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The Surreal Numbers and Combinatorial Games

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Appendix: Definitions

Axiom 1: For any two sets of numbers $L$ and $R$,

\[ \exists \text{ the number } \{L|R\} \iff \exists x \in L : x \geq y, \forall y \in R. \]

Axiom 2: For any two numbers $x = \{X^L|X^R\}$ and $y = \{Y^L|Y^R\}$,

\[ x \leq y \iff \exists x^L \in X^L : x^L \geq y \land \exists y^R \in Y^R : y^R \leq x. \]

\[ x = y \iff x \leq y \land y \leq x. \]

\[ x < y \iff x \leq y \land y \not< x. \]

\[ x \equiv y : \text{ The two numbers } x \text{ and } y \text{ are identical if and only if } X^L = Y^L \text{ and } X^R = Y^R. \]

Birthday: The birthday of a number is day on which it is first constructed.

Simplicity: We say that a number $x$ is simpler than another number $y$ if $x$ has an earlier birthday than $y$.

Natural Form: A number $x$ is in its natural form if it has at most one left option, and at most one right option, and all its options are strictly simpler than $x$.

Arithmetic on the Surreal Numbers

\[ x + y = \{X^L + y, x + Y^L|X^R + y, x + Y^R\} \]

\[ = \{x_1^L + y, x_2^L + y, ..., x + y_1^L, x + y_2^L, ..., |x_1^R + y, x_2^R + y, ..., x + y_1^R, x + y_2^R, ...\} \]

\[ -x = \{-X^R| -X^L\} \]

\[ = \{-x_1^R, -x_2^R, ... | -x_1^L, -x_2^L, ...\} \]

\[ xy = \{X^L y + x Y^L, X^R y + x Y^R|X^L y + x Y^R - X^R Y^R | X^L y + x Y^R - X^R Y^R, X^R y + x Y^L - X^R Y^L\} \]

where for sets $A$ and $B$, $Ax = \{ax : a \in A\}$, and $AB = \{ab : a \in A, b \in B\}$

\[ \frac{1}{x} = \left\{ \frac{0}{x^R} + \frac{1 + (x^R - x)\frac{1}{x}L}{x^L}, \frac{1 + (x^L - x)\frac{1}{x}R}{x^R}, \frac{1 + (x^L - x)\frac{1}{x}L}{x^L}, \frac{1 + (x^R - x)\frac{1}{x}R}{x^R} \right\} \]

where $(1/x)^L$ and $(1/x)^R$ are the already computed elements of $1/x$