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THE GALACTIC MAGNETIC FIELD

by

Belinda Jane Brett

A thesis submitted to the Council for National Academic Awards in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Department of Mathematics and Statistics Plymouth Polytechnic

In collaboration with :

Department of Physics Victoria University of Manchester



September 1986

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DECLARATIONS

While registered as a candidate for the degree of Doctor of Philosophy, the author has not been a registered candidate for another award of the C.N.A.A. or other academic or professional institution. No part of this thesis has been submitted for any award or degree at any other institution.

The following advanced studies were undertaken in connection with the programme of research:

(i) Use of library resources for research.

(ii) Guided reading in astronomy and astrophysics, and in electromagnetism, statistics, and inverse theory.

(iii) Attendance at a lecture course on electrodynamics.

(iv) Computing for research, principally programming in standard FORTRAN 77, including use of numerical, statistical, and graphical subroutine libraries.

(v) External visits, and attendance and participation at relevant conferences.

by

B. J. Brett

ABSTRACT

The magnetic field of the Galaxy is investigated by spherical harmonic analysis of Faraday rotation measures of extragalactic sources and interstellar polarization measurements. Two methods of analysis are used. Initially, these are compared on synthetic data sets, and some problems illustrated.

Faraday rotation measures of extragalactic radio sources are taken primarily from a recent catalogue (Simard-Normandin, Kronberg, and Button, 1981). These show certain trends with position in the sky, somewhat disrupted by random effects. A recently developed inverse theory method of spherical harmonic analysis gave an interesting quantitative indication of these trends, although some problems of analysis remain. A sharp positive peak in value of rotation measures is shown in the region of the North Polar Spur, reaching 300 rad m⁻². Negative values are found in the galactic equator between $\ell = 70^{\circ}$ and 160°, reaching -300 rad m⁻². Positive values up to 200 rad m⁻² are found along the galactic equator between $\ell =$ 170° and 330°. The reversal at $\ell = 70^{\circ}$, b = 0°, is sharp and may be associated with the North Polar Spur. First order spherical harmonics indicate that a longitudinal field in the plane of the Galaxy appears to run from $\ell = 99^{\circ}$.

Measurements of polarization of starlight (Axon and Ellis, 1976) have been difficult to analyse using these methods. Further methods of inverse theory may give better results.

Models of the field are discussed. Some models of Faraday rotation of extragalactic sources due to a local longitudinal field affected by systematic variations in electron density are proposed. One of these is briefly investigated. From this investigation it seems unlikely that variations in electron density due to enhancement in the local spiral arm have much effect on the rotation measures.

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CHAPTER 1

THE GALACTIC MAGNETIC FIELD

1.1. Introduction

A weak magnetic field threads the partly-ionized gas of the interstellar medium of the Galaxy, varying in intensity, direction and regularity. The methods of observation of the field, results of measurements, and theories of the characteristics of the magnetic field and its role in the Galaxy, are reviewed by Heiles (1976) and Verschuur (1979). Thomson (1981) gives detailed discussions of several methods of evaluating the field.

The role of the galactic magnetic field in large-scale and smallscale physical processes in the Galaxy is not well understood. It is important for further progress in galactic research to have more information about the magnitude and morphology of the magnetic field in nearby and distant parts of the Galaxy. The overall purpose of the work described in this thesis is to investigate the large-scale features of the magnetic field within a few kiloparsecs of the Sun. A brief report of preliminary results has already been published (Brett, 1985, in Appendix B).

Distant magnetic fields can be detected by several methods. Linearly polarized radiation passing through thermal electrons in a

magnetic field suffers Faraday rotation, twisting the angle of polarization. High speed electrons spiralling around the lines of force of the magnetic field emit linearly polarized radio waves by the synchrotron process. Interstellar dust grains with paramagnetic properties are aligned by the local magnetic field, and partly polarize the beams of starlight passing through them. Characteristic spectral lines emitted by the excitation of atomic hydrogen undergo Zeeman splitting in the presence of a magnetic Some cloud filaments seen in the interstellar medium are field. aligned with the local magnetic fields. These methods of detection and techniques of measuring the direction and intensity of the field will be discussed in more detail in Section 1.3.

Investigation of the galactic magnetic field has shown that the strength of the field is of the order of a few μ G (Verschuur, This is made up of regular and irregular components, 1979). thought to be of approximately equal magnitude (Thomson, 1981; Phillipps et al, 1981a). The regular component is one which changes only slowly on a scale of kpc, the irregular component varies over distances of 10 to 70 pc (Thomson, 1981; Spoelstra, The regular component is largely parallel to the plane of 1984). the Galaxy. It may run along the local spiral arm, or toroidally around the centre of the Galaxy (Simard-Normandin and Kronberg, 1980; Sofue and Fujimoto, 1983; Vallee, 1984), as predicted by the dynamics and magnetohydrodynamics of the Galaxy (Piddington, 1972; Roberts and Yuan, 1970; Parker, 1979; Sawa and Fujimoto, 1980; Zeldovich et al, 1983).

Observations of the magnetic field in other spiral galaxies are relevant to this investigation. They are not yet conclusive (Sofue et al, 1980; Beck, 1982; Sofue et al, 1985; Vallee, 1986). However it seems likely that the field in other galaxies are in some case bi-symmetric open spirals and in others toroidal. It is particularly difficult to determine the shape of the magnetic field, like other large-scale features of the Galaxy, from the position of the Sun in the disk. As with features such as the spiral appearance of the disk, information about our Galaxy and about other galaxies complement each other.

1.2. The Interstellar Medium

The interstellar medium of the Galaxy consists of hot gas, cool gas, dust grains, and dense clouds, confined to the disk of the Galaxy by gravity, and pervaded by cosmic rays, electromagnetic radiation and the electromagnetic field (McKee and Ostriker, 1977; Spitzer, 1978; Dyson and Williams, 1980). Supernovae explode in the medium, sweeping out a bubble around them, with a shell of swept up material. Stellar winds may also form shells (Weaver et al, 1977). Gas and dust are often blown out of the disk of the Galaxy into the halo in a fountain effect (Bregman, 1980).

Stars, dust and gas move around the Galaxy in orbits determined by gravity. The angular speed of rotation is greatest at the centre of the Galaxy, and decreases with the distance from the centre. The density wave theory of spiral structure proposes that the

pattern of orbits form spiral density waves, where stars, dust and gas are temporarily concentrated in the gravity well (Lin and Shu, 1964). Star formation occurs in spiral strips in the disks of spiral galaxies. A shock associated with the spiral density wave is supposed to promote formation of stars (Roberts and Yuan, 1970; Lin and Bertin, 1985). The local magnetic field is amplified by the shock (Parker, 1979) and is thought to have a crucial role in star formation (Davies, 1981; Elmegreen, 1982). This role may be the formation of molecular clouds and supporting them against gravity, encouraging the formation of stars at a rate at which spiral features appear and persist in grand design spiral arms and in spurs. However the action of the magnetic field may vary according to the conditions prevailing in different clouds (Zeldovich et al, 1985).

The behaviour of magnetic fields in the context of astrophysics is described by Parker (1979) and Zeldovich et al (1983). Here electric and magnetic fields cannot be considered one as cause and one as effect, as in a laboratory experiment where an electric current in a wire sets up a magnetic field. A magnetic field in an ionized gas is accompanied by an electric current, and this is opposed by the resistivity of the gas. The magnetic field decays in driving the current. The time taken to decay is proportional to the capacity of the material to carry a current, and is equal to $l^2\sigma/c^2$ where σ is the electrical conductivity in e.s.u., ℓ is the characteristic dimension of the body of gas in cm, and c is the speed of light in cm sec⁻¹ (Parker, 1979).

In the gas in the disk of the Galaxy, $\ell = 200 \text{ pc}$ roughly, and the electrical conductivity of the intercloud medium is at least 10^{11} e.s.u. (Parker, 1979). The magnetic relaxation time is therefore about 10^{24} years, considerably longer than the estimated age of the Galaxy, which is 10^{10} years. The kinetic energy of the gas far exceeds the magnetic energy of the μ G field (Spitzer, 1968). Consequently the dynamics of the medium are dominated by the motion of the gas, which drags the magnetic field along with it, and so this is described as the 'frozen-in' field. There is, however, a tendency for the magnetic field lines in the plane of the Galaxy to expand out of the plane under their own pressure and the pressure of cosmic rays, leading to a revised time of decay of 10⁸ years (Parker, 1979). It is still tempting to suppose that the present interstellar field is a relic of the primordial magnetic field existing in the material from which the Galaxy formed.

The theory describing forces and motion in the coupled gaseous and magnetic fluid is magnetohydodynamics, developed particularly in investigation of the surface of the Sun, where the magnetic relaxation time is 300 years. It is this theory that shows that the balance between the weight of the interstellar gas and the pressure of the magnetic field and the cosmic rays is not stable (Parker, 1966, 1979). The magnetic field lines tend to buckle, the gas collecting in dense pockets, anchoring the field lines, which in between the pockets tend to bulge outwards (Parker, 1979). The dense clouds contain a condensed magnetic field and this has been measured by means of Zeeman splitting, for example by Davies (1981). The balance will be upset by the passage of a shock wave

and dense clouds will be formed supported by the condensed field (Mouschovias, 1976). Star formation proceeds by further fragmentation and is promoted by stellar winds and supernova explosions from the earlier stars. There is still some controversy over whether the spiral shock wave is needed to set this process in motion (Gerola and Seiden, 1978; Seiden, 1985; Shu, 1985).

A primordial field is destroyed by diffusion and dissipation, but it can be amplified by turbulent diffusion. Parker (1979) considers that amplification by turbulent diffusion is the only acceptable explanation of the existence of the observed astrophysical magnetic fields. The magnetohydrodynamical theory of turbulent diffusion of magnetic fluids is extremely complex. See, for example, Moffatt (1978) and Parker (1979). Amplification of the field is efficient and Parker (1979) suggests that the observed value of $3\,\mu\text{G}$ for the intensity of the galactic magnetic field is due to limitation of the field by magnetic buoyancy and suppression of interstellar turbulence by magnetic stresses. The generation of the magnetic field by differential rotation and small scale turbulence forms a galactic dynamo. This is treated by Stix (1975), Moffatt (1978), Parker (1979), Zeldovich et al (1983), and Such astrophysical dynamos operate in Ruzmaikin et al (1985). different periodic modes, involving increasing numbers of reversals in the generated field. The real dynamos tend to operate in low modes (Parker, 1979).

Parker (1979) described the mode which may operate in the Galaxy, illustrated in Figure 1.1. The field is dominated by a toroidal



Figure 1.1 Magnetic field lines in the lowest even mode of the galactic dynamo (sketch following Parker, 1979). x is the direction of the centre of the Galaxy, y is the direction of rotation. The continuous lines show the field due to the gas flow. The broken lines show the resulting large-scale circulation. component, but is regenerated by turbulent eddies. These always rotate in the same sense because of galactic rotation, and cause the circulation shown, above and below the plane. Away from the plane, the circulating field diffuses into the halo of the Galaxy. This leaves behind the part circulating towards the centre of the Galaxy along the centre of the plane, which is sheared by differential rotation and augments the toroidal component. The toroidal field may be reversed on a scale of kiloparsecs. This mode of galactic dynamo was chosen by Parker (1979) on the basis of observations of Faraday rotation in pulsars, which show the field taking the same direction above and below the plane. In the next mode of dynamo the magnetic field reverses its direction above and below the plane of the Galaxy, and its physical interpretation relies on the isolation of the disk of the Galaxy from the halo.

A galactic dynamo relying on more complex assumptions is proposed by Sawa and Fujimoto (1980). It is inspired by the work of Tosa and Fujimoto (1978), Sofue et al (1980) Simard-Normandin and Kronberg (1980) and Thomson and Nelson (1980). These all suggest that the magnetic fields in M51, M81, and M33 and in our Galaxy, follow the spiral arms into the centre of the disk and out again and are open to intergalactic space. A model of a dynamo has been developed by Sawa and Fujimoto (1980) which supports the field lines in the disk in a spiral configuration without their being twisted into a ring by the shearing action of differential rotation. In this model, the less dense halo is a reservoir of the magnetic field configuration for the disk. The magnetic field and gas of the disk move out across the disk by turbulence, diffusing

into the halo before becoming sheared into a toroidal shape. The halo has a bisymmetric spiral field, preserved by the pattern of rotation of the gas in the halo. The gas diffusing out of the disk relaxes in the halo, eventually drifting in toward the centre of the Galaxy to replenish the field in the disk. Ruzmaikin et al (1985) suggest that the magnetic field may have a bisymmetric pattern in outer regions, but in the inner regions where differential rotation dominates the motion of the gas, the configuration will be toroidal.

Toroidal fields have been observed in external galaxies (Beck, 1982). Beck (1983) and Vallee (1986) discuss the magnetic fields in several galaxies, and both emphasise the need for further investigations. Vallee (1986) speculates that spiral galaxies with ring (toroidal) magnetism have a larger mass of HI gas, and have a limited interaction with other galaxies. Spiral galaxies with spiral magnetic fields, conversely, are likely to have less HI gas and a major interaction with another galaxy.

1.3. Observing the Magnetic Field

The galactic magnetic field can be detected, and its direction and intensity measured, by the methods mentioned in Section 1.1. However, in all of these methods, except measurements of the Zeeman effect, other factors have a great deal of influence on the observations.

Faraday rotation occurs when electromagnetic radiation passes through thermal electrons in a magnetic field. The angle of polarization is rotated by the medium. The effect varies with the square of the wavelength of the radiation. For suitable sources, measurements at several radio frequencies yield a rotation measure which is proportional to the product of the electron density and the line of sight component of the magnetic field, integrated over the whole length of the line of sight to the source. This is described in more detail in Chapter 4. Faraday rotation is biased to give more weight to the magnetic field in regions where the electron density is high. It is also limited to positions in the sky where there are radio sources to probe the intervening medium.

Rotation measures of pulsars can be found fairly easily, and estimates of the electron density along the line of sight are available independently of the rotation measure. Pulsars are found close to the plane of the interstellar medium, which is important to determination of the average magnetic field from the rotation measure. Distances to the pulsars can be estimated. Manchester (1974) analysed the rotation measures of pulsars available at that time. An important model for analysis of data relevant to the magnetic field is a longitudinal local field. This is used to give the direction of the field. Manchester (1974) found that the direction of the regular component of the field was $\ell = 94^{\circ}\pm 11^{\circ}$ (In this equation and in the rest of this work, ℓ refers to galactic longitude and b to galactic latitude.).

More recently, Thomson and Nelson (1980) have carried out an improved analysis on a larger data set, and obtain a field direction of ℓ =107°±7°. They find the pattern of the residuals

(the rotation measures unexplained by the model) unsatisfactory and also fit a model of a longitudinal field including a reversal in field direction toward the centre of Galaxy. The residuals of the latter model are more acceptable. Using it they find the direction of the field to be $\ell=74^{\circ}\pm10^{\circ}$, the intensity of the regular component of the field to be $3.5\pm0.3 \mu$ G, and the perpendicular distance to the reversal to be 170 ± 90 pc.

It is more difficult to obtain rotation measures of the Faraday rotation of extra-galactic sources, and more difficult to analyse the results. There are many sources of this kind, well distributed around the sky, so their analysis can be very rewarding. There is difficulty in obtaining the rotation measures because the sources are less consistently polarized. Material in the source and the intervening intergalactic medium add extra Faraday rotation. The early determinations of rotation measures were hampered by the lack of sufficient measurements of polarization. They can be determined more reliably now that polarization measurements are available in larger numbers. Some recent developments are reviewed by Seymour (1984).

Seymour (1967) carried out a spherical harmonic analysis of rotation measures of extragalactic sources found by Gardner and Davies (1966). He found that the magnetic field was directed towards ℓ =91°±1° in the plane of the Galaxy. He used 65 rotation measures. Wright (1973) calculated the rotation measures of 354 sources, and then found the uniform linear component of the galactic magnetic field to be in the direction ℓ =94°±3°.

Determination and analysis of rotation measures by Vallee and Kronberg (1975) has been superseded by the more recent work of Simard-Normandin et al (1981), whose catalogue of rotation measures have been calculated from many published polarization measurements. Simard-Normandin and Kronberg (1980) analyse the rotation measures in this catalogue, however they do not consider the longitudinal field model, but look at models of spiral and toroidal galactic-scale fields.

Tabara and Inoue (1980) have published a large catalogue of measurements of linear polarization of radio sources, including rotation measures they have calculated, with indications of their quality. Many of these are used by Thomson and Nelson (1982) who fit them to a longitudinal model containing the reversal found with the pulsar rotation measures. Thomson and Nelson (1982) find the direction of the field to be $\ell = 71^{\circ} \pm 13^{\circ}$. Inoue and Tabara (1981) look at the galactic magnetic field using rotation measures from their catalogue. They find the longitudinal local field to run in the direction $\ell = 100^{\circ} \pm 10^{\circ}$, and do not find evidence from these extragalactic sources of a reversal in the field within 3 kpc. Sofue and Fujimoto (1983) also analysed rotation measures from this catalogue, by smoothing the trends in the rotation measures, and comparing them with the rotation measures predicted by a bisymmetric spiral model of the magnetic field in the Galaxy. They observe a maximum in rotation measures at ℓ =56° which they ascribe to the Sagittarius arm, and a minimum of negative rotation measures at ℓ =90°, ascribing this to the longitudinal local field.

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Faraday rotation and the modelling of rotation measures predicted by a configuration of the field have proved to be the most fruitful method of investigating the galactic magnetic field. In the case of extragalactic radio sources, there are problems involved in estimating the rotation measures, as will be described, and other contributions to the rotation measure, once found. Further observations and refinement of calculation techniques can improve the situation.

The galactic background emission, also called the radio continuum emission, is due to the synchrotron process. Relativistic electrons interact with interstellar magnetic field lines, or with the more compressed magnetic lines of force in a supernova remnant. Radiation emitted by the synchrotron process is highly linearly polarized, although it may be depolarized by Faraday effects, or the confusion of different emitting regions along the line of sight. The intensity of the radiation emitted, and the angle of polarization, contain information about the magnetic field in that region. The amount of Faraday rotation occurring along the line of sight depends on the thermal electrons and magnetic fields along the line of sight.

Surveys of the radio continuum emission in large areas of the sky show a high intensity in the plane of the Galaxy. Occasionally a ridge of high intensity reaches out of the plane, and such ridges are called the galactic spurs. They can be traced along ridges of lower intensity to form small circles in the sky, as shown, for example, by Berkhuijsen (1971). Also known as galactic loops, they

are identified with local supernova remnants (Berkhuijsen et al, 1971; Spoelstra, 1972). Cosmic ray electrons, or relativistic electrons from the supernova remnant, are interacting with the magnetic field concentrated in the shell swept up by the supernova explosion.

Much of the radio continuum emission from the disk of the Galaxy is thought to come from distant supernova remnants. Some comes from nearby regions (including the loops), and here patterns can be seen in the polarization of the radio waves. The linear polarization of synchrotron emission is perpendicular to the local magnetic field, although it may be rotated along the line of sight by Faraday rotation. In small regions where the magnetic field is highly organised, a lot of information can be recovered from the radio continuum emission. Spoelstra (1972) was able to evaluate models of the structure of Loop I, the North Polar Spur, and Loop II, the Cetus Arc, as supernova remnants.

Wilkinson and Smith (1974) mapped the Faraday rotation of galactic synchrotron emission over a large region of the sky, between $\ell=100^{\circ}$ and $\ell=180^{\circ}$, and between b=+40° and b=-40°. In this area the emitting region is a thin sheet (Heiles, 1976) at a distance of between 140 and 400 pc. Depolarization is least for such a thin emitting region. Wilkinson and Smith (1974) found that the magnetic field and electron density between the emitting region and the Sun were quite variable, fluctuating on a scale length of between 10 and 50 pc. Spoelstra (1984) finds that the continuum is polarized by features at an average distance of 450 pc, and that

the magnetic field and electron density vary on a typical scale of 10 to 75 pc.

Analysis of the all-sky radio continuum map of Haslam et al (1981a, 1981b), by Phillips et al (1981), leads to some interesting conclusions. They are able to unfold the total emission by using logarithmic spiral sections of the plane. They select a spiral pitch angle of 12°. From their results they suggest that the Sun is between major arms, and that the Galactic magnetic field is made up of a 3μ G regular component, and a 3μ G irregular component. This work apart, the radio continuum is most useful for detailed inspection of small regions of the key. The depth of the region emitting polarized radiation is variable, and so is the amount of polarization. The polarization about the detailed behaviour of the Galactic magnetic field.

Starlight shows a small amount of linear polarization, the fractional intensity of the polarized component being typically one or two per cent. The amount of polarization correlates with the amount of reddening and is caused by the same thing, the interstellar dust. Davis and Greenstein (1951) proposed a mechanism whereby elongated dust grains become aligned spinning in the interstellar magnetic field, and preferentially extinguish light polarized perpendicular to the field. This interstellar polarization is often plotted on 'E-vector' maps, which show the degree of polarization and the orientation of maximum polarization. The most recent catalogue of measurements of stellar polarization

was compiled by Axon and Ellis (1976). They have plotted E-vector maps at different distance ranges. These show the typical features of plots of interstellar polarization, as close as 100 pc to the Sun.

In many places along the galactic plane the polarization angle of starlight is parallel to the plane, showing that the magnetic field is parallel to the plane. This is most noticeable at $\ell=140^{\circ}$, and 180° away from this position, at $\ell=320^{\circ}$. It seems as if the field direction is $\ell=50^{\circ}$, in contrast to the information from extragalactic sources. Numerical analysis leads to the same result, a field direction of about $\ell=50^{\circ}$ (Ellis and Axon, 1978). It now seems likely that local anomalies and irregular distribution of dust lead to this misleading result. Inoue and Tabara (1981) found that away from the galactic plane, and away from the North Polar Spur, interstellar polarization near the South galactic pole shows a direction of $\ell=100^{\circ}$, a figure more in line with Faraday rotation results.

Magnetic fields in HI clouds can be found from the Zeeman splitting of the 21 cm spectral line. Unfortunately the splitting is small, and has proved difficult and time consuming to detect (Verschuur, 1979). Recent detections are described by Davies (1981) who shows that more dense clouds have stronger magnetic fields, and that the field direction is maintained during collapse of the gas clouds under their own gravity, or the magnetohydrodynamical instability described by Parker (1966, 1979).

Other sources of information about the magnetic field of the Galaxy, and its role in galactic dynamics on large and small scales, are the alignment of filamentary clouds along the local field, and flux of cosmic rays measured from the Earth. The information about the field in these sources is very indirect, and relies on information already found by other methods.

1.4. Analysis and Modelling

Mathematical models are an appropriate method of investigating the shape of the galactic magnetic field. On the small scale, the behaviour of gas and field in supernova explosions, supernova remnants, interstellar clouds, and in density wave shocks can be modelled, and compared with the observations of various kinds. A model of the magnetic field in the shell of a supernova remnant was devised by Van der Laan (1962), and used by Spoelstra (1972) on synchrotron emission from the galactic loops. On the large scale galactic and local neighbourhood models may be developed and compared with the measurements of Faraday rotation of extragalactic sources and pulsars, and polarization of starlight, as for example Inoue and Tabara (1981) have done.

The best parameters for the model, such as the direction of the longitudinal field, are found from the measurements by numerical methods such as least squares fitting. The models can be evaluated by inspection of the residuals (the values left when the model estimates of measurements are subtracted from the actual

measurements) for remaining large-scale trends, or by the chi-square test. Some estimate of the original error distribution in the measurements is needed for this test, it can then give the likelihood of that particular weighted sum of squares of residuals being found if the model were true. Clear and appropriate use of these methods can be found in Thomson (1981), or Thomson and Nelson (1980, 1982). Different models can be compared with each other using residuals by the F-test.

In most of the measurements, it is useful to have methods of looking at the trends in the data, in order to suggest useful directions for modelling, and for quantitative and qualitative methods of evaluating models. One way of doing this is looking at some representation of the measurements, such as the diagrams of the directions of the polarization vectors of stars by Mathewson and Ford (1970), and, following them, of Axon and Ellis (1976). Rotation measures are usually shown by plotting symbols representing their sign and size, as in work by Wright (1973), Simard-Normandin et al (1981), Sofue and Fujimoto (1981), Thomson (1981), and many others. These are extremely useful, however they show measurements that are inaccurate, fluctuating with random variations. Consequently, smoothed maps have been developed to show significant trends in the data.

Seymour (1965, 1966, 1967) used spherical harmonic analysis to quantify the trends in Faraday rotation measures for extragalactic radio sources, interstellar polarization and polarization of synchrotron radiation from the galaxy. More recently, Ellis and

Axon (1978) and Tabara and Inoue (1981) averaged the interstellar polarization measurements in distance and direction bins. Kronberg and Simard-Normandin (1980) averaged the rotation measures of extragalactic sources from their catalogue (Simard-Normandin et al, 1981), by finding the average of all sources in a particular solid angle. Sofue and Fujimoto (1981) do the same with the catalogue of Tabara and Inoue (1980), using a Gaussian beam to weight the sources towards the centre of the solid angle.

Spherical harmonic analysis is a powerful and appropriate method of looking at trends in data scattered over the surface of a sphere. It is less useful when the data is very poorly distributed (although recent techniques allow for this). It is not appropriate when the underlying trends are not expected to vary smoothly. The pulsar rotation measures, for example, are not well distributed about the sky (Lyne et al, 1985). Different pulsars, from their position in the galaxy, sample different depths of the interstellar medium, so the underlying trends cannot be expected to be smooth. The results of a spherical harmonic analysis of pulsar rotation measures could not be compared with a galactic magnetic field model. However, the rotation measures of extragalactic sources are good material for spherical harmonic analysis. Interstellar polarization measurements of stars divided into distance groups are also appropriate. Synchrotron radiation now seems more suitable to investigation of limited areas. This is because depolarization of measurements by Faraday rotation or by the coincidence of several emitting regions (or one large one) causes confusion, and emission from thin sheets is the best subject.

1.5. Aims

The aims of this work are to carry out spherical harmonic analysis of Faraday rotation measures of extragalactic sources, and of interstellar polarization measurements in order to quantify significant large-scale trends in these variables, and to relate them to the galactic magnetic field, using them to suggest and evaluate models and ideas about the field.

CHAPTER 2

MATHEMATICAL METHODS I

2.1. Spherical Harmonic Analysis

The best introduction to spherical harmonic functions and spherical harmonic analysis is the following passage by Kaula (1967)

Spherical harmonics arise in a physical context as solutions of Laplace's equation in spherical co-ordinates. Their property of orthogonality, ... suggests, however, that the surface spherical harmonics...constitute the natural spectral representation for any function that varies over the surface of the sphere, regardless of whether the function has anything to do with Laplace's equation. Used in this manner, spherical harmonics are a device for studying variations over a spherical surface analogous to Fourier series for variations in time.

The spherical functions $S_n^m(\Theta)$ used throughout are the Schmidt quasi-normalised functions, described and recommended by Chapman and Bartels (1951). n is called the degree and is zero or a positive integer. m is called the order and is an integer less than or equal to n. The $S_n^m(\Theta)$ are numerical multiples of the associated Legendre functions $P_{n,m}(\Theta)$. Examples of these are $P_{1,1}(\Theta) = \sin \Theta$, $P_{3,1}(\Theta) = 3/4 \sin \Theta$ (5cos2 θ +3).

The surface spherical harmonics are functions of Θ , colatitude, and ϕ , longitude, and take the form $S_n^m(\Theta)\cos m\phi$ and $S_n^m(\Theta)\sin m\phi$. They are orthogonal over the sphere because

$$\int_{S_{n}}^{m} (\Theta) \begin{cases} \cos \\ \sin \end{cases} m \phi S_{n'}^{m'} (\Theta) \begin{cases} \cos \\ \sin \end{cases} m' \phi \sin \Theta d\Theta d\phi = 0$$

unless n=n' and m=m', when the above expression is equal to $\frac{4 \text{ TT}}{n+1}$.

Spherical harmonic analysis is used to model the geomagnetic field from measurements taken at the surface of the Earth. In geomagnetism the use of spherical harmonic functions is physically significant (Barraclough, 1978; Whaler, 1981) because no currents flow between the surface of the Earth and the atmosphere when averaged over 20 years or more. The geomagnetic field therefore obeys Laplace's equation at the surface of the Earth, and so spherical harmonic functions are not used simply as fitting functions but are an appropriate way of expressing the geomagnetic field in spherical polar ∞ -ordinates.

Spherical harmonic analysis is also used to model the variations in surface height of the Earth, the planets and the Moon. In this problem the surface spherical harmonic functions are used to model a scalar quantity, the deviation from a perfect sphere.

Seymour (1965, 1966, 1967, 1969) used spherical harmonics to investigate the galactic magnetic field. He looked at trends over sky of observations of Faraday rotation measures the of extragalactic radio sources, and the measurements of the polarization of starlight, and polarization of the galactic background emission. In this work, spherical harmonic analysis was used as a method of representing the observed data, not the field. The galactic magnetic field does not satisfy Laplace's equation, because currents flow in the interstellar medium. (Representations

of the galactic magnetic field might be better expressed in cylindrical polar co-ordinates). If harmonic functions were used to represent the field it would be important to include other terms to describe the field component induced by currents. The terms of such a model then would be physically significant in the same way that spherical harmonic functions are a physically significant model of the geomagnetic field.

Seymour (1967) analysed sets of rotation measures of extragalactic sources, all containing less than 90 sources. He also analysed 550 measurements of stellar polarization, and selected points from surveys of synchrotron radiation at various frequencies. The examples from geophysics mentioned above were particularly relevant to his work. He carried out the spherical harmonic analysis with a computer program which had been first used to represent the surface of the Moon, by M.E. Davidson. The method used in the program had been developed by Fougere (1963) on geomagnetic data.

Faraday rotation measures give a scalar quantity at the position of each radio source in the sky. Stellar polarization measurements yield Stokes parameters Q and U, which can each be analysed separately with surface spherical harmonics. The same holds for the polarization of the galactic background emission. His method of analysis enabled Seymour (1967) to look at trends over the sky in each of these quantities separately.

Faraday rotation and interstellar polarization depend on the galactic magnetic field and on several other factors. These

relationships will be discussed in Chapters 4 and 5. It is sufficient at this point to emphasise that any underlying dependence on large-scale properties of the interstellar medium (such as the galactic magnetic field) are obscured by little-known factors which are assumed to be random.

In addition to this problem, the data sets are not distributed very evenly around the sky. This makes the experience of geophysicists particularly relevant to analysis of these data sets. Workers in the field of geomagnetism use spherical harmonic analysis, and the measurements available are typically inaccurate and poorly distributed about the Earth's surface. Consequently the area is rich in strategies for carrying out analysis of awkward data.

Spherical harmonic analysis is a method of smoothing data and interpolating values between measured positions. Rotation measures of extragalactic sources have been studied by Simard-Normandin and Kronberg (1980) and by Sofue and Fujimoto (1983). In both cases, the rotation measures were smoothed and interpolated by averaging over small areas of the sky. Spherical harmonic analysis is in principle an economical and appropriate method of averaging, able to represent large scale and small (to choice) scale features. In practice there are problems in using spherical harmonics which will be described. Different methods of smoothing can be evaluated by similar criteria, and essentially are similar problems. The wider theory which studies the problem most generally is 'inverse theory' and is briefly introduced in Chapter 3.

Gubbins (1983) and Whaler (1981) have worked on the application of inverse theory methods to spherical harmonic analysis. Their algorithm developed in geomagnetic field analysis goes some way to solving some particular problems. This is introduced in Chapter 3 and will be used alongside Fougere's method as an alternative.

The mathematical expression of the problem is this: Spherical harmonic analysis seeks the function $f(\theta, \phi)$ which is the best representation of trends of the observed scalar variable over the surface of the sphere, and where

$$f(\theta,\phi) = \sum_{n=0}^{L} \sum_{m=0}^{n} (a_n^m \cos \phi + b_n^m \sin \phi) S_n^m(\theta)$$
(2a)

In equation (2a) above, a_n^m and b_n^m are constants to be determined, and L is the degree and order at which the series of harmonics is truncated.

Looking for the 'best representation' occupies the rest of this chapter, and the whole of Chapter 3.

2.2. Least Squares Solution

Spherical harmonic analysis is a two-dimensional extension of curve-fitting. The method of least squares is a well known procedure for finding a curve which fits well to a set of points when there are fewer parameters to the fitting curve than there are

observed points. It relies on evaluating the misfit between the curve and the observed data points.

It is assumed the reader is familiar with the method of least squares. However, in this section it will be described, or sketched, concisely in order to present notation and specific information about its use with spherical harmonics.

Particular values of the co-efficients or parameters a_n^m and b_n^m in equation (2a) will determine the function $f(\theta, \phi)$. Then $f(\theta', \phi')$ will predict a value of the observed variable at position (θ', ϕ') . If d_i is the observed value of the variable at position (Θ_i, ϕ_i) , the prediction error or residual error is defined to be $e_i = d_i - f(\Theta_i, \phi_i)$.

When there are fewer parameters than observed values, generally there is no exact solution to the problem. The least squares solution is the set of estimates of the parameters which minimizes the sum of squares of the prediction errors. Matrices are used to find the least squares solution in non-trivial cases.

The set of observed data consists of D measurements d_i at (θ_i, ϕ_i) . If the spherical harmonic series in equation (2a) above is truncated at degree and order L, there will be $P=(L+1)^2$ coefficients of the sort a_n^m and b_n^m . These can be ordered as $[x_i]$ i=1,P and the spherical harmonic functions as $[s_i]$ i=1,P.

The 'equations of condition', equations (2b) below, relate

coefficients and harmonics to observed data. Usually they are not all satisfied when D>P.

$$\begin{aligned} & x_1 s_1 (\theta_1, \phi_1) + \ldots + x_i s_i (\theta_1, \phi_1) + \ldots + x_p s_p (\theta_1, \phi_1) = d_1 \\ & \ddots \\ & \ddots \\ & \ddots \\ & x_1 s_1 (\theta_D, \phi_D) + \ldots + x_i s_i (\theta_D, \phi_D) + \ldots + x_p s_p (\theta_D, \phi_D) = d_D \end{aligned}$$
(2b)

Or in matrix form :

 $G \underline{x} = \underline{d}$

where G is DxP matrix, \underline{x} is a vector of length P and \underline{d} is a vector of length D.

Linear algebra shows that if D>P there is, generally, no exact solution. In this case a good $f(\Theta, \phi)$ is the one which minimizes the value of $\sum_{i=1}^{D} e_i^2$, hence called the least squares solution. $\sum_{i=1}^{D} e_i^2$ is called the Euclidean or L_2 norm of the prediction error $e_1 \dots e_D$ or \underline{e} . It is one of several measures of goodness of fit using the prediction error.

Use of the Euclidean norm corresponds to the assumption that the errors affecting the data are random and have a Gaussian distribution about zero (Draper and Smith, 1981). If the errors are Gaussian, the least squares solution is the estimate of the parameter values which maximizes the probability that the given data set was observed if those parameters were in fact correct.
It is often realistic to assume that the errors have a Gaussian distribution. The Central Limit Theorem shows that when random variables are drawn from any particular distribution, the distribution of their sum tends towards a Gaussian distribution as the number drawn increases. So when errors in the data are due to many comparable contributions, they tend to have a Gaussian distribution.

The question of whether a Gaussian distribution of errors is a reasonable assumption in the cases of Faraday rotation and interstellar polarization will be discussed in the relevant chapters. For now it is assumed that the errors in the data are due to effects from many sources of a similar size, and have a Gaussian distribution.

If this assumption were wrong, other norms of the prediction error could be used. Using the L_1 norm, or minimizing $\sum_{i=1}^{D} |e_i|$ is appropriate when the noise has a wider (long tailed) distribution. This norm is good at attaching less weight to the measurements scattered widely from the trend, the outliers. The higher power norms (L_3 and above) are less robust to spurious values as they put more weight on the largest prediction errors.

The least squares solution to the problem posed in (2b) is given in many textbooks, for example by Menke (1984). Expressed in matrix form it is

$$\hat{\underline{x}} = \left[\mathbf{G}^{\mathrm{T}} \mathbf{G} \right]^{-1} \mathbf{G}^{\mathrm{T}} \underline{\mathbf{d}}$$
(2c)

where $\frac{x}{2}$ is the estimate of the parameter vector <u>x</u> which minimises $\sum_{i=1}^{D} e_i^2$. $G^T G$ is called the normal equations matrix.

Least squares analysis was developed by Gauss. He used it to determine coefficients of spherical harmonics to describe the geomagnetic field in 1839. This method was widely used until 1960, the data measurements were interpolated by drawing contour maps by hand so values could be estimated for particular points regularly spaced in colatitude and longitude. This enabled a preliminary harmonic analysis to be carried out, along the parallels of colatitude, which reduced the number of the normal equations in the final least squares analysis. With the aid of a computer, Gauss' method is used directly to calculate \hat{x} , without intermediate interpolation to regularly spaced points and intermediate harmonic analysis.

Use of the computer has generated many problems. For example there are problems of round-off error, especially in the instability of matrix inversion, and of determining when to truncate the series. The need to use the computer economically and accurately has spawned many investigations of different practical methods of least squares analysis, and, specifically of spherical harmonic analysis. Many algorithms for spherical harmonic analysis of the geomagnetic field are presented by Barraclough (1978). Further problems with the analysis of the geomagnetic field are posed by varying combinations of vector measurements of the field.

2.3. Fougere's Method

It is not easy to determine an appropriate truncation level for the series of spherical surface harmonics. Using a very large number of harmonics is unwieldy and may obscure trends in the data with detailed modelling of noise. (Raising the number of parameters to equal or exceed the number of data measurements renders the method of least squares inappropriate).

However it is important that successive terms in the series decrease in size substantially before the series is truncated. If the coefficients do not converge in this way the model may be very poor in some places where higher harmonics are needed. Convergence in spherical harmonic coefficients is investigated by looking at the average 'power' in coefficients of harmonics of the same degree.

This problem was being tackled along with others in the 1960s and 1970 when the increasing use of computers made them important. Fougere's (1963) method was an attempt to improve the least squares solution in spherical harmonic analysis by incorporating a statistical determination of truncation level. The statistical method is a form of regression analysis called forward selection. It is still in use in planetary physics.

Fougere's strategy was taken from statistical techniques new in the 1960s. In this method a least squares solution is found to a predetermined level of truncation. The importance of the

individual independent components of the solution are evaluated, and the least important ones are discarded. This usually means that a smaller number of spherical harmonics are retained.

Fougere's method is based on the conventional least squares solution found by inverting matrix $G^{T}G$ as in equation (2c). In order to be able to carry out the statistical evaluation of the components of the solution, he uses the method of decomposing the matrix G by Gram-Schmidt orthogonalization into the product of a matrix with orthogonal columns and a triangular matrix. This is a standard method of finding the least squares solution and is described by Lawson and Hanson (1974).

Gram-Schmidt orthogonalization derives a set of mutually orthogonal vectors $\underline{q}_1, \ldots, \underline{q}_n$ from any linearly independent set of vectors $\underline{h}_1, \ldots, \underline{h}_n$. These two sets of vectors span the same n-dimensional subspace. The set $\underline{q}_1, \ldots, \underline{q}_n$ are determined by the following procedure:

Let
$$\underline{q}_1 = \underline{h}_1$$

and
$$\underline{\mathbf{q}}_2 = \underline{\mathbf{h}}_2 - \underline{\mathbf{h}}_2^{\mathrm{T}} \underline{\mathbf{q}}_1 - \underline{\mathbf{q}}_1^{\mathrm{T}} \underline{\mathbf{q}}_1$$

and then generally

$$\underline{q}_{j} = \underline{h}_{j} - \underline{r}_{ij} \underline{q}_{i}$$
 where $\underline{r}_{ij} = \frac{\underline{h}_{j} \cdot \underline{q}_{i}}{\underline{q}_{i} \cdot \underline{q}_{i}}$

Then if the \underline{h}_i are the columns of matrix G, and Q is the matrix with (mutually orthogonal) columns \underline{q}_i , and R is an upper triangular matrix with elements \mathbf{r}_{ij} ,

G = QR

Round-off error in the Gram-Schmidt procedure can be minimized by a modification described by Lawson and Hanson (1974). Fougere (1963) originally attempted to reduce round-off errors by repeating the orthogonalization procedure twice on each vector. He found a new orthogonal vector \underline{q}_n from \underline{h}_n and then resubmitted \underline{q}_n for orthogonalization as if it were \underline{h}_n . He also used double precision variables in part of his computer program.

Round-off error has not been a problem in this analysis. Double precision arithmetic has been used throughout. On rotation measure data for example the Gram-Schmidt procedure was found to give the same solutions as the modified Gram-Schmidt procedure to 5 significant figures.

Formation of matrix Q is a major step towards finding the least squares solution of the equations of condition (2b). The matrix form of (2b) can be rewritten as

$$QR_{\underline{X}} = \underline{d}$$

Now the least squares solution to the equation Qy = d can be easily

found. It is

$$\hat{\underline{\mathbf{y}}} = \left[\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right]^{-1}\mathbf{Q}^{\mathrm{T}}\mathbf{d}$$

which collapses to

$$\hat{\mathbf{y}} = \mathbf{Q}^{\mathrm{T}}\mathbf{d}$$

because Q is made up of mutually orthogonal columns.

Then if

$$y = Rx$$

R is strictly upper triangular, so R^{-1} exists and is easily found. So the least squares estimate of the best set of parameters \hat{x} is

$$\hat{\mathbf{x}} = \mathbf{R}^{-1}\hat{\mathbf{y}}$$

The solution is an expansion of the data vector \underline{d} in terms of the base $\{\underline{h}_1, \dots, \underline{h}_n\}$ or the base $\{\underline{q}_1, \dots, \underline{q}_n\}$:

$$d = \sum_{i=1}^{P} x_i h_i + r$$
 (2d)

or

$$d = \sum_{i=1}^{p} y_{i}q_{i} + r$$
 (2e)

where <u>r</u> is a remainder term. It is easy to find the y_i by taking the scalar product of <u>d</u> with each \underline{q}_i :

$$y_{j} = \frac{\underline{q}_{j}}{\underline{q}_{j}} \cdot \underline{\underline{q}}_{j}$$

because the $\underline{q}_{\underline{i}}$ are mutually orthogonal. This is another way of writing the solution

$$\hat{\underline{\mathbf{y}}} = \mathbf{Q}^{\mathbf{T}}\underline{\mathbf{d}}$$

The y_i are coefficients of the orthogonal vectors which are specifically calculated for each set of observations. The x_i however are coefficients of the spherical harmonic functions. After the y_i have been found, and before the x_i are calculated from them, Fougere's method evaluates each y_i statistically. This can be done using the F-test because the y_i are coefficients of an orthogonal set of vectors. Each vector and its coefficient are independent of all the others. The F-test provides a method of comparing two independent models of a set of data. A complicated model, that is one with more parameters, will fit the data better than a simpler one.

Suppose the number of observations is D and the number of parameters of model J is P_J . For each of the two models the following quantity is calculated :

$$W_{J} = \frac{1}{V_{T}} \sum_{i=1}^{D} e_{i}^{2}$$

where e_i is the ith prediction error as usual, and \mathcal{V}_J is the number of degrees of freedom of the Jth model (usually D-P_J). Then the ratio W_2/W_1 can be compared with tables of the F-distriction, which described the ratio of two variances. A significance level \ll is chosen, say 0.05. If the ratio W_2/W_1 exceeds the number given in the table, and if the errors in the data are distributed normally, there is a less than 5% chance that model 2 was in fact worse than model 1. This suggests that model 2 is better than model 1, taking into account the number of parameters in each.

When the set y_i have been calculated, they are tested to see which

are significant components of the solution at a chosen significance level \propto . For each y_i the quantity V_i is found, where

$$v_i = y^2 \underline{q}_i^T \cdot \underline{q}_i$$

This is a sample variance. These are sorted in order of size, so that $V_{i(j)}$ is the 'jth largest' :

$$V_{i(1)} > V_{i(2)} \dots > V_{i(P)}$$

Then models are composed which include successively more of the components $y_i \underline{q}_i$. Each model M_K includes the $y_{i(j)}$ for j=1,K, that is to say, it includes the components corresponding to the K largest of the V_i s,

$$M_{K} = \sum_{j=1}^{K} Y_{i(j)} \quad q_{\underline{i}(j)}$$

The residual variance of this model M_{κ} is

$$W_{K} = \left[d^{2} - \sum_{j=1}^{K} V_{i(j)}\right] / V_{K}$$

The value of including this $y_{i(K)}$ is tested by comparing $V_{i(K)}/W_K$ with the F-distribution. If this $y_{i(K)}$ is significant, the next model M_{K+1} is composed and tested. The value K' is determined at which the remaining $y_{i(K)}, \dots, y_{i(P)}$ are set aside (set equal to zero, for programming purposes). In the case that K'=P+1, all the y_i will be included.

The set $y_{i(1)}, \dots, y_{i(K'-1)}$ are used to calculate all i=1 to P of

the x_i . These will be the required coefficients of the spherical harmonic functions. The set y_1, \ldots, y_p will have gaps (or zeros) in various positions. However, due to orthogonalization procedure, the x_1, \ldots, x_p will not have any gaps up to a certain maximum i corresponding to the highest value of i(j) for j=1, K'-1 for which y_i was included, and after that they will be zero.

Starting with the most significant component as basic model, better models are built up by testing whether the next most significant component is a useful addition to the model already accepted. Instead of using the complete set of components $y_i q_i$ to model <u>d</u>, this method selects a subset which produces a better model, in the sense of having lower variance.

The advantage of the procedure is its determination of a lower truncation level than the preselected truncation level of the preliminary least squares analysis. This preselected level can be raised, as more harmonics can be included without starting from the beginning again.

In practice the series of coefficients of the harmonics may appear not to decrease in value quickly or at all. Then increases in the preselected truncation level lead to increases in the lower truncation level determined by Fougere's method. In this case the method doesn't go far enough in solving the problem of slow convergence.

It has been pointed out by Kaula (1967) that the solution obtained

is the least squares solution in regard to those orthogonalized vectors (the \underline{q}_1) which were accepted. It is not true to say (as this has been taken by, for example, Winch (1966) and Barraclough (1978)), that Fougere's solution is the same as the least squares solution once the lower truncation level has been determined. To see this, suppose orthogonal coefficients $\underline{v}_1, \ldots, \underline{y}_{,o}$ correspond to spherical harmonic coefficients $\underline{x}_1, \ldots, \underline{x}_{10}$ and that \underline{y}_8 and \underline{y}_{10} fail the significance test. Then new spherical harmonic coefficients $\underline{z}_1, \ldots, \underline{z}_9$ will form Fougere's solution because a component corresponding to \underline{y}_8 has been removed, changing $\underline{x}_1, \ldots, \underline{x}_8$ to $\underline{z}_1, \ldots, \underline{z}_8$. However $\underline{y}_1, \ldots, \underline{y}_7, \underline{g}_9$.

Fougere (1963, 1966a, 1966b) made strong replies to criticism of his procedure by Leaton (1963), Malin and Leaton (1966), and Winch (1966). One point made by his critics was that his solution was very similar to the least squares solution, and was therefore not very interesting. The procedure will be used on test data sets and on astrophysical data sets, and its usefulness can be evaluated in the light of the results of the analysis.

The data should be weighted by the estimated errors, when these are available. These are estimates of the variance of each data measurement. This gives more accurate observations a greater weight in the composition of the solution. It is done by dividing each row of the equations of condition (2b) by the square of the error associated with the data measurement, which is equivalent to minimizing a weighted prediction error. See Lawson and Hanson (1974). In the F-test, the ratio of the estimated variances is

replaced by the ratio of χ^2 (or chi-squared) for each model, when

$$\chi_{v}^{2} = \frac{1}{v} \sum_{i=1}^{D} \frac{e_{i}^{2}}{e_{i}^{2}}$$
(2f)

 σ_{i} is the error associated with the ith observation, and \mathcal{V} , D, and e_{i} are as before. Applying this to Fougere's method, no alteration is made to the calculation of the V_{i} in order to carry out the statistical testing.

A computer program was written for this application of Fougere's method to astrophysical data. It was written in FORTRAN and made use of routines from the NAG (Numerical Algorithm Group) library. The procedure was precisely as described, except that no provision was made for weighting the observations according to the estimated errors. The whole procedure is carried out in one run of the program. Each run was given an identity number to help keep track of the solution, and had a predetermined truncation level of the harmonic series, and a significance level for the statistical analysis.

The program was initially tested by runs on data that were functions of spherical harmonics. When used to find the least squares solution of various sets of data, the results were practically identical to least squares results from an independent program.

Modern analysts make use of the convergence to zero of the series of coefficients as a diagnostic tool. If the series is not

converging the representation will be poor. Truncating the series before it converges will mean the omission of terms which have a large contribution to make. Coefficients of the terms which are included will be altered, attempting to make up the deficiency. Representation will be very poor in some areas and specific coefficients will change a lot with variations in data or truncation level.

Convergence is considered in terms of the average power of all the coefficients of harmonics of a particular degree. Suppose that degree is N. Then the properties of spherical harmonics show that there are N^2 harmonics of a lower degree, and 2N+1 of degree N. The average power of the coefficients of harmonics of this degree is given by the following formula :

$$P(N) = \frac{4 \Pi}{2N+1} \sum_{m=0}^{N} \left[\left(a_{N}^{m} \right)^{2} + \left(b_{N}^{m} \right)^{2} \right]$$

A plot of degree (ordinate) against the logarithm base 10 of the power (abscissa) is useful in showing the important information about convergence of the solution.

The example plot drawn in Figure 2.1 shows the 'power spectrum' of a solution that is converging. The dashed line corresponds to the harmonic degree where truncation would result in a negligible amount of truncation effects.



Figure 2.1 Typical 'power spectrum' of a converging spherical harmonic solution. x marks the average power of harmonics of each degree. The dashed line indicates the harmonic degree where truncation is acceptable.

2.4 Fougere's Method and Test Data

Sets of synthetic data were used to investigate the ordinary least squares and Fougere's method solutions on data from a known distribution with known error distribution. They were selected to be similar to the astrophysical data. These test data sets themselves will be referred to as A, B, and C.

The statistical package 'MINITAB' was used to generate random numbers with a Gaussian distribution about a mean of zero, with standard deviation 30 or 130. These were associated with random positions on the sphere, also generated by MINITAB, with some manipulation.

The random selection of positions on the sphere using the procedure available with MINITAB was not straightforward. Longitude values (ϕ) were selected from a uniform distribution of integers between 0 and 360. Values of colatitude (Θ) were found by selecting real numbers from a uniform distribution between 0.0 and 1.0 with 5 decimal places, and reversing the signs of half of these and taking the arc-cosine.

The same set of 600 positions over the sphere were used in each set, while the simulated data values were altered. Because of the method of selection of values of the colatitude, the first 300 values were on one hemisphere and the second 300 on the other. This made the distribution of data points a little better (in the sense of more even) than a completely random distribution would be.

The test data values put into set A were 600 numbers selected randomly from a Gaussian distribution with zero mean and standard deviation 130. This set was designed to mimic the rotation measure dataset in size. (Although the rotation measures do not have a Gaussian distribution, their mean is close to zero, and the standard deviation about 130.) The data set A is illustrated in Figure 2.2.

Set B did not contain random data. The value 150 was assigned to positions where $0^{\circ} < \phi < 70^{\circ}$ and $-90^{\circ} < \Theta < 40^{\circ}$, and the value zero assigned to all other positions. This pattern was chosen to show up the problems of early truncation of the spherical harmonic series. It is shown in Figure 2.3.

A set of numbers chosen randomly from a Gaussian distribution with zero mean and standard deviation 30 were added to the values in set B to form set C. This was designed to show how the response to the pattern can be disrupted by noise, and appears in Figure 2.4.

Initially a least squares analysis was carried out on each dataset. This was done by using the program containing Fougere's procedure, with significance level $\propto =1.0$. This level accepts every component of the least squares solution. Initial runs of the analysis had been done truncating the harmonic series at degree and order 6. Eventually degree and order 12 was selected as the most practical truncation level for the rotation measure data, and this will be described in Chapter 4. Because of this, all the runs on test data were carried out with a truncation level of degree and order 12.





Figure 2.3 Synthetic data set B.

+ve -ve





+ve -ve

Also, bearing in mind the significance levels which were found to be useful with the astrophysical data sets, the test runs were carried out with the significance level of $\ll = 0.1$. This stops the inclusion of components when there is a greater than 10% chance that the model with the next component included is a worse model than the model built up so far.

The results are presented in the form of contour maps of the surface spherical harmonic model. The projection used is the sine projection, also known as the Samson-Flamsteeed projection. The contouring was carried out by routines from the graphics section of the NAG library combined with routines setting up the sinusoidal spherical projection. The projection caused problems with contouring at the poles, so that latitudes displayed are between $+80^{\circ}$ and -80° .

Only two sets of contour intervals are used, to aid interpretation. A contour key appears with each map, showing contour levels and label numbers. Certain contours are labelled with these numbers, these are at zero, ± 150 , ± 300 , ± 450 , ± 600 . The power spectra are presented to show convergence properties of the solutions.

<u>Set A</u> The least squares analysis of set A is No. 5008, Figure 2.5. The values taken by the spherical harmonic model are between +250 and -300. The model is very lumpy, showing many features that are due to accidental conjunctions. The power spectrum in Figure 2.6 shows that the series is not converging.

The Fougere analysis ($\propto = 0.1$) of set A is No. 5006, Figure 2.7. The model values are lower except in a few places. They are close to zero, which is the mean of the underlying Gaussian distribution, in many parts of the sphere. However the power spectrum in Figure 2.8 shows no convergence.

<u>Set B</u> The least squares analysis of set B is No. 5009, Figure 2.9. The difference between the model and the original function is less than 25 except near the sharp edge of the pattern. Here the value dips below -25 in a few places where it should be zero, and is over 200 in two places where 150 would be correct. These are effects of truncating the series early. Figure 2.10 shows that convergence is slow.

The Fougere analysis ($\alpha = 0.1$) of set B is No. 5011, in Figure 2.11. It is very similar to the least squares analysis in No. 5009 above, appearing very slightly smoother. It is a worse model in two places. Figure 2.12 shows that again convergence is slow.

<u>Set C</u> The least squares analysis of set C is No. 5012, Figure 2.13. The model has been disrupted by the addition of random noise. The difference between the model and the original pattern is larger than in 5009 above, but is less than 100 everywhere, and in many places it is less than 50. The power spectrum in Figure 2.14 shows very slow convergence.

The Fougere analysis ($\propto = 0.1$) of set C is run No. 5014, shown in Figure 2.15. This model is smoother than the previous one. Figure

2.16 shows slow convergence.

2.5. Discussion

The analysis of test data illustrates the problems of the least squares method of analysis. The coefficients found by this method do not form what looks like a convergent series, so truncation is always 'too early'.

One effect of this is that the solution takes extreme values in places where the data is sparse. An example is found in Figure 2.5, run No. 5008. It is practical to discuss areas of the sphere in terms of longitude and latitude ℓ and b, instead of longitude and colatitude ϕ and Θ . The high positive values predicted at $\ell = 150^{\circ}$, b = 15° are not a good model of the data, as can be seen in the plot of data set A in Figure 2.2. Fougere's method, Figure 2.7 run No. 5006, smooths out the solution but reduces values in this area by very little.

Another effect appears at the sharp edge of the pattern. This is seen in Figure 2.9 and 2.13. Again, this effect is hardly touched by Fougere's method seen in Figures 2.11 and 2.15. This is like the ringing (Gibbs) phenomenon in Fourier analysis.

These are serious drawbacks to carrying out a least squares analysis and occur when the data is very noisy, causing slow

convergence. They are not affected in these tests by application of Fougere's selection procedure. Fougere (1963) claims for his method that it selects a sensible truncation level by an objective statistical procedure. These tests suggest that Fougere's method will only improve the solution of a least squares solution slightly in some cases.

The reason that Fougere's method does not attack the problems is that it is restricted to rejecting part of the least squares solution. It does not eliminate components which fit very well in many areas but poorly in a few places. Moreover it cannot pull in new components which might form better compromise solutions and give faster convergence, allowing earlier truncation.

Fougere's method will be applied to astrophysical data in Chapters 4 and 5, but it will be seen that the same problems appear. A new approach is needed, and this is offered by recent work in the field of inverse theory described in the next chapter, Chapter 3.



Figure 2.5 Analysis No. 5008 of set A, using least squares method, truncated at degree and order 12.

CONTOUR KEY	
1	-600.0
2	-550.0
3	-500.0
4	-450.0
5	-400.0
6	-350.0
7	-300.0
8	-250.0
9	-200.0
12	-150.0
11 0	-100.0
12	-50.0
13	0.0
14	50.0
15	100.0
16	150.0
17	200.0
18	250.0
19	300.0
20	350.0
21	400.0
22	450.0
23	500.0
24	550.0
25	600.0



Figure 2.6 Power spectrum of analysis No. 5008 (least squares analysis of set A).



Figure 2.7 Analysis No. 5006 of set A, using Fougere's method, truncated at degree and order 12.

CONTOL	IR KEY
1 : -	-600.0
2 .	-550.0
3 .	-500.0
9 -	-450.0
5 .	-400.0
6 .	-350.0
7 -	-300.0
8 .	-250.0
9 .	-200.0
10 .	-150.0
11 -	-100.0
12	-50.0
13	0.0
14	50.0
12	100.0
16	150.0
17	200.0
18	250.0
19	300.0
1 20	350.0
21	400.0
44	500.0
23	500.0
25	500.0
23	020.2



Figure 2.8 Power spectrum of analysis No. 5006 (Fougere analysis of set A).



Figure 2.9 Analysis No. 5009 of set B, using least squares method, truncated at degree and order 12.

54

1	-300.0
2	-275.2
2	-250 0
3	-230.0
4	-223.0
5	-200.0
6	-175.0
7	-150.0
R	-125 2
ä	-100 0
	75 0
10	-75.0
111 .	-20.0
12	-25.0
13	2.0
14	25.0
15	50.0
16	75 2
1.72	100 0
	120.0
IB ·	125.0
19	150.0
20	175.0
21	200.0
22	225.0
23 1	250.0
24	275 9
100	200 0
60	300.0

CONTOUR KEY



Figure 2.10 Power spectrum of analysis No. 5009 (least squares analysis of set B).



Analysis No. 5011 of set B, using Fougere's method, truncated at Figure 2.11 degree and order 12.

56

CONT

CUR KEY - 300.00 - 275.00 - 275.00 - 255.00 - 255.00 - 1750.00 - 250.00 - N9456789911 12345 16789812845 250.0 275.0 300.0



Figure 2.12 Power spectrum of analysis No. 5011 (Fougere analysis of set A).



Figure 2.13 Analysis No. 5012 of set C, using least squares method, truncated at degree and order 12.

1 -300	Same 1
$\begin{array}{c} -275 \\ -275 \\ -226 \\ -226 \\ -226 \\ -175 \\ -226 \\ -175 \\ -156 \\ -176 \\ -176 \\ -176 \\ -176 \\ -176 \\ -176 \\ -26 \\ -176 \\ -26 \\ -176 \\ -26 \\ -176 \\ -26 \\ -176 \\ -26 \\ -176 \\ -26 \\ -176 \\ -26 \\ -176 \\ -26 \\ -176 \\ -26 \\ -176 \\ -26 \\ -26 \\ -176 \\ -26 $	



Figure 2.14 Power spectrum of analysis No. 5012 (least squares analysis of set C).



Figure 2.15 Analysis No. 5014 of set C, using Fougere's method, truncated at degree and order 12.

CONT	OUR KEY
1	-300.0
2	-275.0
3	-250.0
4	-225.0
5	-200.0
6	-175.0
7	-150.0
8	-125.0
9	-100.0
10	-75.0
11	-50.0
12	-25.0
13	0.0
14	25.0
15	50.0
16	75.0
17	100.0
18	125.0
19	150.0
20	175.0
21	200.0
22	225.0
43	250.0
24	273.0
23	300.0



Figure 2.16 Power spectrum of analysis No. 5014 (Fougere analysis of set C).

CHAPTER 3

MATHEMATICAL METHODS II

3.1. Inverse Theory

Inverse theory is the name given to the mathematical techniques required to estimate either the parameters of a model, or a continuous function, from observed data. The word 'inverse' is used in contrast to the 'forward problem' of science, of predicting observations from theory. Hence the 'backward' or 'inverse' problem is the estimation of theoretical parameters or a function from the observations. It invariably requires the inversion of a matrix. Menke (1984) gives this description :

Inverse theory is an organised set of mathematical techniques for reducing data to obtain useful information about the physical world on the basis of inferences drawn from observations.

The matrix form of equation (2b), $G \ge d$, describes any discrete linear inverse problem. The term 'inverse theory' will be used from now on to refer to discrete (parameterized) inverse theory. Spherical harmonic analysis is a typical problem of inverse theory.

In order to find the best representations of the observations by a model, inverse theory uses several measures of how well a model works. One of these is geometrical length, as in the case of the least squares method, where the length of the prediction error vector \underline{e} is minimized. This is only appropriate if the number of parameters P, is less than the number of observations, D. The

principle of minimizing a length can be extended to deal with the other cases.

The D data measurements define a vector in a D-dimensional space, as in the matrix form of equation (2b). The columns of matrix G (in this case $(s_j(\Theta_1, \phi_1), \dots, s_j(\Theta_D, \phi_D))$ forms the jth column) are a set of vectors in D space. There are P of these column vectors. A set of vectors is described as independent if no one vector is a linear combination of others in the set.

If P=D and the set of column vectors is independent, then it will span D-space, which is to say a unique linear combination of the column vectors can be found to equal any vector in D-space. So each set of observations will have a unique expression in terms of the column vectors. This expression is the solution, or the best estimate of the parameters if no other information is available. No minimization has to be applied in this case where P=D, and the problem is called equal-determined (Menke 1984).

If D>P, the column vectors will not span D-space. Provided they are independent, a linear combination of vectors will uniquely describe any vector in a P-dimensional subspace of D-space. This case is called strictly overdetermined by Menke (1984) as the amount of data is more than is needed to determine the parameters. There will be no exact solution to (2b). The least squares criterion is applied to select the vector in the subspace of D-space which is closest to <u>d</u>, the vector of observed data. Jackson (1972) uses 'over constrained' rather than 'overdetermined'.
If D<P then the P vectors in D-space will not be independent and there is an infinite number of ways of constructing a solution vector from a linear combination of them. The problem is described as underdetermined and needs further constraint to select a solution. The counterpart of the least squares solution of the overdetermined problem, minimizing the prediction error norm, is minimizing the 'solution length', which is some norm of the parameters.

The term 'mixed-determined' is used if the P column vectors do not span the whole D-space and are not linearly independent. D<P cannot be rigidly correlated with the underdetermined case, nor can D>P be correlated with the overdetermined case. Part of the solution of an apparently overdetermined problem can be underdetermined or vice versa (Jackson, 1972; Menke, 1984).

When a mixed-determined problem appears to be overdetermined, the least squares solution will be unstable. Some of the column vectors will be composed of a combination of the others, or so close that the independent component is lost amongst round-off errors. The equations of condition matrix is described as degenerate or ill-conditioned. The least squares solution will fluctuate wildly in response to small changes in data values or positions.

In the particular case of spherical harmonic analysis, underdeterminacy can appear in the overdetermined problem if the data positions fail to constrain the high values of a harmonic or

distinguish sufficiently between different harmonics. Fougere's method goes a little way toward tackling this. In his method the P column vectors are orthogonalized, and those which have only a very small independent contribution produce a very small orthogonal component. The sample variances corresponding to these components are very small so that the components are eliminated by a stringent cut-off level.

The underdeterminacy of the mixed-determined problem means that a large number of solutions are nearly equally satisfactory. If the problem appears overdetermined because D>P then the least squares solution is selected arbitrarily. Combining the methods of solving strictly overdetermined and strictly underdetermined solutions can give better solutions to this problem.

Methods of solving an underdetermined problem add extra constraints in order to choose between many exact solutions thus expressing a priori beliefs about the model. This is analogous to the least squares method of minimizing the length of the vector of prediction errors to choose one of many inexact solutions in the overdetermined case.

The solution \underline{x} to the matrix form of equation (2b) $G \underline{x} = \underline{d}$ will be called the solution vector. To constrain an underdetermined problem, such beliefs as simplicity or smoothness are expressed by minimizing some appropriate measure of the solution vector. The exact counterpart of the least squares method is minimizing the Euclidean or L_2 norm of the solution vector.

The solution of the underdetermined problem G $\underline{x} = \underline{d}$ which minimizes the solution length $\underline{x}^{T}\underline{x}$ is

$$\hat{\underline{\mathbf{x}}} = \mathbf{G}^{\mathrm{T}} \left[\mathbf{G} \mathbf{G}^{\mathrm{T}} \right]^{-1} \underline{\mathbf{d}}$$
(3a)

(Menke, 1984). More complex beliefs about the solution may be expressed by using a solution norm $\underline{x}^{T}W_{\underline{x}}\underline{x}$, where $W_{\underline{x}}$ is an appropriate weighting matrix. The more general solution which minimizes $\underline{x}^{T}W_{\underline{x}}\underline{x}$ is

$$\hat{\mathbf{x}} = \mathbf{W}_{\mathbf{x}} \mathbf{G}^{\mathrm{T}} \left[\mathbf{G} \mathbf{W}_{\mathbf{x}} \mathbf{G}^{\mathrm{T}} \right]^{-1} \mathbf{d}$$
(3b)

(Menke, 1984). Carrying this over to the mixed-determined problem, a combination (x) of prediction error and solution length can be minimized, where

$$\overline{\phi}(\underline{x}) = \underline{e}^{\mathrm{T}}\underline{e} + \epsilon^{2}\underline{x}^{\mathrm{T}}\underline{x}$$

The weighting factor ϵ^2 determines the relative importance of the two. The best estimate of the solution is then

$$\hat{\mathbf{x}} = \left[\mathbf{G}^{\mathrm{T}}\mathbf{G} + \boldsymbol{\epsilon}^{2}\mathbf{I}\right]^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{d}$$
(3c)

(Menke, 1984). This is a method now commonly used to stabilize least squares problems (Lawson and Hanson, 1974; Marquardt, 1963), which has appeared since forward selection as used by Fougere (1963). It is described by Menke (1984) as damped least squares and by Draper and Smith (1981) as ridge regression analysis.

66

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The damped least squares method damps the oscillations of the underdetermined part of the solution. The method is equivalent to adding 'white noise' to the data (Gubbins, 1983) and prevents inappropriate values appearing in areas of sparse data.

A value of ϵ^2 is sought which gives an acceptable value for the prediction error norm, and also minimizes the length of the solution. This is done by trial and error, small increases in the prediction error are traded off against decreases in the solution length. (The procedure will be described in more detail in the next section.)

A more general version of solution (3a) contains the prediction error measured by a norm that takes account of the estimated data errors. It uses the matrix W_e , which is diagonal with ith element equal to $1/\sigma_i^2$ where σ_i is the error associated with the *i*th observation. The norm $e^T W_e e$ forms a measure of prediction error which gives more weight to more accurate observations. This is the same as the weighting that was mentioned in section 2.3.

A more general norm of the solution, $\underline{x}^{T}W_{\underline{x}}\underline{x}$, can be included in the function to be minimized:

$$\oint (\underline{x}) = \underline{e}^{\mathrm{T}} W_{\underline{e}} \underline{e}^{\mathrm{T}} + \epsilon^{2} \underline{x}^{\mathrm{T}} W_{\underline{x}} \underline{x}$$
(3d)

The best estimate of the solution using this is

$$\hat{\underline{x}} = \left[\mathbf{G}^{\mathrm{T}} \mathbf{W}_{\mathrm{e}} \mathbf{G} + \boldsymbol{\epsilon}^{2} \mathbf{W}_{\mathrm{e}} \right]^{-1} \mathbf{G}^{\mathrm{T}} \mathbf{W}_{\mathrm{e}} \mathbf{d}$$
(3e)

(Menke, 1984). Other methods can be used to reduce the underdetermined part of the solution. These are restriction of the variance of the model parameters, and inspection of the eigenvalue spectrum of the normal equations matrix.

A second approach to 'best representation' is found in the concepts of data resolution and model resolution. These are included for completeness, because this approach to inverse theory is an important one. However data and model resolution are not of great importance for the rest of this chapter.

A starting point for data resolution is to consider the analysis as forming weighted averages of the data. Weighted local averages were used in the investigation of Faraday rotation of extragalactic sources by Simard-Normandin and Kronberg (1980) and Sofue and Fujimoto (1983). To produce a prediction for a particular position in this way, data measurements nearby are given greater weight than distant measurements. This should be true of spherical harmonic analysis, which also, effectively, forms weighted averages. Analysis of a vector field such as the geomagnetic field has analogous but different requirements. A prediction at a position where a measurement is already available should be heavily biased Inverse theory includes methods of toward that measurement. evaluating how much weight nearby and distant points have in forming the prediction, and uses these to find solutions or evaluate methods of finding solutions.

Model resolution is a corresponding, but more elusive, concept

developed around the resolution of the parameters of the model. It makes use of imaginary 'true' values of the parameters, and asks how much weight these have in forming the actually estimated parameters.

The least squares solution has perfect model resolution and often poor data resolution whereas the solution (3a) to the underdetermined problem has perfect data resolution because the model fits the data perfectly. In this case there are fewer data than model parameters so individual parameters can not be perfectly resolved. Methods of improving the method, for instance by finding a damped least squares solution, can also be seen as improving data resolution at the cost of degrading model resolution. There is not room to go into this further, however this approach is described in more detail by Menke (1984).

A third sense of 'best representation' is distinguished by Menke (1984). This is the maximum likelihood estimate. The best estimate of the parameters should maximize the probability of the specific set of data being observed from those parameters. If the errors in the data have a Gaussian distribution, and if there is no a priori information, the least squares solution is the maximum likelihood estimator. Underdetermined or mixed-determined problems explicitly use a priori information to find the maximum likelihood estimate. There can be no exact solutions and so probability and statistical theory are vital to understanding the powers and limitations of methods of analysis.

The geometrical method, the method of measuring data and model resolution, and the maximum likelihood estimates are presented by Menke (1984) in detail. He points out that the three approaches are parallel, each recovering the least squares and minimum norm solutions under the appropriate circumstances. They provide a range of approaches to difficult problems.

3.2. Minimum Norm Solution

The minimum norm solution (3e) has been used for spherical harmonic analysis of geomagnetic data by Gubbins (1983). Whaler (1981), Whaler and Gubbins (1981), and Gubbins (1983) have derived models of the geomagnetic field at the boundary between the core and the mantle of the Earth, using spherical harmonics. The geomagnetic field is measured at the surface of the Earth, and a model of the field at the surface is extrapolated to the core-mantle boundary. This extrapolation amplifies the size and errors of the harmonics by a factor of the order of 2^{L+2} for harmonics of degree L. Consequently it is important that the coefficients of the spherical harmonic model of the geomagnetic field at the Earth's surface converge to zero quite fast. This and other problems of the least squares method are confronted by Whaler (1981) and Gubbins (1983).

Whaler (1981) considers the application of inverse theory methods to the analysis of the geomagnetic field. She investigates minimum norm methods, and different strategies for tackling difficult least

squares problems. In particular she discusses and uses 'quelling' of higher degree harmonics, which uses norms to take the power out of higher degree harmonics by weighting against them by factors of, for instance, L^2 , where L is the degree.

Gubbins (1983) also describes the use of minimum norm solution (3d) to analyse the geomagnetic field in order to look at the core-mantle boundary. His expression of equation (3d) is derived from non-linear continuous inverse theory. Matrix W_x is used to express a priori beliefs about the model parameters. (Gubbins (1983) uses matrix C_m which is diagonal. C_m^{-1} can always be equated with W_x in (3d).) The ith element of W_x is a function of the degree of the ith harmonic, f(L(i)).

In the problem of the field of the core-mantle boundary, Gubbins (1983) uses a function dominated by the term $(a/c)^{L+2}$, where a is the radius of the Earth and c is the radius of the core. This is the amount by which the harmonics are amplified when the surface field is extrapolated to the core-mantle boundary. This weighting of the solution norm brings about the convergence which is required in a physically satisfying method.

The computer program used to calculate this solution was made available to the author. Advantages of the minimum norm solution are noted by Whaler (1984) principally the great control of convergence of the solution and consequent elimination of truncation effects. Another advantage is the suppression of extreme values in sparse data areas.

The minimum norm method is applied in this work to astrophysical data to attempt to eliminate the truncation effects by finding more convergent solutions. The spherical harmonics used as fitting functions have no physical significance, so the appropriate a priori constraint was an interest in the regular galactic-scale field. The appearance of high degree harmonics reflects small-scale local variations, and therefore it is required that these harmonics are 'quelled' (a term introduced by Parker, 1977).

The program used to calculate minimum norm solutions will be referred to as program GW. It uses Cholesky decomposition as described by Lawson and Hanson (1974), of the normal equations matrix $G^{T}G$, or $G^{T}W_{e}G$ when this is appropriate. Instead of ϵ^{2} program GW uses the the constant $\lambda = 1/\epsilon^{2}$.

Trial and error is used to find a suitable value of λ which reduces the solution norm whilst the prediction error remains acceptably This is done by plotting prediction error norm against low. solution norm to build up a 'trade-off curve' (Backus and Gilbert, 1970). Once the function f(L) has been chosen the solution is calculated for several values of λ and for each value the prediction error norm and solution norm are recorded. An asymptotic, monotonically decreasing curve will be found (Bachus and Gilbert, 1970; Menke, 1984) like the one drawn in Figure 3.1 below. In Figure 3.1 the asterisk marks a position where a low solution norm has been selected without an unacceptable increase in prediction error. The acceptability of the prediction error can be checked by a statistical test, either the chi-squared test, if



solution norm

.

Figure 3.1 Example of a 'trade-off curve'.

The cross marks the position of a useful solution.

error weighting has been used, or the F-test.

The selected solution is tested for convergence by looking at its power spectrum. If the solution is converging too slowly with the increasing degree of the harmonics, it will be necessary to include harmonics of higher degree, or to alter the function f(L) to damp out the fluctuations at high degrees.

The functions used were f(L)=1.0 or $f(L)=L^k$ where k is an integer between 1 and 8. If f(L)=1.0 the solution is the simple damped least squares solution in equation (3c). This is called 'neutral damping' by Gubbins (1983). If this does not bring about convergence at a reasonable degree, stronger damping is brought in. The strength of the damping increases as k is increased.

The question of what is a reasonable degree is decided by the a priori interest in large-scale trends. Harmonics with degree and order 1 to 4 are important in representing large-scale trends in the sky. Harmonics of degree 5 and higher are included in order to reduce truncation effects and allow the appearance of very important small-scale features. The effect of damping is to remove power from these harmonics, forcing the solution to converge. The power in these higher degree harmonics should decrease quite rapidly, and it is regarded as unnecessary to include harmonics of degree and order higher than 15, because the data are reasonably well distributed. The number of harmonics up to and including degree and order 14 is 225.

3.3. Minimum Norm Method and Test Data

Two of the sets of test data described in Chapter 2 were used to demonstrate the results of program GW. These were set A and set C. The runs of program GW have identity numbers as before. The results are presented in the same way as in the previous chapter, in contour maps and power spectra. Each solution was chosen by plotting a trade-off curve and selecting a value of the damping parameter λ . The trade-off curves and chosen values of λ are also given. In each case the series was truncated after degree and order 12.

The following Figures from Chapter 2 are important to this section for comparison with the new solutions: The plots of test data sets in Figures 2.2 and 2.4, the least squares solutions in Figures 2.5 and 2.13, the Fougere solutions in Figures 2.7 and 2.15, and the power spectra of these solutions in Figures 2.5, 2.8, 2.14 and 2.16.

<u>Set A</u> Run No. 0250, Figure 3.2 is the damped least squares solution (this uses the simple damping function of 1.0 for all values of L). The trade-off curve appears in Figure 3.3 where the value of λ chosen was 15.0. This solution is a little away from the 'knee' of the trade-off curve, because a slightly lower solution norm has been selected to improve the convergence of the solution. This has also been done in following solutions. The power spectrum in Figure 3.4 shows no convergence. The amplitude of this solution is considerably reduced from the least squares solution in Figure 2.2.

The heavily damped (L^8) least squares solution of set A is run No. 0266, in Figure 3.5. The trade-off curve appears in Figure 3.6 and the value of λ selected was 1.5×10^{-7} . The power spectrum in Figure 3.7 shows that convergence is slow so that this solution has been truncated too early. The amplitude of this solution is rather larger than the damped least squares solution in Figure 3.2 and still less than the least squares solution in Figure 2.2.

<u>Set C</u> The damped (1.0) least squares solution is run No. 0271, Figure 3.8. The trade-off curve appears in Figure 3.9, and the value of λ selected was 15.0. The power spectrum in Figure 3.10 shows very slow convergence. Even though the solution is for a series that was truncated too early, it is a considerable improvement over Figures 2.13 and 2.15.

The damped (L⁴) least squares solution for set C is run No. 0280 in Figure 3.11. The trade-off curve appears in Figure 3.12, and the value of λ selected was 2.0x10⁻³. Figure 3.13 shows the power spectrum of this solution, which is converging steadily. The solution, in Figure 3.11, has a tidier appearance than the previous solution using 1.0 damping, Figure 3.8, but is in fact further from the original function in a few places.

The heavily damped (L^8) least squares solution is run No. 0286, shown in Figure 3.14. The trade-off curve is shown in Figure 3.15. The value of λ selected was 2.0x10⁻⁷. Convergence is brisk, as the power spectrum in Figure 3.16 shows. The model is better than the least squares solution in Figure 2.13, and the Fougere solution in

Figure 2.15, but some values have been forced slightly higher by the damping than in Figure 3.8 and 3.11.

3.4. Discussion

In these tests the damped least squares solution, with the simple damping function of 1.0, is the smoothest and closest to the original function. For both synthetic datasets convergence is too slow and truncation of the series at harmonics of degree and order 12 is too soon.

The more heavily damped solutions force up the amplitude of the representations and increase the rate of convergence. These solutions were still rather better than solutions found with Fougere's method. Heavy damping corresponds to the a priori assumption that high order harmonics will be rather small and is quite inappropriate to analysis of set A where there are no underlying trends to find. Heavy damping performs quite well in the analysis of set C where there are large-scale trends to find. This suggests that the method will perform better than Fougere's method on rotation measure data where there is known to be at least one large-scale trend (because the absolute value of the rotation measure is proportional to the cosine of the latitude).

The importance of convergence of the power spectrum is not demonstrated in analysis of set C, where the more convergent solutions are not the best in appearance. This may be because even

the least squares solution of set C in Figure 2.13 shows slow convergence in Figure 2.14, and only a light touch is needed. However the damping method of improving the least squares solution always appears much better than Fougere's method.

It is important to be aware that the large random errors in set A caused large amplitude solutions which were not completely damped out. The high values inappropriately appearing in the areas empty of data around $\ell=140^\circ$, b=15°, and $\ell=210^\circ$, b=15°, were also not damped sufficiently.

Far more can be done to investigate and improve solutions using inverse theory. The resolution of the coefficients by the data can be investigated, and error estimates can be found for point values of the data. Analysis of the eigenvalue spectrum of the normal equations matrix $G^{T}G$ described by Jackson (1972) uncovers specific problems with a particular inversion by looking at the variance of individual parameters. More sophisticated methods of damping out the instability can then be devised.



Figure 3.2 Analysis No. 0250 of set A, using simple damping (1.0), truncated at degree and order 12.

CONTOUR KEY -600.0 100.0 200.0 250.0 300.0 350.0 400.0 450.0 500.0 550.0 600.3



Figure 3.3 Trade-off curve for analysis No. 0250 (simple damping analysis of set A). Selected solution No. 0250 is marked with a cross, here $\lambda = 15.0$.



Figure 3.4 Power spectrum of analysis No. 0250 (simple damping analysis of set A).



Figure 3.5 Analysis No. 0266 of set A, using heavy damping (L^8) , truncated at degree and order 12.

 CONTOUR
 KEY

 1
 -600.0

 2
 -550.0

 3
 -500.0

 4
 -460.0

 5
 -400.0

 6
 -350.0

 7
 -300.0

 8
 -250.0

 10
 -150.0

 11
 -100.0

 12
 -50.0

 13
 0.0

 14
 50.0

 15
 120.0

 16
 150.0

 17
 200.0

 20
 350.0

 21
 400.0

 22
 450.0

 23
 500.0

 24
 550.0

 25
 600.0



<u>Figure 3.6</u> Trade-off curve for analysis No. 0266 (heavy damping analysis of set A). Selected solution No. 0266 marked with a cross, here $\lambda = 1.5 \times 10^{-7}$.

· .



Figure 3.7 Power spectrum of analysis No. 0266 (heavy damping analysis of set A).



Analysis No. 0271 of set C, using simple damping (1.0), truncated at Figure 3.8 degree and order 12.

 CONTOUR
 KEY

 1
 -300.0

 2
 -275.0

 3
 -250.0

 4
 -225.0

 5
 -200.0

 6
 -175.0

 7
 -150.0

 9
 -100.0

 10
 -75.0

 11
 -50.0

 12
 -25.0

 13
 0.0

 14
 25.0

 15
 50.0

 16
 75.0

 17
 100.0

 18
 125.0

 20
 175.0

 20
 175.0

 20
 175.0

201223425

200.0

250.0 275.0 300.0



<u>Figure 3.9</u> Trade-off curve for analysis No. 0271 (simple damping analysis of set C). Selected solution No. 0271 marked with a cross, here $\lambda = 15.0$.



Figure 3.10 Power spectrum of analysis No. 0271 (simple damping analysis of set C).





 CONTOUR
 KEY

 1
 -300.0

 2
 -275.0

 3
 -250.0

 4
 -225.0

 5
 -200.0

 6
 -175.0

 7
 -150.0

 8
 -125.0

 9
 -100.0

 10
 -75.0

 11
 -50.0

 12
 -25.0

 13
 0.0

 14
 25.0

 15
 50.0

 16
 75.0

 17
 100.0

 18
 125.0

 19
 150.0

 20
 175.0

 21
 200.0

 22
 225.0

 23
 250.0

 24
 275.0

 25
 300.0



<u>Figure 3.12</u> Trade-off curve for analysis No. 0280 (moderate damping analysis of set C). Selected solution No. 0280 marked with a cross, here $\lambda = 2.0 \times 10^{-3}$.



Figure 3.13 Power spectrum of analysis No. 0280 (moderate damping analysis of set C).

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Figure 3.14 Analysis No. 0286 of set C, using heavy damping (L⁸), truncated at degree and order 12.

$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$
13 0.0 14 25.0 15 50.0 16 75.0 17 100.0 18 125.0 19 150.0 20 175.0 21 200.0 22 225.0 23 250.0 24 275.0



Figure 3.15 Trade-off curve for analysis No. 0286 (heavy damping analysis of set C). Selected solution No. 0286 marked with a cross, here $\lambda = 2.0 \times 10^{-7}$.



Figure 3.16 Power spectrum of analysis No. 0286 (heavy damping analysis of set C).

CHAPTER 4

FARADAY ROTATION

4.1. Introduction

Faraday rotation of the plane of polarization of linearly polarized electromagnetic radiation occurs when the radiation traverses a plasma containing a magnetic field. The amount of rotation from the original angle of polarisation is given by :

$$\Delta \Psi = k\lambda^2 \int_{\mathcal{L}} N_{e} \underline{H} . \underline{d} \mathcal{L}$$
(4a)

where $\Delta \psi$ is the rotation in radians, k is a constant equal to 0.812, λ is the wavelength in metres, L is the line of sight to the source in parsecs, $\underline{d\ell}$ is a small part of the line of sight, <u>H</u> is the magnetic field at $\underline{d\ell}$ in μ G, and N_e is the density of free electrons at $\underline{d\ell}$ in cm⁻³. This formula can be found in Lang (1980), and is derived by Harwit (1973) and Zeldovich et al (1983). It neglects redshift effects on wavelength, and the units are those commonly used in astrophysics.

The rotation measure RM is wavelength independent, and is defined as

$$RM = k \int_{L} N_{e} \underline{H} . \underline{dL}$$
(4b)

The units of rotation measure are radians metre⁻². The initial

plane of polarisation ψ_0 and the angle of rotation $\psi_r(\lambda)$ are related in this way :

$$\psi_{r}(\lambda) = \psi_{o} + RM.\lambda^{2}$$
 (4c)

In principle, a rotation measure can be found from the angles of polarisation of radiation from a source measured at several radio wavelengths. In practice, fitting a straight line of the form (4c) may be very difficult. Measurements of the plane of polarization gives values between 0 and π . However, the actual value of ψ_r will either be the measured value or this value plus or minus integer multiples of π . The value of ψ_r cannot be measured without this ambiguity, known as the 'n π ambiguity'.

A medium containing free electrons and a magnetic field is said to be Faraday active. The interstellar medium of the Galaxy is Faraday active, and affects radiation from all astrophysical sources. Within the Galaxy, Faraday rotation can be found in pulsars and galactic radio continuum emission. Extragalactic radio sources may undergo Faraday rotation all along the line of sight to the source.

The upper atmosphere of the Earth is also Faraday active, varying day by day. The effect is removed from astrophysical observations, using independent observations of the atmospheric contribution at that particular time and place. This is described by Wright (1973), who notes that it is one possible source of errors in the polarization measurements.

Pulsars are good probes of the interstellar medium, because Faraday active material is not found associated with them (Manchester and Taylor, 1977). The Faraday rotation of the pulsar is entirely due to the intervening interstellar medium, and this makes the rotation measure easier to find. Observations of pulsar emissions also give a 'dispersion measure' which can be used either to estimate the density of electrons along the line of sight, or to estimate the distance of the pulsar.

Dispersion occurs when an electromagnetic wave propagates through a plasma whether or not a magnetic field is present. The dispersion measure DM is a constant for a particular source, and its theoretical derivation is described by Harwit (1973) and Thomson (1981). It can be found by measuring the time difference of arrival of wave packets of different frequencies, and is related to electron density along the line of sight as follows :

$$DM = \int_{L} N_{e} ds$$

with N and L as before, and where ds is the length of the line of sight segment $d\underline{\ell}$. DM is measured in cm⁻³ pc.

The dispersion measure can be used to estimate the distance of a pulsar, provided that values for the electron density in the interstellar medium can be estimated independently. The distances to 38 pulsars have been found independently of electron density as described by Lyne et al (1985). Most of these estimates are made by combining measurements of the absorption of 21cm radiation by

neutral hydrogen (HI) in the interstellar medium with a model for the differential rotation of the Galaxy. Two distances were found from annual parallax, and three from known association of the pulsar with a supernova remnant.

The dispersion measures of pulsars at known distances can be used to estimate the average electron density of the interstellar medium. For example Weisberg et al (1980) put the value at between 0.02 and 0.03 cm^{-3} . More complex models of electron distribution can be built up, and an important one of these is the three component model of Lyne et al (1985). In this model the first component is a layer of constant electron density 0.025 cm^{-3} which has a scale height 'much higher' than the 400 pc scale height of the pulsars. The second component is a thin layer of scale height 70 pc representing the ionized regions in the galactic plane. The electron density of this layer is 0.015 cm^{-3} . The third component is a specific model of the electron distribution in the Gum Nebula, and is a modulation of a basic value of 0.28 cm^{-3} in that region.

Such information about the distribution of electrons in the Galaxy can be combined with rotation measure of pulsars and other sources in order to find out more about the magnetic field of the Galaxy. Estimates of the magnitude of the field from pulsar rotation measures are given for example by Manchester and Taylor (1977) and their values of about $2\mu G$ for the regular field and another $2\mu G$ for the irregular component are typical.

The highly organized radio emissions of pulsars may be due to

synchrotron emission, however the mechanism is not understood. Synchrotron emission is responsible for the background radio continuum emission from the Galaxy and the emission from extragalactic radio sources. In these cases the emitting regions are large. The presence of thermal electrons in and around the emitting regions causes Faraday rotation along the line of sight, within these regions and close to them before the radiation even reaches the Galaxy. This has the effect of 'depolarizing' the beam of radio waves, and will be described in more detail below.

Galactic synchrotron emission has information about the interstellar medium to various depths, depending on the position of the emitting region. Radiation from extragalactic radio sources, on the other hand, is able to probe the whole length of the interstellar medium along the line of sight to the source. However, Faraday rotation by material in and around the source and in the intergalactic medium may be included in the calculated rotation measure. Moreover, the rotation measures of extragalactic sources are often difficult to find, and so they can be quite unreliable guides to the amount of Faraday rotation occurring in the interstellar medium of the Galaxy.

Spherical harmonic analysis will be used to find more reliable information about the rotation measures of extragalactic sources. The method is intended to extract the underlying trends in the rotation measures, eliminating the random contribution from sources and the intergalactic medium and possible bad determinations of the rotation measure. The possible contribution of a regular

intergalactic medium, suggested by Sofue et al (1980) among others, can be discounted, following Simard-Normandin and Kronberg (1980), Thomson and Nelson (1980) and Vallee (1983b). The spherical harmonic analysis will estimate the galactic contribution to rotation measures at all positions in the sky. In the next sections the determination of extragalactic rotation measures is considered in some detail.

4.2. Depolarization of Extragalactic Sources

There are considerable problems in associating a rotation measure with an extragalactic source. The sources are radio galaxies or quasars, where conditions are not simple or well-known. Confusing effects within the source and along the line of sight make it impossible to determine meaningful rotation measures in the poorest cases, because of depolarization.

Linear and circular polarization are special cases of elliptical polarization. A source emits a jumble of differently polarized waves. The resultant beam is partly polarized, and can be regarded as the combination of an unpolarized and a polarized component. It is completely described by the four Stokes parameters I, Q, U, and V, which are defined in terms of time averages of the electric field components. I is the total intensity of the beam, Q and U describe the linear polarization and V describes the circular polarization. I, Q, and U, are important in determining Faraday rotation measures.
The polarized component of the beam can in general be regarded as elliptical with elliptical eccentricity β , and with its major axis inclined at an angle ψ to the chosen x direction. The intensity I p of the linearly polarized component is related to I the intensity of the elliptically polarised component by

$$I_{p} = I_{e} \cos 2\psi$$

Stokes parameters Q and U are functions of I and ψ

$$Q = I_p \cos 2\psi$$
 and $U = I_p \sin 2\psi$

so $I_p = (Q^2 + U^2)^{\frac{1}{2}}$ and $\psi = \frac{1}{2} \tan^{-1} U/Q$.

 $F_{\rm p}$, the fractional intensity of linear polarization, is given by

$$F_p = \frac{I}{I}p$$

More detailed background information can be found in Gardner and Whiteoak (1966), Wright (1973) and Thomson (1981).

Measurements of the angle of polarization are subject to errors, which are exhaustively described by Wright (1973). Among these are errors in the calibration of instruments, calculation of ionospheric Faraday rotation, and pointing of aerials. There may also be problems due to gain and phase drifts, resolution effects and confusion from the galactic background emission or unknown sources. However, apart from these errors, the presence of several

differently linearly polarized components in the beam of the telescope is likely to reduce the value of F_p . This is 'depolarization' and is a major problem in determining rotation measures of extragalactic radio sources.

The synchrotron process is responsible for radio emission from extragalactic sources. The radiation is very highly linearly polarized, with fractional polarization $F_p=70$ % when it is emitted. However, extragalactic sources commonly have $F_p<10$ %, because of depolarization. Fractional intensity is usually greatest at short wavelengths (such as 6cm) and monotonically declines with increasing wavelength, although there may be an increase followed by a peak of polarization intensity, before the decline begins.

There are three distinct causes of depolarization. The first is the co-existence in the source of regions with different physical composition and magnetic field strengths. These are called different 'regimes', referring to large-scale variations. They will have different radio spectra and different angles of polarization. The proportion of radiation from each regime will vary at different wavelengths. The effect on the linearly polarized component of a varying reduction the beam is (depolarization) in fractional intensity from the more highly polarized regime. The overall angle of linear polarization of the resultant beam will also vary with wavelength. Very different (by about $\pi/2$ rad) angles of polarization will effectively cancel one another out.

A second cause of depolarization occurs within one regime, when high velocity electrons cause the emission of synchrotron radiation and thermal electrons cause Faraday rotation. This is called 'front-back' depolarization or internal Faraday rotation. Radiation emitted at the further away part of the region (the back) is rotated more by its passage through the region, than radiation emitted by the nearer parts of the region (the front). Differently rotated parts of the beam interfere, causing depolarization.

A third cause of depolarization is differential Faraday rotation across the beam of the telescope by clumpy material in the source intersecting the line of sight. The telescope beam is very small compared to the angular size of similar material in our Galaxy, so is less likely to be depolarized in this way in the interstellar medium.

The depolarizing effects of internal Faraday rotation and differential rotation across the beam increase with increasing wavelength because the amount of rotation increases as described by equation (4a). There is some debate about how much each of these two effects occur. Conway et al (1974, 1977) say that internal Faraday rotation dominates any differential rotation due to the line of sight intersecting a 'Faraday screen'. Wardle (1977) and Wright (1979) suggest that regimes emitting synchrotron radiation rarely contain the thermal electrons that are responsible for Faraday rotation, and Thomson (1981) also considers that internal Faraday rotation is unimportant compared with depolarization caused by material intersecting the line of sight.

If there is a significant amount of depolarization, the polarization angles of the source will not follow a λ^2 law as in (4c). So the determination of rotation measures is only reasonable under certain circumstances. It is necessary that one regime dominates the synchrotron emission of the source, and that the beam should not be much affected by internal or external depolarization.

Several criteria have been used to exclude sources, or ranges of frequencies from a source, that are depolarized. The interference of different regimes is indicated particularly by a peak in polarization intensity F_{D} at some wavelength. Measurements of polarization at shorter wavelengths are excluded to rule out wavelengths where a different regime dominates. To increase the probability that the observed emission is due to one dominant regime, Vallee (1980) and Thomson (1981) recommend that a limited middle range of wavelengths is used in determining the rotation measure, and an upper wavelength limit (25 or 30 cm) is imposed. Vallee (1980) considers that the worst effects of internal Faraday rotation are also avoided by using a middle range of wavelengths. Depolarization by material intersecting the line of sight is recognised when the fractional intensity of polarization drops very fast with wavelengths.

Measures of depolarization have been used to recognise unacceptable wavelengths or sources. One of these is $\lambda_{\frac{1}{2}}$, the wavelength where polarization intensity $F_p(\lambda)$ has dropped to half its peak value. Simard-Normandin et al (1981) exclude wavelengths greater than $\lambda_{\frac{1}{4}}$ (the wavelength where polarization has dropped to a quarter of its

peak value). Wright (1973) recommends a much more stringent value, λ_2 but finds that this excludes many sources (he eventually develops a complex but useful system for classifying rotation measures according to reliability). It is also worth noting that very high rotation measures cause depolarization across the beamwidth as polarization is measured. The effect can be decreased by using narrower bandwidths.

4.3. Calculation of Rotation Measures

Most sources present a very difficult numerical problem in extraction of a reliable Faraday rotation measure, or determining that one does not exist. This is the effect of the $n\pi$ ambiguity in ψ_r in (4c). There is no number of observations of angle of polarization at different wavelengths which guarantee finding an unambiguous rotation measure.

It is possible to find likely values for the rotation measure using several observations, preferably more than three. λ^2 is plotted against ψ . Several values of ψ corresponding to several low values of n are plotted, and a straight line of the form (4c) is sought. This can be done by hand and eye, but with large amounts of data, and looking for greater accuracy and repeatability, computational methods are preferred. The least squares method is appropriate again. It is necessary to minimize the sum of squares of the position angle residuals weighted by the reciprocals of the estimated errors. This is exactly equivalent to the quantity χ^2_{ν}

in equation (2f). It is calculated for many values of RM, giving the slope of the straight line (4c). Unfortunately the n ambiguity leads to multiple minima in χ^2_{ν} . (Here ν is equal to the number of polarization measurements minus two)

By using tables of the chi-squared distribution a probability can be associated with each RM value which is the probability of the observed polarization measurements being seen if that value were the correct RM. So for each minimum of χ_{ν}^2 , a measure of goodness of fit of the corresponding RM is available. Comparison with all other minima is essential to show whether there is any ambiguity in the determination.

Thomson (1981) has investigated the numerical problem in detail. He ran tests using computer generated data from known distributions of 'rotation measures' and 'errors'. His conclusions are summarised as follows:-

(i) It is important to step through possible slopes of the line in rotation measure units of rad m^{-2} , not degrees or radians. Otherwise minima are poorly resolved or missed at higher values of slope angle. The steps should not be larger than 2 rad m^{-2} .

(ii) Considerable inaccuracy arises in the estimate of rotation measure if the errors in polarization measurements have been underestimated at as little as 2/3 of their real value.

(iii) The absolute value of rotation measures should be restricted to less than 200 rad m^{-2} away from the plane of the Galaxy.

(iv) Long wavelengths measurements (>30cm) cause further numerical problems and should not be included.

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(v) The wavelength $\lambda_{\frac{1}{2}}$, where depolarization has reduced the intensity of polarization to $\frac{1}{2}$ of its peak value, is an outer limit for usable measurements.

Point (iii) is the most contentious. A larger region of search reduces the accuracy of the algorithm, but must be used in the plane of the Galaxy in any case. The a priori restriction of rotation measure away from the plane is unfortunate. Thomson (1981) justifies it by considering other information about the size of the galactic magnetic field, and possible intergalactic fields.

Rotation measures may be found more accurately, and large rotation measures detected, using a method developed by Rudnick et al (1983). They make use of the depolarization of the beam by large rotation measures, across a wide beamwidth. For each source they make one measurement at a short wavelength using a narrow waveband, and then they make several measurements at adjacent wide wavebands, of medium wavelength. If the rotation measure is genuinely high, depolarizing rotation occurs across the bands and the fractional intensity drops dramatically from its initial short wavelength value. High rotation measures can thus be detected, and moreover low rotation measures can be estimated more accurately because there is no $n\pi$ ambiguity between the wide waveband measurements. Clearly this is a very useful approach to the problem of determining rotation measures.

4.4. Rotation Measure Catalogues

Two catalogues of rotation measures have been published recently by Tabara and Inoue (1980) and by Simard-Normandin et al (1981). Thomson (1981) describes the former.

Tabara and Inoue's (1980) catalogue is a collection of all measurements of linear polarization of radio sources published before December 1978. It includes galactic and extragalactic sources but not pulsars. Rotation measures and depolarization measures are calculated for many sources. They classify rotation measures into three grades. These are A, unambiguous good fits; B, unambiguous poor fits; and C, ambiguous or very poor fits. The range of search is limited to $|RM| < 200 \text{ rad m}^{-2}$ in higher galactic latitudes, but is considerably wider in low latitudes ($|b| < 26^\circ$). Step length is not mentioned.

The second catalogue was not available to Thomson (1981), however he gives a detailed critique of a catalogue described earlier by Kronberg and Simard-Normandin (1976). He finds that the numerical procedures used by Vallee and Kronberg (1975) and Kronberg and Simard-Normandin (1976) produce, on his synthetic data, many high rotation measures that are quite spurious. This happens because employed constraints by Kronberg and the less severe Simard-Normandin allowed unlooked for answers. The values of rotation measures stepped through are selected by taking regular steps in angular measure of the slope of the line, not in rad m^{-2} , which leads to the problem mentioned in point (i). Thomson (1981)

found that the errors in polarization measurements were underestimated (point (ii)) and the range of search was very wide (point (iii)).

The more recent catalogue published by Simard-Normandin et al (1981) shows one improvement in the algorithm, because rotation measure values are stepped through in intervals of 10 rad m^{-2} , sometimes less. The accuracy of the estimation of errors in the polarization measurements is difficult to evaluate. An interesting test would be to increase them all by 50% and see how much difference this made to the results using the same algorithm.

The range of search used by Simard-Normandin et al (1981) was -1100 to 1100 rad m⁻², much wider than Thomson recommends. The problem in using a wide range of search is that some high value of rotation measure will give an apparently better fit than the original low rotation measure. Thomson (1981) suggests that there are good reasons to suppose that the galactic contribution will be less in absolute value than 200 rad m⁻². A high extragalactic contribution measure $\lambda_{\frac{1}{2}}$. There is no such correlated with a low depolarization measure $\lambda_{\frac{1}{2}}$. There is no such correlation in Kronberg and Simard-Normandin's (1976) catalogue. Therefore it is suspected that the high values of rotation measure away from the galactic plane are not genuine.

In the investigation of galactic rotation measures, conclusions are biased by restricting the range of search away from the galactic plane. This makes it particularly important that the suspect

sources are investigated using the method of Rudnick et al (1983). If better observations are not made available, a wide range of search should be used with care, checking closely whether a rotation measure is ambiguous by the chi-squared test. The difference between ambiguous rotation measures in such cases tends to be large, perhaps 100 rad m^{-2} , which would eliminate many of these sources from the catalogue.

The rotation measure data set that will be analysed is based on the rotation measures published by Simard-Normandin et al (1981). 31 new sources were included, and some corrections, using rotation measures found by Vallee (1983c, 1983d) and Vallee and Bignell (1983), who use the same algorithm. The whole set is referred to as 'RM data' in computer generated headings to the contour maps. It contains 583 rotation measures with galactic coordinates and estimated errors. It would also be useful to analyse the data from the catalogue of Incue and Tabara (1980), when the latest version becomes available.

The 583 rotation measures in the data set are represented in Figure 4.1. Some trends can be easily seen by inspection of this diagram. Large rotation measures are most common in the plane of the Galaxy. The measures are well distributed about the sky but there are some small areas that have very few. One of these includes the centre of the Galaxy.

There are many negative rotation measures in the area bounded by $\ell=70^{\circ}$ and $\ell=150^{\circ}$, and below the plane of the Galaxy. This feature



Figure 4.1 Rotation measures in units of rad m^{-2} .



is associated with Loop II by Simard-Normandin and Kronberg (1980). The positive rotation measures around $\ell=30^{\circ}$ and above the plane of the Galaxy are associated with Loop I, the North Polar Spur. There is a cluster of positive rotation measures around $\ell=240^{\circ}$ and in the plane of the Galaxy. Some of these are associated with the Gum Nebula at $\ell=260^{\circ}$, b=0°, by Vallee (1984). Satisfactory representation of obvious features is a more important criterion (although an intuitive one) of a good spherical harmonic model than obtaining the very lowest sum of squares of residuals.

4.5. Spherical Harmonic Analysis

Preliminary analysis with Fougere's method on the rotation measure data set were carried out with truncation at very low degree and order. Truncation at degree and order 4 or below gave solutions too simplified to show typical features. As the truncation level was moved to degree and order 5 and above, more details appeared but so did areas of very poor representation. The areas changed dramatically when the truncation level was changed. These problems led to use of the Inverse Theory program GW.

Spherical harmonic analysis of the rotation measure data was carried out using program GW up to degree and order 15. At a particular truncation level a solution is sought with low prediction error norm and low solution norm, as described in section 3.2 above. A good solution will be convergent, in the sense of decreasing average power in harmonics of progressively

higher degrees, used in Chapters 2 and 3.

The least squares and weighted least squares solutions were not convergent. Simple and heavier damping in program GW improved the convergence. Eventually L^8 damping, as described in Chapter 3, produced a solution that was reasonably convergent when truncated at degree and order 12. This solution and some others are presented here, with the power spectra that show their convergence properties. The performance of Fougere's method at this truncation level is also shown.

The first diagram, Figure 4.1 shows the data. The least squares solution of the rotation measure data set to degree and order 12 is run No. 5001, Figure 4.2. This shows a rather bitty model, and is a very poor representation of the data in several areas. An area in the centre of the Galaxy from $\ell=330^{\circ}$ to $\ell=360^{\circ}$, and b=-10° to b=+10°, is quite empty of rotation measures but is shown as being distinctly positive in Figure 4.2. An area just above the plane of the Galaxy where $\ell = 120^{\circ}$ to $\ell = 150^{\circ}$ and $b = 0^{\circ}$ to $b = 30^{\circ}$ is also fairly empty of rotation measures but is represented as a very strongly negative anomaly. The generally positive area associated with the North Polar Spur, $\ell = 0^{\circ}$ to $\ell = 70^{\circ}$ and $b = 0^{\circ}$ to $b = 60^{\circ}$ is considerably weakened, and the large negative area sometimes associated with Loop II, at $\ell = 0^{\circ}$ to $\ell = 180^{\circ}$ and $b = 0^{\circ}$ to $b = 70^{\circ}$ hardly appears. These faults may be due to the early truncation of the solution, because the power spectrum in Figure 4.3. shows that there is no convergence. The sum of squares of residuals is 58.25×10^5 .

The Fougere analysis ($\lambda = 0.1$) to degree and order 12 is run No. 5002, Figure 4.4. This is a somewhat better representation - there are improvements in all the areas mentioned above with the exception of the North Polar Spur area. The improvements are not enough to make the solution really useful. The power spectrum in Figure 4.5 shows no convergence.

The least squares analysis weighted by errors, to degree and order 12 is run No. 217, Figure 4.6. This has eliminated the negative area centred on ℓ =140°, b=15°. The other problem areas still appear. The power spectrum in Figure 4.7 shows no convergence.

The weighted sum of squares of residuals is 4.09 x 10^5 . The weighted least squares analysis with simple damping to degree and order 12, is run No. 216, Figure 4.8. The sum of squares of the residuals is 4.30 x 10^5 . The trade-off curve from which the solution was selected is shown in Figure 4.9, the value of λ selected for the solution was 1.25. Convergence is very poor shown in Figure 4.10. and remained weak when the analysis was taken to degree and order 15. The solution does show some improvements, in spite of this. The North Polar Spur and Loop II areas are better represented, although the centre of the Galaxy is still poor.

The weighted least squares analysis with L^8 damping, to degree and order 12, is run No. 126, Figure 4.11. The trade-off curve is shown in Figure 4.12, and the value of λ of 1.0 x 10⁻⁷ was selected in a position where a low solution norm encouraged convergence. The sum of squares of residuals of this solution is

4.12 x 10^5 . The power spectrum in Figure 4.13 shows an acceptable rate of convergence. This slowed down when the analysis was taken to degree and order 15. This very simplified representation in Figure 4.11 is good in the critical areas except that it seems to take up the unnecessary negative representation again, a little lower at (=140°, b=10°.

The weighted least squares analysis with L^8 damping, to degree and order 12, of the same data set with axes changed ('kvvsetec') is run No. 448, and appears in Figure 4.14. The trade-off curve is not shown, nor is the power spectrum, because these are not distinguishable from the same things for solution No. 126. The same value of λ was used, and the sum of squares of residuals is 4.2 x 10^5 . Clearly this method of analysis is unaffected by change of axes.

The Fougere analysis ($\propto =0.1$) of the data set with new axes as above to degree and order 12, is run No. 5030, and Figure 4.15. This solution has a very poor perturbation in the region corresponding to the centre of the Galaxy. It seems that this method is not tolerant to the change of axes.

For comparison, the least squares solutions (not weighted by the errors in rotation measures) of rotation measure data to degree and order 6, 8 and 10 are included, in Figures 4.16, 4.17, 4.18. These are run Nos. 225, 5027, 227.

4.6. Discussion

Program GW has produced a very good solution in run No. 126, the weighted least squares solution damped by the factor L^8 . This solution, appearing in Figure 4.12, is very simple, reflecting the requirement that only large-scale trends should appear. Predicted values are zero in many areas away from the plane of the Galaxy. This is satisfactory, because in many of these areas the rotation measures are small, and positive and negative values appear mixed together, suggesting that they are not due to galactic Faraday rotation.

This solution has some disadvantages. Some features are not well justified by the data, notably a positive peak at $\ell=300^{\circ}$, $b=0^{\circ}$. The negative area around $\ell=140^{\circ}$, $b=+10^{\circ}$, which was eliminated in the weighted least squares solution, has reappeared! The improvement in resolution given by the high truncation level means that the solution appears detailed rather deceitfully because it can only reflect trends.

It has no competition among the other solutions at the truncation level of degree and order 12. However, a surprisingly good solution is the least squares solution with the series truncated at degree and order 6, run No. 0225. This solution reflects all the obvious trends in the data without appearing misleadingly detailed.

Further analysis could be carried out by inspecting the eigenvalues of the normal equations matrix. Small eigenvalues are responsible

for poorly justified solutions. The good solution truncated at degree and order 6 suggests that in higher truncation level solutions, the eigenvalues decrease after degree and order 6. The damping could be more closely tailored to the problem by allowing eigenvalues freedom up to this level, and then increasing the damping very quickly to very heavy values.

It is questionable whether the accuracy of the rotation measure catalogues justifies spending much more time on the problem. Appearance of more accurate rotation measures, found by such a procedure as that of Rudnick et al (1983) would make it a project well worth undertaking.

The coefficients which make up the solutions of run No. 126 are supplied in Appendix A. This spherical harmonic model can be used to predict the galactic rotation measure for extragalactic sources not in the catalogue. However, there are often very small-scale variations in the galactic contribution. It is therefore better to use very specific local information, from foreground objects, or other parts of the same source, to make such predictions. The sum of squares of residuals of this solution leads to a root mean square error of approximately 27 rad m⁻².

These solutions are primarily relevant to investigation of the large-scale magnetic field in the local region of the interstellar medium. They are intended to quantify the trends in the field. They can be used primarily as indicators of the types of models of

the field which would be useful. Evaluation of models of the magnetic field should preferably be carried out by direct comparison with the rotation measures of the catalogue. Residuals should be found and statistical values calculated from them. Statistical evaluation of proposed field models by their relation to the spherical harmonic representation of rotation measures would be rather complicated. The performance of a field model could be compared with a spherical harmonic representation, although this would be a tough test, bearing in mind that the spherical harmonics are not physically significant.

Pulsar rotation measures cannot be compared directly with the spherical harmonic representation of extragalactic rotation measures, because differences may be due to the position of a particular pulsar in the interstellar medium. A general comparison can be useful, indicating which features are local, although the small-scale variations may also cause the rotation measure of an individual pulsar to be unusual. Manchester and Taylor (1977) have plotted the rotation measures of pulsars available at that time. There is a general agreement with the spherical harmonic solutions. More rotation measures of pulsars are available now, not all of which have been published. Again, there is general agreement with the spherical harmonic solutions (Lyne, 1984).



Figure 4.2 Analysis No. 5001 of rotation measures, using least squares method, truncated at degree and order 12.

$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	CONT	OUR KEY
$\begin{array}{c} 2 \\ -550.0 \\ -560.0 \\ -450.0 \\ 5 \\ -450.0 \\ 5 \\ -450.0 \\ 6 \\ -350.0 \\ 0 \\ -350.0 \\ 0 \\ -250.0 \\ 0 \\ -250.0 \\ 0 \\ -250.0 \\ 0 \\ -250.0 \\ 0 \\ -250.0 \\ 0 \\ 0 \\ -150.0 \\ 0 \\ 0 \\ -150.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 $	1	-600.0
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	2	-550.0
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	3	-500.0
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	4	-450.0
	5	-400.0
$\begin{array}{cccccccccccccccccccccccccccccccccccc$	6	-350.0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	7	-300.0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	8	-250.0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	9	-200.0
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	10	-150.0
12 -50.0 13 0.0 14 50.0 15 100.0 16 150.0 17 200.0 19 300.0 20 350.0 21 400.0 22 500.0 23 500.0 24 550.0 25 600.0	11	-100.0
13 Ø.Ø 14 50.0 15 100.0 16 150.0 17 200.0 18 250.0 20 350.0 21 400.0 23 500.0 24 550.0 25 600.0	12	-50.0
14 50.0 15 100.0 16 150.0 17 200.0 18 250.0 20 350.0 21 400.0 23 500.0 24 550.0 25 600.0	13	0.0
15 100.0 16 150.0 17 200.0 18 250.0 20 350.0 21 400.0 22 450.0 23 500.0 24 550.0 25 600.0	14	50.0
16 150.0 17 200.0 18 250.0 20 350.0 21 400.0 22 450.0 23 500.0 24 550.0 25 600.0	15	100.0
17 200.0 18 250.0 19 300.0 20 350.0 21 400.0 22 450.0 23 500.0 24 550.0 25 600.0	16	150.0
18 250.0 19 300.0 20 350.0 21 400.0 22 450.0 23 500.0 24 550.0 25 600.0	17	200.0
19 300.0 20 350.0 21 400.0 22 450.0 23 500.0 24 550.0 25 600.0	18	250.0
20 350.0 21 400.0 22 450.0 23 500.0 24 550.0 25 600.0	19	300.0
21 400.0 22 450.0 23 500.0 24 550.0 25 600.0	20	350.0
22 450.0 23 500.0 24 550.0 25 600.0	21	400.0
23 500.0 24 550.0 25 600.0	22	450.0
24 550.0 25 600.0	23	500.0
25 600.0	24	550.0
	25	600.0



Figure 4.3 Power spectrum of analysis No 5001 (least squares analysis of rotation measures).



Figure 4.4 Analysis No. 5002 of rotation measures, using Fougere's method, truncated at degree and order 12.

120

CUNT	UUR KET
1	-600.0
2	-550.0
3	-500.0
4	-450.0
S	-400.0
6	-350.0
7	-380 8
0	-250 0
00	-200.0
10	-150.0
12	-100.0
1.2	-100.0
12	-30.0
13	0.0
14	50.0
15	100.0
16	150.0
17	200.0
18	250.0
19	300.0
20	350.0
21	400.0
22	450.0
23	500.0
24	550.0
25	620 0
20	000.0



Figure 4.5 Power spectrum of analysis No. 5002 (Fougere analysis of rotation measures).



Figure 4.6 Analysis No. 0217 of rotation measures, using least squares method with error weighting, truncated at degree and order 12.

CONT	OUR KEY
1	-600.0
2	-550.0
3	-500.0
4	-450.0
5	-400.0
6	-350.2
7	-300.0
B	-250.0
9	-200 0
12	-150 0
11	-100.0
12	-50.0
12	0.0
14	50.0
15	199.0
16	150.0
10	100.0
10	200.0
10	230.0
13	300.0
20	300.0
21	400.0
44	B. BCF
23	500.0
24	0.000
25	600.0
5 · ·	



. . .

Figure 4.7 Power spectrum of analysis No. 0217

(weighted least squares analysis of rotation measures).





Figure 4.8 Analysis No. 0216 of rotation measures, using simple damping (1.0) and error weighting, truncated at degree and order 12.

	LONI	DUR KEY
	1	-320.0
	2	-275.0
	3	-250.0
	4	-225.0
	5	-299 9
	6	-175 0
	7	-150 0
	á	-125 0
	d	-100 0
	10	-100.0
	10	-/3.0
	10	-30.0
	12	-23.0
	13	0.2
	14	25.0
	15	50.0
	16	75.0
	17	100.0
1	18	125.0
	19	150.0
	20	175.0
	21	200.0
	22	225.0
	23	250.0
	24	275.0
	25	300.0



Figure 4.9 Trade-off curve for analysis No. 0216 (simple damping analysis of rotation measures). Selected solution No. 0216 marked with a cross, here $\lambda = 1.25$.



Figure 4.10 Power spectrum of analysis No. 0216 (simple damping analysis of rotation measures).



Figure 4.11 Analysis No. 0126 of rotation measures, using heavy damping (L^8) and error weighting, truncated at degree and order 12.

 CONTOUR
 KEY

 1
 -300.0

 2
 -275.0

 3
 -225.0

 4
 -225.0

 5
 -200.0

 6
 -175.0

 7
 -150.0

 8
 -125.0

 9
 -100.0

 10
 -75.0

 11
 -50.0

 12
 -25.0

 13
 0.0

 14
 25.0

 15
 50.0

 16
 75.0

 17
 100.0

 18
 125.0

 19
 150.0

 20
 175.0

 21
 200.0

 22
 225.0

 23
 250.0

 25
 300.0



Figure 4.12 Trade-off curve for analysis No. 0126 (heavy damping analysis of rotation measures). Selected solution No. 0126 marked with a cross, here $\lambda = 1.0 \times 10^{-7}$.



Figure 4.13 Power spectrum of analysis No. 0126 (heavy damping analysis of rotation measures).

.





 CONTOUR KEY

 1
 -300.0

 2
 -275.0

 3
 -250.0

 4
 -225.0

 5
 -200.0

 6
 -175.0

 7
 -150.0

 8
 -125.0

 9
 -100.0

 10
 -75.0

 11
 -50.0

 12
 -25.0

 13
 0.0

 14
 25.0

 15
 50.0

 16
 75.0

 17
 100.0

 18
 125.0

 20
 175.0

 21
 200.0

 22
 225.0

 230.0
 0



Figure 4.15 Analysis No. 5030 of rotation measures with axes rotated, using Fougere's method, truncated at degree and order 12.

CONTO	OUR KEY
1	-600.0
2	-550.0
3	-500.0
4	-450.0
5	-400.0
6	-350.0
	-300.0
8	-250.0
9	-200.0
12	-100.0
12	-50 0
13	0.0
14	50.0
15 1	100.0
16	150.0
17	200.0
18 1	250.0
19	300.0
20	350.0
21	400.0
22	750.0
23	550.0
25	600 0
60	000.0



Figure 4.16 Analysis No. 0225 of rotation measures, using least squares method, truncated at degree and order 6.

CONT	OUR KEY
1	-300.0
2	-275.8
3	-250.0
4	-225.0
5	-200.0
6	-175.0
7	-150.0
8	-125.0
9	-100.0
10 1	-75.0
11	-50.0
12	-25.2
13	2.2
14	25.2
15	50.0
16	75 2
17 1	100 0
18	125 2
19	150 0
20	175 9
21	200 0
22 1	225 0
22	250 0
20	275 0
25	200 0
20	200.0



Figure 4.17 Analysis No. 5027 of rotation measures, using least squares method, truncated at degree and order 8.

•



Figure 4.18 Analysis No. 0227 of rotation measures, using least squares method, truncated at degree and order 10.

134

CUNT	UUR KEI
1	-680.0
2	-550.0
3	-500.0
4	-450.0
5	-400.0
6	-358 2
7	-300 0
à	-250 0
ă	-200.0
10	-150.0
10	100.0
12	-100.0
12	-30.0
1.3	0.0
14	0.00
15	180.0
16	150.0
17	200.0
18	250.0
19	320.0
20	350.0
21	400.0
22	450.0
23	500.0
24	550.0
25	600.0

CONTOUR KEY

CHAPTER 5

POLARIZATION OF STARLIGHT

5.1. Introduction

An important source of information about the galactic magnetic field is the polarization of starlight by interstellar dust particles, which have been aligned by the field. The angle of polarization of the starlight is the projection of the magnetic field on the sky (Davis and Greenstein, 1951), that is, the component of the magnetic field perpendicular to the line of sight. Consequently, the information is complementary to the information about the line of sight component in galactic Faraday rotation measures.

Interstellar polarization of starlight is affected by irregularities in the distribution of interstellar dust grains and by the varying physical properties and chemical composition of the dust grains (Martin, 1978). Like the extragalactic rotation measures, the polarization measurements contain information about the galactic magnetic field overlaid by a considerable amount of confusion from other effects. Unlike the rotation measures, interstellar polarization probes the interstellar medium only as far as the distance of the individual star.

There are 5070 interstellar polarization measurements available in a catalogue collected by Axon and Ellis (1976). The stars can be
divided into groups according to their distance from the Sun, each group forming a spherical shell. Interstellar polarization of stars within such shells is an appropriate subject for spherical harmonic analysis. Seymour (1967, 1969) investigated the trends in interstellar polarization of distance groups of stars from a catalogue published by Behr (1959).

Behr's (1959) catalogue contained 550 stars, most of these were nearer than 500 pc. Axon and Ellis' (1976) catalogue includes 2155 stars which are within 250 pc of the Sun. Clearly it is important to repeat this method of analysis on the new larger database.

5.2. Polarization by Interstellar Grains

The light from stars is plane polarized by interstellar grains to a fractional intensity (F_p) of a few per cent, although a few stars are intrinsically polarized and show a much higher fractional intensity of polarization.

The polarization of starlight occurs by the Davis-Greenstein mechanism (Davis and Greenstein, 1951). According to this theory, the grains of dust are in thermal motion and have paramagnetic properties due to their composition. They tend to line up, after long periods of time, spinning end over end around their minor axes which are parallel to the magnetic field. This is caused by paramagnetic relaxation applying a torque to the spinning grains.

Van der Hulst (1949) analysed the polarizing properties of aligned cylindrical grains. When the electric field vector of the light is parallel to the major axis of the cylinders, extinction is greatest, and when it is perpendicular to the major axis, extinction is least. When this is combined with the effect of the paramagnetic torque on the grains, extinction will be greatest when the electric field vector of the light is perpendicular to the magnetic field. Jones and Spitzer (1967) looked at the polarizing properties of prolate spheroids ('needles' or 'rice grains') and oblate spheroids ('Smarties'), and again extinction is greatest when the electric field vector is parallel to the major axis of the particles. Real dust grains are assumed to approximate these shapes.

The Davis-Greenstein mechanism as outlined above, needs a magnetic field of at least $10 \,\mu$ G. Cugnon (1983) estimates that a realistic value for the required field strength is 45µG or more. Observed values of the field are of the order of 3μ G. Other mechanisms have been investigated, for example diamagnetic torques. This theory predicts that extinction is greatest when the electric field vector of incoming light is parallel to the electric field. The plane of polarization of many stars is parallel to the plane of the Galaxy (Axon and Ellis, 1976). The Davis-Greenstein mechanism predicts from this that the magnetic field in many places is parallel to the plane of the Galaxy, which agrees with the Faraday rotation of galactic and extragalactic sources. Invoking a mechanism of alignment of dust grains involving diamagnetism means that the magnetic field is in many places perpendicular to the

plane of the Galaxy. This is contrary to the Faraday rotation measures, and the mechanism has been rejected for that reason. Alignment by interaction with gas flow was also proposed (Gold, 1952) but this agrees less well with observations than the paramagnetic mechanism (Serkowski, 1962).

In order to reconcile the discrepancies between estimates of the magnitude of the magnetic field, grains with different paramagnetic properties were considered, such as grains with high iron content, grains of dirty ice, and composite grains with a core partly of iron and a mantle of dirty ice, but the magnetic field required to align the grains was still an order of magnitude too high. Purcell (1979) and Spitzer and McGlynn (1979) recently established that high degrees of magnetic alignment were achieved by suprathermal rotation due to accidental irregularities in the shape and the surface of the grain of dust. This explanation is widely accepted (Aanestad and Greenburg, 1983; Cugnon, 1983).

The relation between extinction and polarization of starlight at different frequencies indicates that there are at least two populations of interstellar dust grains. Aanestad and Greenburg (1983) describe a population of $0.1 \,\mu$ m particles causing polarization and extinction at visible frequencies, and one of $0.01 \,\mu$ m particles causing extinction in the far ultraviolet. Kunkle (1979) proposes a model which uses four distinct populations of grains of different composition and size.

The shape and composition of the polarizing grains vary from region

to region (Aanestad and Greenburg, 1983). The scale of the variation may be as little as 2 pc (Martin, 1978). Consequently the relation between the intensity of the field and the fractional intensity of polarization is variable as well as complex.

5.3. History of Analysis

All-sky plots of interstellar polarization intensity and orientation show the large-scale features of these probes of the interstellar medium, particularly when the stars are collected into distance groups. These have been updated most recently by Axon and Ellis (1976). They show features that have appeared and demanded explanation ever since the first detection of interstellar polarization in 1949. The electric vector of the polarized component of the light is often parallel to the plane of the Galaxy. This is most noticeable at $\ell=140^\circ$, b=0°. At longitudes such as $\ell=50^\circ$, $\ell=80^\circ$ in the plane of the Galaxy the electric field vector seems to take random directions, suggesting (if the Davis-Greenstein mechanism is accepted) that the line of sight runs along the magnetic field in these directions.

A major feature of the all-sky plots is the high degree of polarization and alignment of angle of polarization in stars near the North Polar Spur known from radio astronomy. The directions of the electric field vectors arch up out of the plane toward the North galactic pole. This feature inspired the theory of Ireland (1961) and Hoyle and Ireland (1961) which held that a helical

magnetic field ran along the local spiral arm. The theory was rejected largely because of the failure of measurements of Faraday rotation of radio sources to agree with values such as those predicted by the helical models of Hornby (1966) or Mathewson (1968). The North Polar Spur is now regarded as a local supernova remnant or stellar wind bubble (Berkhuijsen, 1973; Weaver, 1979: Spoelstra, 1972; Bruhweiler et al, 1980; Heiles et al, 1980).

Seymour (1967, 1969) used Fougere's method of spherical harmonic analysis on Behr's (1959) catalogue of stellar polarization. He analysed Stoke's parameters Q and U (expressed in stellar magnitudes and galactic co-ordinates) separately for stars within 500 pc of the Sun. The groups used by Seymour were (i) 101 stars between 0 and 30 pc, (ii) 135 stars between 30 and 60 pc, (iii) 105 stars between 60 and 110 pc, (iv) 103 stars between 110 and 260 pc, and (v) 101 stars between 260 and 500 pc. The series of spherical harmonics was truncated after degree and order 3, so that 10 coefficients were calculated.

Seymour (1967) found that between 30 and 110 pc from the Sun, there is a large amount of disorder in the interstellar polarization measurements, reflected by low significance of components of the least squares spherical harmonic model. He found evidence for the presence of the North Polar Spur within 30 pc of the Sun. He also found that between 110 and 260 pc, a longitudinal field running from ℓ =50° was consistent with the spherical harmonic model, although this did not appear in the analysis of the most distant group. He rejected the helical model of the field by Ireland (1961).

The current catalogue of polarization observations compiled by Axon and Ellis (1976) includes all 1800 stars from the previous catalogue of Mathewson and Ford (1970). Ellis and Axon (1978) carry out a detailed analysis of implications of the optical data. They start by taking a smoothed representation of the data. They use all stars within 15° of the plane of the Galaxy, binned in 10 distances intervals of 200 pc, and in intervals of 15° longitude. The average polarization is calculated for each bin.

Stokes parameter Q in galactic co-ordinates represents components in the plane of the Galaxy and perpendicular to the plane. Stokes parameter U represents components inclined at 45° and 135° to the plane of the Galaxy. Ellis and Axon (1978) fit longitudinal models to Q and U data separately, using their smoothed representation of the catalogue. Using Q data only, out to 2 kpc, they find a field of 2.6 μ G, running from ℓ =48°, b=+11°. If the solution is constrained to be in the plane of the Galaxy, the result is a field of $3.2 \,\mu\text{G}$, from the direction $\ell=54^\circ$, b=0°. To remove the effects of local loops and other local features, they subtract a smoothed representation of polarization in the galactic plane of stars up to 500 pc in distance from the smoothed polarization of stars at 2 kpc. For the Q data again, this gives a poorer fit to the longitudinal model, giving a field magnitude of 3.1 µG running from ℓ =54°, b=+12°. The U parameter is affected more by random effects, perhaps because it describes components which are inclined to the galactic plane.

Ellis and Axon (1978) find that there are severe problems with a longitudinal model of the galactic magnetic field beyond 500 pc. They also particularly note that subtraction of the smoothed Stokes parameters amplifies the errors, so that an incremental map, recording the change in polarization of stars at increasing distances, shows largely random effects. They suggest that this map is affected by a selection effect which increases with distance. This effect makes it appear that all polarization occurs within about 750 pc of the Sun. Highly polarized stars at great distances are also highly extinguished, and are likely to be faint and not included in surveys of polarization.

Seymour (1967), Mathewson and Ford (1970) and Ellis and Axon (1978) all comment on the major feature of interstellar polarization catalogues associated with the North Polar Spur. The direction of ℓ =50° regularly found for the local longitudinal magnetic field from optical data is a direction which is strongly affected by the Spur. Faraday rotation suggests that the field is found running from the direction ℓ =100° or thereabouts. So it is important to determine the extent of the North Polar Spur.

Seymour (1967) concludes from the appearance of the spherical harmonic model of stars within 30 pc, and from the greater degree of order in this group of interstellar polarization measurements, than in more distant stars, that this area includes the position of the Spur. Bingham (1967) contradicts this, suggesting that stars closer than 70 pc show little polarization, in this area of the sky, and do not show the influence of the Spur. Ellis and Axon

also obtain the distance of the feature (regarded as a shell structure by 1978) by examining individual stars. They reckon that the North Polar Spur, Loop I, extends from 50 to 300 pc.

Berhuijsen (1971) describes the morphology of the four loops observed in the galactic background radiation, and presents them as 'small circles' on a projection of the sky, giving the direction of the centres of the loops and their angular diameters. In a later paper (Berkhuijsen, 1973) she adopts the value of 70pc+40pc for the tangential distance to Loop I (the North Polar Spur), the value 130pc+75pc for the distance to the centre of the shell, and 230pc+135pc for the diameter of the shell. These values are from workers using evidence from a variety of criteria (Bunner et al, 1972; Seymour 1969; Spoelstra, 1972).

Around 1970 it was recognised that the loops were shells, probably of supernova remnants of exceptionally great age, associated with local groups of stars, and showing up strongly in the magnetic field and the material of the interstellar medium. The North Polar Spur was recognised as a multiple shell, having a detailed internal ridge structure parallel to the main spur. There is far more data available now about the North Polar Spur than there was when Seymour (1967, 1969) discussed its position. The distance is not known more accurately because features of different kinds, HI emission, X-rays, radio, stellar polarization, do not coincide exactly (Heiles et al, 1980). Detailed mapping and modelling of the physical properties of the North Polar Spur would be invaluable to knowledge of the more distant area, because it would enable

account to be taken of the effect of the Spur.

Inoue and Tabara (1980) used the catalogue of Axon and Ellis (1976) to examine the local magnetic field, along with rotation measures from their own catalogue (Tabara and Inoue, 1980). They divide stars into six distance groups, of 100 to 200 pc, 200 to 400 pc, 400 to 600 pc, 600 to 1000 pc, 1000 to 1500 pc and 1500 to 2000 pc. They look at differential polarization, subtracting the average polarization of the inner shell of stars from each star in the next group. They deduce from this that interstellar polarization is more closely correlated with dust clouds than with the component of the regular magnetic field. They therefore reject the evidence from polarized starlight from stars in the plane of the Galaxy and investigate stars around the southern Galactic pole, well away from the plane and from the North Polar Spur. They find that here the regular magnetic field runs from ℓ =100°, a figure which agrees with other evidence.

5.4. Fougere's Method

In carrying out spherical harmonic analysis on interstellar polarization measurements, it is important to look at stars which are all at similar distances. Starlight which has travelled various distances through the interstellar medium will not show a consistent or reasonably smooth variation in polarization angle, in the underlying trends.

Following Seymour (1967) Stokes parameters in galactic co-ordinates were regarded as independent scalar variables and analysed separately. Spherical harmonic analysis of a scalar variable is most appropriate when values are scattered reasonably evenly over the whole sky. This limits analysis to stars which are less than about 200 pc from the Sun.

The stars in the catalogue were combed to produce several groups (or shells) of stars within certain distance ranges, in order to investigate the local interstellar medium and magnetic field. The distance groups selected were 0 to 28 pc, 29 to 53 pc, 54 to 78 pc, 79 to 103 pc, and 104 to 128 pc. The numbers bounding the groups were chosen because at distances of 100 pc and more, distance estimates had often been rounded to the nearest multiple of 5 pc.

The stars at various distances are affected by dust all along the line of sight, cumulatively. In order to look at the trends in the interstellar material in each shell independently, it was planned to subtract the contribution of earlier shells found by the spherical harmonic analysis. The polarization of stars in each group would be scaled to one mean distance. However, this plan proved unworkable in practice, as will be described.

The Q and U parameters of the stars in the first group were analysed twice, once using the values in the catalogue, and once scaling the values to the average distance of all the stars, which was 18.7 pc. The spherical harmonic series was truncated after harmonics of degree and order 4, for this preliminary

investigation, and a significance cut off level of 0.1 was used, although other levels were also investigated. All of the solutions followed a typical pattern. A few, typically three, of the 25 coefficients were ten times the size of the others, dominating the solution. The last of these was also the last harmonic coefficient that was not set to zero by the truncation procedure. This suggests strongly that only the method of selection of the orthogonalized components had kept in other low coefficients, highlighting the arbitrariness of the secondary truncation procedure.

The scaled sets of parameters were found to be initially dominated by one star, which was very highly polarized, at 9 pc distance. However the pattern of domination by a few harmonics persisted after this star was removed from the group. This star suggested a reason for the domination pattern. A very few stars with higher degrees of polarization than the others make a few harmonics sufficiently significant to pass the F-test criterion.

The second and third groups of stars were analysed in the same way, and again stars had to be removed which were dominating the analysis. The same pattern of a few harmonics only being dominant persisted after their removal. Moreover, the predicted values of the coefficients remained low, about 2×10^{-4} . This meant that when the previous contributions were removed, the new values were actually much higher than the analysis of the original trends in the group. The Q parameter of the second shell had predicted values so much lower than the first shell that the appearance after

subtraction of the inner shells contribution, was a reflection of the analysis of the inner shell.

It was clear from the preliminary investigation that little would be gained from carrying on with this method. The stars needed to be combed for the most unlikely values, and the analysis needed to be more robust to the large random variations in the measurements. The large variations may be due to intrinsic polarization in the stars, or to a high level of polarization from small dust clouds.

5.5. Minimum Norm Method

MINITAB, the statistical package, was used to comb the data set of all 2155 stars within 250 pc for outliers. For each star the Q and U parameters were each divided by the distance. The two parameters obtained were converted to a normal distribution by subtracting the mean from each value and dividing by the standard deviation. These two normalized values were each squared, and the two squares were added. The final number should have a chi-squared distribution with two degrees of freedom. Tables of the chi-squared distribution gave the information that 99% of the values should be less than 9.21 and 99.9% should be less than 13.82.

There were 18 stars whose chi-squared value was more than 9.21, which was acceptable, (22 would be expected). However, 8 of these had a value great than 13.82, where 2 were expected. As there were several at these numbers, it was difficult to decide where to draw

the line. There was, however, only one chi-squared value between 8 and 10, this was at 8.02. It therefore seemed sensible to exclude all the stars for which the chi-squared value was more than 9.21, effectively all those over 10.00. There remained 2137 stars within 250 pc of the Sun. This exclusion was perhaps over zealous, but was thought useful to improve the performance of the spherical harmonic analysis.

Four larger distance groups were selected in order to stabilize the analysis. These were a group of 621 stars between 0 and 45 pc, 569 stars between 46 and 90 pc, 410 stars between 91 and 135 pc, and 299 stars between 136 and 182 pc. The last group covers a slightly wider distance, to include more stars. The Stokes parameters were each divided by the distance of the star. Program GW was used for the analysis this time, allowing a range of damping options. Again there proved to be problems carrying out the analysis. Preliminary investigation was carried out on the third set of stars, between 91 and 135 pc from the sun.

Solutions were found for the third set using the damping function L^8 which was used with the rotation measure data. The harmonic series was taken to degree and order 14. Three sets of polarization parameters were analysed, these were Q, U, and $I_p = \left[(Q/d)^2 + (U/d)^2 \right]^{\frac{1}{2}}$. The solutions are presented in the form of contour maps in Figures 5.1, 5.4 and 5.7. The corresponding 'trade-off' curves are shown in Figures 5.2, 5.5 and 5.8, and the power spectra in Figures 5.3, 5.6 and 5.9. The power spectra show that the solutions are converging very slowly. The trade-off

curves show that values were selected not at the knee of the curve but favouring a smaller solution norm at the expense of the residual norm to encourage convergence.

All three solutions are largely featureless with one area of high absolute values of Q, U and Ip. This centres on $\langle =90^{\circ}$, b=15°. The E-vector maps (Axon and Ellis, 1976) show that this feature is due to one star so these solutions are very unsatisfactory. They contain no information that cannot be obtained more accurately from the actual polarization measurements. They confuse rather than inform.

A final attempt was made to find a better set of solutions, in this the Q parameter data was solved, without dividing the value by the distance of the star. This proved to be even less convergent, and no solution is shown.

In view of the results of using heavy damping with synthetic data sets, it was not possible to feel confident that such solutions could be good models of the polarization parameters, and the solutions themselves inspire no confidence.

5.6. Discussion

Both Fougere's method and the inverse theory method using damping have failed to find usable solutions on this set of data. It is possible to bring more powerful armoury to bear on the problem of

spherical harmonic analysis of awkward data sets. The eigenvector/ eigenvalue analysis of Jackson (1972) allows much more detailed inspection of the behaviour of the solution. The two methods used are both 'packages', which do not permit detailed examination of the analysis, although it would be possible to extend them. Further use of spherical harmonic analysis on interstellar polarization measurements should start by considering the methods of Jackson (1972) Damped(L**8) SHA of gleset to d&o 14: #0327



Figure 5.1 Analysis No. 0327 of Q parameters, using heavy damping, truncated at degree and order 14.

CONTOUR KEY
1 -600.0
2 -550.0
3 -500.0
4 -450.0
5 -400.0
6 -350.0
7 -300.0
8 -250.0
9 -200.0
10 -150.0
11 -100.0
12 -50.0
13 0.0
14 50.0
15 120.0
16 150.0
17 200.0
18 250.0
19 320.2
20 350.0
21 400 0
22 458.0
23 1 500.0
24 559 9
25 600.0
23 000.0



Figure 5.2 Trade-off curve for analysis No. 0327 (heavy damping analysis of Q parameters). Selected solution No. 0327 marked with a cross, here $\lambda = 2.0 \times 10^{-8}$.



Figure 5.3 Power spectrum of analysis No. 0327 (heavy damping analysis of Q parameters).



Figure 5.4 Analysis No. 0345 of U parameters, using heavy damping (L^8) , truncated at degree and order 14.

CONT	OUR KEY
1	-600.0 -550.0
3.	-500.0
5	-400.0
6	-350.0
8	-250.0
9	-200.0
11	-100.0
12	-50.0
14	50.0
15	150.0
17	200.0
19	300.0
20	350.0
22	450.0
23	500.0
25	600.0



<u>Figure 5.5</u> Trade-off curve for analysis No. 0345 (heavy damping analysis of U parameters). Selected solution No. 0345 marked with a cross, here $\lambda = 2.0 \times 10^{-8}$.



Figure 5.6 Power spectrum of analysis No. 0345 (heavy damping analysis of U parameters).



Figure 5.7 Analysis No. 0349 of I_p values, using heavy damping (L^8), truncated at degree and order 14.



Figure 5.8 Trade-off curve for analysis No. 0349 (heavy damping analysis of I_p values). Selected solution No. 0349 marked with a cross, here $\lambda = 5.0 \times 10^{-9}$.



<u>Figure 5.9</u> Power spectrum of analysis No. 0349 (heavy damping analysis of I_p values).

CHAPTER 6

MODELLING THE GALACTIC MAGNETIC FIELD

6.1. Early Magnetic Field Models

An early model of the magnetic field was of field lines running along the spiral arms of the Galaxy, described by Hoyle and Ireland (1960a) among others. The magnetic field was seen as tubes of force which had a vital role in maintaining the spiral arms. This role has now been superseded by spiral density waves, however in other respects, these early ideas are very similar to the currently proposed open bisymmetric spiral field (Sawa and Fujimoto, 1980; Sofue and Fujimoto, 1983). Hoyle and Ireland (1960a) suggested then that the field is primordial in origin, and partly wound by differential rotation. The winding leads to instability, so that loops of the field emerge from the plane to the halo of the Galaxy. The gas and magnetic field flow along the spiral arms, outward. Conservation of angular momentum requires that there is an inflow of gas in the halo towards the centre of the Galaxy. Hoyle and Ireland (1960b) discuss this problem, noting that the inflow is not observed.

A new model emerged when Hoyle and Ireland (1961) proposed that the magnetic field lines form a tightly wound helix around the spiral arm. They were dissatisfied with the spiral field model, and wished to account for the polarization of starlight. The model is discussed in detail by Ireland (1961). Measurements of interstellar polarization show a region of maximum polarization parallel

to the galactic plane at $\ell=140^{\circ}$. This region suggests that the field direction is $\ell=50^{\circ}$, and is still found (Axon and Ellis, 1976). These observations are not in line with the spiral field model, but are explained by the helical field model, for stars which are not too close to the mid-plane.

A new impetus was given to the helical magnetic field model by the estimation of Faraday rotation measures of extragalactic sources by Gardner and Davies (1966). Hornby (1966) found evidence in these estimates for the reversal in sign of the rotation measures across the galactic plane. Seymour (1967) found that the equatorially anti-symmetric component of a third order spherical harmonic model of the rotation measures was an order of magnitude larger than the equatorially symmetric component, confirming Hornby's (1966) analysis. Seymour (1967) however, found that analysis of radio continuum polarization measurements was not consistent with the helical magnetic field model.

Mathewson (1968) had measured the polarization of 1400 stars and his results led him to suggest that a tightly wound helix with pitch angle 7° accurately described the magnetic field in the local spiral arm. He then interpreted the spurs observed in radio continuum surveys as radio tracers of the helical field.

Rotation measures found after this time (Wright, 1973) began increasingly to suggest that the magnetic field did not reverse its direction at the galactic plane. This rules out the possibility that the field is a tightly wound helix. A more loosely wound

helix remains a possibility, but this removes from the model its ability to account for the interstellar polarization, which is its raison d'etre. The radio continuum spurs are now regarded as parts of supernova remnants (Berkhuijsen, 1971; Berkhuijsen et al, 1971; Spoelstra, 1971; Haslam et al, 1971). Moreover Berkhuijsen et al (1971) point out that Loops II and III are in places perpendicular to the helices of the tightly wound model.

Fujimoto et al (1971) speculate that 'rotating eddies' along the spiral arm cause helical twisting of the magnetic field. The angle of the helix would be variable. This idea was not taken up, and the interpretation of the radio loops or spurs as supernova remnants marked a defeat for the helical model of the magnetic field, because its explanation of interstellar polarization measurements had been its strength. The pattern of interstellar polarization shows a clear association with the North Polar Spur, and this has led to the assumption that the features of the field described are local and anomalous.

6.2. Toroidal and Spiral Models

Simard-Normandin and Kronberg (1980) use 543 rotation measures from their catalogue of extragalactic rotation measures (Simard-Normandin et al, 1981) to investigate the galactic magnetic field. They have a method of averaging rotation measures over small areas of sky, which gives an approximate value for the galactic contribution to the Faraday rotation at many grid points

over the sky, called the GRM. Particularly unusual rotation measures are excluded and the GRM is not evaluated if there are too few rotation measures in the area. This process is analogous to using spherical harmonic analysis to determine large-scale trends. The GRMs however are not evaluated everywhere (there are many gaps in the plane of the Galaxy), and can be misleading where the rotation measures change sign sharply.

Many features in the spherical harmonic analysis of the rotation measure data set appear also in Simard-Normandin and Kronberg's (1980) set of GRMs. Three regions are particularly important. The first is the concentration of strong negative values below the plane at ℓ =90°, called region A by Simard-Normandin and Kronberg (1980), and associated with Loop II. The second is the region of high positive rotation measures around $\ell=250^\circ$, which is associated by Simard-Normandin and Kronberg (1980) with the direction of the Gum Nebula, and called region B. The third is the region of high positive rotation measures above the galactic plane at ℓ =40°, which they call region C, and which is coincident with the North Polar Spur, Loop I. Although Simard-Normandin and Kronberg (1980) draw attention to the association in direction between three local supernova remnants and the three major features of their GRM map, in each case they reject the possibility of a real physical association, and regard the rotation measure features as large scale features of galactic dynamics. This allows them to construct large scale models of the galactic magnetic field, and compare the rotation measures predicted from these models with the real rotation measures and with the GRMs, in the galactic plane.

Simard-Normandin and Kronberg (1980) consider a set of toroidal models and a set of spiral models. The parameters of the models are chosen from reasonable values, and not selected to give a good match to the rotation measures. Four toroidal models are considered: (1) a toroidal model with no reversals; (2) a toroidal model with one reversal within the orbit of the Sun; (3) a toroidal model with the same reversal, and with a null field in the outer parts of the Galaxy; (4) and lastly a toroidal model with two reversals. The second of these proves to have interesting areas of similarity with both the GRMs and the rotation measures. There are several problem areas for this model.

Simard-Normandin and Kronberg (1980) then look at a set of four spiral field models each of a four armed spiral with pitch angle 14°. These are: (i) a spiral model with no reversals; (ii) a bisymmetric model with sharp reversals; (iii) a spiral model with four reversals and (iv) a bisymmetric model with modulated reversals. The last of these proves to be in better agreement with the GRMs than other models, although it predicts a sharp negative feature at $\langle = 320^{\circ}$ which does not appear. It is not a notably good model of the actual rotation measures.

Simard-Normandin and Kronberg (1980) recommend the bisymmetric spiral model with modulated reversals. There are several unpalatable aspects to this recommendation. The first is that the major features of the GRM map which the models need to reproduce are all three associated with known local anomalies in the local magnetic field, a point that has been made strongly by Vallee

(1983a, 1984). A second problem is that the models have not been evaluated quantitatively. It is important to carry out a statistical evaluation of the merit of different models, which can be done using the F-test and the original rotation measures, not the smoothed ones. The smoothed 'rotation measures' are statistically more complicated, because the distribution of errors can no longer be assumed to be Gaussian.

A bisymmetric spiral magnetic field model is also used by Sofue and Fujimoto (1983). They select rotation measure data from the catalogue of Tabara and Inoue (1980). They use a Gaussian beam to smooth the observed rotation measures, and take the values of this smoothed distribution in the plane of the Galaxy, for comparison with the predictions of the bisymmetric spiral model. The comparison seems to show that the model is interesting but has serious faults. No quantitative evaluation is offered, nor any comparison with the performance of any other model, however Sofue and Fujimoto (1983) conclude that the galactic magnetic field follows this pattern.

6.3. Local Longitudinal Magnetic Field Models

It is relatively simple to determine the direction of the magnetic field in the local area, from whatever kind of observations are being used, although it is unfortunate that such a wide variety of answers are found. This is a valid and basic model, because finding a first order approximation to the field is a practical and

interesting analysis. One or more parameters are calculated, and some estimate of errors and probabilities can be provided.

The field configurations which are spiral or toroidal on a galactic scale are both longitudinal locally, and difficult to distinguish from each other, particularly since the density wave theory, involving a toroidal field, allows for some perturbation in the direction of the field.

Thomson and Nelson (1980) find using pulsar rotation measures that a good longitudinal model of the magnetic field is in the direction of $\ell=74^\circ$, and has a reversal at a distance of 170 pc. Thomson and Nelson (1981) find very similar results using extragalactic rotation measures from the catalogue of Tabara and Inoue (1980). Inoue and Tabara (1981) criticise Thomson and Nelson (1980, 1981) because they find no reversal in the rotation measures, looking at southern galactic latitudes. Inoue and Tabara (1981) feel that it is important to avoid anomalous areas of sky in making the analysis of the data. However, this can be risky in itself because nearly anything could be confirmed, or rejected, in some area of the rotation measure sky. The reversal of rotation measures in the area of the North Polar Spur cannot be avoided, although it seems that this feature is less marked in the catalogue of Tabara and Inoue (1980), considering the plot of the distribution of rotation measures presented by Sofue and Fujimoto (1983). However, if there is a real reversal of rotation measures associated with a local feature, then there is either a real small scale reversal or doubling back, or a real large scale reversal. Thomson and Nelson

(1980), analysing pulsar rotation measures, find that a longitudinal model with a large scale reversal is a better model than a longitudinal model on its own. They exclude six sources associated with the North Polar Spur.

Vallee (1983a) investigates values of the magnetic field in nearby spiral arms assuming a longitudinal field with reversals directed toward ℓ =90°. Vallee (1984) and Broten et al (1985) lay great emphasis on the role of the galactic loops, calling them interstellar magnetic bubbles. A straight forward model of a small region of sky is used by Vallee (1983a) to estimate the value of the magnetic field in the Perseus arm at 1.5 μ G.

Vallee and Bignell (1983) model the effect of the shell of the Gum Nebula on rotation measures in that region but no all sky modelling has been done which includes a model of bubbles. Streitmatter et al (1985) consider that the sun lies in a much larger superbubble around the stars of Gould's belt and promise a model of the effect of this on rotation measures. This superbubble is much larger than the size of the North Polar Spur, and is on a scale of kiloparsecs. The modelling of the solar neighbourhood promises to be interesting.

6.4. Implications of Spherical Harmonic Analysis

The coefficients of individual spherical harmonic functions from the analysis of rotation measures can be used to investigate

features of the rotation measure sky. The harmonic of degree and order 1 are

$$\sin\theta\cos\phi$$
 and $\sin\theta\sin\phi$

The coefficients of these two functions can be combined to give a value for the first order approximation of the direction of the magnetic field $f(\phi)$ (strictly, the direction of highest rotation measures, the maximum of the line of sight component):

 $f(\phi) = a \cos \phi + b \sin \phi$.

Then, at constant latitude, maximum and minimum rotation measures are found at the zeros of $\frac{df}{d\phi}$. If neither a nor b are zero, the turning points occur when $\tan \phi = \frac{b}{2}$.

In solution 126, a=7.3 and b=-44.8, so that turning points of $f(\phi)$ are at $\phi = \langle =99.3^{\circ}$ and 279.3°. The values of $f(\phi)$ at these positions are -45.4 and +45.4 respectively. This value for the direction of the longitudinal field is extremely close to the values found by Inoue and Tabara (1981) who found that the regular field runs in the direction $\langle =100^{\circ}\pm10^{\circ}$, and by Thomson and Nelson (1980) who found from pulsar rotation measures that the best longitudinal field direction with reversals is $\langle =107^{\circ}\pm7^{\circ}$. A maximum can be seen in this region in all the spherical harmonic models of the rotation measure data set.

Inspection of harmonics of the second degree shows that high values in the plane will also be represented by the second degree harmonics of second order :

$$\frac{1}{6}$$
 $3\sin^2\theta\cos 2\phi$ and $\frac{1}{6}$ $\sqrt{3}\sin^2\theta\cos 2\phi$

These are harmonics that simply have two maxima and minima around the plane of the sphere, and are zero at the poles. Using the same method, they are found to have maximum and minimum values of +14.5 at $\langle =30.1^{\circ}$ and 210.1° , and -14.5 at $\ell = 120.1^{\circ}$ and 300.1° . Clearly these harmonics are being used to model the maximum and minimum values either side of the reversal in rotation measures shown by the model at $\ell = 70^{\circ}$, $b = 10^{\circ}$. Other coefficients can be found taking up similar positions, in order to build up the model predictions of -300 rad m^{-2} at $\ell = 100^{\circ}$ and $+300 \text{ rad m}^{-2}$ at $\ell = 40^{\circ}$, in the plane of the Galaxy. The spherical harmonic model, constrained as it was to use low order harmonics, could hardly model a reversal any more sharply than this. The feature deserves further attention.

A reversal in the field between the Sun and the centre of the Galaxy has been found to be a good model when incorporated into bisymmetric spiral and toroidal fields by Simard-Normandin and Kronberg (1980). Sofue and Fujimoto (1983) also include a reversal in their bisymmetric field model. Thomson and Nelson (1981) find that a longitudinal field directed towards $l=74^{\circ}\pm10^{\circ}$, with a reversal of that field, inward of the Sun, at a distance of 170 pc, is a better model than a simple longitudinal model - that is, better than the one already mentioned, in the direction of $l=107^{\circ}\pm7^{\circ}$. No reversal has been found by Inoue and Tabara (1981). The catalogue of Tabara and Inoue (1980) does not contain so many positive rotation measures in the region around $l=30^{\circ}$, $b=10^{\circ}$, nor such large ones. The determination of rotation measures is more strictly controlled by Tabara and Inoue (1980) so that it is possible that the several large positive rotation measures found by

Simard-Normandin et al (1981) failed to make the higher grades of Tabara and Inoue (1980). However it is also possible that extra measurements were available to Simard-Normandin et al (1981). Clarification of this point would be very useful.

Pulsar rotation measures tend to confirm the high positive values found for the extragalactic sources in this region (Manchester and Taylor, 1977). However, Heiles et al (1980) estimate a value of 26 rad m⁻² with an uncertainty of 50% for the Faraday rotation inside the emitting region of the North Polar Spur. They also estimate that the electron density within the shell is a greatly enhanced value of 0.4 cm⁻³. They find the magnetic field to be low, about 1.5μ G, along the line of sight. The distance to the shell is also of the order of 70 pc (Berkhuijsen, 1973), so it is impossible that the large rotation measures come from this region, even if the magnetic field or electron density are anomalous. This suggests that there is a large scale reversal of the field including the North Polar Spur, to bring about positive rotation measures as high as two or three hundred.

It is easy to estimate the rotation measure that would be expected in this region if a longitudinal field pointing away from the Sun in the direction of ℓ =100°, dominated the sky. The North Polar Spur region is found at approximately ℓ =40°. The line of sight makes an angle of 60° with the line of sight, so rotation measures of -300cos60° are expected, which is -150 rad m⁻². To find positive rotation measures here would indicate a reversal. To find large positive rotation measures is extraordinary. From this it

appears certain that there is a large-scale reversal, as well as large or small scale anomalies in this region.

Estimates of a longitudinal field without a reversal being in the direction of ℓ =100° may be due to the reversed area 'pushing' the maximum away from ℓ =90° or ℓ =80°. However, there may really be a field directed away from us toward ℓ =100°, as well as the reversed section running toward us from ℓ =74°. Detailed modelling of a number of possibilities, with statistical tests of the significance of the models, would be of great value in considering the configuration of the field in the local area.

6.5. New Models of Electron Density

Lyne et al (1985) describe a model of the electron density which includes a constant term of 0.025 cm^{-3} for all pulsars, and a term of 0.015 cm^{-3} due to ionized hydrogen in the disc of the Galaxy, with a scale height of 70 pc. This suggests an improvement which can be made in modelling the rotation measures of extragalactic sources, because most models regard the electron density as constant within the Galaxy. It is important to consider local variations in the electron density, and how these affect the rotation measure trends. Models will be presented below which show possible systematic variations in electron density, and one of these is considered in some detail.

The models presuppose that the galactic magnetic field is
longitudinal in the region within 2 kpc of the Sun and do not include effects outside this region. This region was chosen because variations in the interstellar medium with a small scale height are only probed to greater distances by very low latitude sources. Sources with $|b| < 2^{\circ}$ will have a line of sight within 70 pc of the mid-plane for over 2 kpc. Sources with $|b| < 5^{\circ}$ have a line of sight within 200 pc of the plane for over 2 kpc.

The direction of the longitudinal field is allowed to vary within the plane of the Galaxy. Strictly this is not necessary, the field direction could be fixed and only the electron density pattern allowed to vary its position. However, it is clearer to use galactic co-ordinates and consider various magnetic field directions. This allows investigation of the difference between field directions of ℓ =50° and ℓ =110° and the directions in between for the local field.

Electron density patterns which will commonly occur are enhanced density in a spiral arm region, local hot-spots, and ring or shell-structures of high ionization. In the first model, type (i), the electron density is highest along a straight line passing through the Sun. Away from this line it falls off exponentially at a rate described by a parameter q. This is a simplistic model of high ionization in a spiral arm, and improvements could be made by offsetting the Sun from the centre of the arm and away from the mid-plane by a variable amount.

The second model, type (ii), describes a hot-spot of electron

density, which decreases exponentially at a specified rate with increasing distance from its centre. In the third model, type (iii), electron density is greatest on a given sphere, and decreases in the same way with increasing distance from the sphere. Exponential decay is used to simulate a real situation while allowing quite simple mathematical analysis. In the case of a hot-spot, for example, the electron density might reach a maximum at the centre of $N_c \text{ cm}^{-3}$. The electron density at distance d from the centre would be $N_c e^{-(\frac{d}{2})^2} \text{ cm}^{-3}$. At the distance q from the centre, the electron density will have dropped to 37% of its maximum value.

All three models will be described restricted to the plane of the Galaxy. This is another simplification for preliminary analysis, not appropriate for sources in the plane of the Galaxy, but useful for sources quite near the plane. An extension to three dimensions is shown for the type (i) model.

A position in the plane of the Galaxy is described in polar ∞ -ordinates by (r, ℓ) , where r is its distance from the Sun and ℓ is its galactic latitude. The magnetic field comes from a variable galactic longitude, $\ell = \beta$. The spiral arm (in the case of the first model) runs in the direction of $\ell = \infty$. This is shown in Figure 6.1. In model type (i) the variable component of the electron density N_v decays with distance d from the spiral arm. So

$$N_{V}(r, \ell) = N_{C}e^{-(\frac{d}{2})^{2}}$$



Figure 6.1 The plane of the Galaxy showing a spiral arm running in direction α and a longitudinal magnetic field in direction β .

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Figure 6.2 The plane of the Galaxy showing a line of sight in relation to the spiral arm direction \propto .

Figure 6.2 shows a line of sight in relation to the spiral arm direction. Inspection shown that

$$d = r | \sin (\alpha - \ell) |$$

If the intensity of the magnetic field is B, then the component of the magnetic field directed along the line of sight at (r, l) is B cos $(\beta - l)$.

So

$$RM(\ell) = k \int_{C}^{2 k p c} N_{c} e^{-\binom{d}{2}^{2}} B \cos (\beta - \ell) dr$$
$$= k N_{c} B \cos (\beta - \ell) \int_{r=0}^{2 k p c} e^{-\binom{d}{2}^{2}} dr$$

This integral is the error function and is available in tables or, when using the computer, as a FORTRAN function in the NAG library of numerical algorithms.

The two dimensional analysis is very similar for models (ii) and (iii). In type (ii), suppose the centre of maximum electron density is at (r_c, ℓ_c) and the same parameter q describes the rate of decay. Then

$$N_{v}(r, \ell) = N_{c}e^{-(\frac{d}{2})^{2}}$$

as before, where now

$$d^2 = r_1 + r^2 - 2r r_2 \cos(\ell_1 - \ell_1)$$

by the cosine formula, in Figure 6.3.

In model type (iii) suppose the centre of the circle is at (r_c, ℓ_c) , and the radius is R, with parameter q as before. Then

$$N_{v}(\dot{r}, \ell) = N_{c}e^{-(\frac{d-R}{q})^{2}}$$

where d^2 is as in type (ii) above.

The type (i) model can easily be extended to include galactic latitude b, if variations out to a scale height of say 200 pc are to be considered. The line of sight component of the magnetic field becomes $B\cos(\beta - \ell)\cos(b)$ in the three dimensional case. Then consider the point (s, ℓ , b). Figure 6.4 shows a slice perpendicular to the galactic plane which includes (s, ℓ , b) and the Sun. If the perpendicular distance of (s, ℓ , b) from the centre of the spiral arm is t, then clearly

 $t^2 = s^2 sin^2 b + d^2$

where d is the distance of (s $\cos(b)$, ℓ , 0) from the centre of the spiral arm.



Figure 6.3 The plane of the Galaxy showing line of sight in direction l, in relation to electron density hotspot at (r_c, l_c) in polar co-ordinates.



Figure 6.4 A slice through the plane of the Galaxy including (s, ℓ, b) and the Sun.





Then

$$d^2=s^2\cos^2(b) \sin^2(\alpha-\ell)$$

as before substituting $r = s \cos(b)$. So

$$t^2 = s^2 \cos^2(b) \sin^2(\alpha - \ell) + s^2 \sin^2(b)$$

If the electron density in the spiral arm is to be cylindrically symmetrical replace

$$N_v(r, \ell) = N_c e^{-(\frac{d}{4})^2}$$

by

$$N_v(s, \ell, b) = N_c e^{-(\frac{\ell}{2})^2}$$

If it is to have an ellipsoidal cross section replace the single parameter q by two parameters p and q. Then replace

$$(\frac{L}{2})^{2}$$

by

$$s^{2}\left(\frac{\cos^{2}(b) \sin^{2}(\alpha-\ell)}{q^{2}} + \frac{\sin^{2}(b)}{p^{2}}\right)$$

Parameter p describes the decay of electron density out of the plane of the Galaxy. Then

$$RM(l, b) = k B \cos(b)\cos(\beta - l)N_{c} \int_{L} e^{-(\frac{t}{q})^{2}} ds$$

L is the line of sight to the edge of the region being considered, and ds is an element of the line of sight. The limits of integration can be determined by a sharp cut off to the slab of the Galaxy as in Figure 6.5. If $0 < |b| < \tan^{-1}(0.1)$, the limits of integration are 0 to $2000/\cos(b)$ pc. If $|b| > \tan^{-1}(0.1)$, the limits of integration are 0 to $200/\sin(b)$ pc.

The two-dimensional version of the type (i) model was used to predict the variation of rotation measures near the plane of the Galaxy. Five relative directions of the spiral arm $(\ell = \alpha)$ and the longitudinal magnetic field $(\ell = \alpha)$ were considered. The direction α of the spiral arm was kept at 75°, and the width of the arm was described by parameter q as 400 pc (following Elmegreen, 1985), and the direction β of the field was made to be 0°, 45°, 75°, 90° and 110°. The strength of the magnetic field used was 3 μ G, and the maximum electron density was 0.015 cm⁻³. The resulting variation of rotation measure with galactic longitude is shown in Figures 6.6 to 6.10.

The variation of rotation measure along the galactic plane due to a longitudinal magnetic field in constant electron density can be .

RM = 2000 k Ne H cos $(\beta - \ell)$

where l is the galactic longitude, and other symbols have their usual meaning. This describes the rotation measure due to the effect of the local region, within 2 kpc, and is applicable to

sources near the galactic plane. The magnetic field strength used was $3\mu G$, and the electron density was 0.025 cm^{-3} . This variation is shown as a function of galactic longitude in Figure 6.11, for comparison with the model of variable electron density.

The Figures 6.6 to 6.11 show that, with these values of electron density, the rotation measures are dominated by the contribution from the 'constant' background electron density in the interstellar medium. Varying electron density in the local spiral arm segment would show up as a small patch of high rotation measures. This might not be detected by the particular sources sampled, especially considering the probable extragalactic contribution to individual sources. If detected it would be hard to distinguish from other anomalous regions in the local region of the Galaxy.

6.6. Conclusions

Modelling can give useful hints about the behaviour of the galactic magnetic field. Generally, strong conclusions about the shape of the field await further measurements and the use of strict statistical inference from models.

Models of small-scale variations in the local regions are important, because of the clear appearance of local anomalous features in the rotation measure sky. Models were proposed for investigation, which considered the models of electron density used by workers with pulsar measurements. The model of varying electron

density in the local spiral arm proved unhelpful, showing that the approximation of using constant electron density in models is a practical one.

Many more complex models can be devised and considered in this way. The conclusions that can be drawn from such models are limited by the accuracy of observed values such as rotation measures. The area of sky near the North Polar Spur deserves particular attention in future work. In particular it would be useful to know whether this is a local anomaly or a large-scale feature.



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Galactic longitude





Galactic longitude





Galactic longitude





Galactic longitude

Figure 6.9 Rotation measure model (i) with $\alpha = 75^{\circ}$, $\beta = 90^{\circ}$.



Galactic longitude





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Galactic longitude



CHAPTER 7

CONCLUSIONS AND FUTURE DIRECTIONS

7.1. Spherical Harmonic Analysis

In Chapters 2 and 3 two methods of spherical harmonic analysis were described and investigated. These were a method developed by Fougere (1963) and an inverse theory method developed by Gubbins (1983). Both of these find a solution without detailed inspection of the numerical characteristics of a particular analysis. The second method, however, allowed consideration and comparison of a large number of solutions. Use of the two methods of analysis in Chapters 2 to 5 leads to the following conclusions:-

(i) Fougere's method is based on the statistical method of forward selection, which is an important method of inverse theory. However, it has proved to be of limited value in the analysis of extragalactic rotation measures and interstellar polarization.

(ii) The damping (or ridge regression) method developed by Gubbins (1983) has been useful in the analysis of extragalactic rotation measures, but good solutions have not been found from interstellar polarization data. In general, damped weighted least squares solutions are improvements over the least squares solutions. Inspection of the individual analyses using other methods of inverse theory would be profitable.

(iii) Spherical harmonic analysis is the appropriate method of modelling variations over a spherical surface. However, this can be time consuming, and it can be difficult to find a good representation, as shown in Chapters 2 to 5. Therefore other methods of smoothing data on the surface of a sphere are worth considering.

(iv) Future use of spherical harmonic analysis of astrophysical data should investigate the further use of inverse methods, particularly as used in the work of Jackson (1972).

7.2. Rotation Measures

The problems of finding rotation measures were described in Chapter 4. Spherical harmonic analysis was carried out on extragalactic Faraday rotation measures from the catalogue of Simard-Normandin et al (1981). This work suggests the following conclusions:-

(i) It is important to investigate further the accuracy of values and errors of rotation measures of extragalactic sources, following the work of Wright (1973) and Thomson (1981). Ideally, polarization measurements should be made specifically to find the rotation measure, as done for example by Rudnick et al (1983).

(ii) The results of the spherical harmonic analysis in chapter 4 indicate the large-scale trends in extragalactic rotation measures over the sky. They should be considered however in conjunction

with plots of the rotation measures, as some features may be spurious. The solutions indicate the galactic contribution to the rotation measure, but for specific sources a better indication will be obtained from nearby sources and inspection of the rotation measure in different parts of the source, if this is possible.

(iii) The analysis shows an area of high positive rotation measures, up to 300 rad m⁻², at galactic longitude $10^{\circ} < \ell < 70^{\circ}$, and slightly above the plane of the Galaxy. Values switch rapidly to negative of similar magnitude, at $70^{\circ} < \ell < 160^{\circ}$, in the plane of the Galaxy. Subject to further clarification of rotation measures, this indicates a reversal of the galactic magnetic field. The reversal is probably close to the solar neighbourhood (Thomson and Nelson, 1980). Positive rotation measures, up to 200 rad m⁻² are shown in the plane of the Galaxy, at longitudes $170^{\circ} < \ell < 330^{\circ}$. The root mean square error of the spherical harmonic model is 27 rad m⁻².

(iv) An important set of data that should be investigated in the near future is the latest complete set of pulsar rotation measures. Many have been measured recently (Lyne, 1984). They are better probes of the interstellar medium than extragalactic rotation measures, but require more complex analysis because they are at varying distances in the medium. Modelling and statistical analysis as carried out by Thomson and Nelson (1980) and Thomson (1981) is an example of a promising approach.

7.3. Interstellar Polarization

Spherical harmonic analysis was carried out on measurements of interstellar polarization from the catalogue of Axon and Ellis (1976). Stars at distances up to 135 pc were considered, however, results were poor. No satisfactory solutions can be presented, and conclusions are as follows:-

(i) Spherical harmonic analysis of interstellar polarization proved lengthy, and the results were not useful. Both Fougere's method and the damping method failed to produce solutions. Further strategies of inverse theory could be applied to the analysis.

(ii) Nearby shells of stars, out to 135 pc, show very disordered polarization which disrupted the analysis. The interstellar medium is disordered on a very small scale, locally.

Spherical harmonic analysis is only appropriate to the (iii) analysis of shells of stars around the sum, and the comparison of different shells has some numerical and statistical pitfalls. Interstellar polarization measurements may contain as much information about the distribution of interstellar dust as about the interstellar magnetic field, and the E-vector maps of Axon and Ellis (1976) already form valuable and accessible indicators of Therefore it is trends in the polarization measurements. questionable whether further attempts at spherical harmonic analysis would be worth while.

7.4. Models

In Chapter 6 modelling of the galactic magnetic field was discussed. Some implications of spherical harmonic analysis of extragalactic rotation measures were presented. Some models of systematic variation of electron density were suggested, with methods of evaluating their effect on extragalactic rotation measures. One of these was investigated briefly. This work leads to the following conclusions:-

(i) It is not possible to reach firm conclusions about the form of the galactic magnetic field at this stage. Models presented so far by workers in the subject are not conclusive, but have a role which is rather more experimental and inspirational. The application of appropriate statistical tests and acknowledgement of limitations and assumptions are vital in clarifying what conclusions can be drawn from a particular modelling exercise.

(ii) The North Polar Spur, which appears in the radio continuum emission, is a region of particular interest. A sharp large-scale reversal of the extra-galactic rotation measures is found in this region. The Spur is thought to be part of a supernova remnant, and modelling is recommended as an appropriate method of exploring the anomalies which appear in this area of observations of X-rays, radio continuum, Faraday rotation, and starlight.

(iii) First order spherical harmonics show a maximum in Faraday rotation of extragalactic sources in the plane of the Galaxy at

longitude ℓ =99°. The accuracy of this figure is not estimated, it is very rough. The value is undoubtedly affected by the nearby field reversal associated with the North Polar Spur.

(iv) A preliminary investigation of a model of variable electron density in the local spiral arm showed that this would have little effect on the rotation measure sky. Many other general and detailed models of variable electron density in the local neighbourhood deserve attention.

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APPENDIX A

The following numbers are the coefficients of Schmidt normalized spherical harmonics, up to degree and order 12, which appear in the solution of analysis No. 0126. The first three figures are significant, at most. The coefficients appear in the order a_0^o , a_1^o , a_1^i , b_1^i , a_2^o , a_2^i , b_2^i , and so on (to b_{12}^{12}).

J.66845130+01	-0.15563370+01	0.72953920+01	-0.44754450+02	-0.12110730+02
0.52973040+01	0.33424390+02	0.24400970+02	0.43835830+02	-0.12157910+01
-0.02406310+01	0.45101550+02	0.44295070+01	0.29650460+02	-0.16174680+02
0.31403200+02	0.2152159D+C2	-0.17174740+02	-0.38823270+02	-0.61632650+01
-0.23550570+02	-0.28302700+02	0.13728320+02	-0.40453560+02	-0.57603420+01
-0.60595280+01	0.72766100+01	-0.40923850+02	-0.17156170+02	-0.34431390+02
6.33035790+01	-0.33656510+02	-0.22553790+02	0.12459150+01	-0.33965740+02
-0.28496770+02	-0.28169920+02	0.13621460+02	0.22730680+02	0.55559950+01
0.10646110+02	3.23753560+02	-0.11386250+02	0.29409710+02	-0.39227830+01
-0.20541950+02	-0.61666550+01	0.37911960+02	-0.71535860+01	0.72407210+01
-0.1640500D+02	0.31043470+02	0.16482410+02	0.22164780+02	0.30844330+01
0.12487780+02	0.22331080+02	-0.36443230-92	0.43711420+01	-0.40238950+01
0.11529840+02	-0.39275770+01	-0.21664490+01	0.20243090402	0.19553870+02
-0.90858880.01	-0,12398070+02	0.50739569+01	-0.30119650+01	-0.23036330+02
0.31385670+00	-0.10719140+02	0.17419910+02	0.21483330+02	0.10003810+02
-0.83887880+01	-0.17193140+01	-0.45191380+01	-0.12181810+02	-0.14363610+02
-0.30842000+02	-0.72465940+01	0.10229560+02	-0.10779260+02	-0.93990210+01
-0.82058080+01	-0.14372510+02	-0.35682070+01	-0.97196780+01	0.47257580+01
-0.13197840+02	0.63325640+01	-0.20163640+02	0.53586820+01	0.22529500+01
0.15970000+02	-0.36956530+01	0.15532820+01	0.29040030+02	-0.56450080+01
-0.52468420+01	0.45106700+01	0.10395480+01	-0.55031260+01	0.78478070+00
0.78943750+01	-0.22483320+01	0.69646279+01	-0.13659960+02	-0.93776380+01
-0.44291900+01	0.35888500+01	0.28457820+00	0.12633180+01	0.47883530+01
-0.53109940+01	0.26405020+01	-0.499C597D+01	0.74110180+01	-0.58919460+01
0.69445480+01	0.33290230+01	-0.15876570+01	0.27053400+01	0.16973260+01
0.14716250+01	0.49673480+01	-0.25373060+01	0.20334830+01	0.43498840+00
0.70957560+01	-0.20723970+00	0.53259800+01	-0.28085500+01	0.13560430+00
-0.26087720+01	0.47750470+01	-0.11506810+01	-0.53624390+01	+0-48739170+01
0.27996770+01	-0.49427970+01	-ü.4086548D+01	0.10525900+01	0.49443110+00
-0.13979160+01	0.17308300+01	-0.57326150+00	0.62036640+00	-0.29313100+01
0.10632550+01	0.72850320+00	0.35169680+01	0.69444320-01	-0.37455070+00
-0.26313720+01	0.23934650+01	0.55665000-01	0.10447700+01	0.12115260+01
0.19207350+01	0.26768950+01	-0.37532350+00	0.53053810+01	0.10163260+01
0.18367950+01	0.31437290-01	0.14367290+01	0.44213260+00	

APPENDIX B

MODELLING THE GALACTIC CONTRIBUTION TO THE FARADAY ROTATION OF RADIATION FROM EXTRA-GALACTIC SOURCES

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Faraday rotation occurs in the Galaxy according to the formula:

$$RM = k \int_{L} n_{e} H_{||} d\ell$$

.

where: RM is the rotation measure n density of electrons H_{II}^{c} component of the magnetic field parallel to line of sight L distance travelled through interstellar medium k constant depending on units.

We have taken a collection of rotation measures of 552 extra-galactic sources, compiled by Simard-Normandin, Kronberg and Button (Preprint, 1980) and modelled them over the sphere using spherical harmonics. We hope in this way to model the dependence on galactic co-ordinates which will be due to the structure of our Galaxy (position of free electrons and direction and strength of the magnetic field).

The spherical harmonic functions are the solutions of Laplace's equation expressed in spherical polar co-ordinates and are the fitting functions for data on the surface of a sphere. We used a least-squares fitting procedure, starting with first-order harmonics, and extending to second, third, fourth and fifth. (The mean of the data, as a constant 'function', gives a zero-order model.)

We found that the first, second and fourth-order models were statistically most significant, and illustrate the first and second-order models here. The first-order model (Fig. 1) is a significant improvement over the mean model at 0.001 level, using the F-test. The second-order model (Fig. 2) is a significant improvement over the first-order model at 0.001 level using the F-test.

These are preliminary results, more detailed investigations are still in progress. The first-order harmonics support a simple model of a linear field in the solar neighbourhood, parallel to the plane of the 249

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Calaxy, and having a direction from $\ell^{II} = 112^{\circ}$ to $\ell^{II} = 292^{\circ}$. Higherorder models indicate large-scale deviations from this model.



Figure 1: Rotation Measures, First-Order Harmonic Model



Figure 2: Rotation Measures, Second-Order Harmonic Model