						
29. Initially, high school geometry should be hands-on with proofs coming later in the course.						

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In order to determine whether there were any relationships between variables we used the Chi-squared statistic. We cross-tabbed the six Likert statements (Table 1) with the statement from the personal data section: "I have used manipulatives to teach geometric concepts." Each of the tables 3-7 contains the observed frequencies and their totals. The expected frequencies (in parentheses) rounded to the nearest whole number were found by using the Chi-squared tests on the TI-83 and TI-84 plus calculators. We found statistically significant results for each of the statements except for statement 29. Respondents who have used manipulatives agreed significantly more than expected to statement 14, using manipulatives is motivational. Respondents who do not use manipulatives disagreed more than was expected with the statement (Table 2). We obtained similar results with statements 19, 24, 26 and 23 (Tables 3,4,5 and 6). Respondents who have used manipulatives agreed significantly more than expected with these statements. The use of manipulatives and teachers' beliefs about whether a geometry course should be initially hands-on with proof coming later (Table 7) are independent of each other.

Table 2
Crosstabs between statement 14 and manipulative use

	I have used manipulatives to teach geometric concepts		
Using manipulatives is motivational	N	Y	Total
Disagree	14 (5)	10 (19)	24
Agree	83 (92)	395 (386)	478
Total	97	405	502

Chi-squared = 24.61 p = 7.03E-7
Significant

Table 3

Crosstabs between statement 19 and manipulative use

	I have used manipulatives to teach geometric concepts		
Manipulatives help students grasp basic ideas	N	Y	Total
Disagree	10 (2)	3 (11)	13
Agree	84 (92)	400 (392)	484
Total	94	403	497

Chi-squared = 29.29 p = 6.23E-8

Table 4

Crosstabs between statement 24 and manipulative use

	I have used manipulatives to teach geometric concepts		
Beneficial to use manipulatives in my lessons	N	Y	Total
Disagree	25 (7)	14 (32)	39
Agree	66 (84)	387 (369)	453
Total	91	401	492

Chi-squared = 58.44 p = 2.09E-14

Significant

Table 5

Crosstabs between statement 26 and manipulative use

	I have used manipulatives to teach geometric concepts		
Manipulatives makes learning geometry fun	N	Y	Total
Disagree	13 (3)	6 (16)	19
Agree	77 (87)	395 (385)	472
Total	90	401	491

Chi-squared = 33.13 p = 8.63E-19

Significant

Table 6

Crosstabs between statement 23 and manipulative use

	I have used manipulatives to teach geometric concepts		
It is important to use hands-on activities	N	Y	Total
Disagree	14 (4)	5 (15)	19
Agree	81 (91)	396 (386)	477
Total	95	401	496

Chi-squared = 37.94 p = 7.29E-10

Significant

Table 7

Crosstabs between statement 29 and manipulative use

	I have used manipulatives to teach geometric concepts		
Geometry should be hands-on with proofs coming later	N	Y	Total
Disagree	24 (23)	93 (94)	117
Agree	72 (73)	310 (309)	382
Total	96	403	499

Chi-squared = .1597 p = .6894
Not significant

We further investigated whether there was any relationship between gender and manipulative use. Statistically significant results were found when the Chi-squared test was applied to responses to the statements "I have used manipulatives to teach geometrical concepts" and gender (Table 8). For this particular sample we have found that female high school teachers use manipulatives significantly more than males. Additional study is needed to see if this is true in general and if so, why?

Similarly, significant results were found when the Chi-squared test was applied to the statements "I am a member of NCTM, etc." and " I have used manipulatives to teach geometrical concepts (Table 9), the statement "I have attended at least 2 NCTM national meetings and I have used manipulatives to teach geometrical concepts" (Table 10), and the statements "I have used dynamic geometry software with my students" and "I have used manipulatives to teach geometrical concepts" (Table 11). More members of professional organizations use manipulatives than was expected, more respondents who attend professional meetings use manipulatives than was expected, and more teachers than expected who use dynamic geometry software also use manipulatives.

Table 8

Crosstab between gender and manipulative use

	I have used manipulatives to teach geometric concepts		
gender	N	Y	Total
F	35 (52)	230 (213)	265
M	63 (46)	176 (193)	239
Total	98	406	504

Chi squared = 13.8779 $p = 1.9507E\text{-}4$ df = 1
Significant

Table 9

Crosstab between membership to professional organization and manipulative use

	I have used manipulatives to teach geometric concepts.		
I am a member of NCTM, ATM etc.	N	Y	Total
N	71 (54)	204 (221)	275
Y	28 (45)	200 (183)	228
Total	99	404	503

Chi squared = 14.45 $p = 1.44E\text{-}4$ df = 1
Significant

Table 10

Crosstabs between attendance at professional meetings and manipulative use

	I have used manipulatives to teach geometric concepts		
I have attended at least 2 NCTM meetings	N	Y	Total
N	83 (68)	257 (272)	340
Y	16 (31)	140 (125)	156
Total	99	397	496

Chi squared = 13.41 $p = 2.49E\text{-}4$ df = 1
Significant

Table 11

Crosstabs between use of dynamic geometry and manipulatives

	I have used manipulatives to teach geometric concepts		
I have used dynamic geometry software with my students	N	Y	Total
N	81 (61)	226 (246)	307
Y	19 (39)	179 (159)	198
Total	100	405	505

Chi squared = 21.36 $p = 3.8E\text{-}6$ df = 1
Significant

We tested to determine if there was any relationship between use of manipulatives and the type of undergraduate (first degree) or graduate degree respondents had. When considering undergraduate major we divided majors into groups: business majors that includes accounting, finance, marketing, and economics; education that included all content areas; computer majors; and other that included history, art, psychology, etc. We did a similar grouping for graduate degrees adding up group of respondents without degrees and respondents with unspecified graduate degrees. We did not find any statistical significance with this grouping. When we looked at a mathematics related undergraduate major (first degree) which included mathematics education, statistics and computers versus any other undergraduate major we found significance with respect to use of manipulatives (Table 12). Similarly, we found significance with having a graduate degree and manipulative use (Table 13). Significantly, more teachers who majored in a mathematics related area use manipulatives than expected. More teachers that have some type of graduate degree use manipulatives significantly more than expected.

Table 12

Crosstabs between undergraduate major and manipulative use

	I have used manipulatives to teach geometric concepts		
Undergraduate major (first degree)	N	Y	Total
Mathematics	55 (64)	279 (270)	334
Other	40 (31)	124 (133)	164
Total	95	403	498

Chi-squared = 4.47 p = .0344 df = 1
Significant

Table 13

Crosstabs between having a graduate degree and manipulative use

	I have used manipulatives to teach geometric concepts		
I have a graduate degree	N	Y	Total
N	35 (24)	85 (96)	351
Y	65 (76)	316 (305)	150
Total	100	401	501

Chi-squared = 8.37 p = .0038 df = 1
Significant

When we compared the number of years teaching with manipulative use, we found 74% manipulative use by teachers with 5 years or less teaching experience, 87% manipulative use by teachers with 6-10 years experience, 80% use by teachers with 11-15 years experience, 82.5% use with 16-20 years experience, 78.8% use with 21-25 years experience, 85.7% use with 26-30 years experience, and 75.5% use by teachers with over 30 years of experience (Table 14). The use of manipulatives was independent of the number of years the teachers had been teaching.

Table 14
Crosstabs between number of years teaching and manipulative use

	I have used manipulatives to teach geometric concepts		
Number of years teaching	N	Y	Total
0-5	40 (33)	132 (139)	172
6-10	9 (14)	67 (62)	76
11-15	10 (11)	49 (48)	59
16-20	11 (12)	54 (53)	65
21-25	8 (7)	28 (29)	36
26-30	4 (6)	27 (25)	31
Over 30	13 (11)	46 (48)	59
Total	95	403	498
Chi squared = 6.329	p = .352	df = 6	

Not significant

We wanted to know whether school size affects the use of manipulatives. When we applied the Chi-squared test to the variables school size and manipulative use, we did not find a significant relationship between these variables (Table 15).

Table 15
Crosstabs between number of students in school and manipulative use

	I have used manipulatives to teach geometric concepts		
Number of students in my school	N	Y	Total
≤ 1000	16 (23)	98 (91)	114
1001-2000	23 (24)	99 (98)	122
2001-3000	20 (21)	87 (86)	107
3001-4000	19 (14)	52 (57)	71
Over 4000	15 (11)	40 (44)	55
Total	93	376	469
Chi-squared = 6.65	p = .1566	df = 4	

Not significant

18

There was significance when comparing the type of school and manipulative use. Manipulatives are used more that expected in suburban and rural high schools and less than expected in inner city and other types of high schools such as private (Table 16).

Table 16
Crosstab between location of school and manipulative use

Type of high school	I have used manipulatives to teach geometric concepts N	Y	Total
Inner city	69 (63)	248 (254)	317
Suburban	12 (20)	90 (82)	102
Rural	4 (6)	27 (25)	31
Other	10 (6)	20 (24)	30
Total	95	385	480

Chi-squared = 9.31 p = .025 df 3
Significant

We investigated whether there is a relationship between the way geometry is taught, for instance as part of a course or as a year long course, and the use of manipulatives. We found that when geometry is taught as a one year course there is a significant relationship (Table 17). More respondents than were expected used manipulatives when teaching geometry as a full year course. There was no significant difference with respect to the use of manipulatives when respondents taught geometry as a topic in an integrated course (Table 18).

Table 17
Crosstab between geometry as a full year course and use of manipulatives

I have taught geometry as a full year course	I have used manipulatives to teach geometric concepts N	Y	Total
N	51 (36)	131 (146)	182
Y	48 (63)	273 (258)	321
Total	99	404	503

Chi-squared = 12.55 p = 3.965E-4
Significant

19

Table 18

Crosstab between geometry as a topic in an integrated curriculum and use of manipulatives

	I have used manipulatives to teach geometric concepts		
Taught geometry as a topic in an integrated curr.	N	Y	Total
N	25 (22)	86 (89)	111
Y	75 (78)	320 (317)	395
Total	100	406	506

Chi-squared = .68 p = .409
Not significant

We wanted to know what the effects of having taken an undergraduate geometry course or courses in mathematical methods (pedagogy, how-to-teach courses) were on the use of manipulatives. When we applied a Chi-squared test to the variables, we found that there was significance between taking methods courses and the use of manipulatives, but we did not find any significant relationship between taking an undergraduate geometry course and the use of manipulatives (Tables 19 & 20).

Table 19

Crosstabs between taking mathematics methods courses and use of manipulatives

	I have used manipulatives to teach geometric concepts		
I have taken mathematics methods courses	N	Y	Total
N	25 (16)	56 (65)	81
Y	73 (82)	348 (339)	421
Total	98	404	502

Chi-squared = 7.9087 p = .0049
Significant

Table 20

Crosstabs between taking undergraduate geometry courses and use of manipulatives

	I have used manipulatives to teach geometric concepts		
I have taken an undergraduate geometry course	N	Y	Total
N	37 (30)	114 (121)	151
Y	62 (69)	290 (283)	352
Total	99	404	503

Chi-squared = 3.17 p = .075
Not significant

Conclusions

We have found that there are teachers who agreed with the statements that manipulative use is motivational and fun but don't actually use them. Other studies have reported similar findings (Howard, Perry, Tracey, 1997; Moyer, 2001).

For this particular sample we have found that male high school teachers use manipulatives significantly less than female teachers. Further study is needed to see if this is true in general and if so, why? It is interesting to find that membership in professional organizations and attendance at professional conferences is significant with respect to manipulative use. Does membership in a professional organization and/or attendance at conferences provide more awareness of manipulatives and their uses or vice versa, do teachers who believe in using manipulatives join organizations and/or attend meetings to learn more about their profession? Our analysis does not provide us with an answer.

We found significant relationships between use of manipulatives and whether the teacher had a mathematics related undergraduate major, whether the teacher had a graduate degree in any field, and whether the teacher took a mathematics methods course. Our results also showed that new teachers used manipulatives least. These findings beg for further investigation. The new teachers for the most part do not yet have graduate degrees. Are the new teachers in schools with fewer resources than more experienced teachers? Are new teachers coming from alternative certification programs? It appears that graduate programs support the use of manipulatives perhaps by providing further training with manipulatives and/or graduating teachers that have more flexibility in their teaching. We found that suburban teachers use manipulatives significantly more than urban or rural teachers. Is there an equity issue here? Are manipulatives available in all high schools?

The questionnaire did not probe deeply enough into finding out which manipulatives were used and how often they were used. A voluntary sample of respondents will be interviewed and a subset of them will be observed in order to examine whether professed beliefs are indeed practiced and to determine the effectiveness of the use of manipulatives.

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