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The Pareto Principle

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Abstract
The Pareto Principle (also known as the 80-20 rule) states that for many phenomena, about 80% of the consequences are produced by 20% of the causes. In this article we discuss the Pareto Principle and its importance in real life problems, describe some mathematical model related to it and also address the concept of the Lorenz curve and Gini coefficient. We tested two sets of real life data to see if the Pareto principle applies to these aspects. For the Forbes list in 2012, we found that 20% of the richest people own 56.72% of the money. For the world Gross Domestic Product (GDP) in 2011, 20% of the richest countries in the world have 91.62% of the total amount of money. In both cases results have a relatively close resemblance to the Pareto principle.

1 Introduction

More than a hundred years ago the Italian economist and sociologist Vilfredo Pareto made the famous observation that 20% of the population owned 80% of the property in Italy [1]. Later on, he created a mathematical formula to describe the unequal distribution of wealth in his country, which is known as the Pareto distribution. In the late 1940s, business-management consultant J.M. Juran generalised Pareto's findings into the 80-20 rule, which is also known as the Pareto Principle [2].

The Pareto Principle states that for many phenomena 80% of the output or consequences are produced by 20% of the input or causes. It is often used in management, economics and business to improve productivity and make better decisions, but is also used in computer science and human activity. It helps to realize that often the majority of results comes from a minority of inputs. Here are some examples of the Pareto Principle as it applies to various situations: 80% of the revenue comes from 20% of the customers, 20% of products yield 80% of sales, 20% of society hold 80% of its wealth and so on [2]. The Pareto Principle is a simplified version of the mathematics behind the Pareto distribution. It is also not important that the two numbers add up to 100%. The numbers 20 and 80 are not mathematically fixed, but are used as a rule of thumb. It could be 80-20, 90-10, or even 90-20.

In this paper, we will describe some mathematical models related to the Pareto Principle beginning with Section 2 where we also introduce the concepts of the Lorenz curve and Gini coefficient. We show how one can use these tools to evaluate the inequality of wealth distributions. We then analyse two sets of real life data in Section 3, i.e. the 2012 Forbes list of the super-rich and the distribution of the world GDP in 2011. We try to fit them with Pareto distribution models to see if the Pareto Principle applies to these data. To visualise the results, the different distributions are plotted in Pareto charts and histograms. In Section 4 we finally draw conclusions from our results and discuss possible further studies on this topic.

2 Mathematical Models

2.1 Pareto Distribution

The Pareto principle is a special case of the wider phenomenon of Pareto distributions. Pareto stated in his book [1] that there is a simple law which governs the distribution of income in all countries and at all times. Briefly, if N represents the number of people with wealth larger than a certain income limit x, and A and α are constants, then $N = A/x^{\alpha}$, therefore,

$$\log(N) = \log(A) - \alpha \log(x) \ . \tag{1}$$

In other words, if the logarithm of the number of persons with incomes above a definite amount is plotted against the logarithm of these incomes, the resulting graph will be

a straight line. Its slope will be α, which is also known as the Pareto index. A more general description of the statement above is given by the Pareto distribution.

The classical Pareto distribution is defined in terms of its cumulative distribution function

$$F_P(x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^{\alpha} & \text{for} \quad x \geq x_m \ , \\ 0 & \text{for} \quad x < x_m \ , \end{cases} \tag{2}$$

where x_m is a scale which indicates the (necessarily positive) minimum value of x, and α, the Pareto index, is a positive shape parameter. The density function is given by $f_P = F'_P$ or

$$f_P(x) = \begin{cases} \alpha \frac{x_m^{\alpha}}{x^{\alpha+1}} & \text{for} \quad x \geq x_m \ , \\ 0 & \text{for} \quad x < x_m \ . \end{cases} \tag{3}$$

When the Pareto index is $\alpha_0 = \log_4 5$, approximately 1.16, then one has 80% of effects coming from 20% of causes, which leads to the 80-20 rule.

2.2 Lorenz curve and Gini coefficient

In economics, the Lorenz curve is mainly used as a graphical tool for representing the empirical probability distribution of income or wealth, where it shows for the bottom x% of households, what percentage y% of the total income they have.

Given any distribution function (cdf) $F(x)$, the theoretical Lorenz curve corresponding to it is defined by

$$L(p) = \mu^{-1} \int_0^p F^{-1}(t) dt \ , \tag{4}$$

where we assume that $F(x)$ increases on its support, which implies that $F^{-1}(p)$ is well defined and is the population p^{th} quantile, and the mean μ of $F(x)$ exists [3].

Considering the classical Pareto distribution, the corresponding Lorenz curve is given by

$$L_P(p) = 1 - (1-p)^{1-1/\alpha}, \quad 0 < p < 1 \ , \tag{5}$$

provided $\alpha > 1$.

A perfectly equal income distribution would be one in which every person has the same income. In this case, the bottom N% of society would always have the same N% percentage of the income. This can be depicted by the straight line $y = x$; called the “line of equality”. This idea leads to the concept of the Gini coefficient, also known as the Gini index [3].

The Gini coefficient is commonly used as a measure of inequality of income or wealth. Figure 1 shows the representation of a Lorenz curve and the concept of the Gini coefficient in a particular case where Pareto index $\alpha_0 = \log_4 5$ as mentioned previously. The Gini coefficient G is defined as the ratio of the area that lies between the line of equality and the Lorenz curve (marked A in the diagram) over the total area

under the line of equality (marked A and B in the diagram); i.e., $G = A/(A+B)$. Since $A + B = 0.5$, the Gini coefficient is $G = 1 - 2B$. Therefore, a Gini coefficient of zero expresses perfect equality, where all values are the same, while a Gini coefficient of one expresses maximal inequality among values. The world income Gini coefficient was shown to be between 0.6 to 0.7 in recent years [4].

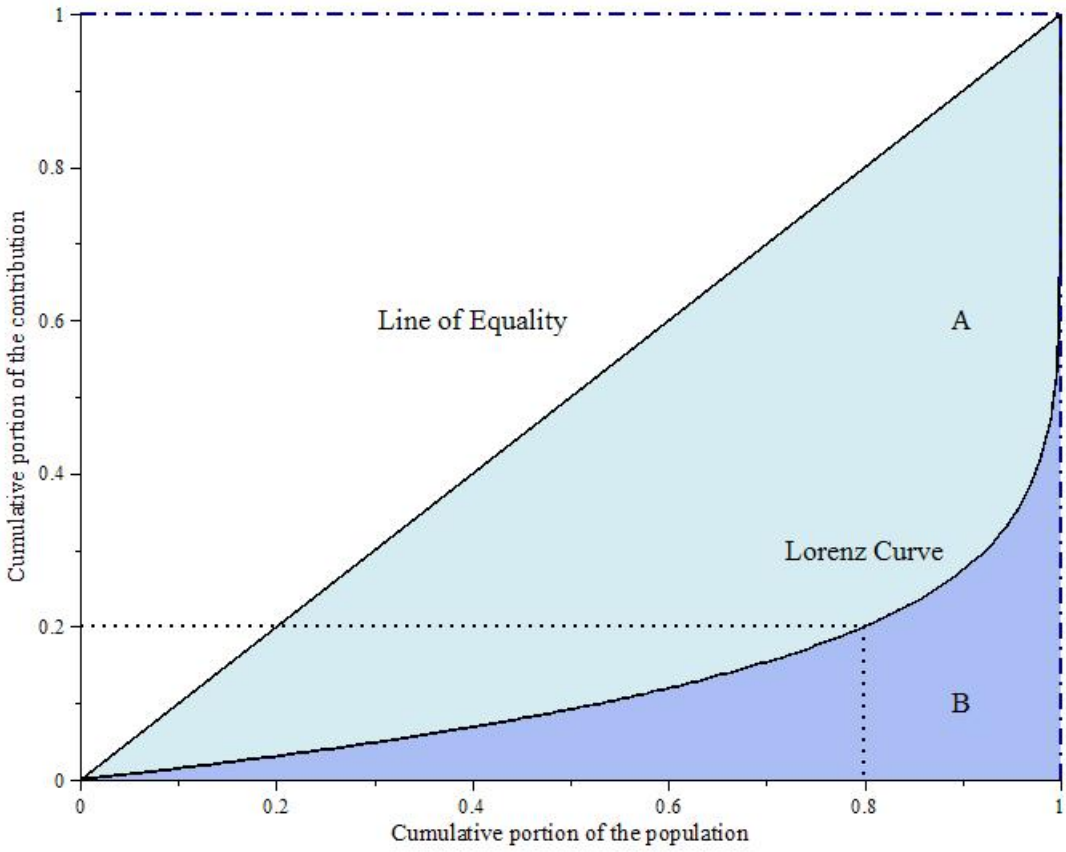

Figure 1: Representation of Lorenz curve and the concept of the Gini coefficient under 80-20 rule.

For our particular case, we can see from the diagram that 80% of the input contributes to 20% of the output, which perfectly obeys the 80-20 rule. Accordingly, the Gini coefficient of this model is

$$G = 1 - 2\int_0^1 [1 - (1-p)^{1-1/\alpha}]dp \approx 0.76 \ . \tag{6}$$

This expresses a substantial inequality as expected from the underlying 80-20 rule.

3 Testing real life data

In this section, the Pareto principle will be tested against two real life data sets which are the 2012 Forbes list of the super-rich [5] and the World's Nominal GDP in 2011 [6], respectively. Our aim is to find the wealth distribution of these two examples and test if they obey the Pareto principle.

3.1 Example 1: Forbes list of the super-rich

The percentages of total wealth held by each 20% quantile of the "population" was calculated from the 1226 richest people in the world [5]. When the principle was tested, it was found that in fact the top 20% of people on the Forbes list own only 56.72% of the money, cf. Fig. 2. However, this does follow the Pareto principle, as the top 20% of the people on the list do have a rather large percentage of the total wealth, it just does not obey the (rather particular) 80-20 case.

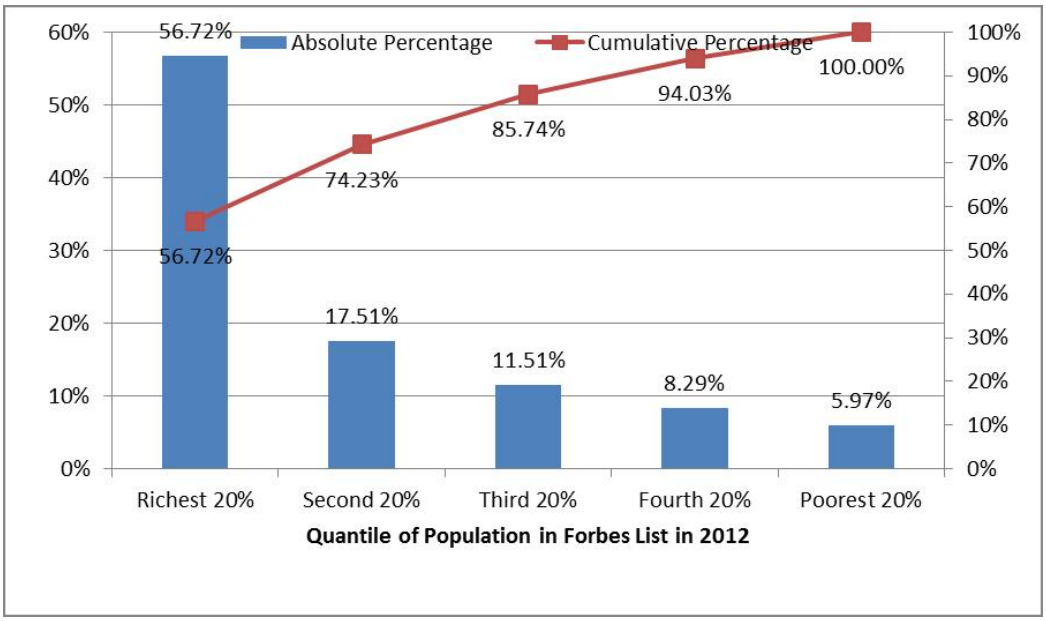

Figure 2: Pareto Chart of the Distribution of the Forbes List.

We then went one further step to fit this data by a Pareto distribution model. By using the results found above in (2) and (3), we calculated the Pareto index from the Lorenz curve and obtained that $\alpha \approx 1.36$. As this is a larger Pareto index than that of the 80-20 case, where $\alpha \approx 1.16$, it shows that a smaller proportion of people have a high income. Figure 3 also demonstrates the density distribution of the data and a density curve of Pareto Distribution with Pareto index 1.36 and $x_m = 1$ (billion dollar).

We find that the density curve fits our data quite well, meaning that the data set is well explained by a Pareto distribution model. The Gini coefficient corresponding

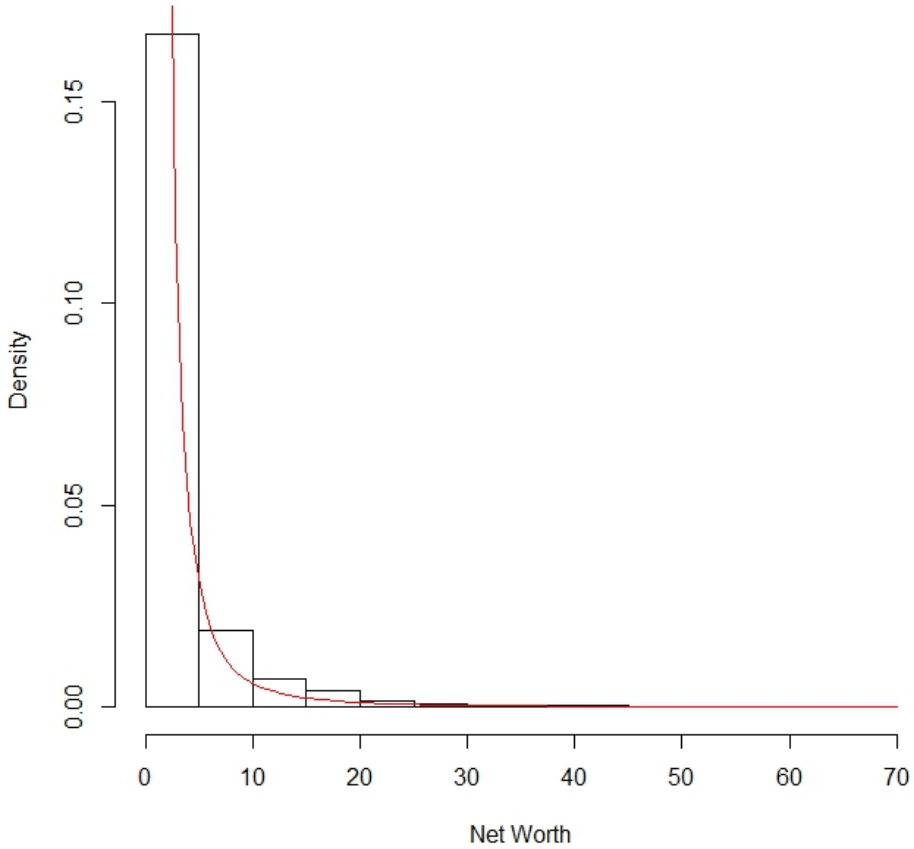

Figure 3: Histogram for the distribution of the Forbes List and density curve of the corresponding Pareto distribution.

to this model is found to be 0.58, which is slightly less than the world average Gini coefficient (0.6 to 0.7), implying that the distribution of the wealth among billionaires is slightly more equal. However, we should notice that the Gini coefficient only fits the particular model based on our "56-20" finding. There are more accurate methods to estimate the Gini coefficient, but this is not our topic.

3.2 Example 2: World GDP

This data set lists countries by their GDP in 2011 [6]; it shows that the top 20% of the richest countries in the world have 91.62% of the total amount of money in the world (see Fig. 4); this does generally follow the Pareto principle.

We also calculated the Pareto index for the world GDP and obtained $\alpha = 1.09$. This is lower than for the 80-20 case, which implies that a higher proportion of the

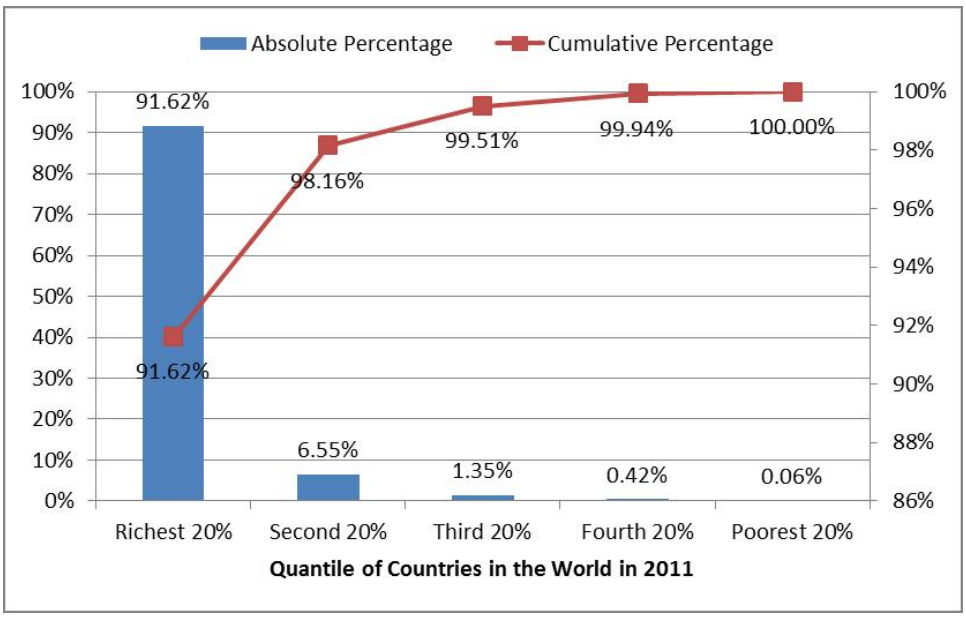

Figure 4: Pareto Chart of the Distribution of World GDP.

money is with the top 20%. This supports what we have seen in the results. When trying to plot a histogram of the data and fitting it with a Pareto distribution model with $x_m = 1$, the result seems unconvincing. The reason is the large number of countries with low GDP that makes the lower tail of the data not well explained by our model.

However, if we set x_m as 100 billion, so that only countries having GDP greater than 100 billion dollars, i.e. about the top 30% (62 countries) of all countries, are taken into account, a better result is obtained. Figure 5 shows our finding. This result is consistent with evidence showing that the Pareto distribution usually produces good fits only of the upper tail of the population [7]. As the GDP is not a measure of personal income or wealth, we did not calculate the Gini coefficient here. But we should notice that the Gini coefficient and the GDP are two main indicators that are often discussed together in economy.

4 Discussion and Conclusion

The examples show that a majority of wealth is with a small percentage of the population. Although the real data considered did not exactly follow the 80-20 rule, they nevertheless support Pareto's principle.

One reason why the Forbes list may not precisely follow the 80-20 rule could be that the less rich people on the Forbes rich list are becoming richer and the ones at the top of the list are earning at a slower pace now that they have become so successful – possibly a saturation effect. With the world GDP, looking at data from 1989 [2] the 80-20 rule works almost perfectly, with the richest 20% of countries having 82% of

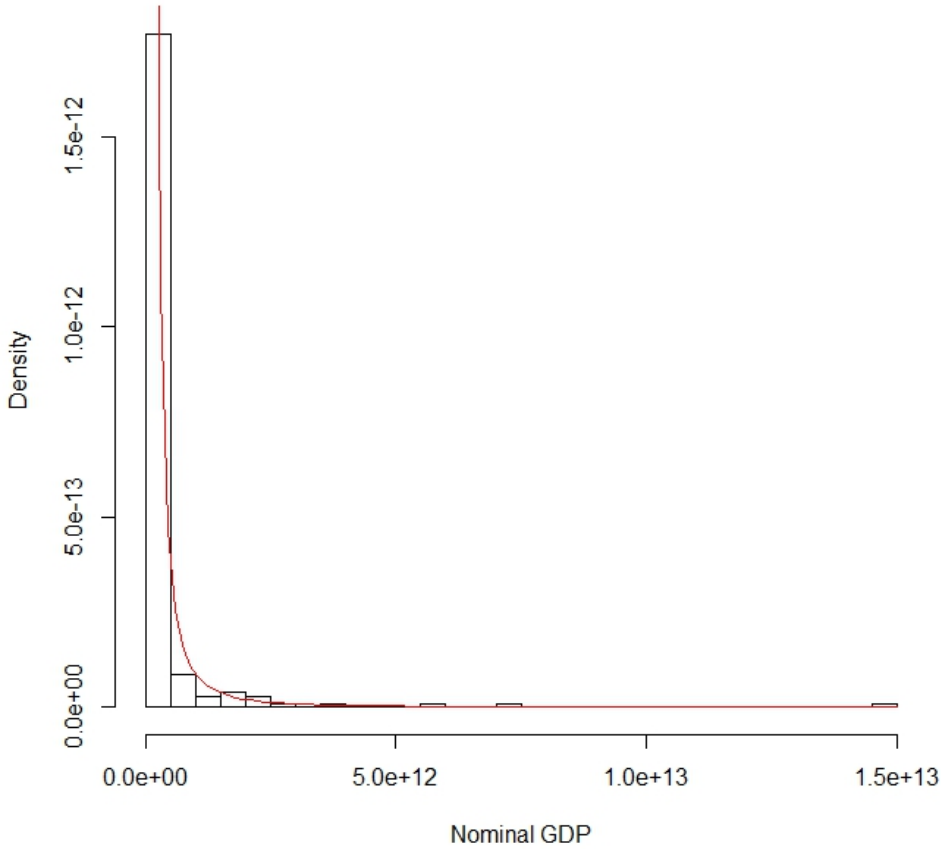

Figure 5: Histogram of the distribution of the World GDP list and density curve of corresponding Parato distribution.

the money; looking at more recent data, however, the top 20% of the countries own 91.62% of the money in the world. Clearly, this suggests that the imbalance has increased – despite the recent financial crisis in the "wealthy" countries.

As mentioned previously, the Pareto distribution sometimes does not yield good fits when considering the whole population. An alternative model to be used in this situation is usually the log-normal distribution. In finance, this is the standard model for prices of financial assets since the development of the Black-Scholes [8] option pricing theory based on geometric Brownian motion. However, because of the inability of the log-normal distribution to account for large-impact low-probability events, i.e. to model heavy tails, its usage also has limitation.

To solve this dilemma, [9] indicated that a "message that frequently arises is that the log-normal distribution may be a model that fits well the body of the distribution but not the tail, where a Pareto distribution provides a better fit. Thus, it makes sense to

consider a composite log-normal Pareto model, namely a mixture of a right-truncated log-normal and a Pareto." Recently, log-normal Pareto models have been adopted in various areas, see e.g. [9, 10, 11].

References

[1] Vilfredo Pareto, *Cours d'conomie Politique Profess a l'Universit de Lausanne*, Vol. I, 1896; Vol. II, 1897.

[2] Wikipedia, *Pareto principle*, http://en.wikipedia.org/wiki/Pareto_principle, accessed 03-02-2013.

[3] Joseph L. Gastwirth, *The Estimation of the Lorenz Curve and Gini Index*, The Review of Economics and Statistics, Vol. 54, No. 3, pp. 306-316, Aug., 1972.

[4] Pinkovskiy, Maxim and Xavier Sala-i-Martin, *Parametric Estimations of the World Distribution of Income*, NBER Working Paper 15433, 2009.

[5] The World's Billionaires, http://www.forbes.com/billionaires/list/, March 2012, accessed 03-02-2013.

[6] United Nations, http://unstats.un.org/unsd/snaama/dnltransfer.asp?fID=2, 2011, accessed 03-02-2013.

[7] Giorgio Fazio, Marco Modica, *Pareto or log-normal? A recursive-truncation approach to the distribution of (all) cities*, Glasgow Economics Department Discussion Paper Series, 2012-10, presented at RSAI 2012 (Timisoara).

[8] F. Black and M. Scholes, The Journal of Political Economy **81**, 637-654 (1973).

[9] Marco Bee, *Statistical analysis of the Lognormal-Pareto distribution using Probability Weighted Moments and Maximum Likelihood*, Department of Economics Working Papers 1208, University of Trento, 2012.

[10] William J. Reed, Murray Jorgensen, *The Double Pareto-Lognormal Distribution A New Parametric Model for Size Distributions*, Communications in Statistics - Theory and Methods, Vol. 33, No. 8, pp. 1733-1753, 2004.

[11] Sandra Teodorescu, *On the Truncated Composite Lognormal-Pareto Model*, Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 2009.