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Analytic and Algebraic Aspects of Integrability for First Order Partial Differential Equations

by

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Abstract

This work is devoted to investigating the algebraic and analytic integrability of first order polynomial partial differential equations via an understanding of the well-developed area of local and global integrability of polynomial vector fields.

In the view of characteristics method, the search of first integrals of the first order partial differential equations

$$P(x,y,z)\frac{\partial z(x,y)}{\partial x} + Q(x,y,z)\frac{\partial z(x,y)}{\partial y} = R(x,y,z), \tag{1}$$

is equivalent to the search of first integrals of the system of the ordinary differential equations

$$\frac{dx}{dt} = P(x,y,z), \quad \frac{dy}{dt} = Q(x,y,z), \quad \frac{dz}{dt} = R(x,y,z). \tag{2}$$

The trajectories of (2) will be found by representing these trajectories as the intersection of level surfaces of first integrals of (1).

We would like to investigate the integrability of the partial differential equation (1) around a singularity. This is a case where understanding of ordinary differential equations will help understanding of partial differential equations. Clearly, first integrals of the partial differential equation (1), are first integrals of the ordinary differential equations (2). So, if (2) has two first integrals $\phi_1(x,y,z) = C_1$ and $\phi_2(x,y,z) = C_2$, where C_1 and C_2 are constants, then the general solution of (1) is $F(\phi_1,\phi_2) = 0$, where F is an arbitrary function of ϕ_1 and ϕ_2.

We choose for our investigation a system with quadratic nonlinearities and such that the axes planes are invariant for the characteristics: this gives three dimensional Lotka–Volterra systems

$$\begin{aligned}
\dot{x} =& \frac{dx}{dt} = P = x(\lambda + ax + by + cz),\\
\dot{y} =& \frac{dy}{dt} = Q = y(\mu + dx + ey + fz),\\
\dot{z} =& \frac{dz}{dt} = R = z(\nu + gx + hy + kz),
\end{aligned}$$

where $\lambda, \mu, \nu \neq 0$.

Several problems have been investigated in this work such as the study of local integrability and linearizability of three dimensional Lotka–Volterra equations with $(\lambda:\mu:\nu)$–resonance. More precisely, we give a complete set of necessary and sufficient conditions for both integrability and linearizability for three dimensional Lotka-Volterra systems for $(1:-1:1)$, $(2:-1:1)$ and $(1:-2:1)$–resonance. To prove their sufficiency, we mainly use the method of Darboux with the existence of inverse Jacobi multipliers, and the linearizability of a node in two variables with power-series arguments in the third variable. Also, more general three dimensional system have been investigated and necessary and sufficient conditions are obtained. In another approach, we also consider the applicability of an entirely different method which based on the monodromy method to prove the sufficiency of integrability of these systems.

These investigations, in fact, mean that we generalized the classical centre-focus problem in two dimensional vector fields to three dimensional vector fields. In three dimensions, the possible mechanisms underling integrability are more difficult and computationally much harder.

We also give a generalization of Singer's theorem about the existence of Liouvillian first integrals in codimension 1 foliations in $\mathbb{C}^n$ as well as to three dimensional vector fields.

Finally, we characterize the centres of the quasi-homogeneous planar polynomial differential systems of degree three. We show that at most one limit cycle can bifurcate from the periodic orbits of a centre of a cubic homogeneous polynomial system using the averaging theory of first order.

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Last, but by no means least, I would like to express my gratitude to my family in Kurdistan for surviving and my absence far from home. Also I am indebted to my wife, Gulla, for her patience, support and encouragement.

Dedication

This dissertation is lovingly dedicated to my flower of a wife, Gulla, for her support, encouragement and constant love which have sustained me throughout my life.

I also dedicate this work to all my family members especially my mother and father.

Last and not least, this work is dedicated to my little girl ’Mi amor’ Alia.

Author's declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award.

Relevant scientific seminars and conferences were regularly attended at which work was presented several times. One paper have been published in refereed journal. Four papers have been submitted for publication. Two papers are in the preparation process.

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Abbreviations

$\mathbb{N}$	The set of all natural numbers
$\mathbb{Z}$	The set of all integer numbers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$\mathbb{P}^1$	Riemann sphere
$\mathbb{CP}^k$	Complex projective variety
2D	Two dimensional system
3D	Three dimensional system
IJM	Inverse Jacobi multiplier

Chapter 1

Introduction

Often much more powerful methods are available for ordinary differential equations than for partial differential equations. So, we are particularly interested of the interaction between partial differential equations and ordinary differential equations. For this purpose, we consider a first order partial differential equation

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z), \tag{1.1}$$

where P, Q and R are known polynomials in x,y,z and $z(x,y)$ is unknown function in x and y.

Let $z=z(x,y)$ be a smooth solution of (1.1) and let

$$S=\{(x,y,z)\in\mathbb{R}^3 : z=z(x,y)\}$$

be a solution surface for (1.1).

It is well-known that the general solution of the partial differential equation (1.1) is $F(\phi_1,\phi_2)=0$, where F is an arbitrary function of $\phi_1(x,y,z)=C_1$, $\phi_2(x,y,z)=C_2$ and C_1,C_2 are constants.

The partial differential equation (1.1) can be solved by using the method of characterstics. The characterstics of (1.1) is the solution of the system of ordinary differential equations

$$\frac{dx}{dt}=P(x,y,z),\quad \frac{dy}{dt}=Q(x,y,z),\quad \frac{dz}{dt}=R(x,y,z), \tag{1.2}$$

because

$$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=\frac{dz}{dt}=R(x,y,z).$$

Thus, the investigation of solutions of partial differential equations can be reduced to investigation of solutions of ordinary differential equations.

Obviously, first integrals of the partial differential equation (1.1), are first integrals of the ordinary differential equations (1.1). Conversely, if $\phi_1 = C_1$ and $\phi_2 = C_2$ are first integrals of (1.1), then the intersection of their level surfaces, representing the trajectories of (1.2).

In this thesis, we are interested in the study of the integrability problem of the partial differential equation (1.1) around a singularity. We choose for our investigation a system with quadratic nonlinearities and such that the axes planes are invariant for the characteristics: this gives

$$P = x(\lambda + ax + by + cz),$$

$$Q = y(\mu + dx + ey + fz),$$

$$R = z(\nu + gx + hy + kz).$$

Thus, the characteristics of (1.1) are therefore the solutions of the following systems of ordinary differential equations

$$\begin{aligned}
\dot{x} =& \frac{dx}{dt} = P = x(\lambda + ax + by + cz),\\
\dot{y} =& \frac{dy}{dt} = Q = y(\mu + dx + ey + fz),\\
\dot{z} =& \frac{dz}{dt} = R = z(\nu + gx + hy + kz),
\end{aligned} \tag{1.3}$$

where $\lambda, \mu, \nu \neq 0$.

This system is known as three dimensional Lotka-Volterra systems. Therefore, our main goal of this work is to investigate the local integrability and linearizability of the three dimensional Lotka-Volterra systems at the origin.

The Lotka–Volterra equations, also known as predator-prey equations, are classical models that describe the evolution of conflicting species in population biology (May, 1974; May and Leonard, 1975). Since the works of Lotka (Lotka, 1920) and Volterra

(Volterra, 1931) and more recent works (Brenig, 1988; Brenig and Goriely, 1989) these systems are widely used with a diverse range of applications like neural networks (Noonburg, 1989), chemical kinetics (Murza and Teruel, 2010), laser physics (Lamb, 1964), plasma physics (Laval and Pellat, 1975), etc. The qualitative properties of these models have been extensively studied, see for instance (Bobienski and Żołądek, 2005; van den Driessche and Zeeman, 1998; Zeeman, 1993) among others. The integrability of some Lotka–Voltera families using Darboux's method was done by several authors, like (Cairó, 2000; Cairó and Llibre, 2000a; Christodoulides and Damianou, 2009; Llibre and Valls, 2011b).

We are interested in the case where the eigenvalues λ, μ and ν have two independent resonances and are in the Siegel domain, that is the convex hull of the eigenvalues with the origin located inside itself or on its boundary. Without loss of generality, this means that we can assume (after a possible scaling of time) that $\lambda, \mu, \nu \in \mathbb{Z}$ with $gcd(\lambda, \mu, \nu) = 1$, and that λ, $\nu > 0$ and $\mu < 0$. We say that the origin has $(\lambda : \mu : \nu)$–resonance in this case.

By the integrability of (1.3), we mean there exists an analytic change of coordinates that bringing system (1.3) to orbitally linearizable. The existence of the two independent analytic first integrals assures that the three quantities λ, μ and ν cannot have the same sign, and that furthermore, ν/μ and λ/μ must be rational.

Restricting (1.3) to $z = 0$ we can see the problem above as a generalization of the problem of classifying the integrability conditions for the system

$$\dot{x} = x(\lambda + ax + by), \quad \dot{y} = y(\mu + dx + ey), \quad \lambda, \mu \in \mathbb{Z}, \quad \lambda\mu < 0.$$

This problem has been considered by several authors: (Christopher and Rousseau, 2004; Liu et al., 2004). The authors in (Cairó and Llibre, 2000b) characterize all polynomial first integrals of two-dimensional Lotka-Volterra equations which have polynomial inverse integrating factors. Gravel and Thibault (2002) discussed the integrability and linearizability of a critical point at the origin of a saddle type for Lotka-Volterra equations with $(1 : -\lambda)$–resonance. They give a complete classification for the case $\lambda = 2/p$ or $\lambda = p/2$ for some $p \in \mathbb{Z}$. They also used the Blow-down method from a saddle to a node.

Recently Giné and Romanovski (2010) derived the necessary and sufficient conditions for a critical point at the origin to be integrable for planar quintic Lotka-Volterra equations with $(1:-1)$–resonance. They proved their sufficiency by finding first integrals or inverse integrating factors.

However, for more general quadratic systems many authors give a simple generalization of the Poincaré centre-focus problem. That is, to find conditions for which a local analytic first integral exists in a planar system. In this case where $n_1\lambda + n_2\mu = 0$, for some $n_1, n_2 \in \mathbb{N}$, we have the classical centre-focus problem in complex variables. For instance, Fronville et al. (1998) investigated the necessary and sufficient conditions for integrability of quadratic planar polynomial differential systems with $(1:-2)$–resonance. They used Blow-up method of a resonant saddle and they reduced it to a different resonance which is orbitally linearizable. The authors in (Hu et al., 2008), classified centres of a class of cubic homogeneous complex differential polynomials. They consider only systems admitting a first integrals of the form $x^3y + r(x,y)$ with $r(x,y)$ a complex polynomial of degree greater than or equal to 5. Llibre and Pantazi (2004) characterize the complex polynomial planar differential systems that have Darboux first integrals, this problem is known as, the inverse problem. They improved the Darboux theory of integrability by taking into account the degree of invariant algebraic curves. In a recent publication (Ferčec et al., 2011) obtain the necessary conditions for the existence of the local analytic first integrals of four subfamilies of a class of quintic polynomial planar differential systems. They proved their sufficiency, as usual, using the Darboux method or considering systems which are time-reversible.

Many works have been devoted to the study of the linearizability problem for Lotka-Volterra systems. Wang and Liu (2008a), gave necessary and sufficient conditions for linearizability of the two dimensional Lotka-Volterra system with $(3:-4)$ and $(3:-5)$ resonance, while in further work (Wang and Liu, 2008b), they obtain a linearizability conditions for Lotka-Volterra quadratic systems via computing periodic constants. They present a recursive algorithm to compute the periodic constants as well. Necessary and sufficient conditions for linearizability of the quartic Lotka-Volterra system having non-homogeneous nonlinearity obtained in a recent study by Giné et al. (2011). Other work on the linearizability problem can be found in (Christopher and Rousseau, 2001; Romanovski et al., 2001, 2003; Dolićanin and Romanovski, 2005; Dolićanin et al., 2007; Chen et al., 2008; Giné and Romanovski, 2009; Wu et al., 2012).

More recent works on integrability and linearizability of Lotka-Volterra type systems with $(p,-q)$–resonance can be found in (Hu et al., 2008; Wang and Liu, 2008a; Giné and Romanovski, 2009, 2010; Giné et al., 2011; Chen et al., 2012). Liu et al. (2004) consider the planar differential systems with $(p,-q)$ where the coefficients are real numbers. They studied the integrability and linearizability problem for the case $p=3$ and $q=4,5$. In (Christopher et al., 2003), Christopher et al. considered complex quadratic vector fields having a saddle or saddle-node type of singularity with $1:-\lambda$ ratio of eigenvalues. They used the Darboux method and other tools for showing normalizability, integrability and linearizability. Other works can be found in (Moulin-Ollagnier, 2002; Romanovski and Shafer, 2008).

Recall that in the classical centre-focus problem, there are currently only two known mechanisms for integrability (Christopher and Li, 2007): the existence of an algebraic symmetry or the existence of a Darboux integrating factor. Other methods appear to be reducible to these: for example, the composition condition is a form of algebraic symmetry. In the general case of $p:-q$ resonance, however, other mechanisms appear to come into play: in particular, blow-down to a node (Christopher et al., 2003, Section 5), where the known linearizing change of coordinates for a node can be pulled back to the saddle, and reduction to a Riccati equation (Żołądek, 1997, p104), where the system has a Darboux-type first integral but in terms of solutions to a second order linear differential equation. For the Lotka-Volterra system, many of these conditions were subsumed in (Christopher and Rousseau, 2004) under a simple monodromy condition, applied to the neighborhoods of the invariant lines $x=0$ and $y=0$, and the line at infinity. These conditions reduced essentially to finite checks on the nature of the singularities (finite and infinite) of the system. We add in passing, that the existence of a formal first integral will imply the existence of an analytic one, so any argument which establishes the former, will also establish the latter.

Our aim here is to see if a generalization of the Poincaré centre-focus problem to higher dimension gives a similarly simple list of integrability mechanisms.

It was a surprise to us that the problems of integrability was much harder in this case, giving rise to new forms of argument which rely less on geometric properties than the form of the power series concerned. This might be due to the fact that such resonant singularities mix the saddle and node-like properties of their two-dimensional counterparts. It would be interesting to know whether there were more geometric ways of obtaining the sufficiency of these conditions.

Other work on 3D Lotka-Volterra equations has been done by Bobienski and Żołądek (2005), who consider the finite singularity away from the axes planes, and give a number of mechanisms for the existence of a centre in the $(i : -i : \lambda)$ case; Cairó (2000); Cairó and Llibre (2000a), who obtain a number of conditions for the existence of Darboux first integrals in terms of the parameters; and Basov and Romanovski (2010), who take one of the eigenvalues equal to zero. There has also been several works devoted to systems which are homogeneous ($\lambda = \mu = \nu = 0$) and hence reducible to a two-dimensional Lotka-Volterra equation (Gao and Liu, 1998; Gonzalez-Gascon and Peralta Salas, 2000; Moulin-Ollagnier, 2001; Christodoulides and Damianou, 2009)). In (Gao, 1999), the author used a direct integration method to find a time-dependent first integrals of three dimensional vector fields. Recently, Murza and Teruel (2010) give a detail of analysis of the qualitative behaviour of three dimensional Lotka-Volterra equations in a restricted region. We cite (Laburnie, 1996; Christodoulides and Damianou, 2009; Angew, 2011) for more results.

There are other works on three dimensional systems which are different from Lotka-Volterra systems. For instant, Mahdi and Valls (2001) consider a three dimensional system which known as a Nosé-Hoover equation for a one dimensional oscillator

$$\dot{x} = -y - xy, \qquad \dot{y} = x, \qquad \dot{z} = \alpha(x^2 - 1).$$

They study the integrability problem for this kind of systems in order to understand its global dynamics. Also in (Valls, 2006), the author studies the local integrability of three-dimensional Halphen system

$$\dot{x} = yz - x(y+z), \quad \dot{y} = xz - y(x+z), \quad \dot{z} = xy - z(x+y).$$

Sometimes, we would like to restrict the solutions of a differential equation to just closed form solutions. For this type of study, we need to use differential algebra and such solutions are called Liouvillian and the corresponding systems are said to be Liouvillian integrable. Prelle and Singer (1992) proved that if a polynomial vector field has a Liouvillian first integral, then this integral can be computed using the invariant algebraic curves of the systems. Moulin-Ollagnier (2001) characterizes all Liouvillian first integrals of the ABC Lotka-Volterra systems. The ABC Lotka-Volterra system is a Lotka-Volterra system that the linear and the diagonal terms are absent of that system. In (Avellar et al., 2007), the authors presented a semi-algorithm to find Liouvillian first integrals of two-dimensional systems. They used Darbouxian method to find integrating factors. In (Durate and da Mota, 2010), the authors provide a semi-algorithm to find elementary first integrals of three dimensional polynomial differential equations. They extended the Prelle and Singer's method for two dimensional systems using the method of Darboux. In recent paper, Llibre and Valls (2010), consider the Lienárd polynomial differential systems and characterize its Liouvillian first integrals. They give an explicit form of that first integrals. Some related study can be fond in (Moulin-Ollagnier, 1996; Christopher, 1999; Cairó et al., 2003; Casale, 2011).

The structure of this thesis is as follows: Chapter two is a general introduction about the notion of integrability and centre-focus problem as well as the relation between them. We also give the basic definitions and theorems in this area. Furthermore, the Darboux theorem for three dimensional vector fields is given in this chapter. In chapter three, we consider the problem of local integrability and linearizability of three dimensional Lotka–Volterra (1.3) at the origin. More precisely, we give a complete set of necessary and sufficient conditions for integrability and linearizability in the case (λ,μ,ν)–resonance such that $\lambda+\mu+\nu \leq 2$. To prove their sufficiency, we mainly use the method of Darboux with esixtence of inverse Jacobi multiplier, and the linearizability of a node in two variables with power-series argument in the third variable. One related and more general three dimensional system is

$$\begin{aligned} \dot{x} &= P = x(\lambda + ax + by + cz), \\ \dot{y} &= Q = \mu y + dx^2 + exy + fxz + gyz + hy^2 + kz^2, \\ \dot{z} &= R = z(\nu + \ell x + my + pz). \end{aligned} \tag{1.4}$$

A complete classification of integrability and linearizability conditions at the origin of system (1.4) where $(\lambda : \mu : \nu) = (1 : -1 : 1)$–resonance are given in Chapter four. We use the same possible mechanisms above to prove their sufficiency and in some cases the solution of a Ricatti equation comes into play to linearize the third variable. In chapter five, we are interested in studying the integrability problems in chapter two and Appendix A via the monodromy method. Additionally, we use this method to give an alternative proof for sufficiency of some of the centre conditions which are found in (Chen et al., 2012; Liu et al., 2012). A brief explanation of Liouvillian integrability with some basic definitions are given in chapter six. We also generalize Singer's theorem to the case of integrable 1–forms in n–dimensions. In addition, we state and prove the extension of Singer's theorem to Liouvillian first integrals of three dimensional vector fields.

Chapters two, four and five are joint work with Colin Christopher. Chapter seven contains work conducted in collaboration with Jaume Llibre and Chara Pantazi. We classify all centres of cubic quasi-homogenous planar polynomial differential systems. We also study the number limit of limit cycles of these kind of systems. In particular, we used the averaging theory of first order to prove that the system has only one limit cycle.

Further study on local integrability and linearizability of three dimensional Lotka-Volterra equations are placed in Appendices. Precisely, local integrability and linearizability of three dimensional Lotka-Volterra equations with $(3 : -1 : 2)$–resonance is in Appendix A and integrability with $(Rank : 1)$–resonance is in Appendix B

Chapter 2

Background

In this introductory chapter, we give a brief overview of the integrability/centre–focus problem, the Darboux method, and related issues. We first explain the notion of integrability and linearizability. Furthermore, we explain the Darboux method of integrability for three dimensional vector fields. As an introduction to these ideas, we first discuss the centre-focus problem in two dimensional vector fields, giving an example of how one can find integrability conditions for quadratic planar differential systems and prove their sufficiency.

2.1 Integrability and linearizability problems

The notion of integrability is basic to the theory dynamical systems (Goriely, 2001). In addition, sometimes, perturbation gives a rich picture of bifurcations for integrable systems. A complete system of integrals for two dimensional vector fields is given by the existence of a single first integral of that system as that first integral determines completely its phase portrait. However, in general, for n-dimensional vector fields, the existence of $(n-1)$ independent first integrals are required in order to the system be completely integrable (Cairó and Llibre, 2000a; Goriely, 2001; Zhang, 2008; Christodoulides and Damianou, 2009). Specifically, for three dimensional vector fields, the existence of two independent first integrals are required to be completely integrable and the intersection of the level curves of that two independent first integrals, say $\phi = c_1$ and $\psi = c_2$, when c_1 and c_2 vary in $\mathbb{R}$, determines its trajectories (Cairó and Llibre, 2000a).

The linearizability problem of a system of differential equations is the problem of investigating when the system can be converted to a linear system via an analytic change of coordinates. In this sense, we can say that the system is linearizable. Note that a linearizable system is also integrable.

In this section, with a view to the application later in this thesis we concentrate on the three dimensional system,

$$\begin{aligned} \dot{x} &= P(x,y,z) = \lambda x + p(x,y,z), \\ \dot{y} &= Q(x,y,z) = \mu y + q(x,y,z), \\ \dot{z} &= R(x,y,z) = \nu z + r(x,y,z), \end{aligned} \tag{2.1}$$

where $\lambda\,\mu\,\nu \neq 0$ and p, q and r are analytic functions in x, y and z. Let we denote by $\mathcal{X}$, the corresponding vector field of (2.1)

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}. \tag{2.2}$$

Definition 1. *System* (2.1) *is integrable at the origin if and only if there is a change of coordinates*

$$(X,Y,Z) = \big(x + o(x,y,z), y + o(x,y,z), z + o(x,y,z)\big), \tag{2.3}$$

which transforms the system (2.1) *into*

$$\dot{X} = \lambda X \zeta(x,y,z), \quad \dot{Y} = \mu Y \zeta(x,y,z), \quad \dot{Z} = \nu Z \zeta(x,y,z), \tag{2.4}$$

where $\zeta = 1 + o(x,y,z)$*. Then* $X^{-\mu}Y^{\lambda}$ *and* $Y^{\nu}Z^{-\mu}$ *are first integrals of system* (2.1).

Definition 2. *System* (2.1) *is linearizable at the origin if and only if the change of coordinates* (2.3) *can be chosen to make* $\zeta \equiv 1$.

This can be shown to be equivalent to asking that all the coefficients in the normal form vanish.

Remark 1. *System* (2.1) *is integrable at the origin if and only if it is orbitally linearizable.*

2.1.1 First integrals and inverse Jacobi multiplier

The main concern in this work, is to study first integrals of vector fields. So we first introduce the concept of a first integrals and after that the related concept of an inverse Jacobi multiplier.

Definition 3. *We say that a non-constant analytic function $\phi(x,y,z)$ of the differential system* (2.1) *or its corresponding vector field* (2.2) *is a first integral if it is constant on all its solutions (trajectories). That is, ϕ satisfy the partial differential equation*

$$\mathcal{X}\phi = P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y} + R\frac{\partial \phi}{\partial z} = 0.$$

In higher dimensions, the inverse Jacobi multiplier can be thought as of a generalization of the inverse integrating factor (Berrone and Giacomini, 2003). In the Darboux method, we usually consider the corresponding reciprocals: inverse integrating factors, and inverse Jacobi multipliers.

Definition 4. *A $\mathcal{C}^1$ function M is an inverse Jacobi multiplier for the vector field* (2.2) *if it satisfies the first order linear partial differential equation*

$$\mathcal{X}(M) = M div(\mathcal{X}),$$

or simply

$$div(\frac{\mathcal{X}}{M}) = 0.$$

In three dimensional systems, the existence of two independent first integrals implies the existence of an inverse Jacobi multiplier. Conversely, given just one first integral, ϕ, and an inverse Jacobi multiplier, M, one can construct another first integral in the following manner.

Suppose that the level surfaces $\phi = c$, where c is a constant, are locally parameterized by some function $z = f_c(x,y)$. Using the x and y coordinates to parameterize $\phi = c$, we obtain a vector field

$$P(x,y,f_c(x,y))\frac{\partial}{\partial x} + Q(x,y,f_c(x,y))\frac{\partial}{\partial y}. \tag{2.5}$$

In (Berrone and Giacomini, 2003), author gives a procedure to computer a Jacobi multiplier of system (2.1) if a first integral of system (2.1) together with an integrating factor of the reduced two dimensional system (2.5) are known. However, here, we give the same procedure but to find inverse Jacobi multiplier when first integral with inverse integrating factor are given. Since $z = f_c(x,y)$, then

$$\frac{\partial f_c}{\partial x} = -\frac{\partial \phi}{\partial x} / \frac{\partial \phi}{\partial z}, \tag{2.6}$$

and

$$\frac{\partial f_c}{\partial y} = -\frac{\partial \phi}{\partial y} / \frac{\partial \phi}{\partial z}. \tag{2.7}$$

On $z = f_c(x,y)$, we have a 1–form associated to a vector filed (2.1)

$$Q\big(x,y,f_c(x,y)\big)dx - P\big(x,y,f_c(x,y)\big)dy = 0. \tag{2.8}$$

By definition of *inverse integrating factor*, a function N is an *inverse integrating factor* if

$$\frac{\partial}{\partial y}\left(\frac{Q(x,y,f_c)}{N}\right) = \frac{\partial}{\partial x}\left(\frac{P(x,y,f_c)}{N}\right),$$

which implies that

$$\frac{\partial}{\partial x}\left(\frac{P}{N}\right) + \frac{\partial}{\partial z}\left(\frac{P}{N}\right)\frac{\partial f_c}{\partial x} + \frac{\partial}{\partial y}\left(\frac{Q}{N}\right) + \frac{\partial}{\partial z}\left(\frac{Q}{N}\right)\frac{\partial f_c}{\partial y} = 0. \tag{2.9}$$

Through the equations (2.6) and (2.7), equation (2.9) takes the form

$$\frac{\partial\phi}{\partial z}\frac{\partial}{\partial x}(\frac{P}{N})-\frac{\partial}{\partial z}(\frac{P}{N})\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial z}\frac{\partial}{\partial y}(\frac{Q}{N})-\frac{\partial}{\partial z}(\frac{Q}{N})\frac{\partial\phi}{\partial y}=0. \tag{2.10}$$

Since

$$\frac{\partial\phi}{\partial z}\frac{\partial}{\partial x}(\frac{P}{N})-\frac{\partial}{\partial z}(\frac{P}{N})\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial z}\frac{P}{N})-\frac{\partial}{\partial z}(\frac{\partial\phi}{\partial x}\frac{P}{N}),$$

and

$$\frac{\partial\phi}{\partial z}\frac{\partial}{\partial y}(\frac{Q}{N})-\frac{\partial}{\partial z}(\frac{Q}{N})\frac{\partial\phi}{\partial y}=\frac{\partial}{\partial y}(\frac{\partial\phi}{\partial z}\frac{Q}{N})-\frac{\partial}{\partial z}(\frac{\partial\phi}{\partial y}\frac{Q}{N}).$$

Then we can write equation (2.10) as

$$\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial z}\frac{P}{N})+\frac{\partial}{\partial y}(\frac{\partial\phi}{\partial z}\frac{Q}{N})-\frac{\partial}{\partial z}\big(\frac{1}{N}(\frac{\partial\phi}{\partial x}P+\frac{\partial\phi}{\partial y}Q)\big)=0. \tag{2.11}$$

Since ϕ is a first integral, then

$$P\frac{\partial\phi}{\partial x}+Q\frac{\partial\phi}{\partial y}=-R\frac{\partial\phi}{\partial z},$$

then equation (2.11) becomes

$$\frac{\partial}{\partial x}(\frac{\partial\phi}{\partial z}\frac{P}{N})+\frac{\partial}{\partial y}(\frac{\partial\phi}{\partial z}\frac{Q}{N})+\frac{\partial}{\partial z}(\frac{\partial\phi}{\partial z}\frac{R}{N})=0. \tag{2.12}$$

It is easy to see that the *inverse Jacobi multiplier* is given by

$$M=\frac{\partial\phi}{\partial z}\frac{1}{N}$$

on $z=f_c(x,y)$. Therefore, if both a first integral of the system and an inverse integrating factor of the reduced system (2.8) are known, then we can construct an inverse Jacobi multiplier for system (2.1) (for more detail see (Berrone and Giacomini, 2003)). Hence, by quadratures along $\phi=c$, we can construct a second first integral $\psi_c(x,y)$ for each value of c. The function $\psi_{\phi(x,y,z)}(x,y)$ gives a second first integral of the system.

2.1.2 Darboux method for integrability

In 1878, Darboux used a new technique based on a sufficient number of invariant algebraic curves to find a first integral for two dimensional polynomial vector fields. In essence, he proved that if the number of invariant algebraic curves exceed $n(n+1)/2$ for planar polynomial differential systems of degree n, then the system is integrable and the first integral can be expressed as the product of these invariant algebraic curves. This method connects the algebraic theory of solutions of differential equations to the search of first integrals of that system. Jouanolou (1979), extended the Darboux method to polynomial differential systems in $\mathbb{C}^n$ and he proved that a polynomial vector field of degree m has a rational first integral whose all its solutions are invariant algebraic curves when the number of invariant algebraic curves goes beyond $\binom{m+n-1}{n}+n$. More recently in (Llibre and Zhang, 2010), authors gave a simple and elementary proof of Jouanolou 1979 about the necessity of the number of invariant algebraic hypersurfaces in order to a polynomial differential system in $\mathbb{R}^n$ or $\mathbb{C}^n$ has a rational first integrals and Corrêa Jr et al. (2011) generalize Jouanolou's theorem to r–forms. If ω is an r–form on k^n, such that the coefficients are polynomials of degree at most d, where k is an algebraically closed field and it has $\binom{d-1+n}{n}\cdot\binom{n}{r+1}+r+1$ invariant irreducible hypersurfaces, then ω possesses a rational first integral.

The invariant algebraic surfaces work a particularly crucial role in the studying of integrability for polynomial differential systems using the Darboux method as it has the main role in constructing first integrals.

Definition 5. *Given a polynomial $F \in \mathbb{C}[x,y,z]$, a surface $F=0$ is called an invariant algebraic surface of the system* (2.1)*, if the polynomial F satisfies the partial differential equation*

$$\dot{F} = \mathfrak{X}F = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} + R\frac{\partial F}{\partial z} = C_F F \tag{2.13}$$

for some polynomial $C_F \in \mathbb{C}$. Such a polynomial is called the cofactor of the invariant algebraic surface $F=0$. One can note that from equation (2.1) *that any cofactor has degree one less than the polynomial vector field.*

The invariant algebraic surface $F=0$ is shaped by trajectories of the vector field

$\mathcal{X}$, as on the points of the curve $F = 0$, the gradient $\nabla F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$ of the curve is perpendicular to the vector field $\mathcal{X}$. That is, the vector field $\mathcal{X}$ is tangent to the curve $F = 0$. This is a justification of the name of invariant algebraic surfaces since it is invariant under the the flow defined by the vector field $\mathcal{X}$.

In general one can find such invariant algebraic surfaces using the method of undetermined coefficients. More precisely, we look for invariant algebraic surface

$$\ell = \xi + \sum_{i+j+k=1}^{n} a_{ijk} x^i y^j z^k, \qquad \xi = 0, 1$$

and with cofactor

$$L_\ell = 1 - \xi + \sum_{i+j+k=0}^{d-1} b_{ijk} x^i y^j z^k,$$

where d is the degree of the system. Then try find coefficients a_{ijk} and b_{ijk} which are to solve $\mathcal{X}\ell = \ell L_\ell$.

Remark 2. *1) invariant algebraic curves for two dimensional systems can be found in the literature under different names such as: Darboux polynomials, second integrals, special integrals, eigenpolynomials, particular algebraic solutions, algebraic particular integrals, special polynomials, Darboux curves, invariant polynomials or stationary solutions (Goriely, 2001; Weil, 1995).*

2) For a class of polynomial differential systems previously known its degree, say d, unfortunately there does not exist a uniform upper bound N(d) of the degree of its invariant algebraic surfaces. For instant in two dimensional systems

$$\dot{x} = rx, \quad \dot{y} = sy,$$

where r and s are positive integers. The system admits a rational first integral $H = \frac{y^r}{x^s}$ *and clearly* $x^s - hy^r = 0$ *is an invariant algebraic surface for all* $h \in \mathbb{C}$ *(Christopher and Llibre, 2002; Llibre, 2011).*

Proposition 1. *Suppose $f \in \mathbb{C}[x,\ y,\ z]$ and let $f = f_1^{n_1}, \cdots, f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{C}[x,\ y,\ z]$. Then for a vector field $\mathcal{X}$, $f = 0$ is an invariant algebraic curve with cofactor C_f if, and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \cdots, r$ with cofactor C_{f_i}. Moreover $C_f = n_1 C_{f_1} + ... + n_r C_{f_r}$.*

Proof. See (Dumortier et al., 2006; Christopher and Li, 2007; Llibre, 2004, 2011). □

Proposition 2. *Suppose f and g are invariant algebraic surfaces with respective cofactors C_f and C_g. Then*

1) fg=0 is an invariant algebraic surface with cofactor $C_f + C_g$.

2) If $C_f = C_g$, then f+g is an invariant algebraic surfaces with cofactor $C_f = C_g$.

3) f^k for $k \in \mathbb{C}$ is an invariant algebraic surfaces with cofactor kC_f.

Proof. It is straightforward. □

To complete the study of integrals of parametric families, we will also need the notion of *exponential factor* which plays the same role of as an invariant algebraic surface in the case when two or more such surfaces coalesce.

Definition 6. *Let $E(x,y,z) = \exp(f(x,y,z)/g(x,y,z))$ where $f, g \in \mathbb{C}[x,y,z]$, then E in an exponential factor if*

$$\mathcal{X}E = C_E E, \tag{2.14}$$

for some polynomial C_E of degree one less than the vector field. The polynomial C_E is called the cofactor of E.

Proposition 3. *Let $E = \exp(g/f)$ be an exponential factor for a vector field $\mathcal{X}$, then f=0 is an invariant algebraic surface and g satisfies the equation*

$$\mathcal{X}(g) = gC_f + fC_E,$$

where C_f is the cofactor of f and C_E is the cofactor of the exponential factor E.

Proof. Clear. □

Definition 7. *A Darboux function is a function of the form,*

$$D = \prod F_i^{\lambda_i} E^{\lambda_0},$$

where the F_i are invariant algebraic surfaces of the system, and $E = \exp(f/g)$ is an exponential factor. Given a Darboux function, D, we can compute

$$\mathcal{X}(D) = D\Big(\sum \lambda_i C_{F_i} + \lambda_0 C_E\Big).$$

Clearly, the function D is a non-trivial first integral of the system if and only if the cofactors C_{F_i} and C_E are linearly dependent.

Remark 3. *1) In some literature an exponential factor is known as degenerate algebraic curves. For instance see (Christopher, 1994; Christohper and Llibre, 2000; Christohper et al., 2007).*

2) All invariant surfaces not passing through a critical point have cofactors which vanish at this point.

3) All exponential factors have cofactors which vanish at the critical point not in denominators of their exponents.

2.1.3 The Darboux theorem

The following theorem is an improved version of the Darboux original theorem of integrability.

Theorem 1. *Consider a three dimensional vector field $\mathcal{X}$ of degree m which possesses p distinct invariant algebraic surfaces $f_i = 0$ for $i = 1, \cdots, p$ with cofactors C_{f_j} and q independent exponential factors E_j for $j = 1, \cdots, q$ with cofactors C_{E_j}. Then the following holds:*

1. *The function*

$$IJM = f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q}$$

is an inverse Jacobi multiplier provided that the condition

$$\sum_{i=1}^{p} \lambda_i C_{f_i} + \sum_{j=1}^{q} \mu_j C_{E_j} = div\mathcal{X}$$

is satisfied for certain complex numbers λ_i, $i = 1, \cdots, p$ and μ_j, $j = 1, \cdots, q$.

2. *There exist λ_i, $\mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i C_{f_i} + \sum_{j=1}^{q} \mu_j C_{E_j} = 0,$$

if and only if the the (multi-valued) function

$$\phi = f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q}$$

is a first integral of the vector field $\mathcal{X}$.

3. *If $p+q \geq N+1$ where $N = \binom{2+m}{3}$, then there exist λ_i, $\mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i C_{f_i} + \sum_{j=1}^{q} \mu_j C_{E_j} = 0.$$

4. *If $p+q \geq N+3$, then the vector field $\mathcal{X}$ admits a rational first integral.*

2.1.4 Normal Forms

Normal form theory is a powerful tool for studying the local behavior of a nonlinear system near a singular point. The basic idea of normal forms is to perform a sequence of transformations of a given differential equation to yield a simpler equation for each successive transformation. The normal form theory can be traced back to Poincaré in his thesis.

Consider the analytic system

$$\dot{x} = Ax + F(x), \quad x \in \mathbb{R}^n, \tag{2.15}$$

where A is an $n \times n$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $F(x) = O(2)$ is an analytic vector-valued function in $(\mathbb{R}^n, 0)$.

Definition 8. *An n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in C^n$ is called resonant, if it satisfies the resonant identity*

$$\alpha_i = \langle \ell, \alpha \rangle = \sum_{i=1}^{n} \ell_i \alpha_i, \quad |\ell| \geq 2, \tag{2.16}$$

for non-negative integers $\ell_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and the natural numbers $|\ell| = \sum_{i=1}^{n} \ell_i$ is the order of the resonance. The coefficient X_i^{ℓ} of the monomial x is called a resonant coefficient and the corresponding term is called a resonant term.

Note that if condition (2.16) does not hold, then $\alpha_1, \ldots, \alpha_n \in C^n$ are called non-resonant and then the differential equation can be formally linearized (i.e. a formal, but in general, non-convergent transformation series exists). For more detail consult (Ilyashenko and Yakovenko, 2008; Romanovski and Shafer, 2009).

Let $\mathcal{H}_s$ denote the vector space of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ each of whose components is a homogeneous polynomial of degree s.

The linear operator (homological operator) $\mathcal{L}$ on $\mathcal{H}_s$ is defined by

$$\mathcal{L}h(y) = d\,h(y)Ay - Ah(y).$$

By performing a sequence of transformations of the form

$$x = H(y) = y + h^s(y), \qquad h^s \in \mathcal{H}_s,$$

we can remove all non-resonant terms of system (2.15) and is formally equivalent to

$$\dot{y} = Ay + G(y). \tag{2.17}$$

The problem of when an analytic system is analytically equivalent to its normal form is a classical and an open problem.

Let

$$\mathcal{M}_\alpha = \Big\{ \ell = (\ell_1, \ldots, \ell_n) : \langle \ell, \alpha \rangle = \sum_{i=1}^{n} \ell_i \alpha_i = 0, \quad |\ell| \geq 1 \Big\}.$$

Let us denote R_α, the rank of vectors in the set $\mathcal{M}_\alpha$ Then $R_\alpha \leq n-1$.

Theorem 2 (Zhang (2008), Theorem 1.1, page 1081)**.** *Suppose that the origin of system* (2.15) *is non-degenerate. Then system* (2.15) *has $n-1$ linearly independent analytic first integrals if and only if $R_\lambda = n-1$, and it is analytically equivalent to its normal form*

$$\dot{y}_i = \lambda_i y_i (1 + G(y)), \quad i = 1, \ldots, n,$$

by an analytic normalization, where $G(x)$ has no constant term and is an analytic function of y^ℓ with $\ell \in \mathcal{M}$ and $\ell = (\ell_1, \ldots, \ell_n) = 1$.

For more detail on the formal and analytic transformations to the normal form using Poincaré-Dulac normal form, one can consult (Walcher, 2004; Dumortier et al., 2006).

2.1.5 Reduction to the Poincaré domain

A singular point whose eigenvalues lie in the Poincaré domain (that is, the convex hull of the eigenvalues does not contain the origin) can be brought to normal form via an analytic change of coordinates (Ilyashenko and Yakovenko, 2008). In particular, a node with two analytic separatrices can have no resonant terms in its normal form and so must be analytically linearizable. For more details (Christopher and Rousseau, 2004).

We use this principle in two ways. Firstly, in many cases we can choose a coordinate system so that two of the variables decouple to give a linearizable node at the origin. If this is so, it just remains to find a linearizing transformation for the third variable via some simple power series arguments. Secondly, and more rarely, we can perform a blow down to a three-dimensional system in the Poincaré domain. Since this new system is linearizable, we can find two first integrals which we can pull back to first integrals of the original system.

2.1.6 Blow-down

The result of Poincaré guarantees that there exists an analytic change of coordinates in a neighbourhood of a singular point of the type of non-resonant node to a linear system. In the case of a singular point of the type of resonant node, the system is linearizable if, and only if the resonant monomials have zeros coefficients. For more detail, see (Romanovski and Shafer, 2009).

2.2 The centre–focus problem

One of the interesting problem in the qualitative theory of planar differential systems is the *centre–focus problem* or known as *Poincaré centre–focus problem*. This problem deals with the distinction between a centre and a focus. In the real case one has to find conditions on the coefficients of the system in the parameter space in order a neighbourhood of the origin filled by periodic solutions except the the origin which is a singular point. In general, it is well-known and unsolved problem. Also this problem has a strong relation with *Hilbert's sixteenth problem* (to estimate the number of an isolated closed orbits which is known as *limit cycles*) and the study of integrability as well. In contrast to the centre–focus problem, the conditions for integrability of three dimensional systems is less known.

We begin with integrability problem in dimension two and illustrate the mechanisms behind the necessary and sufficient conditions that makes the critical point at the origin integrable. In other words, the system admits a local first integral.

We consider the general quadratic two-dimensional differential system of the form

$$\begin{aligned} \dot{x} &= x + ax^2 + bxy + cy^2 &&= P(x,y), \\ \dot{y} &= -y + dx^2 + exy + fy^2 &&= Q(x,y), \end{aligned} \tag{2.18}$$

and let $\mathcal{X}$ denote its vector field

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

We will explain how one can obtain necessary conditions for integrability of system (2.18) and then prove their sufficiency. This problem has been solved completely by Dulac (1908) and Kapteyn (1912, 1911) (See also (Fronville et al., 1998; Wang and Liu, 2008a)). We now find the necessary conditions for the critical point at the origin to be a centre for system (2.18). The conditions for a centre is equivalent the conditions that the system (2.18) has a first integral of the form

$$\phi = xy + \sum_{i+j>3} v_{ij} x^i y^j,$$

where

$$\mathcal{X}\phi = \sum v_{nn}(xy)^{n+1},$$

and v_{nn} are the obstruction of the existence of such a ϕ and are polynomials of the coefficients of (2.18) and the so-called *saddle quantities* (Fronville et al., 1998). A necessary and sufficient conditions for integrability is then obtained by common zeros of all these quantities. According to the Hilbert basis theorem, we only need a finite number of these quantities. Although exactly how many is not known. Having calculated a finite number of these quantities, we then solve them simultaneously by computing a factorized Gröbner basis to obtain necessary integrability conditions and we need to prove that they are sufficient.

For system (2.18), it is suffices to have three of these quantities v_{11}, v_{22} and v_{33} that generate the entire integrability conditions and they have the following expression:

1) $v_{11} = fe - ba.$

2) $$v_{22} = -18ba^2f + 2ca^2e - 3cae^2 - 27b^2ae - 24b^2a^2 + 18f^2ae - 2dbf^2 - 2ce^3 + 24f^2e^2 + 2db^3 + 3dfb^2 + 27e^2fb.$$

3) $$\begin{aligned} v_{33} = & 7236e^3f^3 - 7236a^3b^3 + 16380e^2abf^2 - 16380ea^2fb^2 + 1212fca^3e \\ & + 238fca^2e^2 - 1440fcae^3 + 254bca^3e - 399bca^2e^2 - 2222bcae^3 \\ & - 696bca^4 - 2dab^4 + 696def^4 + 621deb^4 + 399deb^2f^2 + 1440dafb^3 \\ & - 1212dabf^3 - 238dab^2f^2 - 132d^2b^2cf + 28d^2bcf^2 + 890db^2ca^2 \\ & - 890de^2f^2c + 132de^2c^2a - 28dea^2c^2 - 864df^2cae + 1754db^2cae \\ & - 1754de^2fcb + 2222defb^3 - 254debf^3 + 2fce^4 - 621cbe^4 - 73d^2b^3c \\ & + 73de^3c^2 - 6768e^2ab^3 + 10476e^2af^3 + 3888ea^2f^3 - 13824ea^2b^3 \\ & + 6768e^3fb^2 + 13824e^3bf^2 - 10476a^3fb^2 - 3888a^3bf^2 + 864dfca^2b. \end{aligned}$$

Then the integrability conditions (centre conditions) can be summarized in the following theorem:

Theorem 3. *Consider the quadratic system (2.18). The origin is integrable if and only if one of the following conditions are satisfied:*

1) $2f + b = 2a + e = 0.$

2) $2b - f = a - 2e = cd - eb = 0.$

3) $ab - ef = db^3 - ce^3 = a^3c - df^3 = ace^2 - b^2df = a^2ce - bdf^2 = 0.$

4) $e = b = 0.$

Proof.

We now prove the sufficiency of conditions above. We explain each case in detail.

1) $2f+b=2a+e=0$. The system (2.18), reduce to

$$\begin{aligned} \dot{x} &= x+ax^2-2fxy+cy^2 &&= P'(x,y), \\ \dot{y} &= -y+dx^2-2axy+fy^2 &&= Q'(x,y), \end{aligned} \tag{2.19}$$

Since

$$\frac{\partial P'}{\partial x}+\frac{\partial Q'}{\partial y}=0,$$

then the system (2.19) is Hamiltonian. In this case, we have a first integral of the form

$$\phi(x,y)=xy+ax^3y+\frac{b}{2}xy^2+\frac{c}{3}y^3-\frac{d}{3}x^3. \tag{2.20}$$

2) $f-2b=a-2e=be-dc=0$. When $de\neq 0$, the system has an invariant algebraic conic

$$\ell_1=1+2ex-\frac{2dc}{e}y+\frac{d^2c}{e}x^2-2dcxy+cey^2=0,$$

and the invariant algebraic cubic

$$\begin{aligned} \ell_2 =& 1+3ex-\frac{3dc}{e}y+\frac{3}{2}\frac{d^2c+e^3}{e}x^2-\frac{3}{2}\frac{d^4c^2+2e^3cd^2+e^6}{e^3d}xy \\ &+\frac{3}{2}\frac{c(d^2c+e^3)}{e^2}y^2+\frac{1}{2}\frac{d^2c(d^2c+e^3)}{e^3}x^3-\frac{3}{2}\frac{dc(d^2c+e^3)}{e^2}x^2y \\ &+\frac{3}{2}\frac{c(d^2c+e^3)}{e}xy^2-\frac{1}{2}\frac{c(d^2c+e^3)}{d}y^3=0 \end{aligned}$$

with cofactors $L_{\ell_1}=2(ex+\frac{dc}{e}y)$ and $L_{\ell_2}3(ex+\frac{dc}{e}y)$ respectively. It is easy to see that

$$\psi=\ell_1^{-\frac{3}{2}}\ell_2$$

is a first integral. Hence the desired first integral is

$$\varphi=\xi^{-1}\psi-1=xy+\cdots,$$

where $\xi=\frac{3(2cd^2e^3-c^2d^4-e^6)}{de^3}$.

When $de = 0$, we have several cases:

i) $\mathbf{e = 0, d \neq 0}$: In this case, we have $2b - f = a = e = c = 0$ and we get an invariant algebraic conic $\ell_1 = 1 - 2by + bdx^2 = 0$ and invariant algebraic quartic $\ell_2 = 1 - 6by + 3bdx^2 + 12b^2y^2 - 12b^2dyx^2 - 8b^3y^3 + 3b^2d^2x^4 + 3b^3dx^2y^2 = 0$ with respective cofactors $2by$ and $6by$. Hence $\psi = \ell_1^{-3}\ell_2$ is a first integral which can be put into the required form:

$$\phi = -1 - \frac{1}{9db^3}\psi = x^2y^2 + \cdots .$$

ii) $\mathbf{d = 0, e \neq 0}$: In this case, we have $2b - f = a - 2e = d = f = 0$. The system has an invariant algebraic conic $\ell_1 = 1 + 2ey + cey^2 = 0$ and an invariant algebraic quartic $\ell_2 = 1 + 6ex + 12e^2x^2 + 3cey^2 + 8e^3x^3 + 12ce^2xy^2 + 3ce^3x^2y^2 + 3c^2e^2y^4 = 0$ with respective cofactors $2by$ and $6by$. Hence $\psi = \ell_1^{-\frac{3}{2}}\ell_2$ is a first integral so that $\phi = -1 - \frac{1}{9ce^3}\psi = x^2y^2 + \cdots$ is a first integral in the required form.

iii) $\mathbf{e = d = 0}$: In this case, we have $2b - f = a = e = d = 0$. We have an an invariant algebraic line $\ell_1 = 1 - 2by = 0$ and invariant algebraic conic $\ell_2 = 1 - 3by - \frac{3b^3}{2c}xy + \frac{3b^2}{2}y^2 = 0$ with respective cofactors $2by$ and $3by$ that yield a first integral $\psi = \ell^{-\frac{3}{2}}\ell_2$. Thus, $\phi = -1 - \frac{c}{3b^3} = xy + \cdots$ is a first integral of the form required.

3) When $a \neq 0$, the conditions are reduces to $ab - ef = db^3 - ce^3 = 0$. The system has an invariant algebraic line

$$\ell_1 = 1 + \frac{a^2 - df}{a}x + \frac{df^2 - fa^2}{a^2}y = 0,$$

and an invariant algebraic conic

$$\begin{aligned}\ell_2 = 1 &+ \frac{3(a^2 + ea + f)}{a}x - \frac{f(a^2 + ea + df)}{a^2}y + \frac{3edfa + ea^3 + a^2e^2 + 2dfa^2 + 2d^2f^2}{2a^2}x^2 \\ &+ \frac{2d^3f^3 + 5ead^2f^2 - 2da^4f + 4e^2a^2df - a^5e + e^3a^3}{2da^3}xy \\ &+ \frac{f^2(3edfa + ea^3 + a^2e^2 + 2dfa^2 + 2d^2f^2)}{2a^4}y^2 = 0,\end{aligned}$$

with respective cofactors $\frac{a^2-df}{a^2}(ax+fy)$ and $\frac{a^2+ae+df}{a^2}(ax+fy)$.

Then $\psi = \ell_1^{-\frac{(a^2+ae+df)}{a^2-df}}\ell_2$ is a first integral. The required first integral is

$$\phi = -1+\xi^{-1}\psi = xy+\cdots,$$

where $\xi = \frac{2da^4f-2dfea^3-4d^2f^2a^2-6e^2a^2df-11ead^2f^2-6d^3f^3+a^5e-e^3a^3}{2da^3}$.

When $a=0$, we have two subcases:

i) $a = f = b^3d-ce^3 = 0$. We have an invariant algebraic curves

$$\ell_1 = 1-\frac{db}{e^2}(ex-yb) = 0,$$

and

$$\begin{aligned}\ell_2 =& 1+ex-by+\frac{1}{2}(db+e^2)x^2+\frac{6d^3b^3+9e^2d^2b^2+4de^4b+e^6}{2de^3}xy\\ &+\frac{b^2(db+e^2)}{2e^2}y^2-\frac{db(2d^2b^2+3e^2db+e^4)}{2e^3}x^3-\frac{b(2d^2b^2+3e^2db+e^4)}{2e^2}x^2y+\\ &+\frac{b^2(2d^2b^2+3e^2db+e^4)}{2e^3}xy^2+\frac{db^4(2d^2b^2+3e^2db+e^4)}{2e^6}=0,\end{aligned}$$

with respective cofactors $-\frac{db}{e^2}(ex+yb)$ and $(ex+yb)$ which give a first integral of the form $\phi = \ell_1^{\frac{e^2}{bd}}\ell$. Then the required first integral is given by

$$\phi = \xi^{-1}\psi-1 = xy+\cdots,$$

where $\xi = \frac{6bde^4+6b^3d^3+11b^2d^2e^2+e^6}{2de^3}$. Note that when one of b, d or e is zero, we get subcases which already have been considered.

ii) $a = d = e = 0$. If $cf \neq 0$, the system has two invariant curves

$$\ell_1 = 1-fy = 0,$$

and

$$\ell_2 = 1-(b+f)y+\frac{b^3-bf^2}{2c}xy+\frac{b^2+bf}{2}y^2=0,$$

which gives a first integral $\psi = \ell_1^{-1-\frac{b}{f}}\ell_2$. Thus we can get the first integral of the desired form $\phi = \frac{2c}{b^3-bf^2}\psi - 1 = xy+\cdots$. When $cf=0$, already considered.

4) $b=e=0$. In this case, we can arrive to a subsystem of the form

$$\begin{aligned}\dot{x} &= x+ax^2+cy^2,\\ \dot{y} &= -y+dx^2+fy^2.\end{aligned} \tag{2.21}$$

Here, we want to find an invariant algebraic line of the form

$$\ell = 1+rx+sy=0$$

with cofactor

$$L_\ell = A+Bx+Cy,$$

where r,s,A,B and C are constants. After some easy computation, we indicate that ℓ is an invariant algebraic line if and only if

$$ra+sd-r^2=0, \tag{2.22}$$

and

$$rc+sf+s^2=0. \tag{2.23}$$

When $d\neq 0$, we can find the value of s from equation (2.22)

$$s=-\frac{r(a-r)}{d}$$

and put into equation (2.23). Then the resulting equation is

$$\frac{r\big(r^3-2ar^2+(a^2+df)r+cd^2-fad\big)}{d^2}=0.$$

Note that $r \neq 0$, otherwise, ℓ is a constant. Here the cubic equation has three different values of r, say r_1, r_2 and r_3 with discriminant h of the form

$$h = -d^2(27c^2d^2 - 18acdf + 4ca^3 - a^2f^2 + 4df^3).$$

Let s_1, s_2 and s_3 denotes the corresponding values of r_1, r_2 and r_3 respectively. Thus we have three invariant algebraic lines

$$\ell_1 = 1 + r_1x + s_1y,$$

$$\ell_2 = 1 + r_2x + s_2y,$$

$$\ell_3 = 1 + r_3x + s_3y.$$

Then $\psi = \ell_1^{\alpha_1}\ell_2^{\alpha_2}\ell_3^{\alpha_3}$ is a first integral of system (2.21), if

$$\alpha_1r_1 + \alpha_2r_2 + \alpha_3r_3 = 0, \tag{2.24}$$

$$\alpha_1s_1 + \alpha_2s_2 + \alpha_3s_3 = 0. \tag{2.25}$$

Since the first integral ψ should start with the factor xy, then we must show that for some chosen for α_i, $i = 1,2,3$, ψ has the form

$$\psi = xy + \cdots.$$

From equations (2.24) and (2.25), we see

$$\psi_x(0,0) = \alpha_1r_1 + \alpha_2r_2 + \alpha_3r_3 = 0,$$

and

$$\psi_y(0,0) = \alpha_1s_1 + \alpha_2s_2 + \alpha_3s_3 = 0.$$

We evaluate $\psi_{xx}(0,0)$, $\psi_{xy}(0,0)$ and $\psi_{yy}(0,0)$. Again from using (2.24) and (2.25) with

$\psi_{xx}(0,0) = 0$, $\psi_{yy}(0,0) = 0$ and $\psi_{xy}(0,0) \neq 0$, one can observe that

$$\alpha_1 r_1^2 + \alpha_2 r_2^2 + \alpha_3 r_3^2 = 0, \tag{2.26}$$

$$\alpha_1 s_1^2 + \alpha_2 s_2^2 + \alpha_3 s_3^2 = 0, \tag{2.27}$$

and

$$\alpha_1 r_1 s_1 + \alpha_2 r_2 s_2 + \alpha_3 r_3 s_3 \neq 0. \tag{2.28}$$

Remark 4. *Note that, we have put $\psi_{xx}(0,0) = \psi_{yy}(0,0) = 0$ and $\psi_{xy}(0,0) \neq 0$, because the first integral ψ started by xy term.*

Since

$$s_j = \frac{r_j^2 - ar_j}{d}, \qquad j = 1,2,3$$

and substitute in equation (2.28) we have:

$$r_1\left(\frac{r_1^2 - ar_1}{d}\right)\alpha_1 + r_2\left(\frac{r_2^2 - ar_2}{d}\right)\alpha_2 + r_3\left(\frac{r_3^2 - ar_3}{d}\right)\alpha_3 \neq 0. \tag{2.29}$$

Using equation (2.26), we can simplify equation (2.29) and becomes

$$r_1^3\alpha_1 + r_2^3\alpha_2 + r_1^3\alpha_3 \neq 0. \tag{2.30}$$

Since

$$s_j = \frac{r_j^2 - ar_j}{d},$$

and

$$\alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3 = 0,$$

we get

$$\frac{1}{d}(\alpha_1 r_1^2 + \alpha_2 r_2^2 + \alpha_3 r_3^2) - \frac{a}{b}(\alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3) = 0.$$

This means that (2.25) is a linear combination of (2.24) and (2.26).

Since

$$\begin{vmatrix} r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_2^3 \\ r_1^3 & r_2^3 & r_3^3 \end{vmatrix} = -r_1r_2r_3(r_1-r_2)(r_1-r_3)(r_3-r_2)$$

is a non-zero. The only choice of α_1, α_2 and α_3 that satisfy (2.24) and (2.26) but violates (2.30) is $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus for any choice of α_1, α_2 and α_3

$$\psi = \frac{\psi - 1}{r_1^3\alpha_1 + r_2^3\alpha_2 + r_1^3\alpha_3}$$

is a first integral of the required form.

When $d = 0$, $f \neq 0$, the subsystem is

$$\dot{x} = x + ax^2 + cy^2, \qquad \dot{y} = -y + fy^2, \tag{2.31}$$

and has invariant algebraic curves $\ell_1 = 1 - fy$, $\ell_2 = 1 + ax + (-\frac{1}{2}f + \frac{1}{2}\sqrt{f^2-4ac})y$ and $\ell_3 = 1 + ax + (-\frac{1}{2}f - \frac{1}{2}\sqrt{f^2-4ac})y$ which gives a first integral

$$\phi = \ell_1^{-\frac{1}{f}\sqrt{f^2-4ac}}\ell_2^{-1}\ell_3.$$

The desired first integral is then

$$\psi = \frac{\phi - 1}{a\sqrt{f^2-4ac}}.$$

When $f = 0$, the system has the invariant algebraic curves has $\ell_1 = 1 + ax + i\sqrt{4ac}y$, $\ell_2 = 1 + ax - i\sqrt{4ac}y$ and exponential factor $E = \exp(y)$ has first integral

$$\phi = \ell_1^{-1}\ell_2 E^{2i\sqrt{ac}}.$$

Other work is devoted to study a polynomial first integrals of quadratic planar polynomial vector fields. For instance, in (Giné, 2005), the author characterize the quadratic systems of the form

$$\dot{x} = -y - bx^2 - Cxy - dy^2,$$
$$\dot{y} = x + ax^2 + Axy - ay^2, \tag{2.32}$$

which has a polynomial first integral.

Theorem 4. *System (2.32) has a polynomial first integral if and only if one of the following conditions are satisfied:*

$$1) \quad A - 2b = C + 2a = 0 \quad (Hamiltonian).$$
$$2) \quad a = C = 0 \text{ and } A = \frac{2m}{m-2} \quad (Reversible).$$
$$3) \quad b + d = 0,\ A = \frac{3b}{m-3} \text{ and } C^2 = \frac{mb^2}{2(m-3)} \quad (Lotka - Volterra).$$

For proof, one can see (Giné, 2005).

Chapter 3

An analytic approach of integrability problem of three dimensional Lotka–Volterra systems

3.1 Introduction

In this chapter, we present a complete classification of integrability and linearizability conditions at the origin of the three dimensional Lotka–Volterra systems,

$$\begin{aligned}
\dot{x} &= P = x(\lambda + ax + by + cz),\\
\dot{y} &= Q = y(\mu + dx + ey + fz),\\
\dot{z} &= R = z(\nu + gx + hy + kz),
\end{aligned} \tag{3.1}$$

where $\lambda, \mu, \nu \neq 0$. Integrability understood, in general, as the existence of an adequate number of first integrals. For three dimensional systems, two independent first integrals are required for the system be completely integrable. We have selected several choices of λ, μ, and ν in the Siegel domain with $\lambda + \mu + \nu \leq 2$. Our main objective along this chapter is to find conditions on the parameters such that the system possesses two independent first integrals

$$\phi = x^{-\mu} y^{\lambda}(1 + O(x,y,z)), \quad \text{and} \quad \psi = y^{\nu} z^{-\mu}(1 + O(x,y,z)), \tag{3.2}$$

In particular, we give a complete classification of the integrability and linearizability conditions for (3.1) with $(1:-1:1)$, $(2:-1:1)$ and $(1:-2:1)$ resonant critical points at the origin.

In general, even the necessary conditions of integrability are obtained, their sufficiency perhaps not easy to prove. As usual, we have used mainly the Darboux theory of integrability with inverse Jacobi multipliers, but other techniques are also necessary for a complete a classification, for instance, the linearizability of a node in two variables with power-series arguments in the third variable.

Before explain a mechanism to calculate integrability and linearizability conditions of the critical point at the origin, we will prove some theorems which are quite useful for proving the sufficincy of integrability and linearizability conditions.

Theorem 5. *Suppose the analytic vector field*

$$x(\lambda+\sum_{|I|>0}A_{xI}X^I)\frac{\partial}{\partial x}+y(\mu+\sum_{|I|>0}A_{yI}X^I)\frac{\partial}{\partial y}+z(\nu+\sum_{|I|>0}A_{zI}X^I)\frac{\partial}{\partial z},$$

has an analytic first integral $\phi=x^{\alpha}y^{\beta}z^{\gamma}(1+O(x,y,z))$ with at least one of α, β, $\gamma\neq 0$ and a Jacobi multiplier $M=x^ry^sz^t(1+O(x,y,z))$ and suppose that the cross product of $(r-i-1,s-j-1,t-k-1)$ and (α,β,γ) is bounded away from zero for any integers $i,j,k\geq 0$, then the system has a second analytic first integral of the form $\psi=x^{1-r}y^{1-s}z^{1-t}(1+O(x,y,z))$, and hence the system (3.1) *is integrable.*

Proof. Without loss of generality, we assume that $\alpha>0$. After an analytic change of coordinates of the form $x\mapsto x(1+O(x,y,z))$, which will not alter the form of the vector field, we can assume that $\phi=X^{\delta}$ where $\delta=(\alpha,\beta,\gamma)$. Furthermore, by absorbing the factor $(1+O(x,y,z))$ of M into the vector field itself, we can take the inverse Jacobi multiplier M to be X^{θ}, where $\theta=(r,s,t)$. We take $A_I=(A_{xI},A_{yI},A_{zI})$ and write $A_{(0,0,0)}=(\lambda,\mu,\nu)$.

From the hypothesis, we can take $K>0$ such that

$$|(\theta-I-\mathbf{1})\times\delta|>K \tag{3.3}$$

for all I.

Since ϕ is a first integral, then $\mathcal{X}\phi = 0$ gives

$$\delta \cdot A_I = 0 \tag{3.4}$$

for all I.

Since also M is a Jacobi multiplier, then we have

$$\mathcal{X}(\mathcal{M}) = \operatorname{div}(\mathcal{X})\, M. \tag{3.5}$$

Now,

$$\operatorname{div}(\mathcal{X}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \mathbf{1} \cdot \ell + \sum_I (I + \mathbf{1}) A_{xI} X^I,$$

where $\mathbf{1} = (1,1,1)$. Hence from equation (3.5), we have

$$(\theta - I - \mathbf{1}) \cdot A_I = 0 \tag{3.6}$$

for all I. By hypothesis, $(\theta - I - \mathbf{1})$ and δ are linearly independent, and so (3.4) and (3.6) imply that

$$A_I = k_I \, (\theta - I - \mathbf{1}) \times \delta, \tag{3.7}$$

for some k_I.

Now

$$\begin{aligned}
\frac{\Omega}{M} &= \frac{P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy}{X^\theta} \\
&= \sum_I \big(k_I \, (\theta - I - \mathbf{1}) \times \delta\big) \cdot \big(\frac{dydz}{yz}, \frac{dzdx}{zx}, \frac{dxdy}{xy}\big) X^{I-\theta+1} \\
&= \sum_I k_I \big((\theta - I - \mathbf{1}) \cdot (\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z})\big) \wedge \big(\delta \cdot (\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z})\big) X^{I-\theta+1} \\
&= \sum_I k_I \big((\theta - I - \mathbf{1}) \cdot (\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z}) X^{I-\theta+1}\big) \wedge \frac{d\phi}{\phi} \\
&= d\big(\sum_I k_I X^{I-\theta+1}\big) \wedge \frac{d\phi}{\phi}.
\end{aligned} \tag{3.8}$$

Thus, we have a formal first integral of the form

$$\psi = \sum_I k_I X^{I-\theta+1}.$$

From (3.3) and (3.7), we must have that $|k_I| < K|A_I|$ for all I, and so ψ is in fact analytic.

$\square$

Theorem 6. *If the system* (3.1) *is integrable and there exists a function* $\xi = x^\alpha y^\beta z^\gamma (1 + O(x,y,z))$ *such that* $\mathcal{X}(\xi) = k\xi$ *for some constant* $k = \alpha\lambda + \beta\mu + \gamma\nu$*, then the system is linearizable.*

Proof. To see this, suppose (3.1) is integrable, then there exists a change of coordinates

$$(X,Y,Z) = \big(x + o(x,y,z), y + o(x,y,z), z + o(x,y,z)\big), \tag{3.9}$$

bringing system (3.1) to

$$\dot{X} = \lambda X m(x,y,z), \quad \dot{Y} = \mu Y m(x,y,z), \quad \dot{Z} = \nu Z m(x,y,z), \tag{3.10}$$

where $m = 1 + o(x,y,z)$. Note that (3.9) implies that $\xi_0 = \xi X^{-\alpha} Y^{-\beta} Z^{-\gamma} = 1 + O(x,y,z)$ satisfies $\mathcal{X}(\xi_0^{1/k}) = \xi_0^{1/k}(1-m)$. Hence $(\tilde{x}, \tilde{y}, \tilde{z}) = (X\xi_0^{\frac{\lambda}{k}}, Y\xi_0^{\frac{\mu}{k}}, Z\xi_0^{\frac{\nu}{k}})$ is a linearizing change of coordinates. $\square$

The Lotka–Volterra equations have another property which was first noted in (Christopher and Rousseau, 2004) for two dimensional systems.

Theorem 7. *Consider three dimensional Lotka–Volterra system (3.1) for which the three separatrices at the origin* $x = 0$*,* $y = 0$ *and* $z = 0$ *have cofactors* L_x*,* L_y *and* L_z *respectively. If* L_x*,* L_y*,* L_z *and the divergence div*$(\mathcal{X})$ *are linearly independent then the origin is integrable if and only if it is linearizable.*

Proof. Suppose that the origin is integrable. Then there exists two independent first integrals $\varphi = x^{-\mu} y^{\lambda} \varphi_1(x,y,z)$ and $\psi = y^{\nu} z^{-\mu} \psi_1(x,y,z)$ where $\varphi_1(x,y,z) = 1 + O(x,y,z)$

and $\psi_1(x,y,z) = 1 + O(x,y,z)$ are analytic. The functions $\varphi_1(x,y,z)$ and $\psi_1(x,y,z)$ obey the equations

$$\mathcal{X}\varphi_1 = \varphi_1 L_{\varphi_1}, \qquad \mathcal{X}\psi_1 = \psi_1 L_{\psi_1},$$

where

$$L_{\varphi_1} = \mu L_x - \lambda L_y, \qquad L_{\psi_1} = -\nu L_y + \mu L_z.$$

Since φ and ψ are first integrals, then $d\varphi \wedge \Omega = 0$, $d\psi \wedge \Omega = 0$ and $d\varphi \wedge d\psi = M\Omega$ where $\Omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$ as before, and M is a Jacobi multiplier. One can easily show that

$$M = x^{-(\mu+1)} y^{\lambda+\nu-1} z^{-(\mu+1)} \phi(x,y,z)$$

where $\phi(x,y,z)$ satisfies $\phi(0,0,0) = -\mu \neq 0$ and ϕ has a cofactor given by the divergence plus a linear combination of L_x, L_y and L_z. Hence the cofactors L_x, L_y, L_z and ϕ are linearly independent. Note that the cofactors L_{φ_1}, L_{ψ_1} and L_{ϕ_1} have no constant term. The condition on linear independence implies that we can find a change of coordinates $X = x\varphi_1^{\alpha_1}\psi_1^{\alpha_2}\phi^{\alpha_3}$, $Y = y\varphi_1^{\beta_1}\psi_1^{\beta_2}\phi^{\beta_3}$ and $Z = z\varphi_1^{\gamma_1}\psi_1^{\gamma_2}\phi^{\gamma_3}$ which linearizes the system. □

3.2 Mechanisms for integrability and linearizability

In this section, we briefly describe a formulation of a mechanism to compute the integrability conditions for system (3.1). We firstly consider the existence of two independent first integrals of the form (3.2) such that

$$\dot{\phi} = \sum_{n_1,n_2 \geq 0} \eta_{n_1,n_2} x^{n_1} y^{n_1+n_2} z^{n_2} \qquad \dot{\psi} = \sum_{n_1,n_2 \geq 0} \zeta_{n_1,n_2} x^{n_1} y^{n_1+n_2} z^{n_2}$$

and $\lambda n_1 + \mu(n_1+n_2) + \nu n_2 = 0$ for all $n_1, n_2 \in \mathbb{N}$. The coefficients η_{n_1,n_2} and ζ_{n_1,n_2} $\big((\lambda:\mu)$ and $(\mu:\nu)$ resonant terms$\big)$ are polynomials in the parameter space of the system (3.1) which are the obstructions of the existence of ϕ and ψ. Let we denote I, the ideal generated by the polynomials η_{n_1,n_2} and ζ_{n_1,n_2}, that is $I = \langle \eta_{n_1,n_2}, \zeta_{n_1,n_2} \rangle$ and $V(I)$, the set of common zeros of polynomials η_{n_1,n_2} and ζ_{n_1,n_2} which known as the *variety* of I. In fact, Hilbert's Basis theorem guarantees that this ideal can be generated finitely. The necessary conditions were found by computing the conditions for the existence of two independent first integrals up to a given degree in the form (3.2) using Maple. The degree used 6, 10 and 12 for $(1:-1:1)$, $(2:-1:1)$ and $(1:-2:1)$–resonance respectively. A factorized Gröbner basis was then found using Reduce and finally the minAssGTZ algorithm in Singular (Decker et al., 2011; Greuel et al., 2012) was used to check that the conditions found were irreducible.

As is typical with this type of calculation, the generation of a set of obstructions to integrability is fairly straight forward. The time-consuming part lies in the reduction of the the conditions to simpler form. Expression exponentially grown in the computational algebra often leads to intractability at this stage even for relatively simple problems. In the cases considered here, the computations are still tractable though hard, and further generalizations are possible, either for more complex resonances (see Appendex) or for more general systems without three invariant coordinate planes (see next Chapter).

For linearizability, we proceeded similarly: computing the conditions for the existence of a linearizing change of coordinates up to some finite order to find necessary conditions, and exhibiting a linearizing change of coordinates for sufficiency. In this case, the first integrals can be obtained easily by pulling back the first integrals of the linearized system (3.2).

In what follows, we will explain the stands of our mechanism:

Step 1: We seek two analytic first integrals of the form

$$\phi = x^{-\mu}y^{\lambda}(1+\sum_{i+j+k>0} a_{ijk}x^iy^jz^k)$$

and

$$\psi = y^{\nu}z^{-\mu}(1+\sum_{i+j+k>0} b_{ijk}x^iy^jz^k)$$

where $\lambda,\nu>0$ and $\mu<0$.

Step 2: Compute $(\lambda:\mu)$ and $(\mu:\nu)$ resonant terms which are correspond to the obstacle to the existence of ϕ and ψ. This will be done by calculating the successive terms in the power series expansion of the supposed first integrals, that is $\mathcal{X}\phi = 0$ and $\mathcal{X}\psi = 0$.

Step 3: Having calculated a number of these quantities from Step 2, we then solve them simultaneously by computing a factorized Gröbner basis to obtain integrability conditions. The conditions are necessary, but we do not know as yet that they are sufficient. The calculations were performed in computer algebra system Maple and Reduce. Finally the minAssGTZ algorithm in Singular was used to check that the conditions found were irreducible.

Step 4: We need finally to prove sufficiency of these conditions by exhibiting two independent first integrals via the Darboux method together with inverse Jacobi multipliers or some other technique like blow-downs or the existence of a linearizable node.

Remark 5. *The coefficients of the resonant terms are polynomials in the coefficients of the system.*

3.3 Integrability and linearizability conditions

In this section, we will give a complete classifications for the integrability and linearizability conditions for systems (3.1) with $(1:-1:1)$, $(2:-1:1)$ and $(1:-2:1)$–resonance. We note that $x=0$, $y=0$ and $z=0$ are always invariant algebraic surfaces with cofactors $\lambda+ax+by+cz$, $\mu+dx+ey+fz$ and $\nu+gx+hy+kz$ respectively.

3.3.1 $(1:-1:1)$–resonance

In this case, we seek two independent first integrals of (3.1):

$$\phi_1 = xy(1+O(x,y,z)) \quad \text{and} \quad \phi_2 = yz(1+O(x,y,z)).$$

We are interesting to explain the mechanism above in detail for this case only. We first express ϕ_1 and ϕ_2 as power series up to terms of order 6 and compute resonant terms which are obstructions to them and we denote by $\eta_{k,k}$ and $\zeta_{k,k}$ for $i = 1,\cdots,5$, then

$$\dot{\phi}_1 = \frac{\partial \phi}{\partial x}(x+p) + \frac{\partial \phi}{\partial y}(-y+q) + \frac{\partial \phi}{\partial z}(z+r) = \sum_{\substack{j=2,3\\ i=0,\cdots,j}} \eta_{k,k} x^{j-i} y^j z^i, \quad k = 1,\cdots 5,$$

and

$$\dot{\phi}_2 = \frac{\partial \psi}{\partial x}(x+p) + \frac{\partial \psi}{\partial y}(-y+q) + \frac{\partial \psi}{\partial z}(z+r) = \sum_{\substack{j=2,3\\ i=0,\cdots,j}} \zeta_{k,k} x^{j-i} y^j z^i, \quad k = 1,\cdots,5,$$

where p, q and r are polynomials of degree greater than one and $\eta_{k,k}$ and $\zeta_{k,k}$ for $k = 1,\cdots,5$ are polynomials in the coefficients of system (3.1). Since each quantities $\eta_{k,k}$ and $\zeta_{k,k}$ for $k = 1,\cdots,5$ are obstruction of the existence of two analytic first integrals, so they must be equal to zero. These quantities are the following:

1) $\eta_{1,1} = ab - ed$

2) $\eta_{2,2} = bf - hf - hc + ef$

3) $\eta_{3,3} = -6a^2eb - 8a^2b^2 + 6ade^2 - 9adb^2 + 9d^2eb + 8d^2e^2$

4) $\eta_{4,4} = 3ebfa - (9/2)ebca - (3/2)ebgc + 21ebfd + (9/2)ebcd - b^2fa - (9/2)b^2ca -$
$(3/2)b^2gc + 5b^2fd - (3/2)b^2cd - (1/2)b^2gf - h^2fa - (3/2)h^2ca - (1/2)h^2gc -$
$h^2fd - (3/2)h^2cd - (1/2)h^2gf + 4e^2fa + 16e^2fd + 6e^2cd + 2e^2gf +$
$(3/2)ebgf - 4bhfa - 6bhca - 2bhgc - 4bhfd - 6bhcd - 2bhgf - 3ehfa -$
$(9/2)ehca - (3/2)ehgc - 3ehfd - (9/2)ehcd - (3/2)ehgf$

5) $\eta_{5,5} = -3ehf^2 - 3ehkc - 6ehfc - 3ehkf + 6ebfc + 3ebkf - 2bhkc - 4bhfc - 2bhkf +$
$8e^2f^2 + 4b^2f^2 - 2h^2f^2 - 2h^2c^2 + 2e^2kf - 3ehc^2 + 12ebf^2 +$
$4e^2fc + b^2kf - 2bhf^2 - 2bhc^2 + 2b^2fc - 2h^2kc - 4h^2fc - 2h^2kf$

6) $\zeta_{1,1} = -hk + fe$

7) $\zeta_{2,2} = de + dh - db - gb$

8) $\zeta_{3,3} = -3adeb + 2ade^2 - 2adb^2 - 3d^2eb + 8d^2e^2 - 2d^2b^2 - 2bhd^2 - 3ebag - 6ebdg-$

$2bhad - 2bhag - 4bhdg + 3ehad + 6ehdg - 2b^2g^2 + 4h^2d^2 - 3ebg^2 - 2b^2ag-$

$4b^2dg + 2h^2dg - 2bhg^2 + 12ehd^2 + h^2ad + 4e^2dg$

9) $\zeta_{4,4} = -(3/2)ebgc - 3ebfd - (3/2)ebcd - (1/2)b^2gc - b^2fd - (1/2)b^2cd - (3/2)b^2gf-$

$(3/2)h^2gc + 5h^2fd - (1/2)h^2cd - (3/2)h^2gf + 16e^2fd + 2e^2cd + 6e^2gf-$

$(9/2)ebgf - 2bhgc - 4bhfd - 2bhcd - 6bhgf - (3/2)ehgc + 21ehfd + (3/2)ehcd+$

$(9/2)ehgf + 4e^2dk - b^2dk - (3/2)b^2gk - h^2dk - (9/2)h^2gk + 3ehdk-$

$(9/2)ehgk - 3ebdk - (9/2)ebgk - 4bhdk - 6bhgk$

10) $\zeta_{5,5} = 6e^2kf + 8e^2f^2 - 6ehk^2 + 9ehf^2 - 9h^2kf - 8h^2k^2.$

It seems the above quantities generate the entire ideals of coefficients of $\eta_{k,k}$ and $\zeta_{k,k}$ for $k = 1, \cdots, 5$. A factorized Gröbner basis gives the following necessary conditions for integrability.

Theorem 8. *The origin of three dimensional Lotka-Volterra system (3.1) with* $(\lambda, \mu, \nu) = (1, -1, 1)$ *is integrable if and only if one of the following conditions are satisfied:*

1) $ab - de = ac - 2ak + gk = ae + ah - de - eg = af + ak - dk - gk =$
$bd + bg - de - dh = bf - ch - fh + hk = bk - ce + ek - hk =$
$cd + cg - 2dk + fg - gk = ef - hk = 0$

2) $b = d = f = h = 0$

3) $f = g = h = b - e = d - a = 0$

3*) $b = c = d = f - k = e - h = 0$

4) $b = c = d = f = k = 0$

4*) $a = d = g = h = f = 0$

5) $b = e = h = 0.$

Moreover, the system is linearizable if and only if either one of the conditions (2)-(5) or *one of the following holds:*

$$1.1)\, a = c = d = f = g = k = 0$$

$$1.2)\, a = bk - ch = d = e - h = f - k = g = 0$$

$$1.2^*)\, a - d = b - e = c = dh - eg = f = k = 0$$

$$1.3)\, a - g = b - h = c - k = d - g = e - h = f - k = 0.$$

Proof. Cases 3*, 4* and 1.2* are dual to Cases 3, 4 and 1.2 under the transformation $(x,y,z) \mapsto (z,y,x)$, and do not need to be considered separately. The other cases are considered below.

Case 1: If $e \neq 0$, the conditions in this case reduces to

$$ab - ed = hk + ce - ke + bk = kh - ef = ac + gk - 2ak = 0.$$

In addition to the axes, the system has an invariant algebraic plane $\ell = 1 + ax - ey + kz = 0$ with cofactor $L_\ell = ax + ey + kz$, then we have two independent first integrals

$$\phi_1 = xy\ell^{-1-\frac{b}{e}}, \qquad \phi_2 = yz\ell^{-1-\frac{h}{e}}.$$

Now we study the case when $e = 0$. We have several cases.

i) If $b = h = 0$. This is a subcase of Case 5, so we consider the following:

ii) $h = 0, b \neq 0$. In this case, we have $a = f = k = g + d = 0$ and we get an exponential factor $\ell = \exp(dx - by + cz)$ with cofactor $dx + by + cz$, and first integrals $\phi_1 = xy\ell^{-1}$ and $\phi_2 = yz$.

iii) $b = 0, h \neq 0$. In this case, we have $a = d = k = 0$ and $c = f(b/h - 1)$. We get an exponential factor $\ell = \exp(gx - hy + fz)$ with cofactor $gx + hy + fz$. This gives first integrals $\phi_1 = xy$ and $\phi_2 = yz\ell^{-1}$.

iv) $b, h \neq 0$. In this case, we have $a = k = 0$, $c = f(b/h - 1)$ and $g = d(h/b - 1)$. We get an exponential factor $\ell = \exp(dhx - bhy + bfz)$ with cofactor $dhx + bhy + bfz$.

This gives first integrals $\phi_1 = xy\ell^{-\frac{1}{h}}$ and $\phi_2 = yz\ell^{-\frac{1}{b}}$.

Case 1.1: If $e \neq 0$, we have an invariant plane $1 - ey = 0$, and the change of coordinates $(X,Y,Z) = (x(1-ey)^{-\frac{b}{e}}, y(1-ey)^{-1}, z(1-ey)^{-\frac{h}{e}})$ linearizes the systems. When $e = 0$, we replace $(1-ey)^{-\frac{b}{e}}$ and $(1-ey)^{-\frac{h}{e}}$ above by $\exp(by)$ and $\exp(hy)$ respectively.

Case 1.2: When $e \neq 0$, The linearizing change of variables is given by $(X,Y,Z) = (x(1-ey+fz)^{-\frac{b}{e}}, y(1-ey+fz)^{-1}, z(1-ey+fz)^{-1})$. When $e = 0$, then either $b = 0$ and $f \neq 0$, and we have a linearizing change of coordinates $(X,Y,Z) = (x(1+fz)^{c/f}, y(1+fz)^{-1}, z(1+fz)^{-1})$, or $f = 0$ and we linearize by $(X,Y,Z) = (x\exp(by-cz), y, z)$.

Case 1.3: In this case, we have an invariant plane $\ell = 1 + ax - by + fz$ with cofactor $L_\ell = ax + by + fz$ and the linearizing change is $(X,Y,Z) = (x\ell^{-1}, y\ell^{-1}, z\ell^{-1})$.

Case 2: In this case system (3.1) correspond to

$$\dot{x} = x(1+ax+cz), \quad \dot{y} = y(-1+ey), \quad \dot{z} = z(1+gx+kz). \tag{3.11}$$

It is obvious that $\ell = 1 - ey$ is an invariant algebraic plane with cofactor $L_\ell = ey$. Performing the change of coordinates

$$Y = \frac{y}{1-ey},$$

gives

$$\dot{Y} = -Y.$$

Furthermore, the equations

$$\dot{x} = x(1+ax+cz), \qquad \dot{z} = z(1+gx+kz), \tag{3.12}$$

do not depend on y and have a node with two analytic separatrices $x = 0$ and $z = 0$ which is therefore analytically linearizable. Thus, there exists a change of coordinates $X = x(1+O(x,z))$ and $Z = z(1+O(x,z))$ such that

$$\dot{X} = X, \qquad \dot{Z} = Z.$$

The first integrals can be obtained from re-expressing the functions $\phi_1 = XY$ and $\phi_2 = YZ$ in the original x, y and z coordinates.

Remark 6. *Since the eigenvalues of system (3.12) are $\lambda_1 = \lambda_2 = 1$ and $\frac{\lambda_1}{\lambda_2} = n = 1 \in \mathbb{N}$, then the normal form of system (3.12) is*

$$\dot{x} = x,$$
$$\dot{z} = nz + \alpha x^n,$$

If $\alpha = 0$, then the system is linear. If $\alpha \neq 0$, then the curve $x = 0$ is the unique analytic integral curve through the origin, but we have two separatrices $x = 0$ and $z = 0$, so we get $\alpha = 0$ and the system is linearizable. For more detail see (Christopher and Rousseau, 2004).

Case 3: The corresponding system is

$$\dot{x} = x(1 + ax + by + cz), \quad \dot{y} = y(-1 + ax + by), \quad \dot{z} = z(1 + kz). \tag{3.13}$$

The transformation

$$X = \frac{x}{1 + ax - by}$$

gives the subsystem

$$\dot{X} = X(1 + cz - acXz), \qquad \dot{z} = z(1 + kz).$$

As in Case 2, this is a linearizable node, and hence there is a change of coordinates $\tilde{X} = X(1 + O(X,z))$ and $Z = z(1 + O(X,z))$ such that

$$\dot{\tilde{X}} = \tilde{X}, \qquad \dot{Z} = Z.$$

Similarly, by choosing

$$Y = \frac{y}{1 + ax - by},$$

we get

$$\dot{Y} = Y(-1 - acX(\tilde{X}, Z)\, z(Z)).$$

It is sufficient to find a function $\ell(\tilde{X}, Z)$ such that

$$\dot{\ell}(\tilde{X}, Z) = X(\tilde{X}, Z) z(Z),$$

then the substitution

$$\tilde{Y} = Ye^{ac\ell}$$

will gives $\dot{\tilde{Y}} = -\tilde{Y}$.

To complete this case, it remain to show that ℓ is convergent. Write

$$\ell(\tilde{X}, Z) = \sum_{i+j>0} a_{ij} \tilde{X}^i Z^j,$$

then

$$\dot{\ell}(\tilde{X}, Z) = \sum_{i+j>0} (i+j) a_{ij} \tilde{X}^i Z^j = \sum_{i+j>0} d_{ij} \tilde{X}^i Z^j$$

where

$$\sum d_{ij} \tilde{X}^i Z^j = X(\tilde{X}, Z) z(Z).$$

Hence

$$a_{ij} = \frac{d_{ij}}{i+j},$$

and the convergence of ℓ is clear. We obtain two independent first integrals by re-expressing $\phi_1 = \tilde{X}\tilde{Y}$ and $\phi_2 = \tilde{Y}Z$ in x, y, z coordinates.

Case 4: The system can be written as

$$\dot{x} = x(1 + ax), \quad \dot{y} = y(-1 + ey), \quad \dot{z} = z(1 + gx + hy),$$

Apart from the invariant axes, the system has two invariant algebraic planes $1 + ax = 0$ and $1 - ey = 0$ with cofactors ax and ey respectively.Then the linearizing change of coordinates is given by

$$(X,Y,Z)=\left(\frac{x}{1+ax},\frac{y}{1-ey},\frac{z}{(1+ax)^{\frac{g}{a}}(1-ey)^{\frac{h}{e}}}\right).$$

When $a=0$, we have an exponential factor $\exp(x)$, and we replace $(1+ax)^{g/a}$ by $\exp(gx)$. Similarly, when $e=0$, we have an exponential factor $\exp(y)$ and replace $(1-ey)^{h/e}$by $\exp(-hy)$.

Case 5: Then the system reduces to

$$\dot{x}=x(1+ax+cz),\quad \dot{y}=y(-1+dx+fz),\quad \dot{z}=z(1+gx+kz).$$

The subsystem

$$\dot{x}=x(1+ax+cz),\qquad \dot{z}=z(1+gx+kz),$$

is independent of y and gives a linearizable node. Hence there exists a change of coordinates $X=x(1+O(x,z))$ and $Z=z(1+O(x,z))$ such that

$$\dot{X}=X,\qquad \dot{Z}=Z.$$

The remaining equation is

$$\frac{\dot{y}}{y}=(-1+dx(X,Z)+fz(X,Z)).$$

We are looking for a function $\ell(X,Z)$ such that

$$\dot{\ell}(X,Z)=(dx(X,Z)+fz(X,Z)),\tag{3.14}$$

then the transformation $Y=ye^{-\ell}$ will gives $\dot{Y}=-Y$.

Writing

$$\ell(X,Z)=\sum_{i+j>0}b_{ij}X^iZ^j,$$

we have to solve

$$\dot{\ell} = \sum_{i+j>0} (i+j) b_{ij} X^i Z^j = dx(X,Z) + fz(X,Z) = \sum_{i+j>0} a_{ij} X^i Z^j,$$

for which it is clear the solution exists and is analytic. We find two independent first integrals from XY and YZ as in Case 3. □

3.3.2 $(2:-1:1)$–resonance

In this case, we seek two independent analytic first integrals of (3.1):

$$\phi_1 = xy^2(1+O(x,y,z)) \quad \text{and} \quad \phi_2 = yz(1+O(x,y,z)).$$

We express ϕ and ψ as power series up to terms of order 10 and compute the obstructions to them forming first integrals. A factorized Gröbner basis is obtained giving the following necessary conditions for integrability.

Theorem 9. *The origin of system (3.1) with $(\lambda,\mu,\nu) = (2,-1,1)$ is integrable if and only if one of the following conditions holds:*

1) $ef - hk = ab - ah - de + eg = ac - 3ak + 2gk = ae + ah - de - eg =$
$af + ak - dk - gk = bd + bg - de - 2dh + eg = bf - ch - 2fh + 2hk =$
$bk - ce + 2ek - 2hk = cd + cg - 3dk + 2fg - gk = 0$

2) $b + e = c = d = f = k = 0$

3) $b + h = c + k = d = e - h = f - k = 0$

4) $a - d = b - e = f = g = h = 0$

5) $b + e = d = f = h = 0$

6) $a = d = f = g = h = 0$

7) $b = c = e - h = f - k = 0$

8) $b = e = h = 0$

$$9)\, b = f = h = 0$$

$$10)\, b = c = f = k = 0$$

$$11)\, b = c - 4k = e + h = f + k = 0.$$

Moreover, the origin is linearizable if and only if the system satisfied either one of the conditions (2)-(10) or one of the following conditions:

$$1.1)\, a = c = d = f = g = k = 0$$

$$1.2)\, a - d = b - e = c = dh - eg = f = k = 0$$

$$1.3)\, a = bk - ch = d = e - h = f - k = g = 0$$

$$1.4)\, a - g = b - h = c - k = d - g = e - h = f - k = 0.$$

Proof. We treat the cases individually.

Case 1: If $e \neq 0$, the conditions emerge as

$$fe - hk = ae + ah - de - eg = kb + 2ke - 2hk = ab + ae - 2de = 0.$$

In this case, we have an invariant surface $\ell = 1 + \frac{a}{2}x - ey + kz = 0$ with cofactor $L_\ell = ax + ey + kz$. From this we can construct two independent first integrals

$$\phi_1 = xy^2\ell^{-2-\frac{b}{e}} \quad \text{and} \quad \phi_2 = yz\ell^{-1-\frac{h}{e}}.$$

When $e = 0$ and $h \neq 0$ then we have an exponential factor $\ell = \exp((d+g)x/2 - hy + fz)$ with cofactor $(d+g)x + hy + fz$ with corresponding first integrals

$$\phi_1 = xy^2\ell^{-\frac{b}{h}} \quad \text{and} \quad \phi_2 = yz\ell^{-1}.$$

If $e = 0$ and $h = 0$, then we can assume $b \neq 0$ since otherwise we are in Case 8. The conditions then gives an exponential factor $\ell = \exp(gx + by - cz)$ with cofactor $2gx - by - cz$ and corresponding first integrals $\phi_1 = xy^2\ell$ and $\phi_2 = yz$.

Case 1.1: The system is

$$\dot{x} = x(2 + by), \quad \dot{y} = y(-1 + ey), \quad \dot{z} = z(1 + hy),$$

If $e \neq 0$, the linearizing change of coordinates is given by

$$(X,Y,Z) = (x(1-ey)^{-\frac{b}{e}}, y(1-ey)^{-1}, z(1-ey)^{-\frac{h}{e}}).$$

When $e = 0$, we have an exponential factor $\exp(y)$, then replace $(1-ey)^{-\frac{b}{e}}$ and $(1-ey)^{-\frac{h}{e}}$ by $\exp(by)$ and $\exp(hy)$ respectively.

Case 1.2: The system is

$$\dot{x} = x(2 + ax + by), \quad \dot{y} = y(-1 + ax + by), \quad \dot{z} = z(1 + \frac{ah}{b}x + hy).$$

For $b \neq 0$, a linearizing change of variables is given by

$$(X,Y,Z) = (x(1+\frac{a}{2}x-by)^{-1}, y(1+\frac{a}{2}x-by)^{-1}, z(1+\frac{a}{2}x-by)^{-\frac{h}{b}}).$$

When $b = 0$, either $h = 0$ and $a \neq 0$, and we can linearize by

$$(X,Y,Z) = (x(1+\frac{a}{2}z)^{-1}, y(1+\frac{a}{2}z)^{-1}, z(1+\frac{a}{2}z)^{-g/a}),$$

or $a = 0$ and we linearize by $(X,Y,Z) = (x, y, z\exp(by - gx))$.

Case 1.3: This case is exactly similar to case 1.2.

Case 1.4: The system is

$$\dot{x} = x(2 + ax + by + fz), \quad \dot{y} = y(-1 + ax + by + fz), \quad \dot{z} = z(1 + ax + by + fz).$$

In this case we have an invariant surface $\ell = 1 + \frac{a}{2}x - by + fz$ with cofactor $L_\ell = ax + by + fz$ and the linearizing change is give by $(X,Y,Z) = (x\ell^{-1}, y\ell^{-1}, z\ell^{-1})$.

Case 2: The system (3.1) reduces to

$$\dot{x} = x(2+ax+by), \quad \dot{y} = y(-1-by), \quad \dot{z} = z(1+gx+hy). \tag{3.15}$$

In this case we have invariant planes $\ell_1 = 1+\frac{a}{2}x+by$, $\ell_2 = 1+\frac{a}{2}x-\frac{ab}{2}xy$ and $\ell_3 = 1+by$ with cofactors $L_{\ell_1} = ax-by$, $L_{\ell_2} = ax$ and $L_{\ell_3} = -by$.

The change of coordinates

$$\big(X,Y,Z\big) = \big(x\ell_1^{-1}\ell_3^2, y\ell_3^{-1}, z\ell_2^{-\frac{g}{a}}\ell_3^{\frac{h}{b}}\big)$$

linearizes the system (3.15).

When $a = 0$, the system has the form

$$\dot{x} = x(2+by), \quad \dot{y} = y(-1-by), \quad \dot{z} = z(1+gx+hy).$$

which has an invariant algebraic plane $\ell = 1+by$ and an exponential factor $E = \exp(-\frac{g}{2}x(1-by))$ with respective cofactors $-by$ and $-gx$. Then we replace $\ell_2^{-g/a}$ with $\exp(-x(1-by)g/2)$.

When $b = 0$, the system has an invariant algebraic plane $\ell = 2+ax$ and exponential factor $E = \exp(y)$ with cofactors ax and $-y$ respectively. Now, the transformation

$$(X,Y,Z) = (x\ell^{-1}, y, z\ell^{-\frac{h}{a}}E^h)$$

linearizes the system.

Finally, when $a = b = 0$, the system has two exponential factors $\exp(x)$ and $\exp(y)$. The change of variables

$$(X,Y,Z) = (x, y, z\exp(-ax+by))$$

linearizes the system.

Case 3: By the integrability conditions of this case we arrive the system

$$\dot{x} = x(2 + ax - ey - fz), \quad \dot{y} = y(-1 + ey + fz), \quad \dot{z} = z(1 + gx + ey + fz). \tag{3.16}$$

After the change of coordinates $(X,Y,Z) = (x,xy,xz)$, the system (3.16) becomes

$$\dot{X} = 2X + aX^2 - eY - fZ, \quad \dot{Y} = Y(1 + aX), \quad \dot{Z} = Z(3 + (a+g)X). \tag{3.17}$$

The critical point at the origin of (3.17) is in the Poincaré domain and hence is linearizable via an analytic change of coordinates which can be chosen to be of the form

$$\big(\tilde{X}, \tilde{Y}, \tilde{Z}\big) = \big(X - eY + fZ + O(2), Y(1 + O(1)), Z(1 + O(1))\big).$$

The two first integrals $\tilde{\phi} = \tilde{X}^{-1}\tilde{Y}^2$ and $\tilde{\psi} = \tilde{X}^{-2}\tilde{Y}\tilde{Z}$ of the linear system pull back to first integrals of (3.16) in the form

$$\phi_1 = xy^2(1 + O(1)), \quad \text{and} \quad \phi_2 = yz(1 + O(1)).$$

The expression $\xi = x^g y^{a+g} z^{-a}$ satisfies $\dot{\xi} = (a - 2g)\xi$ and so the system is linearizable from Theorem 6

Case 4: The system (3.1) takes the form

$$\dot{x} = x(2 + ax + by + cz), \qquad \dot{y} = y(-1 + ax + by), \qquad \dot{z} = z(1 + kz). \tag{3.18}$$

Perform the transformation $X = x/(2 + ax - 2by)$, the first equation and third equation of (3.20) decouples the terms in X and z to give

$$\dot{X} = X\big(2 + (1 - \frac{a}{2}X)cz\big), \qquad \dot{z} = z(1 + kz), \tag{3.19}$$

which a linearizable node. Thus, there is a linearizing change of coordinates

$$\tilde{X} = \tilde{X}(X, z) = x(1 + O(1)) \quad \text{and} \quad \tilde{Z} = \tilde{Z}(z) = z(1 + O(1))$$

which brings (3.19) to the form

$$\dot{\tilde{X}} = 2\tilde{X}, \qquad \dot{\tilde{Z}} = \tilde{Z}.$$

Similarly, taking $Y = y/(1 + ax - by)$, we get

$$\dot{Y} = Y(-1 - \frac{a}{2}cXz).$$

It is sufficient to find $\ell(X, Y)$ such that

$$\dot{\ell}(X, Z) = X(\tilde{X}, \tilde{Z})z(\tilde{Z}).$$

Then the substitution $\tilde{Y} = Ye^{ac\ell/2}$ will linearize the system. The completion of this case requires the convergence of ℓ. To do so, writing

$$\ell(X, Z) = \sum_{i+j>0} a_{ij}X^iZ^j, \quad x(X, Z) = \sum b_{ij}X^iZ^j, \quad x(X, Z)z(Z) = \sum d_iZ^i,$$

we find that

$$\dot{\ell}(X, Z) = \sum_{i+j>0} (2i + j)a_{ij}X^iZ^j = \sum_{i+j>0} d_{ij}X^iZ^j.$$

Hence, we can take $a_{ij} = d_{ij}/(2i + j)$ to find ℓ. The convergence of $\sum_{i+j>0} d_{ij}X^iZ^j$, clearly implies the convergence of ℓ.

Two independent first integrals of the original system can be obtained by pulling back the first integrals $\phi_1 = \tilde{X}\tilde{Y}^2$ and $\phi_2 = \tilde{Y}\tilde{Z}$ of the linearized system.

Case 5: Using the integrability conditions of this case, system (3.1) gives

$$\dot{x} = x(2 + ax + by + cz), \qquad \dot{y} = y(-1 - by), \qquad \dot{z} = z(1 + gx + kz). \tag{3.20}$$

Performing a change of coordinates

$$(X,Y,Z) = (\frac{x}{1+by}, y, \frac{z}{1+by}),$$

system (3.1) appears as

$$\dot{X} = X(2+aX+cZ)(1+bY), \quad \dot{Y} = -Y(1+bY), \quad \dot{Z} = Z(1+gX+kZ)(1+bY). \quad (3.21)$$

After rescaling the system (3.21) by $(1+bY)$, we have $\dot{Y} = -Y$ and the first and third equation gives a linearizable node. This implies the original system is integrable. However, since $\xi = y(1+by)^{-1}$ satisfies $\dot{\xi} = -\xi$, then the system (3.20) must also be linearizable by Theorem 6.

Case 6: The reduced system is

$$\dot{x} = x(2+by+cz), \qquad \dot{y} = y(-1+ey), \qquad \dot{z} = z(1+kz).$$

A linearizing change of coordinates is given by

$$(X,Y,Z) = (x(1-ey)^{-\frac{b}{e}}(1+kz)^{-\frac{c}{k}}, y(1-ey)^{-1}, z(1+kz)^{-1}).$$

When $k=0$, we have an exponential factor $\exp(z)$, and we replace $(1+kz)^{-\frac{c}{k}}$ by $\exp(-cz)$. Similarly, when $e=0$, we have an exponential factor $\exp(y)$ and replace $(1-ey)^{-\frac{b}{e}}$ by $\exp(by)$.

Case 7: System (3.1) reduces to

$$\dot{x} = x(2+ax), \qquad \dot{y} = y(-1+dx+ey+fz), \qquad \dot{z} = z(1+gx+ey+fz). \qquad (3.22)$$

The system has an invariant algebraic plane $\ell = 1+\frac{a}{2}x$ with cofactor $L_\ell = ax$ yielding a first integral

$$\phi = x^{-1}y^{-1}z\ell^{\frac{d-g+a}{a}}.$$

We also have an inverse Jacobi multiplier

$$IJM = x^{\frac{5}{2}} y^3 \ell^{-\frac{1}{2}-\frac{2d}{a}+\frac{g}{a}}.$$

When $a=0$, an exponential factor $\exp(x)$ will appears and replace $\ell^{\frac{d-g+a}{a}}$ and $(2+ax)^{-\frac{1}{2}-\frac{2d}{a}+\frac{g}{a}}$ by $\exp(\frac{d-g}{2}x)$ and $\exp((\frac{g}{2}-d)x)$ respectively. Theorem 5 therefore guarantees the existence of a second first integral of the form $\psi = x^{-3/2}y^{-2}z(1+O(1))$. From these two integrals it is easy to construct integrals of the form required as $\phi_1 = \phi^2\psi^{-2} = xy^2(1+\cdots)$ and $\phi_2 = \phi^3\psi^{-2} = yz(1+\cdots)$. Since $\xi = x\ell^{-1}$ satisfies $\dot{\xi} = 2\xi$, the system is linear by Theorem 6.

For interest, we show how such a first integral can also be obtained by quadratures.

Let $\phi = C$ where C is a constant. Then

$$z = xy(2+ax)^{-\frac{(d-g+a)}{a}}C.$$

The inverse integrating factor on z is given by

$$IJM\frac{\partial \phi}{\partial z} = x^{\frac{3}{2}}y^2(2+ax)^{\frac{1}{2}-\frac{d}{a}}.$$

Thus on z we have two dimensional system as

$$\begin{aligned}
\dot{x} &= x(2+ax) = p(x,y), \\
\dot{y} &= y\big(-1+dx+ey+fCxyz(2+ax)^{-\frac{d-g+a}{a}}\big) = q(x,y).
\end{aligned} \tag{3.23}$$

We write the system (3.23) as a 1-form

$$x(2+ax)dy - y\big(-1+dx+ey+fCxyz(2+ax)^{-\frac{d-g+a}{a}}\big)dx = 0. \tag{3.24}$$

Dividing both sides of equation (3.24) by the inverse integrating factor, and after some simplification we get:

$$d\Big(-x^{-\frac{1}{2}}y^{-1}(2+ax)^{\frac{d}{a}+\frac{1}{2}}+\int_0^x \frac{e+fC\xi(2+a\xi)^{-\frac{d-g+a}{a}}}{\xi^{\frac{3}{2}}(2+a\xi)^{\frac{1}{2}-\frac{d}{a}}}\,d\xi\Big)=0. \tag{3.25}$$

One can evaluate the value of the integral above by expanding it and then evaluate the integral term-by-term. Substituting back this integral to (3.25) and integrate it, we obtain a second integral of the form $\tilde{\psi}=x^{-1/2}y^{-1}\hat{\psi}$, where $\hat{\psi}$ is analytic and $\hat{\psi}(0,0,0)\neq 0$. Therefore, the desired first integrals are

$$\phi_1=\tilde{\psi}^{-2} \qquad \phi_2=\phi\,\tilde{\psi}^{-2}.$$

Case 8: Then the system reduces to

$$\dot{x}=x(2+ax+cz), \qquad \dot{y}=y(-1+dx+fz), \qquad \dot{z}=z(1+gx+kz).$$

The first and third equations gives a linearizable node, so it suffices to find $\ell(X,Y)$ such that

$$\dot{\ell}(X,Z)=dx(X,Z)+fz(X,Z) \tag{3.26}$$

The transformation $Y=ye^{-\ell}$ then linearizes the second equation.

Let

$$\ell(X,Z)=\sum_{i+j>0} b_{ij}X^iZ^j,$$

and write the right hand side of equation (3.26) as $\sum_{i+j>0}a_{ij}X^iZ^j$. Since

$$\dot{\ell}=\sum_{i+j>0}(2i+j)b_{ij}X^iZ^j,$$

then one can choose $b_{ij}=a_{ij}/(2i+j)$ to find ℓ. The convergence follows from the convergence of $\sum_{i+j>0}a_{ij}X^iZ^j$.

Case 9: The system (3.1) can be written as

$$\dot{x} = x(2+ax+cz), \qquad \dot{y} = y(-1+dx+ey), \qquad \dot{z} = z(1+gx+kz). \tag{3.27}$$

The first and third equations in (3.27) give a linearizable node. To linearize the second equation, we seek an invariant surface of the form $\ell + \chi y = 0$ with cofactor $dx + ey$ where $\ell = \ell(X,Z)$, $\chi = \chi(X,Z)$, and $\ell(0) = 1$. The change of variable $Y = \frac{y}{\ell+\chi y}$ will linearize the second equation.

By equation (2.13), we get

$$\dot{\chi} y - \chi y = \ell(dx+ey) - \dot{\ell}.$$

To find such ℓ and χ to satisfy this equation, we therefore need to solve

$$\dot{\chi} - \chi = e\,\ell, \qquad \dot{\ell} = d\,x\,\ell. \tag{3.28}$$

First to find ℓ, we write $\ell = e^{\psi}$ and solve $\dot{\psi} = dx$. Let $\psi = \sum_{i+j>0} c_{ij} X^i Z^j$, then

$$\sum_{i+j>0} (2i+j) c_{ij} X^i Z^j = dx(X,Z) = dX + \sum_{i+j>1} d_{ij} X^i Z^j,$$

for some d_{ij}. Clearly, $c_{10} = \frac{d}{2}$, $c_{01} = 0$, $c_{ij} = \frac{d_{ij}}{2i+j}$ for $i+j > 1$. The convergence of $\sum_{i+j>1} d_{ij} X^i Z^j$, guarantees the convergence of ψ and hence ℓ. Furthermore, it is clear that ℓ will contain no term in Z.

Now, writing $\ell = \sum b_{ij} X^i Z^j$, and noting that $a_{01} = 0$, we find that $\chi = \sum \frac{e}{2i+j-1} a_{ij} X^i Z^j$ gives a convergent expression for χ.

Case 10: In this case the system (3.1) reduces to

$$\dot{x} = x(2+ax), \qquad \dot{y} = y(-1+dx+ey), \qquad \dot{z} = z(1+gx+hy). \tag{3.29}$$

The transformation

$$X = \frac{x}{1+\frac{a}{2}x}, \qquad Y = \frac{y}{\ell + \chi y} \tag{3.30}$$

would linearize the first and the second equations in (3.29) if $\ell + \chi y = 0$ were the defining equation of an invariant algebraic surface with cofactor $dx + ey$ where $\ell = \ell(X)$, $\chi = \chi(X)$. From equation (2.13) we see that this condition is equivalent to

$$\dot{\chi} y - \chi y = \ell(dx+ey) - \dot{\ell} \quad \Longleftrightarrow \quad \dot{\chi} - \chi = e\ell, \quad \dot{\ell} = dx\ell.$$

The second of these equations is clearly satisfied if we take $\ell = (1+ax/2)^{d/a} = (1 - aX/2)^{-d/a}$ (when $a = 0$ let $\ell = e^{dx/2} = e^{dX/2}$).

To solve

$$\dot{\chi} - \chi = \ell e, \tag{3.31}$$

writing $\chi = \sum a_i X^i$ and $\ell = \sum b_i X^i$, equation (3.31) is satisfied if we set $a_i = b_i/(2i-1)$. The resulting function χ is clearly convergent.

To linearize the third equation, it is suffices to find $\gamma(X,Y)$ such that

$$\dot{\gamma} = gx(X) + hy(X,Y).$$

Thus the substitution $Z = z\exp(-\gamma)$ is the transformation sought. Writing,

$$\gamma = \sum_{i+j>0} c_{ij} X^i Y^j, \quad x(X) = \sum_{i>0} d_i X^i, \quad y(X,Y) = \sum_{i+j>0} e_{ij} X^i Y^j, \tag{3.32}$$

we see that we require

$$\sum (2i-j) c_{ij} X^i Y^j = \sum g d_i X^i + \sum h e_{ij} X^i Y^j = \sum f_{ij} X^i Y^j.$$

If $y(X,Y)$ contains no terms of the form $X^k Y^{2k}$, then we can set $c_{ij} = (gd_i + he_{ij})/(2i-j)$ for $2i \neq j$, and $c_{ij} = 0$ otherwise, to find a convergent expression for γ. Therefore, it just remains to show that the inverse transformation $y = y(X,Y)$ from (3.30) contains no term like $(XY^2)^n$. From (3.30),

$$y = \frac{\ell Y}{1-\chi Y} = \sum \ell \chi^n Y^{n+1},$$

suppose $n+1 = 2m$ for some m. It is suffices to show that $\ell \chi^n$ contains no term X^m or in another word $\ell \chi^{2m-1}$ has no term X^m. Since

$$\dot{\chi} = \frac{d\chi}{dt} = 2X\frac{d\chi}{dX},$$

then, from equation (3.31), we obtain

$$\frac{2X}{2m}\frac{d\chi^{2m}}{dX} - \chi^{2m} = e\,\ell\,\chi^{2m-1}. \tag{3.33}$$

Let A denote the coefficient of X^m in (3.33), then either $A = \frac{2X}{2m}\frac{dX^m}{dX} - X^m = 0$, or $e = 0$, in which case $\chi \equiv 0$. Thus y in (3.30) contains no $(XY^2)^n$ and we have established the existence of a linearizing change of coordinates.

Case 11: After substitution integrability conditions, system (1) becomes

$$\dot{x} = x(2+ax-4fz), \qquad \dot{y} = y(-1+dx+ey+fz), \qquad \dot{z} = z(1+gx-ey-fz).$$

By using the transformation $w = yz$ the system above becomes

$$\dot{x} = x(2+ax-4fz), \qquad \dot{w} = wx\tilde{d}, \qquad \dot{z} = z(1+gx-fz)-ew, \tag{3.34}$$

where $\tilde{d} = d+g$. In this case we seek ψ such that

$$\psi = \psi_0 + w\psi_1 + w^2\psi_2 + \cdots$$

and $\dot{\psi} = x$, where $\psi_i = \psi_i(x,z)$, then $\phi = we^{-\tilde{d}\psi}$ is a first integral.

We write the associated vector field as

$$\mathcal{X} = \mathcal{X}_0 + \mathcal{X}_1 + \mathcal{X}_\omega,$$

where

$$\mathcal{X}_0 = x(2+ax-4fz)\frac{\partial}{\partial x} + z(1+gx-fz)\frac{\partial}{\partial z}, \qquad \mathcal{X}_1 = -ew\frac{\partial}{\partial z},$$

and

$$\mathcal{X}_\omega = wx\tilde{d}\frac{\partial}{\partial w}.$$

From $\mathcal{X}\psi = x$, we get:

$$\mathcal{X}_0\psi_0 = x, \qquad \mathcal{X}_0\psi_k + kx\tilde{d}\,\psi_k = -\mathcal{X}_1\psi_{k-1} \quad (k>0). \tag{3.35}$$

We now solve the equation (3.35). To prove this, we first show that for every $B = \sum_{i+j>0} b_{ij}x^iz^j$, there exists an $A = \sum_{i+j>0} a_{ij}x^iz^j$, such that $(\mathcal{X}_0 + k\tilde{d}x)A = B$. Since

$$(\mathcal{X}_0+k\tilde{d}x)A = \sum_{i+j>0}(2i+j)a_{ij}x^iz^j + \sum_{i+j>0}(ia+jg+k\tilde{d})a_{ij}x^{i+1}z^j - \sum_{i+j>0}(4i+j)fa_{ij}x^iz^{j+1},$$

we find that the a_{ij} must satisfy

$$(2i+j)a_{i,j} + ((i-1)a+jg+k\tilde{d})a_{i-1,j} - (4i+j-1)fa_{i,j-1} = b_{i,j}.$$

There is clearly no obstruction to solving these equations recursively, and standard majorization techniques imply that the resulting series is convergent.

Now in (3.35) we can solve the equations term by term provided that the right hand side of the equations have no constant term. However, this follows from the stronger observation that we can choose ψ_i in (3.35) to be divisible by x. To show this, we can suppose by induction that x divides the right hand side of (3.35) (for $k=0$ this is immediate). This implies that

$$z(1+gx-fz)\frac{\partial\psi_k}{\partial z},$$

is divisible by x and this in turn shows that $\frac{\partial \psi_k}{\partial z}$ is divisible by x. Writing $\psi_k(x,z) = g(z)+xh(x,z)$, we see that $g'(z)=0$, so that g is a constant. Clearly $\psi_k - g$ also satisfies (3.35) and we proceed by induction.

Thus, after standard majorization arguments, we find a first integral $\phi_1 = yze^{-\tilde{d}\psi}$. Now in this case we have an inverse Jacobi multiplier

$$x^{\frac{3}{2}} y^{1+\frac{g+a/2}{\tilde{d}}} z^{\frac{g+a/2}{\tilde{d}}}.$$

Theorem 5 therefore guarantees a second first integral

$$\psi = x^{-\frac{1}{2}} y^{-\frac{g+a/2}{\tilde{d}}} z^{1-\frac{g+a/2}{\tilde{d}}} \big(1+O(1)\big),$$

from which we deduce the following first integral

$$\phi_2 = \phi_1^{\frac{2d-a}{\tilde{d}}} \psi^{-2} = xy^2(1+O(1)).$$

When $\tilde{d}=0$ then $x^{3/2}ye^{(g+a/2)\psi}$ is an inverse Jacobi multiplier and we proceed as before. □

3.3.3 $(1:-2:1)$–resonance

For $(1:-2:1)$–resonance, we seek two independent analytic first integrals of (3.1):

$$\phi_1 = x^2y(1+O(x,y,z)) \quad \text{and} \quad \phi_2 = yz^2(1+O(x,y,z)).$$

We express ϕ_1 and ϕ_2 as power series up to terms of order 12 and compute the obstructions to them forming first integrals. A factorized Gröbner basis is obtained giving the following necessary conditions for integrability.

Theorem 10. *The origin of the three dimensional Lotka–Volterra system (3.1) with* $(\lambda,\mu,\nu) = (1,-2,1)$ *is integrable if and only if on of the following conditions holds:*

1) $ab + ah - de - eg = ac - 2ak + gk =$

$ae + 2ah - de - 2eg = af + 2ak - dk - 2gk =$

$bd + 2bg - de - dh - eg = bf + ce - 2ch - ek - fh + 2hk =$

$bk - ce + ek - hk = cd + 2cg - 2dk + fg - 2gk =$

$ef + ek - 2hk = 0$

2) $a = d = f + k = g = h = 0$

3) $a - 2g = b + e = c - 3k = d = f + k = h = 0$

4) $c = d = f = g = 0$

5) $a = d = f = g = 0$

5^*) $c = d = f = k = 0$

6) $b - h = d = f = 0$

7) $a - g = c - k = d = f = 0$

8) $b + h = c + k = d = e - h = f - k = g = 0$

9) $b + 3h = c - 3k = d = e + 2h = f + 2k = g = 0$

10) $b = c = d = e - h = f - k = 0$

10^*) $a - d = b - e = f = g = h = 0$

11) $b = e = h = 0$

12) $a + d = b = cd - 2dk - gk = f + k = h = 0$

13) $a - d = b - e = f + k = g = h = 0$

13^*) $a + d = b = c = e - h = f - k = 0$

14) $c - 2k = d = f + k = g = h = 0$

14^*) $2a - g = b = c = 2d + g = f = 0$

15) $a + g = b + 3h = c - 3k = d = e + 2h = f + 2k = 0$

16) $3a - g = b = 2c - k = 3d + g = e + h = f = 0$

17) $a + g = b + h = c = d + g = e + h = f = 0$

18) $3a - g = 3b + h = c = 3d + 2g = 3e - 2h = f = 0$

$$19)\; 3a-g=3b+h=c+k=3d+2g=3e-2h=f=0$$

$$20)\; b=d=f+k=g=h=0$$

$$20^{*})\; a+d=b=c=f=h=0$$

$$21)\; a+d=b=c=f=k=0.$$

Moreover, the system is linearizable if and only if either one of the conditions (2)-(8), (10)-(14), (16)-(17), (20)-(21) or one of the following holds:

$$1.1)\; a=bk-ch=d=e-h=f-k=g=0$$

$$1.1^{*})\; a-d=b-e=c=dh-eg=f=k=0$$

$$1.2)\; a-g=b-h=c-k=d-g=e-h=f-k=0.$$

Proof. Cases 1*, 5*, 10*, 13*, 14* and 20.1* are dual to Cases 1, 5, 10, 13, 14 and 20.1 under the transformation $(x,z)\mapsto(z,x)$, and we do not consider them further. The other cases are considered below.

Case 1: If $e\neq 0$, the integrable conditions appears as

$$ac-2ak+gk=ef+ek-2hk=ab+ah-de-eg=ae+de-2ab=0.$$

The system has an invariant algebraic plane $\ell=1+ax-\frac{e}{2}y+kz=0$ with cofactor $L_\ell=ax+ey+kz$ and produces two independent first integrals $\phi_1=x^2y\ell^{-1-\frac{2b}{e}}$ and $\phi_2=yz^2\ell^{-1-\frac{2h}{e}}$.

If $e=0$, we have several sub cases:

i) $\mathbf{k\neq 0}$: We have $b=h=0$ and the system has first integrals $\phi_1=x^2y\ell^{-(\frac{f+2c}{k})}$ and $\phi_2=yz^2\ell^{-2-\frac{f}{k}}$.

ii) $\mathbf{h\neq 0}$: We have $a=k=0$ and we have an exponential factor $E=\exp(-(d+2g)x+hy-fz)$ with cofactor $L_E=-(d+2g)x-2hy-fz$ which yields first integrals $\phi_1=x^2yE^{\frac{b}{h}}$ and $\phi_2=yz^2E$.

iii) $\mathbf{h=k=0}$: If $b=0$, we are in Case 11, so we assume that $b\neq 0$ which implies that $a=d+2g=f=0$. Then there exists an exponential factor $E=\exp(2gx+by-2cz)$ with cofactor $L_E = 2gx - 2by - 2cz$ which yields two first integrals $\phi_1 = x^2 y E$ and $\phi_2 = yz^2$.

Case 1.1: The system appears as

$$\dot{x} = x(1 + by + \frac{bf}{e}z), \quad \dot{y} = y(-2 + ey + fz), \quad \dot{z} = z(1 + ey + fz).$$

If $e \neq 0$, the change of coordinates $(X,Y,Z) = (x(1 - \frac{e}{2}y + fz)^{-\frac{b}{e}}, y(1 - \frac{e}{2}y + fz)^{-1}, z(1 - \frac{e}{2}y + fz)^{-1})$ linearizes the system. When $e = 0$, then either $k = 0$ or $b = 0$. In the first case, we can linearize via $(X,Y,Z) = (x\exp(\frac{1}{2}by - cz), y, z)$, and in the second, taking $k \neq 0$, we can linearize via $(X,Y,Z) = (x(1+kz)^{-\frac{c}{k}}, y(1+kz)^{-1}, z(1+kz)^{-1})$.

Case 1.2: The system (3.1) becomes

$$\dot{x} = x(1 + ax + by + fz), \quad \dot{y} = y(-2 + ax + by + fz), \quad \dot{z} = z(1 + ax + by + fz).$$

In this case we have an invariant plane $\ell = 1 + ax - \frac{b}{2}y + cz$ with cofactor $L_\ell = ax + by + cz$ and the linearizing change is $(X,Y,Z) = (x\ell^{-1}, y\ell^{-1}, z\ell^{-1})$.

Case 2: The system (3.1) reduces to

$$\dot{x} = x(1 + by + cz), \quad \dot{y} = y(-2 + ey + fz), \quad \dot{z} = z(1 - fz).$$

The system has an invariant algebraic plane $\ell = 1 - \frac{e}{2}y - fz = 0$ with cofactor $L_\ell = ey - fz$. The substitution

$$Y = y\ell^{-1}(1 - fz)^2, \qquad Z = \frac{z}{1 - fz}. \tag{3.36}$$

gives $\dot{Y} = -2Y$ and $\dot{Z} = Z$, and the changes of coordinates $X = xe^{-\phi}$ will gives $\dot{X} = X$ if and only if

$$\dot{\phi}(Y,Z) = -2Y\frac{\partial \phi}{\partial Y} + Z\frac{\partial \phi}{\partial Z} = b\,y(Y,Z) + c\,z(Z). \tag{3.37}$$

An obstruction to the existence of ϕ is possible only if $y = y(Y,Z)$ in (3.37) contains terms of the form $(YZ^2)^n$. According to (3.36), one can find

$$y = \frac{Y(1+fZ)}{1+\frac{e}{2}Y(1+fZ)^2} = \sum_{n\geq 1}(-\frac{e}{2})^{n-1}Y^n(1+fZ)^{2n-1},$$

and hence y contains no term of the form $(YZ^2)^n$ because always $2n > 2n-1$.

Case 3: The reduced system is therefore

$$\dot{x} = x(1+2gx-ey-3fz), \quad \dot{y} = y(-2+ey+fz), \quad \dot{z} = z(1+gx-fz).$$

If $g = 0$, then we obtain a sub-case of Case 2 and if $f = 0$, we obtain a subcase of Case 5*. Hence, we shall assume that f and g are both non-zero. If we apply to this system a transformation of the form $(X,Y,Z) = (gx-fz, xy, z^2)$, the resulting system is

$$\dot{X} = X + 2X^2 - f^2Z - geY, \quad \dot{Y} = Y(-1+2X), \quad \dot{Z} = Z(2+2X).$$

Finally, we apply the projective transformation $(\hat{X},\hat{Y},\hat{Z}) = (\frac{X}{Y},\frac{1}{Y},\frac{Z}{Y})$ to get the linear system

$$\dot{\hat{X}} = 2\hat{X} - f^2\hat{Z} - ge, \quad \dot{\hat{Y}} = \hat{Y} - 2\hat{X}, \quad \dot{\hat{Z}} = 3\hat{Z}.$$

This system admit first integrals

$$\phi = \ell_1^{-2}\ell_2, \qquad \psi = \ell_1^{-3}\ell_3,$$

where

$$\begin{aligned}
\ell_1(\hat{X},\hat{Y},\hat{Z}) &= 1 - \frac{1}{ge}\hat{X} - \frac{1}{2ge}\hat{Y} - \frac{f^2}{2ge}\hat{Z}, \\
\ell_2(\hat{X},\hat{Y},\hat{Z}) &= 1 - \frac{2}{ge}\hat{X} - \frac{2f^2}{ge}\hat{Z}, \\
\ell_3(\hat{X},\hat{Y},\hat{Z}) &= \hat{Z}.
\end{aligned}$$

Pulling back the integrals via the combined transformation

$$(\hat{X},\hat{Y},\hat{Z}) = (\frac{gx-fz}{xy},\frac{1}{xy},\frac{z^2}{xy}),$$

the system admits two first integrals:

$$\phi = \frac{xy(gexy-2gx+2fz-2f^2z^2)}{(1-2gexy+2gx-2fz+f^2z^2)^2},$$

and

$$\psi = \frac{x^2y^2z^2}{(1-2gexy+2gx-2fz+f^2z^2)^3}.$$

One can find two independent first integrals of the desirable form

$$\phi_1 = -\frac{\phi}{2g}+\frac{f}{g}\sqrt{\psi} = x^2y(1+\cdots)$$

and

$$\phi_2 = \frac{\psi}{\phi_1} = yz^2(1+\cdots).$$

The expression $\xi = xyz^{-2}$ satisfies $\dot{\xi} = -3\xi$ and so the system is linearizable by Theorem 6.

Case 4: In this case, one can see system (3.1) as

$$\dot{x} = x(1+ax+by), \quad \dot{y} = y(-2+ey), \quad \dot{z} = z(1+hy+kz). \tag{3.38}$$

The change of coordinates

$$Y = \frac{y}{1-\frac{e}{2}y} \tag{3.39}$$

linearizes the second equation. To linearize the first equations, we seek invariant algebraic plane $A(Y)+B(Y)x = 0$ with cofactor $ax+by$, where A and B are analytic with $A(0) = 1$. Thus we seek A and B to satisfy the equation

$$-2YA'(Y) = byA(Y), \qquad -2YB'(Y)+B(Y) = aA(Y). \tag{3.40}$$

The first equation gives $A=(1-ey/2)^{b/e}$ (or $\exp(by/2)$ when $e=0$).

Writing $A=\sum_{i\geq 0}a_iY^i$, we have $B=\sum_{i\geq 0}a\frac{a_i}{1-2i}Y^i$, which is clearly convergent.

The substitution $X=x/(A+Bx)$ linearizes the first equation of (3.38). In the same way, we can find an invariant surface $C(Y)+D(Y)z=0$ with cofactor $hy+kz$ to so that $Z=z/(C+Dz)$ linearizes the third equation of (3.38).

Case 5: According to conditions in this case, system (3.1) becomes

$$\dot{x}=x(1+by+cz),\quad \dot{y}=y(-2+ey),\quad \dot{z}=z(1+hy+kz).$$

The substitution $Y=\frac{y}{1-ey/2}$ linearizes the second equation. As in the previous case, we can find an invariant surface $C(Y)+D(Y)z=0$ with cofactor $hy+kz$ where C and D are analytic with $C(0)=1$. Then $Z=\frac{z}{C+Dz}$ linearizes the third equation. Such a C and D must satisfy

$$-2YC'(Y)=hyC(Y),\qquad -2YD'(Y)+D(Y)=kC(Y). \tag{3.41}$$

We seek a transformation $X=xe^{-\phi}$ which linearizes the first equation. Such a ϕ satisfies

$$\dot{\phi}(Y,Z)=by(Y)+cz(Y,Z). \tag{3.42}$$

Taking $\phi=\sum a_{ij}Y^iZ^j$, we find that $\dot{\phi}=\sum(2i-j)a_{ij}$. It is clear that (3.42) can be solved analytically for ϕ as long as $z(Y,Z)$ contains no term of the form $(YZ^2)^n$. Since

$$z(Y,Z)=\frac{ZC(Y)}{1-ZD(Y)}=\sum_{i>0}CD^{i-1}Z^i,$$

it is sufficient to show that CD^{2n-1} contains no term in Y^n.

However, from (3.41), we have

$$CD^{2n-1}=\frac{1}{k}\Big(-2Y\frac{dD}{dY}+D\Big)D^{2n-1}=\frac{1}{k}\Big(-\frac{Y}{n}\frac{dD^{2n}}{dY}+D^{2n}\Big),$$

which clearly does not contain any term of degree Y^n. When $k=0$ we have $D=0$ from

(3.41) and the corresponding result is trivial.

Case 6: The system (3.1) reduces to

$$\dot{x} = x(1+ax+by+cz), \quad \dot{y} = y(-2+ey), \quad \dot{z} = z(1+gx+by+kz). \tag{3.43}$$

Let $X = xh(y)$ and $Z = zh(y)$ for some analytic function $h(y)$ with $h(0) = 1$, to bring (3.43) to the form

$$\begin{aligned} \dot{X} &= X(1+ax+cz+by+\frac{h'}{h}y(-2+ey)), \\ \dot{y} &= y(-2+ey), \\ \dot{Z} &= Z(1+gx+kz+by+\frac{h'}{h}y(-2+ey)). \end{aligned} \tag{3.44}$$

Choosing h so that

$$1+by+\frac{h'}{h}y(-2+ey) = \frac{1}{h}$$

or, equivalently,

$$-2y(1-\frac{e}{2}y)\frac{h'}{h} = \frac{1}{h}-(1+by), \quad h(0) = 1. \tag{3.45}$$

This equation has an explicit solution,

$$h = -\frac{1}{2}\sqrt{y}(1-ey/2)^{-(\frac{b}{e}+\frac{1}{2})}\int y^{-\frac{3}{2}}(1-ey/2)^{\frac{b}{e}-\frac{1}{2}},$$

which can be seen to be analytic in y with $h(0) = 1$. After scaling by $h(y)$, the system (3.44) becomes,

$$\dot{X} = X(1+aX+cZ), \qquad ,\dot{y} = y(-2+ey)h(y), \qquad \dot{Z} = Z(1+gX+kZ). \tag{3.46}$$

Clearly the first and third equations gives a linearizable node. The second equation can be linearized by a substitution $Y = \ell(y)$, such that

$$y(2-ey)h(y)\frac{d\ell(y)}{dy} = 2\ell(y), \quad \ell(0) = 0, \quad \ell'(0) = 1.$$

This is clearly solvable analytically once we have determined $h(y)$ as above.

Remark 7. *For interest, one can solve equation (3.45) explicitly as follows:*

On can write equation (3.45) as

$$\frac{\dot{h}}{h}+\frac{1}{2y(1-\frac{e}{2}y)}\frac{1}{h}=\frac{1+by}{2y(1-\frac{e}{2}y)}. \tag{3.47}$$

The substitution $z=\frac{1}{h}$ transform equation (3.47) to a Bernoulli equation

$$\dot{z}+\frac{1+by}{2y(1-\frac{e}{2}y)}z=\frac{1}{2y(1-\frac{e}{2}y)}z^2.$$

The change of variable $u=z^{-1}$, transform a Bernoulli equation to a first order differential equation of the form

$$\dot{u}-\frac{1+by}{2y(1-\frac{e}{2}y)}u=-\frac{1}{2y(1-\frac{e}{2}y)}.$$

Thus

$$u=\frac{1}{z}=h=\frac{\int e^{-\int\frac{1+by}{2y(1-\frac{e}{2}y)}dy}(-\frac{1}{2y(1-\frac{e}{2}y)})dy+C}{e^{\int\frac{1+by}{2y(1-\frac{e}{2}y)}dy}}.$$

It is not difficult to see that

$$h=\sqrt{y}(1-\frac{e}{2}y^{\frac{1}{2}-\frac{b}{e}})\Big(\int-\frac{1}{2}y^{-\frac{3}{2}}(1-\frac{e}{2}y)^{\frac{b}{e}-\frac{3}{2}}+C\Big).$$

Now let $m=\frac{b}{e}-\frac{3}{2}$, then

$$\int y^{-\frac{3}{2}}(1-\frac{e}{2}y)^m=-2y^{-\frac{1}{2}}-m\,e\,y^{\frac{1}{2}}+\frac{m(m-1)}{3}y^{\frac{3}{2}}+\cdots.$$

Therefore,

$$h=\sqrt{y}(1-\frac{e}{2}y^{\frac{1}{2}-\frac{b}{e}})\big(y^{-\frac{1}{2}}-\frac{m\,e}{2}y^{\frac{1}{2}}+\frac{m(m-1)}{6}y^{\frac{3}{2}}+\cdots),$$

Clearly $h(0)=1$.

Case 7: System (3.1) becomes

$$\dot{x}=x(1+ax+by+cz),\quad \dot{y}=y(-2+ey),\quad \dot{z}=z(1+ax+hy+cz).$$

In this case, different from the axes the system has an invariant algebraic plane $\ell = -1 + \frac{e}{2}y$ with cofactor $L_\ell = ey$ producing a first integral

$$\phi = xz^{-1}\,\ell^{\frac{h-b}{e}}.$$

It is easy to see that

$$IJM = y^{\frac{3}{2}} z^3 (-2+ey)^{\frac{b}{e}-\frac{2h}{e}+\frac{1}{2}} \tag{3.48}$$

as inverse Jacobi multiplier. Note that when $e = 0$, we take $\phi = xz^{-1}\,E^{\frac{b-h}{2}}$ and $IJM = y^{\frac{3}{2}}\,z^3\,E^{h-\frac{b}{2}}$ appears as a first integral and an inverse Jacobi multiplier respectively where $E = \exp(y)$ is an exponential factor with cofactor $-2y$. By Theorem 5, we can get a second first integral $\psi = xy^{-\frac{1}{2}}z^{-2}\big(1+O(1)\big)$. Now we can construct two independent first integrals of the form desired as

$$\phi_1 = \phi^4\psi^{-2},$$

and

$$\phi_2 = \phi^2\psi^{-2}.$$

Case 8: The system is

$$\dot{x} = x(1+ax+by+cz), \quad \dot{y} = y(-2-by-cz), \quad \dot{z} = z(1-by-cz).$$

This case has invariant algebraic planes $\ell_1 = 1+\frac{b}{2}y - cz = 0$ and $\ell_2 = 1+ax+abxy-\frac{ac}{2}xz = 0$ with cofactors $L_{\ell_1} = -by-cz$ and $L_{\ell_2} = ax$ which allow us to find two independent first integrals

$$\phi_1 = x^2\,y\,\ell_1\ell_2^{-2} \qquad \text{and} \qquad \phi_2 = yz^2\,\ell_1^{-3}.$$

Case 9: System (3.1) can be written

$$\dot{x} = x(1+ax-3hy+3kz), \quad \dot{y} = y(-2-2ky-2ky), \quad \dot{z} = z(1+hy+kz).$$

In this case $\ell = ax(1+\frac{k}{2}z) - ahxy + (1+hy+kz)^2 = 0$ appears as an invariant algebraic surface with cofactor $L_\ell = ax - 4hy + 2kz$, giving rise to the first integrals

$$\phi_1 = x^2 y \ell^{-2} \qquad \text{and} \qquad \phi_2 = yz^2.$$

Case 10: In this case we have the system

$$\dot{x} = x(1+ax), \quad \dot{y} = y(-2+ey+fz), \quad \dot{z} = z(1+gx+ey+fz).$$

Performing the change of variables $(Y,Z) = (y(1+fz-\frac{e}{2}y)^{-1}, z(1+fz-\frac{e}{2}y)^{-1})$ gives a new system

$$\dot{x} = x(1+ax), \quad \dot{Y} = -2Y(1+fgxZ), \quad \dot{Z} = Z(1+gx-2fgxZ).$$

The first and the third equations obviously gives a node and therefore there is a linearizing change of coordinates $\hat{X} = \hat{X}(x)$, $\hat{Z} = \hat{Z}(x,Z)$. To linearize the second equation, it is suffices to find $\psi(\hat{X},\hat{Z})$ such that $\dot{\psi} = fgxZ$ and use the transformation $\hat{Y} = Ye^{2\psi}$.

Setting

$$\psi(\hat{X},\hat{Z}) = \sum_{i+j>0} a_{ij}\hat{X}^i\hat{Z}^j,$$

and

$$x(\hat{X},\hat{Z})Z(\hat{X},\hat{Z}) = \sum b_{ij}\hat{X}^i\hat{Z}^j,$$

we find that

$$\dot{\psi}(\hat{X},\hat{Z}) = \sum_{i+j>0} (i+j)a_{ij}\hat{X}^i\hat{Z}^j = \sum_{i+j>0} fgb_{ij}\hat{X}^i\hat{Z}^j.$$

Hence, we can set $a_{ij} = d_{ij}/(i+j)$ to find a convergent expression for ψ above.

Case 11: Then the system reduces to

$$\dot{x} = x(1+ax+cz), \quad \dot{y} = y(-2+dx+fz), \quad \dot{z} = z(1+gx+kz).$$

The first and third equations gives a linearizable node. We denote the linearizing coordinates by X and Z.

The transformation $Y = ye^{-\phi}$ will linearize the second equation, if ϕ can be chosen so that

$$\dot{\phi}(X,Z) = dx(X,Z) + fz(X,Z). \tag{3.49}$$

Let

$$\phi(X,Z) = \sum_{i+j>0} b_{ij} X^i Z^j,$$

and let the right hand side of equation (3.49) be written as $\sum_{i+j>0} a_{ij} X^i Z^j$, Then we require

$$\sum_{i+j>0} (2i+j) b_{ij} X^i Z^j = \sum_{i+j>0} a_{ij} X^i Z^j.$$

Setting $b_{ij} = a_{ij}/(2i+j)$, we obtain a convergent expression for ϕ.

Case 12: The system (3.1) reduces to

$$\dot{x} = x(1 - dx + cz), \quad \dot{y} = y(-2 + dx + ey - kz), \quad \dot{z} = z(1 + gx + kz).$$

The system has invariant algebraic surfaces $\ell_1 = 1 - dx - \frac{e}{2}y + kz = 0$ and $\ell_2 = 1 - \frac{e}{2}y - \frac{ed}{2}xy + \frac{ke}{2}yz = 0$ with cofactors $L_{\ell_1} = -dx + ey + kz$ and $L_{\ell_2} = ey$. The substitution

$$Y = y\ell_1 \ell_2^{-2}$$

linearizes the second equation, and the first and third equations define a linearizable node.

Case 13: In this case system (3.1) becomes

$$\dot{x} = x(1 + ax + by + cz), \quad \dot{y} = y(-2 + ax + by - kz), \quad \dot{z} = z(1 + kz).$$

The system has an invariant algebraic plane $\ell = 1 + kz = 0$ with cofactor $L_\ell = kz$. By mean of a change of variables

$$(X,Y,Z) = (x(1+kz)^{-\frac{c}{k}}, y(1+kz), z(1+kz)^{-1}),$$

we arrive to the system

$$\begin{aligned}
\dot{X} =& X(1+aX(1-kZ)^{-\frac{c}{k}}+bY(1-kZ)),\\
\dot{Y} =& Y(-2+aX(1-kZ)^{-\frac{c}{k}}+bY(1-kZ)),\\
\dot{Z} =& Z,
\end{aligned} \tag{3.50}$$

with first integral $\phi = X^{-1}YZ^3$ and inverse Jacobi multiplier $IJM = X^2Y$.

We cannot apply Theorem 5, as there are non-negative integer values of i, j and k for which the cross product of $(1-i,-j,-1-k)$ and $(-1,1,3)$ is zero. However, this is only true when, $(1-i,-j,-1-k) = \alpha(-1,1,3)$ for some α. Clearly, the only possibility is when $\alpha = -1$, $i = 0$, $j = 1$ and $k = 2$. However, in this case, the proof of Theorem 5 will still work as long as $A_{(0,1,2)} = 0$ because, in this case, (3.7) will still hold. But it is clear that in (3.50) that there are no terms in YZ^2 in the cofactors of X, Y or Z, so indeed $A_{(0,1,2)} = 0$ and we have a second first integral of the form $\psi = X^{-1}Z(1+O(X,Y,Z))$. We get first integrals in the required form by pulling back the first integrals $\phi_1 = \phi/\psi^3 = X^2Y(1+O(X,Y,Z))$ and $\phi_2 = \phi/\psi = YZ^2(1+O(X,Y,Z))$ to the original coordinates. The substitution $(\hat{X},\hat{Y},\hat{Z}) = (Z/\psi, \phi_2/Z^2, Z)$ linearizes the system.

When $k = 0$, we replace $(1+kz)^{-\frac{c}{k}}$ by $\exp(-cz)$ and proceed similarly.

Case 14: In this case system (3.1) written

$$\dot{x} = x(1+ax+by-2fz), \quad \dot{y} = y(-2+ey+fz), \quad \dot{z} = z(1-fz).$$

We have two invariant algebraic planes $\ell_1 = 1-\frac{e}{2}y-fz = 0$ and $\ell_2 = 1-fz = 0$ with cofactors $L_{\ell_1} = ey-fz$ and $L_{\ell_2} = -fz$. This allow us to construct a first integral

$$\phi = yz^2\ell_1^{-1},$$

and inverse Jacobi multiplier $IJM = x^2 z^{-2} \ell_1^{2-\frac{b}{e}} \ell_2^{\frac{b}{e}-1}$. Now, Theorem 5 guarantees a second first integral $\psi = x^{-1}yz^3(1+O(1))$. We write the first integrals in the desired form as

$$\phi_1 = \phi^3\psi^{-2} = x^2 y\big(1+m(x,y,z)\big) \qquad \text{and} \qquad \phi_2 = \phi = yz^2\big(1+\hat{m}(x,y,z)\big).$$

The following change of coordinates linearizes the system

$$(X,Y,Z) = (x(1+m)^{\frac{1}{2}}(1-fz)^{-1}(1+\hat{m})^{-\frac{1}{2}}, y(1-fz)^2(1+\hat{m}), z(1-fz)^{-1}).$$

Case 15: In this case system (3.1) written

$$\dot{x} = (1-gx-3hy+3kz), \quad \dot{y} = y(-2-2hy-2kz), \quad \dot{z} = z(1+gx+hy+kz).$$

One can see that $\ell_1 = (1+hy+kz)^2 - kgxz = 0$ and $\ell_2 = gx(-2+gx+2hy-2kz) + (1+hy+kz)^2 = 0$ are invariant algebraic surfaces with cofactors $L_{\ell_1} = -4hy+2kz$ and $L_{\ell_2} = -2gx-4hy+2kz$. The two first integrals are given by

$$\phi_1 = x^2 y \ell_1^{-1} \ell_2^{-1} \qquad \text{and} \qquad \phi_2 = yz^2 \ell_1^{-1} \ell_2.$$

Case 16: In this case the corresponding system is

$$\dot{x} = x(1+ax+cz), \quad \dot{y} = y(-2-ax+ey), \quad \dot{z} = z(1+3ax-ey+2cz).$$

The system has invariant algebraic surfaces $\ell_1 = 1+2ax+2cz+a^2x^2-2ceyz = 0$ and $\ell_2 = 2a^2x^2+2cz-ceyz+2ax = 0$ with cofactors $L_{\ell_1} = 2ax+2cz$ and $L_{\ell_2} = 1+2ax+2cz$. This gives a first integrals $\phi = xyz\ell_1^{-\frac{3}{2}}$ and $\psi = x^{-1}\ell_2\ell_1^{-1/2}$, though the latter integral is not of the required form. However, when $z = 0$, we have $\psi = 2a$. Hence, $\xi = (\psi - 2a)x/(2cz) = 1+O(1)$ is analytic and satisfies $\dot{\xi} = \xi(-3ax+ey-cz)$. From this we construct a linearizing change of coordinates $(X,Y,Z) = (x\ell_1^{-1/2}, y\ell_1^{-1/2}\xi^{-1}, z\ell_1^{-1/2}\xi)$.

When $c = 0$ we have invariant surfaces $\ell_3 = 1+ax$ and $\ell_4 = 1+ax-ey/2$ with cofactors ax and $ax+ey$, giving a linearizing change of coordinates $(X,Y,Z)=(x\ell_3^{-1}, y\ell_3^2\ell_4^{-1}, z\ell_3^{-4}\ell_4)$.

Case 17: The system reduces to

$$\dot{x}=x(1+ax+bz), \quad \dot{y}=y(-2+ax+by), \quad \dot{z}=z(1-ax-by+kz).$$

This case has invariant algebraic surfaces

$$\ell_1=1+ax-\frac{b}{2}y=0, \qquad \ell_2=1+kz+\frac{ak}{2}xz-bkyz=0,$$

with cofactors $L_{\ell_1}=ax+by$ and $L_{\ell_2}=kz$. It is easy to obtain two independent first integrals

$$\phi_1=x^2y\ell_1^{-3} \qquad \text{and} \qquad \phi_2=yz^2\ell_2^{-1}.$$

Case 18: The reduced system is

$$\dot{x}=x(1+ax+bz), \quad \dot{y}=y(-2-2ax-2by), \quad \dot{z}=z(1+3ax-3by+kz).$$

and it has an invariant algebraic plane $\ell=(1+ax+by)^2+kz(1+\frac{a}{2}x-by)=0$ with cofactor $L_\ell=2ax-4by+kz$. These give two independent first integrals

$$\phi_1=x^2y \qquad \text{and} \qquad \phi_2=yz^2\ell^{-2}.$$

Case 19: The system appears as

$$\dot{x}=x(1+ax+bz), \quad \dot{y}=y(-2-2ax-2by), \quad \dot{z}=z(1+3ax-3by-cz).$$

In this case we find two invariant algebraic surfaces $\ell_1=(1+ax+by)^2-acxy=0$ and $\ell_2=(1+ax+by)^2-2cz(1+ax-by-\frac{c}{2}z)=0$ with cofactors $L_{\ell_1}=2ax-4by$ and $L_{\ell_2}=$

$2ax-4by-2cz$. One can easily obtain two independent first integrals

$$\phi_1 = x^2 y \ell_1^{-1} \ell_2 \qquad \text{and} \qquad \phi_2 = y z^2 \ell_1^{-1} \ell_2^{-1}.$$

Case 20: The system (3.1) reduces to

$$\dot{x} = x(1+ax+cz), \quad \dot{y} = y(-2+ey+fz), \quad \dot{z} = z(1-fz).$$

The first and third equations gives a linearizable node and the change of coordinates $Y = \frac{y(1-fz)^2}{1-\frac{e}{2}y-fz}$ linearizes the second equation.

Case 21: The system is

$$\dot{x} = x(1+ax), \quad \dot{y} = y(-2+dx+ey), \quad \dot{z} = z(1+gx+hy).$$

The system in this case has three invariant algebraic surfaces $\ell_1 = 1+ax-\frac{e}{2}y = 0$, $\ell_2 = 1+ax = 0$ and $\ell_3 = 1-\frac{e}{2}y+\frac{ae}{2}xy = 0$ with cofactors $L_{\ell_1} = ax+ey$, $L_{\ell_2} = ax$ and $L_{\ell_3} = ey$. It is easy to find two independent first integrals $\phi_1 = x^2 y \ell_1^{-1}$ and $\phi_2 = y z^2 \ell_2^{\frac{a-2g}{a}} \ell_3^{-\frac{2h+e}{e}}$, and a linearizing change of coordinates $(X,Y,Z) = (x\ell_2^{-1}, y\ell_2\ell_3^{-1}, z\ell_2^{-\frac{g}{a}}\ell_3^{-\frac{h}{e}})$.

When $a=0$, we replace $\ell_2^{1/a}$ with the exponential factor $\exp(x)$, while, when $e=0$, we replace $\ell_3^{1/e}$ by the exponential factor $\exp(-y(1-ax)/2)$.

$\square$

Chapter 4

Integrability and linearizability of three dimensional vector fields

4.1 Introduction

In this chapter, we continuous the work that starts in previous chapter which we studied the integrability and linearizability problem for three dimensional Lotka-Volterra equations (3.1) with $(\lambda : \mu : \nu)$–resonance.

A natural way to generalize Lotka–Volterra equations (3.1) is the following three dimensional system

$$\begin{aligned}
\dot{x} &= P = x(\lambda + ax + by + cz),\\
\dot{y} &= Q = \mu y + dx^2 + exy + fxz + gyz + hy^2 + kz^2,\\
\dot{z} &= R = z(\nu + \ell x + my + pz).
\end{aligned} \tag{4.1}$$

This case is closer to the general three dimensional system but still tractable. Under the assumption of $(\lambda : \mu : \nu) = (1 : -1 : 1)$–resonance, in this chapter, we provide the integrability and linearizability conditions and prove their sufficiency through the Darboux method with inverse Jacobi multiplier and by power series arguments. Also should mention that in some cases the solution of a Ricatti equation comes into play to linearize the third variable. Recall that the system is completely integrable if there exists two independent first integrals. In general, this is equivalent to finding one first integral together with an inverse Jacobi multiplier.

The following theorem generalizes Theorem 5 to our context. As the proof is very similar, we do not give all the details.

Theorem 11. *Consider the analytic vector field*

$$x(\lambda+\sum_{|I|>0}A_{xI}X^{I})\frac{\partial}{\partial x}+(\mu y+\sum_{|I|>1}A_{yI}X^{I})\frac{\partial}{\partial y}+z(\nu+\sum_{|I|>0}A_{zI}X^{I})\frac{\partial}{\partial z}, \tag{4.2}$$

with $\lambda,\nu>0$ and $\mu<0$. Suppose that it has a first integral $\phi=x^{\alpha}z^{\gamma}(1+O(x,y,z))$ with at least α or $\gamma\neq 0$ and an inverse Jacobi multiplier $M=x^{r}z^{t}(1+O(x,y,z))$. Suppose the cross product of $(r-i-1,-j-1,t-k-1)$ and $(\alpha,0,\gamma)$ is bounded away from zero for any integers $i,k\geq 0$ and $j\geq -1$, then the system has a second analytic first integral of the form $\psi=x^{1-r}yz^{1-t}(1+O(x,y,z))$, and hence is integrable.

Proof. Without loss of generality, we can rewrite system (4.2) as:

$$x(\lambda+\sum_{|I^*|>0}A_{xI^*}X^{I^*})\frac{\partial}{\partial x}+y(\mu+\sum_{|I^*|>0}A_{yI^*}X^{I^*})\frac{\partial}{\partial y}+z(\nu+\sum_{|I^*|>0}A_{zI^*}X^{I^*})\frac{\partial}{\partial z}, \tag{4.3}$$

where $I^*=(i_1,i_2,i_3)$ with $i_1,i_2+1,i_3\geq 0$, $A_{xI^*}=A_{zI^*}=0$ for $i_2=-1$, $A_{y(1,-1,0)}=A_{y(0,-1,1)}=A_{y(0,-1,0)}=0$ and $(\lambda,\mu,\nu)=A_{(0,0,0)}$. Now the prove is similar as the proof of Theorem 5. □

4.2 Mechanism to find integrability and linearizability conditions

In this section, we will give the integrability and linearizability conditions for the origin of (4.1) and prove their sufficiency by constructing two independent first integrals. The necessary condition were found by computing conditions up to degree 12 for the existence of two independent first integrals of the form $\phi_1=x(y+\cdots)$ and $\phi_2=z(y+\cdots)$ by Maple. A factorized Gröbner basis was then found using Reduce and finally the minAssGTZ algorithm in Singular (Decker et al., 2011) was used to check that the conditions found were irreducible.

For linearizability, we computed the conditions up to degree 12 for the existence of a linearizing change of coordinates and then for sufficiency, we provided a linearizing change of coordinates.

Theorem 12. *Consider the system* (4.1)*. The origin is integrable if and only if one of the following conditions are satisfied:*

$$1)\, a+\ell = b+m = c-3p = d = e+\ell = g+p = h+m = k = 0$$

$$1^{*})\, 3a-\ell = b+m = c+p = d = 3e+\ell = g-p = h-m = k = 0$$

$$2)\, a = b+m = c = d = e = g = k = \ell = p = 0$$

$$3)\, a = b+m = c = d = e = g = h-2m = \ell = p = 0$$

$$3^{*})\, a = b+m = c = e = g = h+2m = k = \ell = p = 0$$

$$4)\, ab-eh = ac-2ap+\ell p = ag+ap-ep-\ell p = ah+am-eh-h\ell =$$
$$be+b\ell-eh-em = bg-cm-gm+mp = bp-ch+hp-mp =$$
$$ce+c\ell-2ep+g\ell-\ell p = d = f = gh-mp = k = 0$$

$$5)\, 4a-\ell = 2b+m = c = d = 4e+\ell = g = 2h-m = k = p = 0$$

$$5^{*})\, a = b+2m = c-4p = d = e = g+p = h+m = k = \ell = 0$$

$$6)\, a-2\ell = b-2m = c = d = e-2\ell = f = g = h-2m = p = 0$$

$$6^{*})\, a = 2b-m = 2c-p = e = f = g-p = h-m = k = \ell = 0$$

$$7)\, 5a-2\ell = b+2m = c = d = 5e+2\ell = f = g = h-2m = p = 0$$

$$7^{*})\, a = 2b+m = 2c-5p = e = f = g+p = h+m = k = \ell = 0$$

$$8)\, a = c = d = e = f = g = h-2m = \ell = p = 0$$

$$8^{*})\, a = 2b-h = c = e = f = g = k = \ell = p = 0$$

$$9)\, a = b-h+m = c = d = e = g = k = \ell = p = 0$$

$$10)\, a-e = b-h = d = g = \ell = m = 0$$

$$10^{*})\, b = c = e = g-p = h-m = k = 0$$

$$11)\, 2a-\ell = b+h = c\ell-2fh = d = 2e+\ell = g = m = 0$$

$$11^{*})\, b = c-2p = e = fm+\ell p = g+p = h+m = k = 0$$

$12)\, b=c=e=f=g=k=p=0$

$12^*)\, a=d=e=f=g=\ell=m=0$

$13)\, b=c=e=f=g=h-2m=p=0$

$13^*)\, a=2b-h=e=f=g=\ell=m=0$

$14)\, b=h=m=0$

$15)\, b=e=g=m=0$

$16)\, a=b-m=c=e=g=h-2m=\ell=p=0$

$17)\, a-\ell=b-m=c-p=e+2\ell=g+2p=2h+m=0.$

Moreover, the system is linearizable if and only if either one of the conditions (2), (3), (6), (8), (9), (10), (13), (16), (17) *or one of the following holds:*

$1.1)\, a+\ell=b+m=c=d=e+\ell=g=h+m=k=p=0$

$1.1^*)\, a=b+m=c+p=d=e=g-p=h-m=k=\ell=0$

$4.1)\, a=c=d=e=f=g=k=\ell=p=0$

$4.2)\, a-e=b-h=c=d=em-h\ell=f=g=k=p=0$

$4.2^*)\, a=bp-cm=d=e=f=g-p=h-m=k=\ell=0$

$4.3)\, a-\ell=b-m=c-p=d=e-\ell=f=g-p=h-m=k=0.$

Proof. Cases $1^*,3^*,5^*,6^*,7^*,8^*,10^*,11^*,12^*,13^*,1.1$ and 4.2^* are dual to Cases 1, 3, 5, 6, 7, 8, 10, 11, 12, 13 1.1 and 4.2 under the transformation $(x,y,z)\mapsto(z,y,x)$, and do not need to be considered separately. The other cases are considered below.

Case 1: The system is

$$\dot{x}=x(1+ax+by+cz),\quad \dot{y}=-y-axy+fxz-\frac{c}{3}yz+by^2,\quad \dot{z}=z(1-ax-by+\frac{c}{3}z),$$

which has an invariant algebraic surface

$$F=1+2ax-2by+\frac{2c}{3}z+a^2x^2+2abxy-(\frac{2}{3}ac+bf)xz+b^2y^2-\frac{2bc}{3}yz+\frac{c^2}{9}z^2=0$$

with cofactor $C_{\mathcal{F}} = 2ax + 2by + \frac{2c}{3}z$ giving a first integral $\phi = x^{-1}z\mathcal{F}$ and inverse Jacobi multiplier $IJM = x^{\frac{7}{6}}z^{-\frac{1}{6}}\mathcal{F}^{\frac{1}{3}}$. Theorem 11 now guarantees the existence of the second first integral $\psi = x^{-\frac{1}{6}}yz^{\frac{7}{6}}(1+\cdots)$. We can construct two independent first integrals of the desired form as $\phi_1 = \phi^{-\frac{7}{6}}\psi = x(y+\cdots)$ and $\phi_2 = \phi^{-\frac{1}{6}}\psi = z(y+\cdots)$.

Case 1.1: The system reduces to

$$\dot{x} = x(1+ax+by), \quad \dot{y} = -y+axy+fxz+by^2, \quad \dot{z} = z(1-ax-by),$$

and it has invariant algebraic surfaces $\mathcal{F}_1 = 1+2ax-2by+a^2x^2+2abxy+bfxz+b^2y^2 = 0$ and $\mathcal{F}_2 = b^2x+z+2axz-2byz+a^2x^2z+2abxyz+bfxz^2+b^2y^2z = 0$ with cofactors $C_{\mathcal{F}_1} = 2ax+2by$ and $C_{\mathcal{F}_2} = 1+ax+by$. Therefore, a linearizing change is given by $(X,Y,Z) = (x\mathcal{F}_1^{-\frac{1}{2}}, \mathcal{F}_1^{-\frac{1}{2}}\mathcal{F}_2, z\mathcal{F}_1^{\frac{1}{2}})$.

Case 2: System (4.1) has the form

$$\dot{x} = x(1+by), \quad \dot{y} = -y+fxz+hy^2, \quad \dot{z} = z(1-by). \tag{4.4}$$

When $h \neq 0$, the system has invariant algebraic surface $\mathcal{F} = 1-2hy+fhxz+h^2y^2 = 0$ with cofactor $C_{\mathcal{F}} = 2hy$ and hence the transformation $(X,Z) = (x\mathcal{F}^{-\frac{b}{2h}}, z\mathcal{F}^{\frac{b}{2h}})$ linearizes the first and third equations. One can note that the second equation

$$\dot{y} = -y+fxz+hy^2 = -y+fXZ+hy^2$$

is a Riccati equation. We seek a solution of the Riccati equation of the form $y = G(XZ) = G(xz)$, then

$$G'(XZ) = \frac{f}{2} - \frac{1}{2XZ}G(XZ) + \frac{h}{2XZ}G^2(XZ),$$

or

$$G'(xz) = \frac{f}{2} - \frac{1}{2xz}G(xz) + \frac{h}{2xz}G^2(xz),$$

which has a particular solution $y_1 = \frac{\sin(\sqrt{fhxz})-\cos(\sqrt{fhxz})\sqrt{fhxz}}{h\sin(\sqrt{fhxz})}$. Note that y_1 is analytic with leading term $\frac{f}{3}xz$. The change of variables $y = Y + y_1$ transforms the second equation from (4.4) to

$$\dot{Y} = Y(-1 + 2hy_1 + hY).$$

Now, we look for an invariant algebraic surface of the form $\alpha(X,Z) + \beta(X,Z)Y = 0$ with cofactor $2hy_1 + hY$ and $\alpha(0,0) = 1$, so that the transformation $\frac{Y}{\alpha+\beta Y}$ will linearize this equation. To find such α and β, we have to solve

$$\dot{\beta} - \beta = \alpha h, \qquad \dot{\alpha} = 2h\alpha y_1. \tag{4.5}$$

Write $\alpha = e^{\psi}$ and we have to solve $\dot{\psi} = 2hy_1$. Suppose $\psi = \sum_{i+j>0} c_{ij} X^i Z^j$, then

$$\sum_{i+j>0} (i+j) c_{ij} X^i Z^j = 2hy_1(X,Z) = 2ha_{11}XZ + \sum_{i+j>2} a_{ij} X^i Z^j.$$

Note that $c_{10} = c_{01} = 0$, $c_{11} = \frac{2}{3} fh$ and $c_{ij} = \frac{a_{ij}}{i+j}$ for $i + j > 2$. Clearly the convergence of $\sum_{i+j>2} a_{ij} X^i Z^j$, guarantees the convergence of ψ and so α. Moreover, in fact, α does not contain X and Z term. Now write $\beta = \sum b_{ij} X^i Z^j$, then from first equation of (4.5) we see that $b_{11} = \frac{2}{3} fh^2$ and $b_{ij} = \frac{h}{i+j-1} a_{ij}$ for $i + j > 2$ and the convergence is clear.

When $h = 0$, the change of variables $(X,Y,Z) = (xe^{b(y-fxz)}, y - \frac{f}{3}xz, ze^{-b(y-fxz)})$ gives

$$\dot{X} = X(1 - bfXZ), \quad \dot{Y} = -Y, \quad \dot{Z} = Z(1 + bfXZ).$$

Clearly, the first and third equations give a linearizable node. Hence there exists a change of coordinates $\tilde{X} = X(1 + o(X,Z))$ and $\tilde{Z} = Z(1 + o(X,Z))$ such that $\dot{\tilde{X}} = \tilde{X}, \quad \dot{\tilde{Z}} = \tilde{Z}$.

Case 3: The system (4.1) reduces to

$$\dot{x} = x(1 + by), \quad \dot{y} = -y + fxz - 2by^2 + kz^2, \quad \dot{z} = z(1 - by). \tag{4.6}$$

Performing the change of variables $(X,Y,Z) = (x^2, x^2y, xz)$, the system (4.6) becomes

$$\dot{X} = 2X + 2bY, \quad \dot{Y} = Y + fXZ + kZ^2, \quad \dot{Z} = 2Z.$$

The critical point at the origin is then in Poincaré domain and is linearizable via an analytic change of coordinates of the form

$$(\tilde{X},\tilde{Y},\tilde{Z}) = (X+2bY+O(2), Y+O(2), Z).$$

The first integrals of the linear system are $\tilde{\phi} = \tilde{X}^{-\frac{1}{2}}\tilde{Y}$ and $\tilde{\psi} = \tilde{X}^{-\frac{3}{2}}\tilde{Y}\tilde{Z}$. Pulling back these first integrals to the original coordinates, we get first integrals of desired form

$$\phi_1 = x(y+\cdots) \quad \text{and} \quad \phi_2 = z(y+\cdots).$$

Case 4: If $h \neq 0$, the integrability conditions reduces to

$$k = f = d = ab-eh = gh-mo = bo-ch+ho-mo = ah+am-eh-h\ell = 0,$$

and the system has an invariant algebraic surface $F = 1+ax-hy+pz = 0$ with cofactor $C_F = ax+hy+pz$. It is not difficult to see that $\phi = xz^{-1}F^{\frac{m-b}{h}}$ is a first integral and $IJM = xF^{2+\frac{m}{h}}$ is an inverse Jacobi multiplier. Now, Theorem 11 gives a second first integral $\psi = yz(1+\cdots)$. The desired first integrals are $\phi_1 = \phi\ \psi = x(y+\cdots)$, and $\phi_2 = \psi = z(y+\cdots)$. When $h = 0$, we have $a = p = 0, bm \neq 0$ (clearly when $b = 0$ or $m = 0$, we are in Case 16) with $bg-cm-gm = 0$ and $be+b\ell-em = 0$. The system has an exponential factor $E = \exp(mex-mby+gbz)$ with cofactor $C_E = mex-mby+gbz$ which allows us to construct two independent first integrals $\phi_1 = xyE^{-\frac{1}{m}}$ and $\phi_2 = yzE^{-\frac{1}{b}}$.

Case 4.1: The system is

$$\dot{x} = x(1+by), \quad \dot{y} = -y+hy^2, \quad \dot{z} = z(1+my).$$

The transformation

$$(X,Y,Z) = (x(1-hy)^{-\frac{b}{h}}, y(1-hy)^{-1}, z(1-hy)^{-\frac{m}{h}})$$

linearizes the equations. When $h = 0$, we replace $(1-hy)^{-\frac{b}{h}}$ and $(1-hy)^{-\frac{m}{h}}$ by $\exp(-by)$

and $\exp(-my)$ respectively.

Case 4.2: The system (4.1) is

$$\dot{x} = x(1+ax+by), \quad \dot{y} = -y+axy+by^2, \quad \dot{z} = z(1+\frac{am}{b}x+my).$$

When $b \neq 0$, a linearizing change of coordinates is given by

$$(X,Y,Z) = (\frac{x}{1+ax-by}, \frac{y}{1+ax-by}, \frac{z}{(1+ax-by)^{\frac{m}{b}}}).$$

When $b = 0$, then we have two subcases:

i) If $a = 0$, $m \neq 0$, then the change of coordinates

$$(X,Y,Z) = (x, y, z\exp(-\ell x)\exp(my))$$

linearizes the system.

ii) If $m = 0$, $a \neq 0$, the transformation

$$(X,Y,Z) = (\frac{x}{1+ax}, \frac{y}{(1+ax)^{\frac{\ell}{a}}}, \frac{z}{(1+ax)^{\frac{\ell}{a}}})$$

linearizes the system.

When $a = m = 0$, we get sub cases which already have been taken into account.

Case 4.3: The system (4.1) appears

$$\dot{x} = x(1+ax+by+cy), \quad \dot{y} = -y+axy+cyz+by^2, \quad \dot{z} = z(1+ax+by+cy).$$

In this case the change of coordinates

$$(X,Y,Z) = (\frac{x}{1+ax-by+cz}, \frac{y}{1+ax-by+cz}, \frac{z}{1+ax-by+cz})$$

linearizes the system.

Case 5: The system (4.1) becomes

$$\dot{x}=x(1+ax+by), \quad \dot{y}=-y-axy+fxz-by^2, \quad \dot{z}=z(1+4ax-2by).$$

In this case, the system has invariant algebraic surface

$$\mathcal{F}=(1+ax+by)^3-\frac{1}{2a}bfz(3ax+3by+2a^2x^2+6abxy-bfxz)=0$$

with cofactor $3ax-3by$. One can find a first integral $\phi=xz^{-1}\mathcal{F}$ and inverse Jacobi multiplier $IJM=z\mathcal{F}^{\frac{1}{3}}$ and hence Theorem 11 guarantees the existence of second first integral $\psi=xy(1+\cdots)$. The desired first integrals are $\phi_1=\psi=x(y+\cdots)$ and $\psi_2=\phi^{-1}\psi=z(y+\cdots)$. When $a=0$, $\phi=xy-\frac{f}{3}x^2z$ and $\psi=xz^{-1}\hat{\mathcal{F}}^3$ are first integrals where $\hat{\mathcal{F}}=1+by-\frac{fb}{2}xz$. Then the desired first integrals are $\phi_1=\phi=x(y-\frac{f}{3}xz)$ and $\phi_2=\phi\,\psi^{-1}=z(y+\cdots)$.

Case 6: The reduced system is

$$\dot{x}=x(1+2\ell x+2my), \quad \dot{y}=-y+2\ell xy+2my^2+kz^2, \quad \dot{z}=z(1+\ell x+my).$$

A linearizing change of coordinates is given by

$$(X,Y,Z)=\left(x\mathcal{F}^{-1},(-3y+kz^2)\mathcal{F}^{-1},z\mathcal{F}^{-\frac{1}{2}}\right),$$

where $\mathcal{F}=1+2\ell x-2my+kmz^2$.

Case 7: The system is

$$\dot{x}=x(1+\frac{2}{5}\ell x-2my),\ \dot{y}=-y-\frac{2}{5}\ell xy+2my^2+kz^2,\ \dot{z}=z(1+\ell x+my). \tag{4.7}$$

The change of variables $\big(w,Z\big) = \big(x(1+\frac{2}{5}\ell x - 2my), xz^2\big)$ transforms the system (4.7) to

$$\dot{x} = w, \quad \dot{w} = (1+\frac{4}{5}\ell x)w - 2mkZ, \quad \dot{Z} = 3Z(1+\frac{4}{5}\ell x). \tag{4.8}$$

Now, we apply the transformation $(\tilde{X},\tilde{W},\tilde{Z}) = (\frac{x}{L},\frac{w}{L^2},\frac{Z}{L^3})$, where $L = 1+\frac{4}{5}\ell x$ and after rescaling the system by L we get

$$\dot{\tilde{X}} = \tilde{W}(1-\frac{4}{5}\ell\tilde{X}), \quad \dot{\tilde{W}} = \tilde{W} - 2mk\tilde{Z} - \frac{8}{5}\ell\tilde{W}^2, \quad \dot{\tilde{Z}} = 3\tilde{Z}(1-\frac{4}{5}\ell\tilde{W}) \tag{4.9}$$

Clearly the second and third equation of (4.9) gives a linearizable node and hence there exists a change of coordinates $\big(\hat{W},\hat{Z}\big) = \big(\tilde{W}+mk\tilde{Z}+O(\tilde{W},\tilde{Z}), \tilde{Z}(1+O(\tilde{W},\tilde{Z}))\big)$ such that $\dot{\hat{W}} = \hat{W}$ and $\dot{\hat{Z}} = 3\hat{Z}$. Then one first integral is given by $\phi = \frac{\hat{W}^3}{\hat{Z}}$. Pull back variables we get $\phi = x^2z^{-2}\big(1+O(2)\big)$. System (4.7) has inverse Jacobi multiplier $IJM = x^{-\frac{2}{3}}z^{\frac{5}{3}}$ and Theorem 11 therefore guarantees the existence of a second first integral of the form $\psi = x^{5/3}yz^{-2/3}(1+O(1))$. From these two integrals it is easy to construct integrals of the form required as $\phi_1 = \phi^{-1/3}\psi = x(y+\cdots)$ and $\phi_2 = \phi^{-5/6}\psi = z(y+\cdots)$.

Case 8: Through the conditions of this case we have the system

$$\dot{x} = x(1+by), \quad \dot{y} = -y+2my^2+kz^2, \quad \dot{z} = z(1+my).$$

The change of coordinates

$$(X,Y,Z) = \big(x\digamma^{-\frac{b}{2m}}, (-3y+kz^2)\digamma^{-1}, z\digamma^{-\frac{1}{2}}\big)$$

linearizes the systems where $\digamma = 1-2my+kmz^2$. When $m=0$, we are in Case 14*, which is a dual with Case 14.

Case 9: The system (4.1) can be written as

$$\dot{x} = x(1+(h-m)y), \quad \dot{y} = -y+fxz+hy^2, \quad \dot{z} = z(1+my).$$

The change of variables $(X,Y,Z) = \big(xF^{\frac{m}{h}-1}, (y-\frac{f}{3}xz)F^{-1}, zF^{-\frac{m}{h}}\big)$ where $F = 1-hy+\frac{fh}{2}xz$, linearizes the system. When $h=0$, the linearizing change is $(X,Y,Z) = (xE^{-m}, y-\frac{f}{3}xz, zE^{m})$ where $E = \exp(y-\frac{f}{2}xy)$.

Case 10: This case system (4.1), takes the form

$$\dot{x} = x(1+ax+by+cz), \quad \dot{y} = -y-axy+fxz+by^2+kz^2, \quad \dot{z} = z(1+pz).$$

The transformation

$$(X,Y) = \left(\frac{x}{1+ax-by+\zeta z}, \frac{y}{1+ax-by+\zeta z}\right),$$

where $\zeta = \frac{1}{2}(1+\sqrt{p^2-4bk})$ yields a new system

$$\begin{aligned}
\dot{X} =& X(1+(c+\zeta)z-\eta\,\zeta Xz),\\
\dot{Y} =& Y(-1-\zeta z-\eta\; Xz+\frac{bkz^2}{1+\zeta z})+fXz+(1-aX)\frac{kz^2}{1+\zeta z},\\
\dot{z} =& z(1+pz),
\end{aligned} \tag{4.10}$$

where $\eta = ac-bf-2a$. The first and third equation gives a linearizable node and then there exists a change of coordinates $\tilde{X} = \tilde{X}(X,z) = x(1+o(1))$ and $\tilde{Z} = \tilde{Z}(z) = z(1+o(1))$ such that $\dot{\tilde{X}} = \tilde{X}, \quad \dot{\tilde{Z}} = \tilde{Z}$. It remains to linearize the second equation in (4.10). We look for a change of variables of the form $\tilde{Y} = A(\tilde{X},\tilde{Z})Y + B(\tilde{X},\tilde{Z})$ with $\dot{\tilde{Y}} = -\tilde{Y}$. Then this equation implies the following differential equations:

$$\dot{A} - A\big(\zeta z+\eta Xz-\frac{bkz^2}{1+\zeta z}\big) = 0, \tag{4.11}$$

$$\dot{B} + A\big((fXz+(1-aX)\frac{bkz^2}{1+\zeta z}\big) = -B. \tag{4.12}$$

To find A, we write $A = e^{\lambda}$ and solve $\dot{\lambda} = F(\tilde{X},\tilde{Z})$ where $F(\tilde{X},\tilde{Z}) = \zeta z+\eta Xz-\frac{bkz^2}{1+\zeta z}$.

Writing $\lambda = \sum_{i+j>0} c_{ij}\tilde{X}^i\tilde{Z}^j$, $F = \sum_{i+j>0} d_{ij}\tilde{X}^i\tilde{Z}^j$, then

$$\sum_{i+j>0} (i+j)c_{ij}\tilde{X}^i\tilde{Z}^j = \sum_{i+j>0} d_{ij}\tilde{X}^i\tilde{Z}^j,$$

and hence $c_{ij} = \frac{d_{ij}}{i+j}$ for $i+j>0$. The convergence of $F(\tilde{X},\tilde{Z})$, guarantees the convergence of $\lambda(\tilde{X},\tilde{Z})$. Again we write $B(\tilde{X},\tilde{Z}) = \sum_{i+j>0} b_{ij}\tilde{X}^i\tilde{Z}^j$, $G(\tilde{X},\tilde{Z}) = \sum_{i+j>0} a_{ij}\tilde{X}^i\tilde{Z}^j$ where $G = fXz + (1-aX)\frac{bkz^2}{1+\zeta z}$. From (4.12), we see $b_{ij} = \frac{a_{ij}}{i+j+1}$ for $i+j>0$. Again the convergence of G, guarantee the convergence of B.

Case 11: If $b=0$, we obtain sub-cases of Case 14. If $b \neq 0$, the system takes the form

$$\dot{x} = x(1+ax+by+cz), \quad \dot{y} = -y-axy-\frac{ac}{b}xz-by^2+kz^2, \quad \dot{z} = z(1+2ax+pz).$$

which has invariant algebraic surfaces $F_1 = 1+ax+by+\frac{1}{2}(p+\sqrt{p^2+4kb})z = 0$ and $F_2 = 1+ax+by+\frac{1}{2}(p-\sqrt{p^2+4kb})z = 0$ with cofactors $C_{F_1} = ax-by+\frac{1}{2}(p+\sqrt{p^2+4kb})z$ and $C_{F_2} = ax-by+\frac{1}{2}(p-\sqrt{p^2+4kb})z$ which producing a first integral

$$\phi = xz^{-1}F_1^{-\frac{2c-p-\sqrt{p^2+4bk}}{2\sqrt{p^2+4bk}}}F_2^{\frac{2c-p+\sqrt{p^2+4bk}}{2\sqrt{p^2+4bk}}},$$

and inverse Jacobi multiplier

$$IJM = zF_1^{\frac{2c+p+\sqrt{p^2+4bk}}{2\sqrt{p^2+4bk}}}F_2^{\frac{-2c-p+\sqrt{p^2+4bk}}{2\sqrt{p^2+4bk}}}.$$

Now, Theorem 11 guarantees the second first integral $\psi = xy(1+\cdots)$. Therefore, the desired first integrals are $\phi_1 = \psi = x(y+\cdots)$ and $\phi_2 = \phi^{-1}\psi = z(y+\cdots)$.
When $\sqrt{p^2+4bk} = 0$ we replace $F_1^{-\frac{2c-p-\sqrt{p^2+4bk}}{2\sqrt{p^2+4bk}}}$ and $F_1^{\frac{2c+p+\sqrt{p^2+4bk}}{2\sqrt{p^2+4bk}}}$ by $(1+ax+by+\frac{1}{2}p)$ as well as $F_2^{\frac{2c-p+\sqrt{p^2+4bk}}{2\sqrt{p^2+4bk}}}$ by $F_3^{\frac{1}{2}p-c}$ and $F_2^{\frac{-2c-p+\sqrt{p^2+4bk}}{2\sqrt{p^2+4bk}}}$ by $F_3^{\frac{1}{2}p+c}$ where $F_3 = \exp(\frac{z}{1+ax+by+\frac{1}{2}p})$.

Case 12: The corresponding system is

$$\dot{x} = x(1+ax), \quad \dot{y} = -y+dx^2+hy^2, \quad \dot{z} = z(1+\ell x+my).$$

When $ah \neq 0$, the system has invariant algebraic surfaces $F_1 = 1 + ax = 0$, $F_2 = 1 + \frac{1}{2}(a + \sqrt{a^2 - 4hd})x - hy = 0$ and $F_3 = 1 + \frac{1}{2}(a - \sqrt{a^2 - 4hd})x - hy = 0$ with cofactors $C_{F_1} = ax$, $C_{F_2} = \frac{1}{2}(a + \sqrt{a^2 - 4hd})x + hy$ and $C_{F_3} = \frac{1}{2}(a - \sqrt{a^2 - 4hd})x + hy$. A linearizing change of coordinates of the first and third equation is given by the transformation

$$(X, Z) = \left(xF_1^{-1}, zF_1^{\frac{m}{2h} - \frac{l}{a}}(F_2F_3)^{-\frac{m}{2h}}\right).$$

Note that

$$\begin{aligned} y_1 = &\frac{(1+ax)^{-\frac{\sqrt{a^2-4hd}}{a}}\left(1 + \frac{1}{2}(a + \sqrt{a^2 - 4hd})x\right) - 1 - \frac{1}{2}(a - \sqrt{a^2 - 4hd})x}{h\left((1+ax)^{-\frac{\sqrt{a^2-4hd}}{a}} - 1\right)} \\ = &\frac{1}{3}dx^2(1 + \cdots) \end{aligned}$$

is a particular solution of the equation $\dot{y} = -y + dx^2 + hy^2$ (or the corresponding Riccati equation $\frac{dy}{dx} = -\frac{y}{x(1+ax)} + \frac{dx}{1+ax} + \frac{hy^2}{x(1+ax)}$). The change of variable $y = Y + y_1$ transform the Riccati equation to $\dot{Y} = Y(-1 + 2hy_1 + hY)$. Note that this equation is exactly as Case 2 and hence is linearizable. When $a = 0$, $h \neq 0$, the system has an exponential factor $F_1 = \exp(x)$ and invariant algebraic surfaces $F_2 = 1 + i\sqrt{hd}x - hy = 0$ and $F_3 = 1 - i\sqrt{hd}x - hy = 0$ with cofactors $C_{F_1} = x$, $C_{F_2} = i\sqrt{hd}x + hy$ and $C_{F_3} = -i\sqrt{hd}x + hy$. A linearizing transformation of the first and third equation is given by

$$(X, Z) = \left(x, zF_1^{-\ell}(F_2F_3)^{-\frac{m}{2h}}\right).$$

In this case the Riccati equation above has a particular solution

$$y_1 = \frac{1}{h} + \frac{i\sqrt{dh}}{h}x\coth(i\sqrt{dh}x) = \frac{1}{3}dx^2(1 + \cdots)$$

and in the same way, the transformation $y = Y + y_1$ linearizes the Riccati equation. When $h = 0$, $a \neq 0$, $E = \exp(dx - ay)$ is an exponential factor with cofactor $dx + ay = 0$. The first and third equations linearize by

$$(X,Z)=\left(xF_1^{-1}, zF_1^{\frac{md}{a^2}-\frac{l}{a}}E^{-\frac{m}{a}}\right).$$

In this case

$$y_1=\frac{d}{a^3}\frac{(-2-2ax)ln(1+ax)+2ax+a^2x^2}{x}=\frac{1}{3}dx^2(1+\cdots)$$

is a particular solution of the linear differential equation $\dot{y}=-y+dx^2$ (or the corresponding linear equation $\frac{dy}{dx}=-\frac{y}{x(1+ax)}+\frac{dx}{1+ax}$). The transformation $y=Y+y_1$ linearizes the equation above. Finally when $a=h=0$, then the linearizing change is given by

$$(X,Y,Z)=(x,3y-dx^2,ze^{-(\ell x-my+\frac{1}{2}mdx^2)}).$$

Case 13: After plugging in the conditions for this case, system (4.1) takes the form

$$\dot{x}=x(1+ax),\quad \dot{y}=-y+dx^2+2my^2+kz^2,\quad \dot{z}=z(1+\ell x+my). \tag{4.13}$$

The transformation $(Y,Z)=(yF^{-1},zF^{-\frac{1}{2}})$ where $F=1-2my+kmz^2+\frac{1}{2}\lambda x=0$, $\lambda=a+\sqrt{a^2-8md}$, converts the system to

$$\begin{aligned}\dot{x}=&x(1+ax),\\ \dot{Y}=&-Y+\frac{dx^2(1+2mY-kmZ^2)}{1+\lambda x}+kZ^2-\lambda xY-km(1-2\ell)\lambda xYZ^2,\\ \dot{Z}=&Z\left(1+\ell x-\frac{1}{2}\lambda xZ-\frac{1}{2}mk(1-2\ell)\lambda xZ^2\right).\end{aligned} \tag{4.14}$$

Clearly, the first and third equations give a linearizable node and hence there is a linearizing transformation $\tilde{X}=x(1+O(x,z))$ and $\tilde{Z}=Z(1+O(x,Z))$ such that $\dot{\tilde{X}}=\tilde{X}$, $\dot{\tilde{Z}}=\tilde{Z}$. To linearize the second equation from (4.14), we looke for a transformation of the form $\tilde{Y}=A(\tilde{X},\tilde{Z})Y+B(\tilde{X},\tilde{Z})$ such that $\dot{\tilde{Y}}=-\tilde{Y}$. This implies that

$$\dot{A}-A\left(\lambda x-2mdx^2(1+\lambda x+\cdots)+\lambda km(1-2\ell)xZ^2\right)=0, \tag{4.15}$$

and

$$\dot{B} + B = -A\big(dx^2(1 - kmZ^2)(1 + \lambda x + \cdots) + kZ^2\big). \tag{4.16}$$

As in Case 10, one can find such A and B which satisfies (4.15) and (4.16).

Case 14: The system is given by

$$\dot{x} = x(1 + ax + cz), \quad \dot{y} = -y + dx^2 + exy + fxz + gyz + kz^2, \quad \dot{z} = z(1 + \ell x + pz).$$

The first and third equations gives a linearizable node, so there exists a change of coordinates $X = x(1 + O(x,z))$ and $Z = z(1 + O(x,z))$ such that

$$\dot{X} = X, \qquad \dot{Z} = Z.$$

Even though it is not easy to find an explicit change of coordinates to linearize the second equation, we might be able to find its linearizing transformation using the power series arguments. To do so, we seek a transformation $Y = \alpha + \beta y$ where $\alpha = \alpha(X,Z)$ and $\beta = \beta(X,Z)$ such that $\dot{Y} = -Y$. We therefore need to prove the following equations

$$\dot{\alpha} + \beta(dx^2 + fxz + kz^2) = -\alpha, \tag{4.17}$$

and

$$\dot{\beta} + \beta(ex + gz) = 0. \tag{4.18}$$

By the same way as Case 10, we can find such α and β satisfies (4.17) and (4.18).

Case 15: The reduces system is given by

$$\dot{x} = x(1 + ax + cz), \quad \dot{y} = -y + dx^2 + fxz + hy^2 + kz^2, \quad \dot{z} = z(1 + \ell x + pz).$$

The first and third equations gives a linearizable node. We denote the linearizing coordinates by X and Z. In order to linearize the second equation, note that the second equation is a Riccati equation and suppose that it has a solution of the form $y=-\frac{\dot{U}}{hU}$ where $\dot{U}=X\frac{\partial U}{\partial X}+Z\frac{\partial U}{\partial Z}$. This gives

$$\ddot{U}+\dot{U}=-hU(dx^2+fxz+kz^2). \tag{4.19}$$

Write U, x and z as a power series

$$U=\sum a_{ij}X^iZ^j,\quad x=\sum b_{ij}X^iZ^j,\quad z=\sum c_{ij}X^iZ^j.$$

By equation (4.19), we have

$$\begin{aligned}\sum(i+j+1)(i+j)a_{ij}X^iZ^j=&-h\sum a_{ij}X^iZ^j\Big(d\big(\sum b_{ij}X^iZ^j\big)^2+\\&f\sum b_{ij}X^iZ^j\sum c_{ij}X^iZ^j+k\big(\sum c_{ij}X^iZ^j\big)^2\Big).\end{aligned}$$

We can now find each a_{ij} uniquely by induction, and the convergence of U can be obtained by standard majorization techniques. As in Case 3 and Case 14, the transformation $Y=y-U$ will linearize the second equation.

Case 16: The reduced system is

$$\dot{x}=x(1+by),\quad \dot{y}=-y+dx^2+fxz+2by^2+kz^2,\quad \dot{z}=z(1+by).$$

This case has invariant algebraic surfaces $\digamma_1=1-2by+bdx^2+fbxz+kbz^2=0$ and

$$\begin{aligned}\digamma_2=&1-6by+3dbx^2+3fbxz+12b^2y^2+3kbz^2-12b^2dx^2y-12b^2fxyz-8b^3y^3\\&-12kb^2yz^2+3b^2d^2x^4+6fdb^2x^3z+3b^3dx^2y^2+(6b^2dk+3f^2b^2)x^2z^2\\&+3b^3fxy^2z+6kb^2fxz^3+3kb^3y^2z^2+3b^2k^2z^4=0.\end{aligned}$$

The change of coordinates $(X,Y,Z)=(x\digamma_1^{-\frac{1}{2}},x\digamma_2^{-\frac{1}{6}},z\digamma_1^{-\frac{1}{2}})$ linearize the system.

Case 17: The system (4.1) can be written as

$$\dot{x}=x(1+ax+by+cz),\ \dot{y}=-y+dx^2-2axy+fxz-2cyz-\frac{m}{2}y^2+kz^2,\ \dot{z}=z(1+ax+by+cz).$$

The system has an invariant algebraic surface

$$\mathcal{F}=y-\frac{1}{3}dx^2+axy-\frac{1}{3}fxz+\frac{1}{2}by^2+cyz-\frac{1}{3}kz^2=0$$

with cofactor $C_{\mathcal{F}}=-(1+ax+by+cz)$ which allows the contraction of two independent first integrals in the desired form $\phi_1=x\mathcal{F}=x(y+\cdots)$ and $\phi_2=x\mathcal{F}=z(y+\cdots)$.

$\square$

Chapter 5

The study of integrability problem via monodromy method

In this chapter, we are interested in studying the integrability problem using a different approach from the methods which have been used in the previous chapters; for ease of references, we shall call this the monodromy method, although it encompasses a variety of ideas. We show that most of the sufficient integrability conditions for Lotka-Volterra equations with $(1:-1:1)$, $(2:-1:1)$, $(1:-2:1)$ and $(3:-1:2)$–resonance can be proved by utilizing the monodromy method. It is possible that some of the remaining cases could also be tackled by this method if we have a stronger geometrical understanding of monodromy.

5.1 The monodromy group

In this section, we will explain the concept of monodromy in two dimensional systems. Generally speaking, the monodromy around a singularity describes the behaviour of trajectories near a non-trivial loop which surrounding a singularity. For each closed loop on a leaf of a foliation, there is an associate *monodromy map* of the loop. It can be visualised as the complex equivalent to the Poincaré return map in the real domain.

We consider the foliation on $\mathbb{CP}^2$ of the 1–form corresponding to a polynomial vector field. Let $\mathcal{L}$ be a leaf of the foliation and let $\tilde{\mathcal{L}} = \mathcal{L} - \{\text{singular points}\}$. Suppose that γ is a non-trivial loop in $\mathcal{L}$ and select a base point p on it. Let Γ_p be a transversal at a point $p \in \gamma$. Then the monodromy will be a map $h_\gamma : (\mathbb{C},0) \to (\mathbb{C},0)$ is defined by lifting the loop γ to the leaves of the foliation near to γ passing through Γ_p. This process is well–defined if we have series of transversals through each point of Γ. Then, for each loop γ at p, h_γ can be identified by the element of a diffeomorphism from neighbourhood of the origin of $\mathbb{C}$ to itself. Let Diff($\mathbb{C}$,0) denotes the set of all such diffeomorphisms, then we get a representation $h : \pi(\tilde{\mathcal{L}}, P) \to$ Diff($\mathbb{C}$,0). The image of h is known as the *monodromy group*. For more detail about monodromy, one can consult (Arnold and Ilyashenko, 1988; Zakeri, 2001; Christopher and Rousseau, 2004) and the references therein.

The following remarks on monodromy map can be found in (Arnold and Ilyashenko, 1988; Christopher and Rousseau, 2004).

Remark 8.
1. *The monodromy of a loop does not change if we choose a different loop homotopic to the original one.*

2. *If we select different family of transversals, we get a monodromy map conjugate to the original one.*

3. *A different initial point on a loop on the leaf produces a conjugate monodromy map to the original one.*

4. *Let S_i be n singular points for $i = 1, \cdots, n$. Let γ_i be non trivial loops surrounding S_i once in the positive direction. Let M_{γ_i} denotes the associated the monodromy maps of γ_i. The composition of loops γ_i, corresponds to the composition of the associated monodromy map. That is, $\gamma_1 \circ \cdots \circ \gamma_n \mapsto M_{\gamma_1} \circ \cdots \circ M_{\gamma_n}$. If γ_1 is homotopic to $\gamma_n^{-1} \circ \cdots \circ \gamma_2^{-1}$, then M_{γ_1} is conjugate to $M_{\gamma_n}^{-1} \circ \cdots \circ M_{\gamma_2}^{-1}$. This will happen if $\mathcal{L}$ is $\mathbb{P}^1$ and the S_i are all the singularities on $\mathcal{L}$.*

5. *It is well known that for two dimensional systems with saddle at the origin, the saddle is integrable if and only if its monodromy map is linearizable.*

6. *A node is linearizable if its resonant monomials have zero coefficients. This is realized if it has two analytic separatrices or one analytic separatrix with the ratio of eigenvalues is natural number and greater than 1. Note that the right angle*

represents the direction of the eigenvalue which is the numerator of the eigenvalue ratio.

5.2 Using the monodromy method

We now describe the monodromy method and how we can use it to prove the sufficiency of integrability conditions of vector fields.

We study the monodromy group of one of the axes which are the separatrices of the origin, together with the monodromy of the line at infinity. Each of these lines as a Riemann sphere $\mathbb{P}^1$.

We examine the monodromy on one axis, say the x-axis. First one can find all singular points on the x-axis including the origin and at infinity. Let us we have three of such points: one at the origin P_0; one at the finite plane P_f; and one at infinity P_∞. Then we evaluate the Jacobian matrix and the eigenvalues at these points. Now, let M_0, M_f and M_∞ be the corresponding monodromy maps of the 0, P_f and P_∞ respectively. As we mentioned in Remark 8, the origin is integrable if and only if its monodromy map M_0 is linearizable. Note that the monodromy of the origin M_0 is equal to $M_\infty^{-1} \circ M_f^{-1}$. Hence the first way to get a sufficient condition for linearizability of M_0 is given by proving M_∞ is linearizable and M_f is the identity. To show M_f is the identity, we can consider whether the associated critical point at p_f is a node with zero coefficients of resonant monomials. That is, the ratio of eigenvalues of P_f is a natural number. Furthermore, M_∞ is linearizable if the critical point P_∞ is a node and hence is linearizable as in this case we have two analytic separatrices. The second way to obtain a sufficient condition is the other way around. We mean by this, the linearizability of M_0 will obtained if M_∞ is identity monodromy and M_f is linearizable. The former is recognized if the ratio of eigenvalues of P_∞ is a natural number. Since, in this case, it has two analytic separatrix, then it is a node with zero coefficients of resonant monomials. The later is understood as it is a non resonant node or it is a node with resonant monomials have zero coefficients.

5.3 Monodromy method for two-dimensional systems

In this section, we give some examples to show the power of the monodromy method to prove the sufficiency of integrability conditions for a family of two dimensional systems as well as to show its ease in applications.

In (Chen et al., 2012), the author considered systems of the form

$$\begin{aligned} \dot{x} &= x(1 - a_{20}x^2 - a_{11}xy - a_{02}y^2), \\ \dot{y} &= y(-q + b_{20}x^2 + b_{11}xy + b_{02}y^2), \end{aligned} \tag{5.1}$$

where $q \in \mathbb{N}$, $q \neq 1$. They found sufficient integrability conditions at the origin for $(1 : -q)$–resonance suggested by computing focus quantities for system (5.1) for selected values of q and they proved integrability in these cases. The following conditions are derived in (Chen et al., 2012, Theorem 3.3, page 11625, condition (b) for even q only, (e) and (f)).

Theorem 13. *Consider system* (5.1). *The origin is integrable if one of the following conditions is satisfied.*

1. $\frac{q}{2} - \frac{b_{20}}{2a_{20}} = n$ *with* $n \in \mathbb{N}$, $4 \leq n < \frac{q+1}{2}$.

2. $a_{11} = b_{11} = 0$ *and* $\frac{q}{2} - \frac{b_{20}}{2a_{20}} = n$ *with* $n \in \mathbb{N}$, $3 \leq n < \frac{q+1}{2}$.

3. $a_{11} = b_{20} = b_{11} = 0$ *and* q *is even.*

Here our aim is therefore to use a simpler method to find integrability conditions (1)–(3) in theorem below. For this purpose we use monodromy method.

Proof. We examine the monodromy map on x–axis. We first find all singular points on the x-axis, then evaluate the Jacobian matrix and the eigenvalues at these points. This will include the line at infinity.

We have four such singular points, one at the origin, $O = (0,0)$, two at the and finite plane, $P_{1xf} = (\frac{1}{\sqrt{a_{20}}}, 0)$ and $P_{2xf} = (\frac{1}{\sqrt{a_{20}}}, 0)$, and they have respective Jacobian matrices

$$J_O = \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}, \quad J_{P_{1xf}} = \begin{pmatrix} -2 & -\frac{a_{11}}{a_{20}} \\ 0 & -q + \frac{b_{20}}{a_{20}} \end{pmatrix}, \quad J_{P_{2xf}} = \begin{pmatrix} -2 & -\frac{a_{11}}{a_{20}} \\ 0 & -q + \frac{b_{20}}{a_{20}} \end{pmatrix}.$$

We also need singular points at infinity. For this purpose, we use a change of coordinates $(u, v) = (\frac{y}{x}, \frac{1}{x})$ which gives the system (after a rescaling of the variables)

$$\dot{u} = u((a_{20} + b_{20}) + (a_{11} + b_{11})u + (a_{02} + b_{02})u^2 - (q+1)v^2),$$
$$\dot{v} = v(a_{20} + a_{11}u + a_{02}u^2 - v^2).$$

Now it is clear that line at infinity, $v = 0$, intersects with the x-axis at $P_{x\infty}|_{(u,v)} = (0, 0)$ and has Jacobian matrix

$$J_O = \begin{pmatrix} b_{20} + a_{20} & 0 \\ 0 & a_{20} \end{pmatrix}.$$

Now let M_0, M_{1xf}, M_{2xf} and $M_{x\infty}$ be the corresponding monodromy map of the origin, P_{1xf}, P_{2xf} and $P_{x\infty}$ respectively. We now apply the monodromy method which has been explained in the previous section. As we mentioned in the first note in Remark 8, the origin is integrable if and only if its monodromy map is linearizable. Note that the monodromy of the origin M_0 is conjugate to $M_{x\infty}^{-1} \circ M_{2xf}^{-1} \circ M_{1xf}^{-1}$. Hence a sufficient condition for the linearizability of M_0 is given by $M_{x\infty}$ being linearizable and M_{1xf}, M_{2xf} being the identity. Noting that $\frac{q}{2} - \frac{b_{20}}{2a_{20}}$ is just the ratio of eigenvalues at p_{1xf} and p_{2xf}, if $\frac{q}{2} - \frac{b_{20}}{2a_{20}} = n$ with $n \in \mathbb{N}$, $n \geq 2$, then M_{1xf} and M_{2xf} are identity because, in this case, we have a node such that all its resonant monomials have zero coefficients. We also need the singular point at infinity, $P_{x\infty}$, to be a node. For that purpose, the ratio of eigenvalues at this point must be positive. That is, $1 + \frac{b_{20}}{2a_{20}} > 0$. In this case, this node is linearizable by Remark 6 because we have two analytic separatrices at $P_{x\infty}$. It is not difficult to see that for $q \geq 4$, the integrability condition is therefore given by $\frac{q}{2} - \frac{b_{20}}{2a_{20}} = n$ with $2 \leq n < \frac{q+1}{2}$. When $q \geq 3$, we need extra conditions to make sure that all the resonant monomials have zero coefficients at the finite singular points P_{1xf} and P_{2xf}. In this case, we need $a_{11} = b_{11} = 0$. Thus the integrability condition will be $a_{11} = b_{11} = 0$ and $\frac{q}{2} - \frac{b_{20}}{2a_{20}} = n$ with $n \in \mathbb{N}$, $3 \le n < \frac{q+1}{2}$. Finally, if $b_{20} = 0$, then clearly the ratio of eigenvalues at p_{1xf} and p_{2xf} is $\frac{q}{2}$ which gives identity monodromy if q is even and at $P_{x\infty}$ is 1 which gives a node. Again $a_{11} = b_{11} = 0$ guarantees that all resonant monomials have zero coefficients.

The diagram 5.1 below show the ratio of eigenvalues where the arrows stands for the direction of the eigenvalues that are on the numerator of the eigenvalue ratio. The sum of these ratios of eigenvalue is 1 by the index formula of Lins Neto (1988).

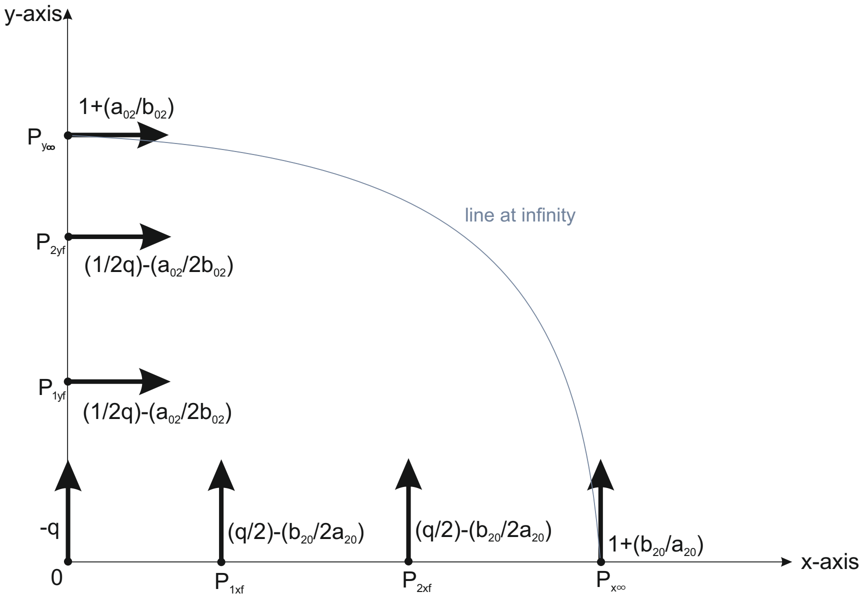


Figure 5.1: Ratio of eigenvalues at the origin, finite plane and infinity for system (5.1)

As a further example, Liu et al. (2012), considered the system

$$\begin{aligned} \dot{x} &= x(1 + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3), \\ \dot{y} &= y(-3q + 1 + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3). \end{aligned} \tag{5.2}$$

They state that the origin is integrability for system (5.2), if $q > 2$ and

$$\prod_{i=1}^{q-2}[(3i-1)a_1 + b_1] = 0.$$

Here we prove this integrability condition for system (5.2) again using monodromy method. The procedure is exactly as the example above. The system (5.2) has five singular points on the x–axis including the origin and infinity. Let us denote the singular points in the finite plane by P_{1xf}, P_{2xf}, P_{3xf} and the one at infinity by $P_{x\infty}$.

The Figure 5.2 below, shows the ratio of eigenvalues which can be found by some easy calculation as above. Note that the ratio of eigenvalues at $P_{x\infty}$ is $1 - \frac{b_1}{a_1}$ and at P_{1xf}, P_{2xf} and P_{3xf} it is $q - \frac{1}{3} + \frac{1}{3}\frac{b_1}{a_1}$. We need $P_{x\infty}$ to be a node, so need $1 - \frac{b_1}{a_1} > 0$, and P_{1xf}, P_{2xf}, P_{3xf} to have identity monodromy, so we need $q - \frac{1}{3} + \frac{1}{3}\frac{b_1}{a_1} = n$ for $n \geq 2$. Therefore the integrability condition by the monodromy method is

$$q - \frac{1}{3} + \frac{1}{3}\frac{b_1}{a_1} = n, \quad 2 \leq n < q \quad q > 2.$$

5.4 Monodromy method for three-dimensional systems

In this section, we want to prove that the majority of the sufficiency proofs of the integrability conditions at the origin that were considered in chapter two and appendix A can be found using the monodromy method.

We now explain the monodromy method for three dimensional Lotka–Volterra systems by examining the monodromy group on, say, the y–axis, together with its intersection with the planes at infinity. Each axis can be considered as $\mathbb{CP}^1$. On the y–axis, we have three singular points: one at the origin, p_0, one at the finite plane, p_f, and one at infinity, p_∞. We find the Jacobian matrix and the eigenvalues at all these points. The corresponding monodromy maps of the singular points p_0, p_f and p_∞ are denoted by M_0, M_f and M_∞ respectively.

Remark 9. *1. A critical point at the origin in the Siegel domain is integrable if and*

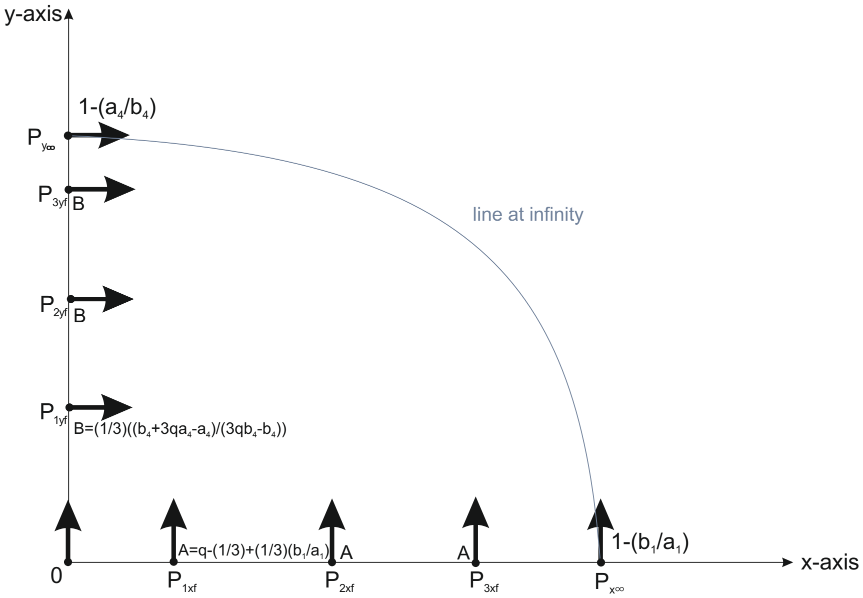


Figure 5.2: Ratio of eigenvalues at the origin, finite plane and infinity for system (5.2)

only if its monodromy, if the direction of the ratio of eigenvalues are positive on y–axis and negative on x–axis and z–axis, is linearizable. This follows from a generalization theorem in (Ilyashenko and Yakovenko, 2008, Theorem 22.7, p. 411).

2. *In three dimensional system, the monodromy map is of the form* $h_\gamma : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$.

3. *A singular point whose eigenvalues lie in the Poincaré domain, can be brought to normal form by analytic change of coordinates. It is linearizable if it has no resonant monomials or it has a finite number of resonant monomials with zero coefficients.*

4. *We can iterate the procedure above to all axes including the ones at infinity. Note that for three dimensional Lotka–Volterra equations, we have three invariant lines at infinity.*

The figure below, explain the ratio of eigenvalues at all singular points.

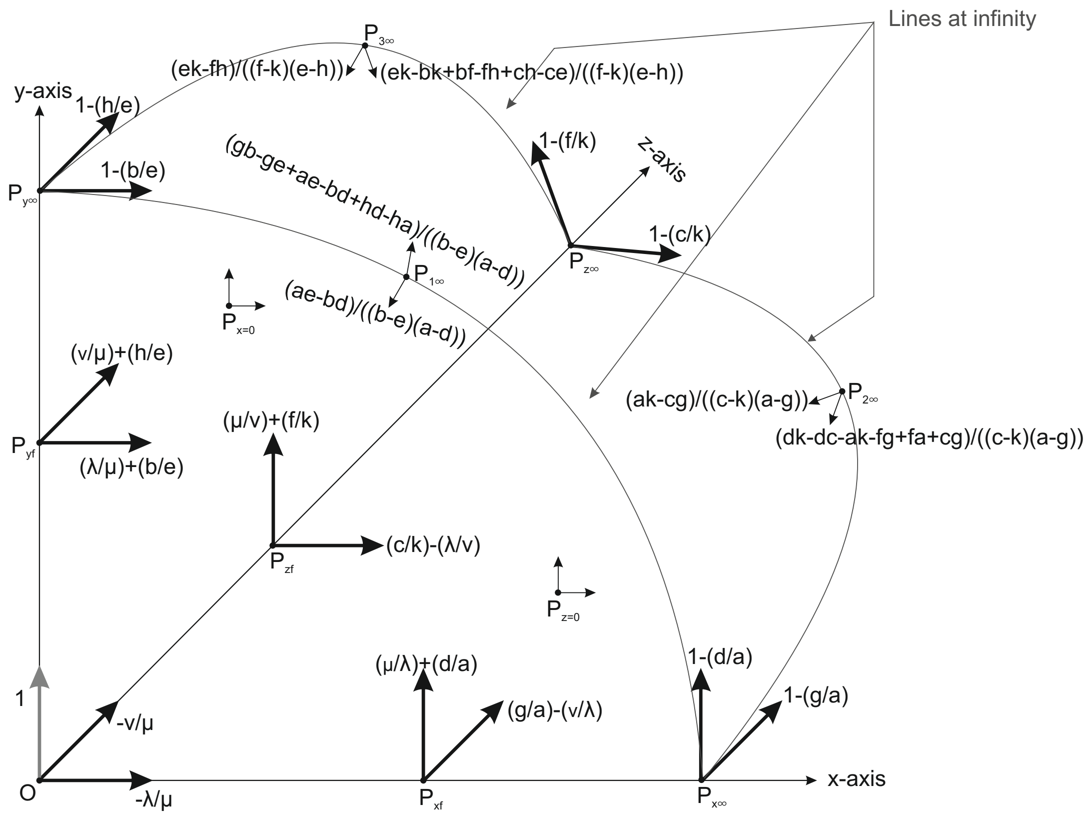


Figure 5.3: Ratio of eigenvalues at the origin, finite plane and infinity for Lotka–Volterra system with $\lambda : -\mu : \nu$.

Before we apply the monodromy method above, we need the following theorem:

Theorem 14 (Blow-up a line of singularities)**.** *Consider the system*

$$\begin{aligned} \dot{x} &= xA(x,y,z) + yB(x,y,z), \\ \dot{y} &= yC(x,y,z), \\ \dot{z} &= xD(x,y,z) + yE(x,y,z), \end{aligned} \tag{5.3}$$

where A, B, C, D and E are polynomials such that one of the following holds:

1. $C(0,0,z) = A(0,0,z)$, $B(0,0,z) = 0$ *and* $A(0,0,0) \neq 0$.

2. $C(0,0,z) = 2A(0,0,z)$ *and* $A(0,0,0) \neq 0$.

If system (5.3) *has a has a line of singularity, then the monodromy is the identity.*

Proof.

1. Suppose system (5.3) has a line of singularities $\mathcal{L}$. We now blow-up this line via the transformation $(x,Y,z) = (x, \frac{y}{x}, z)$ to give a new system

$$
\begin{aligned}
\dot{x} &= x\tilde{A} + xY\tilde{B}, \\
\dot{Y} &= Y(\tilde{C} - \tilde{A}) - Y^2\tilde{B}, \\
\dot{z} &= x\tilde{D} + xY\tilde{E}.
\end{aligned}
\tag{5.4}
$$

Note that $\tilde{\xi}(x,Y,z) = \xi(x,xY,z)$ where ξ can be A, B, C, D or E.

By hypothesis $B(0,0,z) = 0$ and this means that $B(x,y,z)$, can be written as

$$B(x,y,z) = xB_1(x,y,z) + yB_2(x,y,z).$$

Similarly

$$C(x,y,z) - A(x,y,z) = xf_1(x,y,z) + yf_2(x,y,z).$$

Since $y = xY$, then

$$\tilde{B}(x,Y,z) = x(\tilde{B}_1(x,Y,z) + Y\tilde{B}_2(x,Y,z)) = x\tilde{B}_3(x,Y,z).$$

and

$$\tilde{C}(x,Y,z) - \tilde{A}(x,Y,z) = x(\tilde{f}_1(x,Y,z) + Y\tilde{f}_2(x,Y,z)) = x\tilde{F}_3(x,Y,z).$$

Hence system (5.4) can be written as

$$
\begin{aligned}
\dot{x} &= x(\tilde{A} + Y\tilde{B}_3), \\
\dot{Y} &= x(Y\tilde{F}_3 - Y^2\tilde{B}_3), \\
\dot{z} &= x(\tilde{D} + Y\tilde{E}).
\end{aligned}
\tag{5.5}
$$

Clearly we have a surface of singularities $x = 0$ which can be removed by rescaling.

Hence, we obtain a system

$$\begin{aligned}
\dot{x} &= \tilde{A} + Y\tilde{B}_3,\\
\dot{Y} &= Y\tilde{F}_3 - Y^2\tilde{B}_3, \qquad (5.6)\\
\dot{z} &= \tilde{D} + Y\tilde{E}.
\end{aligned}$$

which has no singular point at the origin because $\tilde{A}(0,0,0) \neq 0$. Thus, the monodromy is trivial. However, the topology of the vector field (in particular, the monodromy) is unchanged outside the blow-up locus and hence must be trivial also (See Figure 5.4).

2. We use the same transformation $(x,Y,z) = (x, \frac{y}{x}, z)$ which gives

$$\begin{aligned}
\dot{x} &= x\tilde{A} + xY\tilde{B},\\
\dot{Y} &= Y(\tilde{C} - \tilde{A}) - Y^2\tilde{B}, \qquad (5.7)\\
\dot{z} &= x\tilde{D} + xY\tilde{E}.
\end{aligned}$$

When $C(0,0,z) = 2A(0,0,z)$, still we have $\tilde{C}(0,0,z) = 2\tilde{A}(0,0,z)$. Taking $\hat{C} = \tilde{C} - \tilde{A} - Y\tilde{B}$, we get

$$\begin{aligned}
\dot{x} &= x\tilde{A} + xY\tilde{B},\\
\dot{Y} &= Y\hat{C}, \qquad (5.8)\\
\dot{z} &= x\tilde{D} + xY\tilde{E}.
\end{aligned}$$

Now when $\tilde{C}(0,0,z) = 2\tilde{A}(0,0,z)$, we are in Case 1. Hence the monodromy must be the trivial.

We now apply the monodromy method to prove the sufficiency of the majority of the integrability conditions for $(1:-1:1)$, $(2:-1,1)$, $(1:-2,1)$ and $(3:-1:2)$–resonance.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ denote $(1:-1:1)$–resonant, $(2:-1,1)$–resonant $(1:-2:1)$–resonant and $(3:-1:2)$–resonant cases respectively. So, $\mathcal{A}_1$ refers to Case 1 in $(1:-1:1)$–resonance case. There are many approaches to prove the sufficiency of integrability conditions for these resonances mentioned above. We briefly explain all of them.

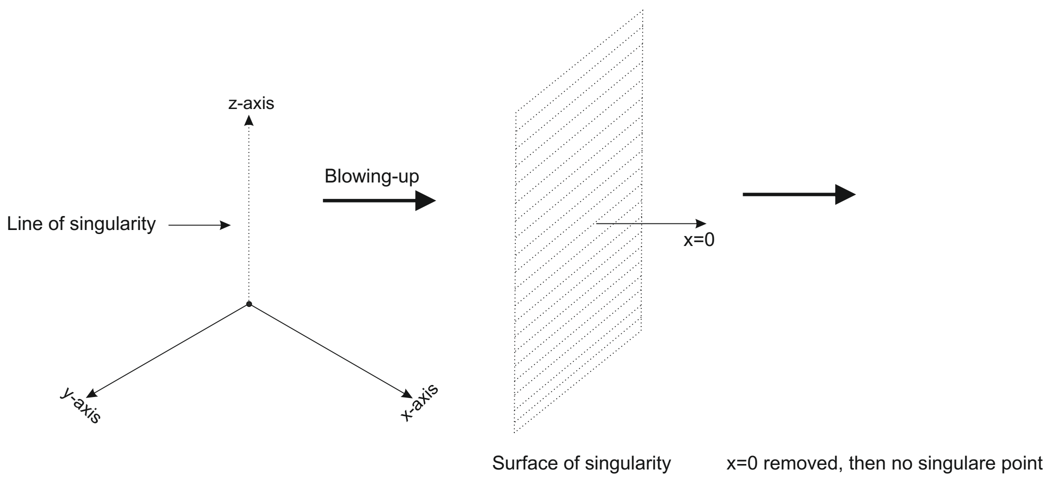


Figure 5.4: Blowing-up of a line of singularity

I) Resonant coefficients zero

- $\mathcal{A}_2$, $\mathcal{B}_5$, $\mathcal{B}_9$, $\mathcal{C}_1$, $\mathcal{C}_{12}$, $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_5$, $\mathcal{D}_{11}$, $\mathcal{D}_{12}$ and $\mathcal{D}_{20}$: In these cases, the integrability conditions imply that the monodromy corresponding to $P_{y\infty}$ is identity and the critical point P_{yf} is in the Poincaré domain and we need only to check that all resonant monomials have zero coefficients and hence M_{yf} linearizable. Thus, the origin is integrable. For example case $\mathcal{A}_2$ is explained in diagram 5.5.

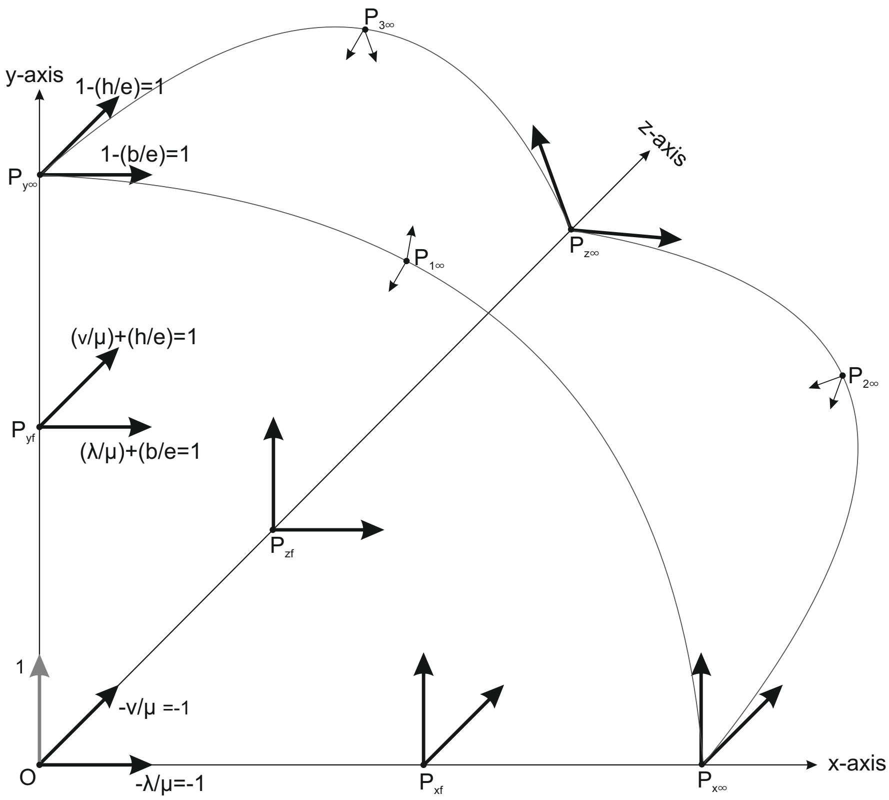


Figure 5.5: Ratio of eigenvalues for the case $\mathcal{A}_2$.

II) Integrability via line of singularities

- $\mathcal{A}_3$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{B}_7$, $\mathcal{D}_6$, $\mathcal{D}_7$, $\mathcal{D}_{15}$, $\mathcal{D}_{16}$, $\mathcal{D}_{17}$ and $\mathcal{D}_{18}$: In these cases, the monodromy of $P_{y\infty}$ is identity via blow-up of the line of singularities (Theorem 14) and the finite critical point of P_{yf} is linearizable. Therefore, the origin in integrable. The case $\mathcal{B}_4$ is shown in Figure 5.6.

- $\mathcal{C}_{10}$ and $\mathcal{C}_{13}$: The integrability conditions for these cases implies that the monodromy of $P_{y\infty}$ is identity via blow-up the line of singularities and the finite critical point of P_{yf} is linearizable with a resonant monomial which has a zero coefficients. Hence, the origin in integrable.

- $\mathcal{B}_{11}$: The monodromy of the critical point of $P_{y\infty}$ is linearizable and the finite critical point P_{yf} has identity monodromy via blow-up the line of singularities. Thus, the origin is integrable.

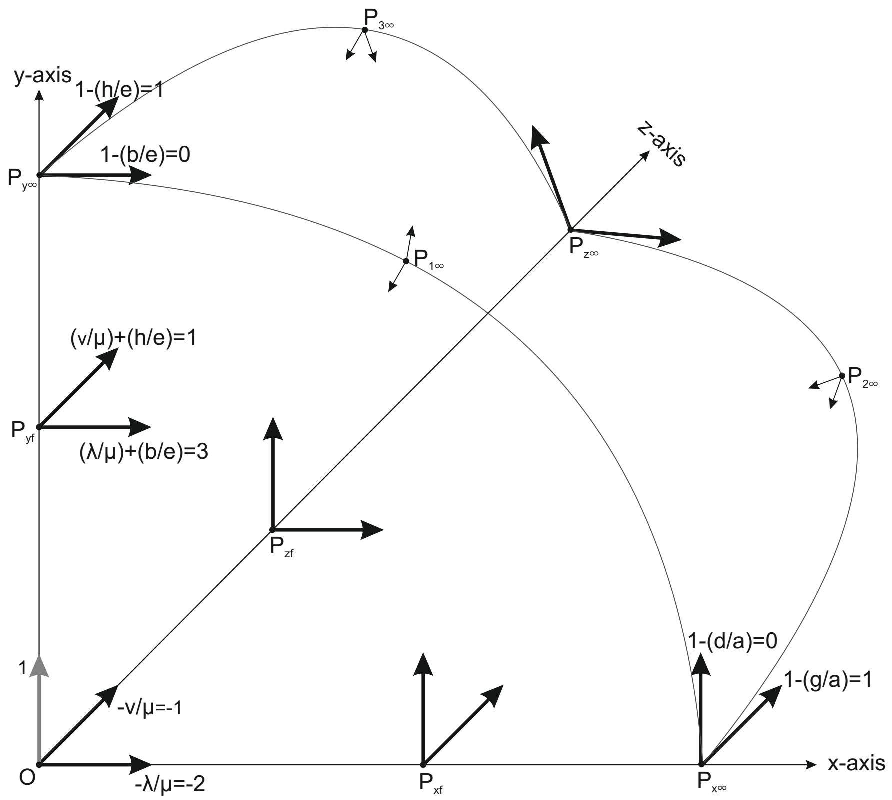


Figure 5.6: Ratio of eigenvalues for the case $\mathcal{B}_4$.

III) Iterative the constructs of case I and case II

In these cases, we need a double or more application of the procedures explained in Case I and Case II above in order to prove that the origin is integrable.

- $\mathcal{C}_9$ and $\mathcal{C}_{15}$: The monodromy of the critical point at infinity $P_{y\infty}$ is conjugate to the inverse of the monodromy of $P_{1\infty}$ and $P_{x\infty}$. The integrability conditions imply that the monodromy corresponding to $P_{1\infty}$ and $P_{x\infty}$ is linearizable and identity respectively, then the monodromy of $P_{y\infty}$ is linearizable. The finite critical point P_{yf} has identity monodromy via blow-up the line of singularities. Thus, the origin in integrable. We show $\mathcal{B}_4$ as an example for this case in Figure 5.7.

- $\mathcal{C}_{18}$ and $\mathcal{C}_{19}$: The monodromy of the critical point at infinity $P_{y\infty}$ is conjugate to the inverse of the monodromy of $P_{3\infty}$ and $P_{z\infty}$. Since the integrability condition in these cases imply that the monodromy corresponding to $P_{3\infty}$ and $P_{z\infty}$ is linearizable and identity respectively, then the monodromy of $P_{y\infty}$ is linearizable. The finite critical point P_{yf} has identity monodromy via blow-up the line of singularities. Hence, the origin in integrable.

- $\mathcal{C}_{8}$: The monodromy of $P_{y\infty}$ is linearizable via line of singularity. The monodromy of the critical point in a finite plane P_{yf} is conjugate to the inverse of the monodromy of $P_{z=0}$ and $P_{x\infty}$. In these cases the integrability conditions implies that the monodromy of $P_{z=0}$ and $P_{x\infty}$ is linearizable and identity respectively and then the monodromy of P_{yf} is linearizable. Thus, the origin is integrable

- $\mathcal{C}_{17}$: The monodromy of $P_{y\infty}$ is linearizable via line of singularity. The monodromy of the critical point in a finite plane P_{yf} is conjugate to the inverse of the monodromy of $P_{x=0}$ and $P_{z\infty}$. In these cases the integrability conditions implies that the monodromy of $P_{x=0}$ and $P_{z\infty}$ is linearizable and identity respectively and then the monodromy of P_{yf} is linearizable. Thus, the origin is integrable

IV) Surface of singularities

- $\mathcal{A}_{1.3}$, $\mathcal{B}_{1.4}$ and $\mathcal{C}_{1.2}$: The integrability conditions implies that the monodromy corresponding to $P_{y\infty}$ is identity and the critical point P_{yf} is in the Poincaré domain with zero resonant monomials and then its linearizable. So the origin is integrable. See Figure 5.8 for the case $\mathcal{A}_{1.3}$.

Remark 10. *With these techniques, we have shown that over half of all the integrability conditions for* $(1:-1:1)$*,* $(2:-1,1)$*,* $(1:-2,1)$ *and* $(3:-1:2)$*–resonance can be found easily using monodromy method.*

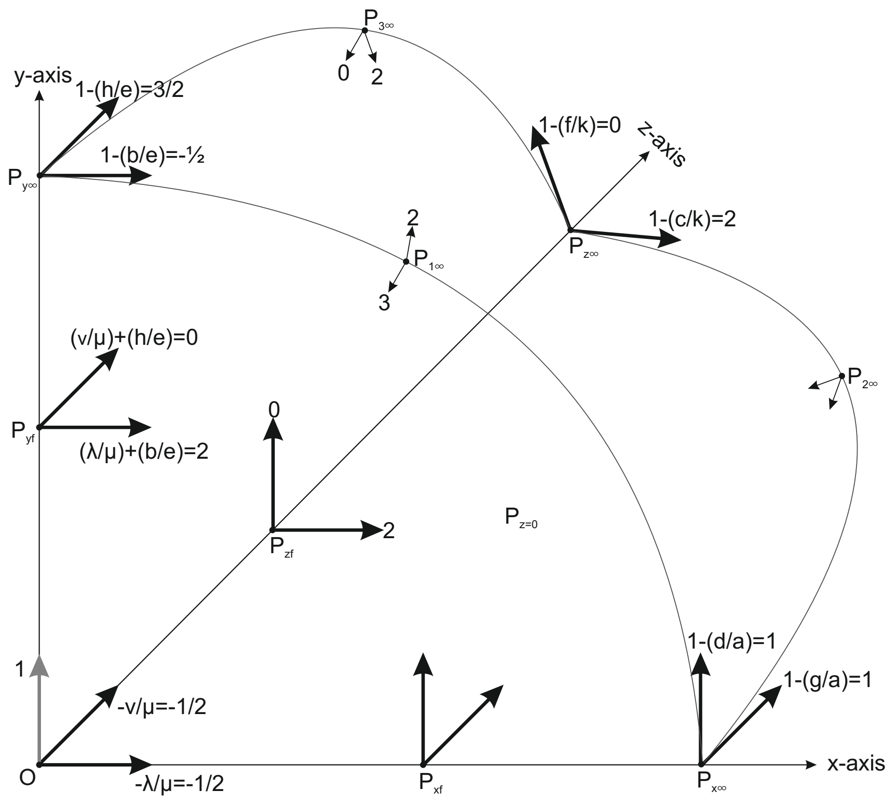


Figure 5.7: Ratio of eigenvalues for the case $\mathcal{C}_9$.

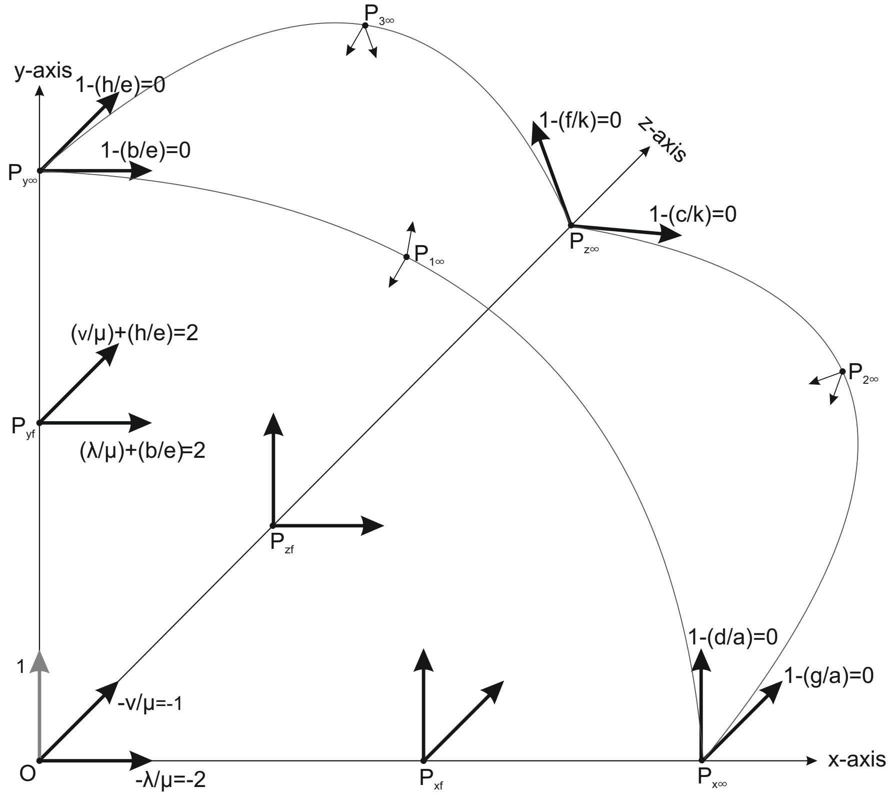


Figure 5.8: Ratio of eigenvalues for the case $\mathcal{A}_{1.3}$.

Chapter 6

Liouvillian integrability of 1–forms and vector fields

In this chapter, we briefly explain the connection between the notion of Darbouxian integrability and the notion of Liouvillian integrability. Liouvillian integrability captures all closed form solutions; that is functions which are rise from rational functions using a finite process of integration, exponentiation and algebraic functions. Surprisingly, for two dimensional system Darbouxian integrability correspond, in some way, to Liouvillian integrability. This means that the Darboux method can be used to find all Liouvillian first integrals.

6.1 Some basic background

We now recall some basic definitions and theorems regarding Liouvillian integrability in order for this section be self-contained.

Definition 9. *Let F be an extension field of K. Then F/K is said to be an algebraic extension of fields if every element of F is algebraic over K. If F/K is not algebraic then we say that it is a transcendental extension.*

Definition 10. *Let K be a field and let $f(x) = a_0 + a_1x + \dots a_nx^n$ be a polynomial in $K[x]$ of degree $n > 0$. An extension field F of K is called a splitting field for $f(x)$ over k if there exist elements $r_1, \dots, r_n$ in F such that*

1. $f(x) = a_n(x - r_1)\dots(x - r_n)$,

2. $F = K(r_1, \ldots, r_n)$.

Example 1. *The splitting field over $\mathbb{Q}$ for the polynomial $x^4 + 4$ is $\mathbb{Q}[i]$ of degree 2 as $x^2 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$ and the roots are $\mp 1 \mp i$.*

Definition 11. *An extension field F/K is normal if F is the splitting field of a family of polynomials in $K[x]$.*

Definition 12. *Let F be an extension field of K. A normal closure of F/K is a field $L \supseteq F$ that is a normal extension of K and is minimal in that respect, that means no proper of L containing F is normal over K.*

Definition 13. *An algebraic extension L of a field K is said to be Galois if it is normal and separable. See (Lang, 2002, page 262) .*

Definition 14. *Let F be an extension field of K. The collection of all K-automorphism of L is said to be a Galois group of L/K and denoted by $Gal(F/K)$.*

Definition 15. *Let G be a group operating on a set A. We denote $s(a) = {}^s a$ for $s \in G$ and $a \in A$. A mapping $s \to a_s$ of G into A is said to be a 1-cocycle of $G \in A$ if the relation $a_{st} = a_s {}^s a_t$ holds for all $s, t \in G$. See (Kneser, 1969, page 1).*

The following theorem (see (Lang, 2002, Chapter VI, page 302)) is a generalization of Hilbert's theorem 90.

Theorem 15. *Let F/K be a finite Galois extension with Galois group G. Then for the operation of G on F^* we have $H^1(G, F^*) = 1$, and for the operation of G on the additive group of F we have $H^1(G, F) = 0$. In other words, the first cohomology group is trivial in both cases.*

Note that $H^1(G, F)$ is the set of equivalence classes of 1-cocycles.

Definition 16. *A differentiable field is a pair (K, Δ) where K is a field together with a derivation $\partial \in \Delta$ such that*

$$\partial(x + y) = \partial x + \partial y, \quad \partial(xy) = (\partial x)y + x(\partial y).$$

Here, we only work with commutative differential fields, that is the derivations in $\partial \in \Delta$ commute: $\partial_i \circ \partial_j = \partial_j \circ \partial_i$.

Note that we assume that all differential fields are of characteristic zero.

Definition 17. *A differential extension of (K,Δ) is a differential field $(\tilde{K},\tilde{\Delta})$ where $\tilde{K}$ is an extension of K and each derivation $\tilde{\partial} \in \tilde{\Delta}$ induces, by restriction, an element $\partial \in \Delta$ and conversely each element $\partial \in \Delta$ extends to an element of $\tilde{\partial} \in \tilde{\Delta}$. Therefore, it is natural to think of $\tilde{\Delta}$ as Δ extended to $\tilde{K}$ and write $(\tilde{K},\Delta)$.*

Definition 18. *An extension $K \supset k$ of fields is called a Liouvillian extension of k if the field of constants of k is $\mathbb{C}$ and if there exists a tower of fields of the form*

$$k = K_0 \subset K_1 \subset \ldots \subset K_n = K$$

such that $K_i = K_{i-1}(t_i)$ for $i = 1,\ldots,n$, where either

1. *$\partial t \in K_{i-1}$, that is t_i is an integral of an element of K_{i-1}, or*
2. *$\partial t_i/t_i \in K_{i-1}$ where $t_i \neq 0$, that is t_i is an exponential of an integral of an element of K_{i-1}, or*
3. *t_i is an algebraic over K_{i-1}.*
   For example $\mathbb{C}(x, e^{x^2}, \int e^{x^2})$ is a Liouvillian extension of $\mathbb{C}(x)$.

Associate to the system (2.1), there is either vector field

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z},$$

or 2-form

$$\Omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$$

Definition 19. *A function $\phi(x,y,z)$ is said to be a Liouvillian first integral if it is a function in some Liouvillian extension of $\mathbb{C}[x,y,z]$ such that $\mathcal{X}\phi = 0$ or $d\phi \wedge \Omega = 0$ where Ω is a corresponding 2–form of a vector field $\mathcal{X}$.*

When does a rational 1–form in $\mathbb{C}^2$ have Liouvillian first integrals? that is a first integral that can be expressed using integration, exponentiation and algebraic functions. The following result is the precise answer and is due to Singer (Singer, 1992).

Theorem 16. *(Singer, 1992) If ω is a rational 1–form on $\mathbb{C}^2$, then ω admits a Liouvillian first integral if and only if there exists a rational closed 1–form α such that $d\omega = \alpha \wedge \omega$.*

For a precise form of the integrating factor of Singer's theorem, Christopher (1999) proved such integrating factor can be written as a Darboux function. That means it takes the form $e^{f/g}\prod L_i^{\ell_i}$ where f, g, L_i are polynomials in x, y and $\ell_i \in \mathbb{C}$.

Theorem 17. *(Christopher, 1999) Let α be a closed rational 1–form and, then $\alpha = \frac{dD}{D}$ where D has a Darboux form. That is $D = e^{\int \alpha}$ is a Darboux integrating factor.*

Note that Theorem 17 shows the study of Liouvillian integrability reduces to the Darboux method. Hence to seek for a Liouvillian first integral, one needs only to seek for invariant algebraic curves and exponential factors.

6.2 Extension of Singer's result to n-dimensional 1–form

In this section, our main result generalizes Singer's theorem above to Liouvillian first integrals of an n-dimension 1–form.

Let we denote by k_n, the field $\mathbb{C}(x_1, \ldots, x_n)$ and by k_n', the the space of 1–forms with coefficients in k_n. This means that a 1–form α belongs to k_n' if and only if α can be written as $\alpha = \sum a_i\, dx_i$ with $a_i \in k_n$. We also denote by K, the extensions of k_n and K' denotes the space of 1–forms with coefficients in K.

Theorem 18 (Extension of Singer's Theorem)**.** *Let ω be a rational 1–form in k_n'. Then ω admits a Liouvillian first integral if and only if there exists a rational closed 1–form $\alpha \in k_n'$ such that $d\omega = \alpha \wedge \omega$.*

Proof. We proceed by induction on the tower of fields. Let K_{i+1} be a Liouvillian extension of K_i. Suppose that there a rational closed 1–form $\alpha \in K_{i+1}'$ such that $d\omega = \alpha \wedge \omega$. We have to show that there exists $\tilde{\alpha} \in K_i'$ such that $d\omega = \tilde{\alpha} \wedge \omega$ with $d\tilde{\alpha} = 0$. Then by Definition 18, K_{i+1} is one of the following three possibilities:

i) K_{i+1} is a finite algebraic extension of K_i.

ii) $K_{i+1} = K_i(t)$, where $dt = \delta t$ with $\delta \in K_i'$ (and necessarily $d\delta = 0$). That is, t represent the exponential of integral of a closed form in K_i.

iii) $K_{i+1} = K_i(t)$, where $dt = \delta$ with $\delta \in K_i'$ (and necessarily $d\delta = 0$). That is, t represent an integral of a closed form in K_i.

We treat each case separately for t.

i) There is no loss of generality in assuming that the extension is a Galois extension. Now apply the Trace of both sides of $d\omega = \alpha \wedge \omega$ and $d\alpha = 0$, we obtain

$$d\omega = \Big(\frac{1}{N}\sum_{\sigma\in\sum}\sigma\alpha\Big)\wedge\omega, \qquad d\big(\frac{1}{N}\sum_{\sigma\in\sum}\sigma\alpha\big) = 0,$$

where $\sum = Gal(K_{i+1}/K_i)$ and N is its order. Thus we can choose $\tilde{\alpha} = \frac{1}{N}\sum_{\sigma\in\sum}\sigma\alpha \in K_i$ and clearly $d\tilde{\alpha} = 0$ since d commute with σ and $d\alpha = 0$.

ii) In this case, suppose that $K_{i+1} = K_i(t)$, where t is transcendental. Then write α as a formal Laurent series

$$\alpha = \sum_{i\geq k} t^i\alpha_i, \quad \alpha_i \in K_i. \tag{6.1}$$

From $\alpha\wedge\omega = d\omega$ and t is transcendental, we can see that

$$\alpha_i\wedge\omega = \begin{cases} d\omega & i=0 \\ 0 & i\neq 0. \end{cases} \tag{6.2}$$

Also, since α is closed, then from (6.1), we have

$$\sum_{i\geq k} t^i(d\alpha_i + i\delta\wedge\alpha_i) = 0. \tag{6.3}$$

From (6.2) and (6.3), it is easy to see that for $i=0$,

$$d\omega = \alpha_0\wedge\omega, \quad d\alpha_0 = 0.$$

Therefore, we can choose $\tilde{\alpha} = \alpha_0 \in K_i$.

iii) As Case (ii), we suppose that $K_{i+1} = K_i(t)$, where t is transcendental. Writing α as a formal Laurent series in decresing power of t

$$\alpha = \alpha_r t^r + \alpha_{r-1} t^{r-1} + \ldots, \quad \alpha_r \in K_i', \quad \alpha_r \neq 0 \quad \text{for some r.}$$

From the hypothesis we have $d\alpha = 0$, and this gives $d\alpha_0 = 0$. Furthermore, from $d\omega = \alpha \wedge \omega$, we have three possibilities depending on r:

1) If $r > 0$, then $\alpha_0 \wedge \omega = 0$. In this case, there exists $h \in K_i$ such that $\alpha_0 = h\omega$. We thus get $d\omega = -\frac{dh}{h} \wedge \omega$ and we can choose $\tilde{\alpha} = -\frac{dh}{h}$.

2) If $k = 0$, we have $d\omega = \alpha_0 \wedge \omega$ and we choose $\tilde{\alpha} = \alpha_0$.

3) If $k < 0$, we see $d\omega = 0$ and we take $\tilde{\alpha} = 0$.

Theorem 19 (Extension of Theorem 17)**.** *Consider a 1–form α in k_n'. If α is a closed rational 1–form and* $\exp(\int \alpha)$ *is an integrating factor, then there exists an integrating factor of the Darboux form. In other words, there exists elements $g, f, f_i \in k_n$ and constants $a_i \in \mathbb{C}$ such that*

$$\alpha = d(\frac{g}{f}) + \sum a_i \frac{df_i}{f_i}.$$

Proof. We proceed by induction. The case $n = 0$ is trivial. Suppose that the theorem holds with k_n replaced by k_{n-1}. Let $\bar{k}$ be the splitting field over k_{n-1} of the denominators of α, then we can write α as a partial fraction expansion in x_n over k_{n-1}

$$\alpha = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \frac{\alpha_{i,j}}{(x_n - \beta_i)^j} dx_n + \sum_{i=0}^{N} \gamma_i x_n^i dx_n + \sum_{i=1}^{\tilde{r}} \sum_{j=1}^{\tilde{n}_i} \frac{\Omega_{i,j}}{(x_n - \beta_i)^j} + \sum_{i=0}^{\tilde{N}} \omega_i x_n^i,$$

where $\Omega_{i,j}$, $\omega_{i,j}$ are 1–forms in k_{n-1}' and $\alpha_{i,j}$, β_i and γ_i are elements of k_{n-1}. Note that, we can ignore the limits of all summations without confusion. Since α is a closed 1–form, then we get the following:

$$
\begin{aligned}
d\gamma_i - (i+1)\omega_{i+1} =& 0, && (6.4)\\
d\alpha_{i,j+1} + j\alpha_{i,j} d\beta_i + j\Omega_{i,j} =& 0, && (6.5)\\
d\omega_i =& 0, && (6.6)\\
d\Omega_{i,j+1} + j d\beta_i \wedge \Omega_{i,j} =& 0. && (6.7)
\end{aligned}
$$

Note that, in particular, $d\alpha_{i,1} = 0$ and $d\Omega_{i,1} = 0$. Moreover, equations (6.4) and (6.5) implies equations (6.6) for $i > 0$ and (6.7) for $j > 0$. From (6.6), clearly $d\omega_0 = 0$ and hence by hypothesis we can write $\omega_0 = d(\frac{\tilde{g}}{\tilde{f}}) + \sum \tilde{a}_i \frac{d\tilde{f}_i}{\tilde{f}_i}$ for some $\tilde{g}, \tilde{f}, \tilde{f}_i \in k_{n-1}$ and $\tilde{a}_i \in \mathbb{C}$. Equations (1)–(4), allow us to write

$$\alpha - \omega_0 = \sum \alpha_{i,1} \frac{d(x_1 - \beta_i)}{(x_1 - \beta_i)} + \sum\sum d\Big(\frac{\alpha_{i,j}}{(x_1 - \beta_i)^{j-1}} \big(\frac{-1}{j-1} \big) \Big) + \sum d\Big(\frac{\gamma_i x_1^{i+1}}{i+1} \Big). \qquad (6.8)$$

Now let Σ be the set of automorphisms of $\bar{k}$ over k_{n-1} and $N = |\Sigma|$. Taking the trace both sides of equation (6.8), we have

$$
\begin{aligned}
\frac{1}{N} \sum_{\sigma \in \Sigma} \sigma(\alpha - \omega_0) \;=& \tfrac{1}{N} \textstyle\sum_{\sigma\in\Sigma} \Big(\sum \alpha_{i,1} \frac{d(x_1 - \sigma\beta_i)}{(x_1 - \sigma\beta_i)} + \sum\sum d\big(\frac{\sigma\alpha_{i,j}}{(x_1 - \sigma\beta_i)^{j-1}} \big(\frac{-1}{j-1} \big) \big) \\
& + \textstyle\sum d\sigma\big(\frac{\gamma_i x_1^{i+1}}{i+1} \big) \Big).
\end{aligned}
$$

Since Σ is the set of all automorphisms of $\bar{k}$ fixing k_{n-1}, then we have α in the desired form. □

Remark 11. *Combing Theorem 18 and Theorem 19, we see that a 1-form ω has Liouvillian first integrals if and only if there exists a Darboux integrating factor.*

6.3 The Singer theorem to a vector fields in three dimensions

We now seek to generalize the work in the previous section to three dimensional vector fields.

For the rest of this section we denote by K, the extension field of $k = \mathbb{C}^3$ and by K', the space of 1–forms with coefficients in K with the usual algebraic properties of the operator d.

Theorem 20 (Extension of Singer's theorem to a vector fields in three dimensions)**.** *Let Ω be a rational 2–form. If there exists Liouvillian first integral then one of the following holds:*

1) *There exist 1–forms $\omega, \alpha \in K'_{i+1}$ such that $\omega \wedge \Omega = 0$, $\alpha \wedge \omega = d\omega$, $d\alpha = 0$.*

2) *There exists a 1–form $\beta \in K_{i+1}$, such that $\beta \wedge \Omega = d\Omega$ with $d\beta = 0$.*

Remark 12. *1. The condition (i) in Theorem 20 means there is a first integral of the form $\phi = \int \frac{\omega}{e^{\int \alpha}}$.*

2. The condition (ii) in Theorem 20 means that Ω has an inverse Jacobi multiplier of the form $e^{\int \beta}$.

Note that both $e^{\int \alpha}$ and $e^{\int \beta}$ are of Darboux type by Theorem 19.

Proof of Theorem 20. We proceed by induction. Let $K_{i+1} = K_i(t)$ be an extension of $K_i \supset k$. The proof falls naturally into three parts depending on the nature of t.

i) $K_{i+1} = K_i(t)$, where $dt = t\,\delta$, $\delta \in K'_i$, $d\delta = 0$. That is, t represents the exponential of integrals of a closed form in K_i.

ii) $K_{i+1} = K_i(t)$, where $dt = \delta$, $\delta \in K'_i$, $d\delta = 0$. That is, t represents an integral of a closed form in K_i.

iii) K_{i+1} is a finite algebraic extension of K_i.

Case 1. Suppose that ω, $\alpha \in K'_{i+1}$ such that $\omega \wedge \Omega = 0$, $\alpha \wedge \omega = d\omega$, $d\alpha = 0$. Then we need to show that for all three cases (i)–(iii) mentioned above, there exists $\tilde{\omega}$, $\tilde{\alpha} \in K'_i$ such that $\tilde{\omega} \wedge \Omega = 0$, $\tilde{\alpha} \wedge \tilde{\omega} = d\tilde{\omega}$, $d\tilde{\alpha} = 0$ or there exists $\tilde{\beta} \in K_i$ such that $\tilde{\beta} \wedge \Omega = d\Omega$ with $d\tilde{\beta} = 0$.

i) Suppose that $K_{i+1} = K_i(t)$, t is transcendental. Then we can write ω, α as a formal Laurent series in t, we have

$$\omega = \omega_r t^r + \omega_{r+1} t^{r+1} \dots, \quad \omega_k \in K_i', k = r, r+1, \dots, \omega_r \neq 0 \text{ for some r,}$$

$$\alpha = \alpha_s t^s + \alpha_{s+1} t^{s+1} \dots, \quad \alpha_m \in K_i', m = s, s+1, \dots, \alpha_s \neq 0 \text{ for some s.}$$

By hypothesis we have $\omega \wedge \Omega = 0$, then $\omega_k \wedge \Omega = 0$ for all k. Since $\alpha \wedge \omega = d\omega$, then $\sum_{i=s}^{\infty} \alpha_i t^i \wedge \sum_{i=r}^{\infty} \omega_i t^i = \sum_{i=r}^{\infty} (d\omega_i + i\,\delta \wedge \omega) t^i$ and we have the following three cases:

1) When $s > 0$, we just have $d(\omega_r t^r) = 0$ and implies that $d\omega_r + r\,\delta \wedge \omega_r = 0$. In this case choose $\tilde{\alpha} = -r\,\delta$ and $\tilde{\omega} = \omega_r$ and clearly $d\tilde{\alpha} = 0$.

2) When $s = 0$, we see $\alpha_0 \wedge \omega_r = d\omega_r + r\,\delta \wedge \omega_r$ from equating the powers of t^r. In this case take $\tilde{\alpha} = \alpha_0 - r\,\delta$ and $\tilde{\omega} = \omega_r$. From $d\alpha = 0$ and $d\delta = 0$, is it clear that $d\tilde{\alpha} = 0$.

3) When $s < 0$, we get $\alpha_s \wedge \omega_k = 0$ for some k and this means that α_s is parallel to ω_k. Since $\omega_k \wedge \Omega = 0$, then $\alpha_s \wedge \Omega = 0$. Clearly $d(\alpha_s t^s) = 0$ and implies that $d\alpha_s + s\,\delta \wedge \alpha_s = 0$. So we choose $\tilde{\alpha} = -s\,\delta$ and $\tilde{\omega} = \alpha_s$. Again it is clear that $d\tilde{\alpha} = 0$.

ii) Also in this case we have $K_{i+1} = K_i(t)$ with t transcendental. Therefore, writing ω, α in decreasing formal Laurent series

$$\omega = \omega_r t^r + \dots, \quad \omega_k \in K_i', \quad k = r, r-1, \dots, \quad \omega_r \neq 0 \quad \text{for some r,}$$

$$\alpha = \alpha_s t^s + \dots, \quad \alpha_m \in K_i', \quad m = s, s-1, \dots, \quad \alpha_s \neq 0 \quad \text{for some s.}$$

Since $\omega \wedge \Omega = 0$, then $\omega_k \wedge \Omega = 0$ for all k. From the assumption $\alpha \wedge \omega = d\omega$, we have several cases:

1) When $s > 0$, we get $\alpha_s \wedge \omega_r = 0$ and hence they are parallel. Since $\omega_r \wedge \Omega = 0$,

then $\alpha_s \wedge \Omega = 0$. Straightforwardly we have $d\alpha_s = 0$ from $d\alpha = 0$. In this case take $\tilde{\alpha} = 0$ and $\tilde{\omega} = \alpha_s$.

2) When $s = 0$, we see $\alpha_0 \wedge \omega_r = d\omega_r$ from equating the powers of t^r. Again from $d\alpha = 0$, clearly $d\alpha_0 = 0$. In this case we choose $\tilde{\alpha} = \alpha_0$ and $\tilde{\omega} = \omega_r$.

3) When $s < 0$, then $d\omega_0 = 0$. Take $\tilde{\alpha} = 0$ and $\tilde{\omega} = \omega$. Immediately $d\tilde{\alpha} = 0$.

iii) Without loss of generality, assume that K_{i+1} is a Galois extension of K_i. We will denote by Σ, the set of automorphism of K_{i+1} fixing K_i. We have two cases either $\sigma(\omega) \wedge \omega = 0$ for all $\sigma \in Gal(K_{i+1}/K_i)$ or there exists $\sigma(\omega)$ such that $\sigma(\omega) \wedge \omega \neq 0$. Let us first assume that $\sigma(\omega) \wedge \omega = 0$ for all $\sigma \in Gal(K_{i+1}/K_i)$, then all σ we have $\sigma(\omega) = k_\sigma \omega$ for some $k_\sigma \in K_{i+1}$. Since $\sigma\tau(\omega) = \sigma(k_\tau) k_\sigma \omega$ and also $\sigma\tau(\omega) = k_{\sigma\tau}\omega$, then we have $k_{\sigma\tau} = \sigma(k_\tau) k_\sigma$. This means k_σ form a 1-cocycle since $k_{\sigma\tau} = k_\sigma \sigma(k_\tau)$ and hence, by Theorem 15, must be of the form $k_\sigma = \sigma(\ell)/\ell$ for some ℓ in the extension field K_{i+1}. It follows easily that $\sigma(\frac{\omega}{\ell}) = \frac{\omega}{\ell}$ for all $\sigma \in Gal(K_{i+1}/K_i)$. This means that $\frac{\omega}{\ell}$ is invariant under the action of σ in $Gal(K_{i+1}/K_i)$. Hence $\frac{\omega}{\ell} \in K_i$. Now

$$d(\frac{\omega}{\ell}) = (\alpha - \frac{d\ell}{\ell}) \wedge \frac{\omega}{\ell}, \tag{6.9}$$

and clearly $d(\alpha - \frac{d\ell}{\ell}) = 0$ from $d\alpha = 0$. Writing $N = [K_{i+1} : K_i]$ and taking the trace of both sides of (6.9), we find

$$d(\frac{\omega}{\ell}) = \frac{1}{N} \sum_{\sigma \in \Sigma} \sigma(\alpha - \frac{d\ell}{\ell}) \wedge \frac{\omega}{\ell}.$$

We see at once that $d\left(\frac{1}{N} \sum_{\sigma \in \Sigma} \sigma(\alpha - \frac{d\ell}{\ell})\right) = 0$ which is clear from $d\alpha = 0$. Hence we can choose $\tilde{\omega} = \frac{\omega}{\ell}$ and $\tilde{\alpha} = \frac{1}{N} \sum_{\sigma \in \Sigma} \sigma(\alpha - \frac{d\ell}{\ell})$.

Suppose now that $\sigma(\omega) \wedge \omega \neq 0$ for some $\sigma \in Gal(K_{i+1}/K_i)$. Then since $\sigma(\omega) \wedge \Omega = 0$ and $\omega \wedge \Omega = 0$ then for some $k \in K_{i+1}$ different from zero we have

$$\omega \wedge \sigma(\omega) = k\Omega.$$

Applying the derivative to the formula above gives

$$d(\omega \wedge \sigma(\omega)) = dk \wedge \Omega + kd\Omega. \tag{6.10}$$

Since

$$d(\omega \wedge \sigma(\omega)) = (\alpha + \sigma(\alpha)) \wedge (\omega \wedge \sigma(\omega)) = (\alpha + \sigma(\alpha)) \wedge k\Omega. \tag{6.11}$$

Combining (6.10) and (6.11) yields

$$(\alpha + \sigma(\alpha) - \frac{dk}{k}) \wedge \Omega = d\Omega. \tag{6.12}$$

Obviously $d(\alpha + \sigma(\alpha) - \frac{dk}{k}) = 0$ as $d\alpha = 0$ and σ commutes with derivative. Applying the trace to the formula (6.12), we have

$$d\Omega = \frac{1}{N} \sum_{\sigma \in \Sigma} \sigma(\alpha + \sigma(\alpha) - \frac{dk}{k}) \wedge \Omega.$$

Clearly $d\big(\frac{1}{N}\sum_{\sigma\in\Sigma}\sigma(\alpha + \sigma(\alpha) - \frac{dk}{k})\big) = 0$ and hence we are in Case 2 when $\beta = \frac{1}{N}\sum_{\sigma\in\Sigma}\sigma(\alpha + \sigma(\alpha)$.

Case 2. Let K_{i+1} be one of the three types (i)–(iii) listed above. Suppose that $\beta \in K'_{i+1}$ such that $\beta \wedge \Omega = d\Omega$ with $d\beta = 0$. Again by induction we have to show that there exists $\tilde{\beta} \in K'_i$ such that $\tilde{\beta} \wedge \Omega = 0$ with $d\tilde{\beta} = 0$ or there exists $\tilde{\omega}$, $\tilde{\alpha} \in K'_i$ such that $\tilde{\omega} \wedge \Omega = 0$, $\tilde{\alpha} \wedge \tilde{\omega} = d\tilde{\omega}$, $d\tilde{\alpha} = 0$.

i) Assume that $K_{i+1} = K_i$ with t transcentental. Consider β as a formal Laurent series in decreasing powers of t, then

$$\beta = \beta_\ell t^\ell + \ldots, \quad \beta_k \in K'_i, \quad k = \ell, \ell-1, \ldots \quad \beta_\ell \neq 0.$$

From $\beta \wedge \Omega = d\Omega$ and t is transcendental, we see

$$\beta_i \wedge \Omega = \begin{cases} d\Omega & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Since β is closed, then

$$\sum_{i=0}^{\infty} t^i (d\beta_i + i\delta \wedge \beta_i) = 0.$$

So we can choose $\tilde{\beta} = \beta_0$ and clearly $d\tilde{\beta} = 0$ from $d\beta = 0$.

ii) As above, we assume that $K_{i+1} = K_i$ with t transcendental and write β as a formal Laurent series in decreasing powers of t:

$$\beta = \beta_\ell t^\ell + \ldots, \quad \beta_k \in K_i', \quad k = \ell, \ell - 1, \ldots \quad \beta_\ell \neq 0.$$

Since it is given $\beta \wedge \Omega = d\Omega$, then we have three possibilities:

1) When $\ell > 0$, we have $\beta_\ell \wedge \Omega = 0$. We take $\tilde{\omega} = \beta_\ell$, $\tilde{\alpha} = 0$ and hence we are in Case 1.

2) When $\ell = 0$, we see $\beta_0 \wedge \Omega = d\Omega$ from equating the powers of t^0. Clearly $d\beta_0 = 0$ from $d\beta = 0$. In this case take $\tilde{\beta} = \beta_0$.

3) When $\ell < 0$, then $d\Omega = 0$. Take $\tilde{\beta} = 0$.

iii) In this case, without loss of generality, we assume that the extension is Galois extension. Denote by Σ the set of all automorphism of Galois extension $Gal(K_{i+1}/K_i)$. Let σ^* be the action of σ on the coefficients of the 1–form $\beta \in K_{i+1}'$. Then from $\beta \wedge \Omega = d\Omega$ and $d\beta = 0$, we see

$$d\Omega = \Big(\frac{1}{N} \sum_{\sigma \in \Sigma} \sigma^* \beta\Big) \wedge \Omega \quad \text{and} \quad \frac{1}{N} d\big(\sum_{\sigma \in \Sigma} \sigma^* \beta\big) = 0.$$

We can therefore choose $\tilde{\beta} = \frac{1}{N} \sum_{\sigma \in \Sigma} \sigma^* \beta \in K_i'$.

$\square$

Chapter 7

Centers of quasi–homogeneous cubics polynomial differential equations

The results of this chapter have been obtained in collaboration with Prof. Jaume Llibre and Dr Chara Pantazi. We all participated equally and actively in all part of this chapter.

7.1 Introduction and statement of the results

Poincaré in (Poincaré, 1951) was the first to introduce the notion of a center for a vector field defined on the real plane. So according to Poincaré a center is a singular point surrounded by a neighborhood filled of closed orbits with the unique exception of the singular point.

Since then the center–focus problem, i.e. the problem to distinguish when a singular point is either a focus or a center is one of the hardest problem in the qualitative theory of planar differential systems, see for instance (Algaba and Reyes, 2003) and the references quoted there. This paper deals mainly with the characterization of the centers problem for the class of quasi–homogeneous polynomial differential systems of degree 3.

In the literature we found classifications of polynomial differential systems having a center. For the quadratic systems we refer to the works of Dulac (1908), Kapteyn (1911, 1912), Bautin (1925) among others. In (Schlomiuk et al., 1990) Schlomiuk, Guckenheimer and Rand gave a brief history of the center problem for quadratic systems.

There are many partial results about centers for polynomial differential systems of degree greater than two. Some of them (closed to our work) are for instance, the classification by Malkin (1964) and Schlomiuk et al. (1990) about the centers for cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree three. Note that for polynomial differential systems of the form linear with homogeneous nonlinearities of degree $k>3$ the centers are not classified. However, there are some results for $k=4,5$ see for instance the works by Chavarriga and Giné (1996, 1997). It seems difficult for the moment to obtain a complete classification of the centers for the class of all polynomial differential systems of degree 3. Actually, there are some subclasses of cubic systems well studied like the ones of Rousseau and Schlomiuk (1995) and the ones of Żołądek (1994, 1996). Some centers for arbitrary degree polynomial differential systems have been studied in (Llibre and Valls, 2011a).

In what follows we denote by $\mathbb{R}[x,y]$ the ring of all polynomials in the variables x and y and coefficients in the real numbers $\mathbb{R}$. In this work we consider polynomial differential systems of the form

$$\dot{x}=P(x,y), \qquad \dot{y}=Q(x,y), \tag{7.1}$$

with $P,Q\in\mathbb{R}[x,y]$ and its corresponding vector field $\mathcal{X}=(P,Q)$. Here the dot denotes derivative with respect to the time t (independent variable). The *degree* of the differential polynomial system (7.1) is the maximum of the degrees of the polynomials P and Q.

System (7.1) is a *quasi–homogeneous polynomial differential system* if there exist natural numbers s_1,s_2,d such that for an arbitrary non–negative real number α it holds

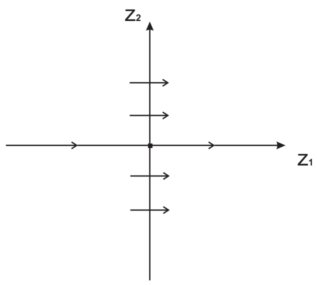
$$P(\alpha^{s_1}x,\ \alpha^{s_2}y)=\alpha^{s_1+d-1}P(x,y), \qquad Q(\alpha^{s_1}x,\ \alpha^{s_2}y)=\alpha^{s_2+d-1}Q(x,y). \tag{7.2}$$

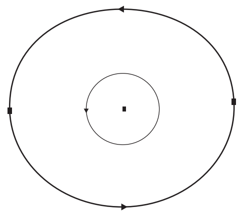
The natural numbers s_1 and s_2 are the *weight exponents* of system (7.1) and d is the *weight degree* with respect to the weight exponents s_1 and s_2. When $s_1=s_2=s$ then we obtain the classical *homogeneous polynomial differential system* of degree $s+d-1$.

It is well known that all quasi–homogeneous vector fields are integrable with a Liouvillian first integral (Garcia, 2003; Garcia et al., 2012; Li et al., 2009).

From Theorem 2 of (Llibre and Pessoa, 2009) we have that there are only two families of cubic polynomial differential homogeneous systems with a center.

In the next result we characterize all the centers of quasi–homogeneous polynomial differential systems.





$(a)\,(z_1,z_2) = (0,0)$ on the $(U_1,F_1.)$ (b) On the Poincaré disk.

Figure 7.1: (a) The local phase portrait at the origin in the local chart U_1. (b) Phase portrait of a cubic quasi–homogeneous non–homogeneous system (7.3) in the Poincaré disk. This system has a global center.

Theorem 21. *The following two statements hold.*

(a) The unique cubic quasi–homogeneous non–homogeneous polynomial differential system (7.1) *with P and Q coprime and $s_1 > s_2$ having a center after a rescaling of the variables can be written as*

$$\dot{x} = y(ax+by^2), \qquad \dot{y} = x+y^2, \tag{7.3}$$

with $(a-2)^2+8b<0$. For all a and b satisfying $(a-2)^2+8b<0$ the phase portrait in the Poincaré disk of system (7.3) *is topologically equivalent to the one* *given in Figure 7.1(b). Moreover, its parameter space (a,b) is described in Figure 7.2(a). Additionally, these centers are not isochronous.*

(b) The unique cubic homogeneous polynomial differential systems having a center after a linear transformation and a rescaling of independent variable can be written in one of the following four forms:

$$\dot{x} = -3\alpha\mu x^2 y - \alpha y^3 + P_3, \quad \dot{y} = \alpha x^3 + 3\alpha\mu x y^2 + Q_3, \tag{7.4}$$

where $\alpha = \pm 1$, $\mu > -1/3$ and $\mu \neq 1/3$;

$$\dot{x} = -\alpha x^2 y - \alpha y^3 + P_3, \quad \dot{y} = \alpha x^3 + \alpha x y^2 + Q_3, \tag{7.5}$$

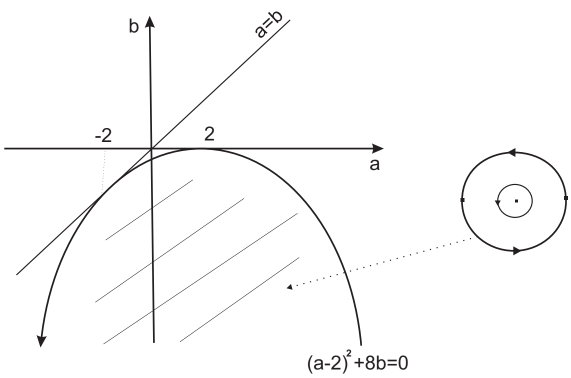
with $\alpha = \pm 1$. Here $P_3 = p_1 x^3 + p_2 x^2 y - p_1 x y^2$ and $Q_3 = p_1 x^2 y + p_2 x y^2 - p_1 y^3$. The phase portraits in the Poincaré disk of systems (7.4) *and* (7.5) *are topologically equivalent to the ones of Figure 7.2(b). Moreover, these centers are not isochronous.*

The proof of Theorem 21 is given in section 7.3.

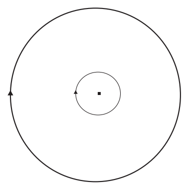
Additional to the classification of centers, another classical problem in the qualitative theory of planar differential systems is the study of their limit cycles. Recall that a *limit cycle* of a planar polynomial differential system is a periodic orbit of the system isolated in the set of all periodic orbits of the system. Thus in what follows we study, using the averaging theory of first order, the limit cycles which bifurcate from the periodic orbits of the centers (7.4) and (7.5) of Theorem 21 when these centers are perturbed inside the class of all cubic polynomial differential systems.

Theorem 22. *Consider the cubic homogeneous system* (7.4) *and* (7.5) *and its perturbation inside the class of all cubic polynomial differential systems. Then, for $|\varepsilon| \neq 0$ sufficiently small one limit cycle can bifurcate from the continuum of the periodic orbits of the center of systems* (7.4) *and* (7.5) *using averaging theory of first order.*

The proof of Theorem 22 is given in section 7.4.



(a) cubic quasi–homogeneous



(b) cubic homogeneous

Figure 7.2: (a) The parameter space (a,b) and the phase portrait of cubic quasi–homogeneous systems (7.3). (b) Cubic homogeneous systems (7.5) having a center.

In section 7.2 we provide the basic results that we shall need for proving Theorems 21 and 22.

7.2 some known results

7.2.1 Classification of quasi–homogeneous non–homogeneous cubic polynomial differential systems

For proving Theorem 21 we should need the following result.

Proposition 4. *A quasi–homogeneous non–homogeneous cubic polynomial differential systems (7.1) with P and Q coprime and $s_1 > s_2$ after a rescaling of the variables can be* *written as one of the following systems.*

(a) $\dot{x} = y(ax+by^2)$, $\dot{y} = x+y^2$, *with* $a \neq b$, *or* $\dot{x} = y(ax \pm y^2)$, $y' = x$, *and both with minimal weight vector* $(2,1,2)$.

(b) $\dot{x} = x^2+y^3$, $\dot{y} = axy$, *with* $a \neq 0$ *and minimal weight vector* $(3,2,4)$.

(c) $\dot{x} = y^3$, $\dot{y} = x^2$, *and minimal weight vector* $(4,3,6)$.

(d) $\dot{x} = x(x+ay^2)$, $\dot{y} = y(bx+y^2)$, *with* $(a,b) \neq (1,1)$,*and minimal weight vector* $(2,1,3)$.

(e) $\dot{x} = axy^2$, $\dot{y} = \pm x^2+y^2$, *with* $a \neq 0$*and minimal weight vector* $(3,2,5)$.

(f) $\dot{x} = axy^2$, $\dot{y} = x+y^3$, *with* $a \neq 0$ *and minimal weight vector* $(3,1,3)$.

(g) $\dot{x} = ax+y^3$, $\dot{y} = y$, *or* $\dot{x} = ax$, $\dot{y} = y$ *with* $a \neq 0$, *and minimal weight vector* $(3,1,1)$.

Proof. See (Garcia et al., 2012). □

7.2.2 Nilpotent center–focus

A singular point is *nilpotent* if both eigenvalues of its linear part are zero but its linear part is not identically zero. Andreev (Andreev, 1958) was the first in characterizing the local phase portraits of the nilpotent singular points. In what follows we summarize the results of the local phase portraits of the nilpotent singular points that we need in this paper, for more details see Theorem 3.5 of (Dumortier et al., 2006).

Theorem 23. *Let* $(0,0)$ *be an isolated singular point of the vector field* $\mathcal{X}$ *given by*

$$\dot{x} = y + A(x,y), \qquad \dot{y} = B(x,y),$$

where A and B are analytic in a neighborhood of the point $(0,0)$ *starting with terms of second degree.*

Let $y = f(x)$ *be the solution of the equation* $y + A(x,y) = 0$ *in a neighborhood of the point* $(0,0)$, *and consider* $F(x) = B(x, f(x))$ *and* $G(x) = (\partial A/\partial x + \partial B/\partial y)(x, f(x))$.

Then the origin can be a focus or a center if and only if one of the following statements holds:

(a) *If* $G(x) \equiv 0$ *and* $F(x) = ax^m + o(x^m)$ *for* $m \in \mathbb{N}$ *with* $m \geq 1$*,* m *odd and* $a < 0$ *then the origin of* $\mathcal{X}$ *is a center or a focus.*

(b) *If* $F(x) = \alpha x^m + o(x^m)$ *with* $\alpha < 0$ *,* $m \in \mathbb{N}$*,* $m \geq 2$*,* m *odd, and* $G(x) = \beta x^n + o(x^n)$ *with* $\beta \neq 0$*,* $n \in \mathbb{N}$*,* $n \geq 1$ *and if either* $m < 2n+1$*, or* $m = 2n+1$ *and* $\beta^2 + 4\alpha(n+1) < 0$*, then the origin of* $\mathcal{X}$ *is a center or a focus.*

7.2.3 Isochronicity

The following result characterizes the isochronous centers.

Theorem 24. *A center of an analytic system is isochronous if and only if there exists an analytic change of coordinates of the form* $u = x + o(x,y)$ *and* $v = y + o(x,y)$ *changing the system to the linear isochronous system*

$$\dot{u} = -kv, \qquad \dot{v} = ku,$$

where k *is a real constant.*

For a proof of Theorem 24, see (Mardešić et al., 1995).

Assume that the origin is an isochronous center for system (7.1). Then Theorem 24 guarantees that there exists an analytic change of coordinates $u = x + o(x,y)$ and $v = y + o(x,y)$ such that $\dot{u} = -kv$, $\dot{v} = ku$. Then since $\ddot{u} + u = 0$, and $\ddot{v} + v = 0$, and doing a rescalling we can take $k = 1$.

7.2.4 Poincaré compactification

In order to plot the global phase portrait of the polynomial vector field (7.1) of degree m we should be able to control the orbits that come or escape at infinity. For this reason we consider the so called *Poincaré compactification* of the polynomial vector field $\mathcal{X}$.

Consider $\mathbb{R}^2$ as the plane in $\mathbb{R}^3$ defined by $(y_1, y_2, y_3) = (x_1, x_2, 1)$. We also consider the *Poincaré sphere* $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 + y_2 + y_3 = 1\}$ (see also (Poincaré, 1891)) and we denote by $T_{(0,0,1)}\mathbb{S}^2$ the tangent space to $\mathbb{S}^2$ at the point $(0,0,1)$. The *Poincaré compactified vector field* $p(\mathcal{X})$ of $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ in the following way: We consider the central projection $f : T_{(0,0,1)} : \mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the northern hemisphere $\{y \in \mathbb{S}^2 : y_3 > 0\}$ and the other in the southern hemisphere. We denote by $\tilde{\mathcal{X}}$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator. We notice that the points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator of $\mathbb{S}^2$, $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ and so we identify $\mathbb{S}^1$ to be the infinity of $\mathbb{R}^2$.

Now we would like to extend the induced vector field $\tilde{\mathcal{X}}$ from $\mathbb{S}^2 \setminus \mathbb{S}^1$ to $\mathbb{S}^2$. It is possible that $\tilde{\mathcal{X}}$ does not stay bounded as we get close to $\mathbb{S}^1$. However, it turns out that if we multiply $\tilde{\mathcal{X}}$ by the factor y_3^{m-1}, namely, if we consider the vector field $y_3^{m-1}\tilde{\mathcal{X}}$ the extension is possible in the whole $\mathbb{S}^2$.

Note that on $\mathbb{S}^2 \setminus \mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$ and knowing the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ at infinity. The *Poincaré disk* D^2 is the projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$. Moreover, $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

We also say that two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(\mathcal{X})$ into orbits of the flow induced by $p(\mathcal{Y})$. The homeomorphism should preserve or reverse simultaneously the sense of all orbits of the two compactified vector fields $p(\mathcal{X})$ and $p(\mathcal{Y})$.

Since $\mathbb{S}^2$ is a differentiable manifold we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1,2,3$ and the diffeomorphisms $F_i : V_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ are the inverses of the central projections from the planes tangent at the points $(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1)$ and $(0,0,-1)$ respectively. Now we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1,2,3$. Then we obtain the following expressions of the compactified vector field $p(\mathcal{X})$ of $\mathcal{X}$ (for more details we refer to chapter V of (Dumortier et al., 2006) and references therein).

$$
\begin{aligned}
z_2^n\Delta(z)\left(Q\Big(\frac{1}{z_2},\frac{z_1}{z_2}\Big)-z_1P\Big(\frac{1}{z_2},\frac{z_1}{z_2}\Big),-z_2P\Big(\frac{1}{z_1},\frac{z_1}{z_2}\Big)\right) & \quad \text{in} \quad U_1,\\
z_2^n\Delta(z)\left(P\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big)-z_1Q\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big),-z_2Q\Big(\frac{z_1}{z_2},\frac{1}{z_2}\Big)\right) & \quad \text{in} \quad U_2,\\
\Delta(z)\big(P(z_1,z_2),Q(z_1,z_2)\big) & \quad \text{in} \quad U_3,
\end{aligned}
$$

where $\Delta(z)=(z_1^2+z_2^2+1)^{-\frac{1}{2(n-1)}}$. Note that in the two sets U_i and V_i the expressions of the vector field $p(\mathcal{X})$ are the same and only difference by the multiplicative factor $(-1)^{n-1}$. In these coordinates $z_2=0$ always denotes the points of $\mathbb{S}^1$. In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(\mathcal{X})$ and so we obtain a polynomial vector field in each local chart.

7.2.5 Separatrix configuration

Let $p(\mathcal{X})$ be the Poincaré compactification of $\mathbb{S}^2$ of a polynomial vector field $\mathcal{X}$ in $\mathbb{R}^2$.

In what follows we consider the definition of parallel flows given by Markus (Markus, 1954) and Neumann in (Neumann, 1975). Let ϕ be a $\mathcal{C}^\omega$ local flow on the two dimensional manifold $\mathbb{R}^2$ or $\mathbb{R}^2\setminus\{0\}$. The flow (M,ϕ) is $\mathcal{C}^k$ *parallel* if it is $\mathcal{C}^\omega$-equivalent to one of the following ones:

strip: $(\mathbb{R}^2,\phi)$ with the flow ϕ defined by $\dot{x}=1,\dot{y}=0$;

annular: $(\mathbb{R}^2\setminus\{0\},\phi)$ with the flow ϕ defined (in polar coordinates) by $\dot{r}=0,\dot{\theta}=1$;

spiral: $(\mathbb{R}^2\setminus\{0\},\phi)$ with the flow ϕ defined by $\dot{r}=r,\dot{\theta}=0$.

It is known that the *separatrices* of the vector field $p(\mathcal{X})$ in the Poincaré disk D are

(i) all the orbits of $p(\mathcal{X})$ which are in the boundary $\mathbb{S}^1$ of the Poincaré disk (recall that $\mathbb{S}^1$ is the infinity of $\mathbb{R}^2$);

(ii) all the finite singular points of $p(\mathcal{X})$;

(iii) all the limit cycles of $p(\mathcal{X})$; and

(iv) all the separatrices of the hyperbolic sectors of the finite and infinite singular points of $p(\mathcal{X})$.

We denote by Σ the union of all separatrices of the flow (D,ϕ) defined by the compactified vector field $p(\mathcal{X})$ in the Poincaré disk D. Then Σ is a closed invariant subset of D. Every connected component of $D \setminus \Sigma$, with the restricted flow, is called a *canonical region* of ϕ.

For a proof of the following result see (Li et al., 2002) and (Neumann, 1975).

Theorem 25. *Let ϕ be a $\mathcal{C}^{\omega}$ flow in the Poincaré disk with finitely many separatrices, and let Σ be the union of all its separatrices. Then the flow restricted to every canonical region is $\mathcal{C}^{\omega}$ parallel.*

The *separatrix configuration* Σ_c of a flow (D,ϕ) is the union of all the separatrices Σ of the flow together with an orbit belonging to each canonical region. The separatrix configuration Σ_c of the flow (D,ϕ) is said to be *topologically equivalent* to the separatrix configuration $\tilde{\Sigma}_c$ of the flow $(D,\tilde{\phi})$ if there exists a homeomorphism from D to D which transforms orbits of Σ_c into orbits of $\tilde{\Sigma}_c$, and orbits of Σ into orbits of $\tilde{\Sigma}$.

Theorem 26. *Let (D,ϕ) and $(D,\tilde{\phi})$ be two compactified Poincaré flows with finitely many separatrices coming from two polynomial vector fields* (7.1). *Then they are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

For a proof of Theorem 26 see (Markus, 1954; Neumann, 1975; Peixoto, 1973).

From Theorem 26 it follows that in order to classify the phase portraits in the Poincaré disk of a planar polynomial differential system having finitely many finite and infinite separatrices, it is enough to describe their separatrix configuration.

7.2.6 Averaging theory and periodic solutions

We consider the system

$$\mathbf{x}'(t) = F_0(t,\mathbf{x}), \tag{7.6}$$

with $F_0 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ a $\mathcal{C}^2$ function, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. We assume that system (7.6) has a submanifold of periodic solutions.

Let ε be sufficiently small and we consider a perturbation of system (7.6) of the form

$$\mathbf{x}'(t) = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{7.7}$$

with $F_1 : \mathbb{R} \times \omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. Averaging theory deals with the problem of the bifurcation of T–periodic solutions of system (7.7), see also for more information on the averaging theory (Sanders and Verhulst, 1985; Verhulst, 1991).

Let $\mathbf{x}(t,\mathbf{z})$ be the periodic solution of the unperturbed system (7.6) satisfying the initial condition $\mathbf{x}(0,\mathbf{z}) = \mathbf{z}$. Now we consider the linearization of system (7.6) along the solution $\mathbf{x}(t,\mathbf{z})$, namely

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t,\mathbf{z}))\mathbf{y},$$

and let $M_{\mathbf{z}}(t)$ be a fundamental matrix of this linear system satisfying that $M(0)$ is the identity matrix.

For a proof of the following theorem see (Buică et al., 2007).

Theorem 27 (Perturbations of an isochronous set)**.** *We assume that there exists an open bounded set V with $Cl(V) \subset \Omega$ such that for each $\mathbf{z} \in Cl(V)$, the solution $\mathbf{x}(t,\mathbf{z})$ is T–periodic, then we consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t,\mathbf{z}) F_1(t, \mathbf{x}(t,\mathbf{z}))dt. \tag{7.8}$$

If there exist $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left((d\mathcal{F}/d\mathbf{z})(a)\right) \neq 0$, then there exists a T–periodic solution $\phi(t,\varepsilon)$ of system (7.7) *such that $\phi(0,\varepsilon) \to a$ as $\varepsilon \to 0$.*

7.3 Proof of Theorem 21

All quasi–homogeneous non–homogeneous cubic polynomial differential systems are given by Proposition 4. Note that all those systems have the origin as the unique singular point.

Now we consider the first system of statement (a) of Proposition 4. This system admits the real first integral

$$\left(by^4+(a-2)xy^2-2x^2\right)\left(\Delta x-2by^2-ax+2x\right)^{\frac{3a+6+\Delta}{\Delta-a-2}}\left(2by^2+ax-2x+\Delta x\right),$$

with $(a-2)^2+8b\geq 0$ and $\Delta=\sqrt{(a-2)^2+8b}$. Note that the real invariant curve $2by^2+ax-2x+\Delta x=0$ passes through the origin. Hence, the origin is not a center.

Now we consider the case where $(a-2)^2+8b<0$. Under the change of coordinates $x\to Y$ $y\to X$ and after renaming (X,Y) by (x,y) we obtain

$$\dot{x}=y+x^2, \qquad \dot{y}=x(ay+bx^2). \tag{7.9}$$

Now we apply Theorem 23 to system (7.9). We have $A(x,y)=x^2$ and $B(x,y)=x(ay+bx^2)$ We have $F(x)=B(x,-x^2)=(b-a)x^3$ and $G(x)=(a+2)x$. Since $a\neq b$ we have that $F\not\equiv 0$. Following the notation of Theorem 23 we have $m=3$, $\alpha=b-a$, $n=1$ and $\beta=a+2$.

For $a=-2$ we have that $G(x)\equiv 0$ and $b<-2$. So $\alpha<0$ and by Theorem 23(a) the origin is a focus or a center. System (7.9) has the real first integral

$$H=\left(y-\left(-1+\frac{1}{2}\sqrt{2(2+b)}\right)x^2\right)\left(y-\left(-1-\frac{1}{2}\sqrt{2(2+b)}\right)x^2\right),$$

well defined at the origin and consequently the origin is a center.

For $a\neq -2$ we have $G(x)\not\equiv 0$. In order that the origin of system (7.9) can be a focus or a center, from Theorem 23(b), we need that $\alpha=b-a<0$ and $(a-2)^2+8b<0$. We notice that system (7.9) under these assumptions admits the real first integral

$$H(x,y)=\frac{\left(16y^2+16x^2y-8x^2ay+8x^4c^2+4x^4-4x^4a+x^4a^2\right)^{2c}}{e^{\sqrt{2}(2+a)\arctan\left(\frac{\sqrt{2}}{4}\frac{-4y-2x^2+x^2a}{x^2c}\right)}},$$

with $c=\sqrt{-2((a-2)^2+8b)}/4$. Since this first integral is defined at the origin, the origin is a center.

The second family of systems of statement (a) of Proposition 4 admits the real invariant curves $\sqrt{a^2+8x} \pm 2y^2 \pm ax = 0$ which pass through the origin. So these systems have no centers.

Easy computations shows that systems (b), (c), (d),(e), (f) and (g) have real invariant curves passing through the origin. Therefore these systems have no centers.

In short, the quasi–homogeneous non–homogeneous cubic polynomial differential systems having a center are the system (7.3) satisfying either $a = -2$ and $b < -2$, or $a \neq -2$, $b-a < 0$ and $(a-2)^2 + 8b < 0$. An easy computation (see Figure 7.2) shows that these conditions for existence of the center in system (7.3) reduces to the unique condition $(a-2)^2 + 8b < 0$.

Now we shall study the phase portrait in the Poincaré disk D and the parameter space of system (7.3). So, we study the infinite singular points of system (7.3) using subsection 2.4. On the local chart U_1 we obtain

$$\begin{aligned} \dot{z}_1 = &\quad {z_2}^2 + (1-a)\,{z_1}^2 z_2 - b{z_1}^4, \\ \dot{z}_2 = &\quad -z_1 z_2 \left(a z_2 + b{z_1}^2\right). \end{aligned} \tag{7.10}$$

Since $8b + (a-2)^2 < 0$ we have that $(z_1, z_2) = (0,0)$ is the only infinite singular point in U_1 and it is linearly zero. In order to classify this infinite singular point we use the standard blow-up techniques, see for instance (Dumortier et al., 2006). Then we obtain that the local phase portrait at the origin (0, 0) of system (7.10) is topologically equivalent to the one described in Figure 7.1(a). Additionally, note that in the chart (U_2, F_2) there are no infinite singular points. Hence, in the Poincaré disk the origin and $\mathbb{S}^1$ are the only separatrices. If we remove the origin and $\mathbb{S}^1$, then we have only one canonical region homomorphic to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ and the flow is locally annular. According to Theorem 25 we obtain that the center is globally defined in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. Hence, the phase portrait of the differential system (7.3) is topologically equivalent to the one of Figure 7.1(b).

The parameter space and phase portrait of system (7.3) is given in Figure 7.2(a).

Now we will study the isochronicity of the center of system (7.3). System (7.3) written in the polar coordinates is

$$\begin{aligned}\dot{r} = &\; P_1(\theta)r + P_2(\theta)r^2 + P_3(\theta)r^3,\\ \dot{\theta} = &\; Q_0(\theta) + Q_1(\theta)r + Q_2(\theta)r^2,\end{aligned}$$

with

$$\begin{aligned}&P_1 = \cos\theta\sin\theta, && P_2 = (\sin^2\theta + a\cos^2\theta)\sin\theta, && P_3 = b\cos\theta\sin^3\theta,\\ &Q_0 = \cos^2\theta, && Q_1 = -(a-1)\sin^2\theta\cos\theta, && Q_2 = -b\sin^4\theta.\end{aligned}$$

Consider the analytic function $H(r,\theta) = \sum\limits_{n=1}^{\infty} H_n(\theta)r^n$ where $H_n(\theta)$ are trigonometric polynomials of degree n. If the condition

$$\ddot{H} + H = 0,$$

is satisfied then in the new variables $(H, -\dot{H})$, system (7.3) could be transformed into the form

$$\dot{u} = -v, \qquad \dot{v} = u.$$

So system (7.3) could have an isochronous center at the origin.

If we expand $\ddot{H} + H = 0$ in power series of r we obtain a recursive system of differential equation. The coefficient of r^n for $n = 1, 2, \cdots$ in this expansion is the differential equation of the form

$$\cos^4\theta H_n''(\theta) + 2(n-1)\sin\theta\,\cos\theta H_n'(\theta) + n\cos^2\theta H_n(\theta)\Big((n-1) - (n-2)\cos^2\theta\Big) + H_n(\theta) = 0,$$

and its general solution for $n = 1$ is

$$H_1(\theta) = \cos\theta\left(C_1\,\sin\left(\frac{\sin\theta}{\cos\theta}\right) + C_2\,\cos\left(\frac{\sin\theta}{\cos\theta}\right)\right).$$

For $n = 2, 3, \cdots$ we have

$$H_n(\theta) = \left(\cos 2\theta + 1\right)^{\frac{n}{2}} \left(C_1 \sin\left(\frac{\sin 2\theta}{\cos 2\theta + 1}\right) + C_2 \cos\left(\frac{\sin 2\theta}{\cos 2\theta + 1}\right)\right).$$

Since these solutions $H_n(\theta)$ must be polynomials of trigonometric functions we have that $H_n \equiv 0$ for all n. Hence we have not an isochronous center and the proof of Theorem 21(a) is completed.

Now we are going to prove Theorem 21(b). The usual forms given in (7.5) for the cubic homogeneous polynomial differential systems having a center were obtained in Proposition 1 and Theorem 2 of (Llibre and Pessoa, 2009). The phase portrait were classified in (Cima and Llibre, 1990). See also Figure 7.2(b).

In order to study the isochronicity of systems (7.4) and (7.5) we can repeat the same mechanism used in the proof of statement (a). In polar coordinates system (7.5) takes the form

$$\dot{r} = P_3(\theta) r^3, \qquad \dot{\theta} = \alpha r^2$$

where $P_3 = p_1(\cos^2\theta - \sin^2\theta) + p_2 \sin\theta\cos\theta$.

We can see that

$$H_1(\theta) = H_2(\theta) = H_3(\theta) = H_4(\theta) = 0,$$

and for $n \geq 5$ we have that

$$H_n(\theta) = -(\alpha^2 H''_{n-4} + 2(n-3)\,\alpha P_3 H'_{n-4} + (n-4) H_{n-4}(\alpha P'_3 + (n-2) P_3^2).$$

Clearly each $H_n \equiv 0$, for all n, so system of (7.5) is not an isochronous center.

System (7.4) can be written in polar coordinates as

$$\dot{r} = P_3(\theta) r^3, \qquad \dot{\theta} = \alpha Q_2(\theta) r^2,$$

where

$$P_3 = p_1(\cos^2\theta - \sin^2\theta) + p_2 \sin\theta\cos\theta,$$

$$Q_2 = \cos^4\theta + 6\mu\cos^2\theta\,\sin^2\theta + \sin^4\theta.$$

Again we obtain

$$H_1(\theta) = H_2(\theta) = H_3(\theta) = H_4(\theta) = 0,$$

and for $n \geq 5$ we have

$$H_n(\theta) = -(\alpha^2 Q_2^2 H''_{n-4} + \left(\alpha^2 Q'^2 Q_2 + 2(n-3)\,\alpha Q_2 P_3\right) H'_{n-4} + (n-4) H_{n-4}(\alpha P'_3 Q_2 + (n-2) P_3^2).$$

Clearly each $H_n \equiv 0$, for all n and therefore system (7.4) is not an isochronous center. This completes the proof of Theorem 21.

7.4 Proof of Theorem 22

System (7.5) in polar coordinates can be written into the form

$$\dot{r} = r^3\left(p_1\cos^2\theta + p_2\sin\theta\,\cos\theta - p_1\sin^2\theta\right), \qquad \dot{\theta} = \alpha r^2,$$

or equivalently

$$\frac{dr}{d\theta} = \frac{r}{\alpha}\left(p_1\cos^2\theta + \sin\theta\, p_2\cos\theta - p_1\sin^2\theta\right),$$

its solution satisfying the initial condition $r(0) = r_0$ is

$$\tilde{r}(\theta,\ r_0) = r_0\exp\left((p_2 + 2p_1\sin(2\theta) - p_2\cos(2\theta))/(4\alpha)\right).$$

Now the fundamental matrix of the linearized equation evaluated on a closed orbit is

$$M_{r_0}(\theta) = M(\theta) = \exp\left(\left(p_2 + 2p_1\sin(2\theta) - p_2\cos(2\theta)\right)/(4\alpha)\right),$$

and satisfies the condition $M(0) = 1$.

Now we perturb system (7.5) inside the class of all cubic polynomial differential systems and we have

$$
\begin{aligned}
\dot{x} &= p_1x^3 + (p_2-\alpha)x^2y - p_1xy^2 - \alpha y^3 + \varepsilon\left(\sum_{0\le i+j\le 3} a_{ij}x^iy^j\right),\\
\dot{y} &= \alpha x^3 + p_1x^2y + (p_2+\alpha)xy^2 - p_1y^3 + \varepsilon\left(\sum_{0\le i+j\le 3} b_{ij}x^iy^j\right).
\end{aligned}
$$

The corresponding differential equation in polar coordinates becomes

$$
\frac{dr}{d\theta} = F_0(\theta,r) + \varepsilon F_1(\theta,r) + O(\varepsilon^2),
$$

with

$$
\begin{aligned}
F_0(\theta,r) &= \frac{r}{\alpha}\left(p_1\left(2\cos^2\theta - 1\right) + p_2\sin\theta\cos\theta\right),\\
F_1(\theta,r) &= \frac{1}{\alpha r^3}(B_4r^4 + B_3r^3 + B_2r^2 + B_1r),
\end{aligned}
$$

where

$$
\begin{aligned}
B_4 = &\ \frac{1}{\alpha}\left(B_{46}\cos^6\theta + B_{45}\sin\theta\cos^5\theta + B_{44}\cos^4\theta + B_{43}\sin\theta\cos^3\theta + B_{42}\cos^2\theta\right.\\
&\quad \left.+B_{41}\sin\theta\cos\theta + B_{40}\right),\\
B_3 = &\ -\frac{1}{\alpha}\left(B_{35}\cos^5\theta + B_{34}\sin\theta\cos^4\theta + B_{33}\cos^3\theta + B_{32}\sin\theta\cos^2\theta + B_{31}\theta\cos\theta + B_{30}\sin\theta\right),\\
B_2 = &\ -\frac{1}{\alpha}\left(B_{24}\cos^4\theta + B_{23}\sin\theta\cos^3\theta + B_{22}\cos^2\theta + B_{21}\sin\theta\cos\theta + B_{20}\right),\\
B_1 = &\ -\frac{1}{\alpha}\left(B_{13}\cos^3\theta + B_{12}\sin\theta\cos^2\theta + B_{10}\sin\theta\right);
\end{aligned}
$$

with

$$
\begin{aligned}
B_{46} = {} & 2\,p_1\,a_{03}+2\,p_1\,b_{12}-2\,p_1\,a_{21}-2\,p_1\,b_{30}+p_2\,a_{12}-p_2\,a_{30}+p_2\,b_{21}-p_2\,b_{03},\\
B_{45} = {} & -2\,p_1\,a_{12}+2\,p_1\,a_{30}+p_2\,a_{03}-p_2\,b_{30}-2\,p_1\,b_{21}-p_2\,a_{21}+p_2\,b_{12}+2\,p_1\,b_{03},\\
B_{44} = {} & -5\,p_1\,a_{03}+3\,p_1\,a_{21}-3\,p_1\,b_{12}-b_{21}\alpha-a_{12}\alpha+a_{30}\alpha+p_2\,a_{30}-p_2\,b_{21}+p_1\,b_{30}\\
& +b_{03}\alpha+2\,p_2\,b_{03}-2\,p_2\,a_{12},\\
B_{43} = {} & -p_2\,b_{12}+3\,p_1\,a_{12}-a_{03}\alpha+p_2\,a_{21}+b_{30}\alpha-3\,p_1\,b_{03}-p_1\,a_{30}-b_{12}\alpha+p_1\,b_{21}\\
& +a_{21}\alpha-2\,p_2\,a_{03},\\
B_{42} = {} & 4\,p_1\,a_{03}+p_1\,b_{12}-p_2\,b_{03}+b_{21}\alpha+a_{12}\alpha-2\,b_{03}\alpha-p_1\,a_{21}+p_2\,a_{12},\\
B_{41} = {} & -p_1\,a_{12}+p_2\,a_{03}+b_{12}\alpha+a_{03}\alpha+p_1\,b_{03},\\
B_{40} = {} & b_{03}\alpha-p_1a_{03},\\
B_{35} = {} & 2\,p_1\,b_{02}-2\,p_1\,a_{11}-2\,p_1\,b_{20}+p_2\,a_{02}+p_2\,b_{11}-p_2\,a_{20},\\
B_{34} = {} & -2\,p_1\,b_{11}-p_2\,b_{20}+2\,p_1\,a_{20}-p_2\,a_{11}-2\,a_{02}p_1+p_2\,b_{02},\\
B_{33} = {} & -a_{02}\alpha+p_1\,b_{20}+a_{20}\alpha+p_2\,a_{20}-2\,p_2\,a_{02}-p_2\,b_{11}-3\,p_1\,b_{02}+3\,p_1\,a_{11}-b_{11}\alpha,\\
B_{32} = {} & -b_{02}\alpha+p_2\,a_{11}-p_1\,a_{20}+b_{20}\alpha+p_1\,b_{11}+3\,a_{02}p_1+a_{11}\alpha-p_2\,b_{02},\\
B_{31} = {} & -p_1\,a_{11}+p_2\,a_{02}+b_{11}\alpha+a_{02}\alpha+p_1\,b_{02},\\
B_{30} = {} & -a_{02}p_1+b_{02}\alpha,\\
B_{24} = {} & -2\,a_{01}p_1-2\,p_1\,b_{10}+p_2\,b_{01}-p_2\,a_{10},\\
B_{23} = {} & -2\,p_1\,b_{01}-p_2\,b_{10}-p_2\,a_{01}+2\,p_1\,a_{10},\\
B_{22} = {} & 3\,a_{01}p_1+a_{10}\alpha+\alpha\,b_{01}+p_2\,a_{10}+p_1\,b_{10}-p_2\,b_{01},\\
B_{21} = {} & p_1\,b_{01}-p_1\,a_{10}+b_{10}\alpha+a_{01}\alpha+p_2\,a_{01},\\
B_{20} = {} & -a_{01}p_1+\alpha\,b_{01},\\
B_{13} = {} & -2\,p_1\,b_{00}-p_2\,a_{00},\\
B_{12} = {} & 2\,a_{00}p_1-p_2\,b_{00},\\
B_{11} = {} & p_1\,b_{00}+p_2\,a_{00}+\alpha\,a_{00},\\
B_{10} = {} & b_{00}\alpha-a_{00}p_1.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathcal{F}(r_0) &= \int\limits_0^{2\pi} M^{-1}(\theta)\, F_1(\theta, \tilde{r}(\theta, r_0))\, d\theta \\
&= \frac{1}{r_0} A_0 I_0 + \frac{2}{r_0^2}\left(A_1 I_1 + A_2 I_2 + A_3 I_3 + A_4 I_4 + \pi\, C_1 + \frac{3\pi}{4} C_2\right) \\
&\quad + 2 r_0 \pi(\alpha b_{03} - p_1 a_{03}) + \frac{5\pi}{8} C_3,
\end{aligned}
$$

where we have

$$
\begin{aligned}
I_0 &= \int_0^{2\pi} E\, d\theta, & I_1 &= \int_0^{2\pi} E \cos\theta\, \sin\theta\, d\theta, \\
I_2 &= \int_0^{2\pi} E\, \cos^2\theta\, d\theta, & I_3 &= \int_0^{2\pi} E \cos^3\theta\, \sin\theta\, d\theta, \\
I_4 &= \int_0^{2\pi} E \cos^4\theta\, d\theta, & E &= \exp\left(-\frac{\sin\theta\,(2 p_1 \cos(\theta) + p_2 \sin\theta)}{\alpha}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
A_0 =&\quad -a_{01} p_1 + \alpha\, b_{01}, \\
A_1 =&\quad -\frac{1}{2}\left((a_{10} - b_{01})\, p_1 - p_2\, a_{01} - \alpha\,(a_{01} + b_{1,0})\right) r_0, \\
A_2 =&\quad \left(\left(\frac{3}{2} a_{01} + \frac{1}{2} b_{10}\right) p_1 + \frac{1}{2}\,(p_2 + \alpha)\,(a_{10} - b_{01})\right) r_0, \\
A_3 =&\quad \left((a_{10} - b_{01})\, p_1 - \frac{1}{2} p_2\,(a_{01} + b_{10})\right) r_0, \\
A_4 =&\quad \left((-a_{01} - b_{10})\, p_1 - \frac{1}{2} p_2\,(a_{10} - b_{01})\right) r_0, \\
C_1 =&\quad \left(\left(2\, a_{03} + \frac{1}{2} b_{12} - \frac{1}{2} a_{21}\right) p_1 + \left(\frac{1}{2} a_{12} - \frac{1}{2} b_{03}\right) p_2 \right. \\
&\quad \left. - \left(-\frac{1}{2} a_{12} + b_{03} - \frac{1}{2} b_{21}\right) \alpha\right) {r_0}^3, \\
C_2 =&\quad \left(\left(\frac{3}{2} a_{21} + \frac{1}{2} b_{30} - 5/2\, a_{03} - 3/2\, b_{12}\right) p_1 + \left(\frac{1}{2} a_{30} - a_{1,2} - \frac{1}{2} b_{21} + b_{03}\right) p_2 \right. \\
&\quad \left. + \frac{1}{2}\alpha\,(b_{03} + a_{30} - a_{12} - b_{21})\right) {r_0}^3, \\
C_3 =&\quad 2\left((b_{12} + a_{03} - a_{21} - b_{30})\, p_1 - \frac{1}{2} p_2\,(b_{03} + a_{30} - a_{12} - b_{21})\right) r_0.
\end{aligned}
$$

In short, the function $\mathcal{F}(r)$ of Theorem 27 is of the form

$$\mathcal{F}(r) = \frac{\alpha r^2 + \beta}{r},$$

so it has at most one real positive root given by $r = \sqrt{-\beta/\alpha}$. Moreover, we have that $\mathcal{F}'\left(\sqrt{-\beta/\alpha}\right) = 2\alpha$. So by Theorem 27 if $-\beta/\alpha > 0$ then there is one limit cycle bifurcating from a periodic orbit of the center of system (7.5). This completes the proof of Theorem 22 for system (7.5).

The rest of the proof of Theorem 22 for system (7.4) is completely analogous to the one done for system (7.5), only changes the computations, and we do not repeat it here.

7.4.1 Examples

First we give an example satisfying the result of Theorem 22 for system (7.5). We consider the system

$$\dot{x} = x^3 + 2x^2y - xy^2 - y^3, \qquad \dot{y} = x^3 + x^2y + 4xy^2 - y^3,$$

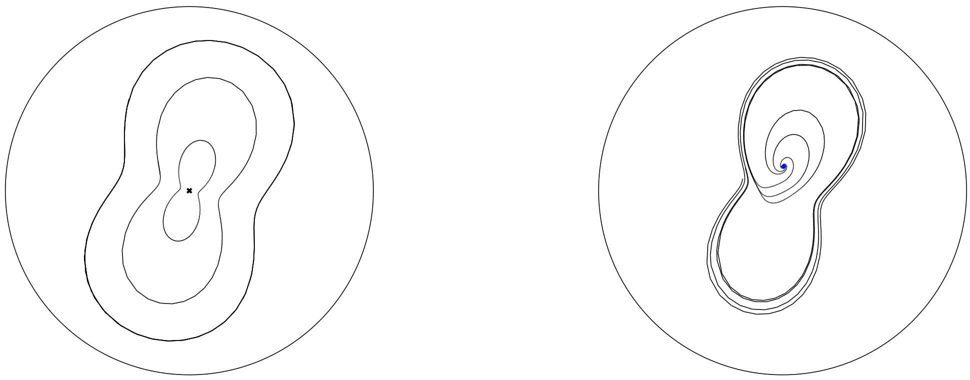
and its perturbation

$$\begin{aligned}
\dot{x} = {} & x^3 + 2x^2y - xy^2 - y^3 \\
& + \varepsilon\left(4y^3 + 3xy^2 + 3x^2y + 5x^3 + 3y^2 + 3xy + 3x^2 - y - x + 2\right), \\
\dot{y} = {} & x^3 + x^2y + 4xy^2 - y^3 \\
& + \varepsilon\left(-3y^3 + xy^2 + x^2y + x^3 + 5y^2 + xy + x^2 + y + 2x + 1\right).
\end{aligned} \tag{7.11}$$

Then

$$\mathcal{F}(r_0) = \frac{11.78097245 r_0^2 - 4.108168642}{r_0}$$

and $\mathcal{F}(r) = 0$ gives $r = 0.5905185728$. So according to Theorem 27 at most one limit cycle can bifurcated from the origin, see also Figure 7.3.



$\varepsilon = 0$ $\qquad\qquad$ $\varepsilon = 0.01$

Figure 7.3: Phase portrait of system (7.11) in the Poincaré disk.

Example-2 Now we give an example satisfying the result of Theorem 22 for system (7.4). For $\varepsilon = 0$ the origin of the system

$$
\begin{aligned}
\dot{x} = \;& x^3 - 6x^2y - xy^2 - y^3 \\
& \quad + \varepsilon\left(2y^3 + 3xy^2 + 3x^2y - 5x^3 + 3y^2 + 10xy + 3x^2 - y - x - 20\right), \\
\dot{y} = \;& x^3 + x^2y + 12xy^2 - y^3 \\
& \quad + \varepsilon\left(-3y^3 + xy^2 - 10x^2y + x^3 + 5y^2 + xy + 1/5x^2 + y + 2x + 100\right),
\end{aligned}
\tag{7.12}
$$

is a center and for $\varepsilon = 0.01$ one limit cycle is produced, see Figure 7.4.

$\varepsilon = 0$ $\qquad\qquad$ $\varepsilon = 0.01$

Figure 7.4: Phase portrait of system (7.12) in the Poincaré disk.

Appendix A

APPENDICES

Local Analytic First Integrals of Three-dimensional Lotka-Volterra Systems with 3:-1:2-resonance

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*A*bstarct

In this paper we study the local integrability at the origin of the three dimensional Lotka-Volterra differential systems of the form

$$\dot{x} = x(3 + ax + by + cz),$$
$$\dot{y} = y(-1 + dx + ey + fz),$$
$$\dot{z} = z(2 + gx + hy + kz).$$

We characterize all the families of such systems and the values of their parameters having two independent local first integrals at the origin of coordinates.

A.1 Introduction and statement of the main results

The integrability of a system of nonlinear differential equations is strongly related with many problems of applied mathematics and physics. In particular, for a three dimensional system the knowledge of a unique first integral reduces the study of the dynamics of the system from dimension 3 to dimension 2, whereas the knowledge of two first integrals determine completely the trajectories of the system. Hence study of the existence of first integrals is an important subject in the qualitative theory of dynamical systems. In the literature we can find different approaches for the existence of first integrals like the ones that use Noether symmetries (Cantrijn and Sarlet (1981), Lie symmetries (Almeida et al. (1995); Olver (1986)), Lax pairs (Audin (1996); Lax (1968)), Painlevé analysis (Bountis et al. (1984)), Differential Galois Theory (Singer (1992); Weil (1995)), Darboux theory of integrability (Darboux (1878b,a))) among others.

Here we use the same method which have been used in (Aziz and Christopher (2012)) for classifying all the Lotka–Voltera three dimensional differential systems of the form

$$\begin{aligned}
\dot{x} &= x(3+ax+by+cz) = P(x,y,z), \\
\dot{y} &= y(-1+dx+ey+fz) = Q(x,y,z), \\
\dot{z} &= z(2+gx+hy+kz) = R(x,y,z),
\end{aligned} \tag{A.1}$$

having two independent local first integrals at the origin of coordinates. The reason for considering the case of (3,-1, 2)–resonance to generalize the work in (Aziz and Christopher (2012)), is that we are still working in the Siegel domain with two independent resonances, but that the nodal behaviour, when restricted to y=0, has no resonance. The hope was therefore to be able to isolate the generic behaviour associated with integrability of these systems, from the more ad-hoc behaviour associated to the resonance of the node. Unfortunately, the situation is still very complex and more analysis of other cases will be needed before we can begin to conjecture the general mechanisms which underlie integrability in these systems for general resonances.

Lotka–Volterra systems are classical models that describe the evolution of conflicting species in population biology (May (1974); May and Leonard (1975)). Since the works of Lotka (Lotka (1920)) and Volterra (Volterra (1931)) and more recent works (Brenig (1988); Brenig and Goriely (1989)) these systems appear in many different topics like neural networks (Noonburg (1989)), chemical kinetics (Murza and Teruel (2010)), laser physics Lamb (1964), plasma physics Laval and Pellat (1975), etc. The qualitative properties of these models has been widely studied, see for instance (Bobienski and Żołądek (2005); van den Driessche and Zeeman (1998); Zeeman (1993)) among others. The integrability of some Lotka–Voltera families using Darboux's method was done by several authors, like (Cairó (2000); Cairó and Llibre (2000a); Christodoulides and Damianou (2009); Llibre and Valls (2011b)).

We associate to system (A.1) the vector field $\mathcal{X}$ given by

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}.$$

Let U be an open neighborhood of the origin of $\mathbb{C}^3$. We say that a non-constant analytic function $H : U \to \mathbb{C}$ is a *local first integral* of system (A.1) if it is constant on all solutions (trajectories) of the system contained in U.

So the function H is a first integral of system (A.1) in U if and only if

$$\mathcal{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + R\frac{\partial H}{\partial z} = 0,$$

in all the points of U.

Two local first integrals H_1 and H_2 defined in U are *independent* if their gradients are linear independent in $\mathbb{C}^3$ except perhaps in a set of measure zero .

A function $M : U \to \mathbb{C}$ is an *inverse Jacobi multiplier* of $\mathcal{X}$ if it satisfies the equation

$$\mathcal{X}(M) = M\mathrm{div}\mathcal{X} \qquad \text{in} \quad U,$$

or equivalently if $\mathrm{div}(\mathcal{X}/M) = 0$, where $\mathrm{div}\mathcal{X} = \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$.

One way of finding two local independent first integrals consists in to find a local first integral and an inverse Jacobi multiplier, see for more details (Goriely (2001)) and for Darboux Jacobi multipliers see (Berrone and Giacomini (2003)).

Another way of finding two local independent first integrals is finding the so called integrable systems (A.1) at the origin or the linearizable systems at the origin, that is, the differential system $(A.1)$ is *integrable at the origin* if and only if there is a change of coordinates of the form

$$(X,Y,Z) = \big(x+o(x,y,z),\, y+o(x,y,z),\, z+o(x,y,z)\big),$$

which transforms system (A.1) into the system

$$\dot{X} = 3X\zeta(x,y,z), \quad \dot{Y} = -Y\zeta(x,y,z), \quad \dot{Z} = 2Z\zeta(x,y,z), \tag{A.2}$$

where $\zeta = 1+o(x,y,z)$. Note that XY^3 and Y^2Z are first integrals of system (A.2) and we can pull them back to the first integrals of system (A.1)

$$\phi_1 = xy^3(1+O(x,y,z)), \qquad \phi_2 = y^2z(1+O(x,y,z)).$$

Conversely, given two independent first integrals of the form ϕ_1 and ϕ_2, it is easily seen that there exits a change of coordinates X, Y, Z so that these first integrals can be written in the form XY^3 and Y^2Z and the new system is of the form (A.2) for some ζ.

We say that system $(A.1)$ is *linearizable at the origin* if and only if the change of coordinates can be chosen to make $\zeta \equiv 1$, or equivalently system (A.1) can be written into the form

$$\dot{X} = 3X, \quad \dot{Y} = -Y, \quad \dot{Z} = 2Z.$$

This can be shown to be equivalent to asking that all the coefficients of the non–linear terms in $\zeta(x,y,z)$ vanish, see (Christopher and Rousseau (2004); Christopher et al. (2004); Darboux (1878a)).

In this paper we study the local integrability at the origin of the three dimensional Lotka-Volterra systems of the form (A.1).

Now we state our main result which characterize all the Lotka–Voltera differential systems (A.1) having two independent local first integrals. So we will provide a complete classification for the integrability conditions of systems (A.1) in $(3:-1:2)$-resonance.

Theorem 28. *Consider the three dimensional Lotka-Volterra system* (A.1) *in* $(3,-1,2)$-*resonance. Then the origin is integrable if and only if one of the following conditions hold.*

1) $2ef-ek-hk = 2cd+cg-5dk+3fg-gk = 2bk-2ce+3ek-3hk =$
$2bf-bk-ch-3fh+3hk = 2bd+bg-2de-3dh+2eg = 2af+ak-2dk-gk =$
$2ae+ah-2de-eg = 2ac-5ak-2cd-cg+5dk-3fg+4gk =$
$2ab-2ah-2bd-bg+3dh = 0$

2) $b = h = 0$

3) $h = b+e = 0$

4) $a = d = g = h = 0$

5) $b+2e = d = h = 0$

6) $a-d = b-e = g = h = 0$

7) $8a-3g = b+3e = 8d+g = h = 0$

8) $b = c = f = k = 0$

9) $b+e = c = f = k = 0$

10) $b+2e = c = d = f = k = 0$

11) $b = e+h = f = 0$

12) $b-h = e+h = f = 0$

13) $b-2h = d = e+h = f = 0$

14) $a = e+h = f = d = g = 0$

15) $a-2d-g = e+h = f = b+h = d+g = 0$

16) $b = 2c-9k = 2e+h = 2f+k = 0$

17) $b = c = e-h = f-k = 0$

18) $2b-h=2c-7k=2e+h=2f+k=0$

19) $b+h=c+k=e-h=f-k=0$

20) $b+2h=c+2k=d=e-h=f-k=0$

Moreover, the system is linearizable if and only if either one of the conditions $2-6$, $8-14$, 17, 19, 20 *or one of the following subcases of case 1) holds.*

1.1) $dh-eg=k=f=c=b-e=a-d$;

1.2) $k=g=f=d=c=a$;

1.3) $f-k=e-h=d-g=c-k=b-h=a-g=0$;

1.4) $bk-ch=g=f-k=e-h=d=a$.

In section A.2 we present the basic notions that we use for the proof of Theorem 28 whereas the complete proof of Theorem 28 is presented in section A.3.

A.2 Preliminaries

An *invariant algebraic surface* of the polynomial differential system (A.1) is an algebraic surface $\ell=0$ which satisfies

$$\dot{\ell}=\mathcal{X}\ell=P\frac{\partial\ell}{\partial x}+Q\frac{\partial\ell}{\partial y}+R\frac{\partial\ell}{\partial z}=\ell L_{\ell}, \tag{A.3}$$

for some polynomial $L_{\ell}\in\mathbb{C}[x,y,z]$. Such a polynomial is called a *cofactor* of the invariant algebraic surface $\ell=0$. Due to equation (A.3) we have that any cofactor has at most degree one because the polynomial vector field has degree two (independently of the degree of ℓ).

In this work we also use the notion of the exponential factor and is related with the multiplicity of the invariant surfaces, see also (Llibre and Zhang (2009b,a)). The exponential factor actually plays a similar role as the invariant algebraic surface and help us to obtain first integrals of the polynomial differential system (A.1). Let

$$E(x,y,z) = \exp(f(x,y,z)/g(x,y,z)),$$

where $f,g \in \mathbb{C}[x,y,z]$ and they are relatively prime. Then E in an *exponential factor* of (A.1) if

$$\mathcal{X}E = EL_E, \tag{A.4}$$

for some polynomial L_E of degree at most one. The polynomial L_E is called the *cofactor* of the exponential factor E.

The following theorems are crucial in our study and they are proved in (Aziz and Christopher (2012)0. In what follows we use the multi–index notation $X^I = x^i y^j z^k$ in order to simplify the presentation of Theorem 29.

Theorem 29. *Suppose the analytic vector field*

$$x\left(\lambda + \sum_{|I|>0} A_{xI}X^I\right)\frac{\partial}{\partial x} + y\left(\mu + \sum_{|I|>0} A_{yI}X^I\right)\frac{\partial}{\partial y} + z\left(\nu + \sum_{|I|>0} A_{zI}X^I\right)\frac{\partial}{\partial z},$$

has a first integral $\phi = x^\alpha y^\beta z^\gamma(1+O(x,y,z))$ with at least one of α, β, $\gamma \neq 0$ and a Jacobi multiplier $M = x^r y^s z^t(1+O(x,y,z))$ and suppose that the cross product of $(r-i-1, s-j-1, t-k-1)$ and (α,β,γ) is bounded away from zero for any integers $i,j,k \geq 0$. Then the vector field has a second analytic first integral of the form $\psi = x^{1-r}y^{1-s}z^{1-t}(1+O(x,y,z))$ and hence the system (A.1) *is integrable.*

Hence for system (A.1) and under conditions from the expression of a first integral and an inverse Jacobi multiplier we can obtain the explicit expression of the other first integral.

Theorem 30. *If the system* (A.1) *is integrable and there exists a function $\xi = x^\alpha y^\beta z^\gamma(1+O(x,y,z))$ such that $X(\xi) = k\xi$ for some constant $k = \alpha\lambda + \beta\mu + \gamma\nu$, then the system is linearizable.*

A singular point whose eigenvalues lie in a Poincaré domain (that is, the convex hull of the eigenvalues does not contain the origin inside or on the boundary) can be brought to normal form (A.2) via an analytic change of coordinates (**?**).

A.3 Proof of Theorem 28

Simple calculations shows that systems (A.1) admit the invariant hyperplanes $x = 0$, $y = 0$, $z = 0$ with cofactors $L_x = 3 + ax + by + cz$, $L_y = -1 + dx + ey + fz$, $L_z = 2 + gx + hy + kz$ respectively.

We seek conditions that guarantees the existence of two independent analytic first integrals of system (A.1) of the form

$$\phi_1 = xy^3(1 + O(x,y,z)) \quad \text{and} \quad \phi_2 = y^2 z(1 + O(x,y,z)). \tag{A.5}$$

So, first we express ϕ_1 and ϕ_2 as power series up to terms of order 18 and then we compute the obstructions to them forming first integrals. A factorized Gröbner basis is obtained giving the following necessary conditions for integrability described in Theorem 28. For this, we have been used Maple and Reduce and finally we used the minAssGTZ algorithm of Singular (Greuel et al. (2012)) to check that the found conditions were irreducible. For more details about the mechanism that we apply see Section 3 of (Aziz and Christopher (2012)).

Proof of Theorem 28. In order to complete the proof of Theorem 28 we shall show below that each of these conditions is also sufficient for the integrability of system (A.1).

Case 1 If $k \neq 0$ the system admits the invariant algebraic plane $\ell = 1 + \frac{a}{3}x - ey + \frac{k}{2}z = 0$ with cofactor $L_\ell = ax + ey + kz$ and using the Darboux theory of integrability (see for more details chapter 8 of (Dumortier et al. (2006))) we obtain the two independent first integrals

$$\phi_1 = xy^3 \ell^{-\frac{3f+c}{k}}, \qquad \phi_2 = y^2 z \ell^{1-\frac{2f}{k}}.$$

If $k = 0$ we consider the following two subcases.

Subcase 1 $k = a = e = 2bd + bg - 3dh = 2bf - ch - 3fh = 2cd + cg + 3fg = 0.$

i) $b \neq 0$ and $h \neq 0$. We get the exponential factor $E = \exp(dhx - bhy + bfz)$ with cofactor $L_E = 3dhx + bhy + 2bfz$. Thus, by the Darboux theory of integrability, system (A.1) admits the two first integrals

$$\phi_1 = xy^3\ell^{-\frac{1}{h}}, \qquad \phi_2 = y^2z\ell^{-\frac{1}{b}}.$$

ii) $h = 0$ and $b \neq 0$. We distinguish the following subcases.

a) $a = 2d + g = e = f = h = k = 0$. System (A.1) admits the exponential factor $E = \exp(dx - by + cz/2)$ with cofactor $L_E = 3d + by + cz$. In this case we obtain the two independent analytic first integrals

$$\phi_1 = xy^3E^{-1}, \qquad \phi_2 = y^2z.$$

b) $2a - 2d - g = c = f = h = k = 2bd + bg - 2de + 2eg$.

If $e \neq 0$ we obtain the invariant hyperplane $\ell = 1 + \frac{a}{3}x - ey$ with cofactor $L_\ell = ax + ey$ and the two independent analytic first integrals

$$\phi_1 = xy^3\,\ell^{-3-\frac{b}{e}}, \qquad \phi_2 = y^2z\ell^{-2}.$$

If $e = 0$ we obtain the exponential factor $E = \exp(dx - by)$ with cofactor $L_E = 3dx + by$. System (A.1) admits the two analytic independent first integrals

$$\phi_1 = xy^3\,E^{-1}, \qquad \phi_2 = y^2\,z.$$

(iii) $a = b = d = e = k = c + 3f = 0$. The exponential factor $E = \exp(1/3g - hy + fz)$ is invariant with cofactor $L_E = gx + hy + 2fz$ and we obtain the two analytic first integrals

$$\phi_1 = xy^3E, \qquad \phi_2 = y^2zE^{-1}.$$

iv) If $b = h = 0$ then we fall in Case 2.

Subcase 2 $k=c=f=2ab-2ae-3ah+3eg=2ae+ah-2de-eg=2bd+bg-2de-3dh+2eg=0$. When $e\neq 0$ the system has the invariant algebraic surface $\ell=1+\frac{a}{3}x-ey=0$ with cofactor $L_\ell=1+ax+by$ and we obtain the two independent first integrals

$$\phi_1=xy^3\,\ell^{-3-\frac{b}{e}},\qquad \phi_2=y^2z\,\ell^{-2+\frac{h}{e}}.$$

When $e=0$ the system (A.1) admits the exponential factor $E=\exp(dx-by)$ with cofactor $L_E=3dx+by$ and then we can construct the two independent first integrals

$$\phi_1=xy^3E^{-1},\qquad \phi_2=y^2zE^{-\frac{h}{b}}.$$

Note that when $b=0$ we can reduce to Case 8.

Case 1.1 For $e\neq 0$ the system has the invariant algebraic surface $\ell=1+\frac{d}{3}x-ey=0$ with cofactor $L_\ell=dx+ey$ and a linearizing change of coordinates is given by

$$(X,Y,Z)=\left(x\ell^{-1},\ y\ell^{-1},\ z\ell^{-\frac{h}{e}}\right).$$

For $e=0$ we distinguish two subcases. First, for $h=0$ and $d\neq 0$ we get a subcase of Case 2. Second, if $d=0$ and $h\neq 0$, a linearizing change of coordinates is given by

$$(X,Y,Z)=\left(x,\ y,\ z\exp\left(hy-\frac{g}{3}x\right)\right).$$

Case 1.2 When $e\neq 0$, the linearizing change of coordinates is

$$(X,Y,Z)=\left(x(1-ey)^{-\frac{b}{e}},\ y(1-ey)^{-1},\ z(1-ey)^{-\frac{h}{e}}\right).$$

When $e=0$, the linearizing change of coordinates is

$$(X,Y,Z)=(x\exp(by),\ y,\ z\exp(hy)).$$

Case 1.3 In this case the linearizing change of coordinates is

$$(X,Y,Z) = \left(x\ell^{-1},\, y\ell^{-1},\, z\ell^{-1}\right),$$

where $\ell = 1 + \frac{a}{3}x - by + \frac{c}{2}z$.

Case 1.4 When $h \neq 0$, the linearizing change of coordinates is given by

$$(X,Y,Z) = \left(x\ell^{-\frac{b}{h}},\, y\ell^{-1},\, z\ell^{-1}\right),$$

where $\ell = 1 - hy + \frac{k}{2}z$. When $h = 0$ we have two possibilities. First for $b = 0$ then we get a subcase of Case 2. Second if $k = 0$ then the linearizing change of coordinates is

$$(X,Y,Z) = \left(x\exp\left(by - \frac{c}{2}z\right),\, y,\, z\right).$$

Case 2 The system can be written into the form

$$\dot{x} = x(3 + ax + cz), \qquad \dot{y} = y(-1 + dx + ey + fz), \qquad \dot{z} = z(2 + gx + kz).$$

Note that the first and third equation yields to a linearizable node. Hence, there exists an analytic transformation of the form $X = x(1 + O(x,z))$ and $Z = z(1 + O(x,z))$ such that the two equations can be written as $\dot{X} = 3X$ and $\dot{Z} = 2Z$. Now in order to linearize the second equation we seek an invariant algebraic surface of the form $\ell = A + By = 0$ with cofactor $L_\ell = dx + ey + fz$ and $B = B(X,Z)$ and $A = A(X,Z)$ such that $A(0,0) = 1$. The change of variable $Y = y/(A + By)$ will then linearize the second equation. In order to find such A and B we have to solve the following system of the two differential equations

$$\dot{B} - B = eA, \qquad \dot{A} = (dx + fz)A. \tag{A.6}$$

We write $A = \exp(\alpha(X,Z))$ and from the second equation of (A.6) we obtain $\dot{\alpha}(X,Z) =$

$dx(X,Z) + fz(X,Z)$. Suppose that $\alpha = \sum_{i+j>0} \alpha_{ij} X^i Z^j$. Then

$$\dot{\alpha}(X,Z) = \sum_{i+j>0} (3i+2j)\alpha_{ij} X^i Z^j = dx(X,Z) + fz(X,Z) = \sum_{i+j>0} c_{ij} X^i Z^j.$$

So $\alpha_{ij} = \frac{c_{ij}}{3i+2j}$ for $i+j>0$ and the convergence of $\sum_{i+j>0} c_{ij} X^i Z^j$ guarantees the convergence of α and hence of A. Now, we write $A = \sum_{i+j>0} a_{ij} X^i Z^j$ and $B = \sum_{i+j>0} b_{ij} X^i Z^j$. Then from the first equation of (A.6) we find that $B = \sum_{i+j>0} \frac{ea_{ij}}{3i+2j-1} X^i Z^j$ which obviously is convergent. Hence, the system is linearizable.

Case 3 The system can be written

$$\dot{x} = x(3+ax+by+cz), \qquad \dot{y} = y(-1+dx-by+fz), \qquad \dot{z} = z(2+gx+kz).$$

The change of variable $Y = xy$ transforms the system into the form

$$\dot{x} = x(3+ax+cz)+bY, \qquad \dot{Y} = Y(2+(a+d)x+(c+f)z), \qquad \dot{z} = z(2+gx+kz).$$

The singular point at the origin is then in the Poincaré domain and therefore there exists a change of coordinates of the form

$$\left(X,\tilde{Y},Z\right) = (x+bY+O(2),\, Y(1+O(1)),\, z(1+O(1))),$$

which linearizes the system. The first integrals of the linear system are then $\psi_1 = X^{-2}\tilde{Y}^3$ and $\psi_2 = \tilde{Y}Z^{-1}$. Pulling back this two first integrals we see that the initial system admits two independent analytic fist integrals of the desired form $\phi_1 = \psi_1 = xy^3(1+\ldots)$ and $\phi_2 = \psi_1 \psi_2{}^{-1} = y^2 z(1+\ldots)$.

Case 4 We provide two different proofs. In the first we prove the existence of two independent local first integrals and in the second we provide one first integral and an inverse Jacobi Multiplier. The system is

$$\dot{x} = x(3+by+cz), \qquad \dot{y} = y(-1+ey+fz), \qquad \dot{z} = z(2+kz).$$

The system has an algebraic invariant surface $\ell = 1 + \frac{k}{2}z = 0$ with cofactor $L_\ell = kz$. Using the change of coordinates $Z = z\ell^{-1}$ we can linearize the last equation and bring it into the form $\dot{Z} = 2Z$. To linearize the second equation we seek an invariant surface of the form $\tilde{\ell} = A + By$ with $A = A(Z)$ and $A(0) = 1$ and $B = B(Z)$ with cofactor $L_{\tilde{\ell}} = ey + fz$. To find A and B we need to solve the system of the two differential equations

$$\dot{A} = Afz, \qquad \dot{B} - B = Ae.$$

For $k = 0$ we have $A = \exp(fZ/2)$ whereas for $k \neq 0$ we have $A = (1 - kZ/2)^{\frac{f}{2k}}$.

Now write $A = \sum_{i>0} a_i Z^i$ and $B = \sum_{i>0} b_i Z^i$. From the relation $\dot{B} - B = Ae$ we have $b_i = e\,a_i/(2i - 1)$ and so B is a convergent series. Hence the second equation is linearized and taking $Y = y/\tilde{\ell}$ we get $\dot{Y} = -Y$. In order to linearize the first equation we set $X = x\exp(-\gamma(Y,Z))$ with γ an analytic function satisfying $\dot{\gamma} = b\,y(Y,Z) + c\,z(Z)$. We set

$$\gamma(Y,Z) = \sum_{i+j>0} \gamma_{ij} Y^i Z^j, \quad y(Y,Z) = \sum_{i+j>0} \eta_{ij} Y^i Z^j, \quad z(Z) = \sum_{j>0} \zeta_j Z^j,$$

and we have that

$$\sum_{i+j>0} (2j - i)\gamma_{ij} Y^i Z^j = b \sum_{i+j>0} \eta_{ij} Y^i Z^j + c \sum_{j>0} \zeta_j Z^j.$$

If $y(Y,Z)$ contains no terms of the form $(Y^2 Z)^n$ we obtain

$$\gamma_{0j} = \frac{b\,\eta_{0j} + c\,\zeta_j}{2j}, \qquad \gamma_{ij} = \frac{b\,\eta_{ij}}{2j - i},\ i > 0.$$

Clearly γ is a convergent series. Now we should show that the inverse transformation $y = AY/(1 - BY)$ contains no term like $(Y^2 Z)^k$. Note that

$$y = \frac{AY}{1 - BY} = \sum A B^k Y^{k+1}.$$

We assume that $k + 1 = 2n$ for some n. We should show that $AB^k = AB^{2n-1}$ contains no term like Z^n. Note that

$$\dot{B} = \frac{dB}{dt} = 2Z\frac{dB}{dZ}.$$

From equation $\dot{B} - B = Ae$ we obtain that

$$\left(\frac{2Z}{2n}\frac{dB^{2n}}{dZ} - B^{2n}\right) = eAB^{2n-1}.$$

Now on the left hand side the coefficient in Z^n in B^{2n} vanishes. So either $e = 0$, that means $B \equiv 0$, or the coefficients of Z^n in B^{2n-1} vanishes. Therefore y given by $Y = y/(A + By)$ contains no term like $(Y^2Z)^n$. Thus we have set up the existence of a linearizing transformation and this completes the first proof.

Now we will provide the second proof. When $bk \neq 0$ the system has the first integral

$$\phi = x^{-2ek}y^{2bk}z^{(b+3e)k}\left(1 + \frac{1}{2}kz\right)^{2ce-3ek-2be-bk},$$

and the inverse Jacobi multiplier

$$IJM = x^{1+\frac{2e}{b}}z^{\frac{1}{2}-\frac{3e}{b}}(1 + \frac{1}{2}kz)^{\frac{3}{2}+\frac{f}{k}+\frac{3e}{b}-\frac{2ce}{bk}}.$$

Theorem 29 guarantees the existence of the second first integral

$$\psi = x^{-\frac{2e}{b}}yz^{\frac{1}{2}+\frac{3e}{b}}(1 + \ldots).$$

The desired first integrals of the initial system are

$$\phi_1 = \phi^{\frac{1}{2}\frac{b+6e}{bke}}\psi^{-3-\frac{b}{e}} = xy^3(1 + \ldots) \quad \text{and} \quad \phi_2 = \phi^{\frac{2}{bk}}\psi^{-2} = y^2z(1 + \ldots).$$

For the remaining cases, we will just give the form for the first integrals and the inverse Jacobi multipliers as the procedures are the same as above.

i) $b=0$ and $k\neq 0$.

$$\begin{aligned}
\phi &= x^{-2k}z^{3k}\left(1+\frac{1}{2}kz\right)^{2ce-3ek-2be-bk}_{,}\\
IJM &= x^2y^2\left(1+\frac{1}{2}kz\right)^{-3k\left(1-\frac{2c}{3k}\right)}_{,}\\
\psi &= x^{-1}y^{-1}z(1+\ldots),\\
\phi_1 &= \phi^{\frac{1}{k}}\psi^{-3}=xy^3(1+\ldots),\\
\phi_2 &= \phi^{\frac{1}{k}}\psi^{-2}=y^2z(1+\ldots).
\end{aligned}$$

ii)$k=0$ and $b\neq 0$.

$$\begin{aligned}
\phi &= x^{-2e}y^{2b}z^{b+3e}\exp\left((ce-bf)z\right),\\
IJM &= x^{1+\frac{2e}{b}}z^{\frac{1}{2}-\frac{3e}{b}}\exp\left(\frac{2ce-bf}{2b}z\right),\\
\psi &= x^{-\frac{2e}{b}}yz^{\frac{1}{2}+\frac{3e}{b}},\\
\phi_1 &= \phi^{\frac{1}{2e}+\frac{3}{b}}\psi^{-3-\frac{b}{e}}=xy^3(1+\ldots),\\
\phi_2 &= \phi^{\frac{2}{b}}\psi^{-2}=y^2z(1+\ldots).
\end{aligned}$$

iii) $b=0$ and $k=0$.

$$\begin{aligned}
\phi &= x^2z^{-3}\exp\left(-cz\right),\\
IJM &= x^{1+\frac{2e}{b}}z^{\frac{1}{2}-\frac{3e}{b}}\exp\left(\frac{2ce-bf}{2b}z\right),\\
\psi &= y^2z^{-\frac{(f-c)x}{2}},\\
\phi_1 &= \phi^2\psi^{-3}=xy^3(1+\ldots),\\
\phi_2 &= \phi\psi^{-2}=y^2z(1+\ldots).
\end{aligned}$$

Since $\xi=z\ell^{-1}$ satisfies $\dot{\xi}=2\xi$, the system is linearizable by Theorem 30.

Case 5 The system is

$$\dot{x}=x(3+ax-2ey+cz),\qquad \dot{y}=y(-1+ey+fz),\qquad \dot{z}=z(2+gx+kz).$$

After the change of coordinates $(X,Y,Z) = (x^{\frac{1}{2}},\ x^{\frac{1}{2}}y,\ z)$ the system becomes

$$\begin{aligned} \dot{X} &= X\left(\frac{3}{2}+\frac{a}{2}X^2+\frac{c}{2}z\right)-eY, \\ \dot{Y} &= Y\left(\frac{1}{2}+\frac{a}{2}X^2+\left(f+\frac{c}{2}\right)z\right), \\ \dot{z} &= z(2+gX^2)+kz. \end{aligned} \tag{A.7}$$

The singular point at the origin of (A.7) is in the Poincaré domain and hence is linearizable via an analytic change of coordinates which can be chosen as

$$(\tilde{X},\ \tilde{Y},\ \tilde{Z}) = (X-eY+O(2),\ Y(1+O(1)),\ z(1+O(1)))\,.$$

We can pull back the two first integrals $\psi_1 = \tilde{X}^{-1}\tilde{Y}^3$ and $\psi_2 = \tilde{X}^{-2}\tilde{Y}^2\tilde{Z}$ of the linear system to obtain the two first integrals of the initial system

$$\phi_1 = xy^3(1+O(1)) \quad \text{and} \quad \phi_2 = y^2z(1+O(1)).$$

Case 6 The system is

$$\dot{x} = x(3+ax+by+cz), \qquad \dot{y} = y(-1+ax+by+fz), \qquad \dot{z} = z(2+kz).$$

For $k \neq 0$ the system admits the algebraic surface $\ell = 1+\frac{k}{2}z = 0$ with cofactor $L_\ell = kz$. In this case we obtain the fist integral

$$\phi = x^{-1}yz^2\,\ell^{\frac{c-f-2k}{k}},$$

and the inverse Jacobi multiplier

$$IJM = x^3\,z^{\frac{-5}{2}}\,\ell^{\frac{9k+2f+4c}{2k}}.$$

For $k = 0$ the system admits the exponential factor $E = \exp(z/2)$ with cofactor $L_E = z$. In this case we obtain the first integral $\phi = x^{-1}yz^2E^{c-f}$ and the inverse Jacobi multiplier

$IJM = x^3 z^{-\frac{5}{2}} E^{f-2c}$. Then Theorem 29 guarantees the existence of the second first integral $\psi = x^{-2} y z^{\frac{7}{2}} (1+O(1))$. Thus we can construct two independent first integrals of the desired form

$$\phi_1 = \phi^7 \psi^{-4} = xy^3(1+O(1)) \quad \text{and} \quad \phi_2 = \phi^4 \psi^{-2} = y^2 z(1+O(1)).$$

Since $\xi = z\ell^{-1}$ satisfies $\dot{\xi} = 2\xi$ then by Theorem 30 the system is linearizable.

Case 7 The system is

$$\dot{x} = x(3-3dx-3ey+cz), \quad \dot{y} = y(-1+dx+ey+fz), \quad \dot{z} = z(2-8dx+kz). \quad \text{(A.8)}$$

The change of variable $(X,Y) = (x^{\frac{1}{3}}, x^{\frac{1}{3}} y)$ transform system (A.8) into

$$\dot{X} = X\left(1-dX^3+\frac{c}{3}z\right) - eY, \quad \dot{Y} = Y\left(\frac{c}{3}+f\right)z, \quad \dot{z} = z(2-8dX^3+kz). \quad \text{(A.9)}$$

We seek an expression $\xi(X,Y,z) = \sum_{i\geq 0} \xi_i(X,z) Y^i$ with $\dot{\xi} = z$. If we found such a ξ, then $\phi = Ye^{-\alpha\xi}$ is a first integral of (A.9) where $\alpha = f + c/3$. Note that for $\alpha = 0$ we have that $f = -c/3$ and then clear $\phi = Y$ is a first integral of system (A.9). Now, we write the vector field as $\mathcal{X} = \mathcal{X}_0 + \mathcal{X}_1 + \mathcal{X}_\alpha$, where

$$\begin{aligned}
\mathcal{X}_0 &= X\left(1-dX^3+\frac{c}{3}z\right)\frac{\partial}{\partial X} + z(2-8dX^3+kz)\frac{\partial}{\partial z},\\
\mathcal{X}_1 &= -eY\frac{\partial}{\partial X},\\
\mathcal{X}_\alpha &= \alpha z Y\frac{\partial}{\partial Y}.
\end{aligned}$$

Since $\mathcal{X}\xi = z$, then

$$\mathcal{X}_0\xi_0 = z, \quad \mathcal{X}_0\xi_\ell + \ell\alpha z\xi_\ell = -\mathcal{X}_1\xi_{\ell-1} \quad (\ell > 0). \quad \text{(A.10)}$$

To solve equation (A.10), we have to show that for each $B = \sum_{i+j>0} b_{ij} X^i z^j$, there exists an $A = \sum_{i+j>0} a_{ij} X^i z^j$, such that $(\mathcal{X}_0 + \ell\alpha z)A = B$. From

$$(\mathcal{X}_0 + \ell\alpha z)A = \sum_{i+j>0} (i+2j) a_{ij} X^i z^j \; - \; d \sum_{i+j>0} (i+8j)\, a_{ij} X^{i+3} z^j + \sum_{i+j>0} \left(i\frac{c}{3} + jk + \ell\,\alpha \right) a_{ij} X^i z^{j+1},$$

the coefficints a_{ij} must satisfy the relation

$$(i+2j)a_{i,j} - ((i-3)+8j)d\, a_{i-3,j} + \left(i\frac{c}{3} + (j-1)k + \ell\,\alpha \right) a_{i,j-1} = b_{i,j}, \quad i > 3,\; j > 1.$$

Thus equation (A.10) has a solution, we can solve term by term, provided that the right hand side of (A.10) has no constant term and this can be done by choosing ξ_ℓ in (A.10) to be divisible by z. To do this, we shall prove by induction that the right hand side of (A.10) is divisible by z. Note that for $\ell = 0$ it is clear. So the part $X\left(1 - dX^3 + (c/3)z\right) \partial\xi_\ell/\partial X$ is divisible by z, and then $\partial\xi_\ell/\partial X$ is also divisible by z. Hence we can write $\partial\xi_\ell/\partial X = z f(X,z)$. From this we get $\xi_\ell = g(X) + z h(X,z)$. Clearly $g'(X) = 0$, so that g is a constant. Still $\xi_\ell - g$ satisfies (A.10) and we are done by induction.

Thus we can solve (A.10) term by term and standard majorization techniques imply that the resulting series are convergent.

Now the system has a first integral $\phi = x^{\frac{1}{3}} y e^{-\alpha\xi}$ and an inverse Jaccobi multiplier

$$IJM = x^{\frac{1}{2}\frac{k+4f+2c}{c+3f}} y^{\frac{1}{2}\frac{4c+6f+3k}{c+3f}} z^{\frac{3}{2}},$$

if $c + 3f \neq 0$. Theorem 29 guarantees a second first integral of the form

$$\psi = x^{1-\frac{1}{2}\frac{k+4f+2c}{c+3f}} y^{1-\frac{1}{2}\frac{4c+6f+3k}{c+3f}} z^{-\frac{1}{2}} \left(1 + O(1)\right),$$

from which we obtain the following two first integrals of system (A.8) and are of the desired form

$$\phi_1 = \phi^3 = xy^3(1+O(1)) \quad \text{and} \quad \phi_2 = \phi^{\frac{3(2f-k)}{c+3f}} \psi^{-2} = y^2 z(1+O(1)).$$

When $c+3f=0$, we set $E=exp(\xi)$ with $X(\xi)=z$ as above. Then the system has a first integral $\phi_1=xy^3$ and and an inverse Jacobi multiplier

$$IJM = xy^2 z^{\frac{3}{2}} E^{3(\frac{1}{2}k-f)\xi},$$

then we proceed as before.

Case 8 The system becomes

$$\dot{x}=x(3+ax), \qquad \dot{y}=y(-1+dx+ey), \qquad \dot{z}=z(2+gx+hy).$$

We will provide two different proofs as we did in Case 4. In the first one the system has an invariant algebraic surface $\ell=1+(a/3)x=0$ with cofactor $L_\ell=ax$. The change of coordinates $(X,Y)=\left(x\ell^{-1},\ y/(A(X)+B(X)y)\right)$ will linearize the first and second equation. We ask that $\tilde{\ell}=A+By=0$ is an invariant algebraic surface with cofactor $L_{\tilde{\ell}}=dx+ey$. Therefore it suffices to find $A(X)$ and $B(X)$ satisfying the equations

$$\dot{A}=dAx, \quad \dot{B}-B=eA, \quad A(0)=1. \tag{A.11}$$

From the first equation of (A.11) we can find $A=(1+ax/3)^{\frac{d}{a}}$ (when $a=0$ we take $A=\exp(dx/3)$). Write $A=\sum_{i>0}a_iX^i$ and $B=\sum_{i>0}b_iX^i$. Clearly B satisfies the second equation if we set $b_i=ea_i/(3i-1)$, and B is convergent. The third equation would be linearizable by the transformation $Z=z\exp(-\gamma(X,Y))$ where $\dot{\gamma}=gx(X)+hy(X,Y)$. To find such γ we write

$$\gamma(X,Y)=\sum_{i+j>0}\gamma_{ij}X^iY^j, \qquad x(X)=\sum_{i>0}\xi_iX^i, \qquad y(X,Y)=\sum_{i+j>0}\eta_{ij}X^iY^j.$$

It is easy to see that from relation $\dot{\gamma}=gx(X)+hy(X,Y)$ we obtain

$$\sum(3i-j)\gamma_{ij}X^iY^j=g\sum\xi_iX^i+h\sum\eta_{ij}X^iY^j.$$

If $y(X,Y)$ contains no terms of the form $(XY^3)^k$, we can set

$$\gamma_{ij} = \frac{g\,\xi_i + h\,\eta_{ij}}{3i-j},$$

to give a convergent expansion of γ. Now it remains to show that the inverse transformation $y = AY/(1-BY)$ contains no term like $(XY^3)^k$. Note that

$$y = \frac{AY}{1-BY} = \sum AB^kY^{k+1}.$$

Assume that $k+1 = 3n$ for some n. We need to show that $AB^k = AB^{3n-1}$ contains no term like X^n. Note that

$$\dot{B} = \frac{dB}{dt} = 3X\frac{dB}{dX}.$$

From Equation (A.11) we obtain that

$$eAB^{3n-1} = \Big(\frac{3X}{3n}\frac{dB^{3n}}{dX} - B^{3n}\Big). \tag{A.12}$$

Clearly, the coefficient in X^n in B^{3n} vanishes on the right hand side so that either $e = 0$, that means $B \equiv 0$, or the coefficients of X^n in B^{3n-1} vanishes. Therefore y in $Y = y/(A+By)$ contains no term like $(XY^3)^k$. Thus we have set up the existence of a linearizing transformation. This completes the first proof.

Now in the second proof we will just give the form of the first integrals and the inverse Jacobi multiplier since this procedure is exactly the same as Case 4.

i) When $ae \neq 0$ we have

$$
\begin{aligned}
\phi &= x^{-a(2e+h)}y^{-3ah}z^{3ae}\left(1+\frac{1}{3}ax\right)^{2ae-3eg+ah+3dh}_{,} \\
IJM &= x^{2+\frac{h}{3e}}y^{2+h/e}\left(1+\frac{1}{3}ax\right)^{\frac{3eg-3ed-ah-3dh}{3ae})}_{,} \\
\psi &= x^{-1-\frac{h}{3e}}y^{-1-\frac{h}{e}}z(1+\ldots), \\
\phi_1 &= \phi^{\frac{1}{ae}}\psi^{-3} = xy^3(1+\ldots), \\
\phi_2 &= \phi^{\frac{3e+h}{3ae^2}}\psi^{-2-\frac{h}{e}} = y^2z(1+\ldots).
\end{aligned}
$$

ii) When $a=0$ and $e \neq 0$ we obtain

$$
\begin{aligned}
\phi &= x^{-2e-h}y^{-3h}z^{3e}\exp\left((dh-eg)x\right), \\
IJM &= x^{2+\frac{h}{3e}}y^{2+h/e}\exp\left(\frac{ge-de-dh}{3e}x\right), \\
\psi &= x^{-1-\frac{h}{3e}}y^{-1-\frac{h}{e}}z(1+\ldots), \\
\phi_1 &= \phi^{\frac{1}{e}}\psi^{-3} = xy^3(1+\ldots), \\
\phi_2 &= \phi^{\frac{3e+h}{3e^2}}\psi^{-2-\frac{h}{e}} = y^2z(1+\ldots).
\end{aligned}
$$

iii) When $e=0$ and $a \neq 0$ we get

$$
\phi = xy^3(1+\frac{a}{3}x)_{,}^{-1-\frac{3d}{a}}, \qquad IJM = x^{\frac{2}{3}}z(1+\frac{a}{3}x)^{\frac{4}{3}+\frac{d}{a}}.
$$

iv) When $a=e=0$ we have

$$
\phi = xy^3e^{-dx}, \qquad IJM = x^{\frac{2}{3}}ze^{\frac{d}{3}x}.
$$

Note that for the subcases (iii) and (iv), Theorem 29 gives the same first integral. However this case is a subcase of Case 9. Finally since $\xi = x\ell^{-1}$ satisfies $\dot{\xi} = 3\xi$, the system is linearizable according to Theorem 30.

Case 9 The system is

$$\dot{x} = x(3+ax+by), \qquad \dot{y} = y(-1+dx-by), \qquad \dot{z} = z(2+gx+hy).$$

The change of variables $(X,Y) = (x,xy)$ transforms the first two equations into

$$\dot{X} = X(3+aX)+bY, \qquad \dot{Y} = Y(2+(a+d)X),$$

which is a linearizable node and hence the linearizable change is given by $(\tilde{X},\tilde{Y}) = (X+bY+O(2), \quad Y(1+O(1)))$ such that $\dot{\tilde{X}} = 3\tilde{X}$ and $\dot{\tilde{Y}} = 2\tilde{Y}$. To linearize the third equation, assume that we have a function $\alpha = \alpha(\tilde{X},\tilde{Y})$ such that $\dot{\alpha} = gx(\tilde{X},\tilde{Y})+hy(\tilde{X},\tilde{Y})$. Then the change of variable $\tilde{Z} = z\exp(-\alpha)$ will give $\dot{\tilde{Z}} = 2\tilde{Z}$. Writing

$$gx(\tilde{X},\tilde{Y})+hy(\tilde{X},\tilde{Y}) = \sum a_{ij}\tilde{X}^i\tilde{Y}^j,$$

it is easy to see that $\alpha(\tilde{X},\tilde{Y}) = \sum \frac{a_{ij}}{3i+2j}\tilde{X}^i\tilde{Y}^j$. The convergence of α is then clear. Then, the first integrals of the linear system are $\tilde{X}^{-2}\tilde{Y}^3$ and $\tilde{Y}^{-1}\tilde{Z}$ and we can pull back them to get $\phi = xy^3(1+\ldots)$ and $\psi = x^{-1}y^{-1}z(1+\ldots)$. The desired first integrals are then

$$\phi_1 = \phi = xy^3(1+\ldots) \quad \text{and} \quad \phi_2 = \phi\,\psi = y^2z(1+\ldots).$$

Case 10 The system becomes

$$\dot{x} = x(3+ax-2ey), \qquad \dot{y} = y(-1+ey), \qquad \dot{z} = z(2+gx+hy).$$

For $ae \neq 0$ this system has the invariant algebraic surfaces $\ell_1 = 1-ey = 0$ and $\ell_2 = 1+(a/3)x+(ae/3)xy+(ae^2/3)xy^2 = 0$ with cofactors $L_{\ell_1} = ey$ and $L_{\ell_2} = a$. Here we obtain the two independent first integrals

$$\phi_1 = xy^3\,\ell_1^2\,\ell_2^{-\frac{1}{3}} \quad \text{and} \quad \phi_2 = y^2z\,\ell_1^{-(2+\frac{h}{e})}\,\ell_2^{-\frac{g}{a}}.$$

When $e=0$ and $a\neq 0$ the system has an invariant algebraic surface $\ell = 1+ax/3$ and an exponential factor $E=\exp(y)$ with cofactors $L_\ell = ax$ and $L_E=-y$ which yield to the two independent first integrals

$$\phi_1 = xy^3\,\ell^{-1} \quad \text{and} \quad \phi_2 = y^2 z\,\ell^{-\frac{g}{a}}\,E^h.$$

While $a=0$ and $e\neq 0$ we have an invariant algebraic surface $\ell = 1-ey$ and an exponential factor $E=\exp(x/(1-ey))$ with cofactors $L_\ell = ey$ and $L_E = 3x$ and we obtain the two independent first integrals

$$\phi_1 = xy^3\,\ell^{-1} \quad \text{and} \quad \phi_2 = y^2 z\,\ell^{-2-\frac{h}{e}}\,E^{-\frac{g}{3}}.$$

Finally when $a=e=0$ there exist exponential factors $E_1=\exp(x)$ and $E_2=\exp(y)$ with cofactors $L_{E_1}=3x$ and $L_{E_2}=-y$ and we have the two independent first integrals

$$\phi_1 = xy^3 \quad \text{and} \quad \phi_2 = y^2 z\,E_1^{-\frac{g}{3}}\,E_2^h.$$

Theorem 30 guarantees the linearizability of the system in this case as $\xi = y\ell^{-1}$ satisfies $\dot{\xi} = -\xi$.

Case 11 The system is

$$\dot{x} = x(3+ax+cz), \qquad \dot{y} = y(-1+dx+ey), \qquad \dot{z} = z(2+gx-ey+kz).$$

The change of coordinates $Y=yz$ transform the system above into

$$\dot{x} = x(3+ax+cz), \quad \dot{Y} = Y\big(1+(d+g)x+kz\big), \quad \dot{z} = z(2+gx+kz)-eY. \qquad \text{(A.13)}$$

Note that the origin of system (A.13) is in the Poincaré domain and so is linearizable by an analytic change of coordinates $(X,\tilde{Y},Z) = (x(1+O(1)), \quad Y(1+O(1)), \quad z(1+O(1)))$. We can now construct two first integrals for the linear system $\phi = x^{-1}Y^3$ and $\psi = x^{-1}Yz^2$.

Pulling back these two first integrals, we get two independent first integrals of the initial system and are of the desired form

$$\phi_1 = \phi^{-2}\psi^3 = xy^3(1+O(1)) \quad \text{and} \quad \phi_2 = \phi\psi^{-1} = y^2z(1+O(1)).$$

Case 12 The system is

$$\dot{x} = x(3+ax+by+cz), \qquad \dot{y} = y(-1+dx-by), \qquad \dot{z} = z(2+gx+by+kz).$$

The change of coordinates $(X,Z) = (x/(1+by)^2, \quad z/(1+by))$ brings the system into the form

$$\begin{aligned}
\dot{X} &= X(3+aX(1+by)+cZ-2bdXy)(1+by),\\
\dot{y} &= y(-1+dX(1+by))(1+by),\\
\dot{Z} &= Z(2+gX(1+by)+kZ-bdXy)(1+by).
\end{aligned} \tag{A.14}$$

After rescaling the system by $(1+by)$ and applying the change of variable $Y = Xy$, system (A.14) becomes

$$\begin{aligned}
\dot{X} &= X(3+aX+(ab-2bd)Y+cZ),\\
\dot{Y} &= Y(2+(a+d)X+(ab+bd)Y+cZ),\\
\dot{Z} &= Z(2+gX+(gb-bd)Y+kZ).
\end{aligned} \tag{A.15}$$

Now the origin of system(A.15) is in the Poincaré domain and hence is linearizable via an analytic change of coordinates which can be chosen as $(\tilde{X},\tilde{Y},\tilde{Z}) = (X(1+O(1)), \quad Y(1+O(1)), \quad Z(1+O(1)))$. The desired first integrals are

$$\phi_1 = \tilde{X}^{-2}\tilde{Y}^3 = xy^3(1+\dots) \quad \text{and} \quad \phi_2 = \tilde{X}^{-2}\tilde{Y}^2\tilde{Z} = y^2z(1+\dots).$$

Since the cofactors of $x=0$, $y=0$, $z=0$ and the divergence $\text{div}(\mathcal{X})$ are linearly independent, then by Theorem 7 the original system is linearizable.

Case 13 The system is

$$\dot{x} = x(3+ax+2hy+cz), \qquad \dot{y} = y(-1-hy), \qquad \dot{z} = z(2+gx+hy+kz).$$

We use the change of variable $(X,Z) = (x/(3(1+hy)), \quad z/(2(1+hy)))$ and rescale the resulting system by $1+hy$ we get

$$\dot{X} = X\left(3+\frac{a}{3}X+2cZ\right), \quad \dot{y} = -y, \quad \dot{Z} = Z\left(2+3gX+\frac{k}{2}Z\right).$$

Clearly the first and third equation give a linearizable node and therefore the system is linearizable. Furthermore, since $\xi = y(1+hy)^{-1}$ satisfies $\dot{\xi} = -\xi$, then the original system must also be linearizable by Theorem 6.

Case 14 The system takes the form

$$\dot{x} = x(3+by+cz), \qquad \dot{y} = y(-1-hy), \qquad \dot{z} = z(2+hy+kz).$$

The system has the invariant algebraic surfaces $\ell_1 = 1+hy = 0$ and $\ell_2 = 1+kz/2-(kh/2)yz = 0$ with cofactors $L_{\ell_1} = -hy$ and $L_{\ell_2} = kz$. In this case we obtain the two independent first integrals

$$\phi_1 = xy^3\ell_1^{\frac{b}{h}-3}\ell_2^{-\frac{c}{k}} \quad \text{and} \quad \phi_2 = y^2z\ell_1^{-1}\ell_2^{-1}.$$

When $h=0$ and $k \neq 0$ then the linearizing change of coordinates is given by

$$(X,Y,Z) = \left(x\exp(by)\left(1+\frac{k}{2}z\right)^{-\frac{c}{k}}, \, y, \, z\left(1+\frac{k}{2}z\right)^{-1}\right).$$

When $k=0$ and $h \neq 0$ the linearizing change can be chosen as

$$(X,Y,Z) = \left(x\exp\left(-\frac{cz}{2(1+hy)}\right)(1+hy)^{\frac{b}{h}}, \, y(1+hy)^{-1}, \, z(1+hy)\right).$$

Finally, when $k = h = 0$ the linearizing change can be taken as

$$(X,Y,Z) = \left(x\exp\left(by - \frac{c}{2}z\right),\ y,\ z\right).$$

Since $\xi = y\ell$ satisfies $\dot{\xi} = -\xi$ we have that the system is linearizable according to Theorem 30.

Case 15 The system is

$$\dot{x} = x(3+ax+by+cz), \qquad \dot{y} = y(-1+ax+by), \qquad \dot{z} = z(2-ax-by+kz).$$

Performing the change of coordinates $(X,Y,Z) = (xz, \quad yz, \quad z)$ the system becomes

$$\dot{X} = X(5+(c+k)Z), \quad \dot{Y} = Y(1+kZ), \quad \dot{Z} = Z(2+kZ) - aX - bY. \tag{A.16}$$

The origin of system (A.16) is in the Poincaré domain and hence is linearizable via a transformation $(\tilde{X},\tilde{Y},\tilde{Z}) = (X(1+O(1)), Y(1+O(1)), Z + \frac{a}{3}X - bY + O(2))$. The first integrals of the linear system are then $\phi = \tilde{X}\tilde{Y}^{-5}$ and $\psi_2 = \tilde{Y}^{-2}\tilde{Z}$. Pulling back these integrals we see that the initial system admits two independent analytic first integrals of the desired form $\phi_1 = \phi\,\psi^4 = xy^3(1+\ldots)$ and $\phi_2 = \psi = y^2z(1+\ldots)$.

Case 16 System (A.1) becomes

$$\dot{x} = x(3+ax-9fz), \quad \dot{y} = y(-1+dx+ey+fz), \quad \dot{z} = z(2+gx-2ey-2fz).$$

The transformation $(Y,Z) = (yz^{\frac{1}{2}}, \quad z^{\frac{1}{2}})$ brings the system into

$$\dot{x} = x(3+ax-9fZ^2), \quad \dot{Y} = Y\left(\frac{9}{2}+d\right)x, \quad \dot{Z} = Z\left(1+\frac{g}{2}x - fZ^2\right) - eY.$$

and we can produce the first integrals exactly as in Case 7.

Case 17 We have the system

$$\dot{x} = x(3+ax), \qquad \dot{y} = y(-1+dx+ey+fz), \qquad \dot{z} = z(2+gx+ey+fz).$$

For $a \neq 0$ the system admits the invariant algebraic surface $\ell = 1+ax/3 = 0$ with cofactor $L_\ell = ax$. Additionally, the system has the first integral

$$\phi = xyz^{-1}\ell^{\frac{g-d-a}{a}},$$

and the inverse Jacobi multiplier

$$IJM = x^{\frac{7}{3}}y^3\ell^{\frac{3g-a-6d}{a}}.$$

For $a = 0$ we obtain the first integral

$$\phi = xyz^{-1}E^{\frac{g-d}{3}},$$

and the inverse Jacobi multiplier

$$IJM = x^{\frac{7}{3}}y^3\ell^{\frac{g-2d}{3}},$$

where $E = \exp(x)$. Then by Theorem 29 the second first integral is $\psi = x^{-\frac{4}{3}}y^{-2}z(1+\cdots)$ and finally we obtain the two independent analytic first integrals to be of the desired form

$$\phi_1 = (\phi\psi)^{-3} = xy^3(1+\cdots), \qquad \phi_2 = \phi^{-4}\psi^{-3} = yz^2(1+\cdots).$$

From Theorem 30 since $\xi = x\ell^{-1}$ satisfies $\dot{\xi} = 3\xi$ we have that the system is linearizable.

Case 18 We obtain the system

$$\dot{x} = x(3+ax-ey-7fz), \quad \dot{y} = y(-1+dx+ey+fz), \quad \dot{z} = z(2+gx-2ey-2fz).$$

A change of coordinate $X = x\ell^{-2}$ where $\ell = 1 - ey - fz$ transforms the above system into

$$\dot{X} = X(3 + aX\ell + 2X(edy + fgz))\,\ell, \quad \dot{y} = y(-1 + dX\ell)\,\ell, \quad \dot{z} = Z(2 + gX\ell)\ell. \quad (A.17)$$

We rescale the system by ℓ and doing the change of variables $X = x$ and $Y = Xy$ we get

$$\begin{aligned}
\dot{X} &= X(3 + aX - (ae - 2ed)Y - (af - 2fg)Xz),\\
\dot{Y} &= Y(-1 + (a+d)X - (de + ae - 2ed)Y - (fd + af - 2fg)Xz),\\
\dot{z} &= z(2 + gX - geY - gfXz).
\end{aligned} \quad (A.18)$$

The origin of (A.18) is in the Poincaré domain and hence is linearizable by a transformation $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X(1 + O(1)), \quad Y(1 + O(1)), \quad z(1 + O(1)))$. The first integrals in the desired forms are

$$\phi_1 = \tilde{X}^{-2}\tilde{Y}^3 = xy^3(1 + \ldots) \quad \text{and} \quad \phi_2 = \tilde{X}^{-2}\tilde{Y}^2\tilde{Z} = y^2z(1 + \ldots).$$

Case 19 In this case system (A.1) can be written as

$$\dot{x} = x(3 + ax + by + cz), \quad \dot{y} = y(-1 + dx - by - cz), \quad \dot{z} = z(2 + gx - by - cz).$$

Under the change of coordinates $(X, Y, Z) = (x, xy, xz)$ we obtain the system

$$\dot{X} = X(3 + aX) + bY + cZ, \qquad \dot{Y} = Y(2 + (a+d)X), \qquad \dot{Z} = Z(5 + (a+g)X),$$

and the singular point is in the Poincaré domain. So a change of coordinates $(\tilde{X}, \tilde{Y}, \tilde{Z}) = (X + bY - \frac{c}{2}Z + O(2), Y(1 + O(1)), Z(1 + O(1)))$ linearizes the system. The first integrals of the linear system are then $\phi = \tilde{X}^{-2}\tilde{Y}^3$ and $\psi_2 = \tilde{Y}^{-\frac{5}{2}}\tilde{Z}$. Pulling back this two first integrals we see that the initial system admits two independent analytic fist integrals of the desired form $\phi_1 = \phi = xy^3(1 + \ldots)$ and $\phi_2 = \phi^{\frac{3}{2}}\,\psi = y^2z(1 + \ldots)$.

Case 20 System becomes

$$\dot{x} = x(3+ax-2hy-2kz), \quad \dot{y} = y(-1+hy+kz), \quad \dot{z} = z(2+gx+hy+kz).$$

After the change of coordinates $(X,Y,Z) = (x^{\frac{1}{2}},\ x^{\frac{1}{2}}y,\ x^{\frac{1}{2}}z)$ we get the system

$$\dot{X} = X\left(\frac{3}{2}+\frac{a}{2}X^2\right)-hY-kZ, \quad \dot{Y} = Y\left(\frac{1}{2}+\frac{a}{2}X^2\right), \quad \dot{Z} = Z\left(\frac{7}{2}+\left(\frac{a}{2}+g\right)X^2\right).$$

Note that the transformation $(\tilde{X},\tilde{Y},\tilde{Z}) = (X-hY+\frac{k}{2}Z(1+O)2)), \quad Y(1+O(1)), \quad Z(1+O(1)))$ linearizes the system. The first integrals of the linear system are then $\phi = \tilde{X}^{-1}\tilde{Y}^3$ and $\psi_2 = \tilde{Y}^7\tilde{Z}^{-1}$. Pulling back these two first integrals we see that the initial system admits two independent analytic fist integrals of the desired form $\phi_1 = \phi = xy^3(1+\ldots)$ and $\phi_2 = \phi^3\,\psi^{-1} = y^2z(1+\ldots)$.

$\square$

Appendix B

APPENDICES

Integrability of Three-Dimensional Lotka-Volterra Equations with Rank-1 Resonances

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*A*bstarct

We investigate the local integrability of three-dimensional Lotka-Volterra equations at the origin with $(p:q:r)$-resonance.

$$\dot{x} = P = x(p + ax + by + cz),$$

$$\dot{y} = Q = y(q + dx + ey + fz),$$

$$\dot{z} = R = z(r + gx + hy + kz),$$

Recent work on this problem has centered on the case where the resonance is of “rank-2". That is, there are two independent linear dependencies of p, q and r over $\mathbb{Q}$. Here, we consider the case where there is only one such dependency. In particular, we give necessary and sufficient conditions for integrability in the case of $(i:-i:\lambda)$-resonance for $\lambda/i \notin \mathbb{Q}$, and also for the case of $(i-1,-i-1,2)$-resonance under the additional assumption that $a = k = 0$.

Our necessary and sufficient conditions for integrability are given via the search for two independent first integrals of the form $x^{\alpha}y^{\beta}z^{\gamma}(1+O(x,y,z))$. However, a new feature in the case of rank-1 resonance is that there is a distinguished choice of analytic first integral, and hence it makes sense to seek conditions for just one (analytic) first integral

to exist. We give necessary and sufficient conditions for just one first integral for the two family of systems mentioned above, except that for the second family some of the cases of sufficiency have been left as conjectural.

B.1 Introduction

The purpose of this paper is to extend some recent work (Aziz and Christopher, 2012; Aziz, Christopher, Llibre, and Pantazi, Aziz et al.) on the local integrability at the origin of the three-dimensional Lotka-Volterra equations

$$\begin{aligned} \dot{x} &= P = x(p+ax+by+cz), \\ \dot{y} &= Q = y(q+dx+ey+fz), \\ \dot{z} &= R = z(r+gx+hy+kz), \end{aligned} \tag{B.1}$$

to more general resonances. Related work on the integrability of Lotka-Volterra and other three dimensional systems can be found in Aziz (Aziz); Basov and Romanovski (2010); Bobienski and Żołądek (2005); Cairó and Llibre (2000a); Cairó (2000); Christodoulides and Damianou (2009); Christopher and Rousseau (2004); Gonzalez-Gascon and Peralta Salas (2000); Gravel and Thibault (2002); Liu et al. (2004); Moulin-Ollagnier (2001)

Recall that the system (B.1) is *integrable* at the origin if there exists a change of coordinates

$$X = x(1+O(x,y,z)), \quad Y = y(1+O(x,y,z)), \quad Z = z(1+O(x,y,z)),$$

bringing (B.1) to a system orbitally equivalent to the linear system:

$$\dot{X} = pXm, \quad \dot{Y} = qYm, \quad \dot{Z} = rZm, \tag{B.2}$$

where $m = m(X,Y,Z) = 1 + O(X,Y,Z)$. This is equivalent to the existence of two first integrals of the form

$$\phi_1 = x^{r_1} y^{s_1} z^{t_1} (1+O(x,y,z)), \qquad \phi_2 = x^{r_2} y^{s_2} z^{t_2} (1+O(x,y,z)),$$

where $(r_1,s_1,t_1) \times (r_2,s_2,t_2) \neq 0$. If the change of coordinates can be chosen so that $m \equiv 1$ then we say the system is *linearizable*.

In the works cited above, the resonances at the origin have all been of “rank-2". That is, there are two independent linear dependencies of p, q and r over $\mathbb{Q}$. This condition is satisfied if and only if we can rescale the system so that p, q and r are in $\mathbb{Z}$. Necessary and sufficient conditions for integrability can therefore be obtained via the search for two independent first integrals of the form $x^{\alpha}y^{\beta}z^{\gamma}(1+O(x,y,z))$, with α, β and γ in $\mathbb{Z}$. In the case where p, q and r do not all share the same sign we can reduce our considerations to the search for two analytic first integrals.

Our aim here is to consider the integrability of the origin of (B.1) in the case where there resonance is of rank-1. That is, there is only one linear dependency of p, q and r over $\mathbb{Q}$. As above, to prove integrability, we still seek two first integrals of the form $x^{\alpha}y^{\beta}z^{\gamma}(1+O(x,y,z))$, but at most one of these can be analytic in this case.

In particular, we give necessary and sufficient conditions for the origin of system (B.1) to be integrable in the case of $(i:-i:\lambda)$-resonance for $\lambda/i \notin \mathbb{Q}$, and also for the case of $(i-1:-i-1:2)$-resonance under the additional assumption that $a=k=0$. In the latter case, the additional assumption is arbitrary, and was imposed purely to bring the computations to manageable size.

A new feature in the case of rank-1 resonances in the Siegel domain is that there is a distinguished choice of analytic first integral of the form $x^{\alpha}y^{\beta}z^{\gamma}(1+O(x,y,z))$, and hence it makes sense to seek conditions for just one (analytic) first integral to exist. We give necessary and sufficient conditions for just one first integral in the cases mentioned above, except that for $(i-1,-i-1,2)$-resonance, a few cases of sufficiency have been left as conjectural.

B.2 Definitions

Let

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z},$$

be the associated vector field to the system (B.1). Given a polynomial $F \in \mathbb{C}[x,y,z]$, a surface $F=0$ is called an *invariant algebraic surface* of the system (B.1), if the polynomial

F satisfies the equation

$$\dot{F} = XF = P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} + R\frac{\partial F}{\partial z} = C_F F \tag{B.3}$$

for some polynomial $C_F \in \mathbb{C}$. Such a polynomial is called the *cofactor* of the invariant algebraic curve $F = 0$. One can note that from equation (B.3) that any cofactor has at most degree one since the polynomial vector field has degree two.

To complete the study of integrals of parametric families, we will also need the notion of *exponential factor* which plays the same role of as an invariant algebraic surface in the case when two such surfaces coalesce. Let $E(x,y,z) = \exp(f(x,y,z)/g(x,y,z))$ where $f,g \in \mathbb{C}[x,y,z]$, then E in an *exponential factor* if

$$XE = C_E E, \tag{B.4}$$

for some polynomial C_E of degree at most one. The polynomial C_E is called the *cofactor* of E.

A *Darboux* function is a function of the form,

$$D = \prod F_i^{\lambda_i} E^{\lambda_0 f/g},$$

where the F_i are invariant algebraic surfaces of the system, and $E = \exp(f/g)$ is an exponential factor. Given a Darboux function, D, we can compute

$$X(D) = D\Big(\sum \lambda_i C_{F_i} + \lambda_0 C_E\Big).$$

Clearly, the function D is a non-trivial first integral of the system if and only if the cofactors C_{F_i} and C_E are linearly dependent.

For Darboux integrability in two dimensions, we seek a Darboux function which is either a first integral or integrating factor for the system. From the latter, it is possible to find a first integral by quadratures.

In higher dimensions, the role of the integrating factor is taken by the Jacobi Multiplier. In the context of Darboux integrability, we usually consider the corresponding reciprocals: inverse integrating factors, and inverse Jacobi multipliers Berrone and Giacomini (2003). A function M is an inverse Jacobi multiplier for the vector field X if it satisfies the equation

$$X(M) = M\mathrm{div}(X) \qquad \Longleftrightarrow \qquad \mathrm{div}(X/M) = 0.$$

A Darboux inverse Jacobi multiplier, D, must satisfy $\lambda_i C_{F_i} + \lambda_0 C_E = \mathrm{div}(X)$.

In three dimensions, the existence of two independent first integrals implies the existence of an inverse Jacobi multiplier. Conversely, given just one first integral, ϕ, and an inverse Jacobi multiplier, M, one can construct another first integral by integrating along the level surfaces of the first integral, noting that M gives rise to an integrating factor on each level surface.

Unfortunately, this prescription breaks down near a critical point, where the leaves become singular. However, the following theorem allows us to construct a second first integral at a critical point in many cases. We use the usual multi-index notation $X^I = x^i y^j z^k$ to simplify the notation.

Theorem 31. *Suppose the analytic vector field*

$$x(\lambda + \sum_{|I|>0} A_{xI} X^I)\frac{\partial}{\partial x} + y(\mu + \sum_{|I|>0} A_{yI} X^I)\frac{\partial}{\partial y} + z(\nu + \sum_{|I|>0} A_{zI} X^I)\frac{\partial}{\partial z}, \qquad \text{(B.5)}$$

has a first integral $\phi = x^\alpha y^\beta z^\gamma(1+O(x,y,z))$ with at least one of α, β, $\gamma \neq 0$ and a Jacobi multiplier $M = x^r y^s z^t(1+O(x,y,z))$ and suppose that the cross product of $(r-i-1, s-j-1, t-k-1)$ and (α,β,γ) is bounded away from zero for any integers $i, j, k \geq 0$, then the system has a second analytic first integral of the form $\psi = x^{1-r}y^{1-s}z^{1-t}(1+O(x,y,z))$, and hence the system (B.5) *is integrable.*

Proof. see **?**. □

We note finally that it is sometimes possible to reduce the critical point to one in the *Poincaré domain*. That is, the origin does not lie in the convex hull of the eigenvalues of the critical point. In this case, it is only necessary to check a finite number of resonant terms vanish in the normal form to conclude that the system is linearizable.

We make use of this fact in two ways: either by decoupling two of the three equations of (B.1) to get one in the Poincaré domain; or performing a blow-down to a critical point in the Poincaré domain. In the former case, ad hoc arguments are used to show that the third equations can also be linearized. In the latter, we can pull back the two first integrals of the linear system to first integrals of the original system.

B.3 Systems with $(i : -i : \lambda)$-resonance.

In this section we study the local integrability of the origin for the three dimensional Lotka-Volterra equations,

$$\begin{aligned} \dot{x} &= P = x(i + ax + by + cz), \\ \dot{y} &= Q = y(-i + dx + ey + fz), \\ \dot{z} &= R = z(\lambda + gx + hy + kz), \end{aligned} \tag{B.6}$$

with $\lambda/i \notin \mathbb{Q}$.

Necessary conditions for the existence of one first integrals were found by searching for a first integral of the form $xy(1+O(x,y,z))$ and, for integrability, a second first integral of the form $y^{\lambda}z^{i}(1+O(x,y,z))$. The computations were carried as far as the resonant terms in $(xy)^2$ to obtain the conditions below. Sufficiency of these conditions was then proved case by case. The computations were carried out in Maple.

Some care needs to be taken over the integrability conditions in this case since polynomials in λ appear in the coefficients of the first integrals. Any value of λ which appears as a root in these denominators will have to be checked separately. However, in the case above, these roots are purely integer multiples of i and so have already been excluded.

Theorem 32. *Consider three dimensional Lotka-Volterra system (B.6).*

1. *The origin has one analytic first integral of the form $\phi = xy(1+O(x,y,z))$ if and only if $ab = ed$.*

2. *The origin is integrable if and only if*

$$ab - de = 0, \qquad dh - b(i\lambda(a-d) + g) = 0.$$

Proof. Necessary conditions are obtained as explained above. To prove sufficiency we make use of Lemma 1 below.

1. If $ab - ed = 0$ then, restricting to $z = 0$ we obtain a vector field

$$X_0 = x(i + ax + by)\frac{\partial}{\partial x} + y(-i + dx + ey)\frac{\partial}{\partial y}.$$

When either a or b are non-zero then this vector field has an invariant algebraic curve $F = 1 - iax + iey$ with cofactor $ax + ey$, and hence a first integral $\phi_0 = xyF^k$, where $k = -1 - d/a$ ($a \neq 0$) or $k = -1 - b/e$ ($e \neq 0$).

If $a = e = 0$ there is an exponential factor $E = e^{i(dx-by)}$ with cofactor $-dx + by$ and hence a first integral $\phi_0 = xyE$.

From Lemma 1, we can therefore construct a first integral ϕ of the original system (B.6).

2. We seek a first integral of the form $y^\lambda z^i e^{-\psi}$ for some analytic ψ. Such an integral exists if and only if we can solve the equation

$$\dot{\psi} = (\lambda d + ig)x + (\lambda e + ih)y + (\lambda f + ig)z.$$

From Lemma 1, we can solve this problem if and only if there exist a $\psi_0(x,y)$ such that

$$X_0(\psi_0) = (\lambda d + ig)x + (\lambda e + ih).$$

If either b or d are non-zero, $X_0(dx - by) = i(dx + by)$ and so we can choose $\psi_0 = k(dx - by)$, where we take $k = g/d - i\lambda$ if $d \neq 0$ or $k = h/b - i\lambda e/b$ if $d = 0$ and $b \neq 0$ (whence $g = a = 0$).

If $b = d = 0$ then we can choose

$$\phi_0 = (\lambda d + ig)(1/a)\log(1 - iax) + (\lambda e + ih)(1/e)\log(1 + iey),$$

where we can replace $(1/a)\log(1-iax)$ by $-ix$ if $a = 0$ and similarly $(1/e)\log(1+iey)$ by iy when $e = 0$ in the expression above.

□

Lemma 1. *Let X be the vector field associated to* (B.6)*, and X_0 the restriction of that vector field onto the plane $z = 0$. If there exists an analytic function $\phi_0(x,y)$ such that $X_0(\phi_0) = f(x,y)$, then, for any analytic function $g(x,y,z)$, there exist an analytic function $\phi(x,y,z) = \phi_0(x,y) + z\tilde{\phi}(x,y,z)$ such that $X(\phi) = f(x,y) + zg(x,y,z)$,*

Proof. We write

$$X = X_0 + z(\lambda + gx + hy)\frac{\partial}{\partial z} + zX_1 + kz^2\frac{\partial}{\partial z},$$

where $X_1 = cx\frac{\partial}{\partial x} + fy\frac{\partial}{\partial y}$. Writing

$$\phi(x,y,z) = \phi_0(x,y) + \sum_{i>0} \phi_i(x,y)z^i, \qquad g = \sum_{i\geq 0} g_i(x,y)z^i,$$

then it is clear that we need to solve

$$(X_0 + (k+1)(\lambda + gx + hy))\phi_{k+1} = -(X_1 + k)\phi_k + g_k,$$

for each $k \geq 0$. It is easy to see that there are no obstructions to obtain a unique formal series solution for ϕ_k in this way and its convergence follows from standard majorization arguments. □

B.4 Systems with $(i-1:-i-1:2)$-resonance.

In this section we shall study the local integrability of the origin for the three dimensional Lotka-Volterra equation,

$$\begin{aligned}\dot{x} &= P = x(i-1+by+cz),\\ \dot{y} &= Q = y(-i-1+dx+ey+fz),\\ \dot{z} &= R = z(2+gx+hy).\end{aligned} \tag{B.7}$$

The assumption that $a=k=0$ from (B.1) is a somewhat arbitrary choice, but was chosen to bring the computations to a manageable form. It would be interesting to compute the integrability conditions for the general case of arbitrary a and k, but this appears to require much more computational power.

As in the previous case, we will give necessary conditions for the origin of (B.7) to have one analytic first integral of the form $xyz(1+O(x,y,z))$ and also for the existence of a second first integral of the form $x^2z^{1-i}(1+O(x,y,z))$. The computations were carried out in Maple up to terms in $(xyz)^5$.

The proof of sufficiency is again handled case by case. However, in the case of one first integral, we have three cases which are are unable to give a complete explanation of why these first integrals exist. This is surprising since the conditions themselves seem very simple. In particular, all three have explicit expressions for an Inverse Jacobi Multiplier.

Theorem 33. *Consider three dimensional Lotka-Volterra system (B.7). the origin is integrable if an only if one of the following conditions hold.*

1) $b=h=2gf+(1+i)d(f+ic)=0$

2) $f=c=0$

3) $g=d=0$

4) $b+(i+1)h=d=e+hi=f=0$

5) $h = b = e = 0$

6) $e = 2bf - (1+i)ch + (1-i)fh = 2cd + (1+i)cg - (1-i)fg$

$= 2bd + (1+i)bg + (1-i)dh = 0$

Furthermore, a necessary condition that the origin has an analytic first integral of the form $\phi = xyz(1 + O(x,y,z))$ *is that either* (1), (2) *or* (3) *hold above, or one of the following conditions hold.*

$4'$) $f = d = 0$

$5'$) $e = 0$

$6'$) $f + c = g + d = 0.$

The sufficiency of conditions (1)-(3) *follows immediately from their integrability.*

Conjecture 1. We conjecture that the conditions ($4'$), ($5'$) and ($6'$) are also sufficient conditions for one analytic first integral, but are unable to prove this at the moment.

Proof. The necessity of the conditions is proved as indicated above. We shall treat the proofs of sufficiency case by case.

Case 1: The equations for $\dot{x}$ and $\dot{z}$ are decoupled from the $\dot{y}$ equation and have eigenvalues in the Poincaré domain with no resonances possible. Hence we can find linearizing transformations $(X,Y) = (x(1+O(x,z), z(1+O(x,z)))$ which bring the system to the form

$$\dot{X} = (i-1)X, \qquad \dot{y} = y(-1-i+dx(X,Z)+eY+fz(X,Z)), \qquad \dot{Z} = 2Z.$$

We now seek an invariant analytic surface of the form $\alpha(X,Z) + y\beta(X,Z) = 0$ such that

$$\frac{d}{dt}(\alpha(X,Z) + \beta(X,Z)y) = (\alpha(X,Z) + \beta(X,Z)y)(dx(X,Z) + eY + fz(X,Z)).$$

The third equation can then be linearized with the substitution $Y = y/(\alpha(X,Z) + y\beta(X,Z)y)$.

These conditions reduce to

$$\dot{\alpha} = \alpha(dx(X,Z) + eY + fz(X,Z)), \qquad \dot{\beta} - (i+1)\beta = \alpha e.$$

Since

$$\frac{d}{dt}\sum_{r,s\geq 0} a_{r,s}X^r Z^s = \sum_{r,s\geq 0}(r(i-1)+2s)a_{r,s}X^r Z^s,$$

The first equation can be solved uniquely term by term and is clearly convergent. The second equation can be solved likewise if and only if the coefficient of XZ in α is zero. However, a small computation shows that this condition is just $2gf + (1+i)d(f+ic) = 0$.

Case 2: In this case the critical point at the origin for the first and second equation are in the Poincaré domain and hence is linearizable using a change of coordinates $(X,Y) = (x(1+O(x,y),y(1+O(x,y))$ that gives

$$\dot{X} = (i+1)X, \qquad \dot{Y} = -(i+1)Y, \qquad \dot{z} = z(2+gx(X,Y)+hy).$$

Suppose there exists a function γ such that $\dot{\gamma} = gx(X,Y) + hy(X,Y)$. The transformation $Z = ze^{-\gamma}$ will linearize third equation. Writing $gx(X,Y) + hy(X,Y) = \sum_{n+m>0} a_{n,m}X^nY^m$, we see that

$$\gamma = \sum_{n+m>0}\frac{a_{n,m}}{(i-1)n-(i+1)m}X^nY^m,$$

which is clearly convergent.

Case 3: This case is effectively the same as the previous case, except that now the second and third equations are in the Poincaré domain and therefore there exists a change of variables $(Y,Z) = (y(1+O(y,z),z(1+O(y,z))$ such that

$$\dot{x} = x(i-1+by(Y,Z)+cz(Y,Z)), \qquad \dot{Y} = -(i+1)Y, \qquad \dot{Z} = 2Z.$$

We seek a function γ such that $\dot{\gamma} = by(Y,Z) + cz(Y,Z)$. Then the transformation $X = xe^{-\gamma}$ gives $\dot{X} = (i-1)X$. To find such a function, we write $by(Y,Z)+cz(Y,Z) = \sum_{n+m>0} a_{n,m}Y^nZ^m$, then we have

$$\gamma = \sum_{n+m>0} \frac{a_{n,m}}{2m-(i+1)n} Y^n Z^m,$$

which gives a convergent expression for γ.

Case 4: When $e \neq 0$

$$\dot{x} = x(i-1-(i+1)hy+cz), \quad \dot{y} = y(-i-1-ihy), \quad \dot{z} = z(2+gx+hy),$$

In this case the system has an invariant algebraic plane $\ell = 1+\frac{i+1}{2}hy = 0$ and an exponential factor $E = \exp(\frac{gx-cy}{1+\frac{(i+1)}{2}hy})$ with cofactors $L_\ell = -ihy$ and $L_E = (i-1)gx - 2cz$ producing a first integral $\phi = x^2 z^{1-i} \ell^{i-3} E$ and inverse Jacobi multiplier $M = xyz\ell$. Theorem 1 then guarantees the existence of a second first integral of the form $\phi' = 1 + O(x,y,z)$, which must be analytic. Consideration of the first non-constant terms in this expansion imply that $\phi' - 1 = (xyz)^k (c + O(x,y,z))$ for some $c \neq 0$ and therefore the system is integrable.

Case 5: The equations for $\dot{x}$ and $\dot{z}$ are decoupled from the $\dot{y}$ equation so we can find linearizing transformations $(X,Y) = (x(1+O(x,z), z(1+O(x,z)))$ which bring the system to the form

$$\dot{X} = (i-1)X, \qquad \dot{y} = y(-1-i+dx(X,Z)+fz(X,Z)), \qquad \dot{Z} = 2Z.$$

We seek a function γ such that $\dot{\gamma} = dx(X,Z) + fz(X,Z)$. Then the transformation $Y = ye^{-\gamma}$ gives $\dot{Y} = -(i+1)Y$.

If $dx(X,Z) + fz(X,Z) = \sum_{n+m>0} a_{n,m} X^n Z^m$, then we have

$$\gamma = \sum_{n+m>0} \frac{a_{n,m}}{(i-1)n+2m} X^n Z^m,$$

which gives a convergent expression for γ.

Case 6: The equations guarantee that the cofactors of x, y and z are linearly dependent, so there exists a first integral of the form $x^\alpha y^\beta z^\gamma$. We also have an inverse Jacobi multiplier

xyz. Theorem 1 therefore guarantees the existence of a first integral of the form $\phi' = 1 + O(x,y,z)$, which must be analytic. Consideration of the first non-constant terms in this expansion imply that $\phi' - 1 = (xyz)^k(c + O(x,y,z))$ for some $c \neq 0$ and therefore the system is integrable.

$\square$

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