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Allen, D.

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The Riemann Hypothesis

Danny Allen, Kyle Bonetta-Martin, Elizabeth Codling and Simon Jefferies

Project Advisor: Tom Heinzl, School of Computing, Electronics and Mathematics, Plymouth University, Drake Circus, Plymouth, PL4 8AA

Abstract
This literature review provides a brief discussion of the Riemann Hypothesis, a conjecture regarding the location of the zeros on the Riemann zeta function. We also look at the implications of solving the hypothesis, and investigate historical attempts to solve the conjecture.

1 Introduction

1.1 The Riemann Hypothesis

The Riemann Hypothesis is a conjecture regarding the zeros of the Riemann zeta function. First proposed by Bernhard Riemann in 1859 [1], the hypothesis has yet to be proved despite great efforts for over 100 years. The hypothesis remains one of Hilbert's unsolved problems [2], as well as being a Millennium Prize Problem [3].

To fully understand the Riemann Hypothesis we must first introduce the Riemann zeta function, $\zeta(s)$. The Riemann zeta function is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots , \tag{1}$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$. An equivalent form of the zeta function is given by the Euler product formula, valid for $\Re(s) > 1$:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdots \frac{1}{1-p^{-s}} \cdots . \tag{2}$$

As a convergent infinite product cannot converge to zero (because its logarithm is a convergent series), this form of the zeta function shows that the zeta function does not vanish for $\Re(s) > 1$.

Another equivalent definition for the zeta function in integral form is given by:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx. \tag{3}$$

In order to use zeta function to explain the Riemann Hypothesis, we must first extend the domain of the function to all complex values of s – through the method of analytic continuation. First, we define the Dirichlet eta function, which converges for any complex number with $\Re(s) > 0$, and is given by the following Dirichlet series:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots . \tag{4}$$

We can now extend the Riemann zeta function to the domain where $\Re(s) > 0$ through the following relation between the eta and zeta functions. The eta function can be represented by

$$\begin{aligned} \left(1 - \frac{2}{2^s}\right) \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{2}{(2n)^s} \\ &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots - \frac{2}{2^s} - \frac{2}{4^s} - \frac{2}{6^s} - \frac{2}{8^s} - \dots \\ &= \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \eta(s). \end{aligned} \tag{5}$$

The series on the right-hand side converges when $\Re(s) > 0$, so we have now extended the domain of the zeta function to values of s with positive real parts aside from the zeros of $1 - 2/2^s$ at $s = (1 + 2\pi i n)/\ln 2$. The zeta function can be extended to these values as well by taking limits leaving only a simple pole at $s = 1$ [4].

To extend the domain as Riemann did himself in his 1859 paper [1], we now establish the following functional equation - and therefore analytically continue the domain of the zeta function to all values of s excluding a simple pole at $s = 1$ with residue 1. The functional equation, as explained by Titchmarsh [4], is:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{6}$$

From the functional equation, we can now see that the zeta function will vanish for negative even integers due to the cyclical nature of the '$\sin(\pi s/2)$' component of the functional equation. These values are known as the trivial zeros of the zeta function. The situation is different for the positive even integers, for which $\Gamma(1-s)$ has a simple pole such that $\sin(\pi s/2)\Gamma(1-s)$ is regular and non-zero for positive even s.

Hadamard in [5] and de la Vallée-Poussin in [6] both proved that there are no zeros on the line $\Re(s) = 1$. From this, we see that all non-trivial zeros must lie in the critical strip where s has real part between 0 and 1. The Riemann Hypothesis is the conjecture which states that the non-trivial zeros of the zeta function all lie on the critical line, where $\Re(s) = 1/2$. As the Riemann zeta function is a map from $\mathbb{C}$

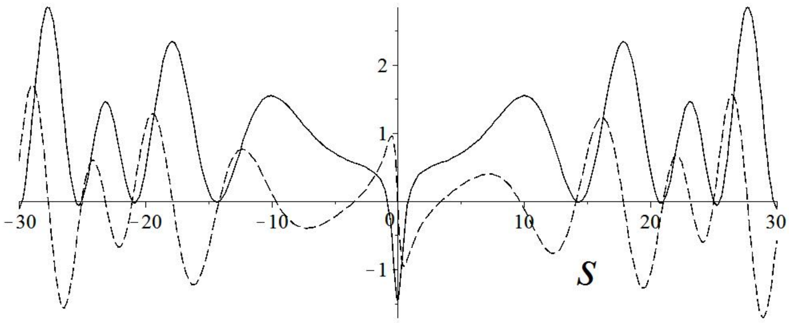

Figure 1: Maple plot of the Riemann zeta function where $\Re(s) = \frac{1}{2}$. Solid and dashed lines represent the real part and imaginary parts of $\zeta(s)$, respectively.

to $\mathbb{C}$, hence returns complex values upon complex input, a plot of the entire function would require a four-dimensional space. Thus, in Figure 1, we separately show the real and imaginary values of the zeta function for $\Re(s) = \frac{1}{2}$. When both lines intersect the horizontal axis, say at t_0, the zeta function is zero at $s = 1/2 \pm t_0$. We can see that the first three zeros have imaginary parts approximately given by ± 14, ± 21, and ± 24. The zeros are symmetric about the real axis as

$$\zeta(s^*) = \sum_{n=1}^{\infty} \frac{1}{n^{s^*}} = \sum_{n=1}^{\infty} \frac{1}{(n^s)^*} = \sum_{n=1}^{\infty} \left(\frac{1}{n^s}\right)^* = \zeta(s)^* \ , \tag{7}$$

where a^* denotes the complex conjugate of a.

1.2 Generalised Riemann Hypothesis and Extended Riemann Hypothesis

The Riemann zeta function is of the form

$$L_k(s,\chi) \equiv \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s}, \tag{8}$$

where $\chi_k(n)$ is a function defined on $\mathbb{Z}$ and has a period k. We see that if $\chi_k(n) = 1$ for all n, then this is equivalent to the Riemann zeta function. If a function has this form, then it is a Dirichlet L-series, see [7]. While the Riemann Hypothesis only relates to the zeroes of the Riemann zeta function, it can be generalised to all Diriclet L-series. The Generalised Riemann Hypothesis (GRH) states that the real parts of all zeroes of any Dirichlet L-series are not larger than $1/2$ [7].

The Extended Riemann Hypothesis (ERH) is the Riemann Hypothesis formulated for Dedekind zeta functions. The Dedekind zeta function is defined as follows:

$$\zeta_K(s) = \sum_{a} \frac{1}{(Na)^s}, \tag{9}$$

where K is a number field where O_K is the integral closure of $\mathbb{Z}$ in K, a is a non-zero ideal of O_K, and Na is the norm of a. The ERH states that when $\zeta_k = 0$ and $0 < \Re(s) < 1$, then $\Re(s) = 1/2$ [8].

It is important to note that either of these implies the Riemann Hypothesis as a consequence.

2 Implications of the Riemann Hypothesis

There are many theorems in mathematics that assume the Riemann Hypothesis: thus, if someone were to prove the hypothesis, it would have large implications throughout mathematics. For example, the Miller-Rabin primality test – an algorithm that determines if a given number is prime – relies on the Extended Riemann Hypothesis being true[9]. If the ERH is not true then this, and many other theorems similar, become null and void. There are also theorems that assume the hypothesis is not true, such as the Gauss class number problem, which relies on the Generalised Riemann Hypothesis being false [10].

Perhaps more importantly, the Riemann Hypothesis is vital in improving the error term in the prime number theorem [11]. This theorem attempts to estimate the rate at which prime numbers appear, or more specifically, the rate at which they become less common. The prime number theorem states that the prime-counting function, $\pi(x)$, can be approximated using $x/\ln(x)$ and was first proposed by Gauss and Legendre towards the end of the 18th century. The function $\pi(x)$ is defined as the number of prime numbers below x, for example $\pi(20) = 8$, as there are 8 primes below 20 [12].

A slightly better approximation of $\pi(x)$ can be given by $\pi(x) \approx Li(x)$, where

$$Li(x) = \int_2^x \frac{1}{\ln t} \mathrm{d}t \tag{10}$$

denotes the (offset) logarithmic integral function. This improvement was demonstrated independently by Hadamard and de la Vallée-Poussin in 1896 using the Riemann zeta function as a main part of the proof, specifically the fact it is non-zero for all complex values of s that have the form $s = 1 + it$ with $t > 0$. Three years later, de la Vallée-Poussin in [13] proved that

$$\pi(x) = Li(x) + \mathcal{O}\left(xe^{-a\sqrt{\ln x}}\right) , \tag{11}$$

and this has since been improved in [14] to:

$$\pi(x) = Li(x) + \mathcal{O}\left(xe^{-A(\ln x)^{0.6}/(\ln \ln x)^{0.2}}\right) . \tag{12}$$

Assuming the Riemann Hypothesis the error term in this equation can be simplified and reduced to

$$\pi(x) = Li(x) + \mathcal{O}(\sqrt{x}\ln x), \tag{13}$$

proved by von Koch in 1901 [15]. Littlewood also used the Riemann Hypothesis to prove that $Li(x) - \pi(x)$ switches from positive to negative infinitely many times for sufficiently large x [11].

Essentially, the Riemann Hypothesis implies a reduced error term for $\pi(x)$ so that we can better understand the distribution of prime numbers.

3 Attempts at Proving the Riemann Hypothesis

3.1 Equivalent Statements

Many have tried to prove the Riemann Hypothesis, but (thus far) without success. One way of looking at the problem is to find statements equivalent to the hypothesis, or statements that would imply the hypothesis. Here, we will show a couple of examples of equivalent statements to the Riemann Hypothesis – all providing a different way to approach the proof (or, indeed, disproof) of the hypothesis.

The asymptotics of the prime counting function given in (13), for instance, is equivalent to the Riemann Hypothesis, but there are many other equivalent statements. Another example is the following [16]: firstly, define the sum of divisors of n as

$$\sigma(n) = \sum_{d/n} d; \tag{14}$$

for example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 = 16$. Introducing Euler's constant,

$$\gamma = \lim_{x\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n \approx 0.577215664 , \tag{15}$$

one can state that the Riemann Hypothesis is equivalent to the inequality

$$\sigma(n) < e^{\gamma} n \ln \ln n \ , \tag{16}$$

when $n > 5040$.

This is Robin's Theorem [17]. The inequality holds for most integers n but is false for 2, 3, 4, 5, 6, 7, 8, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520 and 5040 [18].

Robin's Theorem states that if this inequality holds for all $n > 5040$, then the Riemann Hypothesis is true.

Lagarias continued from Robin's work in [19], and created an inequality equivalent to the Riemann Hypothesis with a smaller bound on n, using the n^{th} *harmonic number*, $H_n = \sum_{j=1}^{n} 1/j$, to state that

$$\sigma(n) \leq e^{H_n} \ln H_n + H_n \tag{17}$$

is equivalent to the Riemann Hypothesis with the weaker condition $n \geq 1$. Nevertheless, Robin's Theorem is often preferred for analysis as it does not contain harmonic numbers.

These and other equivalent statements help us to understand the many facets and implications of the Riemann Hypothesis.

3.2 Other Attempts at Proof

Of course, the finding, and proving, of an equivalent statement is not the only attempt used to determine the validity of the Riemann Hypothesis. As for its disproof, the most obvious tactic is disproof by counterexample: calculate the zeros of the function, and if a zero appears that is neither trivial nor on the critical line, then the Riemann Hypothesis cannot be true. Many – over 10^{13} – zeros of the zeta function have been calculated [20], and as yet all have been trivial, or on the critical line of $\Re(s) = 1/2$.

There has been some progress made on proving the hypothesis by 'narrowing down'. For example, it has been proved that there are infinitely many zeros of the zeta function along the line $\Re(s) = 1/2$ – the critical line – by Hardy, in 1914 [21]. Later, in 1989, Conrey proved that at least 40% of all zeros lie on the critical line [22]. This was continued in 2011, when Bui et al. increased this to 41.05% [23].

Others have seen relations to the Riemann Hypothesis from other, often seemingly unrelated, topics. For example, the Lee-Yang theorem of statistical mechanics implies that (for certain statistical field theories) all the zeros associated with the partition function of the model lie on a critical line [24]. Although there has not been any progress on solving the Riemann Hypothesis, this relation – of all (bar trivial) zeros lying on a particular line – may be one of the tools for finding a proof.

3.3 The Hilbert-Pólya Conjecture

The Hilbert-Pólya Conjecture is an unpublished idea accredited to David Hilbert and George Pólya about a possible way to solve the Riemann hypothesis. Pólya, in a conversation with Landau in 1914, remarked that the Riemann hypothesis would be true "if the non-trivial zeros of [the Riemann zeta function] were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real." [25].

This idea translates to finding a self-adjoint operator A such that the eigenvalues of A are the same as the imaginary parts of the zeros of the zeta function. As any self-adjoint operator has only real eigenvalues, then if $\zeta(s)$ could be mapped to A, then there could not be a zero off of the critical line, and so the Riemann Hypothesis would be proven.

This conjecture was interesting when it was proposed, but there was little evidence in 1914 to support the idea that there would be such a corresponding operator. However, Selberg in [26] showed a connection between the length spectrum of a Riemann surface and the eigenvalues of its Laplacian, typically a self-adjoint operator. The length spectrum of the Riemann surface is the set of closed geodesics (straight lines on the surface) ordered by the length of the line. This connection, the Selberg trace formula, is very similar to the explicit formula connecting the Riemann zeta function and prime powers. This discovery gave credence to the Hilbert-Pólya conjecture, although what the self-adjoint operator could be is still unknown.

The first written instance of the Hilbert-Pólya conjecture appeared in [27], where Montgomery was looking at the spacing between the zeros. In this paper, he notes that the pair correlation function he derives for the spacing between zeros has the same statistical distribution as the eigenvalues of a random complex Hermitian matrix [28]. A Hermitian matrix is one that is self-adjoint, and so this observation also lends credence to the Hilbert-Pólya conjecture.

4 Conclusion

The Riemann Hypothesis is such an important unsolved problem – not only due to its famed difficulty, but also due to the huge impact its proof or disproof would make on mathematics. Many proofs have been attempted from varying fields and disciplines, including many amateur mathematicians trying their hand for the million dollar millennium prize. As of now, there is no definitive proof either way, though with the sheer amount of zeros calculated – none disagreeing with Riemann's hypothesis, and the amount of mathematics that rely on the hypothesis, it seems plausible, at the moment at least, to be true.

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