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THICK SUBCATEGORIES OF DISCRETE DERIVED CATEGORIES

NATHAN T BROOMHEAD

ABSTRACT. We classify the thick subcategories of discrete derived categories. To do this we introduce certain generating sets called ES-collections which correspond to configurations of non-crossing arcs on a geometric model. We show that every thick subcategory is generated by an ES-collection and we describe a version of mutation which acts transitively on the set of ES-collections generating a given thick subcategory.

CONTENTS

1. Background on discrete derived categories and calculation of cones	4
2. Generators for thick subcategories	9
3. A geometric model	10
4. ES-collections	22
5. Morphisms in ES-collections and factoring arcs	25
6. Reduced collections	27
7. Mutation of ES-collections	30
8. The lattice of thick subcategories	34
Appendix A. The Grothendieck group	43

INTRODUCTION

In this article, we study thick subcategories of discrete derived categories $D^b(\text{mod}(\Lambda))$ (introduced by Vossieck in [27]), in the case where Λ has finite global dimension and is not of derived-finite representation type. By a *thick* subcategory, we mean a triangulated subcategory which is closed under taking direct summands. The set of all thick subcategories of any essentially small triangulated category forms a lattice with respect to the partial order given by inclusion, and this is an interesting invariant of the category.

The study of thick subcategories has a long history and descriptions of the lattice exist in the literature in various contexts. For example, Devinatz, Hopkins and Smith treated certain stable homotopy categories [11, 16], Hopkins [17] and Neeman [23] considered the category of perfect complexes over a commutative noetherian ring, and Thomason [26] generalised this to perfect complexes over quasi-compact quasi-separated schemes. More recently there has been further interesting work by many authors; see for example [3, 4, 24]. However, all of these results depend in some way on having a tensor structure, and without such a structure the set of examples is more limited.

If A is a finite dimensional algebra over a field (or more generally for certain connected hereditary Artin algebras), then there is a classification of the thick subcategories of $D^b(\text{mod}(A))$ which are generated by exceptional collections. This was due to Ingalls and Thomas [19] for the path algebra of Dynkin or extended Dynkin type, and generalised by

keywords: Discrete derived category, thick subcategory, geometric model, exceptional collection, ES-collection, mutation.

Igusa, Schiffler, Krause, Antieau and Stevenson [18, 21, 1]. In the case when the algebra A is of Dynkin type, all thick subcategories are generated by exceptional collections, and so this forms a complete classification.

By considering the thick subcategories generated by exceptional collections and those generated by regular objects, there is also a classification of thick subcategories of derived categories of tame hereditary algebras (see work of Brüning, Dichev and Köhler [9, 12, 20]). However, the two types of thick subcategories are not treated in a uniform way, and so this classification doesn't yield the lattice structure. Finally, Takahashi gave a classification of the thick subcategories of the singularity categories of local hypersurfaces, which didn't use a tensor structure [25].

In this paper we add to this rather short list, a three parameter family of examples for which the classification is complete, and the lattice structure is understood. Discrete derived categories form a class of triangulated categories which are complicated enough to exhibit many interesting properties but at the same time are tractable enough that calculations can be done explicitly. Their structure has been studied by Bobiński, Geiß, and Skowroński [5] and more recently by the author with Pauksztello and Ploog [7, 8], and this detailed understanding underpins the proofs of the classification in this paper. I believe however, that many of the techniques will in fact generalise to bounded derived categories of a wider class of finite dimensional algebras, including tame hereditary algebras and examples which are not of finite global dimension.

Motivating Example. The classification presented here, is motivated most strongly by the classification of thick subcategories of $\mathbf{D} = \mathbf{D}^b(\mathbf{mod}(\mathbf{k}A_n))$ where $\mathbf{k}A_n$ is a path algebra of Dynkin quiver of type A_n . We briefly recall this example. Let $\mathbf{Thick}_{\mathbf{D}}$, $\mathbf{Exc}_{\mathbf{D}}^{\text{mut}}$ and $\mathbf{NC}(W_{A_n}, c)$ denote respectively, the lattice of thick subcategories, the lattice of exceptional collections up to mutation, and the lattice of non-crossing partitions. Since the algebra is representation finite, the classification by Ingalls and Thomas yields the following correspondences:

$$\mathbf{Thick}_{\mathbf{D}} \longleftrightarrow \mathbf{Exc}_{\mathbf{D}}^{\text{mut}} \longleftrightarrow \mathbf{NC}(W_{A_n}, c).$$

The non-crossing partitions can be seen naively as follows, by work of Brady (see [6]). We consider a disc with $n + 1$ marked points on the boundary. A partition of the set of marked points is a non-crossing partition, if chords connecting points in one subset of the partition, do not intersect any chords between points in a different subset of the partition. Exceptional collections and their mutations can also be seen in the model by work of Araya [2], which links them to trees of non-crossing chords.

This is the picture that we generalise for discrete derived categories.

Theorem 0.1 (See Theorem 8.1). *Let $\mathbf{D} = \mathbf{D}^b(\Lambda(r, n, m))$ be a discrete derived category with $n > r > 1$. Then there are isomorphisms of partially ordered sets:*

$$\mathbf{Thick}_{\mathbf{D}} \xrightarrow{\cong} \mathbf{ES}_{\mathbf{D}}^{\text{mut}} \xrightarrow{\cong} \mathbf{Arc}_{\mathbf{D}}^{\text{mut}}$$

- $\mathbf{Thick}_{\mathbf{D}}$, the lattice of thick subcategories ordered by inclusion,
- $\mathbf{ES}_{\mathbf{D}}^{\text{mut}}$, the lattice of reduced ES-collections up to mutation,
- $\mathbf{Arc}_{\mathbf{D}}^{\text{mut}}$, the lattice of reduced arc-collections up to mutation.

The structure of the paper is as follows. In Section 1 we recall some of the key properties of discrete derived categories and use these to calculate explicitly the cones of irreducible morphisms between indecomposable objects. In Section 2 we start by finding sets of generating objects for any thick subcategory. It is clear that we can't restrict to

considering exceptional collections as, for example, any discrete derived category contains spherelike objects ([7, Proposition 6.4]) in the sense of [15], and the thick subcategories that these generate contain no exceptional objects. We show however, that the situation doesn't get more complicated than this, as any thick subcategory is generated by a finite set of exceptional and spherelike objects (Corollary 2.3).

In Section 3 we introduce a geometric model, which will play the role of the disc with marked boundary points above. In the model, indecomposable objects of the discrete derived category, up to isomorphism and the action of the suspension functor, correspond to arcs. We can also calculate the dimensions of certain spaces of morphisms in terms of intersection numbers of arcs (Theorem 3.28).

In order to understand the thick subcategories, we need to be able to identify when two sets of exceptional and spherelike objects generate the same thick subcategory. Starting with arbitrary finite sets, this would be an extremely hard problem; to get a clean theory in the case of $D^b(\text{mod}(A))$ for A of finite representation type, one considers exceptional collections, rather than just sets of exceptional objects. Motivated by Araya's description of exceptional collections in terms of non-crossing trees [2], we show in Section 4 that it is possible to restrict our attention to particular sets of exceptional and spherelike objects which we call *ES-collections*. These correspond to certain collections of non-crossing arcs on the geometric model (Lemma 4.4) which we call arc-collections. We prove that every thick subcategory is generated by an ES-collection (Theorem 4.9).

There are two properties of exceptional collections which make them particularly nice to deal with, as generators of thick subcategories:

- i) "Minimality": we don't have more objects than we need to generate a given thick subcategory.
- ii) "Existence of mutations": we can move between different exceptional collections in a prescribed way, such that the generated thick subcategory is preserved.

ES-collections as we defined them do not necessarily satisfy the first of these properties, so we build it into the definition of a *reduced* ES-collection, as an extra condition. We define a reduced arc-collection analogously and in Section 6 we show that the correspondence between the reduced collections holds (Corollary 6.3). In Section 7 we define mutation for reduced ES-collections, so they satisfy both of the nice properties *i*) and *ii*) above. It is these reduced ES- and arc-collections that are analogous to, and significant generalisations of, the exceptional collections and non-crossing trees studied by Araya in the $D^b(\text{mod}(\mathbf{k}A_n))$ case [2]. Finally, in Section 8 we complete the proof of the main theorem.

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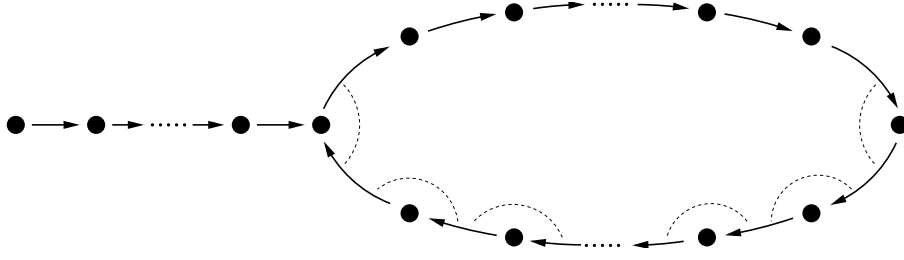


FIGURE 1. The quiver consists of a tail of length m , an oriented cycle of length n and r consecutive zero relations on the cycle.

1. BACKGROUND ON DISCRETE DERIVED CATEGORIES AND CALCULATION OF CONES

We always work over a fixed algebraically closed field \mathbf{k} . In this paper we will consider the thick subcategories of certain discrete derived categories in the sense of Vossieck [27].

Definition 1.1. A derived category \mathbf{D} is *discrete*, if for every map $v: \mathbb{Z} \rightarrow K_0(\mathbf{D})$ there are finitely many isomorphism classes of objects $D \in \mathbf{D}$ with $[H^i(D)] = v(i) \in K_0(\mathbf{D})$ for all $i \in \mathbb{Z}$.

Categories of the form $\mathbf{D} = \mathbf{D}^b(\Lambda)$ where Λ is the path algebra of an ADE type Dynkin quiver are of derived-finite representation type and clearly satisfy this definition. It was shown by Bobiński, Geiß, and Skowroński that for a connected algebra Λ which is not of Dynkin type, then $\mathbf{D}^b(\Lambda)$ is discrete if and only if it is equivalent to a category $\mathbf{D}^b(\Lambda(r, n, m))$, where $\Lambda(r, n, m)$ is the path algebra of a quiver with relations of the form give in Figure 1 [5].

As we mentioned in the introduction, the thick subcategories of derived categories of Dynkin type algebras are already understood, so we will consider only the discrete derived categories of the form $\mathbf{D}^b(\Lambda(r, n, m))$. From now on, we will restrict to the case when $\Lambda(r, n, m)$ has finite global dimension, which happens if and only if $n > r$. In the latter part of this paper, we will add the assumption that $r > 1$. This removes some pathological cases requiring different arguments, while retaining a new three parameter family of examples, for which the classification of thick subcategories is now understood.

Let $\mathbf{D} = \mathbf{D}^b(\Lambda(r, n, m))$ be a discrete derived category of finite global dimension. The Auslander-Reiten quiver of \mathbf{D} was studied in [5]. We recall here without proof, some of the most important facts about its structure, and set up some notation and conventions that will be used in the following.

The AR quiver of $\mathbf{D}^b(\Lambda(r, n, m))$ has $3r$ components [5, Theorem B], which we denote by

$$\mathcal{X}^0, \dots, \mathcal{X}^{r-1}, \quad \mathcal{Y}^0, \dots, \mathcal{Y}^{r-1}, \quad \mathcal{Z}^0, \dots, \mathcal{Z}^{r-1}.$$

Each \mathcal{X}^i and \mathcal{Y}^i is of type $\mathbb{Z}A_\infty$, and each \mathcal{Z}^i is of type $\mathbb{Z}A_\infty^\infty$. We define \mathcal{X} to be the additive subcategory generated by the indecomposable objects in $\bigcup_{i=0}^{r-1} \mathcal{X}^i$, and define \mathcal{Y} and \mathcal{Z} analogously. For each $k = 0, \dots, r-1$, we use the following coordinates on the indecomposable objects in $\mathcal{X}^k, \mathcal{Y}^k, \mathcal{Z}^k$:

$$\begin{aligned} X^k(i, j) &\in \mathcal{X}^k \text{ for } i, j \in \mathbb{Z}, j \geq i; \\ Y^k(i, j) &\in \mathcal{Y}^k \text{ for } i, j \in \mathbb{Z}, i \geq j; \\ Z^k(i, j) &\in \mathcal{Z}^k \text{ for } i, j \in \mathbb{Z}. \end{aligned}$$

Sometimes we will write the coordinates as subscripts to save space, so for example $Z^k(i, j) = Z_{i,j}^k$. The indecomposable objects in any given component which have the same first (respectively second) coordinate are said to lie on a ray (respectively coray).

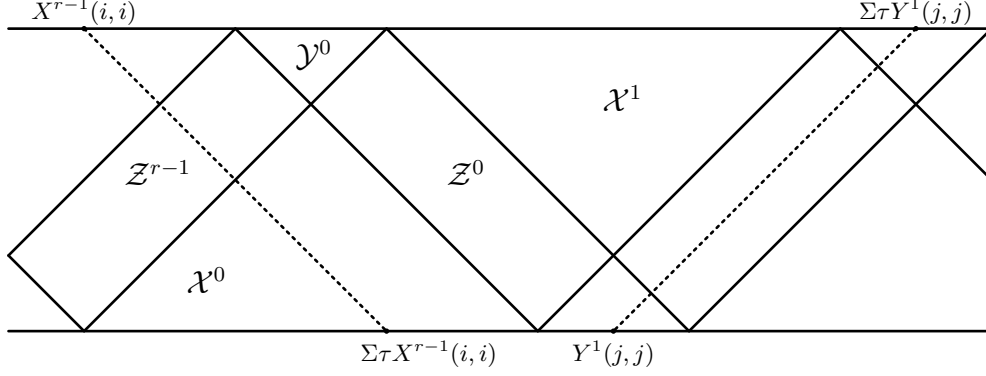
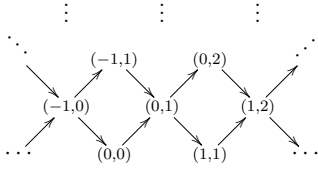
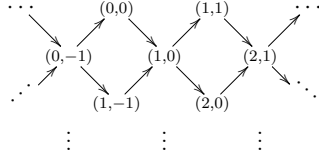


FIGURE 2. Schematic diagram showing how the components of the AR quiver fit together. The dotted lines are two examples of long (co)rays.

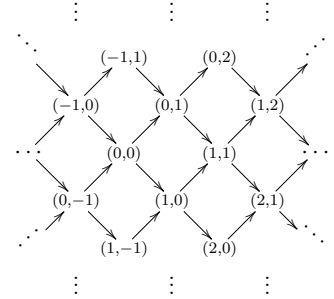
\mathcal{X} component:



\mathcal{Y} component:



\mathcal{Z} component:



Properties 1.2. (1) There are irreducible morphisms from an object with coordinate (i, j) to the objects with coordinates $(i + 1, j)$ and $(i, j + 1)$ in the same component (if such objects exist). For example, there are always irreducible morphisms from $Z^k(i, j)$ to $Z^k(i + 1, j)$ and $Z^k(i, j + 1)$.

(2) The AR translate takes an object with coordinate (i, j) to the object with coordinate $(i - 1, j - 1)$ in the same component, for example $\tau Z^k(i, j) = Z^k(i - 1, j - 1)$.

(3) On objects we have $\Sigma^r|_{\mathcal{X}} = \tau^{-m-r}$ and $\Sigma^r|_{\mathcal{Y}} = \tau^{n-r}$.

(4) There sequences of irreducible morphisms for any $i \in \mathbb{Z}$ and $k = 0, \dots, r - 1$:

$$X_{ii}^k \rightarrow X_{i,i+1}^k \rightarrow \dots \rightarrow Z_{i,i-1}^k \rightarrow Z_{ii}^k \rightarrow Z_{i,i+1}^k \rightarrow \dots \rightarrow \Sigma X_{i-3,i-1}^k \rightarrow \Sigma X_{i-2,i-1}^k \rightarrow \Sigma X_{i-1,i-1}^k,$$

$$Y_{ii}^k \rightarrow Y_{i+1,i}^k \rightarrow \dots \rightarrow Z_{i-1,i}^k \rightarrow Z_{ii}^k \rightarrow Z_{i+1,i}^k \rightarrow \dots \rightarrow \Sigma Y_{i-1,i-3}^k \rightarrow \Sigma Y_{i-1,i-2}^k \rightarrow \Sigma Y_{i-1,i-1}^k,$$

such that the composition of any number of consecutive morphisms from the sequence is non-zero. We say that such morphisms are along a *long (co)ray* (see Figure 2).

(5) There are distinguished triangles, for any $i, j, d \in \mathbb{Z}$ with $d \geq 0$:

$$\begin{aligned} X^k(i, i + d) &\longrightarrow Z^k(i, j) \longrightarrow Z^k(i + d + 1, j), \\ Y^k(j + d, j) &\longrightarrow Z^k(i, j) \longrightarrow Z^k(i, j + d + 1). \end{aligned}$$

(6) Given any two indecomposable objects A, B . If $r \geq 2$, then $\dim \text{Hom}(A, B) \leq 1$ and if $r = 1$, then $\dim \text{Hom}(A, B) \leq 2$.

(7) We define the *height* of an object $X^k(i, j)$ to be $j - i$ and the height of $Y^k(i, j)$ to be $i - j$. Objects of height 0 are said to be on the *mouth* of a component.

We now use these properties to calculate explicitly the cones of a basis of morphisms between indecomposable objects in the category. We start by considering morphisms which factor along a ray or coray within a component.

Lemma 1.3. *Let $X := X^c(i, j)$, $Y := Y^c(i', j')$ and $a, b > 0$. The morphisms $f_1 : X \rightarrow X^c(i, j + b)$ along the ray in \mathcal{X}^c and $g_1 : Y \rightarrow Y^c(i' + a, j')$ along the coray in \mathcal{Y}^c fit into triangles:*

$$(1) \quad X \rightarrow X^c(i, j + b) \rightarrow X^c(j + 1, j + b)$$

$$(2) \quad Y \rightarrow Y^c(i' + a, j') \rightarrow Y^c(i' + a, i' + 1).$$

Suppose additionally that $a \leq j - i = ht(X)$ and $b \leq i' - j' = ht(Y)$. The morphisms $g_2 : X \rightarrow X^c(i + a, j)$ along the coray in \mathcal{X}^c and $f_2 : Y \rightarrow Y^c(i', j' + b)$ along the ray in \mathcal{Y}^c fit into triangles:

$$(3) \quad X \rightarrow X^c(i + a, j) \rightarrow \Sigma X^c(i, i + a - 1)$$

$$(4) \quad Y \rightarrow Y^c(i', j' + b) \rightarrow \Sigma Y^c(j' + b - 1, j').$$

Proof. We prove the first statement for X , but the other statements follow in an analogous way. We use induction on b . The cases where $b = 1$ is exactly that of [7, Lemma 3.2]. Now suppose $b > 1$. We consider f_1 as a composition $f_1 : X \rightarrow X^c(i, j + b - 1) \rightarrow X^c(i, j + b)$ along the ray and construct the following diagram using the octahedral axiom.

$$\begin{array}{ccccc} & & X & \xlongequal{\quad\quad\quad} & X \\ & & f_1 \downarrow & & \downarrow f_1 \\ \Sigma^{-1}X^c(j + b, j + b) & \xrightarrow{h'} & X^c(i, j + b - 1) & \xrightarrow{f_1''} & X^c(i, j + b) \\ & \parallel & \downarrow h'' & & \downarrow \\ \Sigma^{-1}X^c(j + b, j + b) & \xrightarrow{h} & X^c(j + 1, j + b - 1) & \longrightarrow & \text{Cone}(f_1) \end{array}$$

The cone of f_1' and the cocone of f_1'' are known by the induction hypothesis. The morphism h' is non-zero and must be the composition of morphisms along a long (co)ray of the form seen in Properties 1.2(4). (Note that if $r = 1$ there is potentially a second morphism $\Sigma^{-1}X^c(j + b, j + b) \rightarrow X^c(i, j + b - 1)$, but this morphism would factor along the ray in the component and would have a non-zero composition with f_1'' .) Since h'' is also the composition of morphisms along a coray, it follows that h is again a composition of morphisms along a long (co)ray and so is non-zero. Thus $\text{Cone}(f_1)$ is isomorphic to the cone of h where h is the unique (up to scaling) morphism from $\Sigma^{-1}X^c(j + b, j + b)$ to $X^c(j + 1, j + b - 1)$ which is in the infinite radical. Using the induction hypothesis once more, we see that this morphism fits into the following triangle

$$X^c(j + 1, j + b - 1) \rightarrow X^c(j + 1, j + b) \rightarrow X^c(j + b, j + b)$$

and so it follows that $\text{Cone}(f_1) \cong X^c(j + 1, j + b)$ as required. \square

Now we consider morphisms which factor through the irreducible morphisms within a component, but which don't factor along a long (co)ray.

Lemma 1.4. *Let $X := X^c(i, j)$ and let $a, b > 0$ be such that $a \leq j - i = ht(X)$. Then the morphism $f : X \rightarrow X^c(i + a, j + b)$ which factors through irreducible morphisms in the \mathcal{X}^c -component fits into a triangle:*

$$X \rightarrow X^c(i + a, j + b) \rightarrow X^c(j + 1, j + b) \oplus \Sigma X^c(i, i + a - 1).$$

Let $Y := Y^c(i, j)$ and let $a, b > 0$ be such that $b \leq i - j = ht(Y)$. Then the morphism $f : Y \rightarrow Y^c(i + a, j + b)$ which factors through irreducible morphisms in the \mathcal{Y}^c -component fits into a triangle:

$$Y \rightarrow Y^c(i + a, j + b) \rightarrow Y^c(i + a, i + 1) \oplus \Sigma Y^c(j + b - 1, j).$$

Let $Z := Z^c(i, j)$ be an object in some \mathcal{Z} component and let $a, b > 0$. Then the morphism $f : Z \rightarrow Z^c(i + a, j + b)$ which factors through irreducible morphisms in the \mathcal{Z}^c -component fits into a triangle:

$$Z \rightarrow Z^c(i + a, j + b) \rightarrow \Sigma X^c(i, i + a - 1) \oplus \Sigma Y^c(j + b - 1, j).$$

Proof. Again we prove the first statement but the others follow in an analogous way. Using the mesh relations in the component, we see that the morphism $f : X \rightarrow X^c(i + a, j + b)$ factors through $X^c(i, j + b)$. We construct the following diagram, using Lemma 1.3 for the middle row and column.

$$\begin{array}{ccccc} & & X & \xlongequal{\quad\quad\quad} & X \\ & & \downarrow & & \downarrow f \\ X^c(i, i + a - 1) & \xrightarrow{h'} & X^c(i, j + b) & \longrightarrow & X^c(i + a, j + b) \\ & & \downarrow h'' & & \downarrow \\ X^c(i, i + a - 1) & \xrightarrow{h} & X^c(j + 1, j + b) & \longrightarrow & \text{Cone}(f) \end{array}$$

The morphism h' (respectively h'') is a finite composition of irreducible morphisms along a ray (respectively coray) in the component. Using the mesh relations it is clear that the composition h should factor through an object $X^c(j + 1, i + a - 1)$, but this would have height $i - j + a - 2 \leq -2$. Therefore it factors through a zero relation at the mouth of the component and the bottom triangle in the diagram splits as required. \square

Now we consider the cones of morphisms from an \mathcal{X} or \mathcal{Y} component to a \mathcal{Z} component.

Lemma 1.5. *Let $X := X^c(i, j)$, $k \in \mathbb{Z}$ and suppose $0 < a \leq j - i = ht(X)$. The morphism $f : X \rightarrow Z^c(i + a, k)$ fits into a triangle:*

$$X \rightarrow Z^c(i + a, k) \rightarrow Z^c(j + 1, k) \oplus \Sigma X^c(i, i + a - 1).$$

Let $Y := Y^c(i, j)$, $k \in \mathbb{Z}$ and suppose $0 < b \leq i - j = ht(Y)$. Then the morphism $f : Y \rightarrow Z^c(k, j + b)$ fits into a triangle:

$$Y \rightarrow Z^c(k, j + b) \rightarrow Z^c(k, i + 1) \oplus \Sigma Y^c(j + b - 1, j).$$

Proof. We prove the first statement but the other is proved analogously. We note that $\Sigma X^c(i, i + a - 1)$ lies on the ray through ΣX at a height below that of ΣX . In particular there is no morphism from X to $\Sigma X^c(i, i + a - 1)$ which lies in the infinite radical. Using this, together with the triangle from Properties 1.2(5),

$$Z^c(i, k) \rightarrow Z^c(i + a, k) \rightarrow \Sigma X^c(i, i + a - 1)$$

we deduce that f factors through $Z^c(i, k)$. This composition then fits into the following diagram:

$$\begin{array}{ccccc}
& & X & \xlongequal{\quad\quad\quad} & X \\
& & \downarrow & & \downarrow f \\
X^c(i, i+a-1) & \longrightarrow & Z^c(i, k) & \xrightarrow{g} & Z^c(i+a, k) \\
\parallel & & \downarrow h'' & & \downarrow \\
X^c(i, i+a-1) & \xrightarrow{h} & Z^c(j+1, k) & \longrightarrow & \text{Cone}(f)
\end{array}$$

where the middle vertical triangle is again a triangle from Properties 1.2(5). Noting that $j+1 > i+a$ we observe that h'' factors through g and so $h = 0$. Therefore the bottom triangle in the diagram splits as required. \square

Now we consider the cones of morphisms between two indecomposable objects in \mathcal{X} (respectively \mathcal{Y} , \mathcal{Z}) which factor through the infinite radical. For a given pair of indecomposable objects, such morphisms are unique up to scaling so the cones can be read off from the following (non-split) triangles.

Lemma 1.6. *Let $X := X^c(i, j)$ and let $a, b > 0$ be such that $a \leq j - i = ht(X)$. Then there is a triangle:*

$$X \rightarrow X^c(i+a, j) \oplus X^c(i, j+b) \rightarrow X^c(i+a, j+b)$$

Let $Y := Y^c(i, j)$ and let $a, b > 0$ be such that $b \leq i - j = ht(Y)$. Then there is a triangle:

$$Y \rightarrow Y^c(i+a, j) \oplus Y^c(i, j+b) \rightarrow Y^c(i+a, j+b)$$

Let $Z := Z^c(i, j)$ and let $a, b > 0$. Then there is a triangle:

$$Z \rightarrow Z^c(i+a, j) \oplus Z^c(i, j+b) \rightarrow Z^c(i+a, j+b)$$

Proof. Again we give the proof of the first case but the others are analogous. We construct the following diagram, starting with the commuting square in the top right, where f and g are the obvious morphisms along the ray and coray in the component. The middle column of the diagram is a split triangle and the right hand column is calculated using Lemma 1.3.

$$\begin{array}{ccccc}
& & X^c(i+a, j) & \xlongequal{\quad\quad\quad} & X^c(i+a, j) \\
& & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \downarrow & & \downarrow f \\
X^c(i, j) & \longrightarrow & X^c(i+a, j) \oplus X^c(i, j+b) & \xrightarrow{(fg)} & X^c(i+a, j+b) \\
\parallel & & \downarrow & & \downarrow h \\
X^c(i, j) & \longrightarrow & X^c(i, j+b) & \xrightarrow{hg} & X^c(j+1, j+b)
\end{array}$$

The morphism hg is the composition of morphisms along a coray, and so the bottom row of the diagram can be calculated using Lemma 1.3. The middle row is then the desired triangle. \square

Finally we consider the cones of morphisms from a \mathcal{Z} component to an \mathcal{X} or \mathcal{Y} component. Since the Hom-spaces between indecomposable objects in different components are one dimensional, these cones can be read off from the following (non-split) triangles.

Lemma 1.7. *Let $Z := Z^c(i, j)$ be an object in some \mathcal{Z} component and let $a > 0, b \geq 0$. Then there are triangles:*

$$(5) \quad X^c(i - a, i + b) \rightarrow X^c(i, i + b) \oplus Z^c(i - a, j) \rightarrow Z$$

$$(6) \quad Y^c(j + b, j - a) \rightarrow Y^c(j + b, j) \oplus Z^c(i, j - a) \rightarrow Z.$$

Proof. This proceeds in the same way as the proof of Lemma 1.6. \square

2. GENERATORS FOR THICK SUBCATEGORIES

We would like to show that thick subcategories of discrete derived categories are generated by finite sets of exceptional and spherelike objects. We do this in two steps.

Proposition 2.1. *Let T be a thick subcategory of a discrete derived category D . Then at least one of the following holds:*

- i) T is generated by an exceptional collection,
- ii) $\mathsf{T} \subset \mathcal{X} \cup \mathcal{Y}$.

Proof. We show that any thick subcategory of D which intersects \mathcal{Z} is generated by an exceptional collection. Suppose T intersects \mathcal{Z} so it contains some indecomposable object $Z \in \mathcal{Z}$. By [7, Proposition 6.4] we know that this object is exceptional. Let $Z = \mathbf{thick}_{\mathsf{D}}(Z)$ and consider the right orthogonal subcategory Z^\perp . For the semi-orthogonal decomposition $\langle Z^\perp, Z \rangle$ of D there is an equivalence $Z^\perp \simeq \mathsf{D}^b(\mathbf{k}A_{n+m-1})$ by [7, Proposition 7.6]. We consider the intersection $\mathsf{T} \cap Z^\perp$ which is a thick subcategory of Z^\perp . Any thick subcategory of $\mathsf{D}^b(\mathbf{k}A_{n+m-1})$ can be generated by an exceptional collection (see for example [21, Proposition 6.6]), so $\mathsf{T} \cap Z^\perp$ must be generated by an exceptional collection (E_1, \dots, E_a) in Z^\perp . We note that (E_1, \dots, E_a, Z) is an exceptional collection in D and that $\mathbf{thick}_{\mathsf{D}}(E_1, \dots, E_a, Z) \subset \mathsf{T}$. Conversely, for any object $A \in \mathsf{T}$, using the semi-orthogonal decomposition there is a triangle

$$B \rightarrow A \rightarrow C$$

where $B \in Z$ and $C \in Z^\perp$. Since $A, B \in \mathsf{T}$ it follows that $C \in \mathsf{T} \cap Z^\perp$ and so $C \in \mathbf{thick}_{\mathsf{D}}(E_1, \dots, E_a)$. Using the triangle, it is then clear that $A \in \mathbf{thick}_{\mathsf{D}}(E_1, \dots, E_a, Z)$ and so T is generated by an exceptional collection. \square

Now we look at the thick subcategories $\mathsf{T} \subset \mathcal{X} \cup \mathcal{Y}$. Since \mathcal{X} and \mathcal{Y} are fully orthogonal we can consider them separately.

Lemma 2.2. *Any thick subcategory of \mathcal{X} (respectively \mathcal{Y}) can be generated by finitely many exceptional and spherelike objects.*

Proof. We prove the statement for a thick subcategory of \mathcal{X} . The \mathcal{Y} case is analogous. Consider an indecomposable object $X = X^c(i, j) \in \mathcal{X}$ which has height

$$ht(X) = j - i \geq r + m.$$

The r th shift is the object $\Sigma^r X = X^c(i + r + m, j + r + m)$ and since $ht(X) \geq r + m$, there is a non-zero morphism between X and $\Sigma^r X$ [7, Lemma 6.3]. Setting $a = b = r + m$ in Lemma 1.4 we see that the cone of this morphism is

$$X' \oplus \Sigma X'' := X^c(j + 1, j + r + m) \oplus \Sigma X^c(i, i + r + m - 1)$$

The objects both have height $r + m - 1$ and so are spherelike by [7, Proposition 6.4]. We claim that $\mathbf{thick}_{\mathsf{D}}(X) = \mathbf{thick}_{\mathsf{D}}(X', X'')$. Since X', X'' are by definition summands of a self extensions of X , it is clear that $\mathbf{thick}_{\mathsf{D}}(X', X'') \subset \mathbf{thick}_{\mathsf{D}}(X)$.

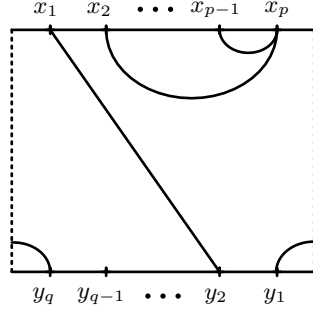


FIGURE 3. The cylinder $C(p, q)$ with marked boundary points and some examples of arcs. The dotted lines on the left and right are identified.

Consider the objects $X(k) := X^c(i, i - 1 + k(r + m))$ for $k \geq 1$. Using Lemma 1.3 there is a triangle,

$$X(k) \rightarrow X(k + 1) \rightarrow \Sigma^{kr} X''.$$

When $k = 1$ the left hand object is X'' , and so by induction, we see that $X(k) \in \text{thick}_{\mathbb{D}}(X', X'')$ for all $k \geq 1$. We choose k to be maximal such that $i - 1 + k(r + m) \leq j$. Note that the assumption on the height of X assures that $k \geq 1$. If $i - 1 + k(r + m) = j$ then $X = X(k)$ and we are done. Otherwise, we may apply Lemma 1.6 with $a = (j - i) - (r + m) + 1 > 0$ and $b = j - (i - 1 + k(r + m)) > 0$ and obtain a triangle

$$X(k) \rightarrow X^c(j + 1 - (r + m), i - 1 + k(r + m)) \oplus X \rightarrow \Sigma^{-r} X'.$$

It follows that $X \in \text{thick}_{\mathbb{D}}(X', X'')$ as required.

The category \mathcal{X} has a countable number of indecomposable objects and it follows that any thick subcategory is generated by a countable number of indecomposable objects. Any such object can be replaced by objects of height $\leq r + m - 1$ using the argument above. There are a finite number of \mathcal{X} -components and Σ^r (which preserves height) can be used to translate any object to one where the first coordinate lies in the interval $[0, m + r)$. Therefore, up to shift there are finitely many objects of height $\leq r + m - 1$ and by [7, Proposition 6.4] they are precisely the exceptional and spherelike objects in \mathcal{X} . \square

Putting together Proposition 2.1 and Lemma 2.2 we get the following corollary.

Corollary 2.3. *Any thick subcategory of a discrete derived category of finite global dimension can be generated by finitely many exceptional and spherelike objects.*

3. A GEOMETRIC MODEL

The geometric model that we introduce in this section will give a way of visualising our categories up to the action of the suspension functor Σ . In the model, indecomposable objects of \mathbb{D} up to isomorphism and the action of Σ , will correspond to ‘arcs’ on a cylinder. We will also see that the dimensions of certain spaces of morphisms are given by intersection numbers of arcs. The arcs in this geometric model are unoriented and ungraded.

3.1. The model. Let $C(p, q) \cong S^1 \times [0, 1]$ be a cylinder, with p marked points on the boundary circle $\delta_X := S^1 \times \{0\}$ and q marked points on the boundary circle $\delta_Y := S^1 \times \{1\}$. We label these points by $\{x_1, \dots, x_p\}$ and $\{y_1, \dots, y_q\}$ as in Figure 3.

We define an arc on the cylinder as follows.

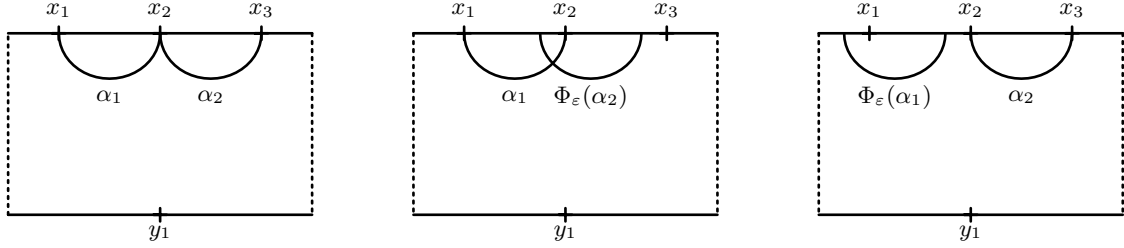


FIGURE 4. The cylinder $C(3,1)$ with arcs α_1, α_2 , chosen such that $\iota(\alpha_2, \alpha_1) = 1$, but $\iota(\alpha_1, \alpha_2) = 0$.

Definition 3.1. Let $I = [0, 1] \subset \mathbb{R}$ be the closed unit interval. An *arc* α is a smooth map $\alpha: I \rightarrow C(p, q)$ with the property that $\alpha(0)$ and $\alpha(1)$ are in the set of marked boundary points.

We say that two arcs α_1, α_2 which have the same end-points are *homotopy equivalent*, written $\alpha_1 \simeq \alpha_2$, if there is an homotopy between them, which fixes the end-points. We will sometimes consider equivalence up to reparametrisation (which may exchange the end-points) and homotopy equivalence, and we will denote this by $\alpha_1 \simeq \alpha_2^{\pm 1}$. An arc which is homotopic to a constant arc at any marked point, such as $\alpha: I \rightarrow C(p, q), t \mapsto x_1$ is called a *trivial* arc.

We say that two arcs are in *minimal position* if they intersect transversally at points which are not self-intersection points of either of the arcs, and one can't decrease the number of such intersection points by taking different representatives in their respective homotopy equivalence classes.

For any $\varepsilon > 0$, there is an ambient isotopy of $C(p, q)$:

$$\Phi_\varepsilon: C(p, q) \times [0, 1] \rightarrow C(p, q) \quad ((\theta, x), t) \mapsto \left(\theta + 2t \left(x - \frac{1}{2} \right) \varepsilon, x \right).$$

This has the effect of rotating the boundary circles by ε in opposite directions, and acts linearly on the rest of the cylinder. Note that this doesn't preserve the marked points. We can however use this small perturbation to define an intersection number for arcs.

Definition 3.2. Let α_1, α_2 be two arcs. We define the number $\iota(\alpha_1, \alpha_2) \in \mathbb{N}$ as follows. Fix some small $\varepsilon > 0$. We can find arcs α'_1 and α'_2 which are homotopy equivalent to α_1 and α_2 respectively, such that $\Phi_\varepsilon(\alpha'_1)$ and α'_2 are in minimal position. Then define

$$\iota(\alpha_1, \alpha_2) := |\Phi_\varepsilon(\alpha'_1) \cap \alpha'_2|.$$

Remark 3.3. i) If two arcs α_1, α_2 are in minimal position and do not share a common end-point then $\iota(\alpha_1, \alpha_2) = |\alpha_1 \cap \alpha_2|$.

ii) This intersection number is in general not symmetric. See Figure 4 for an example.

3.2. The universal cover. We now consider $\widehat{C}(p, q)$, the universal cover of $C(p, q)$. This is homeomorphic to $\mathbb{R} \times [0, 1]$ with covering map $\pi: \mathbb{R} \times [0, 1] \rightarrow S^1 \times [0, 1]: (r, t) \mapsto (2\pi\{r\}, t)$, where $\{-\}$ denotes the fractional part. The fundamental group of the cylinder $\Pi_1(C) \cong H^1(C) = \mathbb{Z}$ acts via deck transformations and for any point $c \in C(p, q)$, we denote the lift of c which lies in the fundamental domain $[i, i+1) \times [0, 1]$ by (c, i) . We denote by σ the generator of the group of deck transformations, with the property that $\sigma(c, 0) = (c, 1)$.

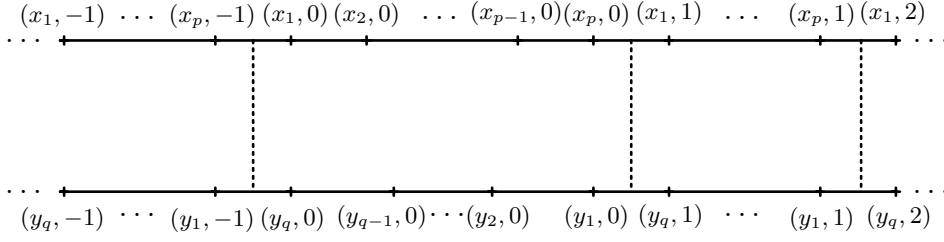
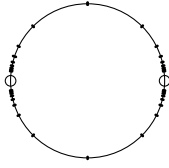


FIGURE 5. The universal cover $\widehat{C}(p, q)$ with marked boundary points.

The strip $\mathbb{R} \times [0, 1]$ is also homeomorphic to the closed unit disc with two boundary points removed. These punctures are accumulation points for the images of the marked points.



Given two arcs in minimal position, we will now show that we can calculate the intersection number by fixing a lift of one of the arcs, and counting the number of lifts of the other arc which it intersects. Let $\widehat{\Phi}_\varepsilon$ be a lift of the ambient isotopy Φ_ε .

Lemma 3.4. *Suppose α_1 and α_2 are arcs in $C(p, q)$ such that $\Phi_\varepsilon(\alpha_1)$ and α_2 intersect transversally at points which are not self-intersection points. Fix some lifts $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ of α_1 and α_2 . Then*

$$|\Phi_\varepsilon(\alpha_1) \cap \alpha_2| = \sum_{i \in \mathbb{Z}} |\widehat{\Phi}_\varepsilon(\widehat{\alpha}_1) \cap \sigma^i \widehat{\alpha}_2|.$$

Proof. Given any intersection point between $\widehat{\Phi}_\varepsilon(\widehat{\alpha}_1)$ and some $\sigma^i \widehat{\alpha}_2$ it is clear that this projects down to an intersection point between $\Phi_\varepsilon(\alpha_1)$ and α_2 . This map is surjective on the intersection points: given an intersection point p of $\Phi_\varepsilon(\alpha_1)$ and α_2 , we consider the corresponding lift of p on $\widehat{\Phi}_\varepsilon(\widehat{\alpha}_1)$. There is a lift of α_2 passing through this point which must be of the form $\sigma^i \widehat{\alpha}_2$ because of the transitive action of deck transformations. Finally we note that if two intersection points on the cover were to project down to the same point, then the intersection point of $\Phi_\varepsilon(\alpha_1)$ and α_2 would also be a self-intersection point of $\Phi_\varepsilon(\alpha_1)$ or α_2 . \square

Lemma 3.5. *Suppose α_1 and α_2 are arcs in $C(p, q)$ such that $\Phi_\varepsilon(\alpha_1)$ and α_2 are in minimal position, and let $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ be lifts to the universal cover. Then $\widehat{\Phi}_\varepsilon(\widehat{\alpha}_1)$ and $\widehat{\alpha}_2$ are in minimal position and intersect in either zero or one point.*

Proof. We look in the strip description of the universal cover and argue the contrapositive statement. Suppose that $\widehat{\Phi}_\varepsilon(\widehat{\alpha}_1)$ and $\widehat{\alpha}_2$ are not in minimal position.

Claim: *There exist arcs $\widehat{\beta}_1$ and $\widehat{\beta}_2$ on the universal cover, which have the same end-points as $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ respectively, such that the pairs of arcs $\widehat{\Phi}_\varepsilon(\widehat{\beta}_1)$ and $\sigma^i \widehat{\beta}_2$ are in minimal position for each $i \in \mathbb{Z}$.*

Assuming this claim, and using the convexity of the strip, we can simultaneously apply linear homotopies from $\widehat{\Phi}_\varepsilon(\widehat{\alpha}_1)$ to $\widehat{\Phi}_\varepsilon(\widehat{\beta}_1)$ and from each $\sigma^i \widehat{\alpha}_2$ to $\sigma^i \widehat{\beta}_2$. We observe that this is compatible with the covering map, so there are induced homotopies from $\Phi_\varepsilon(\alpha_1)$ and α_2 to new representatives $\Phi_\varepsilon(\beta_1)$ and β_2 in their homotopy classes. Looking at the

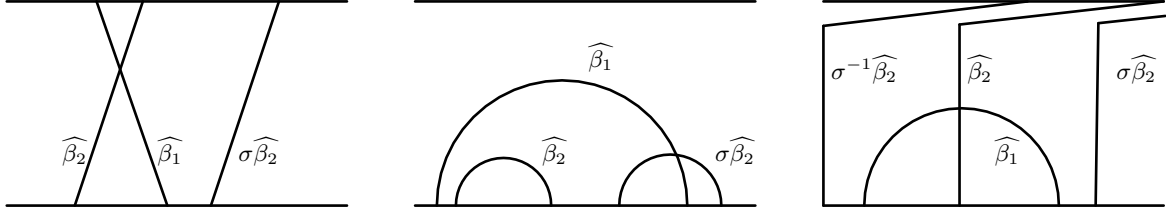


FIGURE 6. Examples of the arcs described in Lemma 3.5. By construction, each pair of arcs intersects in at most one point.

equation in Lemma 3.4, we see that the right hand side must decrease as we apply these homotopies, since $\widehat{\Phi}_\varepsilon(\widehat{\alpha}_1)$ and $\widehat{\alpha}_2$ were not in minimal position. Therefore we have found representatives of $\Phi_\varepsilon(\alpha_1)$ and α_2 with a lower number of intersection points and so they are not in minimal position.

To verify the claim, we write down one choice of $\widehat{\beta}_1$ and $\widehat{\beta}_2$ which we illustrated in Figure 6. A relative version of the bigon criterium [13, Proposition 1.7], shows that simple arcs which intersect transversally in a single point are in minimal position. Therefore, to prove the required minimality, it is sufficient to check that each pair of arcs $\widehat{\Phi}_\varepsilon(\widehat{\beta}_1)$ and $\sigma^i \widehat{\beta}_2$ intersect transversally in at most one point.

If $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ both cross the strip, we take $\widehat{\beta}_1$ and $\widehat{\beta}_2$ to be straight lines between their respective end-points. Each pair of such arcs clearly intersect in at most one point. If $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ both have end-points on one boundary of the strip then, after applying a homeomorphism to make the strip broad enough, we consider semicircles between their respective end-points. These can be treated as geodesics in the hyperbolic upper half-plane and so any two such semicircles intersect in at most one point. Suppose $\widehat{\alpha}_1$ has end-points on one boundary and $\widehat{\alpha}_2$ crosses the strip (or vice-versa). After broadening the strip, we can consider the semicircular arc between the end-points of $\widehat{\alpha}_1$, and a simple (sufficiently smoothed) piecewise-linear arc between the end-points of $\widehat{\alpha}_2$, which is a vertical straight line except in a neighbourhood of the other boundary. Away from this boundary, the arcs can again be considered as geodesics in the hyperbolic upper half-plane and so have at most one intersection point. \square

We define the intersection numbers of arcs on the universal cover in an analogous way to those of arcs on the cylinder, where the small perturbation is given by isotopy $\widehat{\Phi}_\varepsilon$.

Corollary 3.6. *Suppose α_1 and α_2 are arcs in $C(p, q)$. Then*

$$\iota(\alpha_1, \alpha_2) = \sum_{i \in \mathbb{Z}} \iota(\widehat{\alpha}_1, \sigma^i \widehat{\alpha}_2).$$

Proof. We choose representatives for α_1 and α_2 such that $\Phi_\varepsilon(\alpha_1)$ and α_2 are in minimal position. Lemma 3.5 implies that the arcs $\widehat{\Phi}_\varepsilon(\widehat{\alpha}_1)$ and $\sigma^i \widehat{\alpha}_2$ are in minimal position for each $i \in \mathbb{Z}$. The result then follows from Lemma 3.4. \square

We showed in Lemma 3.5 that each term $\iota(\widehat{\alpha}_1, \sigma^i \widehat{\alpha}_2)$ in the sum is either zero or one. We now describe precisely when it is non-zero.

Definition 3.7. We define a total order on the marked points of the universal cover as follows:

$$\begin{aligned} (x_i, s) &< (x_j, t) \text{ if } s < t, \text{ or } s = t \text{ and } i < j, \\ (y_i, s) &< (y_j, t) \text{ if } t < s, \text{ or } s = t \text{ and } i < j, \\ (x_i, s) &< (y_j, t) \text{ for all } (x_i, s) \in \widehat{\delta}_X \text{ and } (y_j, t) \in \widehat{\delta}_Y. \end{aligned}$$

The intersection numbers of lifts of arcs can then be calculated by looking at inequalities relating the end-points of the arcs as follows.

Lemma 3.8. Let $\widehat{\alpha}_i: I \rightarrow \widehat{C}(p, q)$ for $i = 1, 2$ be arcs on the universal cover. Choose a parametrisation such that $\widehat{\alpha}_i(0) < \widehat{\alpha}_i(1)$. Then

$$\iota(\widehat{\alpha}_1, \widehat{\alpha}_2) = \begin{cases} 1 & \text{if } \widehat{\alpha}_1(0) \leq \widehat{\alpha}_2(0) < \widehat{\alpha}_1(1) \leq \widehat{\alpha}_2(1) \\ & \text{or } \widehat{\alpha}_2(0) < \widehat{\alpha}_1(0) \leq \widehat{\alpha}_2(1) < \widehat{\alpha}_1(1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using representatives which are given by chords on the disc model of the universal cover, which are in minimal position, this is an easy exercise. The only small technicality is dealing with the perturbation, which acts by causing a point to decrease slightly in the order. \square

Remark 3.9. The two chains of inequalities $\underline{1}: \widehat{\alpha}_1(0) \leq \widehat{\alpha}_2(0) < \widehat{\alpha}_1(1) \leq \widehat{\alpha}_2(1)$ and $\underline{2}: \widehat{\alpha}_2(0) < \widehat{\alpha}_1(0) \leq \widehat{\alpha}_2(1) < \widehat{\alpha}_1(1)$ which lead to non-zero intersection numbers, can be broken up into ten cases by considering the number of end-points in $\widehat{\delta}_Y$.

3.3. Objects and arcs. We now describe how to go from objects in the discrete derived category, to arcs in the geometric model.

We put a partial order on all indecomposable objects that lie on the mouth of a component. Recall that in coordinates, all objects on the mouth of a component are of the form $X^c(i, i)$ or $Y^c(i, i)$ for some $c \in \{0, \dots, r-1\}$ and $i \in \mathbb{Z}$. For such objects, we simplify the notation and write $X_i^c := X^c(i, i)$ and $Y_i^c := Y^c(i, i)$.

Definition 3.10. We define a partial order on the set of objects at the mouths of components as follows:

$$\begin{aligned} X_i^k &< X_j^l \text{ if } k = l \text{ and } i < j, \\ Y_i^k &< Y_j^l \text{ if } k = l \text{ and } i < j \\ X_i^k &< Y_j^l \text{ for all } k, l \in \{0, \dots, r-1\} \text{ and } i, j \in \mathbb{Z}. \end{aligned}$$

Remark 3.11. Two objects on the mouths of \mathcal{X} -components (respectively \mathcal{Y} -components) are comparable with respect to this partial order if and only if they lie on the mouth of the same component. The partial order restricts to a total order on the objects on the mouth of any given component. We recall that any object A and $\Sigma^r A$ lie in the same component, and we note that $A < \Sigma^r A$ for an object on the mouth of an \mathcal{X} -component, while $A > \Sigma^r A$ for an object on the mouth of a \mathcal{Y} -component.

3.3.1. Identifying the end-points of an arc. We consider the set $\underline{X} = (X_1^0, \dots, X_{m+r}^0)$ of neighbouring objects along the mouth of the \mathcal{X}^0 -component, and $\underline{Y} = (Y_1^0, \dots, Y_{n-r}^0)$ of objects in the \mathcal{Y}^0 -component. The sets \underline{X} and \underline{Y} are examples of exceptional cycles from [7, Lemma 5.1]. We identify objects in these sets with marked points on the cylinder $C(p, q)$.

Definition 3.12. Let \mathcal{L} be the set of all objects in \underline{X} or \underline{Y} , and ℓ the set of boundary points on the cylinder:

$$\mathcal{L} = \{X_1^0, \dots, X_{m+r}^0, Y_1^0, \dots, Y_{n-r}^0\} \quad \ell = \{x_1, \dots, x_{m+r}, y_1, \dots, y_{n-r}\}.$$

Denote by $\eta : \mathcal{L} \rightarrow \ell$ be the bijection sending $X_i^0 \mapsto x_i$ and $Y_i^0 \mapsto y_i$.

We now describe how to associate a pair of points in ℓ to any indecomposable object in \mathbf{D} . These points will be the end-points of the corresponding arc. Let A be an indecomposable object in \mathbf{D} . Since each component is of type $\mathbb{Z}A_\infty$ or type $\mathbb{Z}A_\infty^\infty$, it can be seen from the structure of the AR-quiver that A fits into an AR-triangle

$$(7) \quad A \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} C_1 \oplus C_2 \xrightarrow{\begin{pmatrix} g_1 & g_2 \end{pmatrix}} \tau^{-1}A$$

where the middle term has at most two indecomposable summands. We take C_2 to be zero if it has only one indecomposable summand. Note that these triangles can be read off explicitly from Lemma 1.6 with $a = b = 1$ when $ht(A) \geq 1$ or $A \in \mathcal{Z}$, or Lemma 1.3 with $i = j$ and $a = b = 1$ when $ht(A) = 0$.

Definition 3.13. Define $\varphi_A : A \rightarrow \tau^{-1}A$ to be the composition $g_1 \circ f_1$ factoring through C_1 and let $\mathcal{B}(A)$ be the cocone of φ_A which fits into the triangle

$$(8) \quad \mathcal{B}(A) \xrightarrow{\rho_A} A \xrightarrow{\varphi_A} \tau^{-1}A.$$

Remark 3.14. The mesh relations in the AR quiver ensure that $\varphi_A = g_1 \circ f_1 = g_2 \circ f_2$ and so the definition is independent of the ordering of the summands.

For each indecomposable object A in \mathbf{D} , we can write down $\mathcal{B}(A)$ explicitly using the coordinates on the components.

Lemma 3.15. *Given any object $A \in \text{ind}(\mathbf{D})$, then $\mathcal{B}(A)$ is of the form,*

$$\mathcal{B}(A) = \mathcal{B}^-(A) \oplus \mathcal{B}^+(A).$$

where $\mathcal{B}^-(A)$ and $\mathcal{B}^+(A)$ are indecomposable objects given as follows:

- i) If A is in an \mathcal{X} component, so $A = X^c(i, j)$ for some $i, j, c \in \mathbb{Z}$ such that $j \geq i$, then $\mathcal{B}^-(A) = X_i^c$ and $\mathcal{B}^+(A) = \Sigma^{-1}X_{j+1}^c$.
- ii) If A is in an \mathcal{Y} component, so $A = Y^c(i, j)$ for some $i, j, c \in \mathbb{Z}$ such that $j \leq i$, then $\mathcal{B}^-(A) = Y_j^c$ and $\mathcal{B}^+(A) = \Sigma^{-1}Y_{i+1}^c$.
- iii) If A is in a \mathcal{Z} -component, so $A = Z^c(i, j)$ for some $i, j, c \in \mathbb{Z}$, then $\mathcal{B}^-(A) = X_i^c$ and $\mathcal{B}^+(A) = Y_j^c$.

In particular, each object $\mathcal{B}^\pm(A)$ lies on the mouth of a component.

Proof. For an object in an \mathcal{X} , \mathcal{Y} or \mathcal{Z} component which is not on the mouth, this follows from Lemma 1.4. For an object A on the mouth of a component, the mesh relations mean that the morphism φ_A is zero and the result follows from the split triangle

$$A \oplus \Sigma^{-1}\tau^{-1}A \longrightarrow A \xrightarrow{0} \tau^{-1}A. \quad \square$$

Remark 3.16. By specifying $\mathcal{B}^-(A)$ and $\mathcal{B}^+(A)$ we are ordering the pair of summands of $\mathcal{B}(A)$. We have chosen to do this such that $\mathcal{B}^-(A) < \Sigma\mathcal{B}^+(A)$ with respect to the partial order on the mouths of the components.

For each object on the mouth of a component, we can uniquely identify an object in \mathcal{L} and an integer k using the following lemma.

Lemma 3.17. *For each $B \in \text{ind}(\mathbf{D})$ which lies on the mouth of a component, there exist unique $k(B) \in \mathbb{Z}$ and $L_B \in \mathcal{L}$ such that $B = \Sigma^{k(B)}L_B$.*

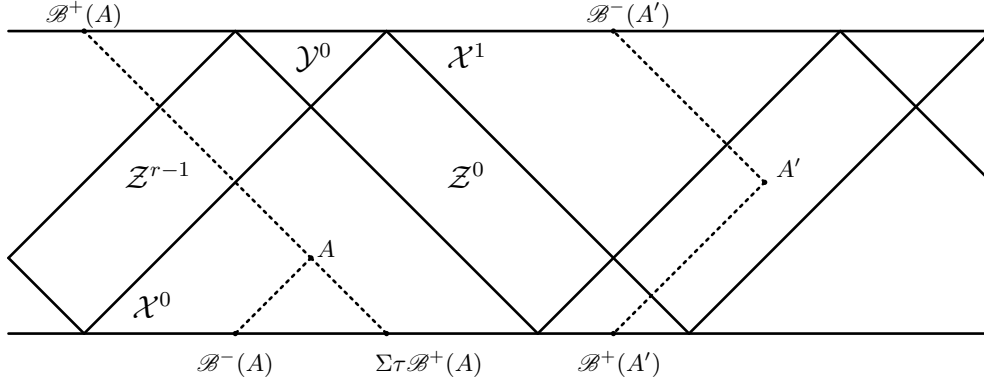


FIGURE 7. A schematic diagram of the AR-quiver of a discrete derived category, showing the relative positions on the mouth of $\mathcal{B}^-(A)$ and $\mathcal{B}^+(A)$ for an object $A \in \mathcal{X}$, and $A' \in \mathcal{Z}$.

Proof. We act with Σ such that $\Sigma^i B$ lies in the same component as the objects of \underline{X} or \underline{Y} . Since this is either an \mathcal{X} or \mathcal{Y} component, then Σ^r acts on objects by τ^{-m-r} or τ^{n-r} respectively. Acting by powers of Σ^r we can always end up in \underline{X} or \underline{Y} since they consist of respectively $m+r$ and $n-r$ objects along the mouth. \square

We use this lemma to make the following definitions.

Definition 3.18. For any $A \in \text{ind}(\mathbb{D})$ we define

$$b^\pm(A) := \eta(L_{\mathcal{B}^\pm(A)}) \in \ell \quad \text{and} \quad k^\pm(A) := k(\mathcal{B}^\pm(A)) \in \mathbb{Z}.$$

For a given indecomposable object A , the marked points $b^\pm(A)$ will be the end-points of the corresponding arc. We will use the integers $k^\pm(A)$ to define how the arc then winds around the cylinder. First we observe that:

Lemma 3.19. *Let $A \in \text{ind}(\mathbb{D})$. Then*

$$k^+(A) - k^-(A) = \begin{cases} -1 \bmod(r) & \text{if } A \text{ is in an } \mathcal{X} \text{ or } \mathcal{Y} \text{ component,} \\ 0 \bmod(r) & \text{if } A \text{ is in an } \mathcal{Z}\text{-component.} \end{cases}$$

We use the quotients to associate an integer to every indecomposable object.

Definition 3.20. Let $w : \text{ind}(\mathbb{D}) \rightarrow \mathbb{Z}$ be the map defined by

$$w(A) := \begin{cases} (k^+(A) - k^-(A) + 1)/r & \text{if } A \text{ is in an } \mathcal{X} \text{ or } \mathcal{Y} \text{ component,} \\ (k^+(A) - k^-(A))/r & \text{if } A \text{ is in an } \mathcal{Z}\text{-component,} \end{cases}$$

Finally we can use the triple $(b^-(A), b^+(A), w(A))$ to associate an arc to any indecomposable object in \mathbb{D} , noting that it is invariant under the action of Σ on $\text{ind}(\mathbb{D})$.

Definition 3.21. For any $A \in \text{ind}(\mathbb{D})$, we define an arc α_A in $C(p, q)$ to be

$$\alpha_A := \pi \circ \widehat{\alpha}_A : [a, b] \longrightarrow C(p, q),$$

where $\widehat{\alpha}_A : [a, b] \longrightarrow \widehat{C}(p, q)$ is the unique arc (up to homotopy) such that

$$\widehat{\alpha}_A(0) = (b^-(A), 0) \text{ and } \widehat{\alpha}_A(1) = (b^+(A), w(A)),$$

and $\pi : \widehat{C}(p, q) \longrightarrow C(p, q)$ is the covering map.

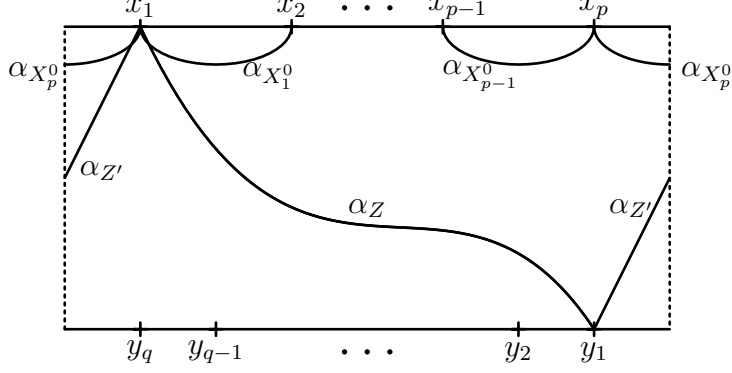


FIGURE 8. The arcs corresponding to some objects in $D^b(\Lambda(r, m, n))$.

Example 3.22. We apply Lemma 3.15 to specific indecomposable objects and determine the associated arcs which are pictured in Figure 8. To simplify notation, we use $p := m+r$ and $q := n-r$.

For each $i = 1, \dots, p-1$ we have $\mathcal{B}^-(X_i^0) = X_i^0$ and $\mathcal{B}^+(X_i^0) = \Sigma^{-1}X_{i+1}^0$, so $k^-(X_i^0) = 0$ and $k^+(X_i^0) = -1$. Therefore, the arc $\alpha_{X_i^0}$ corresponds to the triple $(x_i, x_{i+1}, 0)$, that is, the projection of the arc from $(x_i, 0)$ to $(x_{i+1}, 0)$ in the universal cover.

For X_p^0 we have $\mathcal{B}^-(X_p^0) = X_p^0$ and $\mathcal{B}^+(X_p^0) = \Sigma^{r-1}X_1^0$. Therefore, $k^-(X_p^0) = 0$ and $k^+(X_p^0) = r-1$, and so we can calculate that $w(X_p^0) = 1$. Thus $\alpha_{X_p^0}$ is the projection of the arc from $(x_p, 0)$ to $(x_1, 1)$.

For $Z = Z^0(1, 1)$ we have $\mathcal{B}^-(Z) = X_1^0$ and $\mathcal{B}^+(Z) = Y_1^0$ and we see that $w(Z) = 0$. Therefore α_{X_Z} is the projection of the arc from $(x_1, 0)$ to $(y_1, 0)$.

Finally for $Z' = Z^0(1, q+1)$ we have $\mathcal{B}^-(Z') = X_1^0$ and $\mathcal{B}^+(Z') = Y_{q+1}^0 = \Sigma^{-r}Y_1^0$ so $w(Z') = -1$. Therefore $\alpha_{X_{Z'}}$ is the projection of the arc from $(x_1, 0)$ to $(y_1, -1)$.

3.4. Hom-spaces and intersection numbers. There is a complete description of the Hom-hammocks of indecomposable objects in $D^b(\Lambda)$ in [7, Propositions 3.4, 3.5 and 3.6]. Using Lemma 3.15, we can package this description using the functions $\mathcal{B}^\pm(-)$.

Proposition 3.23. *Let $A, B \in \text{ind}(D)$. In the cases where $r > 1$, then*

$$\text{hom}_D(A, B) = \begin{cases} 1 & \text{if one of the statements } \textcircled{0} \text{ or } \textcircled{1} \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

In the cases where $r = 1$, then

$$\text{hom}_D(A, B) = \begin{cases} 2 & \text{if statements } \textcircled{0} \text{ and } \textcircled{1} \text{ are both satisfied} \\ 1 & \text{if one of the statements } \textcircled{0} \text{ or } \textcircled{1} \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

where

- $\textcircled{0}$ $\mathcal{B}^-(A) \leq \mathcal{B}^-(B) < \Sigma\mathcal{B}^+(A) \leq \Sigma\mathcal{B}^+(B)$,
- $\textcircled{1}$ $\Sigma^{-1}\mathcal{B}^-(B) < \mathcal{B}^-(A) \leq \mathcal{B}^+(B) < \Sigma\mathcal{B}^+(A)$.

In particular, there is a bijection between the set of statements which are satisfied for A and B and a basis of $\text{Hom}_D(A, B)$. If $\text{hom}_D(A, B) = 2$ then under this bijection $\textcircled{0}$ corresponds to a morphism factoring through the irreducible morphisms in a component, and $\textcircled{1}$ to a morphism in the infinite radical.

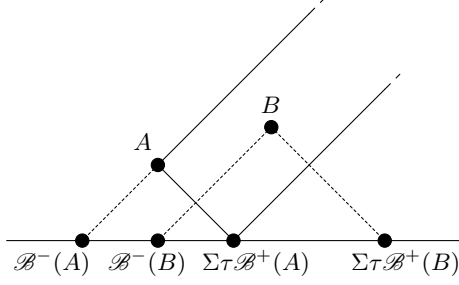


FIGURE 9. Diagram showing hom-hammock from an object $A \in \mathcal{X}^k$.

Proof. This amounts to translating the statements contained in [7, Propositions 3.4, 3.5 and 3.6] into the notation introduced above. For example, if $A, B \in \text{ind}(\mathcal{X}^c)$ and $r > 0$, then we are in the case of the first statement from [7, Proposition 3.4]. By Lemma 3.15, we observe that $\mathcal{B}^-(A)$ and $\Sigma^{-1}\mathcal{B}^-(B)$ are in different components, so they are not comparable and in particular $\textcircled{1}$ can not hold. Comparing the definitions, we see that the object denoted by A_0 in [7, Section 3.2] is $\Sigma\tau\mathcal{B}^+(A)$ in the current notation. The condition that B lies anywhere on a ray through the line segment $\overline{AA_0}$ then translates to the inequality $\mathcal{B}^-(A) \leq \mathcal{B}^-(B) \leq \Sigma\tau\mathcal{B}^+(A)$ or equivalently $\mathcal{B}^-(A) \leq \mathcal{B}^-(B) < \Sigma\mathcal{B}^+(A)$ (see Figure 9). It then lies on one of the positive rays from the line segment if additionally $\Sigma\tau\mathcal{B}^+(A) \leq \Sigma\tau\mathcal{B}^+(B)$. Putting this together, we see that $B \in \text{ray}_+(\overline{AA_0})$ if and only if A and B satisfy statement $\textcircled{0}$. Therefore [7, Proposition 3.4] implies that $\textcircled{0}$ is satisfied if and only if $\text{hom}(A, B) \neq 0$ if and only if $\text{hom}(A, B) = 1$ by [7, Proposition 6.1].

Similar arguments work for all the other cases, using additionally [7, Proposition 6.2] which describes when $\text{hom}(A, B) = 2$ in the $r = 1$ cases. If $r > 1$, then $\mathcal{B}^-(B)$ and $\Sigma^{-1}\mathcal{B}^-(B)$ are in different components and so can not both be comparable with $\mathcal{B}^-(A)$ in the partial order. Therefore at most one of the statements $\textcircled{0}$ or $\textcircled{1}$ can be satisfied. Furthermore, since $\text{hom}(A, B) \leq 1$ [7, Proposition 6.1], if a statement is satisfied then it determines a unique morphism up to scaling. If $r = 1$, then $\textcircled{0}$ and $\textcircled{1}$ could both be satisfied. In this case, tracing through the argument, it can be seen that $\textcircled{0}$ naturally corresponds to a morphism factoring through the irreducible morphisms in a component, and $\textcircled{1}$ to a morphism in the infinite radical. \square

Remark 3.24. If a statement $\textcircled{0}$ or $\textcircled{1}$ holds for a pair of objects A and B , then this can be rewritten in the form of statement $\textcircled{1}$ or $\textcircled{0}$ for the pair of objects B and $\Sigma\tau A$ (using the observation that for any object A' on the mouth of a component, $\tau A'$ is the lower cover of A' in the partial order). In particular, $\text{hom}(A, B) = \text{hom}(B, \Sigma\tau A)$. This is a manifestation of Serre duality and the statements $\textcircled{0}$ and $\textcircled{1}$ are Serre dual to each other.

In Section 4 we will also be interested in morphisms from A to B which factor through $\varphi_A: A \rightarrow \tau^{-1}A$. In other words, morphisms in $\varphi_A^*(\text{Hom}_{\mathbb{D}}(\tau^{-1}A, B)) \subset \text{Hom}_{\mathbb{D}}(A, B)$.

Lemma 3.25. *In the situation of Proposition 3.23, a basis morphism $f \in \text{Hom}_{\mathbb{D}}(A, B)$ is in the image of*

$$\varphi_A^*: \text{Hom}_{\mathbb{D}}(\tau^{-1}A, B) \longrightarrow \text{Hom}_{\mathbb{D}}(A, B)$$

if and only if all of the inequalities in the corresponding statement $\textcircled{0}$ or $\textcircled{1}$ are strict.

Proof. Let f be a basis morphism in the image of φ_A^* . Using Proposition 3.23, there is a corresponding statement $\textcircled{0}$ or $\textcircled{1}$, which is satisfied for A and B . Since $f = g \circ \varphi_A$ for

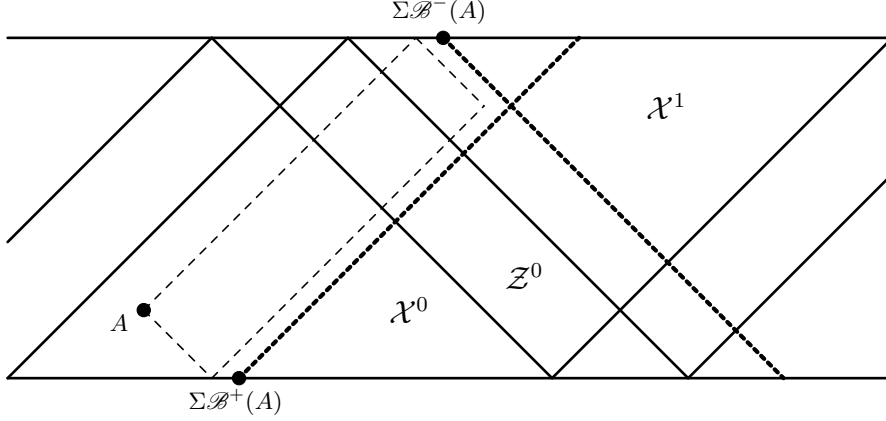


FIGURE 10. A schematic diagram of the AR-quiver of a discrete derived category, showing the long (co)rays from $\Sigma\mathcal{B}^-(A)$ and $\Sigma\mathcal{B}^+(A)$ for an object $A \in \mathcal{X}$.

some non-zero $g \in \text{Hom}_{\mathbb{D}}(\tau^{-1}A, B)$ we see that a statement of the form $\textcircled{0}$ or $\textcircled{1}$ must also be satisfied for $\tau^{-1}A$ and B . Furthermore, these statements must be of the same form $\textcircled{0}$ or $\textcircled{1}$; if $r > 1$, since A and $\tau^{-1}A$ are in the same component then at most one of the types $\textcircled{0}$ or $\textcircled{1}$ could possibly hold. If $r = 1$ then we additionally observe that φ_A is not in the infinite radical, so f is in the infinite radical if and only if g is in the infinite radical, and so the statements must be of the same type. Finally, using the fact that $\mathcal{B}^{\pm}(\tau^{-1}A) = \tau^{-1}\mathcal{B}^{\pm}(A)$ is the upper cover of $\mathcal{B}^{\pm}(A)$ in the partial order, we compare the two statements and deduce that a statement of the same form must be satisfied for A and B , but with strict inequalities.

Conversely, suppose A and B satisfy one of the statements with strict inequalities and let $f \in \text{Hom}_{\mathbb{D}}(A, B)$ be the corresponding basis morphism. Again using the fact that $\mathcal{B}^{\pm}(\tau^{-1}A)$ is the upper cover of $\mathcal{B}^{\pm}(A)$ in the partial order, we see that a statement of the same form holds for $\tau^{-1}A$ and B where the inequalities need not be strict. By Proposition 3.23, this statement corresponds to a non-zero morphism $g \in \text{Hom}_{\mathbb{D}}(\tau^{-1}A, B)$. Suppose to the contrary that $g \circ \varphi_A = 0$. Then g factors through the cone of φ_A , which is $\Sigma\mathcal{B}(A)$ from Equation (8). Since both summands of $\Sigma\mathcal{B}(A)$ are on the mouths of components, and using the knowledge of the Hom-hammocks of such objects (see [7, Lemma 3.1]), it follows that B lies on either the long (co)ray from $\Sigma\mathcal{B}^+(A)$ or the long (co)ray from $\Sigma\mathcal{B}^-(A)$ (see Figure 10). A case analysis then shows that this contradicts the initial assumption that the statement holds. For example, if $\textcircled{0}$ holds with strict inequalities, and A, B are in the same \mathcal{X} -component, then we find that $\mathcal{B}^-(B) = \Sigma\mathcal{B}^-(A)$ which contradicts the assumption that $\textcircled{0}$ holds. Therefore, $g \circ \varphi_A \neq 0$. In the $r > 1$ case, it is then clear that $g \circ \varphi_A$ is equal to f up to scaling and so f is in the image of φ_A^* as required. If $r = 1$, this follows by considering when f and g are in the infinite radical or not. \square

Comparing the statements of Lemma 3.8 and of Proposition 3.23 we see a similarity. We now make this observation more precise, by considering an isomorphism between the ordered sets as follows.

Definition 3.26. i) Let ϖ_X be the isomorphism of totally ordered sets which takes objects on the mouth of the component \mathcal{X}^0 :

$$\cdots < \Sigma^{-r} X_{p-1}^0 < \Sigma^{-r} X_p^0 < X_1^0 < X_2^0 < \cdots < X_p^0 < \Sigma^r X_1^0 < \Sigma^r X_2^0 < \cdots$$

to marked points on $\widehat{\delta}_X$:

$$\cdots < (x_{p-1}, -1) < (x_p, -1) < (x_1, 0) < (x_2, 0) < \cdots < (x_p, 0) < (x_1, 1) < (x_2, 1) < \cdots$$

defined by mapping $\Sigma^{sr} X_i^0$ to (x_i, s) for all $i \in \{1, \dots, p\}$ and $s \in \mathbb{Z}$.

ii) Let ϖ_Y be the isomorphism of totally ordered sets which takes objects on the mouth of the component \mathcal{Y}^0 :

$$\cdots < \Sigma^r Y_{q-1}^0 < \Sigma^r Y_q^0 < Y_1^0 < Y_2^0 < \cdots < Y_q^0 < \Sigma^{-r} Y_1^0 < \Sigma^{-r} Y_2^0 < \cdots$$

to marked points on $\widehat{\delta}_Y$:

$$\cdots < (y_{q-1}, 1) < (y_q, 1) < (y_1, 0) < (y_2, 0) < \cdots < (y_q, 0) < (y_1, -1) < (y_2, -1) < \cdots$$

defined by mapping $\Sigma^{sr} Y_i^0$ to (y_i, s) for all $i \in \{1, \dots, q\}$ and $s \in \mathbb{Z}$.

Lemma 3.27. *The action of the Σ^r and σ are compatible, so for any $A \in \mathcal{X}^0$ and $B \in \mathcal{Y}^0$,*

$$\varpi_X(\Sigma^r A) = \sigma \varpi_X(A) \quad \text{and} \quad \varpi_Y(\Sigma^r B) = \sigma \varpi_Y(B)$$

Proof. This follows straight from the definitions. □

We are now in a position to state the main theorem of this section.

Theorem 3.28. *Let $\mathbf{D} = \mathbf{D}^b(\Lambda(r, n, m))$ be a discrete derived category with $n > r$. There is a bijection:*

$$\{\text{non-trivial arcs in } C(m+r, n-r)\} \longleftrightarrow \text{ind}(\mathbf{D})/\Sigma$$

such that,

$$\iota(\alpha_A, \alpha_B) = \text{hom}_{\mathbf{D}/\Sigma}(A, B) := \sum_{i \in \mathbb{Z}} \text{hom}_{\mathbf{D}}(A, \Sigma^i B)$$

for arcs α_A and α_B associated to any objects $A, B \in \text{ind}(\mathbf{D})$.

Proof. For ease of notation, set $p = m + r$ and $q = n - r$. Let α be any non-trivial arc in $C(p, q)$. For each orientation of the arc, there is a unique lift to $\widehat{C}(p, q)$ with the property that $\widehat{\alpha}(0) = (\alpha(0), 0)$, and precisely one of the orientations satisfies the additional condition that $\widehat{\alpha}(0) < \widehat{\alpha}(1)$. Therefore, there is a bijection between the set of arcs in $C(p, q)$, and the set of arcs in $\widehat{C}(p, q)$ satisfying these two properties.

Suppose we have such a lift $\widehat{\alpha}$, with $\widehat{\alpha}(0) = (\alpha(0), 0)$ and $\widehat{\alpha}(1) = (\alpha(1), w)$ for some $w \in \mathbb{Z}$. We show that there is a unique object $A \in \text{ind}(\mathbf{D}^b(\Lambda))/\Sigma$ such that $\widehat{\alpha}_A = \widehat{\alpha}$. The proof breaks into three cases:

Case 1: $(\alpha(0), \alpha(1) \in \delta_X)$ First we define

$$\begin{aligned} \mathcal{B}^- &:= \varpi_X^{-1}(\widehat{\alpha}(0)) = \eta^{-1}(\alpha(0)) \in \mathcal{L} \\ \mathcal{B}^+ &:= \Sigma^{-1} \varpi_X^{-1}(\widehat{\alpha}(1)) = \Sigma^{rw-1} \eta^{-1}(\alpha(1)). \end{aligned}$$

We note that \mathcal{B}^- and $\Sigma \mathcal{B}^+$ lie on the mouth of the \mathcal{X}^0 -component, so there exist integers i, j such that $\mathcal{B}^- = X^0(i, i)$ and $\Sigma \mathcal{B}^+ = X^0(j+1, j+1)$. Since $\widehat{\alpha}(0) < \widehat{\alpha}(1)$ and ϖ_X is an order preserving bijection, it follows that $\mathcal{B}^- < \Sigma \mathcal{B}^+$ and so $i \leq j$. Therefore, there is a well defined indecomposable object $A = X^0(i, j)$. Lemma 3.15 implies that $\mathcal{B}^-(A) = \mathcal{B}^-$ and $\mathcal{B}^+(A) = \mathcal{B}^+$ and direct calculation shows that $w(A) = ((rw - 1) - 0 + 1)/r = w$. Therefore, $\widehat{\alpha}_A = \widehat{\alpha}$.

We now show that up to shift, A is the unique indecomposable object with this property. Suppose $\widehat{\alpha}_A = \widehat{\alpha}_B$ for some $B \in \text{ind}(\mathbf{D})$. Then, in particular, $b^-(A) = b^-(B)$, $b^+(A) = b^+(B)$ and $w(A) = w(B)$. It follows from the definitions that $\mathcal{B}^-(A) = \Sigma^s \mathcal{B}^-(B) =$

$\mathcal{B}^-(\Sigma^s B)$ for some $s \in \mathbb{Z}$, and so $k^-(A) = k^-(B) + s$. Similarly, $\mathcal{B}^+(A) = \Sigma^t \mathcal{B}^+(B) = \mathcal{B}^+(\Sigma^t B)$ for some $t \in \mathbb{Z}$ and so $k^+(A) = k^+(B) + t$. Rearranging we see that,

$$t - s = (k^+(A) - k^-(A) + 1) - (k^+(B) - k^-(B) + 1) = r(w(A) - w(B)) = 0,$$

so $\mathcal{B}^\pm(A) = \mathcal{B}^\pm(\Sigma^s B)$. Finally, any indecomposable object A' uniquely determines $\mathcal{B}^\pm(A')$ by Lemma 3.15. It follows that $A = \Sigma^s B$ as required.

Case 2: $(\alpha(0), \alpha(1) \in \delta_Y)$ This argument follows in the same way.

Case 3: $(\alpha(0) \in \delta_X, \alpha(1) \in \delta_Y)$ We define

$$\begin{aligned}\mathcal{B}^- &:= \varpi_X^{-1}(\widehat{\alpha}(0)) = \eta^{-1}(\alpha(0)) \in \mathcal{L} \\ \mathcal{B}^+ &:= \varpi_Y^{-1}(\widehat{\alpha}(1)) = \Sigma^{rw} \eta^{-1}(\alpha(1)).\end{aligned}$$

By construction, \mathcal{B}^- and \mathcal{B}^+ are on the mouths of the \mathcal{X}^0 and \mathcal{Y}^0 components respectively. Therefore, there exist integers i, j such that $\mathcal{B}^- = X^0(i, i)$ and $\mathcal{B}^+ = Y^0(j, j)$. We consider the indecomposable object $A := Z^0(i, j) \in \mathcal{Z}^0$ and observe that $\mathcal{B}^-(A) = \mathcal{B}^-$ and $\mathcal{B}^+(A) = \mathcal{B}^+$ using Lemma 3.15. By direct calculation we see that $w(A) = (rw - 0)/r = w$ and so, $\widehat{\alpha}_A = \widehat{\alpha}$. The proof of uniqueness is the same as in Case 1.

Now we prove the statement linking the intersection numbers to the dimensions of the Hom-spaces. Let $A, B \in \text{ind}(\mathbb{D})$ be any objects and for any $a = 0, 1$ define:

$$\begin{aligned}\xi_a(A, B) &= |\{k \in \mathbb{Z} \mid \textcircled{a} \text{ is satisfied for } (A, \Sigma^k B)\}| \\ \zeta_a(A, B) &= |\{k \in \mathbb{Z} \mid \underline{a} \text{ is satisfied for } (\widehat{\alpha}_A, \sigma^k \widehat{\alpha}_B)\}| \end{aligned}$$

We now show that $\xi_a(A, B) = \zeta_a(A, B)$. We consider here the case where $a = 0$ and A and B are both in an \mathcal{X} component, but all the other cases can be proved in the analogous way. In this case, $\mathcal{B}^\pm(A), \mathcal{B}^\pm(B) \in \mathcal{X}$. Note that $\mathcal{B}^-(A)$ and $\mathcal{B}^-(\Sigma^k B)$ are only comparable if they are in the same component, which happens when $k = (k^-(A) - k^-(B)) \bmod(r)$. The action of Σ preserves the partial order so we can rewrite

$$\xi_0(A, B) = |\{t \in \mathbb{Z} \mid \textcircled{0} \text{ is satisfied for } (\Sigma^{-k^-(A)} A, \Sigma^{-k^-(B)+rt} B)\}|.$$

Applying the map ϖ_X , we find:

$$\begin{aligned}\xi_0(A, B) &= |\{t \in \mathbb{Z} \mid (b^-(A), 0) \leq (b^-(B), t) < (b^+(A), w(A)) \leq (b^+(B), w(B) + t)\}| \\ &= |\{t \in \mathbb{Z} \mid \widehat{\alpha}_A(0) \leq \sigma^t \widehat{\alpha}_B(0) < \widehat{\alpha}_A(1) \leq \sigma^t \widehat{\alpha}_B(1)\}| = \zeta_0(A, B).\end{aligned}$$

Finally we see that

$$\text{hom}_{\mathbb{D}/\Sigma}(A, B) = \sum_{a=0}^1 \xi_a(A, B) = \sum_{a=0}^1 \zeta_a(A, B) = \iota(\alpha_A, \alpha_B). \quad \square$$

Remark 3.29. We have actually proved something slightly stronger. Up to scaling, we have uniquely identified a basis morphism of $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{D}}(A, \Sigma^i B)$ for each intersection of the arcs α_A, α_B , rather than just showing that the dimension of this space coincides with the intersection number.

In Section 1, we calculated the cones of irreducible morphisms between indecomposable objects in the discrete derived categories. We now summarise those results in terms of the functions $\mathcal{B}^\pm(-)$.

Lemma 3.30. *Let A and B be indecomposable objects and suppose that $f: A \rightarrow B$ is in the image of*

$$\varphi_A^*: \text{Hom}_{\mathbb{D}}(\tau^{-1} A, B) \longrightarrow \text{Hom}_{\mathbb{D}}(A, B).$$

Then the cone of f has two indecomposable summands:

$$A \rightarrow B \rightarrow C_1 \oplus C_2$$

such that:

Case	$\mathcal{B}^-(C_1)$	$\mathcal{B}^+(C_1)$	$\mathcal{B}^-(C_2)$	$\mathcal{B}^+(C_2)$
①	$\Sigma\mathcal{B}^+(A)$	$\mathcal{B}^+(B)$	$\Sigma\mathcal{B}^-(A)$	$\mathcal{B}^-(B)$
②	$\Sigma\mathcal{B}^-(A)$	$\mathcal{B}^+(B)$	$\mathcal{B}^-(B)$	$\Sigma\mathcal{B}^+(A)$

In particular, we have the following relations between paths of arcs up to homotopy:

if f corresponds to the statement ① then $\alpha_A \cdot \alpha_{C_1} \simeq \alpha_{C_2} \cdot \alpha_B$,

if f corresponds to the statement ② then $\alpha_A \cdot \overline{\alpha_{C_2}} \simeq \alpha_{C_1} \cdot \overline{\alpha_B}$.

Proof. By Lemma 3.25 we know that each f in the image of φ_A^* corresponds to a statement of the form ① or ②, where all of the inequalities are strict. The cones of all morphisms between indecomposable objects are calculated in Section 1. The proposition then follows by a direct calculation. For example, suppose ① is satisfied with strict inequalities and that A and B are both in \mathcal{X} . Then A and B are in the same component and f factors through the component by Proposition 3.23. In particular, there exist $i, j \in \mathbb{Z}$, $c \in \{0, \dots, r-1\}$ and $a, b > 0$ with $a \leq j-i$ such that $A = X^c(i, j)$ and $B = X^c(i+a, j+b)$. We use Lemma 1.4 to calculate the cone of f :

$$A \rightarrow B \rightarrow X^c(j+1, j+b) \oplus \Sigma X^c(i, i+a-1).$$

We can then use Lemma 3.15 to read off the first line of the table:

$$\begin{aligned} \mathcal{B}^-(C_1) &= X^c(j+1, j+1) = \Sigma\mathcal{B}^+(A) & \text{and} & & \mathcal{B}^-(\Sigma^{-1}C_2) &= \mathcal{B}^-(A) \\ \Sigma\mathcal{B}^+(C_1) &= X^c(j+b+1, j+b+1) = \Sigma\mathcal{B}^+(B) & & & \Sigma\mathcal{B}^+(\Sigma^{-1}C_2) &= \mathcal{B}^-(B) \end{aligned}$$

From this we can easily calculate $b^\pm(C_i)$ and $w(C_i)$ for $i = 1, 2$. Lifting the paths $\alpha_A \cdot \alpha_{C_1}$ and $\alpha_{C_2} \cdot \alpha_B$ to the cover, starting at $(b^-(A), 0) = (b^-(C_2), 0)$ we can see that they have common end-point $(b^+(C_1), w(A) + w(C_1)) = (b^+(B), w(B) + w(C_2))$ and so are homotopic. \square

4. ES-COLLECTIONS

Now we return to the main question of the classification of thick subcategories. In Section 2.3, we proved that all thick subcategories of \mathbf{D} are generated by finite sets of exceptional and spherelike objects. However we would like to be able to restrict a smaller, more manageable class of collections of such objects. These will correspond to certain collections of arcs on $C(p, q)$. As a first step, we identify which arcs in the geometric model correspond to exceptional and spherelike objects.

Lemma 4.1. *Let $A \in \mathbf{D}$ be an indecomposable object. Then,*

- i) A is exceptional if arc α_A is not closed and has no self-intersection points,*
- ii) A is spherelike if arc α_A is closed and has no self-intersection points.*

Proof. This follows from Theorem 3.28. We note that A is exceptional if and only if $\text{hom}_{\mathbf{D}/\Sigma}(A, A) = 1$ and spherelike if and only if $\text{hom}_{\mathbf{D}/\Sigma}(A, A) = 2$. \square

We generalise this non-crossing condition to collections of arcs and make the following definition.

Definition 4.2. An *arc-collection* in $C(p, q)$ is a finite collection of arcs

$$\{\alpha_i: [0, 1] \rightarrow C(p, q)\}_{i \in I}$$

such that:

for any $i, j \in I$, there exist representative arcs α'_i, α'_j with the property that

$$\alpha'_i(t) = \alpha'_j(t') \implies t, t' \in \{0, 1\}$$

We call the arc-collection *reduced* if in addition, no arc is homotopy equivalent to a path produced by concatenating other arcs in the collection.

With this definition in mind, we make an analogous definition for an ES-collection in the derived category.

Definition 4.3. An *ES-collection* in \mathbf{D} is a finite collection of indecomposable objects $\{A_i\}_{i \in I}$ such that for any $i, j \in I$, $s \in \mathbb{Z}$ the pullback

$$\varphi_{A_i}^* : \text{Hom}^s(\tau^{-1}A_i, A_j) \longrightarrow \text{Hom}^s(A_i, A_j)$$

is zero. We call the ES-collection *reduced* if in addition, $A_i \notin \text{thick}_{\mathbf{D}}(A_j \mid j \neq i)$.

Lemma 4.4. $\{A_i\}_{i \in I}$ is an ES-collection in \mathbf{D} , if and only if $\{\alpha_{A_i}\}_{i \in I}$ is an arc-collection in $C(p, q)$.

Proof. Suppose $\{A_i\}_{i \in I}$ is not an ES-collection. Then there exists some $f: A_i \rightarrow \Sigma^s A_j$ which factors through the morphism $\varphi_{A_i}: A_i \rightarrow \tau^{-1}A_i$. Lemma 3.25 shows that one of the statements $\textcircled{0}$ or $\textcircled{1}$ holds with strict inequalities and under the correspondence, this implies that one of the statements $\underline{0}$ or $\underline{1}$ holds with strict inequalities for some lifts of α_{A_i} and α_{A_j} , which we assume to be in minimal position. By Lemma 3.8 we see that the lifts have an intersection point in $\widehat{C}(p, q)$ which is not an end-point of the arcs. Therefore $\{\alpha_{A_i}\}_{i \in I}$ is not an arc-collection. Conversely, suppose that α_{A_i} and α_{A_j} are in minimal position and intersect in a point which isn't an end-point. We lift this intersection to an intersection of two lifts $\widehat{\alpha}_{A_i}$ and $\sigma^k \widehat{\alpha}_{A_j}$. In particular, one of the statements $\underline{0}$ or $\underline{1}$ holds with strict inequalities. Under the correspondence this implies that there are shifts of A_i and A_j such that one of the statements $\textcircled{0}$ or $\textcircled{1}$ holds with strict inequalities, and this in turn implies that there is a morphism between A_i and a shift of A_j which factors through $\tau^{-1}A_i$. \square

Corollary 4.5. *The objects in an ES-collection are exceptional or spherelike.*

Proof. Lemma 4.4 and Definition 4.2 together imply that any such object corresponds to an arc with no self-intersections. The result then follows from Lemma 4.1. \square

The class of ES-collections extends the class of exceptional collections.

Lemma 4.6. *An exceptional collection in \mathbf{D} is a reduced ES-collection in \mathbf{D} .*

Proof. Suppose A, B are objects in an exceptional collection which prevent it from being an ES-collection. Then there exists some $s \in \mathbb{Z}$ and $f \in \text{Hom}_{\mathbf{D}}(\tau^{-1}A, \Sigma^s B)$ such that $\varphi_A^*(f) = f \circ \varphi_A \neq 0$. In particular $\text{hom}_{\mathbf{D}}(A, \Sigma^s B) \neq 0$. However, using Serre duality,

$$0 \neq \text{hom}_{\mathbf{D}}(\tau^{-1}A, \Sigma^s B) = \text{hom}_{\mathbf{D}}(\Sigma^{s-1}B, A)$$

but this would contradict the fact that A, B are objects in an exceptional collection. If some object in the exceptional collection were in the thick subcategory generated by the other objects, then the class of this object in the Grothendieck group of \mathbf{D} could be written as a linear combination of the classes of the other object, however this would contradict the fact that the classes of objects in an exceptional collection are linearly independent in the Grothendieck group. \square

The next technical lemma will reduce the amount of work required to check if a set of exceptional and spherelike objects in \mathbf{D} is an ES-collection. In particular it means that for any pair of objects, we only need to verify the condition in one direction between the objects.

Lemma 4.7. *Let A, A' be indecomposable objects in \mathbf{D} . There exists a morphism $f \in \text{Hom}(\tau^{-1}A, A')$ such that $f \circ \varphi_A \neq 0$ if and only if there exists $g \in \text{Hom}(\tau^{-1}A', \Sigma A)$ such that $g \circ \varphi_{A'} \neq 0$.*

Proof. Using Lemma 3.25 we see that f exists if and only if one of the statements $\textcircled{0}$ or $\textcircled{1}$ holds with strict inequalities for A and A' . We can rewrite this as a statement about A' and ΣA which we can check is of the form of the other statement $\textcircled{1}$ or $\textcircled{0}$ with strict inequalities. Finally, g exists if and only if such a statement holds by Lemma 3.25. \square

Remark 4.8. Suppose that the condition for an ES-collection fails between two objects A and A' in \mathcal{X} or in \mathcal{Y} . Then either A and A' are in the same component, and the morphism $f \circ \varphi_A \neq 0$ factors through the component, or A' and ΣA are in the same component, and the morphism $g \circ \varphi_{A'} \neq 0$ factors through this component.

We finish this section, by showing that the set of reduced ES-collections is enough to generate all thick subcategories.

Theorem 4.9. *Any thick subcategory of \mathbf{D} is generated by a reduced ES-collection.*

Proof. Let $\mathbf{T} \subset \mathbf{D}$ be a thick subcategory. If \mathbf{T} intersects one of the \mathcal{Z} components, then by Proposition 2.1, \mathbf{T} is generated by an exceptional collection, which is a reduced ES-collection by Lemma 4.6. It only remains to treat thick subcategories of \mathcal{X} or \mathcal{Y} , which we may do separately, since these components are mutually fully orthogonal. We give the argument for a thick subcategory of \mathcal{X} but the \mathcal{Y} case is proved analogously.

Suppose $\mathbf{T} \subset \mathcal{X}$ is a thick subcategory. By Lemma 2.2 we know that \mathbf{T} is generated by a finite set of exceptional and spherelike objects. We build an ES-collection iteratively from these objects. Suppose that \mathbf{T}' is any thick subcategory generated by an ES-collection and let C be an exceptional or spherelike object in \mathcal{X} . We prove that $\text{thick}_{\mathbf{D}}(\mathbf{T}', C)$ is generated by an ES-collection. We argue by induction on the height of C .

Base case: Suppose $ht(C) = 0$ in \mathcal{X} . Then C is on the mouth of a component and $\mathcal{B}(C) = C \oplus \Sigma^{-1}\tau^{-1}C$ (see Lemma 3.15). There are no objects strictly between $\mathcal{B}^-(C) = C$ and $\Sigma\mathcal{B}^+(C) = \tau^{-1}C$ in the partial order and so Lemma 3.25 implies that $\varphi_C^*(\text{Hom}(\tau^{-1}C, -))$ is zero. Together with the symmetry from Lemma 4.7, this means that if we add C to the ES-collection generating \mathbf{T}' , we obtain an ES-collection as required.

Induction step: For any thick subcategory \mathbf{T}' which is generated by an ES-collection, and any exceptional or spherelike object C of height $ht(C) < h$, we assume that $\text{thick}_{\mathbf{D}}(\mathbf{T}', C)$ is generated by an ES-collection. Now denote by $\{A_i\}_{i \in I}$ an ES-collection generating some thick subcategory \mathbf{T}'' and suppose that D is an exceptional or spherelike object of height $ht(D) = h$. If $\{\{A_i\}_{i \in I}, D\}$ is an ES-collection, we are done. Otherwise we choose an object $A \in \{A_i\}_{i \in I}$ of minimal height such that the defining condition of an ES-collection fails. By Remark 4.8, replacing D with some shift as necessary, we may assume that A and D lie in the same component and there is a morphism f factoring through the component in some direction between A and D which causes the condition to fail.

We consider here the case where $f: A \rightarrow D$; the case where $f: D \rightarrow A$ can be shown using a similar argument. We have that $A = X^c(i, j)$ and $D = X^c(i + a, j + b)$ for some

$a, b > 0$ such that $a \leq j - i$ and so, using Lemma 1.6 we may calculate the cone of f :

$$(9) \quad A \xrightarrow{f} D \longrightarrow X^c(j+1, j+b) \oplus \Sigma X^c(i, i+a-1).$$

We show that $\{\{A_i\}_{i \in I}, X^c(i, i+a-1) = C_2\}$ is an ES-collection. Using Lemma 3.15 we note that $\mathcal{B}^-(C_2) = X^c(i, i) = \mathcal{B}^-(A)$ and $\mathcal{B}^+(C_2) = \Sigma^{-1}X^c(i+a, i+a) < \mathcal{B}^+(A)$ recalling that $i+a < j+1$. We argue by contradiction. If $\{\{A_i\}_{i \in I}, C_2\}$ is not an ES-collection then by Lemma 4.7 there must exist a morphism $g: \tau^{-1}C_2 \rightarrow A'$ where A' is some shift of an object in $\{A_i\}_{i \in I}$, such that $\varphi_{C_2}^*(g) \neq 0$. Lemma 3.25 then implies that one of the statements $\textcircled{0}$ or $\textcircled{1}$ holds for A' and C_2 with strict inequalities. If we substitute the above identities into $\textcircled{1}$ we see that $\Sigma^{-1}\mathcal{B}^-(A') < \mathcal{B}^-(A) < \mathcal{B}^+(A') < \Sigma\mathcal{B}^+(A)$ which would contradict the fact that (up to shift) A, A' are objects in an ES-collection. Therefore, since $\textcircled{1}$ can not hold, it must be that case that $\textcircled{0}$ does, so we have:

$$(10) \quad \mathcal{B}^-(A) < \mathcal{B}^-(A') < \Sigma\mathcal{B}^+(C_2) < \Sigma\mathcal{B}^+(A').$$

If $\mathcal{B}^+(A) < \mathcal{B}^+(A')$ then

$$\mathcal{B}^-(A) < \mathcal{B}^-(A') < \Sigma\mathcal{B}^+(C_2) < \Sigma\mathcal{B}^+(A) < \Sigma\mathcal{B}^+(A')$$

but again this would contradict the fact that A, A' are objects in an ES-collection. The remaining possibility is that $\mathcal{B}^+(A) \geq \mathcal{B}^+(A')$. In this case we see that

$$\mathcal{B}^-(A) < \mathcal{B}^-(A') < \Sigma\mathcal{B}^+(A') \leq \Sigma\mathcal{B}^+(A)$$

and consequently by Lemma 3.15, the height of A' is strictly less than the height of A . Using Lemma 3.15 we see $\mathcal{B}^-(D) = X^c(i+a, i+a) = \Sigma\mathcal{B}^+(C_2)$ and

$$\mathcal{B}^+(D) = \Sigma^{-1}X^c(j+b+1, j+b+1) > \Sigma^{-1}X^c(j+1, j+1) = \mathcal{B}^+(A) \geq \mathcal{B}^+(A').$$

Substituting these into (10) we get

$$\mathcal{B}^-(A') < \mathcal{B}^-(D) < \Sigma\mathcal{B}^+(A') < \Sigma\mathcal{B}^+(D).$$

Lemma 3.25 then implies that the ES-collection condition fails between A' and D , but this contradicts the minimality of the height of A . It follows therefore, that $\{\{A_i\}_{i \in I}, C_2\}$ is an ES-collection.

The object $C_1 = X^c(j+1, j+b)$ has height $b-1 < h$. The induction hypothesis then implies that $\text{thick}_{\mathbb{D}}(\{A_i\}_{i \in I}, C_2, C_1)$ is generated by an ES-collection. Using the triangle (9) it is clear that

$$\text{thick}_{\mathbb{D}}(\{A_i\}_{i \in I}, C_2, C_1) = \text{thick}_{\mathbb{D}}(\{A_i\}_{i \in I}, D)$$

and so $\text{thick}_{\mathbb{D}}(\mathbb{T}'', D)$ is generated by an ES-collection as required. We observe that if this ES-collection is not reduced, then some object is in the thick subcategory generated by the rest of the collection. Removing this object produces a smaller ES-collection which generates the same thick subcategory. Since there are a finite number of objects in the collection, it is clear that after removing a finite number of objects in this way, we obtain a reduced ES-collection. \square

5. MORPHISMS IN ES-COLLECTIONS AND FACTORING ARCS

In this section we use the geometric model to describe all of the indecomposable objects (up to shift) through which a morphism between two objects in an ES-collection factors.

We start by making the following definition, using the order defined in Definition 3.7.

Definition 5.1. Let $\widehat{\alpha}_1, \widehat{\alpha}_2$ and $\widehat{\alpha}_3$ be distinct non-trivial arcs on the universal cover and suppose that they have a common end-point \widehat{v} . We denote the other end-point of each arc $\widehat{\alpha}_i$ by \widehat{v}_i and suppose without loss of generality that $\widehat{v}_1 < \widehat{v}_2$. Then $\widehat{\alpha}_3$ is called a *factoring arc*, between $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ at \widehat{v} if:

$$\widehat{v}_1 < \widehat{v} < \widehat{v}_2 \iff \widehat{v}_3 < \widehat{v}_1 \text{ or } \widehat{v}_3 > \widehat{v}_2.$$

Lemma 5.2. Let A, B be distinct objects in an ES-collection, $\widehat{\alpha}_A$ be any fixed lift of α_A , and C be any other indecomposable object. If $f \in \text{Hom}_{\mathbb{D}}(A, B)$ is a basis morphism corresponding to statement $\textcircled{0}$ then there is a lift $\widehat{\alpha}_B$ of α_B such that either

$$\widehat{\alpha}_A(0) = \widehat{\alpha}_B(0) < \widehat{\alpha}_A(1) < \widehat{\alpha}_B(1) \quad \text{or} \quad \widehat{\alpha}_A(0) < \widehat{\alpha}_B(0) < \widehat{\alpha}_A(1) = \widehat{\alpha}_B(1).$$

If $f \in \text{Hom}_{\mathbb{D}}(A, B)$ is a basis morphism corresponding to statement $\textcircled{1}$ then there is a lift $\widehat{\alpha}_B$ of α_B such that

$$\widehat{\alpha}_B(0) < \widehat{\alpha}_A(0) = \widehat{\alpha}_B(1) < \widehat{\alpha}_B(1).$$

In each case, f factors through a shift $\Sigma^k C$ for some $k \in \mathbb{Z}$ if and only if there is a lift $\widehat{\alpha}_C$ which is a factoring arc between $\widehat{\alpha}_A$ and $\widehat{\alpha}_B$ at their common endpoint.

Proof. Since the deck transformations are order preserving, we assume without loss of generality that $\widehat{\alpha}_A$ is the lift with endpoints $(b^-(A), 0)$ and $(b^+(A), w(A))$. Consider the statement of the form $\textcircled{0}$ or $\textcircled{1}$ corresponding to f and note by Lemma 3.25 that not all of the inequalities are strict. However, A and B are distinct so at most one of the inequalities in the statement is an equality. A case by case calculation then shows that we can translate this statement into one of the above statements in terms of end-points of the arcs. For example, if A and B are in \mathcal{X} , and $\textcircled{0}$ is satisfied with equality $\mathcal{B}^-(A) = \mathcal{B}^-(B)$ then, in particular, $k^-(A) = k^-(B)$ and $b^-(A) = b^-(B)$. Shifting the whole statement by $\Sigma^{k^-(A)}$, so that all of the terms are in \mathcal{X}^0 , and applying the order preserving map ϖ_X from Definition 3.26 we see that

$$(b^-(A), 0) = (b^-(B), 0) < (b^+(A), w(A)) < (b^+(B), w(B)).$$

Therefore $\widehat{\alpha}_A(0) = \widehat{\alpha}_B(0) < \widehat{\alpha}_A(1) < \widehat{\alpha}_B(1)$ as required for the lift $\widehat{\alpha}_B$ with end-points $(b^-(B), 0)$ and $(b^+(B), w(B))$.

Now we consider which indecomposable objects the morphism f factors through. Suppose we are in the case where the statement corresponding to f is

$$\mathcal{B}^-(A) = \mathcal{B}^-(B) < \Sigma \mathcal{B}^+(A) < \Sigma \mathcal{B}^+(B)$$

and the object $S := \mathcal{B}^-(A) = \mathcal{B}^-(B)$ is in \mathcal{X} . The other cases are proved analogously. Using Lemma 3.15, it can be seen that the set of indecomposable objects D such that $\mathcal{B}^-(D) = S$ is precisely the set of objects on the ray from S in the \mathcal{X} -component or on a ray through \mathcal{Z} , which both form part of the long (co)ray from S to $\Sigma \tau S$ (see Figure 2). Furthermore we see that for such objects, the summand $\mathcal{B}^+(D)$ increases as D moves along the ray in \mathcal{X} and then along the ray in \mathcal{Z} . Since $\mathcal{B}^+(A) < \mathcal{B}^+(B)$, it follows that B lies further along the long (co)ray than A and so there is a non-zero morphism from A to B factoring through the indecomposable objects on the (co)ray between them (see Property 1.2(4)). If $\text{hom}_{\mathbb{D}}(A, B) = 1$, then this is clearly f up to scale, and if $\text{hom}_{\mathbb{D}}(A, B) = 2$ this also follows since f corresponds to a statement of the form $\textcircled{0}$ and factors through the component by Proposition 3.23. Therefore f factors through the indecomposable objects D on the (co)ray between A and B . In fact these are the only indecomposable objects through which f factors. In the general case this can be seen by observing that no other indecomposable object admits both a morphism from A and a morphism to B . In the $r = 1$ case, we observe that while such morphisms might exist,

at least one of these two morphisms would have to be in the infinite radical and so f couldn't factor in this way. For the objects D on the (co)ray between A and B we have

$$\mathcal{B}^-(A) = \mathcal{B}^-(B) = \mathcal{B}^-(D) < \Sigma\mathcal{B}^+(A) < \Sigma\mathcal{B}^+(D) < \Sigma\mathcal{B}^+(B).$$

Again rewriting this in terms of arcs, we see that such objects are precisely those for which there exists a lift $\widehat{\alpha}_D$ such that $\widehat{\alpha}_A(0) = \widehat{\alpha}_B(0) = \widehat{\alpha}_D(0) < \widehat{\alpha}_A(1) < \widehat{\alpha}_D(1) < \widehat{\alpha}_B(1)$, that is where $\widehat{\alpha}_D$ is a factoring arc between $\widehat{\alpha}_A$ and $\widehat{\alpha}_B$ at their common endpoint. \square

Now we consider the cones of such morphisms.

Lemma 5.3. *Let A and B be indecomposable objects, and suppose there exists a non-zero basis morphism $f \in \text{Hom}_{\mathcal{D}}(A, B)$ which is not in the image of*

$$\varphi_A^*: \text{Hom}_{\mathcal{D}}(\tau^{-1}A, B) \longrightarrow \text{Hom}_{\mathcal{D}}(A, B).$$

The cone of f has one indecomposable summand C and

$$\alpha'_C \simeq \alpha'_A \cdot \alpha'_B$$

where $\alpha'_X \simeq \alpha_X^{\pm 1}$ and the arcs are concatenated at the common end-point corresponding to f .

Proof. As in the proof of Lemma 5.2 we see that if A and B satisfy one of the circled statements with an equality, then they lie on a long (co)ray. In particular, the morphism f fits into one of the triangles from Lemma 1.3 or one of the triangles from Properties 1.2(5). In each of these cases, the cone is an indecomposable object. Writing these triangles in terms of the coordinates, the second part of the statement can then be shown by direct calculation. \square

Finally in this section we state a technical lemma that will be used as a tool later.

Lemma 5.4. *Let $\alpha = \alpha_0 \cdot \alpha_1 \cdots \alpha_s$ and $\gamma = \gamma_0 \cdot \gamma_1 \cdots \gamma_t$ be paths of arcs in a reduced arc-collection which start at a common vertex v_0 , and suppose $\alpha_0 \neq \gamma_0$. Denote by $\widehat{\alpha} = \widehat{\alpha}_0 \cdot \widehat{\alpha}_1 \cdots \widehat{\alpha}_s$ and $\widehat{\gamma} = \widehat{\gamma}_0 \cdot \widehat{\gamma}_1 \cdots \widehat{\gamma}_t$ the lifts of these paths to the universal cover, starting at $\widehat{v}_0 = (v_0, 0)$. We label the other vertices along the paths by $\widehat{v}_1, \dots, \widehat{v}_{s+1}$ and $\widehat{v}'_1, \dots, \widehat{v}'_{t+1}$ respectively. If*

$$\widehat{v}_i < \widehat{v}_0 < \widehat{v}'_j \text{ or } \widehat{v}'_j < \widehat{v}_i < \widehat{v}_0 \text{ or } \widehat{v}_0 < \widehat{v}'_j < \widehat{v}_i$$

for some $1 \leq i \leq s+1$ and $1 \leq j \leq t+1$ then for each $1 \leq i \leq s+1$ and $1 \leq j \leq t+1$, either

$$\widehat{v}_i < \widehat{v}_0 < \widehat{v}'_j \text{ or } \widehat{v}'_j < \widehat{v}_i < \widehat{v}_0 \text{ or } \widehat{v}_0 < \widehat{v}'_j < \widehat{v}_i.$$

Proof. It is a short exercise to show that if the statement fails, then the two paths on the universal cover must cross, but this would contradict either non-crossing or reducedness. \square

6. REDUCED COLLECTIONS

In Lemma 4.4 we proved that there is a correspondence between ES-collections and arc-collections. When comparing collections however, it will be extremely useful to know that we are dealing with minimal sets of generating objects. This was why we introduced the notion of *reduced* ES-collections and arc-collections. In this section we prove that the correspondence also holds for the reduced collections. The following results will also play a key role in proving the isomorphisms of posets in Section 8.

Theorem 6.1. Let $\mathcal{D} = \mathcal{D}^b(\Lambda(r, n, m))$ be a discrete derived category with $n > r > 1$ and let $\{B_i\}_{i \in I}$ be an ES-collection in \mathcal{D} with corresponding arcs $\{\gamma_i \mid i \in I\}$. Suppose A is an indecomposable object in $\text{thick}(\{B_i\}_{i \in I})$. Then

$$\alpha_A \simeq \gamma'_{i_0} \cdot \gamma'_{i_1} \cdots \gamma'_{i_s}$$

where each arc $\gamma'_{i_j} \simeq \gamma_{i_j}^{\pm 1}$.

Proof. We will say that an indecomposable object A such that $\alpha_A \simeq \gamma'_{i_0} \cdot \gamma'_{i_1} \cdots \gamma'_{i_s}$ as above, is γ -generated. Since by definition $\alpha_{B_i} \simeq \gamma_i$ it is clear that the objects $\{B_i\}_{i \in I}$ are γ -generated. We note that if A satisfies the γ -generation condition, then all shifts of A also satisfy it, since $\alpha_A = \alpha_{\Sigma A}$.

Step 1: Suppose that indecomposable objects A and B are γ -generated, and that C is the cone of a morphism $f: A \rightarrow B$. We prove that any indecomposable summand of C also satisfies the γ -generation condition. Note that if $f = 0$ then this is trivially true so we consider the case where f is non-zero. Since we are assuming that $r > 1$, at most one of the statements of the form $\textcircled{0}$ or $\textcircled{1}$ is satisfied and f is therefore the corresponding basis morphism up to scale. If one of the equalities in the statement is satisfied, then Lemma 5.3 implies immediately that C is indecomposable and is γ -generated. Suppose therefore that the statement is satisfied with strict inequalities. In this case $C = C_1 \oplus C_2$ has two indecomposable summands. As in the proof of Theorem 3.28 we choose lifts $\widehat{\alpha}_A$ and $\widehat{\alpha}_B$ and identifications ϖ_X, ϖ_Y which take the satisfied statement to the corresponding statement in terms of end-points of the lifts. Using the assumption that A and B satisfy the γ -generation condition, we write $\widehat{\alpha}_A \simeq \widehat{\gamma}'_{i_0} \cdot \widehat{\gamma}'_{i_1} \cdots \widehat{\gamma}'_{i_s}$ and $\widehat{\alpha}_B \simeq \widehat{\gamma}''_{j_0} \cdot \widehat{\gamma}''_{j_1} \cdots \widehat{\gamma}''_{j_t}$. Since $\iota(\widehat{\alpha}_A, \widehat{\alpha}_B) = 1$ the paths $\widehat{\gamma}'_{i_0} \cdot \widehat{\gamma}'_{i_1} \cdots \widehat{\gamma}'_{i_s}$ and $\widehat{\gamma}''_{j_0} \cdot \widehat{\gamma}''_{j_1} \cdots \widehat{\gamma}''_{j_t}$ must intersect in at least one point. The non-crossing property implies that the intersection locus must contain a point which is an end-point of arcs in both paths, that is, a point $v = \widehat{\gamma}'_{i_k}(0) = \widehat{\gamma}''_{j_l}(0)$. Using this common point, we can splice together the two paths to produce a path of arcs between any distinct pair of points from the set $\{\widehat{\alpha}_A(0), \widehat{\alpha}_A(1), \widehat{\alpha}_B(0), \widehat{\alpha}_B(1)\}$. By Lemma 3.30, we know $\mathcal{B}^\pm(C_1)$ and $\mathcal{B}^\pm(C_2)$ in terms of $\mathcal{B}^\pm(A)$ and $\mathcal{B}^\pm(B)$ and a calculation in each case, shows that the arcs corresponding to C_1 and C_2 are homotopy equivalent to arcs with endpoints in the set $\{\widehat{\alpha}_A(0), \widehat{\alpha}_A(1), \widehat{\alpha}_B(0), \widehat{\alpha}_B(1)\}$, projected down to the cylinder. Thus C_1 and C_2 satisfy the γ -generation condition.

Step 2: Suppose that A_0, \dots, A_d are indecomposable objects which satisfy the γ -generation condition, and that C is the cone of a morphism $f: A_0 \rightarrow \bigoplus_{i=1}^d A_i$. We prove that any indecomposable summand of C also satisfies the γ -generation condition. We proceed by induction on the number of summands d . In the case $d = 1$, this was done in Step 1. Now suppose the statement holds for any set of such indecomposable objects $A'_0, \dots, A'_{d'}$ where $d' < d$, and any morphism $f': A'_0 \rightarrow \bigoplus_{i=1}^{d'} A'_i$. Consider the triangle

$$A_0 \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}} \bigoplus_{i=1}^d A_i \longrightarrow C$$

If one of the $f_k = 0$ then a calculation shows that

$$C = \text{Cone}((f_1, \dots, f_{k-1}, f_{k+1}, \dots, f_d)^T) \oplus A_k$$

and the result follows using the induction hypothesis. Therefore, we assume that $f_k \neq 0$ for all $k = 1, \dots, d$. For any k we consider the following diagram constructed using the

octahedral axiom.

(11)

$$\begin{array}{ccccc}
& & A_0 & \xlongequal{\quad\quad\quad} & A_0 \\
& & \downarrow f & & \downarrow f_k \\
\bigoplus_{i \neq k} A_i & \longrightarrow & \bigoplus_i A_i & \xrightarrow{(0, \dots, 0, 1, 0, \dots, 0)} & A_k \\
\parallel & & \downarrow & & \downarrow \\
\Sigma^{-1} \text{Cone}(f_k) & \longrightarrow & \bigoplus_{i \neq k} A_i & \longrightarrow & C & \longrightarrow & \text{Cone}(f_k)
\end{array}$$

If $f_k \notin \varphi_{A_0}^* \text{Hom}(A_0, A_k)$, then $\text{Cone}(f_k)$ is indecomposable by Lemma 5.3 and is γ -generated by Step 1. Therefore, considering C as the cone of a morphism in the bottom row of the diagram, the result follows using the induction hypothesis.

This leaves the situation where $f_k \in \varphi_{A_0}^* \text{Hom}(A_0, A_i)$ for all $k = 1, \dots, d$. We break the proof into three cases:

Case 1: A_0 is in an \mathcal{X} component. For each f_k , one of the statements of the form ① or ② is satisfied with strict inequalities. In particular for each k , precisely one of $\mathcal{B}^-(A_k)$ or $\mathcal{B}^+(A_k)$ is contained in the open interval $(\mathcal{B}^-(A_0), \Sigma\mathcal{B}^+(A_0))$. We fix a k such that this element $\mathcal{B}^\bullet(A_k)$ is minimal in the order on the interval. Using Lemma 3.30 we see that one of the two summands of $\text{Cone}(f_k) = C' \oplus C''$ satisfies $\mathcal{B}^-(\Sigma^{-1}C') = \mathcal{B}^-(A_0)$ and $\Sigma\mathcal{B}^+(\Sigma^{-1}C') = \mathcal{B}^\bullet(A_k)$. In particular, the minimality condition ensures that $\mathcal{B}^\pm(A_i) \notin [\mathcal{B}^-(\Sigma^{-1}C'), \Sigma\mathcal{B}^+(\Sigma^{-1}C')]$ for any $i = 1, \dots, d$. Then Proposition 3.23 implies that $\text{Hom}(\Sigma^{-1}C', A_i) = 0$. Therefore, looking at the bottom row of the diagram (11), we have a triangle

$$\Sigma^{-1}C' \oplus \Sigma^{-1}C'' \xrightarrow{\begin{pmatrix} 0 & g_1 \\ \vdots & \vdots \\ 0 & g_{k-1} \\ 0 & g_{k+1} \\ \vdots & \vdots \\ 0 & g_d \end{pmatrix}} \bigoplus_{i \neq k} A_i \longrightarrow C \longrightarrow C' \oplus C''$$

for some morphisms $g_i \in \text{Hom}(\Sigma^{-1}C'', A_i)$. A straightforward calculation shows that

$$C = \text{Cone}((g_1, \dots, g_{k-1}, g_{k+1}, \dots, g_d)^T) \oplus C'$$

and the result follows using the induction hypothesis.

Case 2: A_0 is in a \mathcal{Y} component. The argument is analogous to Case 1.

Case 3: A_0 is in a \mathcal{Z} component.

i) If at least one of the morphisms f_k factors through the \mathcal{Z} component or maps to an object in an \mathcal{X} component, then choose k such that $\mathcal{B}^\bullet(A_k)$ is minimal such that $\mathcal{B}^\bullet(A_k) > \mathcal{B}^-(A_0)$. Again using Lemma 3.30 we see that one of the two summands C' of $\text{Cone}(f_k) = C' \oplus C''$ satisfies $\mathcal{B}^-(\Sigma^{-1}C') = \mathcal{B}^-(A_0)$ and $\Sigma\mathcal{B}^+(\Sigma^{-1}C') = \mathcal{B}^\bullet(A_k)$. In particular C' is in an \mathcal{X} -component and the minimality condition ensures that $\mathcal{B}^\pm(A_i) \notin [\mathcal{B}^-(\Sigma^{-1}C'), \Sigma\mathcal{B}^+(\Sigma^{-1}C')]$ for any $i = 1, \dots, d$. Again using Proposition 3.23 we see that $\text{Hom}(\Sigma^{-1}C', A_i) = 0$. The rest of the argument then works as in Case 1.

ii) If no f_k factors through the \mathcal{Z} component or maps to an object in an \mathcal{X} component but at least one f_k maps to an object in an \mathcal{Y} component, then choose k such that $\mathcal{B}^+(A_k)$ is minimal such that $\mathcal{B}^+(A_k) > \mathcal{B}^+(A_0)$. One of the summands of $\text{Cone}(f_k)$ satisfies $\Sigma\mathcal{B}^+(\Sigma^{-1}C') = \mathcal{B}^+(A_k)$ and $\mathcal{B}^-(\Sigma^{-1}C') = \mathcal{B}^+(A_0)$. Then C' is in an \mathcal{Y} -component and the minimality condition ensures that $\mathcal{B}^\pm(A_i) \notin [\mathcal{B}^-(\Sigma^{-1}C'), \Sigma\mathcal{B}^+(\Sigma^{-1}C')]$ for any $i = 1, \dots, d$. Again using Proposition 3.23 we see that $\text{Hom}(\Sigma^{-1}C', A_i) = 0$ and the rest of the argument then works as in Case 1.

iii) If all the morphisms f_k are in the infinite radical and map into a \mathcal{Z} component, then we choose k such that $\mathcal{B}^-(A_k)$ is minimal. Using Lemma 3.30 we see that one of the summands of $\text{Cone}(f_k)$ satisfies $\mathcal{B}^-(\Sigma^{-1}C') = \mathcal{B}^-(\Sigma^{-1}A_k)$ and $\Sigma\mathcal{B}^+(\Sigma^{-1}C') = \mathcal{B}^+(A_0)$. The minimality condition ensures that $\Sigma\mathcal{B}^-(\Sigma^{-1}C') \leq \mathcal{B}^-(A_i)$ for all $i = 1, \dots, d$ which prevents ① from being satisfied. Condition ② could only be satisfied if $\Sigma^{-1}C'$ and A_i were in the same component, but this isn't the case since $r > 1$. Therefore, $\text{Hom}(\Sigma^{-1}C', A_i) = 0$ and the rest of the argument again works as in Case 1.

Step 3: Suppose that A_0, \dots, A_d and A'_0, \dots, A'_d are indecomposable objects which satisfy the γ -generation condition, and that C is the cone of a morphism $f: \bigoplus_{i=0}^d A_i \rightarrow \bigoplus_{i=0}^d A'_i$. We prove that any indecomposable summand of C also satisfies the γ -generation condition. We do induction on d ; the case $d = 0$ was Step 2 above. The result follows from the following diagram using the induction hypothesis on the middle column and then the bottom row.

$$\begin{array}{ccccc}
A_0 & \xlongequal{\quad} & A_0 & & \\
\downarrow & & \downarrow & & \\
\bigoplus_i A_i & \xrightarrow{f} & \bigoplus_i A'_i & \longrightarrow & C \\
\downarrow & & \downarrow & & \parallel \\
\bigoplus_{i>0} A_i & \longrightarrow & C_0 & \longrightarrow & C
\end{array}$$

Step 4: Any given object in $\text{thick}(\{B_i\}_{i \in I})$ can be generated in finitely many steps by taking shifts, cones and summands. In the previous steps we have shown that the property of γ -generation for any summand is closed under these operations. \square

We also prove the converse.

Lemma 6.2. Let $\{\beta_i\}$ be a set of arcs corresponding to some indecomposable objects $\underline{B} = \{B_i\}$, and let A be an indecomposable object, such that

$$\alpha_A \simeq \beta_1 \cdot \beta_2 \cdots \beta_s.$$

Then $A \in \text{thick}(\underline{B})$.

Proof. We lift this to a path of arcs $\widehat{\beta}_1 \cdot \widehat{\beta}_2 \cdots \widehat{\beta}_s$ in the universal cover. The common end-point of $\widehat{\beta}_1$ and $\widehat{\beta}_2$ uniquely determines a morphism between B_1 and some shift of B_2 . The cone of this morphism is clearly in $\text{thick}(\underline{B})$ and Lemma 5.3 shows that the corresponding arc is homotopy equivalent to the path $\beta_1 \cdot \beta_2$. By iteratively taking cones, we can construct in this way an indecomposable object A' in $\text{thick}(\underline{B})$ whose corresponding arc is homotopy equivalent to α . Theorem 3.28 then implies that A' is isomorphic to A up to shift. Therefore $A \in \text{thick}(\underline{B})$ as required. \square

We immediately get the following corollary of Lemma 4.4, Theorem 6.1 and Lemma 6.2.

Corollary 6.3. $\{A_i\}_{i \in I}$ is a reduced ES-collection in \mathcal{D} , if and only if $\{\alpha_{A_i}\}_{i \in I}$ is a reduced arc-collection in $C(p, q)$.

7. MUTATION OF ES-COLLECTIONS

We now consider how to mutate from one ES-collection to another.

Definition 7.1. Let $\{\alpha_i\}_{i \in I}$ be a reduced arc-collection. Suppose α_a and α_b are distinct arcs in the collection which have a common end-point v , such that there are no factoring arcs between α_a and α_b at v . Let

$$\bar{\alpha}_a \simeq \alpha'_a \cdot \alpha'_b$$

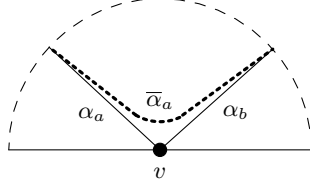


FIGURE 11. Mutation of α_a past α_b .

where $\alpha'_a \simeq \alpha_a^{\pm 1}$ and $\alpha'_b \simeq \alpha_b^{\pm 1}$ and they are concatenated at the common end-point v (see Figure 11). We consider a set of arcs

$$\mu_{a,b}^v(\{\alpha_i\}_{i \in I}) = \{\alpha_i\}_{i \in I \setminus \{a\}} \cup \bar{\alpha}_a$$

and say that this is obtained by *mutating* α_a past α_b at v .

Lemma 7.2. *The set $\mu_{a,b}^v(\{\alpha_i\}_{i \in I})$ is a reduced arc-collection.*

Proof. We lift α_a and α_b to arcs $\hat{\alpha}_a$ and $\hat{\alpha}_b$ on the punctured disc model of the universal cover such that their common end-point v lifts to a point \hat{v} . We denote the other end-points of $\hat{\alpha}_a$ and $\hat{\alpha}_b$ by \hat{v}_a and \hat{v}_b respectively. Without loss of generality we consider representatives in the homotopy classes of $\hat{\alpha}_a$ and $\hat{\alpha}_b$ which are chords on the disc. By construction, there is a lift $\hat{\alpha}_a$ of $\bar{\alpha}_a$ which is a chord between \hat{v}_a and \hat{v}_b . Suppose to the contrary that $\mu_{a,b}^v(\{\alpha_i\}_{i \in I})$ is not an arc-collection. Then there is some lift $\hat{\beta}$ of an arc in the set which intersects $\hat{\alpha}_a$ once on the universal cover. However such an arc must also intersect $\hat{\alpha}_a$ or $\hat{\alpha}_b$ or have an endpoint at \hat{v} and be a factoring arc between $\hat{\alpha}_a$ or $\hat{\alpha}_b$. This would contradict the assumption that $\{\alpha_i\}_{i \in I}$ is an arc-collection, or that there are no factoring arcs between α_a and α_b at v . It is clear that the collection is reduced if and only if the starting collection was reduced. \square

We can do the analogous procedure for ES-collections.

Lemma 7.3. *Let $\underline{A} := \{A_i\}_{i \in I}$ be a reduced ES-collection in \mathcal{D} . For any $a, b \in I$ with $a \neq b$ and basis morphism $f \in \text{Hom}^\bullet(A_a, A_b)$ which doesn't factor non-trivially through any $\Sigma^k A_i$ for some $k \in \mathbb{Z}$ and $i \in I$, then the sets*

$$\mathcal{R}_f \underline{A} := \{A_i\}_{i \in (I \setminus \{a\})} \cup \text{cone}(f), \quad \mathcal{L}_f \underline{A} := \{A_i\}_{i \in (I \setminus \{b\})} \cup \text{cocone}(f)$$

are reduced ES-collections.

Proof. The ES-collection $\underline{A} := \{A_i\}_{i \in I}$ corresponds to an arc-collection $\{\alpha_i\}_{i \in I}$. We consider the intersection corresponding to the morphism $f \in \text{Hom}^\bullet(A_a, A_b)$ which must be at a common end-point v . Lemma 5.2 implies that there are no factoring arcs between α_a and α_b at v . By Lemma 5.3 the object $\text{cone}(f)$ is indecomposable, and corresponds to the arc which is the concatenation of α_a and α_b at v . Therefore, result then follows from Lemma 7.2 using the correspondence from Corollary 6.3. \square

Remark 7.4. It is clear from this proof that the mutations of reduced ES-collections and reduced arc-collections are compatible, so the mutated ES-collection $\mathcal{R}_f \underline{A}$ (respectively $\mathcal{L}_f \underline{A}$) corresponds to the mutated arc-collection $\mu_{a,b}^v(\{\alpha_{A_i}\}_{i \in I})$ (respectively $\mu_{b,a}^v(\{\alpha_{A_i}\}_{i \in I})$).

Definition 7.5. Let $\underline{A} := \{A_i\}_{i \in I}$ be a reduced ES-collection in \mathcal{D} . We call a basis morphism $f \in \text{Hom}^\bullet(A_a, A_b)$ *mutable* if it doesn't factor non-trivially through $\Sigma^k A_i$ for any $i \in I$ and $k \in \mathbb{Z}$. If $f \in \text{Hom}^\bullet(A_a, A_b)$ is a mutable morphism, we call $\mathcal{L}_f \underline{A}$ (respectively $\mathcal{R}_f \underline{A}$) the *left* (respectively *right*) *mutation of \underline{A} along f* .

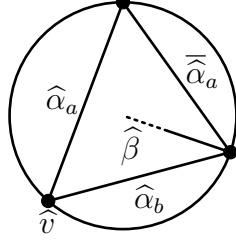


FIGURE 12. Any factoring arc $\hat{\beta}$ would intersect $\hat{\alpha}_a$.

Lemma 7.6. *Suppose $f \in \text{Hom}^k(A_a, A_b)$ is a mutable morphism in \underline{A} which fits into the triangle*

$$\text{cocone}(f) \xrightarrow{f''} A_a \xrightarrow{f} \Sigma^k A_b \xrightarrow{f'} \text{cone}(f).$$

Then f' is mutable in the collection $\mathcal{R}_f \underline{A}$ and f'' is mutable in $\mathcal{L}_f \underline{A}$ and

$$\mathcal{L}_{f'}(\mathcal{R}_f \underline{A}) = \underline{A} = \mathcal{R}_{f''}(\mathcal{L}_f \underline{A}).$$

Proof. We consider the lifts of the corresponding arcs $\hat{\alpha}_a$ and $\hat{\alpha}_b$ with a common end-point \hat{v} corresponding to the morphism f . By Lemma 5.3 the object $\text{cone}(f)$ corresponds to the arc $\hat{\alpha}_a$, and the morphism f' from $\Sigma^k A_b$ to $\text{cone}(f)$ corresponds to the common end-point of $\hat{\alpha}_b$ and $\hat{\alpha}_a$. Suppose to the contrary that this morphism is not mutable. Then there is a factoring arc $\hat{\beta}$ between $\hat{\alpha}_b$ and $\hat{\alpha}_a$ which corresponds to an object in the arc-collection $\mu_{a,b}^v(\{\alpha_i\}_{i \in I})$. Looking at the Figure 12 we see that $\hat{\beta}$ must intersect $\hat{\alpha}_a$ which would contradict the non-crossing property unless $\hat{\beta}$ is also a lift of $\bar{\alpha}_a$. This would imply that $\hat{\alpha}_a(0)$ and $\hat{\alpha}_a(1)$ are covers of the same vertex, so $\bar{\alpha}_a$ corresponds to a sphere-like object. The factoring arc property then implies that \hat{v} is on the same boundary component of the universal cover as $\hat{\alpha}_a(0)$ and $\hat{\alpha}_a(1)$. It then follows that $\hat{\alpha}_a$ must be longer than the spherelike arc $\bar{\alpha}_a$, contradicting the assumption that it is in an arc-collection. Therefore, f' is mutable. Finally, we note that

$$\begin{aligned} \mathcal{L}_{f'}(\mathcal{R}_f \underline{A}) &= \mathcal{L}_{f'}(\{A_i\}_{i \in (I \setminus \{a\})} \cup \text{cone}(f)) \\ &= ((\{A_i\}_{i \in (I \setminus \{a\})} \cup \text{cone}(f)) \setminus \text{cone}(f)) \cup \text{cocone}(f') = \underline{A} \end{aligned}$$

since $\text{cocone}(f') = A_a$. □

Remark 7.7. If (A_a, A_b) forms an exceptional pair such that $\text{hom}_{\mathbb{D}/\Sigma}(A_a, A_b) = 1$, then the mutation of A_a past A_b defined above, coincides with mutation of A_a past A_b as an exceptional pair. If $\text{hom}_{\mathbb{D}/\Sigma}(A_a, A_b) = 2$ then this is not the case. However, using the octahedral axiom, one can see that the mutation of the exceptional pair decomposes as a pair of mutations in the ES-collection.

Definition 7.8. We define an equivalence relation on the set of all reduced arc-collections by saying that $\{\alpha_i\}_{i \in I} \sim_{\text{mut}} \{\beta_j\}_{j \in J}$ if there is a sequence of mutations taking $\{\alpha_i\}_{i \in I}$ to $\{\beta_j\}_{j \in J}$. Similarly, we define an equivalence relation on the set of all reduced ES-collections by saying that $\underline{A} \sim_{\text{mut}} \underline{B}$ if there is a sequence of mutations and suspensions of the objects, taking \underline{A} to \underline{B} .

Definition 7.9. We define

$$\begin{aligned} \text{Arc}_{\mathbb{D}}^{\text{mut}} &:= \{\text{reduced arc-collections}\} / \sim_{\text{mut}} \\ \text{ES}_{\mathbb{D}}^{\text{mut}} &:= \{\text{reduced ES-collections}\} / \sim_{\text{mut}} \end{aligned}$$

to be the sets of reduced arc-collections and reduced ES-collections up to mutation.

Lemma 7.10. *There is a bijective correspondence between the sets $\text{Arc}_D^{\text{mut}}$ and ES_D^{mut} .*

Proof. Corollary 6.3 sets up a bijection between the set of reduced arc-collections and the set of reduced ES-collections. It follows from Remark 7.4 that two reduced ES-collections are related by a sequence of mutations if and only if the corresponding reduced arc-collections are related by a sequence of mutations. \square

We can put partial orders on $\text{Arc}_D^{\text{mut}}$ and ES_D^{mut} as follows. Let \mathcal{C} be the set of connected components of the union of arcs in a reduced arc-collection (when in minimal position). If we write the corresponding reduced ES-collection \underline{A} as a union of a maximal number subsets which are fully orthogonal to each other, we see that these subsets are also indexed by \mathcal{C} , since there is a morphism between two objects if and only if the corresponding arcs intersect. Therefore, $\underline{A} = \bigcup_{c \in \mathcal{C}} \underline{A}^c$ and this in turn corresponds to a decomposition of $\text{thick}(\underline{A})$ into fully orthogonal thick subcategories of D

$$\text{thick}(\underline{A}) = \bigoplus_{c \in \mathcal{C}} \text{thick}(\underline{A}^c).$$

We note that by construction, mutation preserves the connected components \mathcal{C} and the decomposition into orthogonal thick subcategories.

Definition 7.11. We define \leq_{mut} on the set $\text{Arc}_D^{\text{mut}}$, by saying $\{\alpha_i\}_{i \in I} \leq_{\text{mut}} \{\beta_j\}_{j \in J}$ if each subset of arcs $\{\alpha_i\}_{i \in I^c} \subset \{\alpha_i\}_{i \in I}$ whose union is a connected component $c \in \mathcal{C}$, can be extended to a reduced arc-collection $\{\alpha_i\}_{i \in I^c} \cup \{\alpha_{i'}\}_{i' \in I'}$ such that

$$(\{\alpha_i\}_{i \in I^c} \cup \{\alpha_{i'}\}_{i' \in I'}) \sim_{\text{mut}} \{\beta_j\}_{j \in J}.$$

Analogously, we define \leq_{mut} on the set ES_D^{mut} , by saying $\underline{A} \leq_{\text{mut}} \underline{B}$ if each connected component \underline{A}^c of \underline{A} can be extended to a reduced ES-collection $\underline{A}^c \cup \underline{A}'$ such that

$$(\underline{A}^c \cup \underline{A}') \sim_{\text{mut}} \underline{B}.$$

Lemma 7.12. *Let \underline{A} and \underline{B} be any ES-collections. Then*

$$\underline{A} \leq_{\text{mut}} \underline{B} \text{ if and only if } \{\alpha_A\}_{A \in \underline{A}} \leq_{\text{mut}} \{\alpha_B\}_{B \in \underline{B}}.$$

Proof. Suppose $\underline{A} \leq_{\text{mut}} \underline{B}$. Then by definition, for each $c \in \mathcal{C}$, the ES-collection \underline{A}^c can be extended such that $(\underline{A}^c \cup \underline{A}') \sim_{\text{mut}} \underline{B}$. Under the correspondence from Corollary 6.3, and using the compatibility of mutation observed in Remark 7.4, we see that $\{\alpha_A\}_{A \in \underline{A}^c} \cup \{\alpha_{A'}\}_{A' \in \underline{A}'}$ is an arc-collection which is mutation equivalent to $\{\alpha_B\}_{B \in \underline{B}}$. It follows that $\{\alpha_A\}_{A \in \underline{A}} \leq_{\text{mut}} \{\alpha_B\}_{B \in \underline{B}}$. The converse argument is essentially the same. \square

Lemma 7.13. *If $\underline{A} \leq_{\text{mut}} \underline{B}$ then $\text{thick}(\underline{A}) \subseteq \text{thick}(\underline{B})$.*

Proof. It is clear from the definition of mutation of ES-collections, that if \underline{B} and \underline{B}' differ by a mutation, then $\text{thick}(\underline{B}) = \text{thick}(\underline{B}')$. Therefore, for each connected component $\underline{A}^c \subseteq \text{thick}(\underline{A}^c, \underline{A}') = \text{thick}(\underline{B})$. \square

Lemma 7.14. \leq_{mut} is a well defined partial order on $\text{Arc}_D^{\text{mut}}$ and on ES_D^{mut} .

Proof. We prove the result for ES_D^{mut} . The result for $\text{Arc}_D^{\text{mut}}$ then follows from Lemma 7.10 and Lemma 7.12. Reflexivity is clear. To see antisymmetry, let \mathcal{C} and \mathcal{C}' be the indexing sets of the connected components of \underline{A} and \underline{B} respectively. Suppose $\underline{A} \leq_{\text{mut}} \underline{B}$. Then each object in \underline{A}^c is contained in $\text{thick}(\underline{B})$ and in particular their connectedness implies that they are in one of the pieces $\text{thick}(\underline{B}^{c'})$ of the decomposition. We can therefore define a map $\mathcal{C} \rightarrow \mathcal{C}'$. If additionally $\underline{B} \leq_{\text{mut}} \underline{A}$, then we see that each object of $\underline{B}^{c'}$ is in $\text{thick}(\underline{A}^c)$ (and so $\text{thick}(\underline{A}^c) = \text{thick}(\underline{B}^{c'})$). It follows by symmetry that there is a bijection

between \mathcal{C} and \mathcal{C}' . Identifying \mathcal{C} and \mathcal{C}' , and using the fact that mutation preserves the components of the decomposition, we see that $(\underline{A}^c \cup \underline{A}') \sim_{\text{mut}} \underline{B}$ implies that $\underline{A}^c \sim_{\text{mut}} \underline{B}^c$, and this holds for each $c \in \mathcal{C}$. Therefore $\underline{A} \sim_{\text{mut}} \underline{B}$ as required.

To show transitivity, suppose $\underline{A} \leq_{\text{mut}} \underline{B}$ and $\underline{B} \leq_{\text{mut}} \underline{C}$. Given any $c \in \mathcal{C}$ then there exist c' such that $\text{thick}(\underline{A}^c) \subseteq \text{thick}(\underline{B}^{c'})$. Using the orthogonality as above, we can extend \underline{A}^c to a reduced ES-collection such that $(\underline{A}^c \cup \underline{A}') \sim_{\text{mut}} \underline{B}^{c'}$ and we can then extend $\underline{B}^{c'}$ such that $(\underline{B}^{c'} \cup \underline{B}') \sim_{\text{mut}} \underline{C}$. We claim that we can apply a sequence of mutations taking $\underline{B}^{c'} \cup \underline{B}'$ to some collection $\underline{A}^c \cup \underline{A}' \cup \underline{A}''$. We start with the sequence of mutations taking $\underline{B}^{c'}$ to $\underline{A}^c \cup \underline{A}'$ and try to apply the same sequence of mutations to $\underline{B}^{c'} \cup \underline{B}'$. The only problem occurs if one of the morphisms $f: B_i \rightarrow B_j$ along which we would like to mutate, factors through shifts of the extra objects, that is when f factors as $B_i \rightarrow \Sigma^{k_1} B'_1 \rightarrow \cdots \rightarrow \Sigma^{k_s} B'_s \rightarrow B_j$ along a long (co)ray. If this happens, we can first mutate each of these factoring objects in turn past B_j until f becomes mutable and then apply the mutation at f . \square

8. THE LATTICE OF THICK SUBCATEGORIES

Let $\mathbf{D} = \mathbf{D}^b(\Lambda(r, n, m))$ be a discrete derived category with $n > r > 1$ and let $\text{Thick}_{\mathbf{D}}$ denote the lattice of thick subcategories of \mathbf{D} , ordered by inclusion. In this section we prove the main theorem, which allows us to understand this lattice of thick subcategories, in terms of arc-collections and ES-collections up to mutation.

Theorem 8.1. *Let \underline{A} and \underline{B} be reduced ES-collections. The following are equivalent:*

- i) $\text{thick}(\underline{A}) \subseteq \text{thick}(\underline{B})$
- ii) $\underline{A} \leq_{\text{mut}} \underline{B}$.
- iii) $\{\alpha_A\}_{A \in \underline{A}} \leq_{\text{mut}} \{\alpha_B\}_{B \in \underline{B}}$.

In particular, there are isomorphisms of partially ordered sets:

$$\text{Thick}_{\mathbf{D}} \xleftarrow{\cong} \text{ES}_{\mathbf{D}}^{\text{mut}} \xleftarrow{\cong} \text{Arc}_{\mathbf{D}}^{\text{mut}}.$$

Proof. By Theorem 4.9, any thick subcategory is generated by a reduced ES-collection, which corresponds to a reduced arc-collection by Corollary 6.3. Therefore, the final statement of the theorem follows from the first part. We observe that $(ii \iff iii)$ is Lemma 7.12 and $(ii \implies i)$ is Lemma 7.13. The content of this section will therefore be in the proof that $(i \implies ii)$.

We note that the following version of this result is well known for exceptional collections in $\mathbf{D}^b(\mathbf{k}A_n)$ from work of Crawley-Boevey [10].

Lemma 8.2. *Let \underline{A} and \underline{B} be exceptional collections in $\mathbf{D}^b(\mathbf{k}A_n)$ such that $\text{thick}(\underline{A}) \subseteq \text{thick}(\underline{B})$. Then \underline{A} can be extended to an exceptional collection $(\underline{A}', \underline{A})$ which is equivalent to \underline{B} up to mutation and suspension of the objects in the collection.*

Proof. We can use the suspension functor to shift all of the objects in the collection \underline{A} into a standard heart. By [10, Lemma 1], this exceptional collection can be extended to a full exceptional collection in $\text{thick}(\underline{B})$. The braid group then acts transitively via mutation [10, Theorem]. \square

Recall that the discrete derived categories have a semi-orthogonal decomposition where one of the factors is equivalent to $\mathbf{D}^b(\mathbf{k}A_{n+m-1})$ and the other factor is generated by an exceptional object in \mathcal{Z} [7, Proposition 7.6]. We would like to use this decomposition to bootstrap up the result for exceptional collections in $\mathbf{D}^b(\mathbf{k}A_n)$. We start by considering a restricted case when the ES-collections \underline{A} and \underline{B} contain a common object in \mathcal{Z} . As a

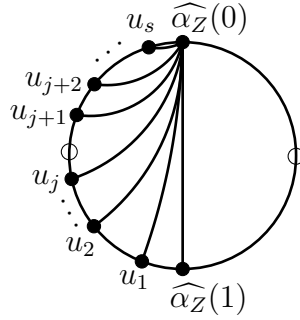


FIGURE 13. Picture of the universal cover showing lifts of those arcs with an end-point at $\alpha_Z(0)$, which contribute to $\iota(\alpha_Z, -) \neq 0$.

first step, we show how these collections can be mutated into exceptional collections in a controlled way.

Lemma 8.3. *Let $\{A_i\}_{i=0\dots t}$ be an ES-collection which contains an object $Z = A_0$ in \mathcal{Z} . By performing a sequence of mutations of objects past Z , we can produce a collection which is exceptional for some choice of ordering of the objects.*

Proof. We consider the objects in the collection for which $\text{Hom}_{\mathcal{D}/\Sigma}(Z, -) \neq 0$. Since the corresponding arcs form part of an arc-collection, they can only intersect α_Z at its end-points. First we look at those arcs which have an intersection at the end-point $\alpha_Z(0) \in \delta_X$ contributing to $\iota(\alpha_Z, -) \neq 0$. We lift them to arcs starting at $\widehat{\alpha_Z}(0) = (\alpha_Z(0), 0)$ in the universal cover and denote their other end-points by u_1, u_2, \dots, u_s . By Lemma 3.8 we see that either $\widehat{\alpha_Z}(1) < u_i$ or $u_i < \widehat{\alpha_Z}(0)$ for each i . Without loss of generality we can relabel so that $\widehat{\alpha_Z}(1) < u_1 < u_2 < \dots < u_j$ and $u_{j+1} < u_{j+2} < \dots < u_s < \widehat{\alpha_Z}(0)$ for some j (see Figure 13). Since there are no end-points between $\widehat{\alpha_Z}(1)$ and u_1 in the order, this implies that there are no factoring arcs between $\widehat{\alpha_Z}$ and the arc with end-point u_1 , so we may mutate this arc past $\widehat{\alpha_Z}$. This was the only factoring arc between $\widehat{\alpha_Z}$ and the arc with end-point u_2 , so we can now mutate this arc past $\widehat{\alpha_Z}$. We proceed until all of the arcs have been mutated past $\widehat{\alpha_Z}$ in turn. Note that by construction, the new common end-point between each mutated arc and α_Z is $\alpha_Z(1)$, but due to the ordering we see that this doesn't contribute to $\iota(\alpha_Z, -) \neq 0$. We now perform the analogous procedure to those arcs in the arc-collection which have an intersection at the end-point $\alpha_Z(1)$ contributing to $\iota(\alpha_Z, -) \neq 0$. The corresponding ES-collection $\{A'_i\}_{i=0\dots t}$ that we produce in this way, is mutation equivalent to $\{A_i\}_{i=0\dots t}$, contains the object Z , and satisfies $\text{Hom}_{\mathcal{D}/\Sigma}(Z, A'_i) = 0$ for each $Z \neq A'_i$. We relabel if necessary so $Z = A'_0$.

We would like to use the equivalence $\text{thick}(Z)^\perp \simeq \mathcal{D}^b(\mathbf{k}A_{n+m-1})$. In the geometric model, the property $\text{Hom}_{\mathcal{D}/\Sigma}(Z, -) = 0$ corresponds to looking at arcs which don't intersect $\Phi_\varepsilon(\alpha_Z)$. The arc $\Phi_\varepsilon(\alpha_Z)$ cuts the cylinder into a disc with $m+n$ marked points on the boundary. Therefore, applying Theorem 3.28 we see that the chords (or arcs up to homotopy equivalence) on this disc form a model for indecomposable objects in $\mathcal{D}^b(\mathbf{k}A_{n+m-1})$ up to shift. Considering how the standard heart of $\mathcal{D}^b(\mathbf{k}A_{n+m-1})$ (corresponding to a consistently oriented A_{n+m-1} quiver) sits inside \mathcal{D} (see for example [7, Figure 4]) we observe that the simple objects in $\text{mod}(\mathbf{k}A_{n+m-1})$ correspond to arcs between neighbouring points on the boundary of the disc. Extensions between simples then correspond to longer arcs obtained by concatenating the arcs of those simples. In [2, Page 3], Araya writes down a bijection Φ between indecomposable objects in $\text{mod}(\mathbf{k}A_{n+m-1})$ and chords on the disc, which he then uses to classify complete exceptional sequences in $\text{mod}(\mathbf{k}A_{n+m-1})$. This

bijection also maps simple objects to chords between neighbouring points on the boundary of the disc and it is a straightforward exercise to see that in fact the two models coincide. Furthermore, since the disc is contractible, a reduced arc-collection admits no closed paths of arcs, so it is a non-crossing tree in the terminology of Araya. Therefore, by [2, Theorem 1.1], this means that some ordering of the collection $\{A'_i\}_{i=1\dots t}$ is exceptional, and it follows that $\{A'_i\}_{i=0\dots t}$ is an exceptional collection, for some choice of ordering as in the proof of Proposition 2.1. \square

We can now prove that $(i \implies iii)$ under the additional assumption that the ES-collections have a common object in \mathcal{Z} .

Lemma 8.4. *Suppose \underline{A} and \underline{B} are reduced ES-collections which have a common object Z in \mathcal{Z} , and such that $\text{thick}(\underline{A}) \subseteq \text{thick}(\underline{B})$. Then $\underline{A} \leq_{\text{mut}} \underline{B}$.*

Proof. We use Lemma 8.3 to produce two exceptional collections $(\tilde{A}_1, \dots, \tilde{A}_s, Z)$ and $(\tilde{B}_1, \dots, \tilde{B}_t, Z)$. By Lemma 8.2 we can extend the exceptional collection $(\tilde{A}_1, \dots, \tilde{A}_s)$ in $\text{thick}(Z)^\perp \simeq \mathbf{D}^b(\mathbf{k}A_{n+m-1})$ to an exceptional collection $(A'_1, \dots, A'_{t-s}, \tilde{A}_1, \dots, \tilde{A}_s)$ which is mutation equivalent to $(\tilde{B}_1, \dots, \tilde{B}_t)$. Note that for objects in $\text{thick}(Z)^\perp$, mutation of exceptional collections and ES-collections coincide. Therefore $(A'_1, \dots, A'_{t-s}, \tilde{A}_1, \dots, \tilde{A}_s, Z)$ and $(\tilde{B}_1, \dots, \tilde{B}_t, Z)$ are mutation equivalent. Since $\text{Hom}^k(\tilde{A}_i, A'_j) = 0$ for all i, j, k , the new objects A'_1, \dots, A'_{t-s} don't affect the mutations between the objects $\tilde{A}_1, \dots, \tilde{A}_s$ and Z . Therefore, we can invert the sequence of mutations from the first step to produce an ES-collection $\{A'_1, \dots, A'_{t-s}, A_1, \dots, A_s, Z\} = \underline{A}' \cup \underline{A}$ which is mutation equivalent to \underline{B} as required. \square

Now suppose that \underline{A} and \underline{B} are reduced ES-collections such that $\text{thick}(\underline{A}) \subseteq \text{thick}(\underline{B})$, and that \underline{A} contains an object in \mathcal{Z} . We will show that \underline{B} is mutation equivalent to an ES-collection \underline{B}' such that \underline{A} and \underline{B}' have a common object in \mathcal{Z} , thus reducing it to the case that we have just proved. We start with a definition.

Definition 8.5. Let $\hat{\alpha}, \hat{\beta}$ be lifts of \mathcal{X} -arcs (respectively \mathcal{Y} -arcs). We say that $\hat{\beta}$ is nested in $\hat{\alpha}$ if both end-points of $\hat{\beta}$ are between the end-points of $\hat{\alpha}$ with respect to the order on $\widehat{\delta}_X$ (respectively $\widehat{\delta}_Y$). We say that a path $\gamma_0 \cdot \gamma_1 \cdots \gamma_s$ is nested if it lifts to a path $\hat{\gamma}_0 \cdot \hat{\gamma}_1 \cdots \hat{\gamma}_s$ such that $\widehat{\gamma}_{i+1}$ is nested in $\hat{\gamma}_i$ for each $i = 0, \dots, s-1$.

Lemma 8.6. *Let γ be any path of arcs in a reduced arc-collection $\{\beta_i\}$ which goes between δ_X and δ_Y . Then there is a arc-collection $\{\beta'_i\}$ which is mutation equivalent to $\{\beta_i\}$, and contains an arc which is homotopy equivalent to γ .*

Proof. Since $\gamma = \gamma_0 \cdot \gamma_1 \cdots \gamma_s$ connects the two boundary components, then

$$\zeta(\gamma) := |\{i \mid \gamma_i(0) \in \delta_X, \gamma_i(1) \in \delta_Y \text{ or } \gamma_i(0) \in \delta_Y, \gamma_i(1) \in \delta_X\}|$$

the number of arcs corresponding to objects in the \mathcal{Z} -component is odd.

Base case: Suppose $\zeta(\gamma) = 1$, so there is a unique arc in the path connecting the two boundary components. Up to a change of orientation and relabelling we can write this in the form $\gamma \simeq \gamma_Y \cdot \gamma_Z \cdot \gamma_X$ where γ_X is a path of \mathcal{X} -arcs, γ_Y is a path of \mathcal{Y} -arcs, and γ_Z is the single \mathcal{Z} -arc.

Claim 1: *By mutating the ES-collection, we can transform γ into a path of the form*

$$\gamma \simeq \rho_0 \cdot \rho_1 \cdots \rho_s \cdot \gamma_Z \cdot \eta \cdots \eta \cdot \alpha_0 \cdot \alpha_1 \cdots \alpha_t$$

where η is a spherelike arc, and $\alpha_0 \cdot \alpha_1 \cdots \alpha_t$ and $\rho_0 \cdot \rho_1 \cdots \rho_s$ are nested paths.

We postpone the proof of this claim until after Lemma 8.7 below.

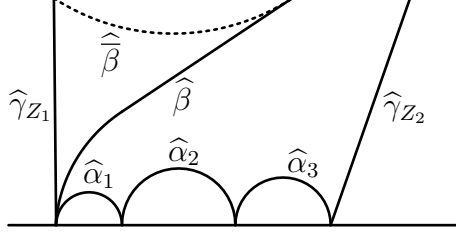


FIGURE 14. Example, showing lifts of the path of arcs $\gamma_{Z_1} \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \gamma_{Z_2}$, a potential factoring arc β and its mutation $\overline{\beta}$.

Any factoring arcs between γ_Z and a spherelike arc must be \mathcal{Z} -arcs and be distinct from γ_Z . Therefore, they do not appear in the path, and we can mutate them in turn past the spherelike arc without effecting the path. We can then mutate γ_Z past the spherelike arc. This has the effect of changing the \mathcal{Z} -arc γ_Z and reducing the number of spherelike arcs in the path. Note that the new arc $\widetilde{\gamma}_Z \simeq \gamma_Z \cdot \eta$ is still the only \mathcal{Z} -arc in the path. In this way, we can remove all the spherelike arcs, so, written in terms of our new arc-collection,

$$\gamma \simeq \rho_0 \cdot \rho_1 \cdots \rho_s \cdot \widetilde{\gamma}_Z \cdot \alpha_0 \cdot \alpha_1 \cdots \alpha_t$$

where $\rho_0 \cdot \rho_1 \cdots \rho_s$ is a nested path of \mathcal{Y} -arcs, $\alpha_0 \cdot \alpha_1 \cdots \alpha_t$ is a nested path of \mathcal{X} -arcs and $\widetilde{\gamma}_Z$ is a \mathcal{Z} -arc. The nesting property implies that the lengths of the arcs are ordered, and so any given arc appears once in the path. Furthermore, no factoring arc between $\widetilde{\gamma}_Z$ and α_0 can appear in the path. Again, this means we can mutate such arcs away without changing the arcs in the path. We then mutate $\widetilde{\gamma}_Z$ past α_0 noting that in terms of the new arc-collection, we have

$$\gamma \simeq \rho_0 \cdot \rho_1 \cdots \rho_s \cdot \widetilde{\gamma}'_Z \cdot \alpha_1 \cdots \alpha_t$$

where the two ends are clearly still nested paths. Proceeding iteratively, we remove all the \mathcal{X} - and \mathcal{Y} -arcs and produce an arc-collection containing an arc which is homotopy equivalent to γ .

Induction step: Now suppose that the result holds for any path γ' of arcs in an arc-collection $\{\beta'_i\}$ such that $1 \leq \zeta(\gamma') < t$. Suppose $\zeta(\gamma) = t$, and let $\gamma_{Z_1} \cdot \alpha_1 \cdots \alpha_s \cdot \gamma_{Z_2}$ be some piece of γ between two \mathcal{Z} -arcs. Without loss of generality, we assume that $\alpha_1, \dots, \alpha_s$ are all \mathcal{X} -arcs or all \mathcal{Y} -arcs. Suppose there is a factoring arc β between γ_{Z_1} and α_1 at their common end-point v in the path. We observe that β must be a \mathcal{Z} -arc, otherwise it would cross the arc γ_{Z_1} or the path $\alpha_1 \cdots \alpha_s \cdot \gamma_{Z_2}$, contradicting either non-crossing or reducedness (see Figure 14). Mutating (in turn) any such factoring arcs past γ_{Z_1} , each occurrence of such an arc β in the path γ is replaced with a path $\gamma_{Z_1}^{-1} \cdot \overline{\beta}$, where the mutated arc $\overline{\beta}$ is not a \mathcal{Z} -arc. Therefore, this mutation doesn't affect the number t of \mathcal{Z} -arcs in the path γ . Since $\beta \not\approx \gamma_{Z_1}^{\pm 1}$ and it is a \mathcal{Z} -arc, β doesn't appear in the section of path $\gamma_{Z_1} \cdot \alpha_1 \cdots \alpha_s \cdot \gamma_{Z_2}$ unless $\beta \simeq \gamma_{Z_2}^{\pm 1}$. Therefore, if $\beta \not\approx \gamma_{Z_2}^{\pm 1}$ then mutating β past γ_{Z_1} doesn't change this section of the path. If $\beta \simeq \gamma_{Z_2}$ the piece of the path would become $\gamma_{Z_1} \cdot \alpha_1 \cdots \alpha_s \cdot \gamma_{Z_1}^{-1} \cdot \overline{\beta}$, however, we can show that this contradicts the assumption that γ connects the two boundary components as follows.

If $\gamma_{Z_1} \cdot \alpha_1 \cdots \alpha_s \cdot \gamma_{Z_1}^{-1}$ is part of γ then $\alpha_1 \cdots \alpha_s$ is a closed cycle of \mathcal{X} -arcs or \mathcal{Y} -arcs, which is not homotopic to the constant path using the reduced property of the collection. Since γ is homotopic to a \mathcal{Z} -arc then there must be at least three \mathcal{Z} -arcs in the path. Without loss of generality, we assume that a piece of γ is of the form $\gamma_{Z_1} \cdot \alpha_1 \cdots \alpha_{s_1} \cdot \gamma_{Z_1}^{-1} \cdot \alpha'_1 \cdots \alpha'_{s_2} \cdot \gamma_{Z_3}$, otherwise there would instead be a γ_{Z_0} . Since γ_{Z_3} has an end-point on both boundaries, it

must intersect the cycle $\alpha_1 \cdots \alpha_s$. Using just the arcs in the cycle, we can then construct a path which is homotopic to $\gamma_{Z_1}^{-1} \cdot \alpha'_1 \cdots \alpha'_{s_2} \cdot \gamma_{Z_3}$, contradicting the reducedness hypothesis. Therefore $\beta \neq \gamma_{Z_2}$.

Since we have mutated away all factoring arcs between γ_{Z_1} and α_1 at v , we can now mutate γ_{Z_1} past α_1 . Again this fixes the number of \mathcal{Z} -arcs in the path γ , but there are now $s - 1$ arcs between $\bar{\gamma}_{Z_1}$ and γ_{Z_2} . Iterating this process, we produce an ES-collection with respect to which γ has two consecutive \mathcal{Z} -arcs $\gamma''_{Z_1} \cdot \gamma''_{Z_2}$. Again, we can mutate any factoring arcs past γ''_{Z_1} without changing the number t of \mathcal{Z} -arcs in the path γ . This also leaves the piece of the path $\gamma''_{Z_1} \cdot \gamma''_{Z_2}$ unchanged. Finally we can mutate γ''_{Z_2} past γ''_{Z_1} . With respect to the new ES-collection, γ is homotopic to a path with $t - 2$ \mathcal{Z} -arcs. The argument follows by induction. \square

In order to complete the proof, it remains to prove Claim 1. We start with the following lemma.

Lemma 8.7. *Let $\gamma = \gamma_0 \cdot \gamma_1 \cdots \gamma_s$ be any path of \mathcal{X} -arcs (respectively \mathcal{Y} -arcs) in an arc-collection $\{\beta_i\}$. Suppose further that γ_1 is nested in γ_0 . Then using mutations between \mathcal{X} -arcs (respectively \mathcal{Y} -arcs) which are nested under γ_0 , we can obtain a homotopic path $\gamma \simeq \gamma_0 \cdot \gamma'_1 \cdots \gamma'_{s'}$ of arcs in a mutation equivalent collection $\{\beta'_i\}$ such that γ'_{i+1} is nested in γ'_i for all $0 \leq i < s'$.*

Proof. We start with a path γ of \mathcal{X} -arcs. The \mathcal{Y} -arcs case is completely analogous. We denote the starting vertex of γ by v_0 and the subsequent vertices, where arcs γ_{j-1} and γ_j intersect, by v_j for each $1 \leq j \leq s$. We lift γ to a path on the universal cover starting at $(v_0, 0)$, and denote by \hat{v}_j the vertex corresponding to v_j on the lifted path. The property that γ_1 is nested in γ_0 means that \hat{v}_2 lies in the open interval I_0 between \hat{v}_0 and \hat{v}_1 . In particular, Lemma 5.4 then implies that \hat{v}_j lies in I_0 for each $j \geq 2$. Now suppose that γ_2 is not nested in γ_1 , so \hat{v}_3 is in the interval between \hat{v}_0 and \hat{v}_2 . We note that no factoring arc between γ_1 and γ_2 appears in the path γ . If it did, then there would be a subpath of γ which is a path of arcs from v_2 to itself. By Lemma 5.4 the two corresponding lifts of v_2 would lie in the interval I_0 and so be equal, but this would contradict the assumption that the collection is reduced. Furthermore, we observe that any factoring arc between γ_1 and γ_2 must be a \mathcal{X} -arc nested in γ_0 . Otherwise it would not satisfy the non-crossing condition with γ_0 . Therefore we can mutate each such factoring arc past γ_2 , changing the \mathcal{X} -arcs in the collection, but leaving the path γ unchanged. Finally we mutate γ_1 past γ_2 , introducing the new arc $\bar{\gamma}_1$ and path $\gamma \simeq \gamma_0 \cdot \bar{\gamma}_1 \cdot \gamma_3 \cdots \gamma_s$. Either γ_3 is nested in $\bar{\gamma}_1$ as desired, or we iteratively applying the above process. This terminates in a finite number of steps since the second arc in the path gets longer at each step, and it is nested inside γ_0 . \square

Proof of Claim 1. Recall the path $\gamma = \gamma_0 \cdot \gamma_1 \cdots \gamma_s$ has a unique \mathcal{Z} -arc, $\gamma_p = \gamma_Z$, and $\gamma_X \simeq \gamma_{p+1} \cdots \gamma_s$ is a path of \mathcal{X} -arcs. We denote the vertex where arcs γ_j and γ_{j+1} intersect by v_j . Take the first \mathcal{X} -arc γ_i in the path which is not spherelike and observe that this implies that $v_p = v_{p+1} = \cdots = v_{i-1} \neq v_i$. We lift γ to a path on the universal cover such that the lift of γ_p ends at $(v_p, 0)$, and denote by \hat{v}_j the vertex corresponding to v_j on the lifted path (see Figure 15). Suppose that $\hat{v}_{i-1} < \hat{v}_i$ - the same argument works in the other case with all inequalities reversed. We note that if $\hat{v}_{i+1} < \hat{v}_{i-1}$ then the arc γ_{i+1} would intersect γ_p contradicting the non-crossing property. If $\hat{v}_{i+1} = \hat{v}_{i-1}$ then $\gamma_i \simeq \gamma_{i+1}^{-1}$ and the pair can be homotoped to a point and thus removed from the path. If $\hat{v}_{i-1} < \hat{v}_{i+1} < \hat{v}_i$, then γ_{i+1} is nested in γ_i and the result follows using Lemma 8.7. If $\hat{v}_{i+1} > \hat{v}_i$ then we consider the set of factoring arcs between γ_i and γ_{i+1} . We start

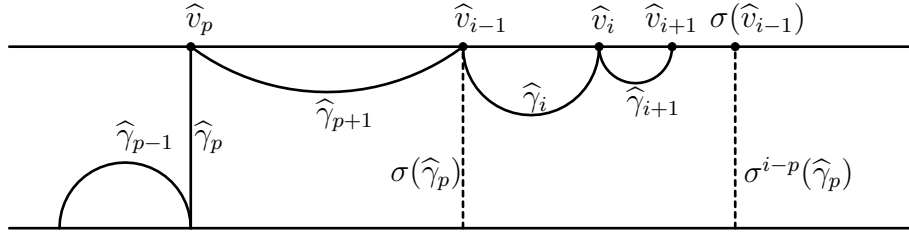


FIGURE 15. Diagram, showing a lift to the universal cover of part of a path of arcs γ . In this case $\hat{v}_{i+1} > \hat{v}_i$ and $i = p + 2$.

by showing that none of these factoring arcs appear in the path γ . By definition, any such factoring arc α would have an end-point at v_i . We lift this to an arc $\hat{\alpha}$ which has end-points \hat{v}_i and \hat{u} . If \hat{u} is on the Y boundary, then since α is not γ_Z , it follows that α doesn't appear in γ . If \hat{u} is on the X boundary, then the non-crossing property and the factoring arc property together imply that $\hat{u} > \hat{v}_{i+1} > \hat{v}_i$. Applying Lemma 5.4 we see that $\hat{u} > \hat{v}_k > \hat{v}_i > \sigma^{-1}\hat{u}$ for all $k > i$. Therefore, \hat{v}_k doesn't equal any shift of u and so α is not an arc in the path.

We mutate in turn each factoring arc in the collection past γ_i until there are no factoring arcs left in the collection. Since they don't appear in the path, this is left unchanged. Then we mutate γ_i past γ_{i+1} and consider the path, written in this new collection. The new arc $\bar{\gamma}_i$ which appears in the path after the spherelike arcs (or after the γ_p if there were no spherelike arcs) is strictly longer. We proceed iteratively. At each step, either the result follows using Lemma 8.7, or this arc $\bar{\gamma}_i^{(d)}$ gets longer. If there were no spherelike \mathcal{X} -arcs in the path then $\bar{\gamma}_i^{(d)}$ could become spherelike for some $d > 0$, at which point we restart the argument with next arc in the path which isn't spherelike. However if there was a spherelike \mathcal{X} -arc in the path, then the reduced non-crossing property means that the length of $\bar{\gamma}_i^{(d)}$ is strictly less than that of a spherelike arc. Therefore the process must stop after a finite number of steps. Applying the same argument to the path γ_Y , we have mutated to get a path of the form claimed. \square

This completes the proof of Lemma 8.6.

We can now prove that $(i \implies iii)$ under the assumption that \underline{B} contains at least one object in \mathcal{Z} .

Lemma 8.8. *Let \underline{A} and \underline{B} be reduced ES-collections such that $\text{thick}(\underline{A}) \subseteq \text{thick}(\underline{B})$ and suppose there exists an object $Z \in \underline{B}$ in \mathcal{Z} . Then $\underline{A} \leq_{\text{mut}} \underline{B}$.*

Proof. Suppose that there exists an object $Z \in \underline{A} \subseteq \text{thick}(\underline{B})$ which is in \mathcal{Z} . Lemma 3.15 implies that α_Z is an arc between the two boundary components and Theorem 6.1 implies that α_Z is homotopy equivalent to a path of arcs in the arc-collection $\{\beta_i\}$ corresponding to \underline{B} . Lemma 8.6 then implies that we can mutate this arc-collection to a collection which contains α_Z and since mutation of ES-collections and arc-collections are compatible (see Remark 7.4), this means that we can mutate \underline{B} to a collection \underline{B}' containing Z . In other words, we can mutate \underline{B} to an ES-collection \underline{B}' such that \underline{A} and \underline{B}' have a common object in \mathcal{Z} . The result then follows from Lemma 8.4.

It remains to consider what happens when \underline{A} has no objects in \mathcal{Z} . In this case the set of objects of \underline{A} which are in \mathcal{X} and \mathcal{Y} are certainly fully orthogonal and so are in different subcategories of the decomposition. We consider a component such that $\text{thick}(\underline{A}^c) \subset \mathcal{X}$ but the other case is proved analogously. If $\underline{A}^c \cup \{Z\}$ is an ES-collection, then it is clearly

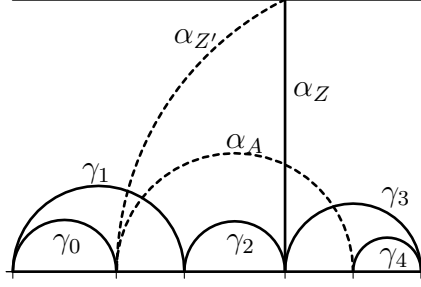


FIGURE 16. Diagram, illustrating part of the proof of Lemma 8.8. In this case, $s = 4$ and $i = 2$. The solid lines denote arcs corresponding to objects in \underline{B} .

reduced and satisfies $\text{thick}(\underline{A}^c, Z) \subseteq \text{thick}(\underline{B})$. By definition, $\underline{A}^c \leq_{\text{mut}} \underline{A}^c \cup \{Z\}$ and using the first part of this proof $\underline{A}^c \cup \{Z\} \leq_{\text{mut}} \underline{B}$. The result then follows by transitivity.

Finally, suppose $\underline{A}^c \cup \{Z\}$ is not an ES-collection. Then the arc α_Z must intersect some of the arcs corresponding to objects in the collection \underline{A}^c . We consider the arc α_A of maximum length with this property. Since $\text{thick}(\underline{A}^c) \subseteq \text{thick}(\underline{B})$, Theorem 6.1 implies that we can write $\alpha_A \simeq \gamma_0 \cdots \gamma_s$ as a path of arcs in the arc-collection corresponding to \underline{B} (see Figure 16). The arc α_Z must intersect this path somewhere, and using the non-crossing property of the arc-collection, this must be the end-point of γ_i for some i . We consider the object Z' corresponding to the path $\gamma_0 \cdots \gamma_i \cdot \alpha_Z$. This is in $\text{thick}(\underline{B})$ by Lemma 6.2. We claim that $\alpha_{Z'}$ doesn't intersect any of the arcs of \underline{A}^c away from common end-points. Otherwise such an arc would intersect either α_A contradicting the non-crossing property, or α_Z and be longer than α_A contradicting maximality. Therefore, $\underline{A}^c \cup \{Z'\}$ is an ES-collection, it is reduced and satisfies $\text{thick}(\underline{A}^c, Z') \subseteq \text{thick}(\underline{B})$, and the result again follows using the first part of the proof and transitivity. \square

In order to complete the proof of Theorem 8.1, it only remains to consider the case when neither \underline{A} nor \underline{B} contain any objects in \mathcal{Z} . Again, we can deal with collections of objects in \mathcal{X} and \mathcal{Y} components separately.

We start with another technical lemma.

Lemma 8.9. *Let $\gamma_0 \cdot \gamma_1 \cdots \gamma_s$ be a path of \mathcal{X} -arcs (respectively \mathcal{Y} -arcs) in an arc-collection, which is homotopy equivalent to an exceptional or spherelike arc γ . Then no arc appears twice in the path with the same orientation and no vertex in the path appears more than twice.*

Proof. We do a proof by contradiction to show the first part of the statement. The other part is proved similarly. Let $\gamma_0 \cdot \gamma_1 \cdots \gamma_s$ be a path as above, and suppose that an arc α appears twice with the same orientation. Consider any lift of the path to the universal cover and label the vertices $\widehat{v}_0, \dots, \widehat{v}_{s+1}$ as before. Let \widehat{u}_1 and \widehat{u}_2 be the end-points of the first occurrence of α and let $\sigma^k \widehat{u}_1$ and $\sigma^k \widehat{u}_2$ be the end-points of the second occurrence. Up to reorientation of the path, we may assume without loss of generality that $\widehat{u}_1 < \widehat{u}_2 \leq \sigma^k \widehat{u}_1 < \sigma^k \widehat{u}_2$. Using Lemma 5.4 we see that all the vertices v_0, v_1, \dots which appear in the path before \widehat{u}_2 , satisfy $\widehat{v}_i < \widehat{u}_2$ or $\widehat{v}_i > \sigma^k \widehat{u}_2$. However since no single arc in the path can have ends in these two different regions and there must be an arc from one of these vertices to \widehat{u}_2 , it follows that $\widehat{v}_0 < \widehat{u}_2$. Similarly we can see that $\widehat{v}_{s+1} > \sigma^k \widehat{u}_1$. The length condition on the arc γ then forces $k = 1$ and $\widehat{u}_1 \leq \widehat{v}_0 < \widehat{u}_2 \leq \sigma \widehat{u}_1 < \widehat{v}_{s+1} \leq \sigma \widehat{u}_2$. Considering the section of the lifted path from $\sigma \widehat{u}_2$ to \widehat{v}_{s+1} , together with the subpath between $\sigma \widehat{v}_0$ and $\sigma \widehat{u}_2$ in the shift of the lift, we apply Lemma 5.4 once more to see that



FIGURE 17. The dotted line represents the lift of the section of the path between the repeated arc. In each case, it follows from the non-crossing property that it includes an arc under which one of the lifts of γ_i is nested.

$\sigma\hat{v}_0 < \hat{v}_{s+1}$. However, this contradicts the fact that γ is an exceptional or spherelike arc. \square

Lemma 8.10. *Let $\{\alpha_i\}$ and $\{\beta_j\}$ be reduced arc-collections consisting of only \mathcal{X} -arcs or only \mathcal{Y} -arcs. Suppose $\alpha \in \{\alpha_i\}$ and that $\alpha \simeq \gamma_0 \cdot \gamma_1 \cdots \gamma_t$, a path of arcs in $\{\beta_j\}$ for some $t > 0$. By mutating the arc-collection $\{\beta_j\}$ at arcs which are not in $\{\alpha_i\}$ we can obtain a new collection $\{\beta'_j\}$ such that $\alpha \simeq \gamma'_0 \cdot \gamma'_1 \cdots \gamma'_s$, a path of arcs in $\{\beta'_j\}$ for some $s < t$.*

Proof. Since $\{\alpha_i\}$ is reduced and $t > 0$ we see that at least one arc in the path $\gamma_0 \cdot \gamma_1 \cdots \gamma_t$ is not contained in $\{\alpha_i\}$.

Step 1: *There exists such an arc γ_i which appears precisely once in the path (with any orientation).* We know from Lemma 8.9 that no arc appears twice with the same orientation. Consider an arc γ_i in the path, which is not contained in the arc-collection $\{\alpha_i\}$. Without loss of generality we assume that γ_i is of maximal length with this property. Suppose to the contrary that this appears in the path twice with the opposite orientations. We lift the path to the universal cover and observe, using the non-crossing property that there must be an arc $\hat{\gamma}$ in the path such that one (but not both) of the lifts of γ_i are nested in $\hat{\gamma}$ (see Figure 17). Note that this means that $\hat{\gamma}$ is longer than $\hat{\gamma}_i$. We denote the end-points of $\hat{\gamma}$ by \hat{v} and \hat{v}' . As a consequence of Lemma 5.4, we see that one of the end-points of the path is in the interval (\hat{v}, \hat{v}') and the other end-point is outside the closed interval $[\hat{v}, \hat{v}']$. Lemma 3.8 then implies that the arcs α and γ intersect away from their end-points, so γ is not in $\{\alpha_i\}$. However, this contradicts the maximality of the length of γ_i . Step 1 then follows.

Since there are at least two arcs in the path α , the arc γ_i must have a predecessor or a successor. By reorienting the path if necessary, we may assume that this is γ_{i+1} . We would like to be able to mutate γ_i past γ_{i+1} , so we look at the set of factoring arcs between γ_i and γ_{i+1} which are in $\{\beta_j\}$.

Step 2: *None of these factoring arcs are in $\{\alpha_i\} \cap \{\beta_j\}$.* We prove this by contradiction. Suppose there was such an arc and denote it by α' . By definition, γ_i, γ_{i+1} and α' have a common end-point which we denote by v . We lift the path $\gamma_0 \cdot \gamma_1 \cdots \gamma_t$ and the arc α' to the universal cover by lifting v to $\hat{v} = (v, 0)$. We denote the vertices along the path by $\hat{v}_0, \dots, \hat{v}_i, \hat{v}_{i+1} = \hat{v}, \hat{v}_{i+2}, \dots, \hat{v}_{t+1}$. Let I be the interval between \hat{v}_i and \hat{v}_{i+2} . The factorising arc property implies that \hat{v} is in the interval I if and only if $\hat{\alpha}'(1)$ is not in the interval. Applying Lemma 5.4 using the three paths in $\{\beta_j\}$,

$$\gamma_{i+1} \cdot \gamma_{i+2} \cdots \gamma_t, \quad \gamma_0 \cdot \gamma_1 \cdots \gamma_i, \quad \alpha'$$

and working through the cases, we can deduce that \hat{v} lies in the interval between \hat{v}_0 and \hat{v}_{t+1} if and only if $\hat{\alpha}'(1)$ does not lie in this interval. Noting that \hat{v}_0 and \hat{v}_{t+1} are the end-points of $\hat{\alpha}$, and \hat{v} and $\hat{\alpha}'(1)$ are the endpoints of $\hat{\alpha}'$, it follows from Lemma 3.8 that $\hat{\alpha}$ and $\hat{\alpha}'$ intersect, but this would contradict the non-crossing property of $\{\alpha_i\}$.

Step 3: *Without loss of generality, we may assume that none of the factoring arcs appear in the path.* Consider the case where there is a factoring arc γ_j in $\{\beta_j\}$ (but not in

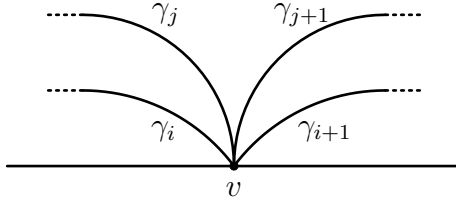


FIGURE 18. Any factoring arc between γ_j and γ_{j+1} is also a factoring arc between γ_i and γ_{i+1} .

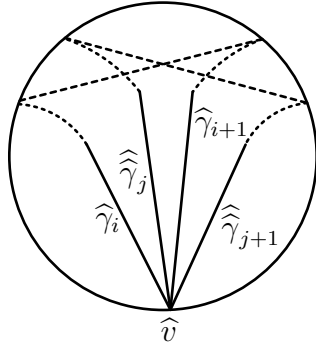


FIGURE 19. If γ_j is a factoring arc between γ_i and γ_{i+1} , but γ_{j+1} is not, then the two lifts of the path are homotopic to chords which intersect on the disc model of the universal cover.

$\{\alpha_i\}$ by Step 2) which appears in the path. Suppose for the moment that $\gamma_j \cdot \gamma_{j+1}$ passes through the vertex v – the other cases can be treated in essentially the same way.

If γ_{j+1} is also a factoring arc, then the inequalities in the definition imply that the set of factoring arcs between γ_j and γ_{j+1} is a proper subset of those between γ_i and γ_{i+1} (see Figure 18). Since the vertex v now appears in the path twice and can't occur again, by Lemma 8.9, it follows that γ_j appears once in the path, and that no factoring arcs between γ_j and γ_{j+1} appear in the path. We may therefore consider γ_j instead of γ_i , noting that it satisfies all the desired properties.

If γ_{j+1} is not a factoring arc, then we lift the path in two ways so that v lifts to $\hat{v} = (v, 0)$ at the two different points it appears in the path. Splitting each of these paths into two pieces, which cover the paths

$$\gamma_{i+1} \cdot \gamma_{i+2} \cdots \gamma_t, \quad \gamma_0 \cdot \gamma_1 \cdots \gamma_i, \quad \gamma_{j+1} \cdot \gamma_{j+2} \cdots \gamma_t, \quad \gamma_0 \cdot \gamma_1 \cdots \gamma_j.$$

and using Lemma 5.4 as above, we show that there is an internal intersection between two different lifts of α , contradicting the fact that it is in an arc-collection (see Figure 19).

Step 4: We have reduced to the case where any factoring arcs between γ_i and γ_{i+1} are not in $\{\alpha_i\} \cap \{\beta_j\}$ and do not appear elsewhere in the path. We can therefore mutate (in turn) any such factoring arcs past γ_i without changing the path or altering the set of arcs $\{\alpha_i\} \cap \{\beta_j\}$. In this new arc-collection, there are no factoring arcs between γ_i and γ_{i+1} , so we can mutate γ_i past γ_{i+1} to produce a new arc $\bar{\gamma}$. Using the fact that γ_i appears once in the path, we see that $\alpha \simeq \gamma_0 \cdots \gamma_{i-1} \cdot \bar{\gamma} \cdot \gamma_{i+2} \cdots \gamma_t$, when written in terms of the arcs in the new arc-collection, which is a strictly shorter path. If necessary, we reduce, removing pairs of arcs which are contractable, but this decreases the number of arcs in the path further. \square

Lemma 8.11. *Let \underline{A} and \underline{B} be reduced ES-collections consisting of objects only in \mathcal{X} or only in \mathcal{Y} and suppose $\text{thick}(\underline{A}) \subseteq \text{thick}(\underline{B})$. Then $\underline{A} \leq_{\text{mut}} \underline{B}$.*

Proof. Let $\{\alpha_i\}$ and $\{\beta_j\}$ denote the arc-collections corresponding to \underline{A} and \underline{B} respectively. We consider any arc $\alpha_A \in \{\alpha_i\}$ which is not in $\{\alpha_i\} \cap \{\beta_j\}$. Then α_A is homotopic to a path of arcs in $\{\beta_j\}$ by Lemma 6.2. We apply Lemma 8.10 repeatedly until this path has length 1, which implies that α_A is in an arc-collection $\{\beta'_j\}$ which is mutation equivalent to $\{\beta_j\}$. Since at each step we don't mutate objects in $\{\alpha_i\} \cap \{\beta_j\}$, this is a subset of $\{\alpha_i\} \cap \{\beta'_j\}$, but $\{\alpha_i\} \cap \{\beta'_j\}$ also contains the arc α_A .

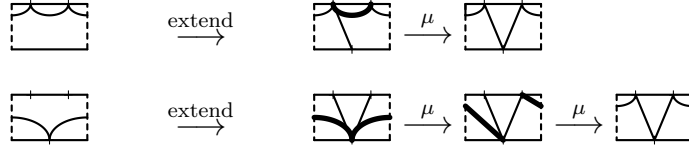
We now consider an arc in $\{\alpha_i\}$ which is not in $\{\alpha_i\} \cap \{\beta'_j\}$ and repeat the procedure. Working inductively, we eventually obtain an arc-collection $\{\beta''_j\}$ which is mutation equivalent to $\{\beta_j\}$ such that $\{\alpha_i\} \subset \{\beta''_j\}$. This happens in a finite number of steps, since the sets are all finite. Then taking $\{\alpha_{i'}\}_{i' \in I'}$ to be the complement of $\{\alpha_i\}$ in $\{\beta''_j\}$, we have $(\{\alpha_i\}_{i \in I^c} \cup \{\alpha_{i'}\}_{i' \in I'}) \sim_{\text{mut}} \{\beta_j\}_{j \in J}$ so $\{\alpha_i\} \leq_{\text{mut}} \{\beta_j\}_{j \in J}$. The result then follows from Lemma 7.12. \square

This completes the proof of Proposition 8.1.

Example 8.12. We conclude with the example $\mathbf{D}^b(\Lambda(2, 3, 0))$. The lattice of thick subcategories is shown in Figure 20, together with a choice of arc-collection for each subcategory. S is empty, and A corresponds to $\mathbf{D}^b(\Lambda(2, 3, 0))$. The thick subcategory corresponding to O is a subcategory of the thick subcategories corresponding to G, K and E . In the first two cases, this is clear, since the single arc in the arc-collection O , is already contained in the collections G and K . In the other case, we can extend the collection O to



and we get the collection E by mutating α past β . To see explicitly, in terms of the arc-collections, that the thick category corresponding to B is a subcategory of the thick subcategory corresponding to A , we consider the two connected components of the arc-collection B separately. These can be extended to arc-collections which are mutation equivalent to A as follows. At each step the bold arc is mutated.



The vertices of Figure 20 marked with a cross, have a representative in the mutation class which is an exceptional collection. The letters E, F, Q, R actually denote \mathbb{Z} -families of thick subcategories. The arc-collections given in these cases generate one member of the family. By performing full rotations of one end of the cylinder (changing the winding numbers of the arcs) produces an arc-collection for each member of the family. The edges in the diagram between these families, should be taken to mean that each element in one family, is less than some element in the other family with respect to the partial order.

APPENDIX A. THE GROTHENDIECK GROUP

We relate the Grothendieck group of the discrete derived category \mathbf{D} to the relative homology $H_1(C, \ell)$ where $C = C(p, q)$ is the cylinder and ℓ is the set of $p + q$ marked points on the boundary.

Let $F_i(C)$ denote the free abelian group with basis given by the set of singular i -simplices in $C(p, q)$, and denote by $\partial_i: F_{i+1}(C) \rightarrow F_i(C)$ the boundary map. We then denote by $F_0(\ell)$ the free abelian group with basis ℓ and by $F_0(C, \ell)$ the quotient group

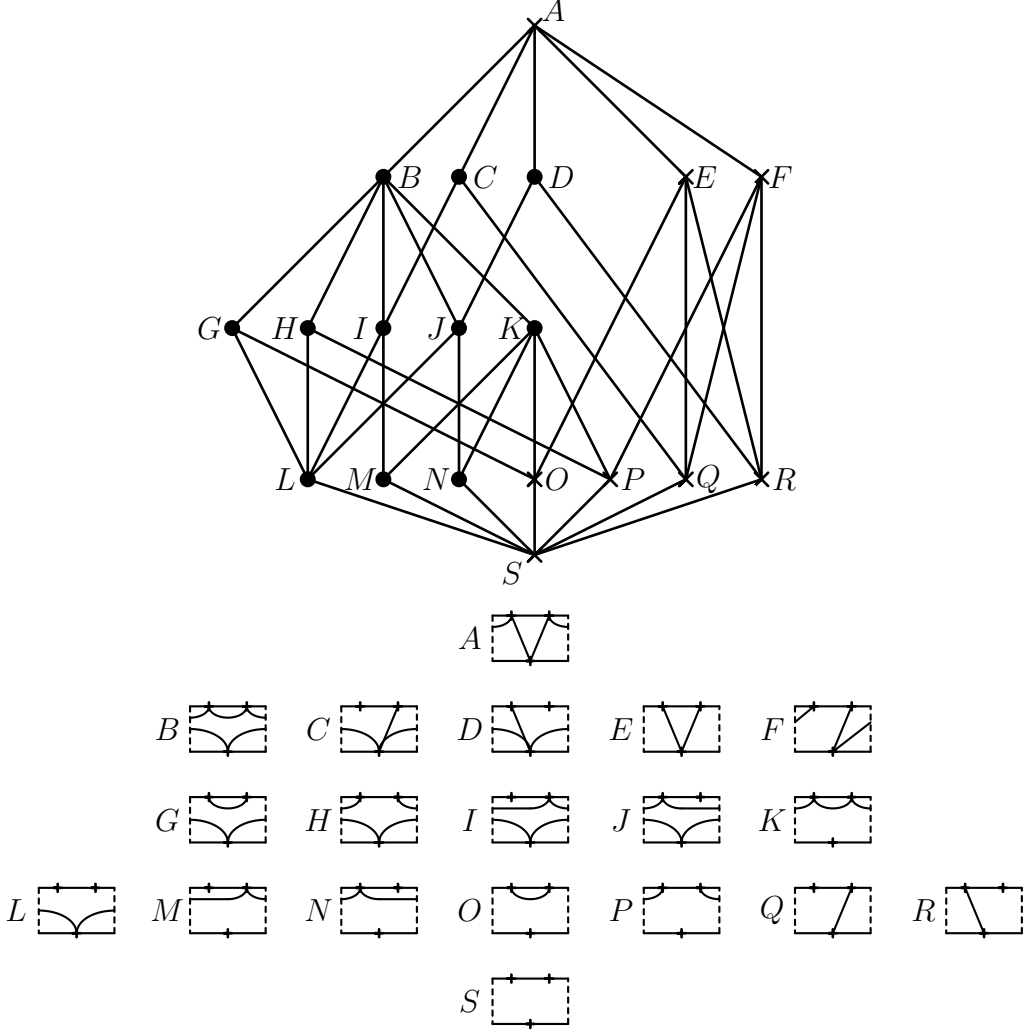


FIGURE 20. Lattice of thick subcategories of $D^b(\Lambda(2, 3, 0))$.

$F_0(C)/F_0(\ell)$. Following the definition of relative homology (see for example [14, Section 2.1]), we construct a commutative diagram with exact columns:

$$(12) \quad \begin{array}{ccccccc} 0 & \longleftarrow & F_0(\ell) & \longleftarrow & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & F_0(C) & \xleftarrow{\partial_0} & F_1(C) & \xleftarrow{\partial_1} & F_2(C) \longleftarrow 0 \\ & & \downarrow & & \parallel & & \parallel \\ 0 & \longleftarrow & F_0(C, \ell) & \xleftarrow{\tilde{\partial}_0} & F_1(C) & \longleftarrow & F_2(C) \longleftarrow 0 \end{array}$$

where $\tilde{\partial}_0$ is the induced quotient boundary map. Then

$$H_1(C, \ell) := \ker \tilde{\partial}_0 / \text{im } \partial_1 = \partial_0^{-1}(F_0(\ell)) / \text{im } \partial_1.$$

There are $p + q$ marked points on the cylinder, so $F_0(\ell) = \mathbb{Z}^\ell$ is a rank $p + q$ lattice with a basis of the marked points. Let $F_0(\ell)^\vee := \text{Hom}_{\mathbb{Z}}(F_0(\ell), \mathbb{Z})$ be the dual lattice with pairing $\langle -, - \rangle: F_0(\ell)^\vee \times F_0(\ell) \rightarrow \mathbb{Z}$. We denote by $\mathbf{1} := \sum_{v \in \ell} v^\vee$ the element in this dual lattice, which is the sum of the elements in the dual basis. Since $\text{im } \partial_1 \subset \ker \partial_0$ we see that the restriction of the boundary map $\partial_0: \partial_0^{-1}(F_0(\ell)) \rightarrow F_0(\ell)$ induces a well-defined

map $\partial_0: H_1(C, \ell) \longrightarrow F_0(\ell)$, and that the image of this map is contained in the sublattice

$$\mathbf{\Lambda} := \underline{1}^\perp = \{u \in H_1(C, \ell) \mid \langle \underline{1}, u \rangle = 0\}.$$

This is isomorphic to an A_{p+q} root lattice. We note that $\mathbf{\Lambda}$ is generated by the following $p + q - 1$ elements,

$$\begin{aligned} \partial_0(\alpha_Z) &= y_1 - x_1 \\ \partial_0(\alpha_{X_{p-1}}) &= x_p - x_{p-1}, \dots, \partial_0(\alpha_{X_1}) = x_2 - x_1, \\ \partial_0(\alpha_{Y_{q-1}}) &= y_q - y_{q-1}, \dots, \partial_0(\alpha_{Y_1}) = y_2 - y_1 \end{aligned}$$

where $Z = Z^0(1, 1)$. In particular, since these are in the image of $\partial_0: H_1(C, \ell) \longrightarrow F_0(\ell)$, we see that the induced map $\partial_0: H_1(C, \ell) \longrightarrow \mathbf{\Lambda}$ is surjective. We observe that the kernel of this map is

$$\partial_0^{-1}(0) / \text{im } \partial_1 = H_1(C) \cong \mathbb{Z}.$$

Therefore, we have seen that the relative first homology group $H_1(C, \ell)$ is an extension of the the first homology $H_1(C)$ by an A_{p+q} root lattice.

Lemma A.1. *There is a short exact sequence of abelian groups:*

$$0 \longrightarrow H_1(C) \longrightarrow H_1(C, \ell) \xrightarrow{\partial_0} \mathbf{\Lambda} \longrightarrow 0.$$

Lemma A.2. *The collection*

$$\underline{E} = \{Z_0, X_{p-1}^0, \dots, X_1^0, Y_{q-1}^0, \dots, Y_1^0, Z\}$$

is a full exceptional collection, where $Z_0 = Z^0(0, q)$.

Proof. Looking at the corresponding arcs (see Example 3.22) and using Lemma 3.28 we can check the Hom-vanishing and deduce that the collection is exceptional. Using the arcs corresponding to objects in the collection, there is a closed path of arcs going through every vertex, whose class is a generator of $H_1(C)$. In particular this means that up to endpoint fixing homotopy, every non-trivial arc on the cylinder can be written by as a path of arcs in the collection. Using Lemma 3.28 and Lemma 6.2 it follows that every indecomposable object in \mathbf{D} is in the the thick subcategory generated by the collection. Therefore the exceptional collection is full. \square

Proposition A.3. *There is an isomorphism of abelian groups*

$$K_0(\mathbf{D}) \cong H_1(C, \ell)$$

Proof. Since \underline{E} is a full exceptional collection, the classes

$$\{[Z_0], [X_{p-1}], \dots, [X_1], [Y_{q-1}], \dots, [Y_1], [Z]\}$$

form a basis for $K_0(\mathbf{D})$. Using Lemma A.1 we see that the classes of the arcs

$$\{\alpha_{Z_0}, \alpha_{X_{p-1}}, \dots, \alpha_{X_1}, \alpha_{Y_{q-1}}, \dots, \alpha_{Y_1}, \alpha_Z\}$$

are linearly independent and generate $H_1(C, \ell)$. \square

Remark A.4. The isomorphism above is not given by $[A] \mapsto [\alpha_A]$ on all indecomposable objects. For example in $\mathbf{D}(\Lambda(3, 4, 2))$, Σ^3 acts as τ^{-1} on the objects in the \mathcal{X} component. Therefore the class of any object A at height 1 in this component is trivial in the Grothendieck group, since A sits in an AR triangle between X and $\Sigma^3 X$ for some object X on the mouth. However the arc $[\alpha_A]$ wraps twice around the cylinder and its class is non-zero in $H_1(C, \ell)$.

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