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The Surreal Numbers and Combinatorial Games

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Abstract

In the first half of this paper we study John H. Conway's construction of the Surreal Numbers, showing it is a proper class that forms the totally ordered Field **No** that extends the real and ordinal numbers, and then explore some of these novel numbers, such as $\omega - 1$, where ω is the first von Neumann ordinal. In the second half we then introduce the notion of Games as a precise expression of two player perfect information sequential games, and analyse several of these Games such as Nim, Brussel Sprouts, and the original Game of Borages.

General Introduction

...a thing once for all done and there you are somewhere and finished in a certain time, be it a day or a year or even supposing, it should eventually turn out to be a serial number of goodness gracious alone knows how many days or years.

– James Joyce, *Finnegans Wake [1, p.118]*

The fundamental concept explored in this paper is precisely what mathematical structures emerge if, sometimes with restrictions, we construct objects called $Games$, which are ordered pairs of independent sets containing already constructed Games, one of which we call the Left set and the other the Right set. These fractal constructions, first discovered and studied by John H. Conway, are able to describe both the games of Game Theory after which they are named, as well the real and ordinal numbers, and more besides. This paper provides an introduction to Games, and specifically to Conway's Surreal Numbers, as introduced in his 1976 book *On Numbers and Games*, a subclass of Games that contains and extends the real and ordinal numbers to a Field [2]. It intends to be readable to the undergraduate student, requiring no prerequisite knowledge except some of the basics of Zermelo-Fraenkel set theory (ZFC), which when needed is refreshed along the way. The paper is split into two parts: in sections 2-5 we study the Surreal Numbers, showing they form a totally ordered field, comparing the surreal constructions of the reals and ordinals to their usual constructions in ZFC (via equivalence relations, Dedekind cuts &c.) and exploring the new numbers that emerge from this method of construction. In sections 6-8 we introduce the concept of combinatorial games and how they relate to the mathematical objects of Games, some of the mechanisms and arguments by which we try to analyse these games, and then work through and prove the complete theory for a few of these games. At the end of the paper there is also an appendix containing the oft-used definitions of the first half for easy reference.

Most of the proofs in the first half of the paper follow those of Conway in *On Numbers and Games*, who is notably terse and leaves much to the reader (to be expected from a 200-page book written in a week), though they are here expanded and explicitly reasoned, and the paper attempts to be comprehensive in the material that it covers, omitting no proofs except those which are obvious repetitions. In contrast, much of the reasoning, notation, and many of the proofs in the second half are, for better or worse, the author's own, and the final section of the paper, on the impartial game of Borages (rhymes with porridges), is, as far as the author can tell, original.

An Introduction to the Surreal Numbers

Loosely, the Surreal Numbers are a totally ordered class that form a Field (a field of a class, rather than a set) that extends the real and ordinal numbers, and is the largest totally ordered Field described yet in history (in von Neumann-Bernays-Gödel set theory (a conservative extension of ZFC to proper classes) it has been shown to be the largest possible ordered Field [3, p.362]). In the next few sections we shall show this rigorously, but in this section we less formally explore the (literally) simplest surreal numbers and arguments to help develop the reader's intuition around this unique construction, as well as defining many key terms.

Construction of Numbers

We begin with two axioms that must be considered in tandem:

Axiom 1: For any two *sets of numbers* L and R,

$$\exists \text{ the number } \{L|R\} \iff \nexists x \in L : x \geqslant y, \forall y \in R.$$

That is, there exists a new number $\{L|R\}$ if and only if no member of L is greater-than-or-equal to any member of R.

We denote the left set of a number a as A^L and the right set as A^R, so $a = \{A^L|A^R\}$. It is important to distinguish between numbers and sets of numbers, so we use lower case letters to denote numbers, and upper case letters to denote sets of numbers. We also denote an arbitrary member of A^L as a^L, and write $a^L = \{A^{LL}|A^{LR}\}$, and similarly write $a^R = \{A^{RL}|A^{RR}\}$ for a member of A^R.

Axiom 2: For any two numbers $x = \{X^L|X^R\}$ and $y = \{Y^L|Y^R\}$,

$$x \leqslant y \iff \nexists x^L \in X^L : x^L \geqslant y \wedge \nexists y^R \in Y^R : y^R \leqslant x.$$

That is, $x \leqslant y$ if and only if no member of X^L (the left set of x) is greater-than-or-equal to y and no member of Y^R (the right set of y) is less-than-or-equal to x. We will also often use the inverse of this:

$$x \nleqslant y \iff \exists x^L \in X^L : x^L \geqslant y \vee \exists y^R \in Y^R : y^R \leqslant x$$

We also define equality here as: $x = y \iff x \leqslant y \wedge y \leqslant x$, define less-than as $x < y \iff x \leqslant y \wedge y \nleqslant x$, and define greater-than as $x > y \iff x \geqslant y \wedge y \ngeqslant x$.

We see directly then that equality is a symmetric relation between two numbers (i.e. $x = y \iff y = x$), that both less-than and greater-than are asymmetric relations (i.e. $x < y \implies y \nless x$), and that both less-than-or-equal and greater-than-or-equal are antisymmetric (i.e. $x \leqslant y \wedge y \leqslant x \implies x = y$).

From these two axioms we can begin to construct and order numbers. First we invoke the Axiom of Existence from ZFC (which states there exists a set containing no elements, called the empty set, written $\varnothing = \{\}$) and consider the empty set. Then by Axiom 1, letting $L = R = \varnothing$, there exists a number $\{\varnothing|\varnothing\}$, since the empty set has no elements and so Axiom 1's requirement is automatically fulfilled. Let us call this number $0 := \{\varnothing|\varnothing\}$ (and we will later show that it is the additive identity for the surreal numbers as we expect from zero), giving us two sets of numbers, $\varnothing$ and $\{0\}$. Then we can construct three more possible numbers:

$$a := \{\{0\}|\varnothing\} \quad b := \{\varnothing|\{0\}\} \quad c := \{\{0\}|\{0\}\}$$

(For ease, since in all our constructions there is only one left set, and only one right set, we usually omit the outmost brackets of these sets, and if one of these sets is the empty set, we leave that side empty. So we rewrite: $a = \{0|\}, b = \{|0\}, c = \{0|0\}, 0 = \{|\}$).

For a, the only member of A^L is 0, and there are no members of A^R, so from Axiom 1, a is a number. Similarly, as there are no members of B^L, b is a number. Generally, any construction $x = \{X^L|X^R\}$ with either $X^L = \varnothing$ or $X^R = \varnothing$ will be a number, as Axiom 1 will hold vacuously. For c, we have $0 \in C^L$ and $0 \in C^R$. But from Axiom 2 we know that $0 \geqslant 0$, so by Axiom 1 we see that c is not a number (it is a $Game$, a more general construction we explore in the second half of this paper).

Now we have three numbers, 0, a, and b. We can use Axiom 2 to order them. First we consider 0 and a:

> Is $0 \leqslant a$? There are no members of 0^L, so $\nexists x \in 0^L : x \geqslant a$. Similarly, there are no members of A^R, so $\nexists a^R \in A^R : a^R \leqslant 0$. So by Axiom 2, $0 \leqslant a$.
>
> Is $a \leqslant 0$? $0 \in A^L$ and $0 \leqslant 0$, so by Axiom 2, $0 \nleqslant a$.
>
> So we have $0 \leqslant a$ and $a \nleqslant 0$, which implies $0 < a$.

Next we consider 0 and b:

> Since $0 \in B^R$ and $0 \leqslant 0$, we have $0 \nleqslant b$. And since $B^L = 0^R = \varnothing$, the conditions for $b \leqslant 0$ hold vacuously, so $0 \nleqslant b \wedge b \leqslant 0 \implies b < 0$.

We can now order our three numbers: $b < 0 < a$, and it is easy to check directly by the same method that $b < a$. We write $1 := a = \{0|\}$ and $-1 := b = \{|0\}$, and will justify these names later. For now we consider the eight sets of numbers we can now form:

$$\emptyset \qquad \{-1\} \qquad \{0\} \qquad \{1\} \qquad \{-1,0\} \qquad \{-1,1\} \qquad \{0,1\} \qquad \{-1,0,1\}$$

Pairing these up into left and right sets, we get 64 candidates for numbers. However, since we have ordered $-1, 0$ and 1, we can quickly show using Axiom 1 that most of them are not numbers, leaving us with the following:

$$\begin{array}{llllll} \{-1|\} & \{-1,0|\} & \{-1,1|\} & \{-1,0,1|\} & \{0,1|\} & \{1|\} \\ \{|-1\} & \{|-1,0\} & \{|-1,1\} & \{|-1,0,1\} & \{|0,1\} & \{|1\} \\ \{-1|0\} & \{-1|0,1\} & \{-1|1\} & \{-1,0|1\} & \{0|1\} & \end{array}$$

However, by using the Truncation Theorem later introduced in section 2 (Theorem 3.8, which tells us we can remove all but the greatest element in the left set, and all but the least in the right set, and this new construction will be equal to the original one), and noting that, for example, $0 \leqslant \{-1|1\}$ and $\{-1|1\} \leqslant 0$ (meaning $\{-1|1\} = 0$), we can show that many of these candidates are equal to each other (we formalise this in the next section after showing $=$ is an equivalence relation), leaving us finally with only these four new numbers created (and ordered using Axiom 2):

$$\{|-1\} < -1 < \{-1|0\} < 0 < \{0|1\} < 1 < \{1|\}$$

Indeed we will show later that we are justified in calling these numbers $-2 := \{|-1\}$, $-1/2 := \{-1|0\}$, $1/2 := \{0|1\}$ and $2 := \{1|\}$, that is that they have the properties we expect, such as $1/2 + 1/2 = 1$.

We began with one number, 0, from which we constructed two more numbers, -1 and 1, from which in turn we constructed four more numbers. Notice that it would be impossible to construct 2 before constructing 1, or 1 before constructing 0. This is because 0 was constructed in a previous step, or day. We call one number simpler than another if it was constructed on an earlier day, and call the day a number is constructed on is its $birthday$. Thus 1 is simpler than 2, but 2 is as simple as -2 or 1/2, since they have the same birthday. We say then that 0 was created on day 0, 1 on day 1, &c. Similarly, if we have a set of numbers, we call the sum of their birthdays their $day\ sum$, and if one set of numbers has a lower day sum than a different set of numbers, we say that set is the simpler set.

Definition. *Birthday: The birthday of a number is day on which it is first constructed.*

Definition. *Simplicity: We say that a number x is simpler than another number y if x has an earlier birthday than y.*

The Surreal Numbers are a Proper Class

We distinguish between lower case $field$ and upper case $Field$ because the Surreal Numbers are not a set. Similarly to how the existence of an ordinal number in ZFC implies the existence of another ordinal [4, p. 108], the existence of a surreal number x implies at the very least that the new surreal number $\{x\,|\}$ exists. Thus the Surreal Numbers are not a set, but a proper class.

Methods of Proof

We will call the elements of X^L the left $options$ of x, and the elements of X^R the right $options$ of x. Often we use induction on the options of a number x to show that a property P holds for that x. In terms of birthdays, we can express this method of proof as follows: suppose that, for some property P, x is the simplest number for which P does not hold. Then if we can show that this implies that P does not hold for one of the options of x, we have a contradiction, since we can always express x in a form where it has only options simpler than x, since this is the only way to express x on its birthday. This means that either P holds for no numbers, or if we show that P holds for the simplest number, 0, that P holds for all numbers. In the first proof below we will repeat this argument explicitly, but after that we will use it implicitly, and use phrases like 'we eventually only have to check the case of 0' or 'we inductively reduce the question down to 0' to refer to this reasoning.

We will in general be proving theorems on all numbers, including infinite ordinals. Usually in transfinite induction we must show that P holds for the base case ($P(0)$ holds), the successor case (for any successor ordinal $\alpha + 1$, $P(\alpha + 1)$ holds if $P(\alpha)$ holds), and the limit case (for any limit ordinal β, $P(\beta)$ holds if $P(\beta')$ holds for all $\beta' < \beta$) [4, p. 114]. However, since here we do induction on the birthday of a number, we treat each day after 0 as a limit case, and only need to show $P(0)$ holds, and then that for any number x, including ordinals, $P(x)$ follows from $P(x')$ for $some$ x' simpler than x, since we assume that $P(x')$ holds for all x' created on an earlier day than x.

Ordering the Surreal Numbers

In this section we build on our two Axioms and show that the Surreal Numbers are totally ordered, and then prove some more general properties of the surreals, which allow us to manipulate them up to equality, which will be very useful for later proofs.

Lemma 0.1. *For any number $x = \{X^L|X^R\}, x \leqslant x$ and $x = x$.*

Proof.

(a) $x \leqslant x$: From Axiom 2, we must show

$$\nexists x^L \in X^L : x^L \geqslant x, \text{ or equivalently, } x \nleqslant x^L, \forall x^L \in X^L \tag{0.1.1}$$

$$\nexists x^R \in X^R : x^R \leqslant x, \text{ or equivalently, } x \ngeqslant x^R, \forall x^R \in X^R \tag{0.1.2}$$

but $x \nleqslant x^L$ if there exists an $\widetilde{x}^L \in X^L$ such that $\widetilde{x}^L \geqslant x^L$ for any x^L, and $x \ngeqslant x^R$ if there exists an $\widetilde{x}^R \in X^R$ such that $\widetilde{x}^R \leqslant x^R$ for any x^R. But we can choose $\widetilde{x}^L = x^L$ and $\widetilde{x}^R = x^R$, and in this way we have reduced the question on x to questions on the options of x, that is we now know that $x \leqslant x$ if both $x^L \leqslant x^L$ and $x^R \leqslant x^R$. Now suppose that the theorem does not hold for x, and that furthermore x is the simplest number for which the theorem does not hold. Then we must also have either that the theorem holds for no numbers, because the theorem holding for simpler numbers would imply it holding for numbers with those simpler numbers as options, or that the theorem holds for all numbers, if the theorem holds for the simplest number, 0. So we only have to consider only whether $0 \leqslant 0$. But this follows from the fact that 0 has no options, so by induction, $x \leqslant x$ for all x. This method of argument will we use often, and from now on, implicitly. See Section 2.3 above for a full explanation of the argument.

(b) $x = x$: This follows directly from our definition of $=$ and (a). □

Theorem 0.2. *For any number $x = \{X^L|X^R\}$, $x^L < x$ for all $x^L \in X^L$.*

Proof.

(a) We first show that $x^L \leqslant x$ for all $x^L \in X^L$. For this to be true we must have from Axiom 2 that both

$$\nexists x^{LL} \in X^{LL} : x^{LL} \geqslant x, \text{ and} \tag{0.2.1}$$

$$\nexists x^R \in X^R : x^R \leqslant x^L, \forall x^L \in X^L. \tag{0.2.2}$$

Note that (0.2.2) is just restating of Axiom 1, so we just need to show (0.2.1), which can be equivalently written as

$$\forall x^{LL} \in X^{LL}, x \nleqslant x^{LL}. \tag{0.2.3}$$

Then by the inverse form of Axiom 2, (0.2.3) is true if

$$\exists x^L \in X^L : x^L \geqslant x^{LL}, \forall x^{LL} \in X^{LL} \tag{0.2.4}$$

but this is the same condition that we wanted to show at the start of (a), replacing x with x^L and x^L with x^{LL}, that is to say, we have $x^L \leqslant x$ only if $x^{LL} \leqslant x^L$, for all $x^{LL} \in X^{LL}$. By repeating this process, we will eventually only have to consider sets whose only left option is 0, so (0.2.1) will hold vacuously. Thus by induction, $x^L \leqslant x$ for all $x^L \in X^L$.

(b) Now we show that $x \nleqslant x^L$ for all $x^L \in X^L$. By the inverse form of Axiom 2, to prove this it is enough to show that

$$\exists \tilde{x}^L \in X^L : \tilde{x}^L \geqslant x^L \tag{0.2.5}$$

and we can choose $\tilde{x}^L = x^L$, and then (0.5.1) holds from Lemma 0.1. Thus $x^L < x$ for all $x^L \in X^L$. □

Theorem 0.3. *For any number $x = \{X^L|X^R\}$, $x < x^R$ for all $x^R \in X^R$.*

Proof.

By a symmetric argument the proof for this is the same as the proof from Theorem 0.2, but considering the right options of x. □

We say that any two numbers $x = \{X^L|X^R\}$ and $y = \{Y^L|Y^R\}$ are $identical$ if and only if $X^L = Y^L$ and $X^R = Y^R$, and write $x \equiv y$ to express identity. We also simplify the notation here by writing 'for all/there exists x^L' instead of the longer 'for all/there exists $x^L \in X^L$', since there is no ambiguity, as every number has only one left set and one right set.

Theorem 0.4. *If two numbers x and y are identical, then they are also equal.*

Proof.

Since x and y are identical, $X^L = Y^L$ and $X^R = Y^R$. Then $x \leqslant y$ if there does not exist any $y^L \geqslant y$ or any $x^R \leqslant x$, and $y \leqslant x$ if there does not exist any $x^L \geqslant x$ or any $y^R \leqslant y$. But from Theorems 0.2/0.3 none of these exist, so $x = y$.

□

Theorem 0.5. *For all numbers x, y, and z, if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.*

Proof.

We will do a proof by contradiction, so assume that the proposition $\pi(x, y, z) : (x \leqslant y \wedge y \leqslant z \wedge x \nleqslant z)$ holds. Then the following all hold by Axiom 2:

$$\nexists x^L : x^L \geqslant y \tag{0.5.1}$$

$$\nexists z^R : z^R \leqslant y \tag{0.5.2}$$

$$\exists x^L : x^L \geqslant z \vee \exists z^R : z^R \leqslant x \tag{0.5.3}$$

If we suppose the first proposition of (0.5.3) holds, then by considering also (0.5.1) the following holds:

$$\exists x^L : y \leqslant z \wedge z \leqslant x^L \wedge y \nleqslant x^L \tag{0.5.4}$$

That is, $\pi(y, z, x^L)$. If we suppose the second proposition of (0.5.3) holds, then by considering also (0.5.2) the following holds:

$$\exists z^R : z^R \leqslant x \wedge x \leqslant y \wedge z^R \nleqslant y \tag{0.5.5}$$

That is, $\pi(z^R, x, y)$. So in either case the truth of $\pi(x, y, z)$ depends on the truth of π with one of x or z replaced by one of their options. Then, since (0.5.3) will not hold for any $X^L = Z^R = \emptyset$, by induction $\pi(x, y, z)$ will not hold. So we must have $x \leqslant y \wedge y \leqslant z \implies x \leqslant z$, that is that numbers are transitive under $\leqslant$. □

It follows directly from Theorem 0.5 that $=$ is transitive, so we have now shown that $=$ is an equivalence relation on numbers, as it is reflexive, symmetric, and transitive. Then equality partitions the surreals into equivalence classes, and in general when we talk of constructing a $new\ number$ we mean a number that is not equal to an already constructed number, that is it does not belong to any already existing equivalence class. We call a construction that is not identical to an already constructed number, but that is equal to one, a new $form$ of that number, so that on day 2 we construct the new number $2 = \{0, 1|\}$, but the new construction $\{-1, 0|\}$ is just a new form of the number 1. In section 5 we define the $natural\ form$ of a number, which is the simplest representation of any equivalence class that can be constructed in a finite number of days.

Theorem 0.6. *For any numbers x and y, either $x \leqslant y$ or $y \leqslant x$.*

Proof.

By contradiction, suppose neither $x \leqslant y$ nor $y \leqslant x$. Then by the inverse of Axiom 2:

$$\exists x^L : x^L \geqslant y \vee \exists y^R : y^R \leqslant x \tag{0.6.1}$$

$$\exists y^L : y^L \geqslant x \vee \exists x^R : x^R \leqslant y \tag{0.6.2}$$

Then there are four combinations of statements that we must show are contradictory:

(a) $\exists x^L : x^L \geqslant y$ and $\exists y^L : y^L \geqslant x$. From Theorem 0.2 we know $x^L \leqslant x$ and $y^L \leqslant y$. It follows from Theorem 0.5 that $y \leqslant x^L \leqslant x$ and $x \leqslant y^L \leqslant y$. But then $x = y$, which contradicts our supposition.

(b) $\exists x^L : x^L \geqslant y$ and $\exists x^R : x^R \leqslant y$. It follows that $\exists x^L, x^R : x^R \leqslant x^L$, but this contradicts Axiom 1.

(c) $\exists y^R : y^R \leqslant x$ and $\exists y^L : y^L \geqslant x$. The argument is as in (b).

(d) $\exists y^R : y^R \leqslant x$ and $\exists x^R : x^R \leqslant y$. Similarly to (a), from Theorem 0.3 it follows that $y \leqslant y^R \leqslant x$ and $x \leqslant x^R \leqslant y$, so $x = y$.

So numbers are total under $\leqslant$.

□

We have now shown that $\leqslant$ is a non-strict total order on numbers, as it is reflexive (from Lemma 0.1), antisymmetric (from the definition of equality), transitive (from Theorem 0.5), and total (from Theorem 0.6).

Theorem 0.7. *$<$ is a strict total order on numbers.*

Proof.

For any numbers x, y, and z, we have:

(a) Irreflexivity: if $x < x$, then both $x \leqslant x$ and $x \nleqslant x$ hold, which is contradictory.

(b) Trichotomy: we show that in all possible cases we have a contradiction. If $x < y$ and $y < x$, then we must have both $x \leqslant y$ and $x \nleqslant y$; if $x = y$ and $x < y$, then $y \leqslant x$ and $y \nleqslant x$; if $x = y$ and $y < x$, then $x \leqslant y$ and $x \nleqslant y$. All three cases produce contradictions so we must have no more than one of $x < y$, $x = y$, and $y < x$. But if $x \nless y$ and $y \nless x$ and $y \neq x$, then we have ($x \nleqslant y$ or $y \leqslant x$) and ($y \nleqslant x$ or x $\leqslant y$) and ($x \nleqslant y$ or $y \nleqslant x$). But for any combination, either ($x \leqslant y$ and $x \nleqslant y$) or ($y \leqslant x$ and $y \nleqslant x$), or ($x \nleqslant y$ and $y \nleqslant x$). The first two are clear contradictions, and if follows from the latter that one of $x^L \geqslant x^R$, $y^R \leqslant y^L$, $y^L \geqslant y$, or $y^R \leqslant y$ holds, all of which are contradictions of Axiom 1 or Theorems 0.2/0.3, so exactly one of $x < y$, $x = y$, and $y < x$ holds.

(c) Transitivity: Suppose that $\pi : (x < y$ and $y < z$ and $x \nless z)$ is true. Then we can rewrite the last inequality as $(x \nleqslant z$ or $z \leqslant x)$. If we have then $x \nleqslant z$, then from the definition of $<$, π implies $(x \leqslant y$ and $y \leqslant z$ and $x \nleqslant z)$, which is false from Theorem 0.5. If we have the instead $z \leqslant x$, then noting that $x < y$ implies $y \nleqslant x$ and $y < z$ implies $z \nleqslant y$, π implies $(y \nleqslant x$ and $z \nleqslant y$ and $z \leqslant x)$. But from Theorem 0.5, $y \nleqslant x$ implies $(y \nleqslant z$ or $z \nleqslant x)$. So either $(y \nleqslant z$ and $y < z)$, or $(z \nleqslant x$ and $z \leqslant x)$, both of which are contradictory, so π must be false, and numbers are transitive under $<$.

Therefore $<$ is a strict total order on numbers.

□

Theorem 0.8. Truncation Theorem: *For any number $x = \{X^L|X^R\}$, such that for some $x_i, x_j \in X^L$ we have $x_i < x_j$, the number $x' = \{X^L \backslash \{x_i\}|X^R\}$ is equal to x. Similarly, for any $y = \{Y^L|Y^R\}$, if for some $y_i, y_j \in Y^R$ we have $y_i > y_j$, the number $y' = \{Y^L|Y^R \backslash \{y_i\}\}$ is equal to y.*

Proof.

For $x = x'$ we need to show that both $x \leqslant x'$ and $x' \leqslant x$. From Axiom 2 these are true if the following are all true:

(a) $\nexists x^L \in X^L : x^L \geqslant x'$

(b) $\nexists x'^R \in X^R : x'^R \leqslant x$

(c) $\nexists x'^L \in X^L \backslash \{x_i\} : x'^L \geqslant x$

(d) $\nexists x^R \in X^R : x^R \leqslant x'$

But (a) follows from the transitivity of $<$ and Theorem 0.2, (c) follows directly from Theorem 0.2, and (b) and (d) follow directly from Theorem 0.3. A similar argument shows that $y = y'$.

□

Corollary 0.8.1. *For any number $z = \{Z^L|Z^R\}$, if Z^L has a greatest member a, we can write $z = \{a\,|Z^R\}$, and if Z^R has a least member b, we can write $z = \{Z^L|\,b\}$.*

Proof.

If a is the greatest element of Z^L, then for all $z^L \neq a$, $z^L < a$, so we can apply the Truncation Theorem on z^L and rewrite $z = \{a\,|Z^R\}$. We can similarly do this for Z^R, and write $z = \{a\,|\,b\}$.

□

A word more on notation: every number has only one left set and one right set. But in specifying that a number's left or right set contains the elements from the union of more than one set, we wish to omit to union sign for ease and esthetics. Thus we read $a = \{A^{L_1}, A^{L_2} | A^{R_1}, A^{R_2}\}$ as $a = \{A^{L_1} \cup A^{L_2} | A^{R_1} \cup A^{R_2}\}$.

Theorem 0.9. Extension Theorem: *For any number $x = \{X^L | X^R\}$, and sets of numbers A and B, if for all $a \in A$, $a < x$, and for all $b \in B$, $x < b$, then $x_e := \{X^L, A | X^R, B\} = x$.*

Proof.

We need to show that $x \leqslant x_e$ and $x_e \leqslant x$. From Axiom 2 then we must show:

(a) $\nexists x^L : x^L \geqslant x_e$

(b) $\nexists x^R : x^R \leqslant x$ and $\nexists b \in B : b \leqslant x$

(c) $\nexists x^L : x^L \geqslant x$ and $\nexists a \in A : a \geqslant x$

(d) $\nexists x^R : x^R \leqslant x_e$

But (a) follows from the fact that $x^L \in x_e^L$ and Theorem 0.2, (b) follows Theorem 0.3 and $x < b$, (c) follows from Theorem 0.2 and $a < x$, and (d) follows from the fact that $x^R \in x_e^R$ and Theorem 0.3.

□

Arithmetic on the Surreal Numbers

Our aim in this section is to show that we can define an arithmetic on the Surreal Numbers such that its equivalence classes have a field structure. That is, for any three surreal numbers x, y, and z, we can define the two operations $+$ (addition) and $\cdot$ (multiplication), such that all the following field axioms hold [5, p. 5]:

(a) Closure under addition and multiplication: $x + y$ is a number and $x \cdot y$ is a number.

(b) Commutativity under addition and multiplication: $x + y = y + x$ and $x \cdot y = y \cdot x$.

(c) Associativity under addition and multiplication: $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(d) Existence of an addition identity 0 such that $x + 0 = x$.

(e) Existence of a multiplicative identity 1 such that $x \cdot 1 = x$.

(f) Existence of an additive inverse $-x$ such that $x + -x = 0$.

(g) Existence of a multiplicative inverse x^{-1} such that $x \cdot x^{-1} = 1$, for all $x \neq 0$.

(h) Distributivity of multiplication over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$.

Note that we will often write $x \cdot y$ as simply xy. We begin now with addition.

Addition

Conway says in *On Numbers and Games*: "The spirit of definitions is to ask what we already know about the object being defined, and to make the answers part of our definition." [2, p. 6] In defining addition then, what we want (if we are to construct an arithmetic that matches our intuitive understanding of the world) is that, for example, $x + y > x^L + y$, for all x and y. Let us then define addition on numbers then as:

$$x + y := \{X^L + y, x + Y^L | X^R + y, x + Y^R\}$$

so as to satisfy all our expectations (recall from Theorems 0.2/0.3 that $(x + y)^L < x + y < (x + y)^R$). However, this definition involves the addition of a number to a set of numbers. So we will define addition between a number and a set of numbers then as:

$$z + A := \{z + a : a \in A\}$$

$$A + z := \{a + z : a \in A\}$$

Clearly then $z + A \equiv A + z$ if addition on numbers is commutative (which we shall show momentarily), and $z + \emptyset \equiv \emptyset + z \equiv \emptyset$. We can now write $x + y$ in an expanded form:

$$x + y = \{x_1^L + y, x_2^L + y, ..., x + y_1^L, x + y_2^L, ... | x_1^R + y, x_2^R + y, ..., x + y_1^R, x + y_2^R, ...\}$$

which is a recursive definition. Inductively however, our definition of addition will cascade down finally into many additions between a number and 0, since 0 is the simplest number, and in evaluating $x_1^L + y$, for example, we have to first know the value of the sum of y and each of the left options of x_1^L. But then we have:

$$z + 0 = \{Z^L + 0, z + \emptyset | Z^R + 0, z + \emptyset\} = \{z_1^L + 0, ... | z_1^R + 0, ...\}$$

$$0 + z = \{\emptyset + z, 0 + Z^L | \emptyset + z, 0 + Z^R\} = \{0 + z_1^L, ... | 0 + z_1^R, ...\}$$

which will themselves reduce down finally to:

$$0 + 0 \equiv \{\emptyset + 0, 0 + \emptyset | \emptyset + 0, 0 + \emptyset\} \equiv 0$$

giving us, eventually, an inductive basis for addition. That is, since $0+0$ is well defined, $0+1$ and $1+0$ are well defined, as 0 is the only option of 1, which in turn means $1+1$ is well defined, which is turn means $2+0, 0+2, 2+1, 1+2$ are well defined, ad infinitum.

We will now show addition on numbers to be commutative, associative, and to have an identity element, as well as that $y \geqslant z$ if and only if $x+y \geqslant x+z$.

Theorem 0.10. *For any numbers x, y and z, we have:*

(a) $0 \equiv \{|\}$ *as the identity element:* $x + 0 \equiv x$

(b) Commutativity: $x + y \equiv y + x$

(c) Associativity: $(x + y) + z \equiv x + (y + z)$

Proof.

(a) $x + 0 \equiv \{X^L + 0|X^R + 0\} \equiv \{x_1^L + 0, ...|x_1^R + 0, ...\}$. Then by induction $x + 0 \equiv x$ if $0 + 0 \equiv 0$, but this follows from $0 + 0 = 0$ as shown above.

(b) We have $x+y \equiv \{X^L+y, x+Y^L|X^R+y, x+Y^R\} \equiv \{x_1^L+y, ..., x+y_1^L, ...|x_1^R+y, ..., x+y_1^R, ...\}$ and $y + x \equiv \{Y^L + x, y + X^L|Y^R + x, y + X^R\} \equiv \{y_1^L + x, ..., y + x_1^L, ...|y_1^R + x, ..., y + x_1^R, ...\}$. The commuativity of x and y then depends on the commutativity of the pairs formed of one of x or y and an option of the other. So inductively we need to check only that $x + 0 \equiv 0 + x$. But $0 + x \equiv \{0 + x_1^L, ...|0 + x_1^R, ...\}$, so inductively $0 + x \equiv x \equiv x + 0$, since $0 + 0 \equiv 0 + 0$.

(c) We have $\{(x_1^L+y)+z, ..., (x+y_1^L)+z, ..., (x+y)+z_1^L, ...\}$ for the left set of $(x+y)+z$, and $\{x_1^L + (y + z), ..., x + (y_1^L + z), ..., x + (y + z_1^L), ...\}$ for the left set of $x + (y + z)$. So again we inductively reduce the question down to associativity on 0, but clearly $(x + y) + z \equiv x + (y + z)$ when one of $x, y,$ or z is equal to 0 from (a). The same argument shows that $((x + y) + z)^R \equiv (x + (y + z))^R$, finishing the proof. □

Theorem 0.11. *For any numbers x, y and z, we have $\pi(x, y, z) : (x \leqslant y \iff x + z \leqslant y + z)$. Furthermore, $x < y \iff x + z < y + z$, and $x = y \iff x + z = y + z$.*

Proof.

(a) First, suppose $x + z \leqslant y + z$. Then we have both

$$\nexists x^L : x^L + z \geqslant y + z \tag{0.11.1}$$

$$\nexists y^R : y^R + z \leqslant x + z \tag{0.11.2}$$

Now further suppose that $\pi(x^L, y, z)$ holds. Then $x^L \leqslant y \iff x^L + z \leqslant y + z$. But from (0.11.1) and Theorem 0.6, we know the right hand side of $\pi(x^L, y, z)$ holds, so we have $x^L \leqslant y$.

Next suppose that $\pi(x, y^R, z)$ holds. Then $x \leqslant y^R \iff x + z \leqslant y^R + z$. But again we know from (0.11.2) and Theorem 0.6 that the right hand side holds, so we have $y^R \geqslant x$

But $x^L \leqslant y$ and $y^R \geqslant x$ implies $x \leqslant y$. So if our assumptions of $\pi(x^L, y, z)$ and $\pi(x, y^R, z)$ hold we have shown one side of the theorem. But by induction this reduces down to showing that $\pi(0, 0, z)$ holds, which it does since we showed in Theorem 0.10 that 0 is the additive identity.

(b) Next suppose that $x \leqslant y$. Then

$$\nexists x^L : x^L \geqslant y \tag{0.11.3}$$

$$\nexists y^R : y^R \leqslant x \tag{0.11.4}$$

By contradiction, assume $x + z \nleqslant y + z$. Then one of the following is true:

$$\exists x^L : x^L + z \geqslant y + z \tag{0.11.5}$$

$$\exists z^L : x + z^L \geqslant y + z \tag{0.11.6}$$

$$\exists y^R : y^R + z \leqslant x + z \tag{0.11.7}$$

$$\exists z^R : y + z^R \leqslant x + z \tag{0.11.8}$$

If (0.11.5) is true, then further suppose that $\pi(x^L, y, z)$ holds. Then $x^L \leqslant y \iff x^L + z \leqslant y + z$. But since $x \leqslant y$, we know $x^L \leqslant y$ from Theorem 0.5. So then $\pi(x, y, z)$ being false depends on $\pi(x^L, y, z)$ being false. But inductively, as in part (a), we know that π holds for the basis case, so we get a contradiction. Similar arguments show that the other possibilities from $x + z \nleqslant y + z$ also give contradictions, so we must have $x \leqslant y$ implies $x + z \leqslant y + z$.

(c) Next, if $x < y$, then $x \leqslant y$ and $y \nleqslant x$. Then it follows directly from (a) and (b) that $x + z \leqslant y + z$ and $y + z \nleqslant x + z$, that is, $x + z < y + z$. Similarly, if $x + z < y + z$, it follows directly that $x < y$.

(d) Finally, if $x = y$, then $x \leqslant y$ and $y \leqslant x$. Then it follows directly from (a) that $x + z \leqslant y + z$ and that $y + z \leqslant x + z$, which implies $x + z = y + z$. Similarly, if $x + z = y + z$ it follows directly that $x = y$. $\square$

Theorem 0.12. $\pi(w, x, y, z)$*: (If* $w \leqslant x$ *and* $y \leqslant z$*, then* $w + y \leqslant x + z$*. If* $w < x$ *and* $y < z$*, then* $w + y < x + z$*)*

Proof.

We have already shown the first part to be true if either $w = x$ or $y = z$ in Theorem 0.11, so we just consider the case when $w < x$ and $y < z$. We need to show that there is no element of $(w + z)^L$ greater-than-or-equal to $y + z$ and no element of $(y + z)^R$ less-than-or-equal to $w + x$, and the same in the other direction. For example, we must show that $w^L + x < y + z$. But since $w^L < w < x$, this is $\pi(w^L, x, y, z)$. Similar arguments reduce π down onto its options in the other cases, so by induction π holds for all w, x, y, z. □

Theorem 0.13. *If* x *and* y *are numbers, then* $x + y$ *is a number.*

Proof.

From Axiom 1 we must show that no element in $(x + y)^L$ is greater-than-or-equal to any element in $(x+y)^R$. But if all of $x^L+y, x+y^L, x^R+y, x+y^R$ are numbers, this follows from Theorems 0.2/0.3/0.5/0.11. For example, since $x^L < x$ and $y < y^R$, we know $x^L + y < x + y < x + y^R$. So inductively we reduce the question on $x + y$ down to questions on the sums of one of x or y and an option of the other. Eventually then we only need to show that for a number z that $z + 0$ and $0 + z$ are numbers, which follows from 0 being the additive identity, so the theorem holds for any numbers x and y. □

It should be noted that when applied to ordinal numbers (defined in the next section), the sum refers to the natural ordinal sum, that is that $\alpha + \beta$ is the least ordinal greater than all $\alpha' + \beta$ and $\alpha + \beta'$ for all $\alpha' < \alpha, \beta' < \beta$. It is also possible to define and use the normal ordinal sum [2, p. 193], but then the field structure of the surreals is lost, since the ordinal sum is not commutative.

Negation

We define the negation of a number x as:

$$-x = \{-X^R | -X^L\}$$

where for any set of numbers A

$$-A = \{-a_1, -a_2, -a_3, ...\} \text{ for all } a_i \in A$$

Then we have the following properties:

Theorem 0.14. *For any numbers x and y, if $-x$ and $-y$ are also numbers, we have $x \leqslant y \iff -y \leqslant -x$, and furthermore $x < y \iff -y < -x$*

Proof.

The first part follows directly from Theorem 0.11, since $x \leqslant y \implies x + -y \leqslant y + -y \implies x + -y + -x \leqslant y + -y + -x \implies -y \leqslant -x$. Similarly $-y \leqslant -x$ implies $x \leqslant y$. The second part follows from the first and Theorem 0.7.

□

Theorem 0.15. *If any x is a number, then so is $-x$*

Proof.

We have $x = \{X^L|X^R\}$ and $-x = \{-X^R| - X^L\}$. We need to show by Axiom 1 that no $-x^R$ is greater-than-or-equal to any $-x^L$. But since x is a number we have $x^L < x^R$, for all x^L, x^R, and if $-x^L$ and $-x^R$ are also numbers, then by Theorem 0.14, we must have that $-x^R < -x^L$, that is that x is a number. So inductively we reduce the question on x to questions of the options of x, and eventually we only have to consider the theorem for 0, but since $0 \equiv -0$, the theorem holds for 0 and therefore inductively for any number x.

□

Theorem 0.16. *For any numbers x and y:*

(a) $-(-x) \equiv x$

(b) $-(x + y) \equiv -x + -y$

(c) $x + -x = 0$

Proof.

(a) $-(-x) \equiv -\{-X^R|-X^L\} \equiv \{-(-X^L)|-(-X^R)\}$. Then by induction, since $-(-0) \equiv -(0) \equiv 0$, we have $-(-x) \equiv x$.

(b) We have $-(x+y) \equiv \{-(X^R+y), -(x+Y^R)| - (X^L+y), -(x+Y^L)\}$ and $-x+-y \equiv \{-X^R+-y, -Y^R+-x| - X^L+-y, -x+-Y^L\}$, so by induction $-(x+y) \equiv -x+-y$ because $-(0 + y) \equiv -0 + -y$ and $-(x + 0) \equiv -x + -0$.

(c) We want to show that both $x + -x \leqslant 0$ and $0 \leqslant x + -x$. $x + -x$ in expanded form is:

$$x + -x = \{X^L + -x, x + -X^R|X^R + -x, x + -X^L\}$$

By contradiction let us assume that $x + -x \ngeqslant 0$. This is the case if there exists an element of $(x + -x)^R$ that is less-than-or-equal to 0, or some element of 0^L is greater-than-or-equal to $x + -x$. But 0^L has no elements so we need either some $X^R + -x \leqslant 0$ or some $x + -X^L \leqslant 0$. But this is only true if there is no element in $(X^R + -x)^L$ greater-than-or-equal to 0 and no element in $(x + -X^L)^L$ greater-than-or-equal to 0. But if we expand these sets we have:

$$(X^R + -x)^L = \{(x_1^R + -x)^L, (x_2^R + -x)^L, ...\}$$

$$(x + -X^L)^L = \{(x + -x_1^L)^L, (x + -x_2^L)^L, ...\}$$

and

$$x_1^R + -x_1^R \in (x_1^R + -x)^L,\ x_2^R + -x_2^R \in (x_2^R + -x)^L, ...$$

$$x_1^L + -x_1^L \in (x + -X^L)^L,\ x_2^L + -x_2^L \in (x + -X^L)^L, ...$$

all of which we need to be not greater-than-or-equal to 0. So by induction, since $0 + -0 \leqslant 0$, we have $x + -x \leqslant 0$. A similar argument shows that $0 \leqslant x + -x$, finishing the proof. $\square$

We have now shown that the equivalence classes of the surreals form an abelian group under addition: they are closed, associative, commutative, with 0 as the identity element and $-x$ as the inverse element of x.

Multiplication

In defining multiplication, we can use the fact that we know, for example, $x - x^L > 0$ and $y - y^L > 0$, and that we want the property that $(x - x^L)(y - y^L) > 0$, and also the property that we can expand this to get $xy - x^L y - xy^L + x^L y^L > 0$. So we want $xy > x^L y + xy^L - x^L y^L$, and similarly we can formulate inequalities for all the other combinations of $x - x^L > 0, y - y^L > 0, x - x^R < 0, y - y^R < 0$, to give us a tentative definition of multiplication as:

$$xy = \{X^L y + xY^L - X^L Y^L, X^R y + xY^R - X^R Y^R | X^L y + xY^R - X^L Y^R, X^R y + xY^L - X^R Y^L\}$$

which we will now check has all the properties we expect of it.

Similarly to addition and negation, we use the notation $Ax = \{ax : a \in A\}$, and $AB = \{ab : a \in A, b \in B\}$ to express multiplication on sets.

Theorem 0.17. *For any numbers x and y:*

(a) $x \cdot 0 \equiv 0$

(b) $x \cdot 1 \equiv x$

(c) $yx \equiv xy$

(d) $(-x)y \equiv x(-y) \equiv -xy$

Proof.

(a) Since $0^L = 0^R = \emptyset$, $x \cdot 0 \equiv \{|\} \equiv 0$.

(b) $x \cdot 1 \equiv \{X^L|X^R\} \cdot \{0|\} \equiv \{X^L \cdot 1|X^R \cdot 1\}$, so inductively, as $0 \cdot 1 \equiv 0$, we have $x \cdot 1 \equiv x$.

(c) $yx = \{Y^Lx + yX^L - Y^LX^L, Y^Rx + yX^R - Y^RX^R|Y^Lx + yX^R - Y^LX^R, Y^Rx + yX^L - Y^RX^L\}$. Inductively then we reduce the question down to whether $Y^Lx \equiv xY^L, yX^L \equiv X^Ly$, &c. But these are all true when one of the terms in 0, which is the basis case.

(d) For the left sets of $(-x)y$ and $-xy$:

$$((-X)Y)^L \equiv \{(-X^R)y + (-x)Y^R - ((-X^R)Y^L), (-X^L)y + (-x)Y^R - ((-X^L)Y^R)\}$$

And

$$(-XY)^L \equiv \{-X^Ly - xY^R + X^LY^R, -X^Ry - xY^L + X^RY^L\}$$

So by induction we reduce the question down to whether $(-X^L)y \equiv -X^Ly, -((-X^L)Y^R) \equiv X^LY^R$. But again these all hold in the basis case, and the proof is similar for the right sets and $x(-y)$. □

Theorem 0.18. *For any numbers x, y and z:*

(a) $\pi(x, y, z) : (x + y)z = xz + yz$

(b) $\Omega(x, y, z) : (xy)z = x(yz)$

Proof.

(a) We have

$$xz + yz \equiv \{(XZ)^L + yz, ...|...\} \equiv \{X^Lz + xZ^L - X^LZ^L + yz, ...|...\}$$

and

$$(x + y)z \equiv \{(X^L + y)z + (x + y)Z^L - (X^L + y)Z^L, ...|...\}$$

Now suppose $\pi(X^L, y, z), \pi(x, y, Z^L), \pi(X^L, y, Z^L)$ all hold. Then $(x+y)z$ becomes, using the equality $x + -x = 0$:

$$(x + y)z \equiv \{X^Lz + yz + xZ^L + yZ^L - X^LZ^L - yZ^L, ...|...\}$$

$$= \{X^Lz + yz + xZ^L - X^LZ^L, ...|...\} \equiv xz + yz$$

So $\pi(x, y, z)$ depends on π holding for the left options of x and z, but if $x^L = 0$ or $z^L = 0$, then clearly π holds, so inductively $(x + y) = xz + yz$.

(b) We have

$$x(yz) \equiv \{X^L(yz) + x(YZ)^L - X^L(YZ)^L, ...|...\}$$
$$\equiv \{X^L(yz) + x(Y^Lz + yZ^L - Y^LZ^L) - X^L(Y^Lz + yZ^L - Y^LZ^L), ...|...\}$$

then using (a), $-(-x) \equiv x$, and $yx \equiv xy$,

$$= \{X^L(yz) + x(Y^Lz) + x(yZ^L) - x(Y^LZ^L) - X^L(Y^Lz) - X^L(yZ^L) + X^L(Y^LZ^L), ..|...\}$$

and

$$(xy)z \equiv \{(XY)^Lz + (xy)Z^L - (xy)^LZ^L, ...|...\}$$
$$\equiv \{(X^Ly + xY^L - X^LY^L)z + (xy)Z^L - (X^Ly + xY^L - X^LY^L)Z^L, ...|...\}$$

then using (a) and $-(-x) \equiv x$,

$$= \{(X^Ly)z + (xY^L)z - (X^LY^L)z + (xy)Z^L - (X^Ly)Z^L - (xY^L)Z^L + (X^LY^L)Z^L, ...|...\}$$

which, if $\Omega(X^L, y, z), \Omega(x, Y^L, z)$, ... all hold,

$$\equiv \{X^L(yz) + x(Y^Lz) - X^L(Y^Lz) + x(yZ^L) - X^L(yZ^L) - x(Y^LZ^L) + X^L(Y^LZ^L), ...|...\}$$
$$\equiv x(yz)$$

So again we reduce $\Omega(x, y, z)$ down to the same proposition on its options, but when one of $x, y,$ or z is equal to 0, clearly Ω holds, so by induction, $x(yz) = (xy)z$. Note that since we evoke the equality $x + -x = 0$ in the proof, the theorem only holds up to equality, not identity. □

Lemma 0.19. *For any x, y and z, we have $\pi(x, y, z) : (x = y \implies xz = yz)$*

Proof.

For $xz = yz$ we must have that both $xz \leqslant yz$ and $yz \leqslant xz$. For the former, this means there is no element of $(xz)^L$ greater-than-or-equal to yz and no element of $(yz)^R$ less-than-or-equal to xz. Suppose then by contradiction that we have some:

$$x^Lz + xz^L - x^Lz^L \geqslant yz > y^Lz + yz^L - y^Lz^L \tag{0.19.1}$$

and further suppose that $\pi(x^L, y^L, z)$, $\pi(x, y, z^L)$, and $\pi(-x^L, -y^L, z^L)$ all hold. From the Extension Theorem we can write x and y so that there exists some $x^L = y^L$, giving us $x^Lz = y^Lz$, $xz^L = yz^L$ and $-x^Lz^L = -y^Lz^L$. Then from Theorem 0.11, we have:

$$x^Lz + xz^L - x^Lz^L = y^Lz + yz^L - y^Lz^L \tag{0.19.2}$$

which clearly contradicts (0.19.1). So we reduce $\pi(x, y, z)$ down to π on their options, and $\pi(a, b, c)$ holds when one of a, b or c is equal to 0. Similar arguments show there is no $(yz)^R \leqslant xz$ and that $yz \leqslant xz$, giving us $xz = yz$. □

Theorem 0.20. $P(x_1, x_2, y_1, y_2)$*: (If* $x_1 \leqslant x_2$ *and* $y_1 \leqslant y_2$*, then* $x_1y_2 + x_2y_1 \leqslant x_1y_1 + x_2y_2$*), and if both premises are strict, the conclusion is.*

Proof.

From the above lemma we have already shown the case when $x_1 = x_2$ and $y_1 = y_2$, as the terms simply cancel to $0 \leqslant 0$ from using Theorem 0.11, so we just look at the case when $x_1 < x_2$ or $y_1 < y_2$. But if $x_1 < x_2$, then $x_2 \nleqslant x_1$, so either $x_1 < x_1^R \leqslant x_2$, or $x_2 > x_2^L \geqslant x_1$, for some x_1^R or x_2^L. But if we have the former, suppose $P(x_1, x_1^R, y_1, y_2)$ and $P(x_1^R, x_2, y_1, y_2)$ hold. Then we have:

$$x_1y_2 + x_1^R y_1 < x_1y_1 + x_1^R y_2$$

$$x_1^R y_2 + x_2y_1 \leqslant x_1^R y_1 + x_2y_2$$

Using Theorems 0.11/0.12, we can add both sides of the inequalities and cancel their common terms, leaving us with:

$$x_1y_2 + x_2y_1 < x_1y_1 + x_2y_2$$

which is what we want to prove, and is strict since $x_1 < x_2$ or $y_1 < y_2$. So in the case that $x_1^R \leqslant x_2$, we can reduce $P(x_1, x_2, y_1, y_2)$ down to $P(x_1^R, x_2, y_1, y_2)$ and $P(x_1, x_1^R, y_1, y_2)$, the former of which is strictly simpler than $P(x_1, x_2, y_1, y_2)$. Similarly, if instead we have $x_2^L \geqslant x_1$, it reduces down to $P(x_1, x_2^L, y_1, y_2)$ and $P(x_2^L, x_2, y_1, y_2)$. Similarly if $y_1 < y_2$ we reduce down to $P(x_1, x_2, y_1^R, y_2)$ and $P(x_1, x_2, y_1, y_1^R)$ if some $y_1^R \geqslant y_2$ or to $P(x_1, x_2, y_1, y_2^L)$ and $P(x_1, x_2, y_2^L, y_2)$ if some $y_2^L \geqslant y_1$, again where all the former propositions have strictly simpler terms than the propositions that they reduced. When the new propositions are simpler, we know by induction that $P(x_1, x_2, y_1, y_2)$. Then for when they are not strictly simpler, consider, for example, when both $P(x_1, x_1^R, y_1, y_2)$ and $P(x_1, x_2, y_1, y_1^R)$. Then we have $P(x_1, x_1^R, y_1, y_1^R)$, which means $x_1y_1^R + x_1^R y_1 \leqslant x_1y_1 + x_1^R y_1^R$. But we can use Theorem 0.11 to rearrange this into $x_1y_1^R + x_1^R y_1 - x_1^R y_1^R \leqslant x_1y_1$, which is always true from Theorem 0.2, since the left hand side is an element of $(X_1Y_1)^L$. The other combinations are similar.

□

Theorem 0.21. *If* x *and* y *are numbers, then* xy *is a number. Furthermore, if* x *and* y *are positive numbers, then* xy *is a positive number.*

Proof.

For the first part, since x and y are numbers, their options are all numbers. For xy to be a number, we need to show that, for example, $x^{L_1}y + xy^L - x^{L_1}y^L \ngeqslant x^{L_2}y + xy^R - x^{L_2}y^R$. From Theorem 0.6, for all x^{L_1}, x^{L_2}, either $x^{L_1} \leqslant x^{L_2}$ or $x^{L_2} \leqslant x^{L_1}$. But if we have the former, then from Theorem 0.20, we have both

$$P(x^{L_1}, x^{L_2}, y^L, y) : x^{L_1}y + x^{L_2}y^L \leqslant x^{L_1}y^L + x^{L_2}y \qquad (0.21.1)$$

$$P(x^{L_2}, x, y^L, y^R) : x^{L_2}y^R + xy^L < x^{L_2}y^L + xy^R \qquad (0.21.2)$$

Then using Theorem 0.11, we can rearrange these and add xy^L to (0.21.1) and add $x^{L_2}y$ to (0.21.2), giving us:

$$x^{L_1}y + xy^L - x^{L_1}y^L \leqslant x^{L_2}y + xy^L - x^{L_2}y^L \qquad (0.21.3)$$

$$x^{L_2}y + xy^L - x^{L_2}y^L < x^{L_2}y + xy^R - x^{L_2}y^R \qquad (0.21.4)$$

which using transitivity give us:

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_2}y + xy^R - x^{L_2}y^R \qquad (0.21.5)$$

which is what we needed to show. Similar arguments show this holds if instead we have $x^{L_2} \leqslant x^{L_1}$, and that every other element of $(xy)^L$ is not greater-than-or-equal to any element of $(xy)^R$. So inductively we have that xy is a number if $x^Ly, xy^L, x^Ly^L, ...$ are all numbers, which holds in the basis case as 0 is a number.

For the second part, since $0 < x$ and $0 < y$, from Theorem 0.20 we have $0 < xy$. □

It should be noted that, like with addition, on ordinals multiplication refers to the natural product, not the ordinal product. Again, the ordinal product can be defined and used [6], but at a loss of the field structure.

Division

The last thing we must show to have a field is how to find the multiplicative inverse of a number, that is, for any $x \neq 0$, if there exists a number y, such that $xy = t$, then we need to show how to find this y. But note that if, for a positive x, we could find a y' such that $xy' = 1$, then we would also know that $t(xy') = t \implies x(ty') = t$, that is that $y = ty'$. If x were instead negative, we would just need to multiply the equation through by -1. So we only need show how to find y for some $xy = 1$, where x is positive.

Lemma 0.22. *For every positive $x = \{X^L | X^R\}$, we can write x in a form with $X^L = \{0, x^L\}$, where all x^L are positive, and this new form is equal to the original one.*

Proof.

Since $0 < x$ is positive, by the Extension Theorem we can append 0 to X^L. Then from the Truncation Theorem we can remove any element of X^L less than 0. □

For the rest of this section when we write x^L we are referring only to the non-zero terms, and since x here is positive, we must have all $x^R > 0$. Now we define y recursively. That is, every element of Y^L generates a new element in Y^L, and similarly for Y^R. We write

$$y = \left\{0, \frac{1+(x^R-x)y^L}{x^R}, \frac{1+(x^L-x)y^R}{x^L} \middle| \frac{1+(x^L-x)y^L}{x^L}, \frac{1+(x^R-x)y^R}{x^R}\right\}$$

which has y^L and y^R in the definition of y! What we mean by this is that we build up these left and right sets by using elements already in them, so that if y_1^L is in Y^L, then, for example, $((1+x^R-x)y_1^L)/x^R$ is also in Y^L. Conway gives the following elucidation [2, p. 21]:

> Let $x = \{0, 2|\}$. Then the only (non-zero) x^L is 2, giving us $1/x^L = 1/2$ and $(x^L - x) = -1$, and there is no x^L, so we have $y = \{0, \frac{1}{2}(1-y^R)|\frac{1}{2}(1-y^L)\}$. Putting in $y^L = 0$ into the right option updates y to $y = \{0, \frac{1}{2}(1-y^R)|\frac{1}{2}, \frac{1}{2}(1-y^L)\}$, and we can now put this new right option into the left set, giving us $y = \{0, \frac{1}{4}, \frac{1}{2}(1-y^R)|\frac{1}{2}, \frac{1}{2}(1-y^L)\}$. We can then repeat this process endlessly.

Theorem 0.23. *For all y^L, y^R, we have $xy^L < 1 < xy^R$*

Proof.

In our recursive definition, every option of y is in the form

$$y'' = \frac{1+(x'-x)y'}{x'}$$

where y' is an already know option of y (the first always one being $y^L = 0$) and x' some non-zero option of x. If we then multiply both sides by x' and take from 1, we get:

$$1 - xy'' = (1-xy')\frac{x'-x}{x'}$$

But since x is positive, $(x^L - x)/x^L < 0$ and $(x^R - x)/x^R > 0$. Then consider

$$y^{L^2} := \frac{1+(x^R-x)y^L}{x^R}$$

$$\implies 1 - xy^{L^2} = (1-xy^L)\frac{x^R-x}{x^R}$$

where y^{L^2} is some recursive member of Y^L. Then if $xy^L < 1$, we know that $1 - xy^L > 0$, and since $(x^R - x)/x^R$ is positive, that $(1-xy^L)((x^R-x)/x^R) > 0$. So $xy^{L^2} < 1$ if $xy^L < 1$. Similarly, consider

$$y^{R^2} := \frac{1+(x^L-x)y^L}{x^L}$$

$$\implies 1 - xy^{R^2} = (1 - xy^L)\frac{x^L - x}{x^L}$$

where y^{R^2} is some recursive member of Y^R. Again, if $xy^L < 1$, then since $(x^L - x)/x^L$ is negative, we have $1 - xy^{R^2} = (1 - xy^L)((x^L - x)/x^L) > 0$, so $xy^{R^2} > 1$ if $xy^L < 1$. A similar argument shows that this is true of the other two forms of the options of y. So by induction, $xy^L < 1 < xy^R$ for all y^L, y^R since vacuously the theorem holds for $y = 0$.

□

Theorem 0.24. *y is a number*

Proof.

This follows directly from Theorems 0.20/0.23, as no $y^L \geqslant y^R$, for all y^R.

□

Theorem 0.25. *For all $(xy)^L$, $(xy)^R$, we have $(xy)^L < 1 < (xy)^R$*

Proof.

Note that if we expand $1 + x^L(y - y^{R^2})$, we get $1 + x^L y - x^L y^{R^2} = 1 + x^L y - (1 + x^L - x)y^L) = x^L y + xy^L - xy^L$, which is a member of $(xy)^L$. But we know that $y - y^{R^2} < 0$ from Theorem 0.3, and that x^L is positive, so therefore $1 - x^L(y - y^{R^2}) = x^L y + xy^L - xy^L < 1$. Similarly we can show that $x^R y - xy^L - x^R y^L = 1 + x^R(y - y^{L^2}) > 1$, since $y - y^{L^2} > 0$ and $x^R > 0$. Similar arguments show this is also the case for the other two options of xy.

□

Theorem 0.26. $xy = 1$

Proof.

We need to check that both $xy \leqslant 1$ and $1 \leqslant xy$. For the former, from Theorem 0.25, we have that no element of $(XY)^L$ is greater-than-or-equal to 1, and we know that $1^R = \emptyset$. For the latter, the only element of 1^L is 0, but since 0 is in X^L and Y^L, it is also in $(XY)^L$, so from Theorem 0.2 we have $xy > 0$, and then from Theorem 0.25 again we have that no element of $(XY)^R \leqslant 1$.

□

So we have shown that we have a multiplicative inverse for any non-zero number. We also showed before that we have an additive inverse, additive and multiplicative identities, that both addition and multiplication are associative, commutative, and closed, and finally that multiplication is distributive under addition. That is, we have defined addition and multiplication on the equivalence classes of the Surreal Numbers in a way that satisfies all the field axioms, as stated in the beginning of the section. We also showed in the previous section that numbers are totally ordered. In sum then, we have shown that the surreals form a totally ordered Field, which Conway calls **No**.

Real, Ordinal, and Surreal Numbers

Having defined arithmetic, we are now in a position to justify our naming of numbers in the first section. Firstly we defined $1 := \{0|\}$ and $-1 := \{|0\}$ (and we shall we show momentarily that these are the $natural\ forms$ of 1 and -1). Clearly, $-(1) = -1$ and $-(-1) = 1$ by our definition of negation. We also want the property that $-1 + 1 = 0$, and this must hold, as we have shown $x + -x = 0$ for all x. We also wanted that $1/2 + 1/2 = 1$. We could show this by simply adding the left hand terms together and showing both $1/2 + 1/2 \leqslant 1$ and $1 \leqslant 1/2 + 1/2$, but we will first introduce a theorem and then prove $1/2 + 1/2 = 1$ as an example of that theorem.

Theorem 0.27. Simplicity Theorem: *If for some $x = \{X^L|X^R\}$ we have $x^L < z < x^R$ for all x^L, x^R, and no option of z has this property, then $z = x$*

Proof.

To show that $z \leqslant x$, we need that no element of $z^L \geqslant x$ and no element of $x^R \leqslant z$. The latter holds by our condition on z, and the former not holding would imply that $x^L < x \leqslant z^L < z < x^R$, that is that $x^L < z^L < x^R$, which is a contradiction since z^L is an option of z. A similar argument shows that $x \leqslant z$, so we have $z = x$. □

We can now apply this to show that $1/2 + 1/2 = 1$. By our definition of addition, we have $1/2 + 1/2 = \{1/2|1 + 1/2\} = \{\{0|1\}|\{1|2\}\}$. Then $1/2 < 1 < 1 + 1/2$ from Theorems 0.2/0.3. But the only option of 1 is 0, which is not greater than $1/2$, so from the Simplicity Theorem, $1/2 + 1/2 = 1$. In general, this means that any number x is the simplest number lying between all x^L and all x^R.

Note that this theorem is well named: if we have a number like $\{2 + \frac{1}{2}|7\}$, we can read off straight away that it must be equal to 3, and we know that any number that has only negatives in its left set and only positives in its right sets must be equal to 0, and this includes the special case when the left or right set is the empty set.

We now introduce the $natural\ form$ of numbers, which will be an essential idea used in later proofs:

Definition. *Natural Form: A number x is in its natural form if it has at most one left option, and at most one right option, and all its options are strictly simpler than x.*

For example, if we express the number 2 as $2 = \{0, 1|\}$, it is not in its natural form because it has two left options. Its natural form is in fact $2 = \{1|\}$. As we shall we later, not all numbers have a natural form, most notably the non-dyadic rationals. But we shall now prove that all numbers constructed on finite day have a natural form:

Theorem 0.28. Natural Form Theorem: *For any number $x = \{X^L|X^R\}$ constructed on a finite day n, we can express x in its natural form, and this form is unique. Furthermore, each equivalence class of the surreals which has a member constructed on a finite day has just one member that is in a natural form.*

Proof.

Since x is constructed on finite day n, it must be able to be expressed with a finite amount of options created by day $n-1$. Then since all the numbers on the $(n-1)$-th day are totally ordered, there must be a greatest left option $\widetilde{x}^L$ and a least right option $\widetilde{x}^R$. Using the Truncation Theorem we can then write x as $x = \{\widetilde{x}^L|\widetilde{x}^R\}$, which is its natural form, and is uniquely determined by $\widetilde{x}^L$ and $\widetilde{x}^R$. The second claim follows from this uniqueness and the transitivity of $=$.

□

Theorem 0.29. *Suppose all the numbers constructed by the finite (n-1)-th day are:*

$$x_1 < x_2 < x_3 < ... < x_m$$

Then the new numbers constructed on the n-th day are:

$$\{|x_1\} < \{x_1|x_2\} < ... < \{x_{m-1}|x_m\} < \{x_m|\}$$

Proof.

Note that from the Natural Form Theorem, we can write any $x = \{X^L|X^R\}$ created on day n in its natural form, with just the greatest x^L in the left set and the least x^R in the right set. First we show that $\{|x_1\}$ and $\{x_m|\}$ are new numbers, then that $\{x_i|x_{i+1}\}$ for all i are all new numbers, and then that $\{x_i|x_j\}$ for $i+1 \neq j$ are all already constructed numbers. Finally we show their ordering.

(a) Since $x_1 < x_k$ for all $k \neq 1$, the number $\{|x_1\}$ must be smaller than any already constructed number, from Theorem 0.3. Similarly $\{x_m|\}$ is greater than any already constructed number.

(b) We know that $x_i < \{x_i|x_{i+1}\} < x_{i+1}$ from Theorems 0.2/0.3, and since we haven't constructed any numbers yet with this property, all numbers in the form $\{x_i|x_{i+1}\}$ must be newly constructed numbers.

(c) Since $i+1 \neq j$, and $i < j$, we have $i+1 < j$. Now let x_k be the simplest number in the range $x_{i+1}, ..., x_{j-1}$, and consider x_k as expressed in its natural form. Then the left options of x_k must be less than x_{i+1} and the right options greater than x_{j-1}. That is, $x_k^L \leqslant x_i$ and $x_j \leqslant x_k^R$. So from the Extension Theorem we can write $\{x_i|x_j\} = \{x_i, x_k^L|x_j, x_k^R\}$. Similarly, since x_k is greater than x_i and less than x_j, we can write $x_k = \{x_k^L|x_k^R\} = \{x_k^L, x_i|x_k^R, x_j\} = \{x_i|x_j\}$, so $\{x_i|x_j\}$ is not a newly constructed number.

(d) The ordering follows directly from Theorems 0.2/0.3. □

Corollary 0.29.1. *The number of numbers constructed by the finite day* n *is* $m_n = 2^{n+1} - 1$

Proof.

From Theorem 0.29 we see that on each day k we construct $m_{k-1} + 1$ new numbers, where m_{k-1} is the number of numbers constructed by day $k - 1$. Then by induction, assume $m_{n'} = 2^{n'+1} - 1$ for all $n' < n$. Then we have that $m_{n-1} = 2^n - 1$. Since we construct $m_{n-1} + 1$ new numbers on day n, the total number of numbers constructed by day n is $m_n = m_{n-1} + m_{n-1} + 1 = 2^n - 1 + 2^n - 1 + 1 = 2^{n+1} - 1$ as required. Then on day 0 we have only $2^{0+1} - 1 = 1$ number constructed, which gives the basis of our induction. □

The Natural Numbers, Integers, and Dyadic Rationals

Let us consider the numbers that are constructed in finite days when we have the condition that either the left or right sets must be empty. On the zeroth day we have $0 = \{|\}$, then on the first day $-1 = \{|0\}$ and $1 = \{0|\}$, on the second day $-2 = \{|0, -1\}$ and $2 = \{0, 1|\}$, etc. On each new day then we create two new numbers, one that is the greater than every other number created, and one that is less than every number already created, and these two being additive inverses of each other. That is, on the n-th day we create the numbers $-n = \{| - 0, -1, ..., -(n-1)\} = \{| - (n-1)\}$ and $n = \{0, 1, ..., n-1|\} = \{n-1|\}$, and all the numbers already created are strictly ordered by $<$.

If we limit ourselves further to only the numbers where the right set is empty, then we construct $0 = \{|\} < 1 = \{0|\} < 2 = \{0, 1|\} < ... < n = \{0, 1, 2, ..., n-1|\}$ by the n-th day. Then these numbers are well-ordered, since we showed that all the Surreal Numbers are totally ordered, so any finite non-empty subset will have a least element in $<$. We call all the numbers constructed in this way in finite days the natural numbers, and justify this name by noting there is a one-to-one correspondence with the natural numbers as constructed in ZFC. In ZFC, using the von Neumann construction, we use the axiom of existence and define $0 = \emptyset$ as the first natural number, then define the successor of an already created natural number n as the union $S(n) = n \cup \{n\}$. Finally using the axiom of infinity we define the set of natural numbers as the set closed under the successor function that contains 0 [4, chap. 3]. We say $n_z \in \mathbb{N}_z$ for these numbers. In Conway's construction, we have instead that $0 = \{|\}$ is the first natural number, and then the successor function is $S(n)^L = n \cup n^L, S(n)^R = n^R = \emptyset$, and that the natural numbers are all those constructed in this way. We say $n_s \in \mathbb{N}_s$ for these numbers. The constructions are essentially the same, and we can explicitly give the bijection between them as:

$$\pi : n_z \to n_s : \pi(n_z) = \{n_z|\}$$

with inverse:

$$\pi^{-1} : n_s \to n_z : \pi^{-1}(n_s) = n_s^L$$

We then get the integers by returning to the condition that either a number's left or right set is empty, since then every number is either equal to a natural number or the negative of a natural number [7]. Note that clearly from our construction we can also inductively define the set of integers $\mathbb{Z}$ as:

$$0 \in \mathbb{Z}$$

$$\text{If } n_0, n_1, ... \in \mathbb{Z} \text{ then } \{n_0, n_1, ...|\} \in \mathbb{Z} \text{ and } \{| -n_0, -n_1, ...\} \in \mathbb{Z}$$

This is in contrast to the standard ZFC construction of the integers, where we would first create ordered pairs $(a, b) = \{a, \{b\}\}$ for every combination of two natural numbers a and b, and then partition these into the equivalence classes of the relation $(a, b) \sim (c, d) \iff a + d = b + c$ [4, chap. 10]. In this method of constructing the integers we must first construct all the natural numbers, then define addition between any pair of these, and then reconstruct them while constructing the negative integers, none of which can be constructed without the axiom of infinity. Conway's construction instead uses symmetry, and does not require having constructed all the positive integers before we can construct any negative integers. Instead, we introduce the negative of a number as an inductive definition, as in section 3, and then in one stroke we create both 1 and -1, then 2 and -2, &c, and every integer is created in a finite amount of steps, or days. We do not need to define addition, and in this manner of constructing integers there is no primacy of the positive integers over the negatives, there is just a symmetry around 0. Indeed, if we wanted we could construct the negative integers $before$ the positives! This symmetry in construction extends beyond just natural numbers, and in general if we can construct a new positive number x on a given day, then we must also have that we can construct $-x$ on that same day.

The next step in ZFC would be to construct the rational numbers as the equivalence classes of the relation $(a, b) \sim (c, d) \iff ad = bc$ for integers a, b, c, d [4, chap. 10]. In this way all the rational numbers are created at the same time, and since they require all the natural numbers having already being constructed, none can be constructed without the axiom of infinity. In contrast, Conway's construction has that every dyadic rational, that is every number of the form $y/2^z$, where y is a integer and z is a natural number, is created in a finite amount of days, and only the non-dyadic rationals require the axiom of infinity before any can be constructed. We formalise all we have said here in the following theorems.

Theorem 0.30. *For positive integers x_m, we have $\{x_m|\}$ is an integer and equals x_m+1. For negative integers x_n, we have $\{|-x_n\}$ is an integer and equal to $-x_n - 1$*

Proof.

We have $x_m + 1 = \{x_m^L + 1, x_m|\}$. Suppose that $x_m = \{x_m^L|\}$ where $x_m^L = x_m - 1$ is an integer. Then $x_m^L + 1 = x_m$, so $x_m + 1 = \{x_m|\}$. So by induction, since $1 = \{0|\} = \{1-1|\}$ is an integer, the first part of the theorem holds. The second part follows directly from the first and our definition of negation.

□

Theorem 0.31. *For any rational number x whose denominator divides 2^n, we have that $x = \{x - 1/2^n \,|\, x + 1/2^n\}$*

Proof.

For $n = 0$, x is an integer. If $x = 0$, then we have from the Simplicity Theorem that $0 = \{-1|1\}$. If x is the positive integer n, then we have from Theorem 0.30 that $n = \{n-1|\}$. Now let $y = \{n-1|n+1\}$. Then we have $y^L < n < y^R$, and the only option of n is $n-1$, which is equal to the left option of y, so from the Simplicity Theorem, $n = y = \{n-1|n+1\}$. A similar argument shows this is also true when x is a negative integer.

For $n > 0$, let $y = \{x - 1/2^n|x + 1/2^n\}$. Then we have $2y = y + y = \{y + y^L|y + y^R\} = \{y + x - 1/2^n|y + x + 1/2^n\}$, so from Theorems 0.2/0.3 we know that both $y + x - 1/2^n < 2y < y + x + 1/2^n$ and $x - 1/2^n < y < x + 1/2^n$. If we then multiply the latter by 2, we get $2x - 1/2^{n-1} < 2y < 2x + 1/2^{n-1}$. We can also add $x - 1/2^n$ to both sides of $x - 1/2^n < y$ to get $2x - 1/2^{n-1} < y + x - 1/2^n$, and similarly add $x + 1/2^n$ to both sides of $y < x + 1/2^n$ to get $y + x + 1/2^n < 2x - 1/2^{n-1}$. Then putting these all together we get the string of inequalities for $2y$:

$$2x - 1/2^{n-1} < y + x - 1/2^n < 2y < y + x + 1/2^n < 2x + 1/2^{n-1}$$

Now by induction, assume that the theorem holds for rational numbers that have denominators that divide 2^{n-1}, and since x is a rational number that has denominator that divides 2^n, $2x$ must be a rational number that has denominator which divides 2^{n-1}. Then we have that $2x = \{2x - 1/2^{n-1}|2x + 1/2^{n-1}\}$. It follows from our inequality $x - 1/2^n < y < x + 1/2^n$ that both $2x < y + x + 1/2^n$ and $y + x - 1/2^n < 2x$ so we also have the string of inequalities for $2x$:

$$2x - 1/2^{n-1} < y + x - 1/2^n < 2x < y + x + 1/2^n < 2x + 1/2^{n-1}$$

Then it is clear that $2x$ is greater than the left option of $2y$ and less than the right option of $2y$, and that neither option of $2x$ also has this property, so from the Simplicity Theorem we must have that $2x = 2y$, which implies $x = y = \{x - 1/2^n | x + 1/2^n\}$. Since we have already shown the basis case when $n = 0$, the theorem must hold for all n.

□

Theorem 0.32. *Suppose that all the positive numbers constructed by the finite n-th day are:*

$$x_1 < x_2 < x_3 < ... < x_m$$

Then any number created on the (n+1)-th day that is not an integer is a dyadic rational. Similarly, all the negative numbers created on the (n+1)-th day that not integers are dyadic rationals.

Proof.

We know from Theorems 0.29/0.30 that all the non-integer numbers created on day $n + 1$ are in the form $\{x_i | x_{i+1}\}$, so let $x = \{x_i | x_{i+1}\}$, where $x_i \geqslant 0$ (we show the theorem holds for positive numbers, and then the negatives follows from our definition of negation). We will now show that we must have $x = (x_i + x_{i+1})/2$, from which the theorem follows.

Firstly, we must have that if $x = \{x_i | x_{i+1}\}$, then from Theorems 0.29/0.30, for some integer a, both $a \leqslant x_i \leqslant a+1$ and $a \leqslant x_{i+1} \leqslant a+1$ hold. It follows that $a < x < a+1$, and also that we cannot have that both x_i and x_{i+1} are created until at least day $\alpha + 1$, where α is the birthday of a.

Now suppose by induction for any $y < y'$ created before day $n+1$, such that on the first day by which they have both been created they directly follow each other, that is there is no number $y*$ such that $y < y* < y'$, that we have $\{y|y'\} = (y + y')/2$.

Then we have both $a \leqslant x_i \leqslant a + 1$ and $a \leqslant x_{i+1} \leqslant a + 1$ for some integer a created before day $n + 1$. Note that if both x_i and x_{i+1} are created by day $\alpha + 1$, we must have both $a = x_i$ and $a + 1 = x_{i+1}$, so we have $x_{i+1} - x_i = a + 1 - a = 1 = 1/2^0$. If instead they are created by day $\alpha + 2$, then we have either that $a = x_i$ and $\{a|a + 1\} = x_{i+1}$ or instead that $\{a|a + 1\} = x_i$ and $a + 1 = x_{i+1}$. In either case we have $x_{i+1} - x_i = (a + 1 - a)/2 = 1/2^1$ by our induction. This motivates us to want to show that if x_i and x_{i+1} are created on day $n = \alpha + k$, then $x_{i+1} - x_i = 1/2^{k-1}$. We have already shown the base case when $n = \alpha + 1$. Now by induction assume that for all $z < z'$ created on day $\alpha + k - 1$ (such that there is no $z*$ with property $z < z* < z'$ also created by day $\alpha + k - 1$) we have $z' - z = 1/2^{k-2}$. Then if we created the numbers x_i and x_{i+1} on day $n = \alpha + k$ we have either that both $x_i = z$ and $x_{i+1} = \{z|z'\}$ or that both $x_i = \{z|z'\}$ and $x_{i+1} = z'$ for some z, z'. If we have the former, then $x_{i+1} - x_i = \{z|z'\} - z = (z + z')/2 - z = (z' - z)/2 = (1/2^{k-2})/2 = 1/2^{k-1}$. Similarly if we have the latter we get the same result. So by induction, for whatever

a we have such that $a \leqslant x_i \leqslant a+1$ and $a \leqslant x_{i+1} \leqslant a+1$, we have $x_{i+1}-x_i = 1/2^{k-1}$, where $k = n - \alpha$.

Similarly, if x_i and x_{i+1} are created on day $\alpha+1$, then we must have that $x_i+x_{i+1} = 2a+1$, which has denominator 1, which divides 2^0. We want to show that if x_i and x_{i+1} are created on any day $n = \alpha + k$, then the denominator of their sum divides 2^{k-1}, so by induction assume for our z and z' as above that we have that $z+z'$ has denominator that divides 2^{k-2}. We have that $x_{i+1}-x_i = 1/2^{k-1}$, so then in the case that $x_{2i+1} = \{z|z'\}$, we have $x_{i+1}+x_i = 2x_{2i+1}+1/2^{k-1} = z+z'+1/2^{k-1}$. If instead we have that $x_{2i+3} = \{z|z'\}$, then we have $x_{i+1}+x_i = 2x_{2i+3}-1/2^{k-1} = z+z'-1/2^{k-1}$, so in either case using our induction assumption the sum must have denominator that divides 2^{k-1}.

We now use Theorem 0.31 and write $x_i + x_{i+1} = \{x_i + x_{i+1} - 1/2^{k-1}|x_i + x_{i+1} + 1/2^{k-1}\}$. We also have that $2x = \{x + x_i|x + x_{i+1}\}$, and that $x_i + x < x_i + x_{i+1} < x + x_{i+1}$. Finally, since $x_{i+1} - 1/2^{k-1} = x_i < x$, using Theorem 0.11, we have $x_i + x_{i+1} - 1/2^{k-1} < x + x_i$, and again since $x < x_{i+1} = x_i + 1/2^{k-1}$ we have $x + x_{i+1} < x_i + x_{i+1} + 1/2^{k-1}$. Putting these all together we get the string of inequalities:

$$(x_i + x_{i+1})^L < (2x)^L < x_i + x_{i+1} < (2x)^R < (x_i + x_{i+1})^R$$

Then it follows from the Simplicity Theorem that $2x = x_i + x_{i+1}$, since $(2x)^L < x_i + x_{i+1} < (2x)^R$, and neither option of $x_i + x_{i+1}$ satisfies this property. Therefore $x = (x_i + x_{i+1})/2$, which suffices to prove the theorem for positive numbers. The negatives follow from our definition of negation.

□

Corollary 0.32.1. *If on some day n we have already constructed the numbers x_i and x_{i+1} such that $x_i < x_{i+1}$ and there does not exist any y such that $x_i < y < x_{i+1}$, then on the day n we construct the number $\{x_i|x_{i+1}\} = \frac{x_i+x_{i+1}}{2}$*

Proof.

Follows directly from Theorems 0.29/0.32

□

So unlike in the standard ZFC construction of numbers, seemingly simple rationals such as $1/3$ cannot be constructed in a finite amount of days (if we try and calculate the multiplicative inverse of 3, we find that we need a countably infinite amount of dyadic rationals, where as, for example, the inverse of 2 quickly resolves to $1/2 = \{0|1\}$). To form the rest of the rationals, and the real numbers, we need to consider numbers that have countably infinitely many options, which are exactly those constructed on day ω.

The Numbers Constructed on Day ω

Suppose that by some day ω we have constructed all the natural numbers. Then consider the number $\{1, 2, 3, ...|\}$. This is a valid number from Axiom 1, and clearly it is greater than every natural number. In ZFC, to construct infinite ordinals we continue to use the von Neumann construction as we had for the natural numbers, and have $\omega = \{0, 1, 2, ...\}$ as the first infinite ordinal, which contains in it every finite ordinal, and every other infinite ordinal is strictly greater than it. In Conway's construction we have that $\omega = \{0, 1, 2, ...|\}$, but we also construct at the same time $-\omega = \{|0, -1, -2, ...\}$, due to the symmetry of construction. This number $-\omega$ has no place in ZFC, but in Conway's construction it is given naturally as long as the negative of every natural number is defined, that is as naturally as ω is given, for we showed that there is a parity of esteem between the positive and negative naturals. It would be more destructive to not have it (and every other negative ordinal) in the class of Surreal Numbers, as this would break the Field structure, since then no ordinal would have an additive inverse. It should also be noted that these two numbers are the first constructed that cannot be truncated into natural forms, with only one element in their left and right sets, since in ω, the left set has no greatest element, and in $-\omega$, the right set has no least element. This is why the Natural Form Theorem demands construction on a finite day, so only integers and dyadic rationals have a natural form. We can instead express ω in an infinite amount of ways, as long as the left set is uncountably large and has no natural number upper bound, such as $\omega = \{\text{all primes}\,|\}$.

Another number we can create on day ω is the infinitesimal number $\epsilon = \{0|\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\}$, which is again a valid construction from Axiom 1. We must have then that:

$$0 < \epsilon < \frac{1}{n}$$

for all natural numbers n. But if ϵ were real, then $1/\epsilon$ would also be real, so we would have the inequality $1/\epsilon > n$ for all natural numbers n, which is a contradiction, since every real number is bounded by some natural number. So ϵ is not a real number, indeed it is a $surreal$ number! Again, by symmetry we also construct $-\epsilon$, but we can also construct numbers such as $1+\epsilon = \{1|1+\frac{1}{2}, 1+\frac{1}{4}, ...\}$ and $1-\epsilon = \{1-\frac{1}{2}, 1-\frac{1}{4}, ...|1\}$, &c.. It is because we can construct new numbers like this, all while still in an ordered Field, that we say the Surreal Numbers $extend$ the real numbers (which we define and show form a subclass of the surreals in the next section).

We can also show that $\epsilon = \frac{1}{\omega}$ as follows. Consider the product

$$\epsilon\omega = \left\{\epsilon, 2\epsilon, ... \,\middle|\, \left\{\frac{\omega}{2}, \frac{\omega}{4}, ...\right\} + \{\epsilon, 2\epsilon, ...\} - \left\{\frac{1}{2}, \frac{1}{4}, ...\right\} \cdot \{1, 2, ...\}\right\}$$

Then clearly all the left options are less than 1 but greater than 0, and by using the

Truncation Theorem and writing the right options as

$$\frac{\omega}{n} + \epsilon - \frac{m}{2} = \frac{2\omega + 2n\epsilon - mn}{2n}$$

for some natural numbers m and n, and then letting $l = mn$, $k = 2n$, which are both still natural numbers, we see that

$$\frac{2\omega + 2n\epsilon - l}{k} > \frac{\omega}{k} > 1$$

So by the Simplicity Theorem we must have that $\epsilon\omega = 1$, which implies $\epsilon = \frac{1}{\omega}$, that is that in our Field of surreals ω has a multiplicative inverse, as we would expect.

In ZFC, after constructing the rational numbers, one way to construct the irrationals is through Dedekind cuts, where we split the rationals into two sets, one closed downwards where every element satisfies some less than inequalitie, and the other which contains all the other rationals [4, chap. 5]. For example, if we split the rationals into the two sets $\{x \in \mathbb{Q} : x^2 < 2 \text{ or } x < 0\}$ and $\{x \in \mathbb{Q} : x^2 \geqslant 2 \text{ and } x > 0\}$, we construct the Dedekind cut representing $\sqrt{2}$. In Conway's construction we do similarly, though we start instead with having all the dyadic rationals constructed by day ω, and then by splitting them into constructions of left and right sets we construct the rest of the real numbers in one sweep, since the dyadic rationals are dense in the reals. For example, let us tentatively say that, for the set of non-negative dyadic rationals $\mathbb{D}$:

$$\frac{1}{3} = \{x \in \mathbb{D} : 3x < 1 \,|\, x \in \mathbb{D} : 3x \geqslant 1\} = \left\{0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \ldots \,\middle|\, \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \ldots\right\}$$

then we want $\frac{1}{3}$ to be the multiplicative inverse of 3, so let $3y = 1$ for some y. Then since $3 = \{2|\}$, by our algorithm for calculating multiplicative inverses we have:

$$y = \left\{0, \frac{1 - y^R}{2} \,\middle|\, \frac{1 - y^L}{2}\right\}$$

where y^R and y^L are already calculated elements of the left and right sets of y respectively. So the first steps become:

$$y = \left\{0, \frac{1 - y^R}{2} \,\middle|\, \frac{1}{2}, \frac{1 - y^L}{2}\right\}$$

$$y = \left\{0, \frac{1}{4}, \frac{1 - y^R}{2} \,\middle|\, \frac{1}{2}, \frac{1 - y^L}{2}\right\}$$

$$y = \left\{0, \frac{1}{4}, \frac{1 - y^R}{2} \,\middle|\, \frac{1}{2}, \frac{3}{8}, \frac{1 - y^L}{2}\right\}$$

and after ω iterations:

$$y = \left\{0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \ldots \,\middle|\, \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \ldots\right\}$$

which is to say that $y = \frac{1}{3}$, as we expected. In a similar manner we can construct every other real number on day ω. For example, we have:

$$\pi = \left\{3, \frac{25}{8}, \frac{201}{64}, \ldots \,\middle|\, \frac{13}{4}, \frac{101}{32}, \frac{3217}{1024}, \ldots\right\}$$

The Real Numbers

We have just shown that on day ω we can construct any real number, though we have not yet formally defined the real numbers in Conway's construction. Thus:

We say that a number $x = \{X^L|X^R\}$ is a real number if both:

$$-m < x < m \text{ for some integer } m, \text{ and}$$

$$x = \{x-1, x-\frac{1}{2}, x-\frac{1}{4}, ...|x+1, x+\frac{1}{2}, x+\frac{1}{4}, ...\}$$

Theorem 0.33. *If x and y are real numbers, then so are $-x$, $x+y$, and xy*

Proof.

(a) We have $-x = \{-X^R|-X^L\} = \{-x-1, -x-\frac{1}{2}, -x-\frac{1}{3}, ...|-x+1, -x+\frac{1}{2}, -x+\frac{1}{3}, ...\}$, and $-m < x < m$ implies $-(-m) < -x < -(m)$, so $-x$ is a real number if x is.

(b) We have $x = \{x-1, x-\frac{1}{2}, x-\frac{1}{3}, ...|x+1, x+\frac{1}{2}, x+\frac{1}{3}, ...\}$ and $y = \{y-1, y-\frac{1}{2}, y-\frac{1}{3}, ...|y+1, y+\frac{1}{2}, y+\frac{1}{3}, ...\}$, so $x+y = \{xy-1, xy-\frac{1}{2}, xy-\frac{1}{3}, ...|xy+1, xy+\frac{1}{2}, xy+\frac{1}{3}, ...\}$ and $-m_x < x < m_x$ and $-m_y < y < m_y$ imply that $-(m_x+m_y) < x+y < m_x+m_y$, so $x+y$ is a real number if x and y are.

(c) We write $x = \{x-1/n|x+1/n\}$ and $y = \{y-1/n|y+1/n\}$, where n ranges over all the positive integers, and then we have $xy = \{xy-1/n^2|xy+1/n^2\} = \{xy-1, xy-1/4, ...|xy+1, xy+1/4, ...\}$, which then using the Extension Theorem becomes $xy = \{xy-1, xy-1/2, xy-1/3, ...|xy+1, xy+1/2, xy+1/3, ...\}$. From Theorem 0.20, since $-\alpha < x < \alpha$ and $-\alpha < y < \alpha$ for some integer α, we have $-\alpha^2 < xy < \alpha^2$ so xy is a real number if x and y are. □

Theorem 0.34. *All integers and all dyadic rationals are real numbers.*

Proof.

For the integers, from Theorem 0.30 every positive integer $a = \{a-1|\}$, which is a's natural form. Let $\alpha = \{a-1, a-1/2, ...|a+1, a+1/2, ...\}$. Then we have that $a \leqslant \alpha$, since the only left option of a is also a left option of α, and every right option of α is strictly greater than a. Similarly, we also have $\alpha \leqslant a$, since every left option of α is strictly less than a, and there are no right options of a. So $a = \{a-1, a-1/2, ...|a+1, a+1/2, ...\}$ and therefore a is a real number for every positive integer a, and then the negative integers follow from Theorem 0.33.

For the dyadics, from Theorem 0.31, we have that for a dyadic rational x whose denominator divides $2^{n'}$, that $x = \{x-1/2^{n'}\,|\,x+1/2^{n'}\}$. But if the denominator of x divides $2^{n'}$, then it must also divide 2^n for all $n > n'$, and so, using also the

Extension Theorem, we can write x as $x = \{x - 1/2^n | x + 1/2^n\}$, where n ranges over all the integers greater than n'. Then again by the Extension Theorem we can we can add in the remaining terms we need such as $x - 1$ in the left set, to obtain $x = \{x - 1, x - 1/2, x - 1/4, ... | x + 1, x + 1/2, x + 1/4, ...\}$ as required. □

Theorem 0.35. *No infinite ordinal numbers or infinitesimal numbers are real numbers.*

Proof.

Any infinite ordinal number is greater than every integer, so we cannot have $-m < \alpha < m$ for an ordinal α. For any infinitesimal number ϵ', the number $\{\epsilon' - 1, \epsilon' - 1/2, \epsilon' - 1/3, ... \,|\, \epsilon' + 1, \epsilon' + 1/2, \epsilon' + 1/3, ...\} = 0$ from the Simplicity Theorem, as every option of the left set must be less than 0, and every option of the right set greater than 0. □

Day $\omega + 1$ and Onwards

On day $\omega+1$, we can use ω as an option and construct the numbers $\omega+1 = \{1, 2, ..., \omega|\}$, then on day $\omega + 2$ construct $\omega + 2 = \{1, 2, ..., \omega, \omega + 1|\}$ &c., which match up with their von Neumann constructions $\omega + 1 = \{1, 2, ..., \omega\}, \omega + 2 = \{1, 2, ..., \omega, \omega + 1\}$. In general we call a surreal number an $ordinal$ if it can be written in the form $\gamma = \{\text{All ordinals less than } \gamma|\}$, which gives a correspondence between the ordinals in Conway's construction and their von Neumann construction in ZFC, where we have $\gamma = \{\text{All ordinals less than } \gamma\}$, from which it is clear that the natural numbers are all the finite ordinals. We can also construct the additive inverses of the ordinals, as well as numbers like $x = \{1, 2, 3, ..|\omega\}$ and $-x = \{-\omega| -1, -2, -3, ...\}$. Then what is x? Observe that

$$x + 1 = \{2, 3, 4, ..., x|\omega + 1\}$$

But then all the left options are less than ω, and the ω is the simplest number less than $\omega + 1$, so by the Simplicity Theorem we must have that $x + 1 = \omega$, so then $x = \omega - 1$, which is smaller than ω but larger than any natural number. We call numbers such as $\omega - 1$ and $\omega + \epsilon$ that are not ordinals in ZFC but are larger than every natural number $surreal\text{-}ordinal$ numbers. Similarly, on the days following ω we can construct $\omega+2$, $\omega+3, \dots, \omega+n = \{\omega, ..., \omega+(n-1)|\}$, and $\omega-2$, $\omega-3, \dots, \omega-n = \{1, 2, 3, ...|\omega, ..., \omega-(n-1)\}$ and all their negatives. But what about after another ω days? Then we would have

$$2\omega = \omega + \omega = \{\omega + 1, \omega + 2, \omega + 3, ...|\}$$

$$\frac{\omega}{2} = \{1, 2, 3, ... \,|\, \omega - 1, \omega - 2, \omega - 3, ...\}$$

created on day 2ω (the above equations being easily verified in our arithmetic). We can also use ϵ as an option on day $\omega + 1$ and construct $\frac{2}{\epsilon} = \{0 \,|\, \epsilon\}$, an even smaller infinitesimal, $2\epsilon = \{\epsilon \,|\, \epsilon + \frac{1}{2}, \epsilon + \frac{1}{4}, ...\}$, a slightly larger infinitesimal, and $\frac{\epsilon}{3}$, 3ϵ, &c. So Conway's construction extends the ordinals and infinitesimals, introduces new numbers between any two real numbers such as $1 + \epsilon$, and places all these numbers along with the reals into one single huge Field, which acts just like the field of real numbers as constructed in ZFC, in an inductive construction that fundamentally uses the symmetry of the positives and negatives around 0.

An Introduction to Combinatorial Games

We will now study the similar constructs of $Games$, as introduced in $Winning\,Ways$ [8], so called because they are the mathematical expressions of certain games played by humans. Specifically, we will be studying only games that fulfill the following properties:

(a) Are played by two players, who move alternatively from one position (or one Game) to another position (or Game) according to well-defined rules;

(b) Always come to an end, with the first player who cannot move being the loser;

(c) Are fully deterministic with no random elements;

(d) Both players have perfect information of the game - nothing is known to only one of the players.

Thus we do not consider games such as Chess(fails (b)), Battleships(fails (d)) or Go(fails (b)) to be Games, but do consider Nim, Hackenbush, Brussels Sprouts, and Mathematical Go to be. We are also only interested in the outcomes of games where both players have perfect play, where they never play a sub-optimal move. To help explain the notation used and basic properties of Games, we will here introduce a card flipping game, played as follows: on a table cards are placed face-down in a grid, in any shape, so long as every card has at least one cardinally neighboring card. This is the $starting\ position$. One player, called Left, flips on their turn any two horizontally adjacent cards, and the other player, called Right, on their turn flips any two vertically adjacent cards. This continues until one player has no two adjacent cards to flip, and is declared the loser.

This game fulfills our properties above, so let us consider the particular starting position given in Figure 1.

Figure 1: The position $* \equiv \{0|0\}$

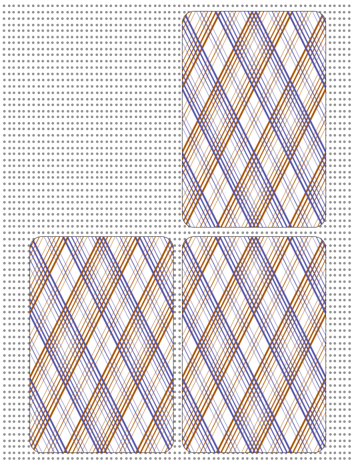
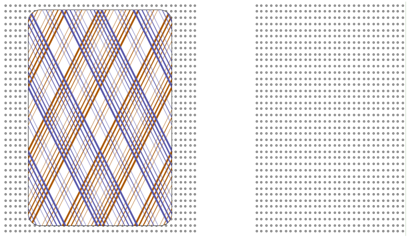


Figure 2: Two forms of the position $0 \equiv \{|\}$



What happens if Left starts? There is only one move, to flip the two lower cards, leaving just one unflipped card. Similarly, if it is Right's turn, the only move is to flip the two rightmost cards, leaving only one lone card. But in that position, there is no move for either player. We call this position the $endgame$, $0 \equiv \{|\}$, aptly named as it cannot be a legal starting position, and it is lost by whoever plays first, as they cannot find a move. Since both players' moves lead to the endgame, Figure 1 is won by the first player to move. Note that this is only one of the forms of the endgame for this card game, the other being when there are no unflipped cards remaining, as in Figure 2.

We call games that are won by the first player $fuzzy\ Games$, and in Figure 1 we have the particular fuzzy game called $star$, $* \equiv \{0|0\}$, which later we shall examine further. Similarly, we call games that are lost by the first player $zero\ Games$, such as the endgame. The notation here is as with numbers, that for a game $g = \{G^L|G^R\}$, the left options $g^L \in G^L$ are the values of the positions that left can move to, if it is their turn, and the right options $g^R \in G^R$ those that right can move to on their turn. However, unlike numbers (where we must respect Axiom 1), we can have left options that are not strictly less than all the right options, such as in $*$, and vice versa.

What about the positions in Figure 3 and Figure 4? We can see that in Figure 3 there is one move, to the zero position, for Left, and none for Right, so this position (or Game) is $\{0|\,\}$. Similarly, Figure 4 is the opposite, with one move for Right to 0, and none for Left, so we can express it as $\{\,|0\}$. This means Left will always win Figure 3, so we call it a $positive\ Game$, and Right will always win Figure 4, a $negative\ Game$.

For a general Game $g = \{G^L|G^R\}$, we write $g = 0$ if it is a zero Game, $g > 0$ if it is a positive Game, $g < 0$ if it is a negative Game, and $g \parallel 0$ if it is a fuzzy Game. Similarly, we write $g \geqslant 0$ if it is a zero or positive Game (Left always wins if Right starts), $g \leqslant 0$ if it is a zero or negative Game (Right always wins if Left starts), $g \parallel\!> 0$ if it is a positive

Figure 3: The position $1 \equiv \{0|\}$

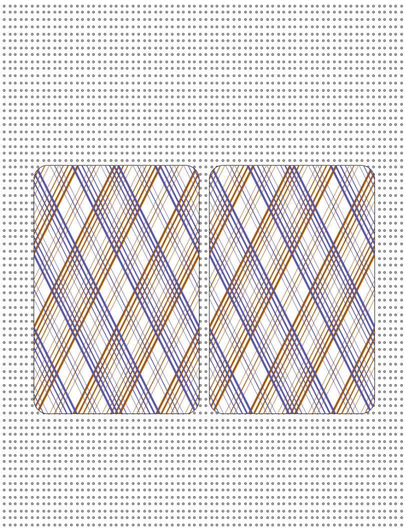
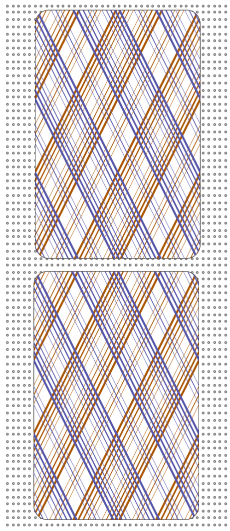


Figure 4: The position $-1 \equiv \{|0\}$



or fuzzy Game (Left always wins if Left starts) and $g <\| \ 0$ if it is a negative or fuzzy Game (Right always wins if Right starts).

More specifically, since in Figure 3, the Game $\{0|\}$, Left has a one move advantage over Right, we say this position has value $1 = \{0|\}$. Similarly, in Figure 4, Right has a one move advantage, so this position has value $-1 = \{|0\}$. Note that these are not specific to this card flipping Game, but describe positions in any Game where one of the players has a single move to a zero position, and the other player has no moves. Also note that any property that we showed for the $numbers$ 1 and -1 will hold for the Games 1 and -1, as in fact these actually are the numbers 1 and -1, in the sense that, for example, both the number $\{0|\}$ and the Game $\{0|\}$ are the same mathematical object, and all the properties we proved in the first section about numbers only use the fact that $\{0|\}$ has only one Left option, $0 \equiv \{|\}$, and has no Right options. Since any Game that has exactly the same options as a number is the same object as that number, all the properties we proved in the first section about numbers, such as being totally-ordered, will hold for all the Games that satisfy our first two axioms, and indeed our class of Surreal Numbers is just the subclass of Games that we obtain by enforcing our two axioms.

Theorem 0.36. *Every Game g is either a positive, negative, zero or fuzzy Game.*

Proof.

We show that we have either $g \leqslant 0$ or $g \ \|> 0$, and that we also have either $g \geqslant 0$ or $g <\| \ 0$. Then the four possible combinations define precisely when g is positive, negative, zero or fuzzy.

Suppose that this is true for all g^L and all g^R. Now if Left starts in g, and there exists some $g^L \geqslant 0$, Left can move to that position, and since it is then Right's move, and that $g^L \geqslant 0$, Left has a winning strategy from g if they go first, that is to say that $g \;\|\!> 0$. If instead all $g^L <\!\|\; 0$, then any move by Left will move to a position that is winning for Right, since in this new position it will be Right's move, which is to say that $g \leqslant 0$. So we must have either $g \leqslant 0$ or $g \;\|\!> 0$. Similarly, by symmetry, if Right begins and there exists some $g^R \leqslant 0$, then $g <\!\|\; 0$, and if instead all $g^R \;\|\!> 0$, then $g \geqslant 0$, so we have either $g \geqslant 0$ or $g <\!\|\; 0$. Thus if the theorem holds for all g^L and all g^R, then it holds for g.

It remains to show that there is a basis case for which the theorem holds. But since we require all Games to come to an end, we must eventually only have to consider the position before the final move, where one of the left or right sets will only contain 0, which is clearly a zero game. So by induction, all Games are either positive, negative, zero or fuzzy Games.

$\square$

For our card flipping game, we can construct complicated starting positions, even when all the cards have direct neighbors. For instance, it is easy to see that a 3×3 square is a fuzzy game (by symmetry, the first player makes sure to flip the middle card, leaving them with two pairs on the table to their opponent's one, and the turn advantage), but a 4×4 one requires some analysis, and we can see how this only grows in complexity. However, for many other Games, such as Nim, single positions are always trivial, so we want to examine what happens when we play multiple Games at once, with each player on their turn making a move in only one of the Games. This leads us to define the disjoint sum of two games:

For any two games $g = \{G^L|G^R\}$ and $h = \{H^L|H^R\}$ their disjoint sum is

$$g + h = \{g^L + h, g + h^L|g^R + h, g + h^R\}$$

for all $g^L \in G^L, g^R \in G^R$, and $h^L \in H^L, h^R \in H^R$.

This is of course the same definition as we had for numbers, and so we know that game addition is commutative, associative, and has the identity element 0, from the proofs for Theorem 0.10, since they do not require that every left option is strictly less than every right option, which is true for all numbers, but not necessarily all Games.

So if we play the games g and h simultaneously, and for instance, Left moves first, then they can either move to some g^L and leave h alone, or move to some h^L and leave g alone. Similarly for Right, and this is exactly what our disjoint sum describes. What happens then if we play the sum of Figure 3 and Figure 4? That is, what is $1 + -1 = \{0|\} + \{|0\}$? If Left starts, their only move is to $0 + -1 = 0 + \{|0\}$, to which Right will reply with their only move, to $0 + 0 = 0$, and Left is the loser. By symmetry, we see that Right must lose if instead they begin. So we have a zero Game, that is that $1 + -1 = 0$, as expected, as these Games are also numbers.

If we play the sum of Figure 1 and Figure 3, we have $*+1 = \{0|0\}+\{0|\}$, which is clearly not a number. Here Right's only move is to $0 + 1 = 1$, where Left wins by a move. If left starts, they can move to either $*$, and promptly lose, or $0 + 1 = 1$, and punctually win. So $* + 1 > 0$, as Left always wins with optimal play (Left should never play to $*$, because it is a strictly worse position for them than 1), and similarly $* + -1 < 0$, as Right always wins. We already know the values of $1 + 1$, $-1 + -1$, $1 + 1 + 1$, &c. since the Games 1 and -1 also belong to subclass of Games which we obtain by enforcing our two axioms from section 2, namely the Surreal Numbers, so since as $numbers$ $1+1 = 2$, it follows that as $Games$ $1+1 = 2$, since the definition of addition is the same on numbers as it is on Games. Similarly, we already know the value of n cards in a row is $\pm n/2$ rounded down to the nearest integer for any integer n.

What is the value of $* + * = \{0|0\} + \{0|0\}$? Since this is a sum of two identical Games in which both players have the same options, it must be a zero Game, as if the first player can find a move, then necessarily the second player can, namely to play the same move but in the other summand, leaving the first player to play in the new position which has this same property again. This leads us to define the negative of a Game, such that the sum of a Game and its negative is always a zero Game.

For any Game $g = \{G^L|G^R\}$, its negative is

$$-g = \{-g^R| - g^L\}$$

for all $g^L \in G^L, g^R \in G^R$.

So any option for Left in g is an option for Right in $-g$, and vice versa. This justifies our naming of the Games 1 and -1, and we must also have that $g + -g = 0$ for any Game g, since the second player can always mimic the first player, by playing the same move in the other summand, and be guaranteed to have a legal move if the first player does. It follows that $*$, and any other Game where both players have the same options must be its own negative, and we call these $impartial\ Games$, such as Nim or Cloves, which we shall study more in the next section. Games that are not impartial we call $partizan$ $Games$, such as Borages, are generally more complicated, so we will study them after impartial Games.

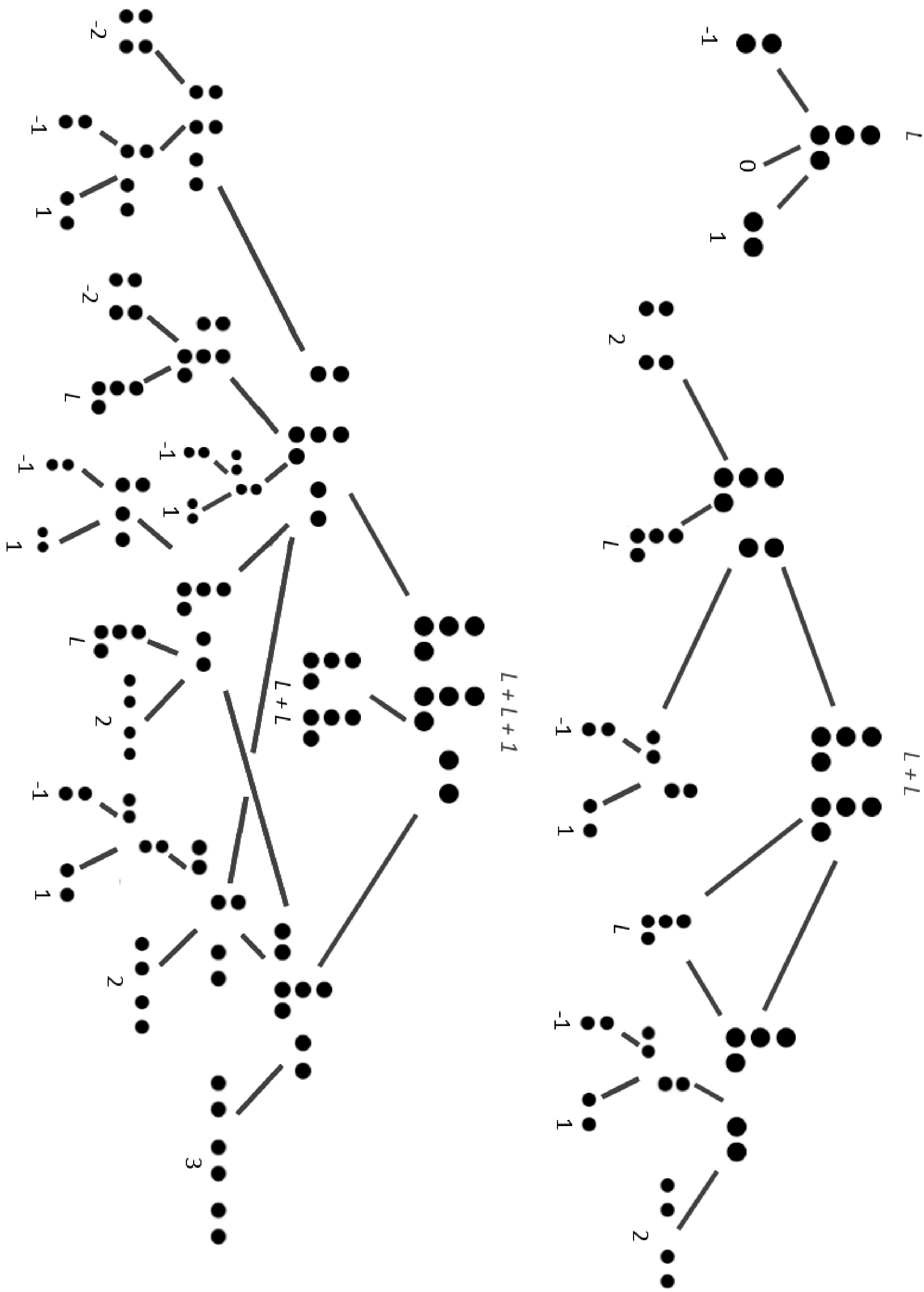
Clearly, we must also have that if $g < 0$, then $-g > 0$, since the winning player in one must lose in the other, and similarly, a zero Game must remain a zero Game, and a fuzzy must remain a fuzzy Game, under negation.

What if, instead of playing the disjoint sum of Figure 3 and Figure 4, we move Figure

4 so that is it directly connected to Figure 3? The only way to do this to make some permutation of an L shape, which has either a line of three vertical cards or three horizontal cards. We can already see from symmetry that these two Games will be negatives of each other, so we will just consider the one with the vertical cards, making an upright L shape, and we shall call this game L. Then by considering each players' options, we have $L = \{-1 \,|\, 0, 1\} < 0$, since Right wins with perfect play regardless of which player begins. Indeed, we already know that the value of the Game L must be $-1/2$, since L is also a number, but then Right must have half a moves advantage in this position. This means that if we were to play the sum of 1 and L, we would still have a positive game, but if we were to play the sum of 1 and two Games of L, we would have a zero game. We can also show this by creating an inverted tree of positions as in Figure 5, starting from $1 + L + L$, and at each node drawing a left arrow to another node for each of Left's moves, and similarly a right arrow for each of Right's moves (and it is this method we use to evaluate Games that are not also numbers). Then by following the branches, we see that there is no path which leads to a positive position if Left begins, or a negative position if Right begins, so $1 + L + L = 0$, which shows that L is half a moves advantage for Right, as we expected.

In Figure 5 we first made the tree of L, from which we could see that it must be a negative Game, and then the tree for $L + L$, in which we use the knowledge that $L < 0$, and finally the tree for $1 + L + L$, where we used both that $L < 0$ and $L + L < 0$, to see that it was a zero Game. Since more complicated positions always collapse into simpler ones, when analysing Games we create a dictionary of positions, so we can quickly truncate branches which already know the values of. For example, in our tree of $1 + L + L$, we can ignore Left's opening move to $L + L$, as we already know it is a winning position for Right.

Figure 5: Position trees for the Games L, $L + L$, and $L + L + 1$



Impartial Games

Earlier we introduced the impartial Game $* = \{0|0\}$, though we don't know much about it yet, except that it is fuzzy, which means it is its own negative, or that $* + * = 0$. Since it is not a number, we must examine it using its sum with other Games. For example, who wins the Game $* + 1 = \{0|0\} + \{0|\}$? It must be Left, since they can play in $*$ and move to 1 if they begin, which is also Right's only move if they start. So we must have $* + 1 > 0$ which implies $-1 < * < 1$, since we already showed that impartial Games are their own negatives. Similarly we have $*+1/2 = \{0|0\}+\{0|1\} > 0$, $*+1/4 = \{0|0\}+\{0|1/2\} > 0$, ..., $*+1/n = \{0|0\}+\{0|2/n\} > 0$, for all natural numbers n, since Left can always plays in $*$ and move to a positive position, and Right must either move to the same position, or play in the other summand to $* + 2/n$, where Left begins and wins. So we have:

$$-1 < -\frac{1}{2} < -\frac{1}{4} < ... < * < ... < \frac{1}{4} < \frac{1}{2} < 1$$

but we also know that it is not equal to 0, since the first player will always win $*$. It follows that $*$ is less than any positive real, and greater than any negative real, since for any positive real we can find a reciprocal power of 2 less than it, and $*$ will be less than that, and similarly for the negative reals. This means that if a Game g is winning for one player, then $g + *$ must also be winning for that player.

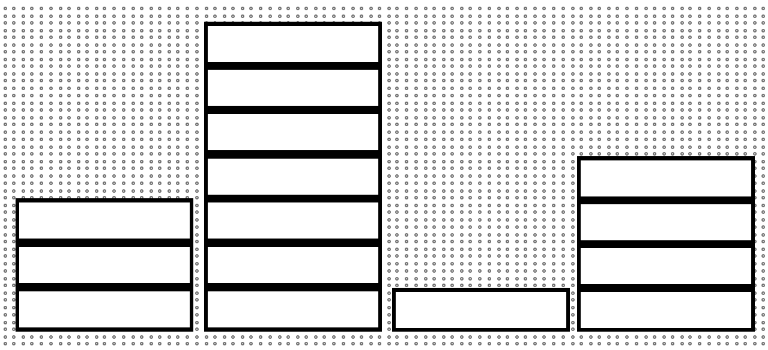
From this point on we will not bother to write out both the left and right sets for impartial Games, as both these sets are always equal, so we just write $\{a, b, c, ...\}$ to mean $\{a, b, c, ...|a, b, c, ...\}$, and not a single set. For example, we write $* = \{0\}$.

The Game of Nim is played with stacks of counters on a table, with either player on their turn removing as many counters as they wish from any one stack. The loser as always is the first player who cannot make a move, as there are no counters left to remove. It is an impartial Game then, as there are no moves that are restricted to only one of the players. We will express Nim positions as $[a, b, c, ...]$, where each variable $a, b, c, ...$ is a natural number greater than 0 denoting how many counter are in that stack. Clearly, then, a single stack $[a]$ is a fuzzy Game, for any a, as the first player can simply remove all a counters from the table. However, it is not necessarily equal to $*$, for if it were we should expect then that $[a, b] = [a] + [b] = 0$ for all a and b. But consider the Game $[3, 2]$. The first player can move to $[2, 2]$, which is a zero Game, as whoever starts in $[2, 2]$ will have their moves copied by the following player, which is to say that $[2] = -[2]$, but $[3] \neq [2]$, even though if they were played alone they would have the same result. So $[3, 2]$ is in fact another fuzzy Game, and in general, $[a, b] = 0$ when $a = b$, and $[a, b] \parallel 0$ when $a \neq b$, since the first player can move to the position where $a = b$, which by the same argument as with $[2, 2]$ is a zero Game but now with the second player beginning.

What are then the values of $[1], [2], [3], ...$? We know that $[0] = \{|\} = \{\} = 0$, so we must have $[1] = \{[0]\} = \{0\} = *$. Then $[2] = \{[1], [0]\} = \{*, 0\}$, and more generally $[n] = \{[0], [1], [2], ..., [n-1]\}$. We call these Games $nimbers$, and we shall shortly show they form a finite field of their own. We generally denote nimbers as $*n = \{0, *1, *2, ..., *(n-1)\}$ for any non-negative integer n, and say that n is the number $associated$ with the nimber $*n$. Then the value of Nim positions with just one stack $[a]$ is exactly $*a$.

Nim and Nimbers

Figure 6: The Nim Game $[3, 7, 1, 4]$



Any position in Nim then is a sum of nimbers, and we have shown enough already to provide a complete theorem for any sum of binary Nim Games, that is for any $[a, b, c, ...]$ where each $a, b, c, ...$ is either equal to 1 or 2. Since $[1, 1] = [2, 2] = 0$, we can remove any pairs of stacks of the same height, without affecting overall value of the game. Since every Game must be of the form $n[1] + m[2]$, where $n[1]$ denotes n stacks of height 1, if both n and m are even that Game is a zero Game, and if one of n or m is odd it is the fuzzy Game $*n$ or $*m$ respectively, and if both are odd we have the Game $*1 + *2 \parallel 0$.

To study more complicated Nim positions we should first work out a general theory of addition on nimbers. For example, the game $[1, 2, 3]$ is a zero game, since any starting play must move to a position in either the form $[a, b]$ (by removing any whole stack), which is a fuzzy Game, or a position in the form $[a, a, b]$ (by removing a portion from either $[2]$ or $[3]$), which is equal to $[b]$, a fuzzy Game. So we have that $*1 + *2 + *3 = 0$, and since nimbers are their own negatives, we know that:

$$*1 + *2 = *3, \text{ and } *3 + *1 = *2, \text{ and } *2 + *3 = *1$$

We can also show this more explicitly using our definition of Game addition. We have that $*1 = \{0\}$, $*2 = \{0, *1\}$, and $3 = \{0, *1, *2\}$, so $*1+*2 = \{*1, *1+*1, *2\} = \{0, *1, *2\} = *3$. However, trying the other two equations gives us $*3 + *1 = \{0, *1, *3\}$ and $*2 + *3 = \{0, *2, *3\}$, for which we need to develop some more theory to verify equal $*2$ and $*1$ respectively.

We define the $minimal\ excluded$ number, or mex, for a list non-negative integers $a, b, c, ...$ as the least non-negative integer that is not included in that list. For example, $mex(0, 1, 2, 4, 5) = 3$. We now use this concept to prove the following theorem:

Theorem 0.37. *Any impartial Game $G = \{*a, *b, *c, ...\}$ that has only nimber options $*a, *b, *c, ...$ is equal to the nimber $*mex(a, b, c, ...)$, where $a, b, c, ...$ are all the numbers associated with the nimbers $*a, *b, *c, ...$.*

Proof.

We have $G = \{*a, *b, *c, ...\}$, and let $*m = *mex(a, b, c, ...)$. Then we must have that the player who wins the Game $*m$, must also win the Game G. For if the first player wins $*m$, whatever their winning move is in $*m$ is also their winning move in G, because every move from $*m$ must be to some $*l$ where $l < m$, and this $*l$ must also be an option of G, otherwise m would not the mex of $a, b, c, ...$. If instead the second player wins $*m$, then either the first player moves in G to some position $*l$ such that $l < m$, or they move to some position $*n$ such that $m < n$. For the former, since $*l$ is an option of $*m$, it is as if they had started in $*m$, and the second player plays their winning strategy in $*m$. For the latter, $*m$ must be an option of $*n$, so the second player plays to $*m$ and wins. So we have that $*m = G$, and that any impartial Game with only nimber options is itself equal to some nimber. $\square$

We call the nimber $*m$ the $nim\text{-}value$ of G, and we call the moves to $*n$ where $m < n$ $reversible\ moves$, since as we showed, the following player can always reverse the Game back to $*m$, unlike in the moves to $*l$ where $l < m$. In Nim no such moves exist, since we only remove counters, but we could play an altered version where either player could on their turn either decrease any stack as usual or increase any stack by any amount of counters they have removed on previous turns. The outcome of this Game from any starting position must be exactly the same as the Nim Game from that position, since if any player chooses to increase a stack by n counters, the other player can instantly reverse to the previous position by removing those same n counters. Note that this game would still be finite, since the player with a winning Nim strategy will never want to increase the amount of counters in any stack, and the other player can only finitely many times increase a stack, until they run out of counters and have to reduce a stack.

It also follows directly from Theorem 0.37 that the sum of two nimbers must be another nimber, and we can now evaluate $*3 + *1$ and $*2 + *3$ explicitly. We had that the former was equal to the impartial Game $\{0, *1, *3\}$, and the latter equal to $\{0, *2, *3\}$. But using Theorem 0.37 the minimum excluded number associated with $0, *1$, and $*3$ is 2, so we have $*3 + *1 = *2$, and similarly $mex(0, 2, 3) = 1$ implies that $*2 + *3 = *1$. To simplify, we will also write $mex\{*a, *b, *c, ...\}$ to mean $*mex(a, b, c, ...)$, so, in this notation, we have simply that $*1 = mex\{*0, *2, *3\}$.

We now define the nim-sum of any amount of nimbers and show that the nim-sum of a Nim position is 0 if and only if the first player wins that position. The nim-sum of any nimbers $*a, *b, *c, , ...$ is defined as the nimber associated with $digital\ sum$ of their associated non-negative integers. That is to say that it is the nimber associated with the sum without carrying over of the binary representations of $a, b, c, ...$. For example, we have:

$$\begin{aligned} 3 +_d 4 +_d 5 = \quad & 011_2 \\ & +_d 100_2 \\ & +_d 101_2 \\ & \qquad = 010_2 = 2 \end{aligned}$$

where x_2 is a number expressed in binary, and $+_d$ is the digital sum. This is clearly commutative and associative, with additive identity 0, and it's also easy to see that the additive inverse of any number is itself. Also note that the digital sum of any natural numbers must also be a natural number. So the natural numbers together with $+_d$ form a group, and so do the nimbers together with the nim-sum. From our example above, we can read off $*3 + *4 + *5 = *2$, which is a fuzzy Game, as long as the nim-sum represents playing multiple Nim stacks at once, which we shall now prove.

Theorem 0.38. *Any Nim game* $[a, b, c, ...]$ *is a zero Game if and only if the nim-sum* $*a + *b + *c + ...$ *is equal to* 0*.*

Proof.

We write each of $a, b, c, ...$ in binary, with a column for each power of 2, so for example, $12 = 0 \cdot 16 + 1 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 0 \cdot 1 = 01100_2$. Then each of $a, b, c, ...$ has, for each power of 2, either a 1 or 0 in that column, and there is a largest power of 2 for each $a, b, c, ...$. We can then write the binary forms of each of $a, b, c, ...$ one under the other, so that for each power of 2 there is a column which contains some amount of 1s and 0s in every row that doesn't have a 1.

Now suppose there are an even amount of 1s in each column. Then the first player can change any one row, and, since the number of counters must decrease in that stack, can only change the column of the largest power of 2 from a 1 to a 0, though if do then they may change any column of a smaller power of 2 to either a 1 or a 0, and thus make those column they change have an odd number of 1s. If they do change the column of the largest power of 2, then the second player can choose another row which has a 1 in that column, and change that 1 to a 0, and then in all the lower power 2 columns change the number so that each column has an even amount of 1s. If the first player does not change the largest power of 2, then they cannot change the next highest power of 2 from a 0 to a 1, as this would increase the number. But of the columns they do change, there must be a highest power of 2 which they change from a 1 to a 0. Then the second player can choose to play in a row that has that power of 2 as its heighest power, and similarly change all the lower powers to make sure every column has an even amount of 1s. So whatever the first player does, if the position starts with an even amount of 1s in each column, the second player can always play to make the position keep an even amount of 1s in each column.

Since each move must reduce the value of a row, after a finite amount of moves we must get to the position were there are no counters left, that is that every column has no 1s, and the player whose turn it is loses. But since in this position there are zero 1s, that is an even amount of 1s in each column, if original position had an even amount of 1s, then with perfect play the second player can always win by the strategy shown above. Since the nim-sum of a position is 0 when there are an even amount of 1s in each column, if the nim-sum of a position is 0, the Nim game $[a, b, c, ...]$ is a zero Game.

It remains to show that if the nim-sum is not equal to zero, that is not every column has an even amount of 1s, then we cannot have a zero Game. But in this case there must be a highest power of 2 that has an odd number of 1s in its column. Then the first player can play in a row with that power of 2 as its highest power, change that 1 to a 0, and then change all the lower powers of 2 so that every column has an even amount of 1s. Then the second player must start on a zero Game, so the first player can always win. Thus if the nim-sum is non-zero the Nim game cannot be a zero Game.

$\square$

With this we can now calculate if any Nim Game is a zero Game or not, without having to play out every possible variation. The digital sum of two non-negative integers is equivalent to just expressing both numbers as their sum of powers of 2, and then adding only the powers of 2 that are in just one of the two numbers, this computation being easier if, by some chance, one wants to actually play a game of Nim. In that case, the optimal strategy is to move to a zero position if possible, and then after each of the opponent's replies revert back to a zero position as we did in Theorem 0.38. If it is not possible to move to a zero position, then position must already a zero position, and so the optimal strategy must be psychological, attempting to trick the opponent by means of some comical expressions or sounds into moving to a non-zero position, as otherwise with perfect play they will always win.

We have for any two nimbers $*a$ and $*b$, that $*a + *b = *a + *c$ if and only if $b = c$, since otherwise either $b < c$ or $c < b$, and if the former then $*a + *b$ is in the mex of $*a + *c$, that is to say $*a + *b$ would be an option of $*a + *c$, and so by Theorem 0.37 $*a + *b \neq *a + *c$, and similarly for the latter. For example, we have that $*3 + *2 = \{0, *1, *2\} + \{0, *1\} = mex\{*3, *3 + *1, *2, *2 + *1, 0\}$, which has $*3 + *1$ in the mex so clearly $*3 + *2 \neq *3 + *1$.

This then allows us to write the sum of $*a$ and $*b$ inductively as:

$$*a + *b = mex\{*a' + *b, *a + *b'\}, \text{ for all } a' < a, b' < b$$

since $*a' + *b \neq *a + *b \neq *a + *b'$. With this expression we can now explicitly show the properties that nim-addition has:

Theorem 0.39. *For any three nimbers $*a$, $*b$, and $*c$, we have:*

(a) $*a + 0 = *a$*;*

(b) $*a + *b = *b + *a$*;*

(c) $*a + *a = 0$ *and* $*a + *b \neq 0$ *for all* $*b \neq *a$*;*

(d) $(*a + *b) + *c = *a + (*b + *c)$*;*

(e) $*a + *b$ *is a nimber.*

Proof.

(a) There are no $0'$ terms, so $*a + 0 = mex\{*a' + 0\}$. So we reduce the question down to the elements of $*a$. Then by induction, since $0 + 0 = mex\{\} = 0$, we have $*a + 0 = mex\{*a'\} = *a$.

(b) We have $*a + *b = mex\{*a' + *b, *a + *b'\}$ and $*b + *a = mex\{*b' + *a, *b + *a'\}$, so we reduce the question down to the elements of $*a$ and $*b$. But clearly $*x + 0 = 0 + *x = *x$ for all nimbers $*x$, so by induction $*a + *b = *b + *a$.

(c) Suppose the property holds for all simpler nimbers. Then we have by induction that $*a + *a = mex\{*a + *a', *a' + *a\} = mex\{*a + *a'\} = 0$, since none of $*a + *a' = 0$, and the inductive basis follows from (a).

(d) By induction suppose the associative property holds when we replace one of $*a$, $*b$ or $*c$ with $*a'$, $*b'$ or $*c'$ respectively. Then we have $(*a+*b)+*c = mex\{(*a+*b)'+*c, (*a+*b)+*c'\} = mex\{(*a'+*b)+*c, (*a+*b')+*c, (*a+*b)+*c'\} = mex\{*a'+(*b+*c), *a+(*b'+*c), *a+(*b+*c')\} = mex\{*a'+(*b+*c), *a+(*b+*c)'\} = *a+(*b+*c)$. It remains to show the inductive basis, but clearly the associative property holds when one of $*a$, $*b$ or $*c$ is equal to 0.

(e) Follows directly from our inductive definition, (a), (b), and Theorem 0.37. □

So we see that the position $[3, 7, 1, 4]$ in Figure 6 is equal to $*1$, since we can express the sum of the stacks in terms of powers of two, and cancel pairs of the same power, as follows: $*3 + *7 + *1 + *4 = (*2 + *1) + (*4 + *2 + *1) + *1 + *4 = (*4 + *4) + (*2 + *2) + (*1 + *1) + *1 = *1$.

Next in our quest of a nimber field we must define multiplication. For any two nimbers $*a$ and $*b$, we have the inductive definition:

$$*a \cdot *b = mex\{*a' \cdot *b + *a \cdot *b' + *a' \cdot *b'\}$$

for all $*a'$ options of $*a$ and $*b'$ options of $*b$. Remembering that nimbers are their own negatives, we see that this definition is just our definition for multiplication on surreals, if we use the Simplicity Theorem in place of mex and limit ourselves to just the naturals. Thus we have the following:

Theorem 0.40. *For any three nimbers $*a$, $*b$ and $*c$, we have:*

(a) $*a \cdot 0 = 0$ *and* $0 \cdot *a = 0$*;*

(b) $*a \cdot *1 = *1$*;*

(c) $*a \cdot *b = *b \cdot *a$*;*

(d) $(*a + *b) \cdot *c = *a \cdot *c + *b \cdot *c$

(e) $(*a \cdot *b) \cdot *c = *a \cdot (*b \cdot *c)$*;*

(f) $*a \cdot *b$ *is a nimber.*

Proof.

(a) Since there are no options of 0, that is $0'$ is not defined, we have directly $*a \cdot 0 = 0 \cdot *a = mex\{\} = 0$

(b) Firstly, we know that $0 \cdot *1 = 0$ from (a). Now by induction assume that $*\alpha \cdot *1 = *\alpha$ for all $\alpha < a$. Then we, since the only option of $*1$ is 0, we have $*a \cdot *1 = mex\{0, *1 \cdot *1, *2 \cdot *1, ... * (a-1) \cdot *1\}$ and so using our induction step $*a \cdot *1 = mex\{0, *1, *2, ..., *(a-1)\} = *a$.

(c) We have $*a \cdot *b = mex\{*a' \cdot *b + *a \cdot *b' + *a' \cdot *b'\}$ and $*b \cdot *a = mex\{*b' \cdot *a + *b \cdot *a' + *b' \cdot *a'\}$. Since the nim-sum is commutative, we have $*a \cdot *b = *b \cdot *a$ if and only if their options are commutative. So inductively we reduce the question on any two nimbers down to their options, and eventually we only have to calculate the basis case, whether $0 \cdot *x = *x \cdot 0$ for some nimber $*x$, which we showed in (a).

(d) By induction suppose that this distributive property holds if we replace any one of $*a$, $*b$ or $*c$ with $*a'$, $*b'$ or $*c'$ respectively. Then we have that $(*a + *b) \cdot *c = mex\{(*a+*b)' \cdot *c + (*a+*b) \cdot *c' + (*a+*b)' \cdot *c'\} = mex\{(*a'+*b) \cdot *c + (*a+*b) \cdot *c' + (*a'+*b) \cdot *c', (*a+*b') \cdot *c + (*a+*b) \cdot *c' + (*a'+*b') \cdot *c'\} = mex\{(*a' \cdot *c + *a \cdot *c' + *a' \cdot *c') + (*b \cdot *c + *b \cdot *c' + *b \cdot *c'), (*a \cdot *c + *a \cdot *c' + *a \cdot *c') + (*b' \cdot *c + *b \cdot *c' + *b' \cdot *c') = mex\{(*a \cdot *c)' + *b \cdot *c, *a \cdot *c + (*b \cdot *c)'\} = *a \cdot *c + *b \cdot *c$. It remains to show the induction basis, but clearly the distributive property holds when any of $*a$, $*b$ or $*c$ equals 0.

(e) By induction suppose that this associative property holds if we replace any one of $*a$, $*b$ or $*c$ with $*a'$, $*b'$ or $*c'$ respectively. Then we have that $(*a \cdot *b) \cdot *c = mex\{(*a \cdot *b)' \cdot *c + (*a \cdot *b) \cdot *c' + (*a \cdot *b)' \cdot *c'\} = mex\{(*a' \cdot *b + *a \cdot *b' + *a' \cdot *b') \cdot *c + (*a \cdot *b) \cdot *c' + (*a' \cdot *b + *a \cdot *b' + *a' \cdot *b') \cdot *c'\} = mex\{(*a' \cdot *b \cdot *c + *a \cdot *b' \cdot *c + *a' \cdot *b' \cdot *c + *a \cdot *b \cdot *c' + *a' \cdot *b \cdot *c' + *a \cdot *b' \cdot *c' + *a' \cdot *b' \cdot *c'\} = mex\{*a' \cdot (*b \cdot *c) + *a \cdot (*b \cdot *c)' + *a' \cdot (*b \cdot *c)'\} = *a \cdot (*b \cdot *c)$. It remains to show the inductive basis, but clearly the associative property holds when one of $*a$, $*b$ or $*c$ is equal to 0.

(f) We have $*a \cdot *b = mex\{*a' \cdot *b + *a \cdot *b' + *a' \cdot *b'\}$. Since the nim-sum is closed, $*a \cdot *b$ is nimber if and only if all of $*a' \cdot *b$, $*a \cdot *b'$, and $*a' \cdot *b'$ are. So inductively we reduce the question on $*a \cdot *b$ down to its options, and eventually only have to show the basis cases, that $0 \cdot *x$ and $*x \cdot 0$ are nimbers, which we did in (a). □

So we can step-by-step calculate the nim-product of any two finite nimbers by starting with $0 \cdot *n = 0$ and $*1 \cdot *n = *n$, then calculating $*2 \cdot *2$, $*2 \cdot *3$, $*3 \cdot *3$, $*3 \cdot *4$ &c., and using the commutative property proved above. Here is the multiplication table for the first nine nimbers (we don't bother writing the $*$s):

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3	1	8	10	11	9	12	14
3	3	1	2	12	15	13	14	4	7
4	4	8	12	6	2	14	10	11	15
5	5	10	15	2	7	8	13	3	6
6	6	11	13	14	8	5	3	7	1
7	7	9	14	10	13	3	4	15	8
8	7	12	4	11	3	7	15	3	5
9	9	14	7	15	6	1	8	5	12

It can be shown by rigorous calculation that the nim-product of two finite nimbers obeys the following rules, where A is a Fermat power (2, 4, 16, 256, ...), that is an integer in the form 2^{2^n} for some non-negative integer n:

$$*A \cdot *b = *(A \cdot b) \quad \text{for } b < A$$
$$*A \cdot *A = *\left(\frac{3A}{2}\right)$$

So for example $*16 \cdot *3 = *(16 \cdot 3) = *48$, $*4 \cdot *4 = *6$, and

$$\begin{aligned} *5 \cdot *9 &= (*4 + *1) \cdot (*4 \cdot *2 + *1) \\ &= *4 \cdot *4 \cdot *2 + *4 + *4 \cdot *2 + *1 \\ &= *6 \cdot *2 + *4 + *8 + *1 \\ &= (*4 + *2) \cdot *2 + *4 + *8 + *1 \\ &= *8 + *3 + *4 + *8 + *1 \\ &= *3 + *4 + *1 = *2 + *1 + *4 + *1 = *6 \end{aligned}$$

We have already said how nimbers are their own negatives, so we clearly have an additive inverse that has the properties we expect. It remains to define a multiplicative inverse, and again we do this similarly to how we did it on the surreals, and letting $*b = *\frac{1}{a}$, we recursively define

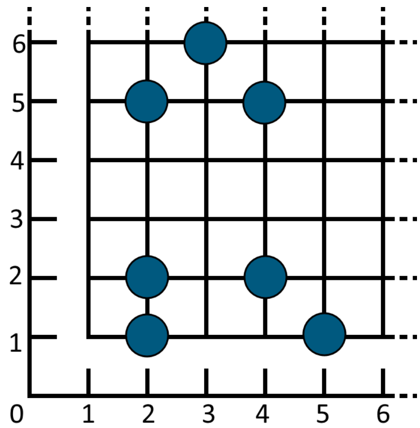
$$*b = mex\left\{0, *1 + \frac{(*a' + *a) \cdot *b'}{*a'}\right\}$$

were $*b'$ is an already previously calculated element in the mex. Then the proof that $*b$ is the multiplicative inverse of $*a$, that is that $*a \cdot *b = *1$, is similar to the the proof given for the multiplicative inverse of surreals, but we do not expand it here.

So we have shown now that nimbers form their own field, but unlike the Field of surreals, since we require that all games come to an end, and have that the only numbers associated with nimbers are non-negative integers, this field of nimbers has a finite amount of elements, so it is a finite field. We can further note that it is of characteristic 2, since we only have to sum two terms of the multiplicative identity, $*1$, to equal the additive identity, 0, since $*1 + *1 = *1 - *1 = 0$.

Turning Corners

Figure 7: The Turning Corners Game $\{(2,1),(2,2),(2,5),(3,6),(4,2),(4,5),(5,1)\}$



In the previous section we defined multiplication on nimbers, but we didn't give any of its applications to Games. So let us consider the Game of Turning Corners, which is an expansion of Nim into two dimensions. The Game is played on a grid board, as in Figure 7, where each intersection may hold a stone, and a $zero$ axis which can never hold any stones. Then on each player's turn they pick a stone that is already on the board and construct a square on the grid (including the zero axis) which has that stone as it right-upmost corner. The move is then to $turn$ all the corner of that square, so that on the corners where there is a stone it is removed, and on the corners where there is not a stone one is placed (but remembering that the zero axis can never hold any stones). Then we can express any position as just a set of the intersection that hold a stone, for example $\{(1,1),(1,2),(2,1),(2,2)\}$ is the position which stones in a square in the left-bottom most corner (and so one possible move from this position would be to turn all the stones over, leaving an empty board). As ever the objective of the Game is to be the last player to move, and since neither player has any moves the other does not, it is an impartial Game. In order to analyse this Game we first prove the following vital theorem of impartial Games:

Theorem 0.41. Sprague-Grundy Theorem: *Every impartial Game is equal to some nimber. Specifically, for an impartial game G with options $a, b, c, ...$, $G = *mex(a, b, c, ...)$*

Proof.

Let $G = \{a, b, c, ...\}$ be an arbitrary impartial Game. Then by induction, since all of $a, b, c, ...$ are simpler impartial Games, suppose they are all equal some the nimbers $*\alpha, *\beta, *\gamma, ...$ respectively. Then from Theorem 0.37 we must have that G is also equal to a nimber, specifically $G = mex\{\alpha, \beta, \gamma, ...\} = *mex(a, b, c, ...)$. Finally since we require that all Games come to an end, we must have that all impartial Games reduce down to the endgame $0 = \{\}$, which is equal to the nimber $*0$, so by induction every impartial game is equal to some nimber.

□

This theorem helps us immensely in solving impartial Games, as we can use all that we have already proved about nimbers, and we just have to work out how to describe any given position of an impartial Game using nimbers. In Turning Corners, let us first consider positions where only the first row and first column hold stones. In such positions, the only moves are to remove a stone entirely, or to remove a stone and turn a lower intersection on the same row or column as the removed stone. For example, if we had the position $\{(1, 1), (1, 2), (1, 4)\}$, where all the stones lie on the first column, we could choose to remove the stones at $(1, 4)$ and $(1, 2)$, to get $\{(1, 1)\}$ or we could remove the stone at $(1, 4)$ and place one at $(1, 3)$, to get $\{(1, 1), (1, 2), (1, 3)\}$. Since all the stones are in the first column, the only moves that can be played will move to positions where still all the stones are in the first column, and so in fact the value of $\{(1, 1), (1, 2), (1, 4)\}$ is equal to the value of the Nim position $[1, 2, 4]$, because we can show inductively that every position from these single column Turning Corners Games has a Nim positions of equal value. For example, the move from $\{(1, 1), (1, 2), (1, 4)\}$ which removes both the $(1, 1)$ and $(1, 4)$ stones and leaves the position $[(1, 2)]$ is equivalent to moving in the Nim Game $[1, 2, 4]$ to the position $[1, 2, 1]$. So in these first row and column positions, we see that the value of the position is just the nim-sum of greatest coordinate of each stone on the board.

Next lets consider the position with just a single stone at the $(2, 2)$ intersection. Then there are four possible moves for the first player: they can: $a)$ remove the stone, so the board is empty and in the zero position; $b)$ turn the corners of the square $\{(0, 1), (0, 2), (1, 2), (2, 2)\}$ so that there remains a single stone at $(1, 2)$, with value $*2$; $c)$ symmetrically, turn the corners of the square that leaves just the single stone $(2, 1)$, again of value $*2$; or (d) turn the corners of the square that leaves the position $\{(1, 1), (1, 2), (2, 1)\}$, of value $*1 + *2 + *2 = *1$. Clearly this is a fuzzy Game, since the first player can move to the zero position, and from the Sprague-Grundy Theorem we know it must be equal to some nimber, and specifically since we know the values of all the options, we can directly read off that the Game $\{(2, 2)\}$ is equal to $*mex(0, 1, 2) = *3$.

Let us write $G(n, m)$ for the value of a general stone at (n, m), also known as its Grundy

Value, which we know will be a nimber. Then to calculate $G(n, m)$ we need to know all values of having a stone at all the intersections further to the left of n and lower than m, since we can construct a square with any of these intersections and (n, m) as two of its corners. That is, we must have already calculated each of $G(n', m), G(n, m'), G(n', m')$, for all $n' < n$ and $m' < m$. Then in turning (n, m) we choose some fixed n' and m' and also turn the intersections $(n', m), (n, m'), (n', m')$, and from the Sprague-Grundy Theorem we must have that (n, m) has value $G(n, m) = mex\{G(n', m) + G(n, m') + G(n', m')\}$ for all n', m'. But note that this is the same as our definition of multiplication between two nimbers given above, since clearly $G(0, 0) = mex\{\} = 0 = *0 \cdot *0$, and then $G(a, b)$ for all $a > 0$ or $b > 0$ is defined recursively in the same way as nim multiplication is, namely $*a \cdot *b = mex\{*a' \cdot *b + *a \cdot *b' + *a' \cdot *b'\}$. So in fact the value of any given stone at (n, m) is equal to the nim-product $*n \cdot *m$.

Finally we consider positions with multiple stones that are not all on the first row and column. Suppose we choose the stone at (n, m) as the top-rightmost stone to turn, then the other three corners of the square we turn either hold a stone or do not. If any do, then we remove it, removing a stone of some definite value $*v_1$ from the board, and if any don't, we add a stone of another definite value $*v_2$ to the board. Now consider instead if we played the same position over multiple boards, with each board holding only one stone. Then in turning the stone at (n, m), we would add a stone to a board for every other corner of the square that we turn. Then if no other board has a stone at that same intersection, we add a stone of definite value $*v_2$ to the sum of boards, and if instead another board does have a stone at intersection, then since it has the same value $*v_1$ as this stone that we have just added, and nimbers are their own negatives, the two stones cancel out, and we have effectively removed a stone of value $*v_1$ from the sum of boards. This means that the total value of any Turning Corners position is just the sum of the values of all the separate stones on the board, so the arbitrary position $\{(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)\}$ has value $*a_1 \cdot *b_1 + *a_2 \cdot *b_2 + ... + *a_n \cdot *b_n$.

So, using our rules of nim multiplication as above, we can calculate that the position in Figure 7 has value:

$$
\begin{aligned}
& *2 \cdot *1 + *2 \cdot *2 + *2 \cdot *5 + *3 \cdot *6 + *4 \cdot *2 + *4 \cdot *5 + *5 \cdot *1 \\
= \; & *2 + *3 + *2 \cdot (*4 + *1) + (*2 + *1) \cdot (4 + *2) + *4 \cdot *2 + *4 \cdot (*4 + *1) + *5 \\
= \; & *2 + *3 + *8 + *2 + *8 + *3 + *4 + *2 + *8 + *6 + *4 + *4 + *1 \\
= \; & *2 + *8 + *6 + *4 + *1 \\
= \; & *9
\end{aligned}
$$

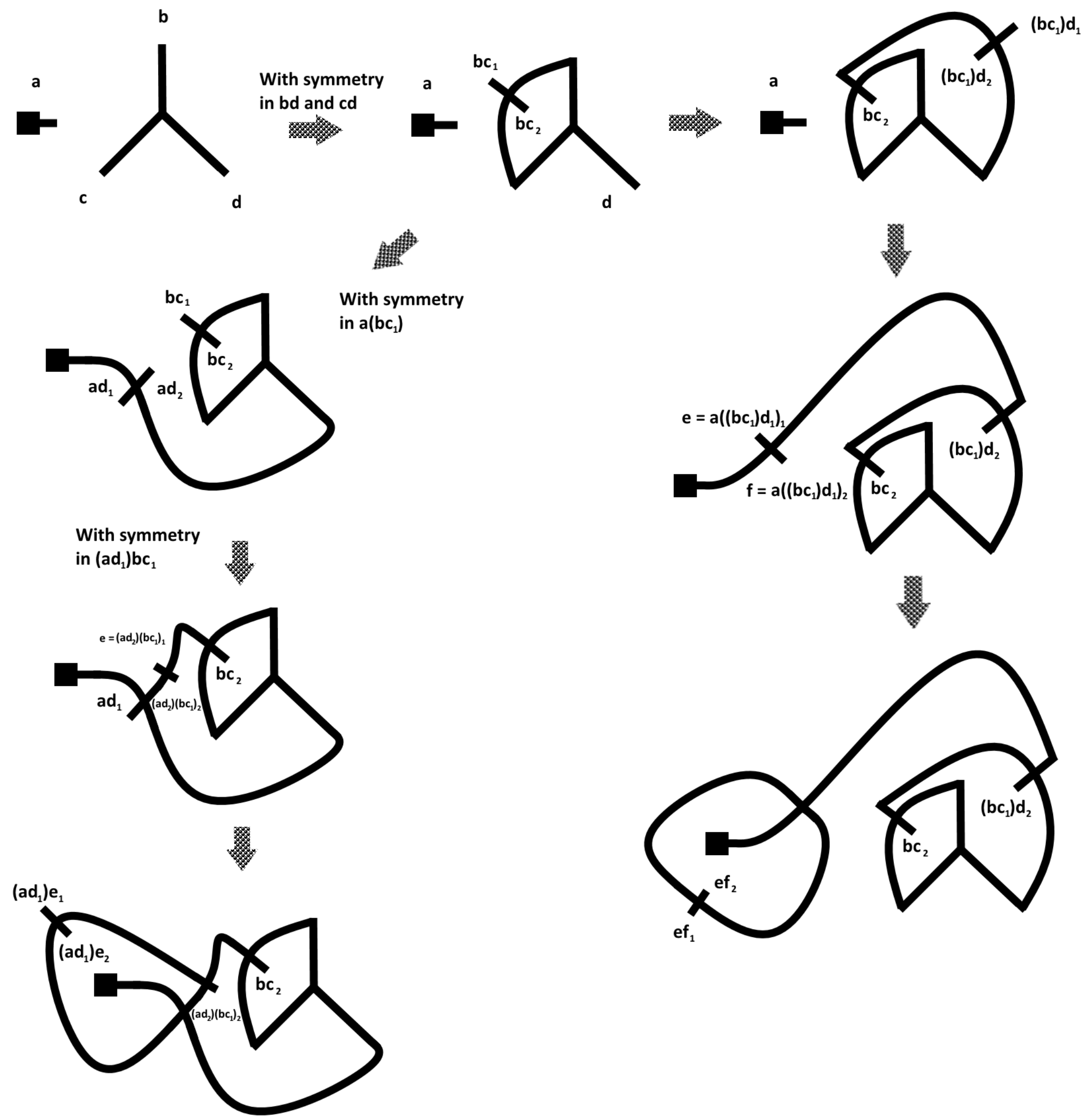
So the position is winning for the first player. The best move is to turn the square $\{(2, 5), (2, 2), (0, 5), (0, 2)\}$, as this removes the stones at $(2, 5)$ and $(2, 2)$, which have a total value of $*2 \cdot *5 + *2 \cdot *2 = *2 \cdot *4 + *2 \cdot *1 + *2 \cdot *2 = *8 + *2 + *3 = *9$, leaving the second player to make the first move in a zero position.

Cloves and Brussels Sprouts

To end this section we will use what we have shown so far about impartial Games to give a complete theory of the impartial Game Cloves, which is a generalized version of the game Brussels Sprouts. The standard version of Brussels sprouts is played with a pen and a piece of paper (a plane), with the initial position being a page that has a number of crosses drawn on it, each of which we consider as two lines, both open at both ends, bisecting each other. Then on each players move, they must connect any two open ends of lines with a new line (which will necessarily be closed on both ends), such that this new line doesn't intersect any other line already on the page, and then make a cross through this line, that is draw through this line another line that is open at both ends. As ever the Game ends when one player cannot find a move, because there is no possible way to connect any two open ends of lines without intersecting an already present line, and they lose. This is then an impartial Game, as every move allowed to one player is also allowed to the other. The only difference in Cloves is that we do not necessarily require that the initial position be a number of crosses, just that it be a number of vertices, each with a number of open lines (or $prongs$) coming out of it, which we'll call the $pieces$ of the position, and write P_n for a piece with n open ends. If we draw over an open end with a square, then we consider it a closed end, so that no other open end can connect to it. We furthermore write $P_{n_1,n_2,\dots,n_k}$ for a page that has the pieces P_{n_1}, P_{n_2}, ..., P_{n_k} on it, and write $P_{n_1} + P_{n_2} + ... + P_{n_k}$ for the Game where each piece is a page by itself, so that we can never connect the open ends of two different pieces to each other, which is to say $P_{n_1} + P_{n_2} + ... + P_{n_k}$ is the disjoint sum of P_{n_1}, P_{n_2}, ..., P_{n_k}. Figure 8 gives some of the reachable positions from the initial position with a vertex of one prong (a P_1 piece, and note the square closing of one end of the line) labeled a, and a vertex of three prongs (a P_3 piece), labeled b, c, and d respectively.

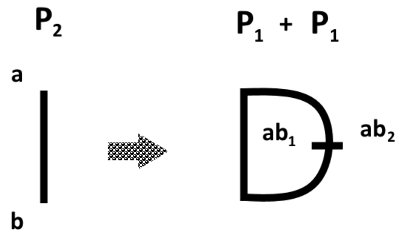
In Figure 8 from our starting position $P_{1,3}$ we connect the ends b and c, and then draw a line through this connecting line, so that the open ends bc_1 and bc_2 appear either side of it, and since b and c are no longer open ends, we remove their labels. Note that if we connected b and d clockwise, or connected c and d anticlockwise, so that the connecting line did not form a loop around c or b respectively, then by symmetry these positions are equivalent to the position in Figure 8, since they all move to positions which have one prong in a loop, and a one prong and a two prong piece outside that loop, which is the meaning of the text above the arrow from our starting position, and the same idea is meant by the other arrows. Then from this second position there are

Figure 8: Just two of the possible move paths in the Cloves Game $P_{1,3}$



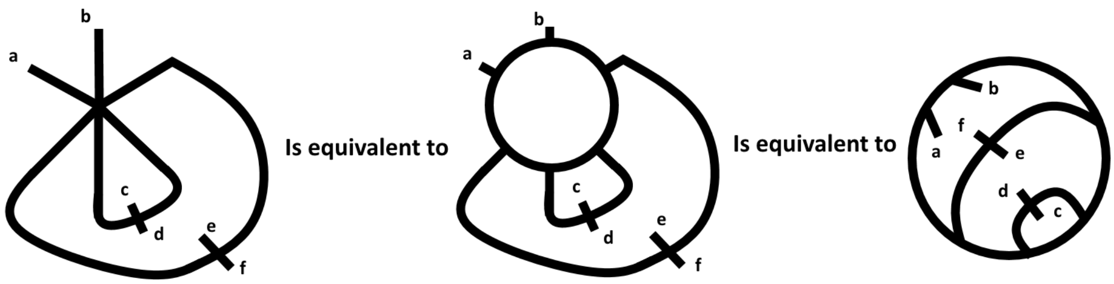
two distinct moves (accounting for symmetry), one pointed to by the rightward arrow where we connect the ends bc_1 and d, and one pointed to by the leftmost arrow where we connect d and a. Then from both of these positions (accounting for symmetry), there are two more moves to be played until we come to positions where there are no legal moves, which are the final graphs in the figure.

Before we can analyse positions like $P_{1,3}$ which have more than one starting piece, we must work out the value of single piece positions. If we consider the simplest piece, P_1, which has just one open end, we see that from this position there are no legal moves, so the second player always wins, that is $P_1 = 0$. Then what about P_2? This piece has two open ends, so a legal move is to these connect these two ends to each other, and then draw an open end either side of the new connecting line. But this connecting line makes a loop, so one of the new open ends will be inside the loop, and the other outside. Clearly then these two open ends can never be connected with a line without cutting through the loop, so from this position there are no moves, that is it is a zero position. Alternatively we can see this by noting that the position is the disjoint sum of P_1 and P_1, so has value $P_1 + P_1 = 0$. Then P_2 must then be a fuzzy position, and we can use the Sprague-Grundy Theorem to precisely show that it has value $P_2 = *mex(0) = *$, since the only option from P_2 is to move to a zero position.



It's obvious that for a single open end there are no moves to be made, regardless of whether the end is inside a loop or not. However it is not so clear in general if the value of n open ends inside the loop is equal to the value of n open ends outside of any loop. But by considering Figure 9 we can see that any position outside a loop can be morphed into an equivalent position inside a loop, where the options from the position are not changed by this morphing, and vice versa. We see that in first position of Figure 9 the legal moves are to make one of the following connections ab, af, bf, de. Then we add an empty loop in the centre of the piece, and the possible moves are unchanged. Finally, we fold the lines on the outside of the circle 'through the page' into the inside the circle, and we see that the legal moves are still to connect one of ab, af, bf, de as before. So we are now justified in also writing P_n for n open ends inside a loop.

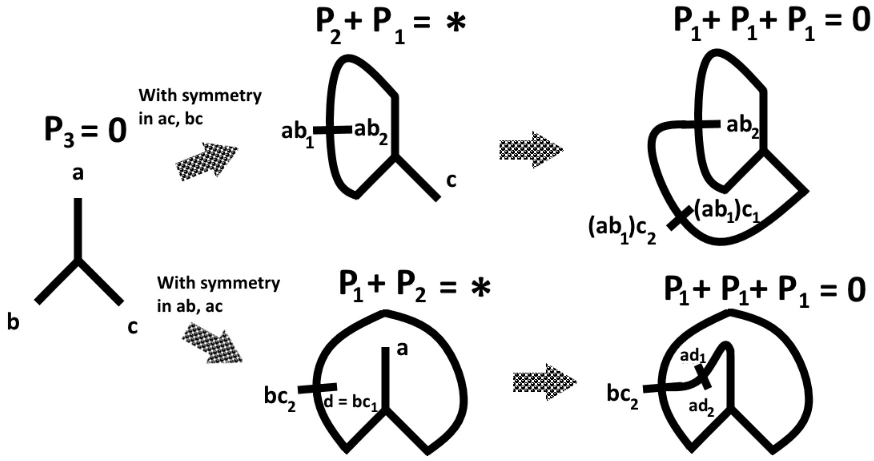
Figure 9: Morphing a position outside a loop to an equivalent position inside a loop



What about P_3? If we label to open ends a, b, and c, then from this position there are 6 distinct moves for the first player: to connect a and b with a line that loops around c, to connect a and b with a line that doesn't loop around c, to connect a and c (the line either looping or not around b), and to connect b and c (the line either looping or not around a.) But since there is nothing distinct about a, b or c, we can use symmetry to reduce this down to two moves, either a connection between two ends that doesn't loop around the last end, or a connection between two points that does loop around the last end.

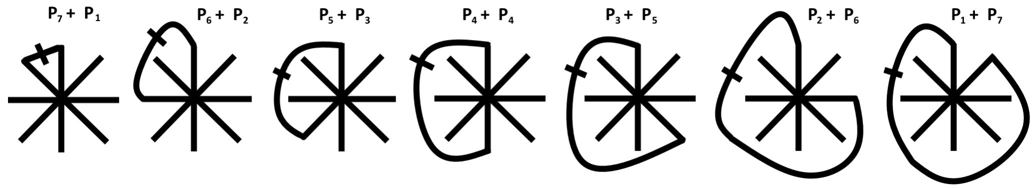
In the first case we move to a position with one open end inside a loop, and two ends outside any loop. But again the ends inside the loop will never be able to connect with those outside, and the single end inside the loop will never connect to another end, so we can consider this position as the disjoint sum of P_1 and P_2. Then there are no moves from P_1, but there is one move to a zero position from P_2, which means the Game $P_1 + P_2 = *$, that is the first player wins, as we'd expect, since $P_2 = *$. So we know that one option from P_3 has value $*$.

In the second case, we move to a position which has two open ends inside the loop, and one outside. Again we see they are disjoint, and there is only one legal move from this position, to join the two ends inside the loop, in which case there remain three disjoint open ends, which is a zero position. Alternatively we can just note that the two open ends inside the loop have value P_2. So we see that the other option of P_3 also has value $*$.



Then we can use the Sprague-Grundy Theorem to see that $P_3 = mex\{*\} = *mex(1) = 0$.

Let us now generalise, and consider the options of an arbitrary P_n. Any move from P_n must connect one of its prongs to another, which will be sure to make a loop, though this loop may contain some of the other prongs of P_n. If we connect a prong to its directly adjacent prong, either clockwise or anticlockwise, then we will make a loop that surrounds no other prong, and so when we draw the bisecting line through the loop, we end up in a position with one open end inside the loop, and $n - 1$ open ends outside the loop (we lose the two that are connected, but gain one from the bisecting line). If instead we make make the loop surround one of the prongs, then we get the position $P_2 + P_{n-2}$, since the bisecting line adds one open end both inside and outside the loop. Similarly, when we surround two prongs with the loop we move to the position $P_3 + P_{n-3}$, and in general, if we surround k prongs, where necessarily $k < n-2$ (considering the two prongs that are joined together), we move to the position $P_{k+1} + P_{n-k-1}$ (the loop contains k prongs plus one side of the bisecting line, and the outside of the loop looses k prongs, plus the two that where joined together, but gains one from the bisecting line). Below we have drawn (accounting for symmetry) each of the possible moves from P_8:



With this in mind, and what we already know about the addition of nimbers, we can prove the following lemma:

Lemma 0.42. *$P_n = *$ for all even n, and $P_n = 0$ for all odd n.*

Proof.

By induction, suppose that the theorem holds for all $n' < n$. Now every move from P_n must form a loop, and so must split the position into two disjoint Games, which have values P_{k+1} and P_{n-k-1} respectively, where $k < n-2$ is the number of prongs surrounded by the loop. Then every option of P_n is in the form $P_{k+1} + P_{n-k-1}$, and we have both $k + 1 < n$ and $n - k - 1 < n$, so we have four possibilities: (i) both n and k are even, so both $k + 1$ and $n - k - 1$ are odd; (ii) n is even and k is odd, so both $k + 1$ and $n - k - 1$ are even; (iii) n is odd and k is even, so $k + 1$ is odd and $n - k - 1$ is even; and (iv) both n and k are odd, so $k + 1$ is even and $n - k - 1$ is odd. Then for (i) we have $P_{k+1} + P_{n-k-1} = 0 + 0 = 0$, for (ii) we have $P_{k+1} + P_{n-k-1} = * + * = 0$, for (iii) we have $P_{k+1} + P_{n-k-1} = 0 + * = *$, and for (iv) we have $P_{k+1} + P_{n-k-1} = * + 0 = *$. In summary then, for all k, if n is even, $P_{k+1} + P_{n-k-1} = 0$, and if n is odd, $P_{k+1} + P_{n-k-1} = *$. Finally, we use the Sprague-Grundy Theorem to see that $P_n = mex\{P_{k+1} + P_{n-k-1}\}$, so $P_n = *$ if n is even, and $P_n = 0$ if n is odd. Then it just remains to show that $P_1 = 0$ and $P_2 = *$, which gives a basis for our induction, and we have already shown this above. □

Now we can move on to considering positions with more than one piece. In positions $P_{n,m}$, with only two pieces, there are only two kinds of moves: to connect the two pieces, so that we reach the one-piece position P_{n+m}, (which has $n + m$ prongs because we connect two ends into one piece, don't form a loop, and add two new ends to that piece), or to connect one piece to itself and leaving the other alone, so that we move either to $P_{n-k-1,m} + P_{k+1}$ or to $P_{n,m-j-1} + P_{j+1}$, for $k < n - 2$, $j < m - 2$. Using this and the Sprague-Grundy Theorem we can prove the following interesting result, in a similar manner to how we proved the above lemma, however using two inductions, one nested within the other. The idea with nested induction is that we assume some property holds for all $P_{n,m'}$ where $m' < m \geqslant n$, and then to show that the same property holds for all $P_{n,m}$ we prove it follows if that property holds for all $P_{n',m}$, where $n' < n$. So we first conside the case when $m = 1$, and the only possible n is $n = 1$, then we increase m to $m = 2$, and show that if the property holds for $n = 1$ then it must hold for $n = 2$, then increase m to $m = 3$, and then show that if the property holds for $n = 1$ it must hold for $n = 2$, and then $n = 3$, &c.

Lemma 0.43. *The value of two pieces played jointly is equal to the sum of their values when played disjointly. That is,* $P_{n,m} = P_n + P_m$

Proof.

First we show by nested induction that $P_{n,m} = *$ if only one of m, n is even, and 0 otherwise. Without loss of generality let $n \leqslant m$. Then by a first induction, for $m' < m$, suppose that $P_{n,m'} = *$ only if only one of n, m' is even, and equals 0 otherwise. We want to show then that this property also holds for n and m. So by a second induction, for fixed m and all $n' < n$, suppose $P_{n',m} = *$ if only one of n', m is even, and equals 0 otherwise. Now we know that the options of $P_{n,m}$ are (i) P_{n+m}, (ii) $P_{n-k-1,m} + P_{k+1}$ for $k \leqslant n - 2$, and (iii) $P_{n,m-j-1} + P_{j+1}$ for $j \leqslant m - 2$.

For (i), since the sum $n+m$ is odd only if only one of n, m is even, using Lemma 0.42 we see that $P_{n+m} = 0$ if only one of n, m is even, and equals $*$ otherwise.

For (ii), first note that $n - k - 1 < n$, so we can use our induction assumption that $P_{n-k-1,m} = *$ only if only one of $n - k - 1, m$ is even, and equals 0 otherwise. Now consider the case when m is even. Then, in the same way as we did in Lemma 0.42, if we consider the combinations of n and k being even or odd, we get the following results: if both are even, then $k + 1$ is odd and $n - k - 1$ is odd, so $P_{n-k-1,m} + P_{k+1} = * + 0 = *$; if both are odd, then $k + 1$ is even and $n - k - 1$ is odd, so $P_{n-k-1,m} + P_{k+1} = * + * = 0$; if n is even and k is odd, then $k + 1$ is even and $n - k - 1$ is even, so $P_{n-k-1,m} + P_{k+1} = 0 + * = *$; finally when n is odd and k is even, then $k + 1$ is odd and $n - k - 1$ is even, so $P_{n-k-1,m} + P_{k+1} = 0 + 0 = 0$. In summary we find that if m is even, $P_{n-k-1,m} + P_{k+1} = 0$ only if n is odd, and equals $*$ otherwise. If we instead consider the case when m is odd, then we similarly find that $P_{n-k-1,m} + P_{k+1} = 0$ only if n is even, and equals $*$ otherwise. Therefore $P_{n-k-1,m} + P_{k+1} = 0$ only if only one of n, m is even, and equals $*$ otherwise.

For (iii), note that $m - j - 1 < m$, so we can use our inductive assumption that $P_{n,m-j-1} = *$ only if only one of $n, m - j - 1$ is even and equals 0 otherwise. Then, by the same method as we did in (ii), we find that $P_{n,m-j-1} + P_{j+1} = 0$ only if one of n, m is even, and equals $*$ otherwise.

Then since each option of $P_{n,m}$ equals 0 only if only one of n, m is even, and equals $*$ otherwise, using the Sprague-Grundy Theorem we find that $P_{n,m} = *$ only if one of n, m is even, and equals 0 otherwise. Then for our second induction it only remains to show the base cases, that is the claim holds for $P_{1,m}$ and $P_{2,m}$, but this is easy (if laborious) to show by applying Lemma 0.42 and the Sprague-Grundy Theorem. We must show the base cases for our first induction, that is that the claim holds for $P_{n,1}$ for $n = 1$ and $P_{n,2}$ for $n \leqslant 2$, but again these are easily shown by applying Lemma 0.42 and the Sprague-Grundy Theorem. This completes our inductive argument, and we have shown that $P_{n,m} = *$ if only one of m, n is even, and equals 0 otherwise.

Finally using Lemma 0.42 we note that $P_n + P_m = *$ only if only one of n, m is even, and equals 0 otherwise, and therefore $P_{n,m} = P_n + P_m$, completing the proof. □

So we see that with two pieces, the outcome of the Game is independent of whether we play the pieces together on one piece of paper or separately on two. This of course begs the question as to whether this is true for an arbitrary amount of pieces, the answer to which is yes, as we prove:

Theorem 0.44. *The value of j pieces played jointly is equal to the sum of their values when played disjointly. That is, $P_{n_1,n_2,...,n_j} = P_{n_1} + P_{n_2} + ... + P_{n_j}$.*

Proof.

We do another nested induction, and show that $P_{n_1,n_2,...,n_j} = *$ only if there is an odd number of even $n_1, n_2, ..., n_j$, and equals 0 otherwise. For the first induction, assume that for $j-1$ pieces $P_{n_1,n_2,...,n_{j-1}} = *$ only if there are an odd number of even $n_1, n_2, ..., n_{j-1}$, and equals 0 otherwise. We want to show this property also holds for any position with j pieces. Then for the second induction, assume that for a position with j pieces $P_{n'_1,n'_2,...,n'_j}$, which have a total sum of prongs $n'_1 + n'_2 + ... + n'_j < s$ for some integer s, then $P_{n'_1,n'_2,...,n'_j} = *$ only if there are an odd number of even $n'_1, n'_2, ..., n'_{j-1}$, and equals 0 otherwise.

Now from the position $P_{n_1,n_2,...,n_j}$ with j pieces and total sum of prongs s, there are two kinds of options. The first, for pieces with more than one prong, is to connect that piece to itself. The second is to connect two distinct pieces, and therefore reduce the position to one of $j - 1$ prongs.

In the first case, we must have some piece P_{n_l} with at least two prongs, and we change that one piece to the disjoint sum of two pieces. So if choose to surround k prongs of P_{n_l}, we move from the position $P_{n_1,n_2,...,n_l,...,n_j}$ to the position

$P_{k+1} + P_{n_1,n_2,...,n_l-k-1,...,n_j}$. Now suppose $P_{n_1,n_2,...,n_l,...,n_j}$ has an even amount of even $n_1, n_2, ..., n_l, ..., n_j$. Then there are four possible combinations of n_l and k being even or odd: (i) n_l is even and k is even; (ii) n_l is even and k is odd; (iii) n_l is odd and k is even; and (iv) n_l is odd and k is odd. Then we have: for (i) $n_l - k - 1$ is odd and $k+1$ is odd; for (ii) $n_l - k - 1$ is even and $k+1$ is even; for (iii) $n_l - k - 1$ is even and $k+1$ is odd; and finally for (iv) $n_l - k - 1$ is odd and $k+1$ is even. Then using our second induction assumption and Lemma 0.42, we have: (i) $P_{k+1} = 0$ and $P_{n_1,n_2,...,n_l-k-1,...,n_j} = *$; (ii) $P_{k+1} = *$ and $P_{n_1,n_2,...,n_l-k-1,...,n_j} = 0$; (iii) $P_{k+1} = 0$ and $P_{n_1,n_2,...,n_l-k-1,...,n_j} = *$; and lastly (iv) $P_{k+1} = *$ and $P_{n_1,n_2,...,n_l-k-1,...,n_j} = 0$. So we see that if $P_{n_1,n_2,...,n_l,...,n_j}$ has an even amount of even $n_1, n_2, ..., n_l, ..., n_j$, then connecting one piece to itself always leads to a position with value $*$. A similar argument shows that if $P_{n_1,n_2,...,n_l,...,n_j}$ has an odd amount of even $n_1, n_2, ..., n_l, ..., n_j$, then all its options have instead value 0.

In the second case, in connecting two distinct pieces, we move from $P_{n_1,n_2,...,n_j}$ to a position of $j-1$ pieces, and without loss of generality let us write $P_{n_1+n_2,...,n_j}$ for this position. Now again suppose $P_{n_1,n_2,...,n_j}$ has an even amount of even $n_1, n_2, ..., n_j$. Then there are again four possible combinations of n_1 and n_2 being even or odd: (i) n_1 is even and n_2 is even, and so n_1+n_2 is even; (ii) n_1 is even and n_2 is odd, and so $n_1 + n_2$ is odd; (iii) n_1 is odd and n_2 is even, and so $n_1 + n_2$ is odd; and (iv) n_1 is odd and n_2 is odd, and so $n_1 + n_2$ is even. But in all of these cases $P_{n_1+n_2,...,n_j}$ has an odd amount of even $n_1 + n_2, ..., n_j$, so using our first induction assumption we see that if $P_{n_1,n_2,...,n_j}$ has an even amount of even $n_1, n_2, ..., n_j$, then every option of $P_{n_1,n_2,...,n_j}$ has value $*$. A similar argument shows that if $P_{n_1,n_2,...,n_j}$ has an odd amount of even $n_1, n_2, ..., n_j$, then every option of $P_{n_1,n_2,...,n_j}$ has instead value 0.

Then using the Sprague-Grundy Theorem we see that $P_{n_1,n_2,...,n_j} = *$ only if there are an odd number of even $n_1, n_2, ..., n_{j-1}$, and equals 0 otherwise. In remains to show the inductive base cases. For our first induction, by Lemma 0.42, if $j = 1$, then $P_{n_j} = *$ only if n_j is even, and equals 0 otherwise. For our second induction, for any number of pieces j the minimum amount of prongs is j, with each piece just having one prong. But then the only move is to connect two distinct pieces, and therefore move to a position of $j - 1$ pieces, where one piece has 2 prongs and $j - 1$ have 1 prong. Then by our first induction this position has value $*$, and so by the Sprague-Grundy Theorem our base case for the second induction has value 0 as expected, since it has 0 pieces with an even amount of prongs.

Finally we note that from Lemma 0.42, $P_{n_1} + P_{n_2} + ... + P_{n_j} = *$ only if there is an odd amount of even $n_1, n_2, ..., n_j$, and equals 0 otherwise, so therefore $P_{n_1,n_2,...,n_j} = P_{n_1} + P_{n_2} + ... + P_{n_j}$, completing the proof.

□

In these proofs we showed that $P_{n_1,n_2,...,n_j} = *$ only if there is an odd number of even $n_1, n_2, ..., n_j$, and equals 0 otherwise. Specifically, if we have $n_1 = n_2 = ... = n_j = 4$, then this is a Game of Brussels Sprouts, and the first player wins only if j is odd, which matches the result attained by analysing Brussels Sprouts in the usual way, using graph theory [9, p. 47]. These results then give a complete theory of Cloves, and since every move from $*$ is to a zero position, the player who begins in a position with an odd number of even $n_1, n_2, ..., n_j$ will always win, regardless of what moves are played by either player. Therefore Cloves is completely deterministic, and isn't much of a game at all.

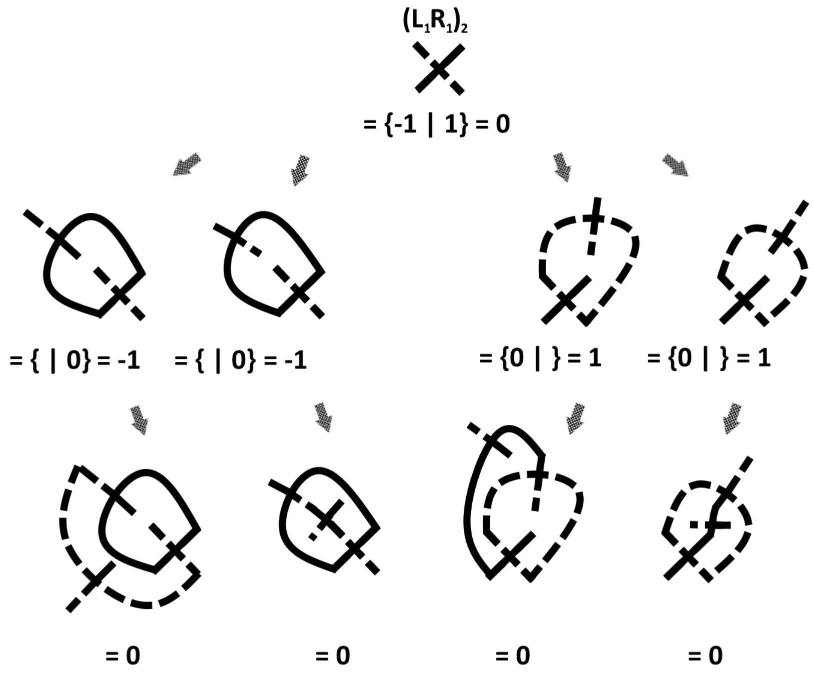
The Partizan Game of Borages

To end this paper we will study a Partizan generalisation of Cloves, which as far as the author is aware has not yet been named, so we shall call it Borages, after the flower.

Borages is played just like Cloves, except that each prong is either filled or dotted, with the left player being only allowed connect filled prongs and the right player being only allowed to connect dotted prongs. When a connecting line is drawn, the player who drew it picks one side of the line to put a new filled line, and one side to put a new dotted line. For example, the position $(L_1R_1)_2$, which is a single piece with 4 prongs that alternate being filled or dotted has the move tree shown in Figure 10 (where a left arrow indicates a legal move for Left (filled), and a right arrow one for Right (dotted), with symmetries omitted). We notate pieces as a list of Left prongs and Right prongs in their anticlockwise order, and conventionally start on a Left prong if there is one. We also condense repeated sequences by writing the sequence once with the number of repeats in a subscript, so for example we shorten $LLLRLRRRR$ to $L_3R_1L_1R_4$, and we call each set of successive prongs a $group$ of prongs (not to be confused with the algebraic structure of groups), for example L_5R_1 is a group of five filled prongs followed by a group of one dotted prong. Finally, we call prongs that can be connected to each other (because there are no lines in the way) $connectable$ prongs.

As ever in trying to analyse Games, we start by considering the simplest positions. What is the value of a general L_n? Any move in L_n will have to join and therefore remove two left prongs, and then add one new Left and one new Right prong. So if Left can manage to never let Right have a move, they should be able to keep reducing the number of left prongs until there is only one left, which will be a zero position. Is there a way to do this? Yes, if Left makes sure to not split up any of their prongs with the connecting move, then the position will be split by the loop created into two disjoint positions, one which contains $n-2$ filled prongs and one which is empty. Then Left can choose to put the new dotted prong in the empty position, and the filled prong in

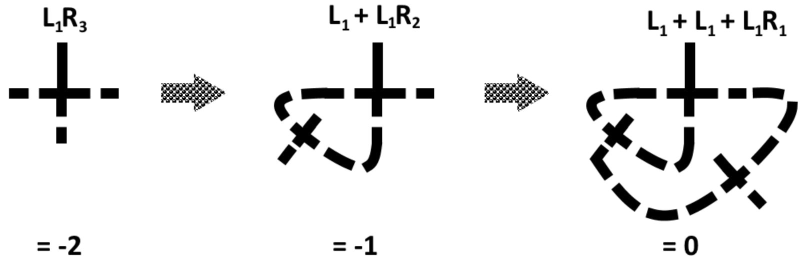
Figure 10: The move tree for the Borages position $(L_1R_1)_2$



the other position, so that they move from L_n to $L_{n-1} + R_1 = L_{n-1}$. Then they can keep repeating this until they have only one prong left, which will be after $n-1$ moves, so the position has value $n-1$. By symmetry we can also see that any R_n has value $-n+1$.

The next positions to consider are those in the form L_nR_1 or L_1R_n, since only Left can move in the former, and only Right can move in the latter. But from L_nR_1, Left can just connect two adjacent prongs, and place the new filled line on the outside of the created loop, and the dotted line on the inside. In this way they move to the position $L_{n-1}R_1 + R_1 = L_{n-1}R_1$. This must be the best move for Left as it moves to another position where Right has no moves, and reduces the amount of connectable filled prongs by 1, which is the minimum (we always lose the two that are connected, and add a new one in). Then Left can repeat this tactic $n-1$ times until they reach the position L_1R_1, where neither player has a move. So again $L_nR_1 = n-1$ and similarly $L_1R_n = -n+1$. The tree of optimal moves (omitting symmetries) for the position L_1R_3 is shown below.

Now we can consider any general L_nR_m, where $n, m > 0$. The key thing to notice is that from this position, since there is no way for either player to split up their opponent's prongs, the best move for either player is to connect any two adjacent prongs, and place the new prongs such that their opponent's is inside the created loop, and theirs is outside. This guarantees that the number of connectable prongs for the opponent remains the same, and the number for that player only reduces by one. So we can



write $L_nR_m = \{L_{n-1}R_m + R_1 | L_nR_{m-1} + L_1\} = \{L_{n-1}R_m | L_nR_{m-1}\}$. It follows that from the position L_nR_m, if both players play optimally, Left will play $n-1$ moves and Right will play $m-1$ moves, so we have $L_nR_m = n-1-(m-1) = n-m$, which matches up our values of L_nR_1 and L_1R_n, and is a zero Game when $n = m$, as we'd expect from a symmetrical position where neither player wants to play, since they can only reduce the number of their connectable prongs, and not the opponent's.

After this point, it becomes increasingly clunky to use diagrams to analyse positions, since there is no limit to the amount of starting prongs we may have. Luckily, we can work out some rules to directly manipulate our notation to express possible moves from positions. In general, we can write positions with a single piece in the form $L_{n_1}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t}$, and we must have n_s and m_t be even numbers (otherwise, for example, $L_{n_{s-1}}$ and L_{m_1} would join to form $L_{n_{s-1}+m_1}$). Then there are two kinds of possible moves for either player (we will just consider those for Left here, but those for Right are symmetrically the same). To connect two prongs that have none of the opponent's prongs in between them (without loss of generality, two prongs in L_{n_1}, if there exists some $n_1 \geqslant 2$), which is possible as long as there is a Left group with more than one prong, or to connect two prongs that do have some of the opponent in between (again without loss of generalisation, a prong in L_{n_1} and one in L_{m_1}), which is always possible as long as there is more than one Left group.

In the first kind of move, we can choose $0 \leqslant k \leqslant n-2$ prongs to surround, and move to the position $L_{k\_} + L_{n_1-k-2}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t\_}$, where one of the underscores is R_1 and the other L_1. In the second type of move, we can choose any prong from L_{n_1} and any from L_{m_1} to be the connecting prongs, and this will split the n_1 and m_1 prongs into two groups each, one of which will be in the surrounded loop, and the other which will not. So, if we choose to have $n_1' < n_1 - 1$ prongs from L_{n_1} and $m_1' < m_1 - 1$ prongs from L_{m_1} to be inside the loop, we should first rewrite our original position as $L_{n_1-n_1'}L_{n_1'}R_{n_2}L_{n_3}...R_{n_s}L_{m_1'}L_{m_1-m_1'}R_{m_2}...R_{m_t}$. Then we connect the one prong from $L_{n_1-n_1'}$ to the one in $L_{m_1-m_1'}$, and move to the position $L_{n_1'}R_{n_2}L_{n_3}...R_{n_s}L_{m_1'\_} + L_{n_1-n_1'-1}L_{m_1-m_1'-1}R_{m_2}...R_{m_t\_}$, which we can then simplify to $L_{n_1'}R_{n_2}L_{n_3}...R_{n_s}L_{m_1'\_} + L_{(n_1+m_1)-(n_1'+m_1')-2}R_{m_2}...R_{m_t\_}$.

With this machinery we can now prove the position $P \equiv L_{n_1}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t}$ has value $V = n_1+n_3+...+n_{s-1}+m_1+m_3+...+m_{t-1}-(n_2+n_4+...+n_s+m_2+m_4+...+m_t)$. As with Cloves we will use a nested induction in our proof, the first over the number of groups of prongs, that is the number $s+t$ (which we know must be even), and the second, for each fixed $s+t$, over the number of prongs, that is $n_1+...+n_s+m_1+...+m_t$. We find all the options from our initial position, and show that the Game P is a number, then use the properties of surreals we proved in earlier sections to show that $P = V$.

Theorem 0.45. *The Game* $P := L_{n_1}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t}$*, where there is at least one* L *term and one* R *term, has value* $V = n_1+n_3+...+n_{s-1}+m_1+m_3+...+m_{t-1}-(n_2+n_4+...+n_s+m_2+m_4+...+m_t)$*, that is the number of Left prongs minus the number of Right prongs.*

Proof.

By a first induction assume that the theorem holds for all n'_s and m'_t where $n'_s+m'_t < n_s+m_t$. Then we consider positions with n_s+m_t pieces, and do another induction on the total number of prongs $\sigma = n_1+...+n_s+m_1+...+m_t$, and assume the theorem holds for all $\sigma' < \sigma$. Now we consider the options for a position with σ prongs.

First we consider Left's options. We have already shown that there are two kinds of moves: to a position in the form (a) $L_{k\_\_} + L_{n_1-k-2}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t\_\_}$, where one of the underscores is R_1 and the other L_1, for some $0 \leqslant k \leqslant n-2$, or to a position of the form (b) $L_{n'_1}R_{n_2}L_{n_3}...R_{n_s}L_{m'_1\_\_} + L_{(n_1+m_1)-(n'_1+m'_1)-2}R_{m_2}...R_{m_t\_\_}$, for some $n'_1 \leqslant n_1-1$ and $m'_1 \leqslant m_1-1$.

In the case of (a), if the L_1 term is in the left summand we get the position $L_{k+1} + L_{n_1-k-2}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t+1}$. At this point there are two cases, when $k=0$ and when $k>0$. For the first case, we have the two pieces L_1 and $L_{n_1-2}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t+1}$. Using our second induction assumption, since this second piece has n_s+m_t groups has but one fewer prong, we know it has value $n_1-2+n_3+...+n_{s-1}+m_1+m_3+...+m_{t-1}-(n_2+n_4+...+n_s+m_2+m_4+...+m_t+1) = V-3$. Then since $L_1 = 0$ the two pieces sum together to the value $V-3$. In the second case we know that $L_{k+1} = k$ and by our second inductive assumption, since the second summand has n_s+m_t groups but fewer prongs, we know its value to be $n_1-k-2+n_3+...+n_{s-1}+m_1+m_3+...+m_{t-1}-(n_2+n_4+...n_s+m_2+m_4+...+m_t+1)$. Adding this to k we get the value $n_1+n_3+...+n_{s-1}+m_1+m_3+...+m_{t-1}-(n_2+n_4+...+n_s+m_2+m_4+...+m_t)-3 = V-3$ again.

If instead the L_1 goes in the right summand we get $L_kR_1 + L_{n_1-k-1}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t}$. Again there are two cases, when $k=0$

and when $k > 0$. If $k = 0$, we get the two pieces $L_0R_1 = R_1 = 0$ and $L_{n_1-1}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t} = V-1$, which sums to $V-1$. Note that this move is always possible for Left, unless all their groups have only one prong. If instead $k > 0$, we get $L_kR_1 = k-1$ and $L_{n_1-k-1}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t} = V-k-1$, which sums to $V-2$.

In the case of (b), we have the sum of two positions with less than $n_s + m_t$ groups, so we'll use our first inductive assumption. If we put L_1 in the left summand we get a position with two pieces of value $n'_1 + n_3 + ... + n_{s-1} + m'_1 + 1 - (n_2 + n_4 + ... + n_s)$ and $n_1 + m_1 - (n'_1 + m'_1) - 2 + m_3 + ... + m_{t-1} - (m_2 + m_4 + ... + m_t + 1)$, which summed together give the option the value of $V-2$. Similarly if we instead put the L_1 term in the right summand we get again that the option has the value $V-2$.

None of the above reasoning depended on anything unique to the Left player. As such, by symmetry we can see straight away that Right's options will be to $V+1$ (if at least one of their groups has more than one prong), $V+2$ and $V+3$. Then there are three cases: when both player have at least one group with more than one prong, when only one player has only groups with one prong, and when both players have only groups with one prong. The third case is the basis for our second induction, and is considered below. In the first case, we have $P = \{V-3, V-2, V-1|V+1, V+2, V+3\}$. We know that V is an integer, so we must have $V \pm 1$, $V \pm 2$, $V \pm 3$ being all integers, and therefore P is a number. So by the Truncation Theorem we can just write $P = \{V-1|V+1\}$. Then by the Simplicity Theorem, since $V-1 < V < V+1$ and no option of V satisfies this (because if V is positive, $V = \{V-1|\}$ in its natural form, and similarly we have $V = \{|V+1\}$ when V is negative), we have that $P = V$. In the second case, we have that if it is Right who only has groups with one prong, then V must be positive (because n_s and m_t are always even), and so $P = \{V-3, V-2, V-1|V-2\}$ still equals V by the Simplicity Theorem. Similarly, if it is Left who has groups of only one prong, then V must be negative, so the Simplicity Theorem still gives $P = \{V-3, V-2|V+1, V+2, V+3\} = V$. So in both cases we have $P = V$ as required.

It remains to show the basis cases. For our first induction, the basis case is with two groups, that is positions of the form $L_{n_1}R_{m_1}$, which we have already shown is equal to $n_1 - m_1$ as we require. For our second induction, the smallest σ for a fixed n_s and m_t is $\sigma = n_s + m_t$, where each of $n_1, ..., n_s, m_1, ..., m_t$ is equal to 1. Then from the position $L_{n_1}R_{n_2}L_{n_3}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t}$, Left can't connect any group to itself, so must connect L_{n_1} to L_{m_1} and move to $R_{n_2}L_{n_3}...R_{n_s}\_ + R_{m_2}L_{m_3}...R_{m_t}\_$. Then regardless of which summand we put the L_1 term in, both of the summands have less than $n_s + m_t$ groups so by our first induction Left's option has value

$n_3+...+n_{s-1}+m_3+...+m_{t-1}+1-(n_2+...n_s+m_2+...+m_t+1) = n_3+...+n_{s-1}+m_3+...+m_{t-1}-(n_2+...n_s+m_2+...+m_t)$. Then since every $n_2, ..., n_s, m_2, ..., m_t$ equals 1, the value of this option is $s/2-1+t/2-1-(s/2+t/2) = -2$. Similarly, by symmetry Right's option has value 2, so the position for our second basis has value $\{-2|2\} = 0$ by the Simplicity Theorem, which is equal to $n_1+...+n_{s-1}+m_1+...+m_{t-1}-(n_2+...+n_s+m_2+...+m_t) = (n_1-n_2)+...+(n_{s-1}-n_s)+(m_1-m_2)+...+(m_{t-1}-m_t)$ as required since n_s and m_t are always even, finishing the proof. □

From the above proof we see two interesting properties. The first is that it is only the number of prongs for each player that affects the value of the Game, and not the configuration they are in. Because of this we will write $V_1 + V_2$ to express the disjoint sum of two pieces of value V_1 and V_2. The second is that, unlike in Cloves, it is possible to make a non-optimal move. The best strategy is to always connect two adjacent prongs if possible, and put the opponent's prong in the loop, which nullifies it since it will be never be able to connect to another prong, and the worst move is to instead put one's own prong into that loop, as again it will be nullified. These moves go to the positions $V \pm 1$ and $V \pm 3$ respectively. However, all the moves apart from these two have the same value, moving to $V \pm 2$.

For two piece positions we write $L_{n_1}R_{n_2}...R_{n_s}L_{m_1}R_{m_2}...R_{m_t} \oplus L_{x_1}R_{x_2}...R_{x_u}L_{y_1}R_{y_2}...R_{y_v}$ where $\oplus$ indicates the joint sum, but for brevity we'll also write $P_a \oplus P_b$, and we say that disjointly the first piece has value V_a and the second V_b. Then there are two kinds of moves: to connect the two pieces together, or ignore one piece, and just play a move that connects two prongs of the same piece (which may or may not surround the other piece, but just as in Figure 9, for each move that surrounds the other piece there is an equivalent move that doesn't, and vice versa). Since we know that connecting two adjacent prongs in one piece then putting the opponent's new prong inside the loop is always better than connecting two non-adjacent prongs, we will only consider this move when a player decides to play in a single piece.

Then consider the sum $P_a \oplus P_b$ where $V_a + V_b = 0$, and were each piece has at least one Left and one Right prong. Then the first player will want to avoid connecting the two pieces, because they will have to remove two of their prongs from the board, and then add a new prong for them and their opponent. But since there is no way to create a loop in connecting the two pieces, by Theorem 0.45, they will just move to a position of value $V_a + V_b - 2 = -2$ if it is Left who begins or $V_a + V_b + 2 = 2$ if it is Right who begins, which will be a winning position for the second player. The other possible option from $P_a \oplus P_b$ for the first player is to ignore one of the pieces and just play a move in the other piece. But since $V_a + V_b = 0$, that is the sum of the two pieces played disjointly is zero, if the first player can find a move in just one piece, then so can the second, because whatever number of moves advantage one player has in P_a the other player must have in P_b. Therefore, eventually, since our Games must be finite, if neither player chooses to connect the two pieces, we will end up in the position $L_1R_1 \oplus L_1R_1$, with the first player to move. Then their only move is to connect the two pieces, and move to $L_1R_3 = -2$ if it is Left who begins or $L_3R_1 = 2$ if it is Right who begins, both of which are again winning for the second player. So we have for positions with $V_a + V_b = 0$, that $P_a \oplus P_b = 0$, that is the value of the joint sum is the same as the value of the disjoint sum.

What about positions with $V_a + V_b = 1$? Played disjointly, the position is winning by one move for Left, so if they begin, they can just play a move in one of the pieces, and move to a position with $V_a + V_b = 0$, which they will then win since they will be the second player. If instead Right begins, they can either connect the two pieces and move to $V_a+V_b+2 = 3$, or play in one piece, and move to a position with $V_a+V_b = 2$, to which Left can reply, again playing in just one piece, and move back to a position with $V_a+V_b = 1$. Eventually, then, we will get to the position $L_2R_1 \oplus L_1R_1$, where it is Right's turn move. They will have to move to $L_4R_1 = 3$. So the Game has value $\{0\,|3\} = 1$. Similarly, if $V_a + V_b = -1$, then the Game will have value -1. So again we have that the disjoint sum is equal to the joint sum.

In general, by the same arguments as above, positions where $V_a + V_b = n$ for $n > 0$ when played disjointly will give Left n moves advantage, and as we have seen, this doesn't change if we allow the players to connect up the two pieces, that is play them jointly, since it is never favourable for either play to play the connecting move. Similarly, when $n < 0$, Right will have n moves advantage whether the game is played disjointly or jointly. So we have for positions with two pieces that their value is the same as the value of the two piece individually, that is $P_a \oplus P_b = V_a + V_b$. We will now use these arguments to prove that this is true for positions with any number of pieces, and use the case of two pieces as an induction basis.

Theorem 0.46. *The value of $m \geqslant 2$ pieces, each with at least one Left and one Right prong, played jointly is equal to the sum of their individual values.*

Proof.

We do another nested induction. Our first induction is on the number of pieces. Suppose that the theorem holds for positions of m' pieces, where $m' < m$. Then for positions with m pieces, where the difference in prongs, $d = V$, is the number of Left's prongs minus the number of Right's (which by Theorem 0.45 is exactly sum of the values of all the pieces played individually), we do another induction on d, and assume the theorem holds for all $d' < d$ (again we just show the case when $d > 0$, and then the case for $d < 0$ is symmetrical, since we just swap the Left and Right prongs). Then from a position with m pieces and $d = V$, one of Left's options will be to connect two of those pieces, and move to a position of $m-1$ pieces, with $d = V - 2$. Similarly one of Right's options is to move to a position of $m-1$ pieces with $d = V + 2$. If they do not play these moves, then the other kind of move is to a position with m pieces, but with either $d = V - 1$ or $d = V + 1$, for Left's or Right's move respectively.

Now consider the case when $d = 0$. The first player will still want to avoid connecting any two pieces, but if they play in single piece, since $d = 0$, the second player will always be able to respond in a single piece, and move to a new position with $d = 0$ but with fewer total prongs. Eventually then, the first player will have to play a connecting move, and reach a position with $m-1$ pieces and $d = -2$ or $d = 2$, for Left or Right beginning respectively, and so the Game is a zero Game, since the first player will always lose.

Next consider the case when $d = V > 0$. If Right begins, they still don't want to connect any two pieces, as this would lead to a position with $m-1$ pieces and $d = V + 2 > 0$, and so by our first induction assumption of value $V + 2$. But again for any single-piece move Left can find a single piece response, that will keep d constant, so eventually Right will have to connect, and still move to a position of value $V + 2$. If instead Left begins, They will move to position with m pieces and $d = V - 1 \geqslant 0$. If $V - 1 = 0$, that is $d = V = 1$, then this position is a zero position, so our original Game has value $\{0|2\} = 1$. If $V - 1 > 0$, then we use our second inductive assumption, and have that Left moves to a position of value $V - 1$, so again our original Game has value $\{V - 1|V + 2\} = V$ by the Simplicity Theorem, since $V > 0$.

Lastly we must show our inductive base cases. But we have already shown the basis for the second induction, which is the case above when $d = 0$, and we have also already considered the case when we have only two pieces, which is the basis for our first induction.

□

Finally we must consider positions which may have pieces with only Left or only Right prongs. By themselves, if they have n prongs, they have value $n - 1$ or $-n + 1$ for Left and Right respectively, because there is no way to use the last remaining prong, that is $L_1 = R_1 = 0$. However, in positions with other pieces, this final prong can be used, and connected to one of those pieces, and this will keep the difference in prongs as one less than before the move was played. So each piece with n Left prongs in a position just gives Left n more moves than in that same position if we removed the piece, and similarly for Right.

So to evaluate the value of a Borages position which includes single-player pieces, we still just calculate the difference in prongs, but in this case the value will not be the same as the sum of each of the pieces played separately.

This then is a complete theory of Borages.

Conclusion

This paper explores a few disparate, yet related topics. We first introduced the Surreal Numbers, and showed how they relate to and extend the real numbers: they form a totally ordered field that contains the real numbers, as well as the standard von Neumann ordinals (ω, $\omega + 1$, &c.), and also some newly created numbers, such as the infinitesimal $\epsilon = \frac{1}{\omega}$, and what we called the surreal-ordinals, such as $\omega - 1$. We then compared the construction of the surreals against the standard construction of numbers in ZFC, and noted some of the interesting symmetries present in the surreal construction.

Next, in the second part of the paper, we introduced the concept of Games, of which the surreals are just a subclass. We explored the way the concept of Games relate to two player perfect information combinatorial games, such as Nim, and discussed methods of anaylsing such games using the concept of Games. Finally we proved complete theories of a few games, including the original game of Borages.

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Appendices are available as supplementary files (please see download area)