2018-03-22

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10.1109/TAC.2018.2816262
IEEE Transactions on Automatic Control
Institute of Electrical and Electronics Engineers

Guaranteed state estimation for nonlinear discrete-time systems via indirectly implemented polytopic set computation

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***Abstract*—This paper proposes a new set-membership technique to implement polytopic set computation for nonlinear discrete-time systems indirectly. The proposed set-membership technique is applied to solve the guaranteed state estimation problem for nonlinear discrete-time systems with a bounded description of noises and parameters. A common practice for this problem in the literature is to search an optimal zonotope to bound the intersection of the evolved uncertain state trajectory and the region of the state space consistent with the observed output at each observation update. The new approach keeps the polytopic set resulting from the intersection intact and computes the evolution of this intact polytopic set for the next time step through representing the polytopic set exactly by the intersection of zonotopes. Such an approach avoids the over-approximation bounding process at each observation update and thus a more accurate state estimation can be obtained. An illustrative example is provided to demonstrate the effectiveness of the proposed guaranteed state estimation via indirectly implemented polytopic set computation.**

***Index Terms*—State estimation, set computation, zonotope, polytope, nonlinear systems.**

I. Introduction

State estimation is formulated as the problem of estimating the real state of the system given the mathematical model of the system and also noise-corrupted measurements of the system output [1]. There are roughly two main types of approaches to tackle the state estimation problem: the stochastic approaches and the deterministic approaches. The most notable stochastic approach for state estimation is the Kalman filter, which is an efficient and recursive procedure to estimate the system state in a way that minimizes the mean of the squared estimation error [2]. If all noises and perturbations are Gaussian, the Kalman filter turns out to be an optimal estimator. However, such probabilistic assumptions on the system can be unrealistic or be hard to validate in practice.

Instead of describing the uncertainties or noises of a system by probability distributions, deterministic approaches to state estimation use various kinds of sets such as ellipsoids, polytopes, intervals and zonotopes to bound unknown perturbations, noises and estimation errors [3], [4], [5], [6]. The deterministic approaches are also called the set-membership approaches and they are particularly useful in case of lacking probabilistic information on the systems concerned. The state estimators in these set-membership approaches use a compact set or a union of sets to contain all the state variables that are consistent with available measurements and disturbance specifications [7]. Acting as the worst-case techniques for estimation, the set-membership approaches to state estimation were studied long before in [8] where ellipsoids were first used to bound the set of all possible states for linear systems with noise-corrupted observations. Due to the shape restriction of ellipsoids, the ellipsoidal bounding of uncertain states can be quite conservative and polytopes are more preferable for the bounding tasks as they can approximate any compact convex set as closely as desired [9]. The use of polytopes for state estimation was studied in [10], [3] where the uncertain states of linear discrete-time systems were recursively bounded by polytopes or the simplified parallelotopes. These set-membership approaches of using polytopes for state estimation are often computationally demanding and they are also restricted to linear systems or piece-wise affine systems [11].

Set-membership state estimation for nonlinear systems was studied in [12], [13] where the admissible state space was bisected and selected into

This work was supported by EPSRC of the U.K. under Grant EP/R005532/1. Corresponding author: J. Wan (E-mail: jian.wan@plymouth.ac.uk).

subsets to test their consistency with the observations via interval set computation. The main hurdles for interval-based state estimation are the so-called wrapping effect that makes the solution conservative and also the curse of dimensionality that makes the computation burden grow exponentially with the dimensionality of the state space. Similar to interval set computation, the dynamic evolution of a nonlinear system with a zonotopic set as the initial state can also be computed directly via zonotopic set computation and the wrapping effect can be reduced greatly with comparison to interval set computation [14]. Except for the reduced wrapping effect, zonotopes as a special kind of polytopes are also more flexible in shape than intervals. Therefore, zonotopic set computation has been increasingly used for state estimation of nonlinear systems [4], [5], [15]. Nevertheless, zonotopic set computation can be used for state estimation of linear discrete-time systems as well [1]. A new class of sets called constrained zonotopes was also proposed in [16] for set-membership state estimation of linear discrete-time systems.

A common practice within these zonotope-based state estimation approaches for nonlinear discrete-time systems is to search an optimal zonotope to bound the intersection of the evolved uncertain state trajectory and the region of the state space consistent with the observed output at each observation update. Then the optimized zonotope is to be propagated for computing the dynamic evolution of this nonlinear system via zonotopic set computation. The set resulting from the intersection of the evolved uncertain state trajectory and the region of the state space consistent with the observed output at each observation update is essentially a polytopic set therein. This polytopic set is often over-approximated by one single zonotope. The repetitive bounding processes of the polytopic set by one single zonotope and the following propagations of the over-approximated zonotopes impact negatively on the accuracy of state estimation although great efforts have been made to obtain a tighter zonotopic bound of the intersection in [15].

The over-approximating bounding of a polytopic set by one single zonotope at each observation update in [4] becomes unnecessary if the dynamic evolution of a polytopic set can be computed for a nonlinear discrete-time system. Currently, there is no direct method to compute the dynamic evolution of a nonlinear discrete-time system with a polytopic set as the initial set. This paper proposes a novel idea to implement polytopic set computation for nonlinear discrete-time systems indirectly, which is to represent the polytope exactly by the intersection of zonotopes at first and then to compute the dynamic evolutions of these individual zonotopes whose intersection forms the polytope. The proposed idea in this paper is originated from the perspective of extending existing interval or zonotopic set computation for nonlinear discrete-time systems into polytopic set computation for nonlinear discrete-time systems. The intersection of zonotopes was called a zonotope bundle in [17] and it was used for the efficient computation of reachable sets. The intersection of ellipsoids was also used to bound the reachable set of singular systems in [18]. However, the problem of researchable set computation is different from state estimation because reachable set computation does not involve the intersection with the observation update at each step as in [19], [20].

The rest of the paper is organized as follows. Section II provides the mathematical formulation of the state estimation problem to be solved. Section III describes the proposed idea of indirectly implemented polytopic set computation for nonlinear discrete-time systems. The procedure of guaranteed state estimation via indirectly implemented polytopic set computation for nonlinear discrete-time systems is given in Section IV. An illustrative example to demonstrate the effectiveness of the proposed technique is provided in Section V. Finally, some conclusions and future work are drawn in Section VI.

II. Problem formulation

Consider the following nonlinear uncertain discrete-time system [4]:

$$\begin{cases} \mathbf{x}_k = f(\mathbf{x}_{k-1}, \omega_{k-1}) \\ \mathbf{y}_k = g(\mathbf{x}_k, \upsilon_k), \end{cases} \tag{1}$$

where $\mathbf{x}_k \in \Re^n$ and $\mathbf{y}_k \in \Re^p$ are the system state and the observed system output at time instant k, respectively; $\omega_k \in \Re^{n_\omega}$ represents the time-varying process parameters and process perturbations; $\upsilon_k \in \Re^{n_\upsilon}$ represents the observation noises. The state function $f(\mathbf{x}_{k-1}, \omega_{k-1})$ is assumed to be nonlinear while the output function $g(\mathbf{x}_k, \upsilon_k)$ is assumed to be linear as in [4], [15] with the format

of $g(\mathbf{x}_k, \upsilon_k) = \mathbf{c}^T\mathbf{x}_k + \mathbf{d}^T\upsilon_k$. It is also assumed that the initial state and all the uncertainties are bounded by known compact sets: $\mathbf{x}_0 \in X_0$, $\omega_k \in W_k$ and $\upsilon_k \in V_k$.

Starting from the initial set X_0 for the system state, the problem to be considered is to estimate recursively the set $X_k(k = 1, 2, \cdots)$ for the system state at future time instants. The estimated set X_k should guarantee to bound any feasible system state under all the uncertainties $W_k(k = 0, 1, 2, \cdots)$ and $V_k(k = 1, 2, \cdots)$. Furthermore, the estimated set X_k should also be consistent with the observed system output $\mathbf{y}_k(k = 1, 2, \cdots)$.

The set of the system state consistent with the observed output $\mathbf{y}_k$ at time instant k can be denoted as $X_{\mathbf{y}_k} = \{\mathbf{x} \in \Re^n : \mathbf{y}_k \in g(\mathbf{x}, V_k)\}$. Then the set X_k for the system state at time instant k can be computed recursively as follows:

$$X_k = f(X_{k-1}, W_{k-1}) \cap X_{\mathbf{y}_k}, k = 1, 2, \cdots \quad (2)$$

It can be seen that the state estimation problem considered here involves the set computation for the dynamic evolution of the past system state X_{k-1} and also the intersection of two sets $f(X_{k-1}, W_{k-1})$ and $X_{\mathbf{y}_k}$. Existing methods for this problem search an optimal zonotope to bound the polytopic set resulting from this intersection at each time instant and thus the initial state for the dynamic evolution in (1) is always a zonotope. The over-approximating bounding process at each time instant facilitates the computation of the dynamic evolution in (1) as the dynamic evolution of a nonlinear system with a zonotopic set as the initial state is straightforward [14]. However, the set resulting from the intersection is essentially a polytopic set and it would be more accurate to compute the dynamic evolution of this exact polytopic set rather than to compute the dynamic evolution of an over-approximating zonotopic set. Since there is no direct method to implement polytopic set computation for nonlinear discrete-time systems, an indirectly implemented polytopic set computation technique is to be proposed for the first time in the following section.

III. Polytopic set computation

Taking a set as the input for a function, set computation returns another set as the output of the function. Polytopic set computation involves the computation of the dynamic evolution of a nonlinear discrete-time system with a polytopic set as the initial state. Polytopic set computation can be implemented indirectly through zonotopic set computation, which is to be introduced in the following subsections.

A. Zonotopic set computation

A zonotope is a centrally symmetric convex polytope and it is closely related to interval analysis in terms of set computation. Given a vector $\mathbf{p} \in \Re^n$ and a matrix $H \in \Re^{n\times m}$, the zonotope $\mathcal{Z}$ of order $n \times m$ is the set:

$$\mathbf{p} \oplus H\mathbf{B}^m = \{\mathbf{p} + H\mathbf{z} | \mathbf{z} \in \mathbf{B}^m\}, \quad (3)$$

where $\mathbf{B}^m$ is a box composed of m unitary intervals $\mathbf{B} = [-1, 1]$ and $\oplus$ is the Minkowski sum of sets, which is to add each member in one set to each member in the other set so as to obtain a new set. Representing the matrix H by its column vectors, i.e., $H = [\mathbf{h}_1 \cdots \mathbf{h}_m]$, then the zonotope can also be regarded as a set spanned by the column vectors of H, which are called line segment generators:

$$\mathcal{Z} = \{\mathbf{p} + \sum_{i=1}^{m} \alpha_i \mathbf{h}_i | -1 \leq \alpha_i \leq 1\}. \quad (4)$$

Geometrically, the zonotope $\mathcal{Z}$ is the transferred Minkowski sum of the line segments defined by the columns of the matrix H to the central point $\mathbf{p}$. Specifically, the zonotope $\mathcal{Z}$ degenerates to be an interval vector as well as a box when H is a diagonal matrix or when $m = 1$. The list of line segment generators is an efficient implicit representation of a zonotope in terms of which set computations such as the Minkowski sum and difference are trivial. The explicit representation of a zonotope or the representation of a zonotope in the format of a polytope is the zonotope construction problem aiming to list all extreme points of a zonotope defined by its line segment generators. A relatively efficient algorithm was proposed in [21] to address the zonotope construction problem, where the addition of line segments was replaced by the addition of convex polytopes. Standard algorithms for polytope geometry such as vertex enumeration for a polytope and the intersection of polytopes have been implemented in Multi-Parametric Toolbox [22].

Using zonotopes, Kühn developed a procedure to bound the dynamic evolution of a nonlinear discrete-time system with a guaranteed sub-

exponential over-estimation [14]. The following theorem introduces the zonotope inclusion operator of Kühn's method [14]:

Theorem 1 (Zonotope Inclusion) Consider a family of zonotopes represented by $\mathbb{Z} = \mathbf{p} \oplus \mathbb{M}\mathbf{B}^m$ where $\mathbf{p} \in \Re^n$ is a real vector and $\mathbb{M} \in \mathbb{I}^{n\times m}$ is an interval matrix. A zonotope inclusion, denoted by $\diamond(\mathbb{Z})$, is defined by:

$$\diamond(\mathbb{Z}) = \mathbf{p} \oplus [\mathtt{mid}(\mathbb{M}) \quad G] \begin{bmatrix} \mathbf{B}^m \\ \mathbf{B}^n \end{bmatrix}, \tag{5}$$

where $\mathtt{mid}(\mathbb{M})$ is the centered-point matrix of $\mathbb{M}$ and $G \in \Re^{n\times n}$ is a diagonal matrix that satisfies:

$$G_{ii} = \sum_{j=1}^{m} \frac{\mathtt{diam}(\mathbb{M}_{ij})}{2}, i = 1, \cdots, n \tag{6}$$

where $\mathtt{diam}(\mathbb{M}_{ij})$ is the length of the interval $\mathbb{M}_{ij}$. Under these definitions, it results that $\mathbb{Z} \subseteq \diamond(\mathbb{Z})$.

Given a function $\mathbf{f}(\mathbf{x}) : \Re^n \to \Re^n, \mathbf{x} \in \mathcal{Z} \subset \mathbb{X} \in \mathbb{I}^n$, where $\mathcal{Z} = \mathbf{p} \oplus H\mathbf{B}^m$ and $\mathbb{X}$ is the bounding box for $\mathcal{Z}$, the centered inclusion function $\mathbf{F}_c(\mathcal{Z}) : \mathbf{f}(\mathcal{Z}) \subseteq \mathbf{F}_c(\mathcal{Z})$ can be deduced by the mean-value theorem [14], [4], i.e.,

$$\mathbf{F}_c(\mathcal{Z}) = \mathbf{f}(\mathbf{p}) + \nabla_{\mathbf{x}}\mathbf{f}(\mathbb{X})(\mathcal{Z} - \mathbf{p}), \tag{7}$$

where $\mathcal{Z} - \mathbf{p} = H\mathbf{B}^m$. Thus the centered inclusion function $\mathbf{F}_c(\mathcal{Z})$ of $\mathbf{f}(\mathbf{x})$ turns out to be a family of zonotopes represented by $\mathbb{Z} = \mathbf{q} \oplus \mathbb{M}\mathbf{B}^m$, where $\mathbf{q} = \mathbf{f}(\mathbf{p})$ and $\mathbb{M} = \nabla_{\mathbf{x}}\mathbf{f}(\mathbb{X})H$, which can be further bounded by its corresponding zonotope inclusion $\diamond(\mathbf{Z})$. This is the primary principle of Kühn's method to bound the dynamic evolution of a nonlinear discrete-time system by zonotopes, where the centered inclusion function is applied instead of the natural inclusion function.

Kühn's method has been generalized to bound the dynamic evolution of nonlinear uncertain discrete-time systems in [4]. According to [4], the evolution of a zonotopic set as the initial state for a nonlinear uncertain function $\mathbf{f}(\mathbf{x}, W)$ can be bounded by the following centered inclusion function:

$$\mathbf{F}_c(\mathcal{Z}, W) = \mathbf{f}(\mathbf{p}, W) + \nabla_{\mathbf{x}}\mathbf{f}(\mathbb{X}, W)(\mathcal{Z} - \mathbf{p}), \tag{8}$$

where W is the uncertain set. Assume that $\mathbf{f}(\mathbf{p}, W) \subseteq \mathbf{p}_w \oplus J\mathbf{B}^w$, then the centered inclusion function $\mathbf{F}_c(\mathcal{Z}, W)$ can be further bounded as follows:

$$\mathbf{F}_c(\mathcal{Z}, W) \subseteq \mathbf{p}_w \oplus J\mathbf{B}^w \oplus \mathbb{M}_w\mathbf{B}^m, \tag{9}$$

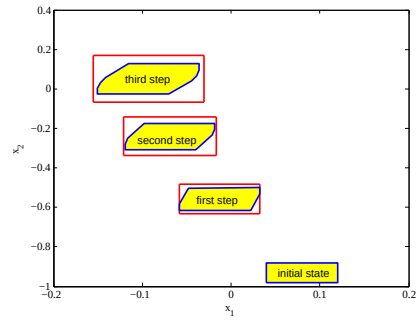

Fig. 1. Zonotopic set computation vs. interval set computation

where $\mathbb{M}_w = \nabla_{\mathbf{x}}\mathbf{f}(\mathbb{X}, W)H$. Based on (5), $\mathbf{p}_w \oplus J\mathbf{B}^w \oplus \mathbb{M}_w\mathbf{B}^m$ can be further bounded by a zonotopic set. Therefore the dynamic evolution of a nonlinear uncertain discrete-time system with a zonotopic set as the initial state returns a zonotopic set as well, which is the essence of zonotopic set computation.

The above discussion shows that the dynamic evolution of a nonlinear discrete-time system with a zonotopic set as the initial state can be computed directly through zonotopic set computation. Compared to interval set computation where each variable is represented by an interval and interval arithmetic is used for set computation [23], zonotopic set computation has the benefit of a reduced wrapping effect. The reduced wrapping effect can be demonstrated by an illustrative example shown in Figure 1, where the dynamic evolution of a nonlinear uncertain discrete-time system studied in [24] is computed for three steps via interval set computation and zonotopic set computation, respectively. These two approaches starts from the same initial state and it can be seen that zonotopic set computation is less conservative with comparison to interval set computation.

B. Polytopic set computation

The dynamic evolution of a nonlinear discrete-time system with a polytopic set as the initial state cannot be computed directly due to its mathematical format involving inequality constraints. However, using the proposed idea of representing a polytope exactly by the intersection of zonotopes, polytopic set computation can be implemented indirectly through computing the dynamic evolution of these individual zonotopes whose intersection forms the

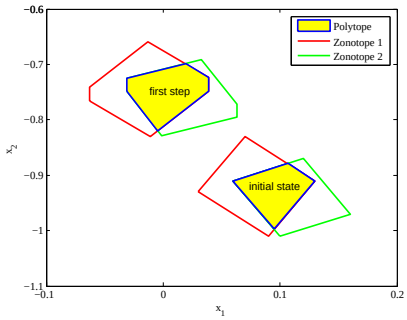

Fig. 2. The principle of indirectly implemented polytopic set computation

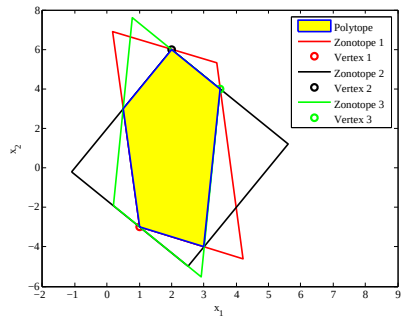

Fig. 3. Representing a polytope exactly as the intersection of zonotopes

polytope. The principle of indirectly implemented polytopic set computation can be illustrated by Figure 2, where the polytope $\mathcal{P} = \mathcal{Z}_1 \cap \mathcal{Z}_2$ and thus $\mathbf{f}(\mathcal{P}) = \mathbf{f}(\mathcal{Z}_1 \cap \mathcal{Z}_2) \subseteq \mathbf{f}(\mathcal{Z}_1) \cap \mathbf{f}(\mathcal{Z}_2)$ according to set theory. So the key for indirectly implemented polytopic set computation is to represent the polytope exactly by the intersection of zonotopes. The following theorem provides the guideline to represent a 2-D polytope exactly by the intersection of parallelograms, which are simple zonotopes in 2-D space:

Theorem 2 (Represent a polytope $\mathcal{P}$ in $\Re^2$ exactly by the intersection of zonotopes) Assume that the polytope $\mathcal{P} \subset \Re^2$ has n_c inequality constraints, then the convex polygon $\mathcal{P}$ can be represented exactly by the intersection of $\frac{n_c}{2}$ zonotopes if n_c is even or exactly by the intersection of $\frac{n_c+1}{2}$ zonotopes if n_c is odd.

Proof. As the polytope $\mathcal{P} \subset \Re^2$ has n_c inequality constraints, it has n_c edges associated to these n_c inequality constraints. At each vertex of the polytope, there are two edges that start from this vertex. Making use of these two edges that start from the vertex, a zonotope or a parallelogram can be constructed to contain the polytope. The polytope can be represented by the intersection of these constructed parallelograms if all its edges have been used up to construct the parallelograms. As each parallelogram uses two edges, the number of parallelograms needed to represent the polytope exactly is $\frac{n_c}{2}$ if n_c is even or $\frac{n_c+1}{2}$ if n_c is odd.

The construction of a parallelogram to contain the 2-D polytope can be transformed to be a linear programming (LP) problem that minimizes the sum of the base length and the side length for the parallelogram to be minimal in volume. Assume that at the vertex V^j, the corresponding two edges starting from the vertex are $ax_1 + bx_2 = p_1$ and $cx_1 + dx_2 = p_2$ according to the associated two inequality constraints. The constructed parallelogram should satisfy: $q_1 \leq ax_1 + bx_2 \leq p_1$ and $q_2 \leq cx_1 + dx_2 \leq p_2$ where q_1 and q_2 determine the size of the parallelogram and they are to be optimized through the following LP problem:

$$(q_1^*, q_2^*) = \arg \min_{(q_1,q_2)} [p_1 + p_2 - q_1 - q_2], \tag{10}$$

subject to

$$\begin{cases} q_1 \leq aV_x^i + bV_y^i, i = 1, \cdots, n_v \\ q_2 \leq cV_x^i + dV_y^i, i = 1, \cdots, n_v, \end{cases} \tag{11}$$

where $V^i = (V_x^i, V_y^i)$ is the ith vertex of the polytope and n_v is the total number of vertices for the polytope. These linear constraints are to guarantee that the constructed parallelogram contains the whole 2-D polytope. Once the parallelogram is constructed with the optimized q_1^* and q_2^*, it can be re-represented in the format of a zonotope $\mathcal{Z} = \mathbf{p} \oplus H\mathbf{B}^2$ where $\mathbf{p}$ is the center of this parallelogram and H is algebraically determined from the vertices of this parallelogram.

Taking the 2-D polytope with five vertices $(1, -3)$, $(0.5, 3)$, $(2, 6)$, $(3.5, 4)$ and $(3, -4)$ as an example, it can be represented exactly as the intersection of three zonotopes as shown in Figure 3, where these three zonotopes are $\mathcal{Z}_1 = \begin{bmatrix} 2.1957 \\ 1.1522 \end{bmatrix} \oplus \begin{bmatrix} -1.6087 & -0.4130 \\ 0.8043 & 4.9565 \end{bmatrix} \mathbf{B}^2$, $\mathcal{Z}_2 = \begin{bmatrix} 2.25 \\ 0.5 \end{bmatrix} \oplus \begin{bmatrix} -1.8 & 1.55 \\ 2.4 & 3.1 \end{bmatrix} \mathbf{B}^2$ and $\mathcal{Z}_3 = \begin{bmatrix} 1.8462 \\ 1.0385 \end{bmatrix} \oplus \begin{bmatrix} -1.3558 & 0.2981 \\ 1.8077 & 4.7692 \end{bmatrix} \mathbf{B}^2$, respectively. These three

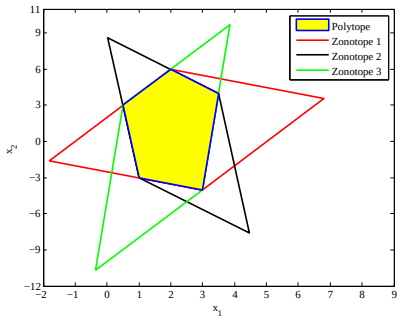

Fig. 4. Constructing zonotopes using inequality constraints

zonotopes are obtained from the formulated LP problem at the vertices of $(1, -3)$,$(2, 6)$ and $(3.5, 4)$, respectively. So a polytope in 2-D space can be represented exactly by the intersection of zonotopes through solving the corresponding LP problems.

It is worthy to note that the two edges with the associated two inequality constraints for formulating the LP problem in (10-11) do not necessarily come from the same vertex. In fact, any two inequality constraints can be sequentially selected from the pool of all inequality constraints for the polytope to formulate the LP problem for finding the optimal parallelogram to contain the polytope as long as the edges from these two inequality constraints are not parallel. Using random inequality constraints for the 2-D polytope shown in Figure 3 to formulate the corresponding LP problems, the constructed parallelograms to represent the polytope exactly are shown in Figure 4. The obtained parallelogram from two random edges in Figure 4 may not be as compact as those obtained from two specified edges coming from the same vertex as shown in Figure 3. However, such an approach of using inequality constraints directly instead of using two specified edges coming from the same vertex to formulate the LP problem can be easily extended into higher dimensional spaces, i.e., the sequential selection of certain inequality constraints from the pool of all inequality constraints for the higher dimensional polytope to formulate the LP problem for finding the optimal zonotope to contain the polytope until all inequality constraints are used up.

IV. Guaranteed State Estimation via Indirectly Implemented Polytopic Set Computation

Based on the problem formulation in Section II and the proposed technique for indirectly implemented polytopic set computation in Section III, the general procedure of guaranteed state estimation for nonlinear discrete-time systems via indirectly implemented polytopic set computation can be listed as follows:

- Step 1: Represent the past system state of a polytopic set X_{k-1} exactly by the intersection of zonotopes $X_{k-1} = \mathcal{Z}_1 \cap \cdots \cap \mathcal{Z}_{n_z}$, where n_z is the number of zonotopes whose intersection forms the polytopic set;
- Step 2: Compute the dynamic evolution of these zonotopic sets $\mathbf{f}(\mathcal{Z}_1, W_{k-1})$,$\cdots$,$\mathbf{f}(\mathcal{Z}_{n_z}, W_{k-1})$ individually via zonotopic set computation as formulated in (9);
- Step 3: Compute the set of the system state $X_{\mathbf{y}_k}$ that is consistent with the observed system output $\mathbf{y}_k$ and $X_{\mathbf{y}_k}$ is a convex set due to the linearity of $g(\mathbf{x}_k, \upsilon_k)$;
- Step 4: Compute the current system state of a new polytopic set $X_k = \mathbf{f}(\mathcal{Z}_1, W_{k-1}) \cap \cdots \cap \mathbf{f}(\mathcal{Z}_{n_z}, W_{k-1}) \cap X_{\mathbf{y}_k}$ where the zonotopes $\mathbf{f}(\mathcal{Z}_1, W_{k-1})$,$\cdots$,$\mathbf{f}(\mathcal{Z}_{n_z}, W_{k-1})$ are to be transformed into the format of polytopes as described in [21] before their intersection with the convex set $X_{\mathbf{y}_k}$;
- Step 5: Return to Step 1;

According to set theory, $\mathbf{f}(X_{k-1}, W_{k-1}) \cap X_{\mathbf{y}_k} \subseteq \mathbf{f}(\mathcal{Z}_1, W_{k-1}) \cap \cdots \cap \mathbf{f}(\mathcal{Z}_{n_z}, W_{k-1}) \cap X_{\mathbf{y}_k} = X_k$ and thus the system states are guaranteed to be contained in the computed polytopic sets $X_k (k = 1, 2, \cdots)$. The computed polytopic set X_k can still be an over-approximation of the real state. However, such over-approximation is mainly from the limited wrapping effect of zonotopic set computation rather than the extra over-approximation of a polytopic set by one single zonotopic set and the following propagation of such an over-approximation as in [4], [5], [15]. Furthermore, the intersection of zonotopes can also contribute to the reduction of conservativeness and the convergence of the algorithm since more constraints have been propagated during the evolution process.

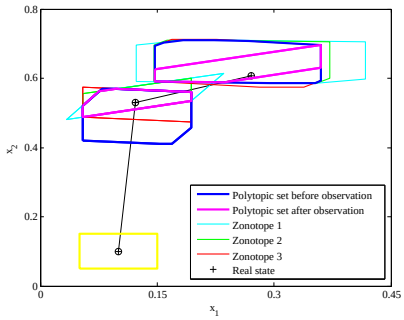

Fig. 5. The process of guaranteed state estimation

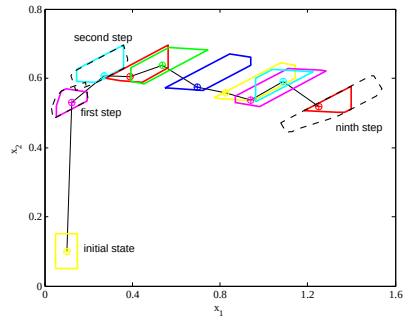

Fig. 6. State estimation via indirectly implemented polytopic set computation

V. An Illustrative Example

A modified nonlinear uncertain discrete-time system studied in [25] is adopted as the illustrative example for the proposed set-membership state estimation of nonlinear discrete-time systems via indirectly implemented polytopic set computation. The system is described as follows:

$$\begin{cases} x_1(k+1) = 0.99x_1(k) + \delta(k)x_2(k) \\ x_2(k+1) = -0.1x_1(k) + \frac{0.5x_2(k)}{1+x_2^2(k)} + \omega(k), \end{cases} \tag{12}$$

$$y(k) = x_1(k) - 3x_2(k) + \upsilon(k), \tag{13}$$

where $\delta(k) \in [0.2, 0.3]$ is the uncertain parameter; $\omega(k) \in [0.4, 0.5]$ is the process perturbation; $|\upsilon(k)| \leq 0.1$ is the bounded measurement noise. According to Section IV, the process of guaranteed state estimation via indirectly implemented polytopic set computation for this system is shown in Figure 5. The initial state is assumed to be within a box $x_1(0) \in [0.5, 0.15]$ and $x_2(0) \in [0.5, 0.15]$. For this particular simulation, the real initial state is set to be $x_1(0) = 0.1$ and $x_2(0) = 0.1$.

As shown in Figure 5, a polytopic set is obtained from the dynamic evolution of this initial set and this polytopic set is then to be intersected with the convex set $X_{\mathbf{y}_1}$ that is consistent with the first observation. The renewed polytopic set from the intersection with this observation update is a hexagon and it is represented by three zonotopes obtained from the LP formulation as described in B of Section III. The dynamic evolution of these three zonotopic sets for the system is computed individually via zonotopic set computation as discussed in A of Section III. The polytopic set before observation at the second step is the intersection of these three propagated zonotopic sets and this polytopic set before observation is to be intersected with the set $X_{\mathbf{y}_2}$ that is consistent with the second observation. The same procedure of representing the new polytopic set after observation exactly by the intersection of zonotopes is performed at the second step and therefore the state for any future steps can all be bounded by polytopic sets.

Repeating the processes in Figure 5, Figure 6 shows an example of nine steps for state estimation of this nonlinear discrete-time system with comparison to an existing method of approximating the polytopic set by one single zonotope with the minimized segments at each observation update [4]. The dashed polytopes in Figure 6 are those polytopes obtained after observation update using the method proposed in [4] while the solid polytopes are those polytopes obtained after observation update using the method proposed here. To avoid too many overlapping polytopes in Figure 6, the dashed polytopes are plotted for only the first two steps and the ninth step of the simulation. It can be seen that the real states plotted by $\oplus$ are all within the obtained polytopic sets of the proposed approach and the polytopic sets from state estimation have various kinds of shape as well. The intersection operation of the propagated zonotopes as well as the intersection with observation update at each step can potentially reduce the complexity of the obtained polytopic set as well as the number of zonotopes needed to represent the polytope exactly. This can be seen directly from the ninth step in Figure 6 as only two zonotopes are needed to represent the obtained polytope exactly. The volume of the set after observation update for the existing method in [4] and the proposed approach is listed in Table I:

Overall, the obtained polytopic sets after obser-

TABLE I
THE COMPARISON BETWEEN THE EXISTING METHOD IN [4] AND THE PROPOSED METHOD IN TERMS OF THE VOLUME OF THE OBTAINED SET AFTER OBSERVATION UPDATE

The volume of the set	Method in [4]	New method
Average volume for 9 steps	0.0210	0.0139
Specific volume at the 9th step	0.0247	0.0081

vation update for the proposed approach have an average volume of 0.0139, which is much smaller than the average volume of 0.0210 for the obtained polytopic sets after observation update from the existing method in [4]. Therefore the average accuracy for state estimation has been improved by 33.81% for these nine steps. Particularly, the proposed approach has a larger improvement of 67.21% at the 9th step as shown in Table I, which shows the greater benefit of using indirectly implemented polytopic set computation for state estimation. The computation involves the converted LP problems and thus efficient algorithms are available.

VI. CONCLUSIONS

This paper has proposed the novel idea of representing a polytope exactly by the intersection of zonotopes, which enables polytopic set computation for a nonlinear discrete-time system with a polytopic set as the initial state. Such an extension of set computation for nonlinear discrete-time systems from interval and zonotopic set computation to polytopic set computation opens new research directions for set-membership methods. The paper has applied the proposed idea to solve the guaranteed state estimation problem for nonlinear uncertain discrete-time systems. The resulting set-membership state estimation via indirectly implemented polytopic set computation avoids the over-approximating processes of bounding the polytopic set at each observation update by a single zonotope and thus a more accurate state estimation can be obtained.

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