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The Surreal Numbers and Combinatorial Games

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Appendix: Definitions

Axiom 1: For any two sets of numbers \( L \) and \( R \),
\[ \exists \text{ the number } \{L|R\} \iff \frac{1}{2} x \in L : x \geq y, \forall y \in R. \]

Axiom 2: For any two numbers \( x = \{X^L|X^R\} \) and \( y = \{Y^L|Y^R\} \),
\[ x \leq y \iff \frac{1}{2} x^L \in X^L : x^L \geq y \land \frac{1}{2} y^R \in Y^R : y^R \leq x. \]
\[ x = y \iff x \leq y \land y \leq x. \]
\[ x < y \iff x \leq y \land y \not\leq x. \]
\[ x \equiv y : \text{The two numbers } x \text{ and } y \text{ are identical if and only if } X^L = Y^L \text{ and } X^R = Y^R. \]

Birthday: The birthday of a number is the day on which it is first constructed.

Simplicity: We say that a number \( x \) is simpler than another number \( y \) if \( x \) has an earlier birthday than \( y \).

Natural Form: A number \( x \) is in its natural form if it has at most one left option, and at most one right option, and all its options are strictly simpler than \( x \).

Arithmetic on the Surreal Numbers

\[ x + y = \{X^L + y, x + Y^L|X^R + y, x + Y^R\} \]
\[ = \{x_1^L + y, x_1^L + y, \ldots, x + y^L, x + y^R, \ldots|x_1^R + y, x_2^R + y, \ldots, x + y^R, x + y^R, \ldots\} \]
\[ - x = \{-X^R| - X^L\} \]
\[ = \{-x_1^R, -x_2^R, \ldots, -x_1^L, -x_2^L, \ldots\} \]

\[ xy = \{X^L y + xY^L - X^LY^L, X^R y + xY^R - X^RY^R|X^L y + xY^R - X^LY^R, X^R y + xY^L - X^RY^L\} \]
where for sets \( A \) and \( B \), \( Ax = \{ax : a \in A\} \), and \( AB = \{ab : a \in A, b \in B\} \)

\[ \frac{1}{x} = \left\{ \frac{1}{y} x^L, \frac{1 + (x^R - x)\frac{1}{x}}{x^L}, \frac{1 + (x^R - x)\frac{1}{x}}{x^L} \right\} \]
where \((1/x)^L\) and \((1/x)^R\) are the already computed elements of \(1/x\)