2019-05-14

The icosahedra of edge length 1

Brakhage, K-H

http://hdl.handle.net/10026.1/13825

10.1016/j.jalgebra.2019.04.028
Journal of Algebra
Elsevier

The icosahedra of edge length 1

Karl-Heinz Brakhage*, Alice C. Niemeyer†, Wilhelm Plesken†, Daniel Robertz‡, Ansgar Strzelczyk†

March 21, 2019

Abstract

Retaining the combinatorial EUCLIDean structure of a regular icosahedron, namely the 20 equiangular (planar) triangles, the 30 edges of length 1, and the 12 different vertices together with the incidence structure, we investigate variations of the regular icosahedron admitting self-intersections of faces. We determine all rigid equivalence classes of these icosahedra with non-trivial automorphism group and find one curve of flexible icosahedra. Visualisations and explicit data for this paper are available under `http://algebra.data.rwth-aachen.de/Icosahedra/visualplusdata.html`.

Keywords: Icosahedron, combinatorial geometry, rigidity

1 Introduction

The regular icosahedron has already fascinated the ancient Greeks. In this paper we investigate variations of the regular icosahedron as follows: We keep the combinatorial part of the EUCLIDean structure of the regular icosahedron, namely the 20 equiangular (planar) triangles, the 30 edges of length 1, and the 12 different vertices with the incidence structure. We drop the assumption of convexity and even allow that the triangles penetrate each other. This results into a system of 30 quadratic equations over the real numbers for $3 \cdot 12$ indeterminates. The real solutions with 12 different vertices we simply call icosahedra. Having tried to solve these equations for some time, we have come to the conclusion that solving them is a hard problem indeed.

We dedicate this paper to the late Charles Sims. His main contribution to mathematics was in group theory, where, for example, his contributions to group theoretic algorithms were a crucial ingredient in the quest for finite sporadic

[1]Institut für Geometrie und Praktische Mathematik, RWTH Aachen University, Templergraben 55, 52062 Aachen, Germany. `brakhage@igpm.rwth-aachen.de`

[2]Lehrstuhl B für Mathematik, RWTH Aachen University, Pontdriesch 10-16, 52062 Aachen, Germany. `Alice.Niemeyer@mathb.rwth-aachen.de`, `Wilhelm.Plesken@mathb.rwth-aachen.de`, `Ansgar.Strzelczyk@math.rwth-aachen.de`

[3]School of Computing, Electronics and Mathematics, University of Plymouth, 2-5 Kirkby Place, Drake Circus, Plymouth, Devon, PL4 8AA, United Kingdom. `daniel.robertz@plymouth.ac.uk`

simple groups. Inspired by his perseverance of tackling hard problems we too have not given up and, using group theory, have at least come close to a classification of all icosahedra allowing a non-trivial symmetry group.

To be more specific, we may and do assume that the edge lengths of the triangles are all 1. To get rid of the translations of 3-space moving around the icosahedra, we assume that the vertices sum up to 0, i.e. the coordinate origin is the center of mass of the twelve (equilibrated) vertices. Hence an icosahedron is given by a 3×12-matrix M whose columns $V_1, \ldots, V_{12}$ give the standard coordinates for the 12 vertices. Having numbered the vertices in some fixed way these icosahedra still allow the operation of the orthogonal group of EUCLIDean 3-space: With M also OM is a solution for every $O \in O_3(\mathbb{R})$. Finally to get rid of this group action as well, we pass over to the GRAM-matrix $G := M^{tr}\cdot M$, which is a 12×12-matrix with the following obvious properties: It is real, symmetric, positive semidefinite of rank at most 3. Its three non-negative eigenvalues give some idea of how the vertices are distributed around their center of mass. Also some choice of eigenvectors for the three non-negative eigenvalues yields a normalized choice for the coordinate matrix M of the icosahedron. Since the action of the orthogonal group has been factored out by the passage to the GRAM-matrices, equivalence, isometry, or isomorphism of the latter is just conjugacy by permutation matrices.

It is well known that the combinatorial automorphism group A of the icosahedron has order 120 and is isomorphic to $C_2 \times A_5$, where the generator d of the center $Z(A) = C_2$ is generated by the permutation interchanging combinatorially opposite vertices of the icosahedron. Here is a summary of our results.

Theorem 1. *The subgroups U of A with more than one element that arise as symmetry group of an icosahedron fall into 11 conjugacy classes of subgroups of A. For each U we list the number of equivalence classes of icosahedra.*

Automorphism group $U \leq A = C_2 \times A_5$	*Number of icosahedra*
$C_2 \times A_5$	2
$C_2 \times D_{10}$	4
$C_2 \times D_6$	2
$D_{10} \quad (\not\leq A_5)$	3
$D_6 \quad (\not\leq A_5)$	2
${C_2}^2 \quad (\ni d)$	1
${C_2}^2 \quad (\not\ni d, \not\leq A_5)$	5
${C_2}^2 \quad (\leq A_5)$	1
$C_2 \quad (\leq A_5)$	5
$C_2 \quad (\not\ni d, \not\leq A_5)$	10
$C_2 \quad (= \langle d \rangle)$	∞

For the finite cases we shall produce formulas for the GRAM-matrices, called formal GRAM-matrices, most of which usually contributing more than one real

GRAM-matrix. This will be discussed in Section 3. Further it turns out, most cases of subgroups U split up into several subcases distinguished by the linear action of U on 3-space. This splitting is an essential step for this classification. Details are discussed in Section 2, cf. also the final Table 1. The final case with infinitely many solutions is investigated in Section 5. We only present existence proofs with some numerical approximations. In particular we have a simulation of a curve of deformable icosahedra, which is obtained by solving a certain ordinary differential equation numerically.

The problem addressed in this paper can also be viewed as the problem of classifying embeddings into Euclidean 3-space of the graph that is defined by the incidence structure of the icosahedron, with prescribed edge lengths and symmetry. The question whether continuous families of embeddings exist is known as the question whether the considered framework is rigid. A common approach to investigate rigidity is to check infinitesimal rigidity and conclude in the affirmative case that the framework is rigid for sufficiently generic embeddings. In [6] this approach was adapted to frameworks with symmetry. Infinitesimal rigidity implies rigidity also for sufficiently generic embeddings with specified symmetry. The method of [6] would be an alternative to show the existence of the curve of deformable icosahedra with symmetry, but the question whether prescribing the edge lengths is compatible with the genericity assumption would still have to be addressed.

As a consequence of the easier parts of our computations we mention:

Proposition 2. *There are exactly four isometry classes of icosahedra* (*with all edge lengths* 1) *such that the midpoints of combinatorially opposite vertices have the center of mass of the icosahedron as their midpoint.*

2 Symmetry and linear action on 3-space

The combinatorial automorphism group $A \cong C_2 \times A_5$ of the abstract icosahedron can be generated by the permutations

$$\begin{aligned}
a &= (1,2)(3,4)(5,7)(6,8)(9,11)(10,12),\\
b &= (1,10)(3,9)(2,12)(4,11)(5,6)(7,8),\\
c &= (1,7)(2,3)(4,11)(5,12)(6,8)(9,10),\\
d &= (1,12)(3,9)(2,10)(4,11)(5,7)(6,8).
\end{aligned}$$

In particular d generates the center C_2 and interchanges combinatorially opposite vertices, whereas the first three generators a, b, c form a minimal generating set of involutions for A_5. The triangles are obtained as the orbit of $\{1,2,3\}$, the 30 edges as orbit of $\{1,2\}$, the 30 diagonals of combinatorial distance 2 as the orbit of $\{3,4\}$, and finally the 6 diagonals of combinatorial distance 3 as the orbit of $\{1,12\}$, which one also finds in the 2-cycles of the generator d of the center of A. There is a geometric bijection called "orthogonal diagonal" between the two orbits $\{1,2\}^A$ and $\{3,4\}^A$ as follows

$$\omega : \{1,2\}^A \to \{3,4\}^A : \{i,j\} \mapsto \{s,t\} \text{ iff } \{i,j,s\}, \{i,j,t\} \in \{1,2,3\}^A.$$

Finally let $\pi : A \to \mathrm{GL}(12, \mathbb{R})$ denote the natural representation of A by permutation matrices.

Definition 3. *The combinatorial automorphism group A acts on the set $\mathcal{G}$ of all* Gram*-matrices of icosahedra by*

$$A \times \mathcal{G} : (g, G) \mapsto \pi(g)^{tr} \cdot G \cdot \pi(g) = (G_{ig,jg})_{1 \leq i,j \leq 12}$$

and the stabilizer of $G \in \mathcal{G}$ is the automorphism group of G or the associated icosahedron. Note, $\pi(g)^{tr} = \pi(g)^{-1}$ so that the trace is an invariant for this action.

We now demonstrate a method splitting up the case of an automorphism group U into more manageable subcases by using the linear action of U on 3-space. To this end we have the following elementary lemma from linear algebra:

Lemma 4. *Let $G \in \mathbb{R}^{n \times n}$ be a symmetric, positive semidefinite matrix of rank k.*
1.) There exists a real $k \times n$-matrix M with $M^{tr}M = G$.
2.) G and M have the same row space.
3.) If $L \in \mathbb{R}^{k \times n}$ also satisfies $L^{tr}L = G$, then there is a unique matrix $g \in \mathbb{R}^{k \times k}$ with $L = gM$. Moreover g is orthogonal.

Proof. 1.) Let $(E_1, \ldots, E_n)$ be an orthonormal basis of eigenrows of G with eigenvalues $\lambda_1 > 0, \ldots, \lambda_k > 0, \lambda_{k+1} = \ldots = \lambda_n = 0$. Then M can be chosen with the rows $\frac{1}{\sqrt{\lambda_i}} E_i$ for $i = 1, \ldots, k$, because $E_i^{tr} E_i$ represents the orthogonal projection of $\mathbb{R}^{1 \times n}$ onto $\langle E_i \rangle$.
2.) Obvious. 3.) Clear from 2.). $\square$

Applying this to our situation we get the following:

Lemma 5. *Let $G \in \mathcal{G}$ be the* Gram*-matrix of an icosahedron and $U \leq A$ its automorphism group. Then there is a faithful orthogonal representation $\delta : U \to O_3(\mathbb{R})$ and a matrix $M \in \mathbb{R}^{3 \times 12}$ called coordinate matrix such that*

$$\delta(g) \cdot M = M \cdot \pi(g) \text{ for all } g \in U \text{ and } G := M^{tr} \cdot M$$

The name coordinate matrix is chosen because its columns represent the vertices of the icosahedron in 3-space. Since this matrix is unique only up to orthogonal transformations, the field generated by its entries is not canonical unlike the corresponding field F_G for the Gram-matrix G. But from a computational point of view the coordinate matrices are more accessible than the Gram-matrices. We therefore go for them as follows: For each minimal subgroup U (up to conjugacy) of A and each faithful orthogonal representation $\delta : U \to O_3(\mathbb{R})$ compute the relevant coordinate matrices M among the intertwining matrices of δ and the restriction $\pi_{|U}$. Find a set of representatives of the A-orbits of the Gram-matrices $G := M^{tr} \cdot M$. Most of these calculations can be done with the formal matrices, as we shall see in the next section.

3 Formal Gram-matrices

Assume $\{1\} \neq U \leq A$. Let $e_1, \ldots, e_{12}$ be the standard basis of $\mathbb{R}^{12\times 1}$. In order to determine the U-invariant Gram-matrices G, we define an ideal I generated by $(e_i - e_j)^{tr} G(e_i - e_j) - 1$ for all $\{i, j\} \in \{1, 2\}^A$ and the 4×4-minors of G, where the entries $G_{i,j}$ of G are equated to variables $y_1, \ldots, y_n$ in correspondence with the U-orbits on the positions in the symmetric 12×12-matrix G. Then I is an ideal in $R := \mathbb{Q}[y_1, \ldots, y_n]$. It will turn out in all but one case, R/I is finite dimensional over $\mathbb{Q}$ so that we concentrate on maximal associated primes first. Here is a complete list of all necessary conditions for the case of zero dimensional I.

Definition 6. *A maximal ideal $m \trianglelefteq R$ associated to I (in the primary decomposition) is called relevant if the following three conditions are satisfied:*
1.) The rank of the image matrix $(G_{i,j} + m) \in (R/m)^{12\times 12}$ is at most 3;
2.) $U = \{g \in A \mid \pi(g)^{tr}(G_{i,j} + m)\pi(g) = (G_{i,j} + m)\}$;
3.) there exists a relevant real embedding $\iota : R/m \to \mathbb{R}$, i.e. $(\iota(G_{i,j} + m))$ is positive semidefinite.
$G(m) := (G_{i,j} + m)$ is called the associated formal Gram-matrix, the residue class field R/m (as well as its abstract isomorphism type) is called the field of definition $F_{G(m)}$ of $G(m)$. The field degree $[R/m : \mathbb{Q}]$ is denoted by $d_{G(m)}$, the number of real embeddings of R/m by $r_{1,G(m)}$, the number of relevant real embeddings of R/m by $r_{G(m)}$, and finally $r_{f,G(m)}$ denotes the number of relevant real embeddings with $(\iota(G_{i,j} + m))$ pairwise inequivalent. We refer to $r_{G(m)}$ as generosity and to $r_{f,G(m)}$ as contribution of $G(m)$.

Given a real Gram-matrix representing an icosahedron whose entries are algebraic numbers, then of course the field generated by all its entries is finite over $\mathbb{Q}$, has a primitive element, and at least one real embedding. So one might hope, that it defines a formal Gram-matrix. This, however, need not be the case. One can compute its automorphism group and relate it to the ideal I above. Though one finds a maximal ideal yielding this Gram-matrix, it might be so that the maximal ideal is embedded rather than associated to I. Such cases exist, because later on, we shall exhibit an example of a one dimensional equational ideal I, indeed we find a curve of icosahedra.

With this definition the reader can already interpret most of the final Table 1. Here are some further comments:

Remark 7. *1.) There are only finitely many formal Gram-matrices up to equivalence.*
2.) For each formal Gram-matrix $G(m)$ the normalizer $N_A(U)$ permutes the maximal ideals above. The stabilizer $D(G(m))$ (called decomposition group) of m in $N_A(U)$ permutes the relevant embeddings of R/m and $D(G(m))/U$ acts faithfully on R/m by field automorphisms.
3.) In particular the length of the orbit of $G(m)$ under $N_A(U)$ is of the form $[N_A(U) : D(G(m))] \cdot [D(G(m)) : U]$, where the first factor is the length of the $N_A(U)$-orbit of m.

It is convenient for checking the relevance of real embeddings as well as for tabulating formal GRAM-matrices to choose a primitive element of R/m. In case the trace of $G(m)$ generates R/m, it is an ideal candidate because it gives the formal GRAM-matrix a canonical form. The last column in Table 1 indicates the minimal polynomial of the trace of the GRAM-matrix over $\mathbb{Q}$. In particular one can read off the degree of R/m over the field generated by the trace. In seven cases the trace is a primitive element. Five of these cases, namely the simplest ones, where the field degree is two, can be explained easily in geometric terms by starting from the regular icosahedron from ancient Greece. In all these cases R/m is isomorphic to $\mathbb{Q}[x]/(x^2-5)$ and both real embeddings are relevant. The following construction will explain this phenomenon:

Remark 8. *If the 5 neighbour vertices of a vertex V lie in a plane, one obtains a new icosahedron from the given one by applying the orthogonal reflection with the fixed plane spanned by the five neighbour vertices of V and the five triangles adjacent to V and keeping the rest of the icosahedron. Though this operation changes the center of mass, it can be adjusted by a translation and does not change the field of definition, nor r or r_f. We call this operation* <u>*denting*</u> *at V.*

So the regular icosahedron with automorphism group $C_2 \times A_5$ yields the one with automorphism group D_{10} by denting at one vertex V. Adding another dent at combinatorial distance 2 away from V yields the one with automorphism group ${C_2}^2$ and with distance 3 away the one with automorphism group $C_2 \times D_{10}$. Finally there is the one with three dents of pairwise distance 2 with automorphism group D_6.

Theorem 9. *There are up to equivalence 14 formal* GRAM*-matrices with non-central automorphism group resulting into 35 isomorphism classes of real* GRAM*-matrices as specified in Table 1.*

The final item in this table to be explained is the strange symbol in the second column for proper subgroups that are direct products. It shows two symbols above and one symbol below a horizontal line. The two symbols on top indicate the trace values of the two generators of the group, and the one below of their product. Here $+, -, \bullet$ stands for $1, -1, -3$ resp.. These symbols are also used for the trace of the generator of C_2 in the last 6 cases.

A final comment on the results: The denominators of the coefficients in the trace relations involve only very few prime factors, which already show up in the truncated version in the last column, e.g. for $d_G = 172$ the primes are $2, 3, 5, 29, 79$, whereas the regular icosahedron and $C_2 \times D_6$-case have an empty set of primes for the denominators.

We finish these introductory sections with some comments on the computations. In our first approach we used the system BERTINI, cf. [1], which was very helpful indeed, since it created some idea of an answer. Due to the complexity of the problem it did not produce a completeness proof. Later we found out that it missed one isomorphism class, which is distinguished by the existence of coplanar neigbouring triangles. Further, BERTINI gives decimal expansions of the

$S{:=}\mathrm{Stab}_A$	$\mathrm{Syl}_2(S)$	d_G	$r_{1,G}$	r_G	$r_{f,G}$	Trace relation
${C_2}^2$	$\langle a,d\rangle \frac{\bullet -}{+}$	8	4	2	1	$\lambda^4 - \frac{76}{3}\lambda^3 + 238\lambda^2 - \frac{4964}{5}\lambda + \frac{23767}{15}$
$C_2 \times A_5$	$\langle a,b,d\rangle$	2	2	2	2	$\lambda^2 - 15\lambda + 45$
$C_2 \times D_{10}$	$\langle a,d\rangle \frac{- \bullet}{+}$	2	2	2	2	$\lambda^2 - 15\lambda + \frac{269}{5}$
${C_2}^2$	$\langle a,bd\rangle \frac{- \pm}{+}$	2	2	2	2	$\lambda^2 - \frac{71}{5}\lambda + \frac{10561}{225}$
$C_2 \times D_{10}$	$\langle a,d\rangle \frac{- \pm}{+}$	4	2	2	2	$\lambda^4 - 18\lambda^3 + \frac{583}{5}\lambda^2 - \frac{1658}{5}\lambda + \frac{9101}{25}$
$C_2 \times D_6$	$\langle a,d\rangle \frac{- \pm}{+}$	4	4	2	2	$\lambda^4 - 26\lambda^3 + 243\lambda^2 - 970\lambda + 1397$
${C_2}^2$	$\langle a,bd\rangle \frac{- \pm}{+}$	24	10	6	3	$\lambda^{12} - \frac{5179\cdot 2^2}{3^2\cdot 5^2}\lambda^{11} \pm \cdots$
${C_2}^2$	$\langle a,b\rangle \frac{- -}{-}$	30	18	6	1	$\lambda^5 - \frac{117}{2}\lambda^4 \pm \cdots$
C_2	$\langle a\rangle \quad -$	172	48	20	5	$\lambda^{43} - \frac{73\cdot 7\cdot 11\cdot 461687}{2^2\cdot 3^3\cdot 5^2\cdot 29\cdot 79}\lambda^{42} \pm \cdots$
D_{10}	$\langle ad\rangle \quad +$	2	2	2	2	$\lambda^2 - \frac{44}{3}\lambda + \frac{2131}{45}$
D_6	$\langle ad\rangle \quad +$	2	2	2	2	$\lambda^2 - \frac{68}{5}\lambda + \frac{1111}{25}$
C_2	$\langle ad\rangle \quad +$	36	12	8	4	$\lambda^{18} - \frac{1106}{9}\lambda^{17} \pm \cdots$
C_2	$\langle ad\rangle \quad +$	168	40	24	6	$\lambda^{42} - \frac{2\cdot 719\cdot 1223}{3^3\cdot 5\cdot 43}\lambda^{41} \pm \cdots$
D_{10}	$\langle ad\rangle \quad +$	4	2	2	1	$\lambda^2 - \frac{26}{3}\lambda + \frac{149}{9}$

Table 1: List of all formal Gram-matrices with symmetry with a, b, c, d as defined in Section 2

solution which for our purposes still have to be turned into algebraic numbers. The number theoretic methods available for this often failed for field degrees above 40. The degrees we found by the methods of this paper were sometimes considerably larger. Also at the time we had no upper bound for these degrees. Moreover we had no good way to reduce the number of solutions by a priori splitting into cases guided by symmetry. So we had to deal with more than 10^4 solutions. But on the positive side BERTINI gave us approximations of two solutions with trivial symmetry, which easily can be turned into an existence proof. Our present approach, presented in this paper, relies on formal computations using the MAGMA [4] GROEBNER-basis functionality and the involutive basis functionality of GINV, cf. [3, 2]. Anyone interested in the actual formal GRAM-matrices can find them in MAPLE-readable form on our homepage [5] for this paper. Anyone interested in the actual computations leading to a proof can find MAPLE-worksheets also on this homepage. Possibly we shall also produce detailed comments on the properties of the various solutions on these pages.

4 Classifying the formal GRAM-matrices

Instead of working with the definition of the formal GRAM-matrices in Definition 6, we go via the coordinate matrices M giving us the GRAM-matrices as $M^{tr}M$. We get rid of the operation of the three dimensional orthogonal group on the coordinate matrices by fixing certain entries. Also we do not go through all subgroups of A, but look at the minimal subgroups and compute the full automorphism groups of their fixed GRAM-matrices. Note, the automorphism groups of the formal GRAM-matrices are equal to ones obtained by relevant real embeddings, and are easily obtained by a stabilizer computation.

As for the minimal subgroups, they are generated by an element of prime order. Here is a complete list up to conjugacy with a, b, c, d as defined at the beginning of Section 2:

$$\langle abc\rangle \cong C_3, \quad \langle ac\rangle \cong C_5, \quad \langle a\rangle \cong C_2, \quad \langle d\rangle \cong C_2, \quad \langle ad\rangle \cong C_2.$$

By the next lemma, the first two cases can be discarded.

Lemma 10. *If the* GRAM*-matrix of an icosahedron is fixed by an element of order* 3 *or* 5*, then its automorphism group also has an element of order* 2*.*

Proof. 1.) As for the case of order 3 we may assume that the GRAM-matrix is fixed by abc. There is (up to conjugacy under the orthogonal group) just one faithful orthogonal representation δ of $U := \langle abc\rangle \cong C_3$ of degree 3. δ is the orthogonal sum of $\delta_1 : abc \mapsto 1$ and an irreducible representation δ_2 of degree 2, i.e. $\delta = \delta_1 \bigoplus \delta_2$. Character theory tells us that the first row of the searched coordinate matrix M lies in a space of dimension 4, and the remaining submatrix consisting of the last two rows lies in a space of dimension $4 \cdot 2$, so that we have

11 indeterminates for our equations to be solved. But we can do better: The 4-dimensional space for the first row reduces to a 3-dimensional space because the sum of the entries is 0 (center of mass condition). The 8-dimensional space for the remaining two rows reduces to a 7-dimensional space since the centralizer of δ_2 in $\mathrm{O}_2(\mathbb{R})$ can be applied to make one entry in the, say, first row 0. Now comes the computation of the solutions for the matrices thus composed, which amounts to solving a system of 30 (= number of edges of the icosahedron) quadratic equations in 9 indeterminates. The involutive basis gives us a residue class algebra of dimension 128 over the ground field $\mathbb{Q}[\sqrt{3}]$ which has exactly 42 residue class fields, of which 18 yield a formal GRAM-matrix with 12 different rows. Computing the orbits under the action of A yields orbit length 1 (four times), 20 (ten times), and 10 (four times), implying that each stabilizer has order 6. Note this calculation was purely on the formal level without computing real embeddings.
2.) As for the case of order 5 we may assume that the GRAM-matrix is fixed by ac. There is up to conjugacy under the orthogonal group and algebraic conjugacy just one irreducible representation δ_2 of degree 2 expressed in the roots of the $\mathbb{Q}$-irreducible polynomial $\lambda^4 - (5/4)\lambda^2 + 5/16$, which has both $\sin(2\pi/5)$ and $\sin(4\pi/5)$ amongst its roots. We again have $\delta = \delta_1 \bigoplus \delta_2$, obtain a vector space of dimension $3+3$ containing the (formal) coordinate matrix R. The quadratic equations lead to 24 solutions. This time all solutions yield 12 different columns for the GRAM-matrix. The lengths of the orbits under A are 1 (four times), 6 (eight times), and 12 (twelve times). Hence again all stabilizers have an order divisible by 2. □

Hence we are left with the task of finding and analyzing the GRAM-matrices fixed by one of the three groups of order 2 above. Each one of them has (up to equivalence) three faithful orthogonal representations of degree 3: the generator might get mapped onto one of

$$\mathrm{diag}(1,-1,-1), \quad \mathrm{diag}(-1,1,1), \quad \mathrm{diag}(-1,-1,-1),$$

so that we are left with $3 \cdot 3 = 9$ cases, summarized in Table 2. The first two numbers in each box give the number of isomorphism classes of formal and of real GRAM-matrices, which come out in the end for that particular case. This is followed by a reference to the corresponding lemma. Of course the cases are not disjoint because many GRAM-matrices have automorphism groups of orders bigger than 2.

One of these cases is ruled out immediately:

Lemma 11. *There is no solution in the case of* $\delta : \langle ad \rangle \to \mathrm{O}_3(\mathbb{R}) : ad \mapsto \mathrm{diag}(-1,-1,-1)$.

Proof. The permutation ad has fixed points, whereas $\mathrm{diag}(-1,-1,-1)$ has no nonzero fixed vectors in $\mathbb{R}^3$. □

	diag$(1,-1,-1)$	diag$(-1,1,1)$	diag$(-1,-1,-1)$
a	$8, 19$ cf. 17	$0, 0$ cf. 13	$1, 1$ cf. 15
d	? cf. 20	$2, 4,$ cf. 19	$2, 4$ cf. 16
ad	$0, 0$ cf. 12	$12, 29$ cf. 18	$0, 0$ cf. 11

Table 2: Cases distinguished by linear action

Slightly more complicated, but still very easy are the next two cases:

Lemma 12. *There is no solution in the case of* $\delta : \langle ad \rangle \to \mathrm{O}_3(\mathbb{R}) : ad \mapsto \mathrm{diag}(1, -1, -1)$.

Proof. To set up the coordinate matrix $M \in \mathbb{R}^{3\times 12}$ and the equations for its entries is done in the obvious way: The first row must be fixed under the permutation ad and the entries must add up to 0. Since ad has 4 fixed points, this leaves $8-1$ indeterminates for the first row. The second and third row lie in the eigenspace of the permutation matrix for ad for the eigenvalue -1, which is of dimension 4, so that we end up with $7+4+4$ indeterminates and 6 zero entries in M. We could still get rid of one more entry, but now already there is no solution, as a short calculation with Involutive shows. □

Lemma 13. *There is no solution in the case of* $\delta : \langle a \rangle \to \mathrm{O}_3(\mathbb{R}) : a \mapsto \mathrm{diag}(-1, 1, 1)$.

Proof. Computing the eigenspaces for the permutation matrix for a leads to the 6-dimensional vector space for the first row of the coordinate matrix M, and a common 6-dimensional space for the second and third row. This second space can be reduced to a 5-dimensional space, because the sum of the columns is zero. Hence we get 16 indeterminates occurring in M. We can reduce by 1, because the first two columns have distance 1, which yields one quadratic equation for one indeterminate. Plugging that solution in yields solutions for M, but they produce only three different columns for the Gram-matrix. Hence there is no solution with 12 different vertices in this case. □

Concerning the results of the computations in case there are solutions, we note the following.

Remark 14. *The icosahedra invariant under some element $x \in A$ are permuted by the action of the normalizer of $\langle x \rangle$ in A. In particular, the ones invariant under d come in orbits under A. The ones fixed by a or ad distribute themselves in orbits under the centralizer of a in A, which is the Sylow 2-subgroup of A containing a. In particular the lengths of the latter orbits divide 4, and the former ones 60.*

Lemma 15. *In case $\delta(a) = \mathrm{diag}(-1,-1,-1)$ there is one formal* Gram*-matrix and one real* Gram*-matrix, with details listed in the following table:*

Stabilizer in A	$d_G, r_{1,G}$	r_G	$r_{f,G}$	*cf. also La.*
$\langle a,d\rangle \cong {C_2}^2$	8, 4	2	1	*20, 18*

The field of definition F_G has four real embeddings. The Galois *group of the normal closure of F_G is the symmetric group S_8. The element $b \in A$ interchanges the two relevant real embeddings of the formal* Gram*-matrix.*
The linear action of the stabilizer can be chosen as $\delta(d) = \mathrm{diag}(1,-1,-1)$, $\delta(ad) = \mathrm{diag}(-1,1,1)$.

Proof. The three rows of the coordinate matrix $M \in \mathbb{R}^{3\times 12}$ lie in the eigenspace of $\pi(a)$ for the eigenvalue -1, so that we have 18 indeterminates. The center of mass condition is automatically fulfilled. Because of the action of the 3-dimensional orthogonal group we may assume that the first column is of the form $(*,0,0)^{tr}$. Since a interchanges 1 and 2, the second column is of the same shape. Since the first three columns form a face of the icosahedron, the stabilizer of the first column in $\mathrm{O}_3(\mathbb{R})$ can be used to force the third column to be of shape $(0,*,0)^{tr}$ so that the first three columns are essentially uniquely determined. After this the relations generate a maximal ideal with residue class field of degree 16 over the rationals. The associated Gram-matrix is then defined over a field of degree 8, which has 4 real embeddings. But for two of them the Gram-matrix becomes indefinite. The automorphism group for the formal Gram-matrix is $\langle a,d\rangle \leq A$, in particular isomorphic to V_4. But the centralizer of a in A is of order 8. And indeed, $b \in A$ interchanges the two real embeddings so that we end up with exactly one icosahedron in this case. One easily checks that the two other automorphisms d and ad have trace -1 resp. 1 in their 3-dimensional associated real representation, so that we know already where we shall encounter this icosahedron again. □

In three of the nine cases of Table 2 one can predict a solution right away: If the representation δ of the C_2 is restriction of the natural matrix representation of A connected to the realization of A as geometric automorphism group of the regular icosahedron. These are the cases
$\delta(a) = \mathrm{diag}(1,-1,-1), \delta(ad) = \mathrm{diag}(-1,1,1), \delta(d) = \mathrm{diag}(-1,-1,-1)$.
We start with the third case because it relates to Proposition 2 of the introduction.

Lemma 16. *In case $\delta(d) = \mathrm{diag}(-1,-1,-1)$ there are two formal* Gram*-matrices corresponding to the two lines of the following table yielding four real* Gram*-matrices:*

Stabilizer in A	$d_G, r_{1,G}$	r_G	$r_{f,G}$	*cf. also La.*
$C_2 \times A_5$	2, 2	2	2	*17, 18*
$C_2 \times D_{10}$	2, 2	2	2	*17, 18*

Their field of definition F_G is $\mathbb{Q}[\sqrt{5}]$.
In both cases $\delta(a) = \mathrm{diag}(1, -1, -1)$ and $\delta(ad) = \mathrm{diag}(-1, 1, 1)$.

Proof. 1.) (Setting up the equations and solving them formally) Let $M \in \mathbb{R}^{3\times 12}$ be the coordinate matrix. If the first column is zero, then so is the twelfth, since d interchanges 1 and 12. But we are dealing with the case that all twelve vertices are pairwise different. Hence the first column is not zero. Because of the action of $\mathrm{O}_3(\mathbb{R})$ we may assume that it is of the form $(*, 0, 0)^{tr}$. There are two cases to be discussed: the special case, where the first two columns are linearly dependent, and the generic case, where the first two columns are linearly independent. In the first case the second column is of the same type as the first and their equations, the intertwining condition (linear!) and the edge length condition (quadratic), quickly lead to a contradiction. So we only have to deal with the generic case. Again because of the action of the orthogonal group, we may assume that the second column is of the form $(*, *, 0)^{tr}$. Taking the action of d into account, we obtain a system of 30 quadratic equations in 16 indeterminates. The primary decomposition of the relation ideal yields the following list of residue class fields: 28 of degree 4, 20 of degree 8, 4 of degree 12, and 8 of degree 24.
2.) (Analysis of the solutions) The 8 residue class fields of degree 24 have no real embeddings and must therefore be discarded. The same applies to the 4 residue class fields of degree 12. Each of the 20 residue class fields of degree 8 does have real embeddings, but none of them leads to a positive semidefinite Gram-matrix by the following argument: We look at the difference of the 3rd and 4th column of M. These two columns are the remaining vertices of the two triangles of the icosahedron formed by the columns or rather vertices 1,2,3 and 1,2,4. Since the triangles are equiangular of edge length 1 the squared distance of 3 and 4 must lie between 0 and 3. It so happens that for all embeddings of the present cases this distance gets a negative value or a value bigger than 3, which is impossible.
Finally the 28 solutions of degree 4 all have real embeddings and lead to formal Gram-matrices over $\mathbb{Q}[x]/(x^2 - 5)$, which for each of the two real embeddings yield positive semidefinite Gram-matrices. More precisely, the traces of the 24 formal Gram-matrices have minimal polynomial $\lambda^2 - 15\lambda + 45$ (four times) and $\lambda^2 - 15\lambda + 269/5$ (24 times). Writing them over the same field makes the 4 equal, i.e. gives one Gram-matrix over $\mathbb{Q}[x]/(x^2 - 5)$, and the other 24 solutions yield 6 Gram-matrices. The first one is fixed by the operation of A, the other six form one orbit. In both cases the two real embeddings lead to positive semidefinite Gram-matrices with distinct real traces. □

Lemma 17. *In case $\delta(a) = \mathrm{diag}(1, -1, -1)$ there are 8 formal Gram-matrices and, up to equivalence, 19 real Gram-matrices, with details listed in the follow-*

ing table:

Stabilizer in A	$d_G, r_{1,G}$	r_G	$r_{f,G}$	*cf. also La.*
$C_2 \times A_5$	2, 2	2	2	*16, 18*
$C_2 \times D_{10}$	2, 2	2	2	*16, 18*
C_2^2	2, 2	2	2	*18*
$C_2 \times D_{10}$	4, 2	2	2	*19, 18*
$C_2 \times D_6$	4, 4	2	2	*19, 18*
${C_2}^2$	24, 10	6	3	*18*
${C_2}^2$	30, 18	6	1	–
C_2	172, 48	20	5	–

Proof. Since this is a difficult case, it is necessary to prepare the computation of the coordinate matrix $M \in \mathbb{R}^{3\times 12}$ well. First of all the first row lies in the 6-dimensional eigenspace of $\pi(a)$ for the eigenvalue 1, and the second and third row in the 6-dimensional eigenspace for the eigenvalue -1. Hence we start with 18 indeterminates. The first column of M cannot be of the form $(*,0,0)^{tr}$, because otherwise it would be equal to the second column contradicting our assumption of having 12 different vertices. By applying a two dimensional rotation to modify the last two rows without changing the Gram-matrix, we may assume that the first row is of the form $(*,*,0)^{tr}$. Since $\{1,2\}$ represents an edge and hence is of square length 1, we conclude that the first column can be chosen to be of the form $(*,1/2,0)^{tr}$. This reduces the number of variables from 3 to 1. The difference of columns 3 and 4 represents the diagonal orthogonal to the edge $\{1,2\}$, proving that column 3 is of the form $(*,0,*)^{tr}$ and hence reducing the number of variables by 1. The reason why this worked so well is that the first two cycles of a, namely $(1,2),(3,4)$ represent an edge and its orthogonal diagonal. The same applies to the last two cycles $(10,12),(9,11)$. Since the action of the two dimensional orthogonal group is no longer available, we get a reduction by 1 only for the number of variables. Finally, the center of mass condition gives us that the sum of the coefficients of the first row is 0, so that we have another reduction by one variable and end up with 13 variables.
The rather time consuming computation of the associated prime ideals with MAGMA leads to 72 maximal ideals with residue class fields of degree 2 (48 times), 4 (10 times), 6 (4 times) , 8 (4 times), 16 (2 times), 48 (2 times), 120 (once), and 688 (once). We shall treat each degree separately.
Degree 2: Of the 48 residue class fields only 20 yield a solution with 12 different vertices. The traces of the 20 formal Gram-matrices have three different minimal polynomials over the rationals with the following roots: $15/2 \pm (3/2)\sqrt{5}$ (4 times), $15/2 \pm (7/10)\sqrt{5}$ (8 times), and $71/10 \pm (5/6)\sqrt{5}$ (8 times). The formal Gram-matrices with the same minimal polynomial also have the same orbit lengths under the action of A, namely $1, 6, 30$. Since the centralizer $C_A(a) = \langle a, b, d\rangle$ of a in A acts on the set of all formal a-invariant Gram-matrices, its orbits must be of length 1 in the first case, and of length 2 in the second and third cases. Now writing the Gram-matrices in terms of polynomials of their traces, we can compare them and find that our list of 20 reduces to

a set of 5 different GRAM-matrices, namely one with automorphism group A, an orbit of 2 with automorphism group isomorphic to $C_2 \times D_{10}$, and an orbit of 2 with automorphism group ${C_2}^2$. The first two GRAM-matrices have shown up already in Lemma 16, where their occurrence was already predicted via the linear action of d which is by multiplication by -1, as one easily checks. Also the linear action of the other elements in the SYLOW 2-subgroups of the stabilizers are easily computed, cf. the comments on the automorphism groups, thus justifying the entries in the last column of the table.

Degree 4: Of the 10 residue class fields only 8 yield a solution with 12 different vertices. Similarly as in the degree-2-case one finds exactly two non-isomorphic GRAM-matrices, but unlike in the previous case none of them is positive semidefinite.

Degree 6: None of the four residue class fields has a real embedding.

Degree 8: All four residue class fields lead to solutions with 12 different vertices. All of them have the same minimal polynomial for the trace and the orbits under A are all of length 6. The coefficients of the GRAM-matrices can be represented as polynomials in the traces of the GRAM-matrices. Another variable co, which can be interpreted as a cosine of a certain angle, has minimal polynomial over the rationals of degree 2. This can be used to factor the minimal polynomial of the trace t in two quadratic factors and to see which of the two can be discarded as discussed below. At this stage one sees that there are only two different formal GRAM-matrices, which fall in one orbit under the centralizer of a in A. So we have to deal with only one formal GRAM-matrix. The minimal polynomial of co has two roots, namely $-2 \pm \sqrt{5}$. We conclude $co = -2 + \sqrt{5}$, because the other root has absolute value bigger than 1. This makes the minimal polynomial of t of degree 2 with solutions $\frac{9}{2} + \frac{7}{10}\sqrt{5} \pm \frac{2}{5}\, 5^{3/4}$, both of which give valid solutions (with different eigenvalues). A SYLOW 2-subgroup of the automorphism group is given by $\langle a, d\rangle$. Unlike in the other case with automorphism group $C_2 \times D_{10}$ the central element d acts with eigenvalues $1, 1, -1$ on the 3-space.

Degree 16: Both residue class fields lead to solutions with 12 different vertices. All of them have the same minimal polynomial for the trace and the orbits under A are all of length 10, proving that the two formal GRAM-matrices fall in one orbit under the centralizer of a in A, so that we only have to look at one of them. Again the entries of GRAM-matrices can be written as polynomials of the GRAM traces, which generate a totally real field extension of the rationals of degree 4. However only two of the real embeddings lead to a positive semi-definite GRAM-matrix. The SYLOW 2-subgroup of the automorphism group is $\langle a, d\rangle$, however d acts not as scalar matrix on the real 3-space but with eigenvalues $-1, 1, 1$.

Degree 48: Both residue class fields lead to solutions with 12 different vertices. All of them have the same minimal polynomial for the trace and the orbits under A are all of length 30. The two minimal polynomials are equal and of degree 12. In both cases, the $1,1$-entry of the GRAM-matrix is a primitive element of the field F_G generated by all entries. Again the minimal polynomials over the rationals are equal and both of degree 24. Writing the entries of both GRAM-matrices as polynomials over these primitive elements makes them equal, so that we have

only one formal Gram-matrix for the present case. We have 10 real embeddings of the field F_G, of which only 6 lead to a positive semidefinite real Gram-matrix. The stabilizer of the formal Gram-matrix in A is $\langle a, bd\rangle \cong V_4$. Clearly, a and bd fix each one of the resulting real Gram-matrices, but d permutes them in three 2-cycles. Note, $N_A(\langle a, bd\rangle)/\langle a, bd\rangle$ is isomorphic to the Galois group of F_G over the field generated by the trace of the Gram-matrix, the latter acting equivalently on the set of the 6 relevant real roots. So we end up with 3 inequivalent embeddings. The eigenvalues of bd and abd in the action on real 3-space can be read off from $a \mapsto (1,-1,-1), bd \mapsto (1,1,-1), abd \mapsto (1,-1,1)$.
Degree 120: The residue class field leads to one formal solution with 12 different vertices and the orbit of the formal Gram-matrices under A is of length 30 with stabilizer $\langle a, b\rangle \cong V_4$. The $1,7$-entry of the Gram-matrix already generates the field F_G generated by all entries, which turns out to be of degree 30 over the rationals. This field has 18 real embeddings of which only 6 lead to positive semidefinite real Gram-matrices. The trace t of the formal Gram-matrix generates a field extension of degree 5, so that the field extension $(F_G/\mathbb{Q}[t])$ is of degree 6. It turns out to be a Galois extension with Galois group isomorphic to $N_A(\langle a, b\rangle)/\langle a, b\rangle \cong C_6$. Indeed, we end up with one real Gram-matrix up to equivalence. On 3-space all three elements a, b, ab of the stabilizer act linearly with trace -1.
Degree 688: The residue class field leads to one formal solution with 12 different vertices. The orbit of the formal Gram-matrices under A is of length 60 with stabilizer $\langle a\rangle \cong C_2$. The sum of the $1,1$-entry and the $5,5$-entry of the Gram-matrix already generates the field F_G generated by all entries. It has degree 172 and has 48 real embeddings of which only 20 lead to positive semidefinite real Gram-matrices. The trace t of the formal Gram-matrix generates a field extension of degree 43, so that the field extension $(F_G/\mathbb{Q}[t])$ is of degree 4. It turns out to be a Galois extension with Galois group isomorphic to $N_A(\langle a\rangle)/\langle a\rangle \cong V_4$. Indeed, the centralizer of a in A is $\langle a, b, d\rangle \cong C_2^3$ and subdivides the twenty Gram-matrices into 5 orbits of length $4 = |V_4|$. □

Whenever in the proofs of the subsequent lemmas a Gram-matrix shows up which is invariant under a, we know by the preceding lemmas that this Gram-matrix occurred already in one of the Lemmas 17, 15 and its isomorphism type can easily be read off from the various invariants used. On the other hand, the previous lemmas also tell us, in which of the subsequent lemmas such Gram-matrices will occur.

Lemma 18. *In case $\delta(ad) = \mathrm{diag}(-1,1,1)$ there are 12 formal Gram-matrices and, up to equivalence, 29 real Gram-matrices, with details listed in the follow-*

ing table[1]*:*

Stabilizer in A	$d_G, r_{1,G}$	r_G	$r_{f,G}$	*cf. also La.*
$C_2 \times A_5$	2, 2	2	2	*16, 17*
$C_2 \times D_{10}$	2, 2	2	2	*16, 17*
${C_2}^2$	2, 2	2	2	*17*
D_{10}	2, 2	2	2	–
D_6	2, 2	2	2	–
${C_2}^2$	24, 10	6	3	*17*
C_2	36, 12	8	4	–
C_2	168, 40	24	6	–
$C_2 \times D_6$	4, 4	2	2	*17, 19*
$C_2 \times D_{10}$	4, 2	2	2	*17, 19*
${C_2}^2$	8, 4	2	1	*20, 15*
D_{10}	4, 2	2	1	–

Proof. The coordinate matrix has its first row in the (row) eigenspace of the permutation matrix for ad for the eigenvalue -1, which is of dimension 4, the second and third row lie in the eigenspace for the eigenvalue 1, which is of dimension 8. Hence we have 20 variables for the coordinate matrix. The fourth column is of the form $(0, *, *)^{tr}$ and can be assumed to be of the form $(0, *, 0)^{tr}$. Because the cycle $(3, 11)$ of ad consists of the vertices of an edge of the icosahedron, the first entry x_3 of the third column satisfies $x_3^2 = 1/4$. W.l.o.g. we may assume $x_3 = 1/2$. (This makes the first entry of the eleventh column $-1/2$.) Also $(4, 9)$ is a cycle of ad and represents an edge of the icosahedron at the same time. Since we are no longer allowed to multiply the first row of the coordinate matrix by -1, we now have two possibilities for the first entry x_4 of the forth column, namely case 1: $x_4 = 1/2$ and case 2: $x_4 = -1/2$. In both cases the number of remaining variables is 18. This number can be reduced to 16, since we have the center of mass condition by which the sum of the entries in the second and third row is zero, which are two new equations unlike to the first row. From here on we treat the two cases separately.

Case 1: We get 63 prime ideals in the primary decomposition. All except for 4 prime ideals lead to icosahedra with 12 different vertices. The residue class fields in these 59 relevant cases have degree 2 (20 times), 4 (28 times), 6 (4 times), 8 (2 times), 48 (2 times), 96 (once), 144 (2 times).

Degree 2, case 1 All Gram-matrices which come up are also fixed by a or a conjugate of a such as b or ab. Hence they are known already. To be more precise, if one writes the Gram-matrices in terms of their traces, one ends up with 5 different formal Gram-matrices coming in three orbits under the centralizer of ad in A. Since we have treated already all a-invariant Gram-matrices, these orbits must correspond to the first three Gram-matrices in Lemma 17.

Degree 4, case 1 In all 28 cases the entries of the formal Gram-matrices can be written in terms of the traces of their Gram-matrices so that equivalence

[1] The subdivision of the table reflects the two cases in the proof.

becomes easy to check. The rational minimal polynomials of the traces are all of degree 2. There are 8, 8, 8, resp. 4 cases where the A-orbit length of the Gram-matrices is 30, 10, 12, resp. 20. The 8 of orbit length 30 yield two $C_A(ad)$-orbits of length 2, one with stabilizer $\langle ad, b\rangle$, hence again isomorphic to the third in Lemma 17, the second orbit yielding only indefinite Gram-matrices. The 8 Gram-matrices with A-orbit length 10 boil down to an orbit of two, which, however, are indefinite. The 8 of orbit length 12 give 4 different Gram-matrices, which form one orbit under $C_A(ad)$. Both real embeddings are positive semidefinite, thus giving a new formal Gram-matrix whose stabilizer is isomorphic to D_{10}. Finally, the 8 of orbit length 20 give 4 different Gram-matrices, which form one orbit under $C_A(ad)$. Both real embeddings are positive semidefinite, thus giving a new formal Gram-matrix whose stabilizer is isomorphic to D_6.

Degree 6, case 1 There are no real solutions. More precisely the traces of the two Gram-matrices have a rational minimal polynomial of degree 2 with negative discriminant.

Degree 8, case 1 This case yields two real Gram-matrices fixed under ab. Both are not positive semi-definite and hence ruled out.

Degree 48, case 1 The Gram-matrices are fixed under b, which is conjugate to a. Hence, this case was treated earlier. Indeed, the two Gram-matrices are equivalent to the ones obtained in Lemma 17 with automorphism group ${C_2}^2$ and $d_G = 24$.

Degree 96, case 1 Here no real solution exists, because the rational minimum polynomial of the trace of the Gram-matrix is of degree 6 without real roots.

Degree 144, case 1 There are two Gram-matrices over a field of degree 36 over the rationals. The stabilizer in A is generated by ad in both cases. Since the normalizer of $\langle ad\rangle$ is of order 8, one of these two solutions gives us already an orbit of four Gram-matrices. This can only be the case if a subgroup of order 2 of $N_A(\langle ad\rangle)/\langle ad\rangle$ is induced by field automorphisms. In any case, it suffices to look at the first formal Gram-matrix. Its $(7, 1)$-entry can be chosen as primitive element of the field. It allows us to look at the real completions, of which there are 12 with 8 of them yielding a positive definite Gram-matrix. It turns out that $b \in A$ induces the expected field automorphism because it permutes the 8 real places in 4 transpositions, whereas d for instance does not. (Ordering the 8 real roots according to their increasing size, b yields the permutation $(1, 2)(3, 6)(4, 5)(7, 8)$.) Hence we end up with 4 real Gram-matrices up to equivalence.

Case 2: Using the MAGMA Groebner-basis routine and the Involutive basis routine we see that the residue class algebra of our equations is of $\mathbb{Q}$-dimension 928. It seems to be very hard to obtain automatically the associated prime ideals. Therefore we compute the minimal polynomial of the trace of the Gram-matrix, which turns out to be of degree 70 only. It factors into irreducible factors of degrees 2, 12, 42, 4, 4, 4, 2. The first factor has no real roots and has therefore to be discarded. The remaining ones have 6, 12, 4, 2, 2, resp. 2 real roots. We discuss each factor separately. Note, at this stage we do not yet know whether each factor leads to a maximal ideal.

Degree-12-factor case 2 The residue class algebra is of $\mathbb{Q}$-dimension 96, leads to a Gram-matrix over a field of degree 24 with stabilizer $\langle ad, ab\rangle \cong {C_2}^2$ under the action of A. Clearly, this was treated earlier in Lemma 17. Note this form also came up in Case 1, degree 48 above in this proof.

Degree-42-factor case 2 The residue class algebra is of $\mathbb{Q}$-dimension 672, leads to a Gram-matrix over a field F_Q of degree 168 as follows: The sum of the first and third diagonal element of the Gram-matrix turns out to have an irreducible minimal polynomial of degree $168 = 672/4$ and all the other entries of the Gram-matrix are polynomials in this element. The stabilizer of the Gram-matrix is $\langle ad\rangle \cong C_2$ under the action of A. The field F_Q has 40 real embeddings, 24 of which lead to a positive semidefinite real Gram-matrix. Since there is only one formal Gram-matrix in this case, the factor group $N_A(\langle ad\rangle)/\langle ad\rangle \cong V_4$ can be embedded into the automorphism group of F_Q over the rationals and distributes the relevant real embeddings into orbits of isometric real Gram-matrices, namely 6 orbits of 4 embeddings each. Indeed, these real matrices have 6 different traces, we end up with 6 classes of real Gram-matrices in this case. We note that this case was computationally hard.

First degree-4-factor case 2 The residue class algebra is of $\mathbb{Q}$-dimension 32 and decomposes into a direct sum of two fields of degree 16 over the rationals. Both fields lead to a Gram-matrix over a field F_Q totally real of degree 4 and have $C_2 \times D_6$ as stabilizers in A. Of the four real embeddings only two lead to positive semidefinite Gram-matrices. The two specializations of one formal Gram-matrix are inequivalent. Therefore, it is clear that the two formal Gram-matrices form an orbit under the normalizer of the stabilizer $\langle ad, d\rangle$ in A. In particular only one Gram-matrix has to be considered. The traces of the linear actions on 3-space are -1, 1, 1, so that this form should come up in Lemmas 17 and 20.

Second degree-4-factor case 2 The residue class algebra is of $\mathbb{Q}$-dimension 32 and decomposes into a direct sum of four fields of degree 8 over the rationals. Each field leads to a Gram-matrix over a field $F_Q \cong \mathbb{Q}[5^{1/4}]$ of degree 4 and have $C_2 \times D_{10}$ as stabilizers in A, all of them containing a and therefore have come up earlier (in the case $a \mapsto (1, -1, -1)$). From the earlier case all the formal Gram-matrices must be equivalent.

Third degree-4-factor case 2 The residue class algebra is of $\mathbb{Q}$-dimension 32. It is already a field and leads to a Gram-matrix over a field F_Q of degree 8 and has $\langle a, d\rangle \cong {C_2}^2$ as stabilizer in A. Therefore, it has come up earlier in the case $a \mapsto (-1, +1, +1)$, cf. Lemma 15.

Degree-2-factor case 2 The residue class algebra leads to four residue class fields of degree 4 over the rationals. The four resulting Gram-matrices all have stabilizer isomorphic to D_{10} and form four different fixed points for ad in accordance with the table of marks of A. They fall into two orbits under the action of $\langle a\rangle$, but lie in one orbit under A. The field F_Q, which is isomorphic to $\mathbb{Q}[5^{1/4}]$, has two real embeddings. These yield isometric icosahedra, the normalizer of the stabilizer modulo the stabilizer acts on the field interchanging the two real embeddings. □

Lemma 19. *In case $\delta(d) = \mathrm{diag}(-1, 1, 1)$ there are two formal* Gram*-matrices and four real* Gram*-matrices, with details listed in the following table:*

Stabilizer in A	$d_G, r_{1,G}$	r_G	$r_{f,G}$	*cf. also La.*
$C_2 \times D_{10}$	4, 2	2	2	*17, 18*
$C_2 \times D_6$	4, 4	2	2	*17, 18*

Proof. The coordinate matrix has its first row in the (row) eigenspace of the permutation matrix for d for the eigenvalue -1, which is of dimension 6, the second and third row lie in the eigenspace for the eigenvalue 1, which is also of dimension 6. Hence, we have 18 variables for the coordinate matrix. The center of mass condition reduces this number by 2. As usual the first column can be assumed to be of the form $(*, *, 0)^{tr}$. However, this leads to an infinite number of solutions for the coordinate matrices. The computation shows that the infinity occurs already if the first column is of the form $(*, 0, 0)^{tr}$. This is a case where the action of the 2-dimensional orthogonal group is still in operation, allowing us to assume that the second column is of the form $(*, *, 0)$, which then results into 4 solutions only and just one Gram-matrix. As for the other case where the second entry of the first column is non-zero, one gets also only finitely many Gram-matrices. All of them fall into A-orbits of lengths 6 and 10. This shows that the automorphism group must contain an element conjugate to a under A. Since all possibilities for linear actions of a have already been discussed, we can look up there, which cases for a contain d in its presently discussed linear action. Note, since d is central in A, the full A-orbits show up in our computation. This explains why the issue with the infinite number of solutions for the coordinate matrices did not show up at the two relevant places. □

Lemma 20. *In case $\delta(d) = \mathrm{diag}(1, -1, -1)$ the equational ideal I of Definition 6 is of dimension 1.*

Proof. The usual ansatz with one entry in the third row set zero to make the equations for the coordinate matrix rigid leads to an ideal of which a Groebner-basis can be computed by MAGMA. This can be turned into a Janet-basis yielding the following Hilbert series:

$$1 + 16t + 121t^2 + 576t^3 + 1625t^4 + 1987t^5 + 1540t^6 + 1371t^7 + 1323t^8 + 1320t^9 + \frac{1320t^{10}}{1-t}$$

telling us that the dimension is 1. We could not prove that the ideal was prime. As was to be expected, the Gram-matrix of Lemma 15 corresponds to a maximal ideal containing I. The assumption that two vertices coincide leads in each case to finitely many maximal ideals containing I so that generically we have twelve vertices. However, it is not clear at this stage whether we have infinitely many real solutions leading to positive semidefinite Gram-matrices. This will be proved in the next section. □

All these lemmas of this section taken together prove Theorem 9.

5 A curve of icosahedra

In this section we prove the remaining part of Theorem 1, namely that there are infinitely many $\langle d\rangle$-invariant icosahedra. The previous computations allow us to define a vector field, at least one of its integral curves consists of icosahedra:

Proposition 21. *There exist $\varepsilon > 0$ and a non-constant real analytic map $\Phi : [0,\varepsilon) \to \mathbb{R}^{3\times 12}$ such that $\Phi(t)$ is the coordinate matrix of a $\langle d\rangle$-invariant icosahedron for all but finitely many $t \in [0,\varepsilon)$. In particular, there exist infinitely many isometry types of $\langle d\rangle$-invariant icosahedra.*

Proof. Let

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_{15} = 0 \tag{1}$$

be the (linearly independent) quadratic equations in the unknown entries y_1, y_2, …, y_{16} of the coordinate matrix M taking into account the $\langle d\rangle$-symmetry (cf. also Section 3 as well as the comments about factoring out the action of the 3-dimensional orthogonal group at the beginning of Section 4 making the fibres of $M \mapsto M^{tr}M$ finite). Let $\mathrm{D}p$ be the Jacobian matrix of $p = (p_1, p_2, \ldots, p_{15})$. Due to the Laplace-expansion for determinants, the vector

$$\tau(y_1,\ldots,y_{16}) := \big(\det \mathrm{D}p|_1,\ \ldots,\ (-1)^{i-1}\det \mathrm{D}p|_i,\ \ldots,\ -\det \mathrm{D}p|_{16}\big)$$

satisfies

$$\tau(y_1,\ldots,y_{16})\,(\mathrm{D}p)^{tr} = 0,$$

where $\mathrm{D}p|_i$ is the square submatrix of $\mathrm{D}p$ that is obtained by omitting the i-th column. Hence, for each real solution $y^0 = (y^0_1,\ldots,y^0_{16})$ of (1) the evaluation of the vector τ at y^0 is tangent to the algebraic curve defined by (1) at the point y^0. Given any such solution y^0 such that $\tau(y^0)$ is non-zero, we consider the following initial value problem for $\phi(t) = (\phi_1(t),\ldots,\phi_{16}(t))$ on some interval containing 0:

$$\begin{cases} \phi'(t) &=& \tau(\phi_1(t),\ldots,\phi_{16}(t)), \\ \phi(0) &=& y^0. \end{cases}$$

By standard theorems on ordinary differential equations, there exist $\varepsilon > 0$ and a real analytic map $\phi : [0,\varepsilon) \to \mathbb{R}^{1\times 16}$ satisfying the above initial value problem. By substituting ϕ_i for y_i in M we obtain a candidate for a real analytic map $\Phi : [0,\varepsilon) \to \mathbb{R}^{3\times 12}$ as required. Note that it still needs to be checked whether the columns of $\Phi(t)$ are pairwise distinct and whether $\Phi(t)^{tr}\Phi(t)$ is a positive semi-definite Gram-matrix of rank 3. As for the first condition, Groebner basis computations (cf. also proof of Lemma 20) show that there are only finitely many (real) solutions of (1) such that two columns of the corresponding coordinate matrices are equal. (Hence, even if y^0 is chosen as a solution of (1) defining a degenerate icosahedron with two coincident vertices, the resulting $\Phi(t)$ will define an icosahedron with pairwise distinct vertices for sufficiently small $t > 0$.) Moreover, for sufficiently small $t > 0$ the conditions on $\Phi(t)^{tr}\Phi(t)$ follow from the corresponding ones satisfied by $\Phi(0)^{tr}\Phi(0)$ by continuity. Hence, by choosing a smaller ε if necessary, a real analytic map Φ as required is obtained.

Finally, since the map $t \mapsto \Phi(t)^{tr}\Phi(t)$ is not constant and since only finitely many isometry types of icosahedra correspond to a GRAM-matrix $\Phi(t)^{tr}\Phi(t)$, we conclude that Φ defines infinitely many pairwise inequivalent $\langle d\rangle$-invariant icosahedra. $\square$

We used the numeric ODE solver in Maple 2017 to find an approximate solution Φ starting with a $\langle d\rangle$-invariant icosahedron given in terms of algebraic numbers. The computation was carried out with a precision of 20 digits, resulting in approximate icosahedra with a residual error of at most 0.14×10^{-8} for 1000 time steps in the interval $[0, 0.00018]$.

6 Some geometric and combinatoric invariants

Beyond the automorphism group of an icosahedron we propose some geometric invariants which might give some idea of how the triangles of the icosahedron are arranged in 3-space.

Definition 22. *1.) Let Δ_1, Δ_2 be two equiangular triangles in* EUCLIDEAN *3-space. A point P in the affine space spanned by the vertices of Δ_1 and Δ_2 is called <u>central</u> for Δ_1, Δ_2 if one of the following three equivalent conditions is satisfied:*
i.) (Circumsphere) There is a sphere with midpoint P passing through all vertices of Δ_1 and Δ_2.
ii.) (Insphere) There is a sphere with midpoint P passing through all the midpoints of the edges of Δ_1 and Δ_2.
iii.) (Centersphere) There is a sphere with midpoint P intersecting the convex hulls of Δ_1 and Δ_2 tangentially in their centers of their incircles.
2.) Let X be an icosahedron in 3-space. A point P in 3-space is called <u>significant of strength</u> $k \geq 2$ for X, if there are k triangles of X such that P is central for any pair of them with the same insphere.

If two equiangular triangles with different midpoints have a central point, it is unique: In case the two triangles do not lie in parallel planes, it is the intersection of the orthogonal middle lines; in case they lie in different parallel planes, it is the midpoint of the centers of their incircles. Note, if the orthogonal middle lines of the triangles intersect, the point of intersection need not be a central point of the two triangles. If two equiangular triangles share exactly one edge, they are either coplanar or have a central point. The central points of an icosahedron can be viewed as a substitute of the midpoint of the regular icosahedron, as the example below will demonstrate. Since the defining equations for the significant points (together with the generators of the maximal ideal defining a formal GRAM-matrix) are all over the rationals, we have an obvious remark:

Remark 23. *1.) If two icosahedra belong to the same formal* GRAM*-matrix, they have the same number of k-significant points for any k.*

2.) Let s be the sum of the strengths of all significant points of an icosahedron. Then $20 \leq s$ *with equality if and only if the spheres of the significant points form a partition of the faces of the icosahedron.*
3.) There is a map from the set of bend edges of the icosahedron to its set of spheres or equivalently to its set of significant points, which takes the edge to the intersection of the two lines orthogonal to the faces passing through their centers.

Example 24. *1.) The regular icosahedron and its* GALOIS *conjugate have exactly one significant point. It is of strength 20.*
2.) The icosahedra with symmetry group $C_2 \times D_{10}$ *have three significant points. These have strengths* $5, 5, 10$*. The first two fall in one orbit under the symmetry group. The sets of triangles with the same central point form a partition of the set of faces of* X*.*
3.) Two neighbouring triangles in an icosahedron are either coplanar or give rise to a significant point of strength at least 2. This is the simplest way in which significant points arise. Significant points of bigger strength are clearly more interesting. We call a significant point trivial, if it is of strength 2 and its two associated triangles share an edge.
4.) There is a unique icosahedron with symmetry group D_{10} *and field degree* $d_G = 4$*. Here one computes easily that there is one significant point of strength 10, two of strength 5, and ten trivial ones. The five triangles associated to a significant point of strength 5 share a common vertex. These two vertices have combinatorial distance 3, i.e. form a cycle of d, the ten remaining triangles belong to the strength-10-point. The 10 trivial significant points bind together one triangle of the 10-belt with one triangle of the two 5-caps so that in the end each triangle belongs to exactly two significant points.*
5.) There is a unique icosahedron with symmetry group ${C_2}^2$ *and field degree* $d_G = 30$*. In this case all significant points are trivial: They are in obvious bijection with the edges of the icosahedron.*
6.) If a face has two equal face angles $\neq \pi$ *to its neighbours then it gives rise to a significant point of strength at least 3. In case it is not bigger we call it a 3-trivial significant point. If it has all three face angles to its neighbours equal and* $\neq \pi$*, it gives rise to a significant point of strength at least 4. In case it is not bigger we call it a 4-trivial significant point.*

Acknowledgement: The calculations were partly done with Maple[2]. We thank Bernd Schulze for helpful comments.

References

[1] Daniel J. Bates, Jonathan D. Hauenstein, Andrew J Sommese, and Charles W. Wampler. Bertini: Software for Numerical Algebraic Geometry. Available at `bertini.nd.edu` with permanent doi: dx.doi.org/10.7274/R0H41PB5.

[2]Maple is a trademark of Waterloo Maple Inc.

[2] Y.A. Blinkov, C.F. Cid, V.P. Gerdt, W. Plesken, D. Robertz. *The MAPLE Package "Janet": I. Polynomial Systems.* In *Proc. of Computer Algebra in Scientific Computing CASC 2003*, edited by V. G. Ganzha, E. W. Mayr, E. V. Vorozhtsov, 31–40. Garching, Germany: Institut für Informatik, TU München, 2003. Also available together with the package from `http:///algebra.data.rwth-aachen.de/software/Janet`.

[3] Y.A. Blinkov, V.P. Gerdt. The specialised computer algebra system GINV. *Programmirovanie* **34** (2), 67–80 (2008). `http://invo.jinr.ru`

[4] W. Bosma, J. Cannon, C. Playoust. The MAGMA algebra system. I. The user language, *J. Symbolic Comput.* **24** (3-4), 235–265 (1997).

[5] Karl-Heinz Brakhage, Alice C. Niemeyer, Wilhelm Plesken, Daniel Robertz, Ansgar Strzelczyk. Documentation of Paper: The icosahedra of edge length 1, accessible via webpage: `http://algebra.data.rwth-aachen.de/Icosahedra/visualplusdata.html`, (2019).

[6] Bernd Schulze. Symmetry as a sufficient condition for a finite flex. *SIAM J. Discrete Math.* 24 (2010), no. 4, 1291–1312.