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Lange-Hegermann, M

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THOMAS DECOMPOSITION AND NONLINEAR CONTROL SYSTEMS

MARKUS LANGE-HEGERMANN AND DANIEL ROBERTZ

ABSTRACT. This paper applies the Thomas decomposition technique to nonlinear control systems, in particular to the study of the dependence of the system behavior on parameters. Thomas' algorithm is a symbolic method which splits a given system of nonlinear partial differential equations into a finite family of so-called simple systems which are formally integrable and define a partition of the solution set of the original differential system. Different simple systems of a Thomas decomposition describe different structural behavior of the control system in general. The paper gives an introduction to the Thomas decomposition method and shows how notions such as invertibility, observability and flat outputs can be studied. A Maple implementation of Thomas' algorithm is used to illustrate the techniques on explicit examples.

1. INTRODUCTION

This paper gives an introduction to the Thomas decomposition method and presents first steps in applying it to the structural study of nonlinear control systems. It extends and refines our earlier work [28].

Symbolic computation allows to study many structural aspects of control systems, e.g., controllability, observability, input-output behavior, etc. In contrast to a numerical treatment, the dependence of the results on parameters occurring in the system is accessible to symbolic methods.

An algebraic approach for treating nonlinear control systems has been developed during the last decades, e.g., by M. Fliess and coworkers, J.-F. Pommaret and others, cf., e.g., [13], [20], [37], and the references therein. In particular, the notion of flatness has been studied extensively and has been applied to many interesting control problems (cf., e.g., [14], [2], [31]). The approach of Diop [10, 11] builds on the characteristic set method (cf. [24], [47]). The Rosenfeld-Gröbner algorithm (cf. [7]) can be used to perform the relevant computations effectively; implementations of related techniques are available, e.g., as Maple packages `DifferentialAlgebra` (by F. Boulier and E. S. Cheb-Terrab), formerly `diffalg` (by F. Boulier and E. Hubert), and `RegularChains` (by F. Lemaire, M. Moreno Maza, and Y. Xie) [29]; cf. also [46] for alternative approaches. As an example of an application of the Rosenfeld-Gröbner algorithm we refer to [34], where it is demonstrated how to compute a block feedforward form and a generalized controller form for a nonlinear control system.

So far the dependence of nonlinear control systems on parameters has not been studied by a rigorous method such as Thomas decomposition. This paper demonstrates how the Thomas decomposition method can be applied in this context. In particular, Thomas' algorithm can detect certain structural properties of control systems by performing elimination and it can separate singular cases of behavior in control systems from the generic case due to splitting into disjoint solution sets. We also consider the Thomas decomposition method as a preprocessing technique for the study of a linearization of a nonlinear system (cf. [42, Sect. 5.5]), an aspect that we do not pursue here.

Dependence of control systems on parameters has been examined, in particular, by J.-F. Pommaret and A. Quadrat in [38], [37]. For linear systems, stratifications of the space of parameter values have been studied using Gröbner bases in [30].

In the 1930s the American mathematician J. M. Thomas designed an algorithm which decomposes a polynomially nonlinear system of partial differential equations into so-called simple systems. The algorithm uses, in contrast to the characteristic set method, inequations to provide a disjoint decomposition of the solution set (cf. [44]). It precedes work by E. R. Kolchin [24] and A. Seidenberg [43], who followed J. F. Ritt [40]. Recently a new algorithmic approach to the Thomas decomposition method has been developed (cf. [17, 4, 41]), building also on ideas of the French mathematicians C. Riquier [39] and M. Janet [23]. Implementations as Maple packages of the algebraic and differential parts of Thomas' algorithm are available due to work by T. Bächler and M. Lange-Hegermann [5]. The implementation of the differential part is available in the Computer Physics Communications library [19] and has also been incorporated into Maple's standard library since Maple 2018. An earlier implementation of the algebraic part was given by D. Wang [45].

Section 2 introduces the Thomas decomposition method for algebraic and differential systems and discusses the main properties of its output. The algorithm for the differential case builds on the algebraic part. Section 3 explains how the Thomas decomposition technique can be used to solve elimination problems that occur in our study of nonlinear control systems. Finally, Section 4 addresses concepts of nonlinear control theory, such as invertibility, observability, and flat outputs, possibly depending on parameters of the control system, and gives examples using a Maple implementation of Thomas' algorithm.

2. Thomas decomposition

This section gives an introduction to the Thomas decomposition method for algebraic and differential systems. The case of differential systems, discussed in Subsection 2.2, builds on the case of algebraic systems which is dealt with in the first subsection. For more details on Thomas' algorithm, we refer to [4], [17], [35], [3], [26], and [41, Sect. 2.2].

2.1. Algebraic systems.

Let K be a field of characteristic zero and $R = K[x_1, \ldots, x_n]$ the polynomial algebra with indeterminates $x_1, \ldots, x_n$ over K. We denote by $\overline{K}$ an algebraic closure of K.

Definition 2.1. An *algebraic system* S, defined over R, is given by finitely many equations and inequations

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0, \tag{1}$$

where $p_1, \ldots, p_s, q_1, \ldots, q_t \in R$ and $s, t \in \mathbb{Z}_{\geq 0}$. The *solution set* of S in $\overline{K}^n$ is

$$\mathrm{Sol}_{\overline{K}}(S) := \{\, a \in \overline{K}^n \mid p_i(a) = 0 \text{ and } q_j(a) \neq 0 \text{ for all } 1 \leq i \leq s,\ 1 \leq j \leq t \,\}.$$

We fix a total ordering $>$ on the set $\{x_1, \ldots, x_n\}$ allowing us to consider every non-constant element p of R as a univariate polynomial in the greatest variable with respect to $>$ which occurs in p, with coefficients which are themselves univariate polynomials in lower ranked variables, etc. Without loss of generality we may assume that $x_1 > x_2 > \ldots > x_n$. The choice of $>$ corresponds to a choice of projections

$$\begin{array}{rcllcr}
\pi_1\colon \overline{K}^n & \longrightarrow & \overline{K}^{n-1}\colon & (a_1, a_2, \ldots, a_n) & \longmapsto & (a_2, a_3, a_4, \ldots, a_n), \\
\pi_2\colon \overline{K}^n & \longrightarrow & \overline{K}^{n-2}\colon & (a_1, a_2, \ldots, a_n) & \longmapsto & (a_3, a_4, \ldots, a_n), \\
& \vdots & & & \vdots & \\
\pi_{n-1}\colon \overline{K}^n & \longrightarrow & \overline{K}\colon & (a_1, a_2, \ldots, a_n) & \longmapsto & a_n.
\end{array}$$

Thus, the recursive representation of polynomials is motivated by considering each $\pi_{k-1}(\mathrm{Sol}_{\overline{K}}(S))$ as fibered over $\pi_k(\mathrm{Sol}_{\overline{K}}(S))$, for $k = 1, \ldots, n-1$, where $\pi_0 := \mathrm{id}_{\overline{K}^n}$ (cf. also [35]). The purpose of a Thomas decomposition of $\mathrm{Sol}_{\overline{K}}(S)$, to be defined below, is to clarify this fibration structure. The solution set $\mathrm{Sol}_{\overline{K}}(S)$ is partitioned into subsets $\mathrm{Sol}_{\overline{K}}(S_1), \ldots, \mathrm{Sol}_{\overline{K}}(S_r)$ in such a way that, for each $i = 1, \ldots, r$ and $k = 1, \ldots, n-1$, the fiber cardinality $|\pi_k^{-1}(\{ a \})|$ does not depend on the choice of $a \in \pi_k(\mathrm{Sol}_{\overline{K}}(S_i))$. In terms of the defining equations and inequations in (1), the fundamental obstructions to this uniform behavior are zeros of the leading coefficients of p_i or q_j and zeros of p_i or q_j of multiplicity greater than one.

Definition 2.2. Let $p \in R \setminus K$.

a) The greatest variable with respect to $>$ which occurs in p is referred to as the *leader* of p and is denoted by $\mathrm{ld}(p)$.
b) For $v = \mathrm{ld}(p)$ we denote by $\deg_v(p)$ the degree of p in v.
c) The coefficient of the highest power of $\mathrm{ld}(p)$ occurring in p is called the *initial* of p and is denoted by $\mathrm{init}(p)$.
d) The *discriminant* of p is defined as

$$\mathrm{disc}(p) := (-1)^{d(d-1)/2}\,\mathrm{res}\left(p, \frac{\partial p}{\partial\,\mathrm{ld}(p)}, \mathrm{ld}(p)\right) / \,\mathrm{init}(p), \qquad d = \deg_{\mathrm{ld}(p)}(p),$$

where $\mathrm{res}(p, q, v)$ is the resultant of p and q with respect to the variable v. (Note that $\mathrm{disc}(p)$ is a polynomial because $\mathrm{init}(p)$ divides $\mathrm{res}(p, \partial p/\partial\,\mathrm{ld}(p), \mathrm{ld}(p))$, since the Sylvester matrix, whose determinant is $\mathrm{res}(p, \partial p/\partial\,\mathrm{ld}(p), \mathrm{ld}(p))$, has a column all of whose entries are divisible by $\mathrm{init}(p)$.)

Both $\mathrm{init}(p)$ and $\mathrm{disc}(p)$ are elements of the polynomial algebra $K[x \mid x < \mathrm{ld}(p)]$. The zeros of a univariate polynomial which have multiplicity greater than one are the common zeros of the polynomial and its derivative. The solutions of $\mathrm{disc}(p) = 0$ in $\overline{K}^{n-k}$, where $\mathrm{ld}(p) = x_k$, are therefore those tuples $(a_{k+1}, a_{k+2}, \ldots, a_n)$ for which the substitution $x_{k+1} = a_{k+1}$, $x_{k+2} = a_{k+2}$, …, $x_n = a_n$ in p results in a univariate polynomial with a zero of multiplicity greater than one.

Definition 2.3. An algebraic system S, defined over R, as in (1) is said to be *simple* (with respect to $>$) if the following three conditions hold.

a) For all $i = 1, \ldots, s$ and $j = 1, \ldots, t$ we have $p_i \notin K$ and $q_j \notin K$.
b) The leaders of the left hand sides of the equations and inequations in S are pairwise different, i.e., $|\{ \mathrm{ld}(p_1), \ldots, \mathrm{ld}(p_s), \mathrm{ld}(q_1), \ldots, \mathrm{ld}(q_t) \}| = s + t$.
c) For every $r \in \{ p_1, \ldots, p_s, q_1, \ldots, q_t \}$, if $\mathrm{ld}(r) = x_k$, then neither of the equations $\mathrm{init}(r) = 0$ and $\mathrm{disc}(r) = 0$ has a solution $(a_{k+1}, a_{k+2}, \ldots, a_n)$ in $\pi_k(\mathrm{Sol}_{\overline{K}}(S))$.

Subsets of non-constant polynomials in R with pairwise different leaders (i.e., satisfying a) and b)) are also referred to as triangular sets (cf., e.g., [1], [21], [46]).

Remark 2.4. A simple algebraic system S admits the following solution procedure, which also shows that its solution set is not empty. Let $S_{<k}$ be the subset of S consisting of the equations $p = 0$ and inequations $q \neq 0$ with $\mathrm{ld}(p) < x_k$ and $\mathrm{ld}(q) < x_k$. The fibration structure implied by c) ensures that, for every $k = 1, \ldots, n-1$, every solution $(a_{k+1}, a_{k+2}, \ldots, a_n)$ of $\pi_k(\mathrm{Sol}_{\overline{K}}(S)) = \pi_k(\mathrm{Sol}_{\overline{K}}(S_{<k}))$ can be extended to a solution $(a_k, a_{k+1}, \ldots, a_n)$ of $\pi_{k-1}(\mathrm{Sol}_{\overline{K}}(S))$. If S contains an equation $p = 0$ with leader x_k, then there exist exactly $\deg_{x_k}(p)$ such elements $a_k \in \overline{K}$ (because zeros with multiplicity greater than one are excluded by the non-vanishing discriminant). If S contains an inequation $q \neq 0$ with leader x_k, all $a_k \in \overline{K}$ except $\deg_{x_k}(q)$ elements define a tuple $(a_k, a_{k+1}, \ldots, a_n)$ as above. If no equation and no inequation in S has leader x_k, then $a_k \in \overline{K}$ can be chosen arbitrarily.

Definition 2.5. Let S be an algebraic system, defined over R. A *Thomas decomposition* of S (or of $\mathrm{Sol}_{\overline{K}}(S)$) with respect to $>$ is a collection of finitely many simple algebraic systems $S_1, \ldots, S_r$, defined over R, such that $\mathrm{Sol}_{\overline{K}}(S)$ is the disjoint union of the solution sets $\mathrm{Sol}_{\overline{K}}(S_1), \ldots, \mathrm{Sol}_{\overline{K}}(S_r)$.

We outline Thomas' algorithm for computing a Thomas decomposition of algebraic systems.

Remark 2.6. Given S as in (1) and a total ordering $>$ on $\{x_1, \ldots, x_n\}$, a Thomas decomposition of S with respect to $>$ can be constructed by combining Euclid's algorithm with a splitting strategy.

First of all, if S contains an equation $c = 0$ with $0 \neq c \in K$ or the inequation $0 \neq 0$, then S is discarded because it has no solutions. Moreover, from now on the equation $0 = 0$ and inequations $c \neq 0$ with $0 \neq c \in K$ are supposed to be removed from S.

An elementary step of the algorithm applies a pseudo-division to a pair p_1, p_2 of non-constant polynomials in R with the same leader x_k and $\deg_{x_k}(p_1) \geq \deg_{x_k}(p_2)$. The result is a pseudo-remainder

$$r = c_1 \cdot p_1 - c_2 \cdot p_2, \tag{2}$$

where c_1, $c_2 \in R$ and r is constant or has leader less than x_k or has leader x_k and $\deg_{x_k}(r) < \deg_{x_k}(p_1)$. Since the coefficients of p_1 and p_2 are polynomials in lower ranked variables, multiplication of p_1 by a non-constant polynomial c_1 may be necessary in general to perform the reduction in R (and not in its field of fractions). The choice of c_1 as a suitable power of $\mathrm{init}(p_2)$ always achieves this.

In order to turn S into a triangular set, the algorithm deals with three kinds of subsets of S of cardinality two. Firstly, each pair of equations $p_1 = 0$, $p_2 = 0$ in S with $\mathrm{ld}(p_1) = \mathrm{ld}(p_2)$ is replaced with the single equation $r = 0$, where r is the result of applying Euclid's algorithm to p_1 and p_2, considered as univariate polynomials in their leader, using the above pseudo-division. (If this computation was stable under substitution of values for lower ranked variables in p_1 and p_2, then r would be the greatest common divisor of the specialized polynomials.)

The solution set of the system is supposed not to change, when the equation $p_1 = 0$ is replaced with the equation $r = 0$ given by the pseudo-reduction (2). Therefore, we assume that the polynomial c_1, and hence $\mathrm{init}(p_2)$, does not vanish on the solution set of the system. In order to ensure this condition, a preparatory step splits the system into two, if necessary, and adds the inequation $\mathrm{init}(p_2) \neq 0$ to one of them and the equation $\mathrm{init}(p_2) = 0$ to the other. The algorithm then deals with both systems separately. These case distinctions also allow to arrange for the part of condition c) in Definition 2.3 which concerns initials.

Secondly, let $p = 0$, $q \neq 0$ be in S with $\operatorname{ld}(p) = \operatorname{ld}(q) = x_k$. If $\deg_{x_k}(p) \leq \deg_{x_k}(q)$, then $q \neq 0$ is replaced with $r \neq 0$, where r is the result of applying the pseudo-division (2) to q and p. Otherwise, Euclid's algorithm is applied to p and q, keeping track of the coefficients used for the reductions as in (2). Given the result r, the system is then split into two, adding the conditions $r \neq 0$ and $r = 0$, respectively. The inequation $q \neq 0$ is removed from the first new system, because $p = 0$ and $q \neq 0$ have no common solution in that case. The assumption $r = 0$ and the bookkeeping allows to divide p by the common factor of p and q (modulo left hand sides of equations with smaller leader). The left hand side of $p = 0$ is replaced with that quotient in the second new system. Not all of these cases need a closer inspection. For instance, if p divides q, then the solution set of S is empty and S is discarded.

Thirdly, for a pair $q_1 \neq 0$, $q_2 \neq 0$ in S with $\operatorname{ld}(q_1) = \operatorname{ld}(q_2)$, Euclid's algorithm is applied to q_1 and q_2 in the same way as above. Keeping track of the coefficients used in intermediate steps allows to determine the least common multiple m of q_1 and q_2, which again depends on distinguishing the cases whether the result of Euclid's algorithm vanishes or not. The pair $q_1 \neq 0$, $q_2 \neq 0$ is then replaced with $m \neq 0$.

The part of condition c) in Definition 2.3 regarding discriminants is taken care of by applying Euclid's algorithm as above to p and $\partial p / \partial \operatorname{ld}(p)$, where p is the left hand side of an equation or inequation. Bookkeeping allows to determine the square-free part of p, which depends again on case distinctions.

Expressions tend to grow very quickly when performing these reductions, so that an appropriate strategy is essential for dealing with non-trivial systems. Apart from dividing by the content (in K) of polynomials, in intermediate steps of Euclid's algorithm the coefficients should be reduced modulo equations in the system with lower ranked leaders. In practice, subresultant computations (cf., e.g., [32]) allow to diminish the growth of coefficients significantly.

Termination of the procedure sketched above depends on the organization of its steps. One possible strategy is to maintain an intermediate triangular set, reduce new equations and inequations modulo the equations in the triangular set, and select among these results the one with smallest leader and least degree, preferably an equation, for insertion into the triangular set. If the set already contains an equation or inequation with the same leader, then the pair is treated as discussed above. Since equations are replaced with equations of smaller degree and inequations are replaced with equations if possible or with the least common multiple of inequations, this strategy terminates after finitely many steps.

For more details on the algebraic part of Thomas' algorithm, we refer to [4], [3], and [41, Subsect. 2.2.1].

An implementation of Thomas' algorithm for algebraic systems has been developed by T. Bächler as Maple package `AlgebraicThomas` [5].

In what follows, variables are underlined to emphasize that they are leaders of polynomials with respect to the fixed total ordering $>$.

Example 2.7. Let us compute a Thomas decomposition of the algebraic system

$$x^2 + y^2 - 1 = 0$$

consisting of one equation, defined over $R = \mathbb{Q}[x, y]$, with respect to $x > y$. We set $p_1 := x^2 + y^2 - 1$. Then we have $\operatorname{ld}(p_1) = x$ and $\operatorname{init}(p_1) = 1$ and

$$\operatorname{disc}(p_1) = -4\, y^2 + 4.$$

We distinguish the cases whether or not $p_1 = 0$ has a solution which is also a zero of $\operatorname{disc}(p_1)$, or equivalently, of y^2-1. In other words, we replace the original algebraic system with two algebraic systems which are obtained by adding the inequation $y^2 - 1 \neq 0$ or the equation $y^2 - 1 = 0$. The first system is readily seen to be simple, whereas the second one is transformed into a simple system by taking the difference of the two equations and computing a square-free part. Clearly, the solution sets of the two resulting simple systems form a partition of the solution set of $p_1 = 0$. We obtain the Thomas decomposition

$$\boxed{\begin{array}{rcl} \underline{x}^2 + y^2 - 1 & = & 0 \\ \underline{y}^2 - 1 & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} \underline{x} & = & 0 \\ \underline{y}^2 - 1 & = & 0 \end{array}}$$

In this example, all points of $\operatorname{Sol}_{\overline{K}}(\{\, p_1 = 0 \,\})$ for which the projection π_1 onto the y-axis has fibers of an exceptional cardinality have real coordinates, and the significance of the above case distinction can be confirmed graphically.

As a further illustration let us augment the original system by the equation which expresses the coordinate t of the point of intersection of the line through the two points $(0, 1)$ and (x, y) on the circle with the x-axis (stereographic projection):

$$\left\{ \begin{array}{rcl} x^2 + y^2 - 1 & = & 0 \\ (1 - y)\, t - x & = & 0 \end{array} \right.$$

A Thomas decomposition with respect to $x > y > t$ is obtained as follows. We set $p_2 := x + t\, y - t$. Since $\operatorname{ld}(p_1) = \operatorname{ld}(p_2)$, we apply polynomial division:

$$p_1 - (x - t\, y + t)\, p_2 = (1 + t^2)\, \underline{y}^2 - 2\, t^2\, \underline{y} + t^2 - 1 = (\underline{y} - 1)\, ((1 + t^2)\, \underline{y} - t^2 + 1)\,.$$

Replacing p_1 with the remainder of this division does not alter the solution set of the algebraic system. It is convenient (but not necessary) to split the system into two systems according to the factorization of the remainder:

$$\left\{ \begin{array}{rcl} \underline{x} + t\, y - t & = & 0 \\ (1 + t^2)\, \underline{y} - t^2 + 1 & = & 0 \\ \underline{y} - 1 & \neq & 0 \end{array} \right. \qquad \left\{ \begin{array}{rcl} \underline{x} + t\, y - t & = & 0 \\ & & \\ \underline{y} - 1 & = & 0 \end{array} \right.$$

Another polynomial division reveals that the equation and the inequation with leader y in the first system have no common solutions. Therefore, the inequation can be omitted from that system. The initial of the equation has to be investigated. In fact, the assumption $1 + t^2 = 0$ leads to a contradiction. Finally, the equation with leader y can be used to eliminate y in the equation with leader x:

$$(1 + t^2)\, (\underline{x} + t\, y - t) - t\, ((1 + t^2)\, \underline{y} - t^2 + 1) = (1 + t^2)\, \underline{x} - 2\, t\,.$$

A similar simplification can be applied to the second system above. We obtain the Thomas decomposition

$$\boxed{\begin{array}{rcl} (1 + t^2)\, \underline{x} - 2\, t & = & 0 \\ (1 + t^2)\, \underline{y} - t^2 + 1 & = & 0 \\ \underline{t}^2 + 1 & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} \underline{x} & = & 0 \\ \underline{y} - 1 & = & 0 \\ & & \end{array}}$$

from which a rational parametrization of the circle can be read off.

Remark 2.8. A Thomas decomposition of an algebraic system is not uniquely determined. It depends on the chosen total ordering $>$, the order in which intermediate systems are dealt with and other choices, such as whether factorizations of left hand sides of equations are taken into account or not.

According to Hilbert's Nullstellensatz (cf., e.g., [12]), the solution sets V in $\overline{K}^n$ of systems of polynomial equations in $x_1, \ldots, x_n$, defined over R, are in one-to-one correspondence with their vanishing ideals in R

$$\mathcal{I}_R(V) := \{\, p \in R \mid p(a) = 0 \text{ for all } a \in V \,\},$$

and these are the radical ideals of R, i.e., the ideals I of R which equal their radicals

$$\sqrt{I} := \{\, p \in R \mid p^r \in I \text{ for some } r \in \mathbb{Z}_{\geq 0} \,\}.$$

The solution sets V can then be considered as the closed subsets of $\overline{K}^n$ with respect to the Zariski topology.

The fibration structure of a simple algebraic system S allows to deduce that the polynomials in R which vanish on $\mathrm{Sol}_{\overline{K}}(S)$ are precisely those polynomials in R whose pseudo-remainders modulo $p_1, \ldots, p_s$ are zero, where $p_1 = 0, \ldots, p_s = 0$ are the equations in S. If E is the ideal of R generated by $p_1, \ldots, p_s$ and q the product of all $\mathrm{init}(p_i)$, then these polynomials form the saturation ideal

$$E : q^\infty := \{\, p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \,\}.$$

In particular, simple algebraic systems admit an effective way to decide membership of a polynomial to the associated radical ideal (cf. also Proposition 2.30 below).

Proposition 2.9 ([41], Prop. 2.2.7)**.** *Let S be a simple algebraic system as in (1), E the ideal of R generated by $p_1, \ldots, p_s$, and q the product of all* $\mathrm{init}(p_i)$*. Then $E : q^\infty$ consists of all polynomials in R which vanish on* $\mathrm{Sol}_{\overline{K}}(S)$*. In particular, $E : q^\infty$ is a radical ideal. Given $p \in R$, we have $p \in E : q^\infty$ if and only if the pseudo-remainder of p modulo p_1, $\ldots$, p_s is zero.*

2.2. **Differential systems.**

Definition 2.10. A *differential field* K with commuting derivations $\delta_1, \ldots, \delta_n$ is a field K endowed with maps $\delta_i \colon K \to K$, satisfying

$$\delta_i(k_1 + k_2) = \delta_i(k_1) + \delta_i(k_2), \quad \delta_i(k_1\, k_2) = \delta_i(k_1)\, k_2 + k_1\, \delta_i(k_2) \quad \text{for all } k_1, k_2 \in K,$$

$i = 1, \ldots, n$, and $\delta_i \circ \delta_j = \delta_j \circ \delta_i$ for all $1 \leq i, j \leq n$.

In what follows, let K be the differential field of (complex) meromorphic functions on an open and connected subset Ω of $\mathbb{C}^n$. The derivations on K are given by the partial differential operators $\delta_1, \ldots, \delta_n$ with respect to the coordinates of $\mathbb{C}^n$. Moreover, let $R = K\{u_1, \ldots, u_m\}$ be the differential polynomial ring in the differential indeterminates $u_1, \ldots, u_m$. These indeterminates give rise to symbols $(u_k)_J$, where $J = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$, which represent the partial derivatives of m infinitely differentiable functions. More precisely, R is the polynomial algebra $K[(u_k)_J \mid 1 \leq k \leq m,\, J \in (\mathbb{Z}_{\geq 0})^n]$ over K in infinitely many indeterminates $(u_k)_J$, endowed with commuting derivations $\partial_1, \ldots, \partial_n$ such that

$$\partial_j\left((u_k)_J\right) = (u_k)_{J+1_j}, \qquad \partial_j|_K = \delta_j \quad \text{for all } j = 1, \ldots, n,$$

where 1_j is the j-th standard basis vector of $\mathbb{Z}^n$. For $k \in \{1, \ldots, m\}$, we identify $(u_k)_{(0,\ldots,0)}$ and u_k. We set $\Delta := \{ \partial_1, \ldots, \partial_n \}$, and for any subset $\{ \partial_{i_1}, \ldots, \partial_{i_r} \}$ of Δ we define the free commutative monoid of all monomials in $\partial_{i_1}, \ldots, \partial_{i_r}$

$$\mathrm{Mon}(\{ \partial_{i_1}, \ldots, \partial_{i_r} \}) := \{ \partial_{i_1}^{e_1} \ldots \partial_{i_r}^{e_r} \mid e \in (\mathbb{Z}_{\geq 0})^r \}.$$

Definition 2.11. A *differential system* S, defined over $R = K\{u_1, \ldots, u_m\}$, is given by finitely many equations and inequations

$$(3) \qquad p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0,$$

where $p_1, \ldots, p_s, q_1, \ldots, q_t \in R$ and $s, t \in \mathbb{Z}_{\geq 0}$. The *solution set* of S is

$$\begin{aligned} \mathrm{Sol}_\Omega(S) := \{ f = (f_1, \ldots, f_m) \quad | \quad & f_k \colon \Omega \to \mathbb{C} \text{ analytic}, \ k = 1, \ldots, m, \\ & p_i(f) = 0, \ q_j(f) \neq 0, \ i = 1, \ldots, s, \ j = 1, \ldots, t \}. \end{aligned}$$

Remark 2.12. Since each component f_k of a solution of (3) is assumed to be analytic, the equations $p_i = 0$ and inequations $q_j \neq 0$ (and their consequences) can be translated into algebraic conditions on the Taylor coefficients of power series expansions of $f_1, \ldots, f_m$ (around a point in Ω). An inequation $q \neq 0$ then turns into a disjunction of algebraic inequations for all coefficients which result from substitution of power series expansions for $u_1, \ldots, u_m$ in q. (This approach leads to the definition of the differential counting polynomial, a fine invariant of a differential system [27]).

An appropriate choice of $\Omega \subseteq \mathbb{C}^n$ can often only be made after the formal treatment of a given differential system by Thomas' algorithm (as, e.g., singularities of coefficients in differential consequences will only be detected during that process). In general, we assume that Ω is chosen in such a way that the given systems have analytic solutions on Ω.

Clearly, by neglecting the derivations on $R = K\{u_1, \ldots, u_m\}$, a differential system can be considered as an algebraic system in the finitely many variables $(u_i)_J$ which occur in the equations and inequations. The same recursive representation of polynomials as in the algebraic case is employed, but the total ordering on the set of variables $(u_i)_J$ is supposed to respect the action of the derivations.

Definition 2.13. A *ranking* $>$ on $R = K\{u_1, \ldots, u_m\}$ is a total ordering on the set

$$\mathrm{Mon}(\Delta)\, u := \{ (u_k)_J \mid 1 \leq k \leq m, \ J \in (\mathbb{Z}_{\geq 0})^n \}$$

such that for all $j \in \{1, \ldots, n\}$, $k, k_1, k_2 \in \{1, \ldots, m\}$, $J_1, J_2 \in (\mathbb{Z}_{\geq 0})^n$ we have

a) $\partial_j\, u_k > u_k$ and
b) $(u_{k_1})_{J_1} > (u_{k_2})_{J_2}$ implies $\partial_j\, (u_{k_1})_{J_1} > \partial_j\, (u_{k_2})_{J_2}$.

Remark 2.14. Every ranking $>$ on R is a well-ordering (cf., e.g., [24, Ch. 0, Sect. 17, Lemma 15]), i.e., every descending sequence of elements of $\mathrm{Mon}(\Delta)\, u$ terminates.

Example 2.15. On $K\{u\}$ (i.e., $m = 1$) with commuting derivations $\partial_1, \ldots, \partial_n$ the *degree-reverse lexicographical ranking* (with $\partial_1\, u > \partial_2\, u > \ldots > \partial_n\, u$) is defined for u_J, $u_{J'}$, $J = (j_1, \ldots, j_n)$, $J' = (j'_1, \ldots, j'_n) \in (\mathbb{Z}_{\geq 0})^n$, by

$$u_J > u_{J'} \quad :\Longleftrightarrow \quad \left\{ \begin{array}{l} j_1 + \ldots + j_n > j'_1 + \ldots + j'_n \quad \text{or} \\ \\ \big(\ j_1 + \ldots + j_n = j'_1 + \ldots + j'_n \quad \text{and} \quad J \neq J' \quad \text{and} \\ \\ j_i < j'_i \quad \text{for} \quad i = \max\{ 1 \leq k \leq n \mid j_k \neq j'_k \} \ \big). \end{array} \right.$$

For instance, if $n = 3$, we have $u_{(1,2,1)} > u_{(1,2,0)} > u_{(2,0,1)}$.

In what follows, we assume that a ranking $>$ on $R = K\{u_1, \ldots, u_m\}$ is fixed.

Remark 2.16. Let $p_1, p_2 \in R$ be two non-constant differential polynomials. If p_1 and p_2 have the same leader $(u_k)_J$ and the degree of p_1 in $(u_k)_J$ is greater than or equal to the degree of p_2 in $(u_k)_J$, then the same pseudo-division as in (2) yields a remainder which is either zero, or has leader less than $(u_k)_J$, or has leader $(u_k)_J$ and smaller degree in $(u_k)_J$ than p_1.

More generally, if $\operatorname{ld}(p_1) = \theta \operatorname{ld}(p_2)$ for some $\theta \in \operatorname{Mon}(\Delta)$, then this pseudo-division can be applied with p_2 replaced with $\theta\, p_2$. Note that, by condition b) of the definition of a ranking, we have $\operatorname{ld}(\theta\, p_2) = \theta \operatorname{ld}(p_2)$, and that, if $\theta \neq 1$, the degree of $\theta\, p_2$ in $\theta \operatorname{ld}(p_2)$ is one, so that the reduction can be applied without assumption on the degree of p_2 in $\operatorname{ld}(p_2)$. Then c_1 in (2) is again chosen as a suitable power of $\operatorname{init}(\theta\, p_2)$. In case $\theta \neq 1$ we have

$$\operatorname{init}(\theta\, p_2) = \frac{\partial p_2}{\partial \operatorname{ld}(p_2)} =: \operatorname{sep}(p_2),$$

and this differential polynomial is referred to as the *separant* of p_2.

In order not to change the solution set of a differential system, when $p_1 = 0$ is replaced with $r = 0$, where r is the result of a reduction of p_1 modulo p_2 or $\theta\, p_2$ as above, it is assumed that $\operatorname{init}(p_2)$ and $\operatorname{sep}(p_2)$ do not vanish on the solution set of the system. By definition of the separant and the discriminant (cf. Definition 2.2 d)), non-vanishing of $\operatorname{sep}(p_2)$ follows from non-vanishing of $\operatorname{disc}(p_2)$, as ensured by the algebraic part of Thomas' algorithm (cf. Remark 2.6).

We assume now that the given differential system is simple as an algebraic system; it could be one of the systems resulting from the algebraic part of Thomas' algorithm.

Remark 2.17. The symmetry of the second derivatives $\partial_i\, \partial_j\, u_k = \partial_j\, \partial_i\, u_k$ (and similarly for higher order derivatives) imposes necessary conditions on the solvability of a system of partial differential equations. Taking identities like these into account and forming linear combinations of (derivatives of) the given equations may produce differential consequences with lower ranked leaders. In order to obtain a complete set of algebraic conditions on the Taylor coefficients of an analytic solution, the system has to be augmented by these integrability conditions in general. If a system of partial differential equations admits a translation into algebraic conditions on the Taylor coefficients such that no further integrability conditions have to be taken into account, then it is said to be *formally integrable*.

A simple differential system, to be defined in Definition 2.24, will be assumed to be formally integrable. The construction of simple differential systems, and therefore, the computation of a Thomas decomposition, as presented in [4], [41], employs techniques which can be traced back to C. Riquier [39] and M. Janet [23]. The main idea is to turn the search for new differential consequences (i.e., integrability conditions) into a systematic procedure by singling out for each differential equation those derivations (called "non-admissible" here) which need to be applied to it in this investigation. The notion of Janet division, as discussed next, establishes a sense of direction in combining the given equations and deriving consequences. It is a particular case of an involutive division on sets of monomials, a concept developed by V. P. Gerdt and Y. A. Blinkov and others (cf., e.g., [18]).

Definition 2.18. Given a finite subset M of $\operatorname{Mon}(\Delta)$, *Janet division* associates with each $\theta \in M$ a subset of *admissible derivations* $\mu(\theta, M)$ of $\Delta = \{\partial_1, \ldots, \partial_n\}$ as follows. Let $\theta = \partial_1^{i_1} \ldots \partial_n^{i_n}$. Then $\partial_k \in \mu(\theta, M)$ if and only if

$$i_k = \max\{\, j_k \mid \partial_1^{j_1} \ldots \partial_n^{j_n} \in M \text{ with } j_1 = i_1,\, j_2 = i_2, \ldots, j_{k-1} = i_{k-1} \,\}.$$

The subset $\overline{\mu}(\theta, M) := \Delta \setminus \mu(\theta, M)$ consists of the *non-admissible derivations* for the element θ of M.

Example 2.19. Let $\Delta = \{ \partial_1, \partial_2, \partial_3 \}$ and $M = \{ \partial_1^2 \, \partial_2, \, \partial_1^2 \, \partial_3, \, \partial_2^2 \, \partial_3, \, \partial_2 \, \partial_3^2 \}$. Then Janet division associates the sets $\mu(\theta, M)$ of admissible derivations to the elements $\theta \in M$ as indicated in the following table, where we replace non-admissible derivations in the set Δ with the symbol '$*$'.

$$\begin{array}{rl} \partial_1^2 \, \partial_2, & \{\partial_1, \partial_2, \partial_3\} \\ \partial_1^2 \, \partial_3, & \{\partial_1, \, * \, , \partial_3\} \\ \partial_2^2 \, \partial_3, & \{ \, * \, , \partial_2, \partial_3\} \\ \partial_2 \, \partial_3^2, & \{ \, * \, , \, * \, , \partial_3\} \end{array}$$

Definition 2.20. A finite subset M of $\mathrm{Mon}(\Delta)$ is said to be *Janet complete* if

$$\bigcup_{\theta \in M} \mathrm{Mon}(\mu(\theta, M)) \, \theta = \bigcup_{\theta \in M} \mathrm{Mon}(\Delta) \, \theta,$$

i.e., if every monomial which is divisible by some monomial in M is obtained by multiplying a certain $\theta \in M$ by admissible derivations for θ only. (Recall that the left hand side of the above equation is a disjoint union.)

Example 2.21. The set M in Example 2.19 is not Janet complete because, e.g., the monomial $\partial_1 \, \partial_2^2 \, \partial_3$ is not obtained as a multiple of any $\theta \in M$ when multiplication is restricted to admissible derivations for θ. By adding this monomial and the monomial $\partial_1 \, \partial_2 \, \partial_3^2$ to M, we obtain the following Janet complete superset of M in $\mathrm{Mon}(\Delta)$.

$$\begin{array}{rl} \partial_1^2 \, \partial_2, & \{\partial_1, \partial_2, \partial_3\} \\ \partial_1^2 \, \partial_3, & \{\partial_1, \, * \, , \partial_3\} \\ \partial_1 \, \partial_2^2 \, \partial_3, & \{ \, * \, , \partial_2, \partial_3\} \\ \partial_1 \, \partial_2 \, \partial_3^2, & \{ \, * \, , \, * \, , \partial_3\} \\ \partial_2^2 \, \partial_3, & \{ \, * \, , \partial_2, \partial_3\} \\ \partial_2 \, \partial_3^2, & \{ \, * \, , \, * \, , \partial_3\} \end{array}$$

Remark 2.22. Every finite subset M of $\mathrm{Mon}(\Delta)$ can be augmented to a Janet complete finite set by adding certain monomials which are products of some $\theta \in M$ and a monomial which is divisible by at least one non-admissible derivation for θ.

For more details on Janet division, we refer to, e.g., [18], [4], [41].

Each equation $p_i = 0$ in a differential system is assigned the set of admissible derivations $\mu(\theta_i, M_k)$, where $\mathrm{ld}(p_i) = \theta_i \, u_k$ and

$$M_k := \{ \, \theta \in \mathrm{Mon}(\Delta) \mid \theta \, u_k \in \{ \, \mathrm{ld}(p_1), \ldots, \mathrm{ld}(p_s) \, \} \, \} \tag{4}$$

is the set of all monomials which define leaders of the equations $p_1 = 0, \ldots, p_s = 0$ in the system involving the same differential indeterminate u_k. We refer to $d \, p_i$ for $d \in \mathrm{Mon}(\mu(\theta_i, M_k))$ as the *admissible derivatives* of p_i.

Formal integrability of a differential system is then decided by applying to each equation $p_i = 0$ every of its non-admissible derivations $d \in \overline{\mu}(\theta_i, M_k)$ and computing the pseudo-remainder of $d \, p_i$ modulo $p_1, \ldots, p_s$ and their admissible derivatives. The restriction of the pseudo-division to admissible derivatives requires M_k to be Janet complete. If one of these pseudo-remainders is non-zero, then it is added as a new equation to the system, and the augmented system has to be treated by the algebraic part of Thomas' algorithm again.

Definition 2.23. A system of partial differential equations $\{ p_1 = 0, \ldots, p_s = 0 \}$, where $p_1, \ldots, p_s \in R \setminus K$, is said to be *passive* if the following two conditions hold for $\mathrm{ld}(p_1) = \theta_1 \, u_{k_1}, \ldots, \mathrm{ld}(p_s) = \theta_s \, u_{k_s}$, where $\theta_i \in \mathrm{Mon}(\Delta)$, $k_i \in \{1, \ldots, m\}$.

a) For all $k \in \{1, \ldots, m\}$, the set M_k defined in (4) is Janet complete.
b) For all $i \in \{1, \ldots, s\}$ and all $d \in \overline{\mu}(\theta_i, M_{k_i})$, the pseudo-remainder of $d \, p_i$ modulo $p_1, \ldots, p_s$ and their admissible derivatives is zero.

Definition 2.24. A differential system S, defined over R, as in (3) is said to be *simple* (with respect to $>$) if the following three conditions hold.

a) The system S is simple as an algebraic system (in the finitely many variables $(u_i)_J$ which occur in the equations and inequations of S, totally ordered by $>$).
b) The system $\{ p_1 = 0, \ldots, p_s = 0 \}$ is passive.
c) The left hand sides of the inequations $q_1 \neq 0, \ldots, q_t \neq 0$ equal their pseudo-remainders modulo $p_1, \ldots, p_s$ and their derivatives.

Definition 2.25. Let S be a differential system, defined over R. A *Thomas decomposition* of S (or of $\mathrm{Sol}_\Omega(S)$) with respect to $>$ is a collection of finitely many simple differential systems $S_1, \ldots, S_r$, defined over R, such that $\mathrm{Sol}_\Omega(S)$ is the disjoint union of the solution sets $\mathrm{Sol}_\Omega(S_1), \ldots, \mathrm{Sol}_\Omega(S_r)$.

Remark 2.26. Given S as in (3) and a ranking on R, a Thomas decomposition of S with respect to $>$ can be computed by interweaving the algebraic part discussed in Subsection 2.1 and differential reduction and completion with respect to Janet division.

First of all, a Thomas decomposition of S, considered as an algebraic system, is computed. Each of the resulting simple algebraic systems is then treated as follows. Differential pseudo-division is applied to pairs of distinct equations with leaders $\theta_1 \, u_k$ and $\theta_2 \, u_k$ such that $\theta_1 \mid \theta_2$ until either a non-zero pseudo-remainder is obtained or no such further reductions are possible. Non-zero pseudo-remainders are added to the system, the algebraic part of Thomas' algorithm is applied again, and the process is repeated. Once the system is auto-reduced in this sense, then it is possibly augmented with certain derivatives of equations so that the sets M_k defined in (4) are Janet complete. Then it is checked whether the system is passive. If a non-zero remainder is obtained by a pseudo-division of a non-admissible derivative modulo the equations and their admissible derivatives, then the algebraic part of Thomas' algorithm is applied again to the augmented system. Otherwise, the system is passive. Finally, the left hand side of each inequation is replaced with its pseudo-remainder modulo the equations and their derivatives, in order to ensure condition c) of Definition 2.24. The main reason why this procedure terminates is Dickson's Lemma, which shows that the ascending sequence of ideals of the semigroup $\mathrm{Mon}(\Delta)$ formed by the monomials θ defining leaders of equations (for each differential indeterminate) becomes stationary after finitely many steps.

For more details on the differential part of Thomas' algorithm, we refer to [4], [26], and [41, Subsect. 2.2.2].

An implementation of Thomas' algorithm for differential systems has been developed by M. Lange-Hegermann as Maple package `DifferentialThomas` [5].

We also use a simpler notation for the indeterminates $(u_k)_J$ of the differential polynomial ring. In case $m = 1$ we use the symbol u as a synonym for u_1. In addition, if the derivations ∂_1, ∂_2, ∂_3 represent the partial differential operators with respect to x, y, z, respectively, then we write

$$u_{\underbrace{x, \ldots, x}_{i}, \underbrace{y, \ldots, y}_{j}, \underbrace{z, \ldots, z}_{k}}$$

instead of $u_{(i,j,k)}$.

When displaying a simple differential system we indicate next to each equation its set of admissible derivations.

Example 2.27. Let us consider the ordinary differential equation (which is discussed in [22, Example in Sect. 4.7])

$$\left(\frac{\partial u}{\partial x}\right)^3 - 4\, x\, u(x)\, \frac{\partial u}{\partial x} + 8\, u(x)^2 = 0.$$

The left hand side is represented by the element $p := u_x^3 - 4\, x\, u\, u_x + 8\, u^2$ of the differential polynomial ring $R = K\{u\}$ with one derivation ∂_x, where $K = \mathbb{Q}(x)$ is the field of rational functions in x, endowed with differentiation with respect to x.

The initial of p is constant, the separant of p is $3\, u_x^2 - 4\, x\, u$. The algebraic part of Thomas' algorithm only distinguishes the cases whether the discriminant of p vanishes or not. We have

$$\operatorname{disc}(p) = -\operatorname{res}(p, \operatorname{sep}(p), u_x) = -64\, \underline{u}^3\, (27\, \underline{u} - 4\, x^3).$$

This case distinction leads to the Thomas decomposition

$\underline{u_x}^3 - 4\, x\, u\, \underline{u_x} + 8\, u^2 \;=\; 0, \quad \{\partial_x\}$ $(27\, \underline{u} - 4\, x^3)\, \underline{u} \;\neq\; 0$	$(27\, \underline{u} - 4\, x^3)\, \underline{u} \;=\; 0, \quad \{\partial_x\}$

Since both systems contain only one equation, no differential reductions are necessary. The second simple system could be split into two with equations $27\, u - 4\, x^3 = 0$ and $u = 0$, respectively. The solutions of the first simple system are given by $u(x) = c\, (x - c)^2$, where c is an arbitrary non-zero constant. The solutions $u(x) = 0$ and $u(x) = \frac{4}{27}\, x^3$ of the second simple system are called *singular solutions*, the latter one being an envelope of the general solution.

Example 2.28. Let us compute a Thomas decomposition of the system of (nonlinear) partial differential equations

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} & = & 0, \\[2ex] \dfrac{\partial u}{\partial x} - u^2 & = & 0 \end{array} \right.$$

for one unknown function $u(x, y)$. The left hand sides are expressed as elements $p_1 := u_{x,x} - u_{y,y}$ and $p_2 := u_x - u^2$ of the differential polynomial ring $R = \mathbb{Q}\{u\}$ with commuting derivations ∂_x, ∂_y. We choose the degree-reverse lexicographical ranking $>$ on R with $\partial_x\, u > \partial_y\, u$ (cf. Example 2.15).

Since the monomial ∂_x defining the leader of p_2 divides the monomial ∂_x^2 defining the leader of p_1, differential pseudo-division is applied and p_1 is replaced with

$$p_3 := p_1 - \partial_x\, p_2 - 2\, u\, p_2 = -u_{y,y} + 2\, u^3.$$

Janet division associates the sets of admissible derivations to the equations of the resulting system as follows:

$$\left\{ \begin{array}{rcll} \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\[1em] \underline{u_{y,y}} - 2\,u^3 & = & 0, & \{\,*\,, \partial_y\} \end{array} \right.$$

The set of monomials $\{\, \partial_x, \partial_y^2 \,\}$ defining the leaders u_x and $u_{y,y}$ is Janet complete. The check whether the above system is passive involves the following reduction:

$$\partial_x\, p_3 + \partial_y^2\, p_2 - 6\, u^2\, p_2 - 2\, u\, p_3 = -2\, (\underline{u_y} + u^2)\, (\underline{u_y} - u^2).$$

This non-zero remainder is a differential consequence which is added as an equation to the system. In fact, the system can be split into two systems according to the given factorization. For both systems a differential reduction of p_3 modulo the chosen factor is applied because the monomial ∂_y defining the new leader divides the monomial $\partial_{y,y}$ defining $\mathrm{ld}(p_3)$. In both cases the remainder is zero, the sets of monomials defining leaders are Janet complete, and the passivity check confirms formal integrability. We obtain the Thomas decomposition

$$\boxed{\begin{array}{rcll} \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\[1em] \underline{u_y} + u^2 & = & 0, & \{\,*\,, \partial_y\} \\[1em] & & & \end{array}} \qquad \boxed{\begin{array}{rcll} \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\[1em] \underline{u_y} - u^2 & = & 0, & \{\,*\,, \partial_y\} \\[1em] \underline{u} & \neq & 0. & \end{array}}$$

If the above factorization is ignored, then the discriminant of $p_4 := u_y^2 - u^4$ needs to be considered, which implies vanishing or non-vanishing of the separant $2\,u_y$. This case distinction leads to a different Thomas decomposition.

A Thomas decomposition of a differential system is not uniquely determined, as the previous example shows (cf. also Remark 2.8 for the algebraic case). In the special case of a system S of *linear* partial differential equations no case distinctions are necessary, and the single simple system in any Thomas decomposition of S is a Janet basis for S (cf., e.g., [23], [36], [18], [41]). Pseudo-reduction of a differential polynomial modulo the equations of a simple differential system and their derivatives decides membership to the corresponding saturation ideal (cf. also Proposition 2.9).

Proposition 2.29 ([41], Prop. 2.2.50)**.** *Let S be a simple differential system, defined over R, with equations $p_1 = 0$, …, $p_s = 0$. Moreover, let E be the differential ideal of R generated by $p_1, \ldots, p_s$ and define the product q of the initials and separants of all $p_1, \ldots,$ p_s. Then $E : q^\infty$ is a radical differential ideal. Given $p \in R$, we have $p \in E : q^\infty$ if and only if the pseudo-remainder of p modulo $p_1, \ldots, p_s$ and their derivatives is zero.*

Similarly to the algebraic case, the Nullstellensatz for analytic functions (due to J. F. Ritt and H. W. Raudenbush, cf. [40, Sects. II.7–11, IX.27]) establishes a one-to-one correspondence of solution sets $V := \mathrm{Sol}_\Omega(S)$ of systems of partial differential equations $S = \{\, p_1 = 0, \ldots, p_s = 0 \,\}$ for m unknown functions, defined over R, and their vanishing ideals in $R = K\{u_1, \ldots, u_m\}$

$$\mathcal{I}_R(V) := \{\, p \in R \mid p(f) = 0 \text{ for all } f \in V \,\}.$$

These are the radical differential ideals of R. The Nullstellensatz implies that, with the notation of Proposition 2.29, we have $\mathcal{I}_R(\mathrm{Sol}_\Omega(S)) = E : q^\infty$.

The following proposition allows to decide whether a given differential equation $p = 0$ is a consequence of a (not necessarily simple) differential system S by applying pseudo-division to p modulo each of the simple systems in a Thomas decomposition of S. It follows from the previous proposition and the Nullstellensatz and it also applies to algebraic systems by ignoring the separants.

Proposition 2.30 ([41], Prop. 2.2.72)**.** *Let S be a (not necessarily simple) differential system as in (3) and $S_1, \ldots, S_r$ a Thomas decomposition of S with respect to any ranking on R. Moreover, let E be the differential ideal of R generated by $p_1, \ldots, p_s$ and define the product q of $q_1, \ldots, q_t$. For $i \in \{1, \ldots, r\}$, let $E^{(i)}$ be the differential ideal of R generated by the equations in S_i and define the product $q^{(i)}$ of the initials and separants of all these equations. Then we have*

$$\sqrt{E : q^{\infty}} = \left(E^{(1)} : (q^{(1)})^{\infty}\right) \cap \ldots \cap \left(E^{(r)} : (q^{(r)})^{\infty}\right).$$

3. Elimination

Thomas' algorithm can be used to solve various differential elimination problems. This section presents results on certain rankings on the differential polynomial ring $R = K\{u_1, \ldots, u_m\}$ which allow to compute all differential consequences of a given differential system involving only a specified subset of the differential indeterminates $u_1, \ldots, u_m$. In other words, this technique allows to determine all differential equations which are satisfied by certain components of the solution tuples. We adopt the notation from the previous section.

Definition 3.1. Let $I_1, I_2, \ldots, I_k$ form a partition of $\{1, 2, \ldots, m\}$ such that $i_1 \in I_{j_1}$, $i_2 \in I_{j_2}$, $i_1 \le i_2$ implies $j_1 \le j_2$. Let $B_j := \{u_i \mid i \in I_j\}$, $j = 1, \ldots, k$. Moreover, fix some degree-reverse lexicographical ordering $>$ on $\mathrm{Mon}(\Delta)$. Then the *block ranking* on R *with blocks* $B_1, \ldots, B_k$ (with $u_1 > u_2 > \ldots > u_m$) is defined for $\theta_1 u_{i_1}, \theta_2 u_{i_2} \in \mathrm{Mon}(\Delta)\, u$, where $u_{i_1} \in B_{j_1}$, $u_{i_2} \in B_{j_2}$, by

$$\theta_1 u_{i_1} > \theta_2 u_{i_2} \quad :\Longleftrightarrow \quad \left\{ \begin{array}{l} j_1 < j_2 \quad \text{or} \quad \Big(\; j_1 = j_2 \quad \text{and} \quad \big(\theta_1 > \theta_2 \quad \text{or} \\ \quad (\,\theta_1 = \theta_2 \quad \text{and} \quad i_1 < i_2\,)\big) \; \Big). \end{array} \right.$$

Such a ranking is said to satisfy $B_1 \gg B_2 \gg \ldots \gg B_k$.

Example 3.2. With respect to the block ranking on $K\{u_1, u_2, u_3\}$ with blocks $\{u_1\}$, $\{u_2, u_3\}$ (and $u_1 > u_2 > u_3$) we have $(u_1)_{(0,1)} > u_1 > (u_2)_{(1,2)} > (u_3)_{(1,2)} > (u_2)_{(0,1)}$.

In the situation of the previous definition, for every $i \in \{1, \ldots, k\}$, we consider $K\{B_i, \ldots, B_k\} := K\{u \mid u \in B_i \cup \ldots \cup B_k\}$ as a differential subring of R, endowed with the restrictions of the derivations $\partial_1, \ldots, \partial_n$ to $K\{B_i, \ldots, B_k\}$.

For any algebraic or differential system S we denote by $S^{=}$ (resp. $S^{\neq}$) the set of the left hand sides of all equations (resp. inequations) in S.

Proposition 3.3 ([41], Prop. 3.1.36)**.** *Let S be a simple differential system, defined over R, with respect to a block ranking with blocks $B_1, \ldots, B_k$. Moreover, let E be the differential ideal of R generated by $S^{=}$ and q the product of the initials and separants of all elements of $S^{=}$. For every $i \in \{1, \ldots, k\}$, let E_i be the differential ideal of $K\{B_i, \ldots, B_k\}$ generated by $P_i := S^{=} \cap K\{B_i, \ldots, B_k\}$ and let q_i be the product of the initials and separants of all elements of P_i. Then, for every $i \in \{1, \ldots, k\}$, we have*

$$(E : q^{\infty}) \cap K\{B_i, \ldots, B_k\} = E_i : q_i^{\infty}.$$

In other words, the differential equations implied by S which involve only the differential indeterminates in $B_i \cup \ldots \cup B_k$ are precisely those whose pseudo-remainders modulo the elements of $S^{=} \cap K\{B_i, \ldots, B_k\}$ and their derivatives are zero.

Example 3.4. The Cauchy-Riemann equations for a complex function of $z = x + i\, y$ with real part u and imaginary part v are

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} & = & 0, \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & = & 0. \end{array} \right.$$

The left hand sides are represented by the elements $p_1 := u_x - v_y$ and $p_2 := u_y + v_x$ of the differential polynomial ring $R = \mathbb{Q}\{u, v\}$ with derivations ∂_x and ∂_y. Choosing a block ranking on R satisfying $\{u\} \gg \{v\}$, the passivity check yields the equation

$$\partial_x\, p_2 - \partial_y\, p_1 = v_{x,x} + v_{y,y} = 0.$$

Similarly, the choice of a block ranking on R satisfying $\{v\} \gg \{u\}$ yields the consequence $u_{x,x} + u_{y,y} = 0$. These computations confirm that the real and imaginary parts of a holomorphic function are harmonic functions.

Corollary 3.5 ([41], Cor. 3.1.37)**.** *Let S be a (not necessarily simple) differential system, defined over R, and $S_1, \ldots, S_r$ a Thomas decomposition of S with respect to a block ranking with blocks $B_1, \ldots, B_k$. Moreover, let E be the differential ideal of R generated by $S^{=}$ and q the product of all elements of $S^{\neq}$. Let $i \in \{1, \ldots, k\}$ be fixed. For every $j \in \{1, \ldots, r\}$, let $E^{(j)}$ be the differential ideal of $K\{B_i, \ldots, B_k\}$ generated by $P_j := S_j^{=} \cap K\{B_i, \ldots, B_k\}$ and let $q^{(j)}$ be the product of the initials and separants of all elements of P_j. Then we have*

$$\sqrt{E : q^{\infty}} \,\cap\, K\{B_i, \ldots, B_k\} = (E_1 : q_1^{\infty}) \,\cap\, \ldots \,\cap\, (E_r : q_r^{\infty})\,.$$

4. Control-theoretic applications

In order to apply the Thomas decomposition method to nonlinear control systems, we assume that the control system is given by differential equations and inequations whose left hand sides are polynomials. Structural information about certain configurations of the control system is obtained from each simple system of a Thomas decomposition of the given differential equations and inequations. The choice of ranking on the differential polynomial ring depends on the question at hand, although a Thomas decomposition with respect to any ranking, e.g., the degree-reverse lexicographical ranking, may give hints on how to adapt the ranking for further investigations in a certain direction.

Let $R = K\{U\}$ be the differential polynomial ring in the differential indeterminates $U := \{u_1, \ldots, u_m\}$ over a differential field K of (complex) meromorphic functions on an open and connected subset Ω of $\mathbb{C}^n$ (cf. Subsection 2.2). (No distinction is made a priori between state variables, input, output, etc.)

We assume that S is a simple differential system, defined over R, with respect to some ranking $>$. Let E be the differential ideal of R generated by the set $S^{=}$ of the left hand sides of the equations in S and define the product q of the initials and separants of all elements of $S^{=}$.

Definition 4.1. Let $x \in U$ and $Y \subseteq U \setminus \{x\}$. Then x is said to be *observable with respect to Y* if there exists $p \in (E : q^{\infty}) \setminus \{0\}$ such that p is a polynomial in x (not involving any proper derivative of x) with coefficients in $K\{Y\}$ and such that neither its leading coefficient nor $\partial p/\partial x$ is an element of $E : q^\infty$.

Remark 4.2. Let p be a polynomial as in the previous definition. Then the implicit function theorem allows to solve $p = 0$ locally for x in the sense that the component of $(f_1, \ldots, f_m) \in \mathrm{Sol}_\Omega(S)$ corresponding to x can locally be expressed as an analytic function of the components corresponding to the differential indeterminates in Y.

If $>$ satisfies $U \setminus (Y \cup \{x\}) \gg \{x\} \gg \{Y\}$, then by Proposition 3.3, there exists a polynomial p in $(E : q^\infty) \setminus \{0\}$ as above if and only if there exists such a polynomial in $S^= \cap K\{Y \cup \{x\}\}$. For a not necessarily simple differential system S, a Thomas decomposition with respect to a ranking as above allows to decide the existence of such a polynomial among the left hand sides of the differential consequences of S by inspecting each simple system (cf. Corollary 3.5).

Definition 4.3. A subset Y of U is called a *flat output* of S if $(E : q^\infty) \cap K\{Y\} = \{0\}$ and every $x \in U \setminus Y$ is observable with respect to Y.

Remark 4.4. Let $>$ satisfy $U \setminus Y \gg Y$. Then Proposition 3.3 allows to decide whether the conditions in Definition 4.3 are satisfied by checking that $S^= \cap K\{Y\} = \emptyset$ holds and that for every $x \in U \setminus Y$ there exists a polynomial $p \in S^= \cap K\{Y \cup \{x\}\}$ satisfying the conditions in Definition 4.1.

If the differential ideal $I := E : q^\infty$ is prime, then the field of fractions $\mathrm{Quot}(R/I)$ can be considered as a differential extension field of K. Let us assume that Y is a flat output of S and let L be the differential subfield of $\mathrm{Quot}(R/I)$ which is generated by $\{\, y + I \mid y \in Y \,\}$. Then, by Definition 4.3, L/K is a purely differentially transcendental extension of differential fields, and for every $x \in U \setminus Y$, the element $x + I$ of $\mathrm{Quot}(R/I)$ is algebraic over L. Hence, $\{\, y + I \mid y \in Y \,\}$ is a differential transcendence basis of $\mathrm{Quot}(R/I)/K$, and the system is flat in the sense of [14, Sect. 3.2].

Remark 4.5. Following [14], a system which is defined by a differential field extension is called flat if it is equivalent by endogenous feedback to a system which is defined by a purely differentially transcendental extension of differential fields. As opposed to checking whether Y is a flat output of S using the method described above, deciding whether S is flat is a difficult problem in general.

As a first illustration of how differential elimination methods can be applied to nonlinear control systems, we consider inversion, i.e., the problem of expressing the input variables in terms of the output variables (and their derivatives).

Remark 4.6. Using the same notation as above, we assume that disjoint subsets Y and Z of U are specified, where the differential indeterminates in Y and Z are interpreted as input and output variables of the system, respectively. We achieve an *inversion* of the system S if and only if we can exhibit for each $z \in Z$ a $p \in (E : q^\infty) \setminus \{0\}$ such that p is a polynomial in z (not involving any proper derivative of z) with coefficients in $K\{Y\}$ and such that neither its leading coefficient nor $\partial p/\partial z$ is an element of $E : q^\infty$.

If $>$ satisfies $U \setminus (Y \cup \{z\}) \gg \{z\} \gg Y$, then by Proposition 3.3, there exists such a polynomial p in $(E : q^\infty) \setminus \{0\}$ if and only if there exists such a polynomial in $S^= \cap K\{Y \cup \{z\}\}$. A block ranking $U \setminus (Y \cup Z) \gg Z \gg Y$ may allow to find such polynomials p for all $z \in Z$ by computing only one Thomas decomposition (cf. the following example), for instance, if all these polynomials p have degree one.

For displaying simple differential systems resulting from Thomas decompositions in a concise way, we use the following command `Print`, which makes use of both the Maple packages `Janet` [6] and `DifferentialThomas` [5], where `ivar` and `dvar` are the lists of independent and dependent variables, respectively.

```
> with(Janet):
> Print := S->Diff2Ind(
> PrettyPrintDifferentialSystem(S), ivar, dvar):
```

The sets of admissible derivations for the equations in a simple system are not reproduced here. Note that the implementation uses factorization and may, for convenience, return simple systems containing several inequations with the same leader (thus, not strictly complying with condition b) of Definition 2.3).

Example 4.7. The following system of ordinary differential equations models a unicycle as described in [9, Examples 3.20, 4.18, 5.10] (cf. also, e.g., [33, Example 2.35]).

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & \cos(x_3)\, u_1, \\ \dot{x}_2 & = & \sin(x_3)\, u_1, \\ \dot{x}_3 & = & u_2. \end{array} \right.$$

Here x_1, x_2, x_3 are considered as state variables, where (x_1, x_2) is the position of the middle of the axis in the plane and x_3 the angle of its rotation, and the velocities u_1, u_2 are considered as inputs. Moreover, the following outputs y_1, y_2 are given:

$$\left\{ \begin{array}{rcl} y_1 & = & x_1\,, \\ y_2 & = & x_2\,. \end{array} \right.$$

The task is to try to invert the system, i.e., to express u_1, u_2 in terms of y_1, y_2 and their derivatives.

In order to translate the given equations into differential polynomials, we represent $\cos(x_3)$ and $\sin(x_3)$ by differential indeterminates $cx3$ and $sx3$ and add the generating relations

$$cx3^2 + sx3^2 = 1, \quad cx3_t = -sx3\,(x_3)_t, \quad sx3_t = cx3\,(x_3)_t$$

to the system. More precisely speaking, we adjoin to the differential polynomial ring $\mathbb{Q}\{x_1, x_2, x_3, u_1, u_2, y_1, y_2\}$ with derivation ∂_t the differential indeterminates $cx3$ and $sx3$ and define the differential ideal E of the resulting differential polynomial ring which is generated by $(x_1)_t - cx3\,u_1$, $(x_2)_t - sx3\,u_1$, $(x_3)_t - u_2$, $cx3^2 + sx3^2 - 1$, $cx3_t + sx3\,(x_3)_t$ and $sx3_t - cx3\,(x_3)_t$. We then apply elimination properties of the differential Thomas decomposition method to $\sqrt{E}$ (see Proposition 2.30 and Corollary 3.5).

(Alternatively, if one accepts neglecting the particular case of movement of the unicycle in the direction of the x_2-coordinate axis without rotation, one could assume that $\cos(x_3)$ is not the zero function, multiply both sides of the equation $\dot{x}_3 = u_2$ by $\cos(x_3)$, read the resulting left hand side, using the chain rule, as the derivative of $\sin(x_3)$, and obtain the equation $sx3_t = cx3\,u_2$. This would allow to dispose of the differential indeterminate x_3 and the computation of the Thomas decomposition below would essentially yield the first five of the seven simple systems below.)

In the following computations concerning the model of a unicycle the differential polynomial ring is $\mathbb{Q}\{x_1, x_2, cx3, sx3, x_3, u_1, u_2, y_1, y_2\}$ with one derivation ∂_t.

```
> with(DifferentialThomas):
> ivar := [t]:
> dvar := [x1,x2,cx3,sx3,x3,u1,u2,y1,y2]:
```

We specify the block ranking $>$ satisfying $\{x_1, x_2, cx3, sx3, x_3\} \gg \{u_1, u_2\} \gg \{y_1, y_2\}$ as well as $x_1 > x_2 > cx3 > sx3 > x_3$ and $u_1 > u_2$ and $y_1 > y_2$.

```
> ComputeRanking(ivar,
> [[x1,x2,cx3,sx3,x3],[u1,u2],[y1,y2]]):
```

If the left hand sides of the system are written in jet notation, then a conversion into the format expected by the package `DifferentialThomas` is accomplished by the following sequence of commands.

```
> L := [x1[t]-cx3*u1, x2[t]-sx3*u1, x3[t]-u2,
> y1-x1, y2-x2, cx3^2+sx3^2-1, cx3[t]+sx3*x3[t],
> sx3[t]-cx3*x3[t]];
> LL := Diff2JetList(Ind2Diff(L, ivar, dvar));
```

$$LL := [(x_1)_1 - cx3_0\,(u_1)_0,\quad (x_2)_1 - sx3_0\,(u_1)_0,\quad (x_3)_1 - (u_2)_0,\quad (y_1)_0 - (x_1)_0,\quad (y_2)_0 - (x_2)_0,\quad {cx3_0}^2 + {sx3_0}^2 - 1,\quad cx3_1 + sx3_0\,(x_3)_1,\quad sx3_1 - cx3_0\,(x_3)_1]$$

We compute a Thomas decomposition with respect to $>$ of the given system of ordinary differential equations.

```
> TD := DifferentialThomasDecomposition(LL, []);
```

$$TD := [\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}]$$

The first simple differential system is given as follows.

```
> Print(TD[1]);
```

$$[\underline{x_1} - y_1 = 0,\quad \underline{x_2} - y_2 = 0,\quad u_1\,\underline{cx3} - (y_1)_t = 0,\quad u_1\,\underline{sx3} - (y_2)_t = 0,\quad (y_1)_t^2\,\underline{(x_3)_t} + (y_2)_t^2\,\underline{(x_3)_t} - (y_1)_t\,(y_2)_{t,t} + (y_2)_t\,(y_1)_{t,t} = 0,\quad \underline{u_1}^2 - (y_1)_t^2 - (y_2)_t^2 = 0,\quad (y_1)_t^2\,\underline{u_2} + (y_2)_t^2\,\underline{u_2} - (y_1)_t\,(y_2)_{t,t} + (y_2)_t\,(y_1)_{t,t} = 0,\quad \underline{(y_2)_t} \neq 0,\quad \underline{(y_1)_t} \neq 0,\quad \underline{(y_1)_t}^2 + {(y_2)_t}^2 \neq 0,\quad (y_2)_t\,\underline{(y_1)_{t,t}} - (y_1)_t\,(y_2)_{t,t} \neq 0]$$

```
> collect(%[7], u2, factor);
```

$$\left((y_1)_t^2 + (y_2)_t^2\right) u_2 - (y_1)_t\,(y_2)_{t,t} + (y_2)_t\,(y_1)_{t,t} = 0$$

Thus, the equations with leader u_1 and u_2 in $TD[1]$ allow to express u_1 and u_2 in terms of y_1 and y_2. (Up to solving these equations for u_1 and u_2 explicitly, it is the same result as in [9, Example 5.12].)

The remaining six simple differential systems describe particular configurations, which exhibit obstructions to invertibility.

```
> Print(TD[2]);
```

$$[\underline{x_1} - y_1 = 0,\quad \underline{x_2} - y_2 = 0,\quad u_1\,\underline{cx3} - (y_1)_t = 0,\quad u_1\,\underline{sx3} - (y_2)_t = 0,\quad (x_3)_t = 0,\quad \underline{u_1}^2 - (y_1)_t^2 - (y_2)_t^2 = 0,\quad \underline{u_2} = 0,\quad (y_2)_t\,\underline{(y_1)_{t,t}} - (y_1)_t\,(y_2)_{t,t} = 0,\quad \underline{(y_2)_t} \neq 0,\quad \underline{(y_1)_t} \neq 0,\quad \underline{(y_1)_t}^2 + {(y_2)_t}^2 \neq 0]$$

The vanishing of the Wronskian determinant of $(y_1)_t$ and $(y_2)_t$ expresses that one of the velocities $\dot{x}_1$ and $\dot{x}_2$ is a constant multiple of the other. Hence, no rotation is allowed, which forces the input u_2 to be the zero function. Due to the inequations, the vector $(\dot{x}_1, \dot{x}_2)$ is non-zero and not parallel to any of the x_1- or x_2-coordinate axes.

```
> Print(TD[3]);
```

$$[\underline{x_1} - y_1 = 0, \quad \underline{x_2} - y_2 = 0, \quad \underline{cx3} + 1 = 0, \quad \underline{sx3} = 0, \quad \underline{(x_3)_t} = 0,$$
$$\underline{u_1} + (y_1)_t = 0, \quad \underline{u_2} = 0, \quad \underline{(y_2)_t} = 0, \quad \underline{(y_1)_t} \neq 0]$$

```
> Print(TD[4]);
```

$$[\underline{x_1} - y_1 = 0, \quad \underline{x_2} - y_2 = 0, \quad \underline{cx3} - 1 = 0, \quad \underline{sx3} = 0, \quad \underline{(x_3)_t} = 0,$$
$$\underline{u_1} - (y_1)_t = 0, \quad \underline{u_2} = 0, \quad \underline{(y_2)_t} = 0, \quad \underline{(y_1)_t} \neq 0]$$

The previous two simple systems describe cases in which only movement in any of the two directions defined by the x_1-coordinate axis is allowed and no rotation.

```
> Print(TD[5]);
```

$$[\underline{x_1} - y_1 = 0, \quad \underline{x_2} - y_2 = 0, \quad \underline{cx3}^2 + sx3^2 - 1 = 0, \quad \underline{sx3_t} - u_2\, cx3 = 0,$$
$$\underline{(x_3)_t} - u_2 = 0, \quad \underline{u_1} = 0, \quad \underline{(y_1)_t} = 0, \quad \underline{(y_2)_t} = 0, \quad \underline{sx3} + 1 \neq 0, \quad \underline{sx3} - 1 \neq 0]$$

The fifth simple system describes configurations which only allow rotation, and the input u_1 is forced to be the zero function. (Similarly to Example 2.7, the inequation $sx3^2 - 1 \neq 0$ is introduced here to ensure that $cx3^2 + sx3^2 - 1$ has no multiple roots as polynomial in $cx3$. It is included in the simple system in factorized form.)

```
> Print(TD[6]);
```

$$[\underline{x_1} - y_1 = 0, \quad \underline{x_2} - y_2 = 0, \quad \underline{cx3} = 0, \quad \underline{sx3} + 1 = 0, \quad \underline{(x_3)_t} = 0,$$
$$\underline{u_1} + (y_2)_t = 0, \quad \underline{u_2} = 0, \quad \underline{(y_1)_t} = 0]$$

```
> Print(TD[7]);
```

$$[\underline{x_1} - y_1 = 0, \quad \underline{x_2} - y_2 = 0, \quad \underline{cx3} = 0, \quad \underline{sx3} - 1 = 0, \quad \underline{(x_3)_t} = 0,$$
$$\underline{u_1} - (y_2)_t = 0, \quad \underline{u_2} = 0, \quad \underline{(y_1)_t} = 0]$$

The last two simple systems cover the cases of movement in any of the two directions defined by the x_2-coordinate axis and no rotation.

Next we consider the detection of flat outputs.

Example 4.8. A model of a 2-D crane is given by the following system of ordinary differential equations (cf. [14, Sect. 4.1] and the references therein), where $x(t)$ and $z(t)$ are the coordinates of the load of mass m, $\theta(t)$ is the angle between the rope and the z-axis, $d(t)$ the trolley position, $T(t)$ the tension of the rope, $R(t)$ the rope length, and g the gravitational constant.

$$\left\{ \begin{array}{rcl} m\,\ddot{x} & = & -T\,\sin\theta, \\ m\,\ddot{z} & = & -T\,\cos\theta + m\,g, \\ x & = & R\,\sin\theta + d, \\ z & = & R\,\cos\theta. \end{array} \right.$$

The task is to decide whether $\{x, z\}$ is a flat output of the system.

Similarly to the previous example, we represent $\cos\theta$ and $\sin\theta$ by differential indeterminates c and s and add the generating relation $c^2 + s^2 = 1$ to the system. In this example the given equations depend on θ only through $\cos\theta$ and $\sin\theta$. Therefore, we do not include θ as a differential indeterminate and do not need to add the relations $c_t = -s\,\theta_t$ and $s_t = c\,\theta_t$ to the system. (Note that, if I is the differential ideal of $\mathbb{Q}\{\theta, c, s\}$ with derivation ∂_t which is generated by $c^2 + s^2 - 1$ and $c_t + s\,\theta_t$ and $s_t - c\,\theta_t$, then $I \cap \mathbb{Q}\{c, s\}$ is the differential ideal which is generated by $c^2 + s^2 - 1$.)

```
> with(DifferentialThomas):
> ivar := [t]:
> dvar := [T,c,s,d,R,x,z]:
```

We set up the block ranking $>$ which satisfies $\{T, c, s, d, R\} \gg \{x, z\}$ as well as $T > c > s > d > R$ and $x > z$.

```
> ComputeRanking(ivar, [[T,c,s,d,R],[x,z]]):
```

We compute a Thomas decomposition with respect to $>$. (As is customary in Maple, the symbols m and g are treated here as algebraically independent over $\mathbb{Q}$. More precisely, the ground field for the following computation is the differential field $\mathbb{Q}(m, g)$ with trivial derivation.)

```
> TD := DifferentialThomasDecomposition(
> [m*x[2]+T[0]*s[0], m*z[2]+T[0]*c[0]-m*g,
> x[0]-R[0]*s[0]-d[0], z[0]-R[0]*c[0],
> c[0]^2+s[0]^2-1], []);
```

$$\begin{aligned} TD := [&\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem},\\ &\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem},\\ &\mathit{DifferentialSystem}] \end{aligned}$$

The second simple differential system is given as follows.

```
> Print(TD[2]);
```

$$\begin{aligned} [&z\,\underline{T} + m\,z_{t,t}\,R - m\,g\,R = 0, \quad R\,\underline{c} - z = 0, \quad z_{t,t}\,R\,\underline{s} - g\,R\,\underline{s} - z\,x_{t,t} = 0,\\ &z_{t,t}\,\underline{d} - g\,\underline{d} + z\,x_{t,t} - x\,z_{t,t} + g\,x = 0,\\ &{z_{t,t}}^2\,\underline{R}^2 - 2\,g\,z_{t,t}\,\underline{R}^2 + g^2\,\underline{R}^2 - z^2\,x_{t,t}^2 - z^2\,z_{t,t}^2 + 2\,g\,z^2\,z_{t,t} - g^2\,z^2 = 0,\\ &\underline{z} \neq 0, \quad \underline{z_{t,t}} - g \neq 0, \quad \underline{x_{t,t}} \neq 0, \quad \underline{x_{t,t}}^2 + {z_{t,t}}^2 - 2\,g\,z_{t,t} + g^2 \neq 0] \end{aligned}$$

```
> collect(%[5], R, factor);
```

$$(z_{t,t} - g)^2\,R^2 - z^2\left({x_{t,t}}^2 + {z_{t,t}}^2 - 2\,g\,z_{t,t} + g^2\right) = 0$$

We observe that this simple system S contains no equation involving derivatives of x and z only. Moreover, the equations in S show that T, c, s, d, R are observable with respect to $\{x, z\}$. Hence, $\{x, z\}$ is a flat output of S.

The remaining six simple differential systems describe particular configurations for which $\{x, z\}$ is not a flat output. In fact, the movement of the load is restricted by some constraint in these cases (e.g., $x_{t,t} = 0$ or $z = 0$, one reason being, e.g., that vanishing rope tension implies constant acceleration of the load, another being a constant rope length of zero allowing no vertical movement of the load). We do not consider the system to be controllable under these conditions.

```
> Print(TD[1]);
```

$$[\underline{T}=0, \quad R\,\underline{c}-z=0, \quad R\,\underline{s}+d-x=0, \quad \underline{d}^2-2\,x\,\underline{d}+x^2-R^2+z^2=0,$$
$$\underline{x_{t,t}}=0, \quad \underline{z_{t,t}}-g=0, \quad \underline{z}\neq 0, \quad \underline{R}\neq 0, \quad \underline{R}+z\neq 0, \quad \underline{R}-z\neq 0]$$

```
> Print(TD[3]);
```

$$[\underline{T}-m\,z_{t,t}+m\,g=0, \quad \underline{c}+1=0, \quad \underline{s}=0, \quad \underline{d}-x=0, \quad \underline{R}+z=0,$$
$$\underline{x_{t,t}}=0, \quad \underline{z}\neq 0]$$

```
> Print(TD[4]);
```

$$[\underline{T}+m\,z_{t,t}-m\,g=0, \quad \underline{c}-1=0, \quad \underline{s}=0, \quad \underline{d}-x=0, \quad \underline{R}-z=0,$$
$$\underline{x_{t,t}}=0, \quad \underline{z}\neq 0]$$

```
> Print(TD[5]);
```

$$[s\,\underline{T}+m\,x_{t,t}=0, \quad x_{t,t}\,\underline{c}+g\,s=0, \quad g^2\,\underline{s}^2+x_{t,t}^2\,\underline{s}^2-x_{t,t}^2=0, \quad \underline{d}-x=0, \quad \underline{R}=0,$$
$$\underline{z}=0, \quad \underline{x_{t,t}}\neq 0, \quad \underline{x_{t,t}}^2+g^2\neq 0]$$

```
> Print(TD[6]);
```

$$[\underline{T}+m\,g=0, \quad \underline{c}+1=0, \quad \underline{s}=0, \quad \underline{d}-x=0, \quad \underline{R}=0, \quad \underline{x_{t,t}}=0, \quad \underline{z}=0]$$

```
> Print(TD[7]);
```

$$[\underline{T}-m\,g=0, \quad \underline{c}-1=0, \quad \underline{s}=0, \quad \underline{d}-x=0, \quad \underline{R}=0, \quad \underline{x_{t,t}}=0, \quad \underline{z}=0]$$

We give two examples which demonstrate how the Thomas decomposition technique can be used to study the dependence of structural properties of a nonlinear control system on parameters.

Example 4.9. A model of a continuous stirred-tank reactor (cf. [25, Example 1.2]) is given by the differential system

$$\begin{cases} \dot{V}(t) &=& F_1(t)+F_2(t)-k\,\sqrt{V(t)}, \\ \overline{c(t)\,V(t)}^{\,\cdot} &=& c_1\,F_1(t)+c_2\,F_2(t)-c(t)\,k\,\sqrt{V(t)}. \end{cases}$$

A dissolved material has concentration $c(t)$ in the tank and it is fed through two inputs with constant concentrations c_1 and c_2 and flow rates $F_1(t)$ and $F_2(t)$, respectively. There exists an outward flow with a flow rate proportional to the square root of the volume $V(t)$ of liquid in the tank. Moreover, k is an experimental constant.

In order to eliminate the square root of the volume in the given equations, we represent $\sqrt{V(t)}$ as a differential indeterminate sV and substitute other occurrences of $V(t)$ by sV^2. We investigate the dependence of the behavior on parameter configurations by considering c_1 and c_2 as differential indeterminates as well and adding the conditions $\dot{c}_1 = 0$ and $\dot{c}_2 = 0$.

```
> with(DifferentialThomas):
> ivar := [t]:
> dvar := [F1,F2,sV,c,c1,c2]:
```

We define $R=\mathbb{Q}\{F_1,F_2,sV,c,c_1,c_2\}$ and choose the block ranking $>$ on R with blocks $\{F_1,F_2\}$, $\{sV,c\}$, $\{c_1,c_2\}$, i.e., satisfying $\{F_2,F_2\}\gg\{sV,c\}\gg\{c_1,c_2\}$ and $F_1>F_2$ and $sV>c$ and $c_1>c_2$.

```
> ComputeRanking(ivar, [[F1,F2],[sV,c],[c1,c2]]):
> L := [2*sV[t]*sV-F1-F2+k*sV,
> c[t]*sV^2-c2*F2+c*k*sV-c1*F1+2*c*sV[t]*sV,
> c1[t], c2[t]]:
> LL := Diff2JetList(Ind2Diff(L, ivar, dvar));
```

$$LL := [2\,sV_1\,sV_0 - (F_1)_0 - (F_2)_0 + k\,sV_0,$$
$$c_1\,sV_0{}^2 - (c_2)_0\,(F_2)_0 + c_0\,k\,sV_0 - (c_1)_0\,(F_1)_0 + 2\,c_0\,sV_1\,sV_0, \quad (c_1)_1, \quad (c_2)_1]$$

We compute a Thomas decomposition with respect to $>$ of the given system of ordinary differential equations, to which we add the inequations $\sqrt{V} \neq 0$, $c_1 \neq 0$, $c_2 \neq 0$ to exclude trivial cases.

```
> TD := DifferentialThomasDecomposition(LL,
> [sV[0],c1[0],c2[0]]);
```

$$TD := [\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}]$$

The first simple differential system is given as follows.

```
> Print(TD[1]);
```

$$[c_2\,\underline{F_1} - c_1\,\underline{F_1} + 2\,c\,sV\,sV_t - 2\,c_2\,sV\,sV_t + c_t\,sV^2 + c\,k\,sV - c_2\,k\,sV = 0,$$
$$c_1\,\underline{F_2} - c_2\,\underline{F_2} + 2\,c\,sV\,sV_t - 2\,c_1\,sV\,sV_t + c_t\,sV^2 + c\,k\,sV - c_1\,k\,sV = 0,$$
$$\underline{(c_1)_t} = 0, \quad \underline{(c_2)_t} = 0, \quad \underline{c_2} \neq 0, \quad \underline{c_1} \neq 0, \quad \underline{c_1} - c_2 \neq 0, \quad \underline{sV} \neq 0]$$

```
> collect(%[1], F1);
```

$$(c_2 - c_1)\,F_1 + 2\,c\,sV\,sV_t - 2\,c_2\,sV\,sV_t + c_t\,sV^2 + c\,k\,sV - c_2\,k\,sV = 0$$

```
> collect(%%[2], F2);
```

$$(c_1 - c_2)\,F_2 + 2\,c\,sV\,sV_t - 2\,c_1\,sV\,sV_t + c_t\,sV^2 + c\,k\,sV - c1\,k\,sV = 0$$

The first two equations in the first simple system S show that F_1 and F_2 are observable with respect to $\{c, sV\}$. (Although c_1 and c_2 are represented by differential indeterminates here, we consider these still as parameters.) Let E be the differential ideal of R generated by $S^=$ and q the product of the initials (and separants) of all elements of $S^=$. Due to the choice of the block ranking, we conclude that we have $(E : q^\infty) \cap \mathbb{Q}\{sV, c\} = \{0\}$ (cf. Proposition 3.3). Hence, $\{c, sV\}$ is a flat output of S.

The remaining two simple systems describe configurations of the system in which the two concentrations c_1 and c_2 are equal. Since both input feeds are identical and constant, this condition precludes control of the concentration in the tank. These particular systems do not admit $\{c, sV\}$ as a flat output. In fact, by inspecting the equations of these systems, we observe that we have $(E : q^\infty) \cap \mathbb{Q}\{sV, c\} \neq \{0\}$.

```
> Print(TD[2]);
```

$$[c\,\underline{F_1} - c_2\,\underline{F_1} + c\,F_2 - c_2\,F_2 + c_t\,sV^2 = 0,$$
$$2\,c\,\underline{sV_t} - 2\,c_2\,\underline{sV_t} + c_t\,sV + c\,k - c_2\,k = 0, \quad \underline{c_1} - c_2 = 0, \quad \underline{(c_2)_t} = 0,$$
$$\underline{c_2} \neq 0, \quad \underline{c} - c_2 \neq 0, \quad \underline{sV} \neq 0]$$

```
> Print(TD[3]);
```

$$[\underline{F_1} + F_2 - 2\,sV\,sV_t - k\,sV = 0, \quad \underline{c} - c_2 = 0, \quad \underline{c_1} - c_2 = 0, \quad \underline{(c_2)_t} = 0,$$
$$\underline{c_2} \neq 0, \quad \underline{sV} \neq 0]$$

Example 4.10. Let us consider the following system of linear partial differential equations for functions ξ_1, ξ_2, ξ_3 of $\mathbf{x} = (x_1, x_2, x_3)$ involving a parametric function $a(x_2)$

$$\left\{\begin{array}{rcl} -a(x_2)\dfrac{\partial \xi_1(\mathbf{x})}{\partial x_1} + \dfrac{\partial \xi_3(\mathbf{x})}{\partial x_1} - \left(\dfrac{\partial}{\partial x_2} a(x_2)\right) \xi_2(\mathbf{x}) + \dfrac{1}{2}\, a(x_2)\, (\nabla \cdot \xi(\mathbf{x})) & = & 0, \\ -a(x_2)\dfrac{\partial \xi_1(\mathbf{x})}{\partial x_2} + \dfrac{\partial \xi_3(\mathbf{x})}{\partial x_2} & = & 0, \\ -a(x_2)\dfrac{\partial \xi_1(\mathbf{x})}{\partial x_3} + \dfrac{\partial \xi_3(\mathbf{x})}{\partial x_3} - \dfrac{1}{2}\, (\nabla \cdot \xi(\mathbf{x})) & = & 0, \end{array}\right.$$

which describe infinitesimal transformations associated to a certain Pfaffian system [38, Example 4]. In order to study the influence of the parametric function a on the system using the package `DifferentialThomas`, a is included in the list of dependent variables and its dependence on merely x_2 is taken into account by adding the following two equations to the system:

$$\frac{\partial}{\partial x_1} a(x_1, x_2, x_3) = 0, \qquad \frac{\partial}{\partial x_3} a(x_1, x_2, x_3) = 0.$$

Let R be the differential polynomial ring $\mathbb{Q}\{\xi_1, \xi_2, \xi_3, a\}$, endowed with the partial differential operators ∂_1, ∂_2, ∂_3 with respect to x_1, x_2, x_3.

```
> with(DifferentialThomas):
> ivar := [x1,x2,x3]:
> dvar := [xi1,xi2,xi3,a]:
```

We choose a block ranking $>$ on R with blocks $\{\xi_1, \xi_2, \xi_3\}$, $\{a\}$.

```
> ComputeRanking(ivar, [[xi1,xi2,xi3],[a]]):
> L := [-a*xi1[x1]+xi3[x1]-a[x2]*xi2
> +(1/2)*a*(xi1[x1]+xi2[x2]+xi3[x3]),
> -a*xi1[x2]+xi3[x2], -a*xi1[x3]+xi3[x3]
> -(1/2)*(xi1[x1]+xi2[x2]+xi3[x3]), a[x1], a[x3]]:
> LL := Diff2JetList(Ind2Diff(L, ivar, dvar));
```

$$LL := [-a_{0,0,0}\,(\xi_1)_{1,0,0} + (\xi_3)_{1,0,0} + \tfrac{1}{2}\, a_{0,0,0}\,((\xi_1)_{1,0,0} + (\xi_2)_{0,1,0} + (\xi_3)_{0,0,1}) \\ -a_{0,1,0}\,(\xi_2)_{0,0,0}, \quad -a_{0,0,0}\,(\xi_1)_{0,1,0} + (\xi_3)_{0,1,0}, \\ -a_{0,0,0}\,(\xi_1)_{0,0,1} + \tfrac{1}{2}\,(\xi_3)_{0,0,1} - \tfrac{1}{2}\,(\xi_1)_{1,0,0} - \tfrac{1}{2}\,(\xi_2)_{0,1,0}, \quad a_{1,0,0}, \quad a_{0,0,1}]$$

We compute a Thomas decomposition with respect to $>$ of the given system of partial differential equations.

```
> TD := DifferentialThomasDecomposition(LL, []);
```

$$TD := [\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}]$$

The resulting three simple differential systems are given as follows.

```
> Print(TD[1]);
```

$$[a\,\underline{(\xi_1)_{x_2}} - (\xi_3)_{x_2} = 0, \quad a^2\,\underline{(\xi_1)_{x_3}} + (\xi_3)_{x_1} = 0, \quad \underline{\xi_2} = 0, \\ a\,\underline{(\xi_1)_{x_1}} - 2\,(\xi_3)_{x_1} - a\,(\xi_3)_{x_3} = 0, \quad \underline{a_{x_1}} = 0, \quad \underline{a_{x_3}} = 0, \quad \underline{a} \neq 0]$$

```
> Print(TD[2]);
```

$$[a\,\underline{(\xi_1)_{x_2}} - (\xi_3)_{x_2} = 0, \quad a^2\,\underline{(\xi_1)_{x_3}} + a\,(\xi_2)_{x_2} - a_{x_2}\,\xi_2 + (\xi_3)_{x_1} = 0,$$
$$a\,\underline{(\xi_1)_{x_1}} - a\,(\xi_2)_{x_2} + 2\,a_{x_2}\,\xi_2 - 2\,(\xi_3)_{x_1} - a\,(\xi_3)_{x_3} = 0, \quad \underline{a_{x_1}} = 0,$$
$$\underline{a_{x_2,x_2}} = 0, \quad \underline{a_{x_2,x_3}} = 0, \quad \underline{a_{x_3}} = 0, \quad \underline{a} \neq 0, \quad \underline{\xi_2} \neq 0]$$

```
>  Print(TD[3]);
```

$$[\underline{(\xi_1)_{x_1,x_1}} + (\xi_2)_{x_1,x_2} = 0, \quad \underline{(\xi_1)_{x_1,x_2}} + (\xi_2)_{x_2,x_2} = 0, \quad \underline{(\xi_3)_{x_1}} = 0, \quad \underline{(\xi_3)_{x_2}} = 0,$$
$$\underline{(\xi_1)_{x_1}} + (\xi_2)_{x_2} - (\xi_3)_{x_3} = 0, \quad \underline{a} = 0]$$

With regard to the parametric function a, the first simple system is the most generic one, in the sense that $a = a(x_2)$ is only assumed to be non-zero, whereas in the second and third simple systems a is subject to further equations. In particular, the additional condition $a_{x_2,x_2} = 0$ derived in [38] to ensure formal integrability of the system is exhibited in the second simple system of the Thomas decomposition.

5. CONCLUSION

In this paper the Thomas decomposition technique for systems of nonlinear partial differential equations and inequations has been applied to nonlinear control systems. The method splits a given differential system into a finite family of simple differential systems which are formally integrable and define a partition of the solution set of the original differential system. This symbolic approach allows to deal with both differential equations and inequations, which may involve parameters.

Using elimination properties of the Thomas decomposition technique, structural properties of nonlinear control systems have been investigated. In particular, notions such as invertibility, observability and flat outputs can be studied. In the presence of parameters, different simple systems of a Thomas decomposition in general represent different structural behavior of the control system. A Maple implementation of Thomas' algorithm has been used to illustrate the techniques on explicit examples.

At the time of this writing it is unclear how to adapt or generalize the techniques to nonlinear differential time-delay systems or even systems of nonlinear difference equations in full generality. An analog of the notion of Thomas decomposition is not known for systems of nonlinear difference equations. However, note that, generalizing work of J. F. Ritt [40] and R. M. Cohn [8] and others, characteristic set methods have been developed for ordinary difference polynomial systems and differential-difference polynomial systems (cf., e.g., [15], [16]).

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