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THERMAL BUCKLING ANALYSIS OF AXIALLY LOADED COLUMNS OF THIN-WALLED OPEN SECTION WITH NON-UNIFORM SECTIONAL PROPERTIES

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This paper presents an analytical study on the thermal buckling analysis of axially loaded columns of thin-walled open section with non-uniform sectional properties. Critical loads related to flexural, torsional and flexural-torsional buckling of an I-section column subjected to an axial compressive load applied at the geometric centroid, under linearly varied non-uniform temperature distribution scenarios are derived. The analysis is accomplished using traditional energy methods. The influences of thermal strain, non-uniform distribution of pre-buckling stresses, and variation of pre-buckling stresses along the longitudinal axis of the column on critical buckling loads are examined. The present results highlight the importance of non-uniform sectional properties in the buckling analysis of columns of doubly symmetric section.

Keywords: Thin-walled, flexural, torsional, buckling, non-uniform, temperature, column.

1. Introduction

It is well-known that any axially loaded members of doubly symmetric section may have three distinct buckling modes, namely flexural, torsional and flexural-torsional modes, among which the flexural buckling load about the weak axis is almost always the lowest. Hence, in the design of doubly symmetric sections the torsional buckling load is usually disregarded. In non-symmetric sections, however, buckling will be always in the flexural-torsional mode regardless of its shape and dimensions. Thin-walled open mono-symmetric sections, such as angles and channels, can buckle in the flexural and flexural-torsional modes. Which of these two modes is critical depends on the shape and dimensions of the cross-section. Hence, flexural-torsional buckling must be considered in their design. This is normally done by calculating an equivalent slenderness ratio and using the same column strength curve as for flexural buckling.

Note that the definition of "symmetry" used above to characterize the buckling behaviour of a member cannot be based purely on the geometry of the section but also need consider the mechanical properties of the section. For example, an I-section made of composite materials is doubly symmetric in terms of its geometry but may not be doubly symmetric in terms of its mechanical properties.[1-4] Another example is when a doubly symmetric section is exposed to a fire on its one side, which causes a non-uniform distribution of temperature in the cross-section.[5-6] The non-uniform temperature leads to a non-uniform mechanical property, which can turn the column from a flexural buckling to a flexural-torsional buckling. An excellent work was provided by Pi and Bradford in describing the lateral-torsional buckling of I-section beams subjected to transverse loading under non-uniform temperature distribution. [6]

The theory of torsional and flexural-torsional buckling of thin-walled open section members subjected to axial compressive loads can be found from literature of textbooks and research articles.[7-9] Apart from the theoretical study of torsional and flexural-torsional buckling, pre-buckling[10] and post-buckling[11] of thin-walled open section members subjected to axial compressive loads were also discussed. The work involves the use of not only analytical methods[9] but also finite element methods.[12-13]

However, despite the considerable amount of work published in literature, there is very little work on the influence of non-uniform mechanical properties on the torsional and flexural-torsional buckling of thin-walled open sections subjected to axial compressive loads. It is expected that if the mechanical property is not uniform in the cross section of a member, the bending centre of the member will not be at the geometric centroid of the section. In this case compressive loads applied at geometric centroid may cause the member to bend. The combined action of the compression and bending can lead the member to have a torsional or a flexural-torsional buckling. In this paper, an analytical study on the buckling analysis of axially loaded columns of thin-walled open section with non-uniform sectional properties is reported. Critical loads related to flexural, torsional and flexural-torsional buckling of an I-section column subjected to an axial compressive load applied at the geometric centroid, under linearly varied non-uniform temperature distribution scenarios are derived. The analysis is accomplished using traditional energy methods. The non-uniform mechanical properties are assumed to be induced by the non-uniform temperature distribution in the section. The influences of thermal strain, non-uniform distribution of pre-buckling stresses, and variation of pre-buckling stresses along the longitudinal axis of the column on critical buckling loads are examined. The present results highlight the importance of non-uniform sectional properties in the buckling analysis of columns.

2. Pre-buckling analysis

Consider an I-section column subjected to an axial compressive load as shown in Fig. 1. Let b_f and t_f be the width and thickness of the flange, h_w and t_w be the depth and thickness of the web, respectively. Under a uniform temperature the Young's modulus of the column is also uniform although its value may be dependent on the temperature. In this case the pre-buckling stress of the column can be obtained using the traditional theory of axially loaded members. However, if the temperature in its cross-section is not uniformly distributed, the Young's modulus of the column will be different at different points on the cross-section. In this case not only can the axial compressive load applied at geometric centroid cause the compression of the column but also it can lead to the bending of the column about its geometric principal axes.

Let o be the geometric centroid of the I-section, oy and oz be the two corresponding geometric principal axes (see Fig. 2). Since for most cases the temperature distribution on the cross-section is symmetric about the web, for example when a protected I-section column is exposed to a fire on its one side, it is assumed here that the temperature varies only with the y-axis. Let T_2 and T_1 be the temperatures of upper and lower flanges, and E_2 and E_1 be the corresponding Young's moduli of them (see Fig. 2). By using Euler-Bernoulli beam's assumption, the axial strain at any coordinate point (y, z) of the cross-section can be expressed as the sum of a membrane strain and a bending strain about z-axis as follows,

$$\varepsilon(y,z)=\varepsilon_o-y\kappa_{xy} \tag{1}$$

where ε_o is the membrane strain and κ_{xy} is the curvature of the column in the xy-plane (see Fig. 2). On the other hand, the total axial strain can also be decomposed in terms of the strain components generated by individual actions,

$$\varepsilon(y,z)=\frac{\sigma}{E}+\varepsilon_{th} \tag{2}$$

where σ is the axial stress, E is the temperature-dependent Young's modulus, and ε_{th} is the thermal strain. Solve σ from Eqs. (1) and (2), yielding,

$$\sigma=E(\varepsilon_o-y\kappa_{xy}-\varepsilon_{th}) \tag{3}$$

The resultant force and moment on the cross-section requires the following equilibrium equations,

$$N_x=\int_A\sigma dA=\int_A E(\varepsilon_o-y\kappa_{xy}-\varepsilon_{th})dA=\varepsilon_o\int_A EdA-\kappa_{xy}\int_A yEdA-\int_A E\varepsilon_{th}dA \tag{4}$$

$$M_z=-\int_A y\sigma dA=-\int_A yE(\varepsilon_o-y\kappa_{xy}-\varepsilon_{th})dA=-\varepsilon_o\int_A yEdA+\kappa_{xy}\int_A y^2EdA+\int_A yE\varepsilon_{th}dA=N_x v \tag{5}$$

where N_x is the axial membrane force, M_z is the bending moment about z-axis, and v is the transverse deflection of the column in y-direction. Let,

$$S_o=\int_A EdA=(E_1+E_2)\left(A_f+\frac{A_w}{2}\right) \tag{6}$$

$$S_1=\int_A yEdA=(E_2-E_1)\left(\frac{hA_f}{2}+\frac{h_wA_w}{12}\right) \tag{7}$$

$$S_2=\int_A y^2EdA=(E_1+E_2)\left[\left(\frac{h^2}{4}+\frac{t_f^2}{12}\right)A_f+\frac{h_w^2A_w}{24}\right] \tag{8}$$

$$S_{To}=\int_A E\varepsilon_{th}dA=\alpha A_f\left[E_1(T_1-T_o)+E_2(T_2-T_o)\right]+\frac{\alpha A_w}{6}\left[(E_1+E_2)(T_1+T_2-3T_o)+(E_1T_1+E_2T_2)\right] \tag{9}$$

$$S_{T1}=\int_A yE\varepsilon_{th}dA=\frac{\alpha A_f h}{2}\left[(T_2-T_o)E_2-(T_1-T_o)E_1\right]+\frac{\alpha A_w h_w}{12}\left[E_2(T_2-T_o)-E_1(T_1-T_o)\right] \tag{10}$$

where $A_f=t_fb_f$ and $A_w=t_wh_w$ are the cross-sectional areas of the flange and web, $h=h_w+t_f$ is the distance between the midlines of upper and lower flanges, α is the thermal expansion coefficient, T_o is the ambient temperature. The above cross-section integrations are accomplished under the assumptions that the flanges have constant temperatures and the web has a linearly varied temperature from T_1 to T_2 and a linearly varied Young's modulus from E_1 to E_2, respectively (see Fig. 2). By using the notations defined in Eqs. (6)-(10), Eqs. (4)-(5) can be expressed as follows,

$$S_o\varepsilon_o-S_1\kappa_{xy}=N_x+S_{To} \tag{11}$$

$$-S_1\varepsilon_o+S_2\kappa_{xy}-N_xv=-S_{T1} \tag{12}$$

By eliminating ε_o from Eqs. (11) and (12) and noting that $\kappa_{xy}=\dfrac{d^2v}{dx^2}$, one obtains

$$\left(S_2-\frac{S_1^2}{S_o}\right)\frac{d^2v}{dx^2}-N_xv=\frac{S_1}{S_o}(N_x+S_{To})-S_{T1} \tag{13}$$

The transverse deflection of the column for simply supported ends governed by Eq. (13) can be expressed as

$$v(x)=\frac{S_oS_{T1}-S_1(S_{To}-P)}{S_oP}\left(\frac{1-\cos kL}{\sin kL}\sin kx+\cos kx-1\right) \tag{14}$$

where L is the column length, $P = -N_x$ is the axial compressive load, and k is defined as

$$k=\sqrt{\frac{S_oP}{S_oS_2-S_1^2}} \tag{15}$$

The maximum deflection occurs at $x = L/2$ and is given by

$$v\Big|_{x=L/2}=\frac{S_oS_{T1}-S_1(S_{To}-P)}{S_oP}\left[\sec\left(\frac{kL}{2}\right)-1\right] \tag{16}$$

It is apparent from Eq. (16) that, when $kL\rightarrow\pi$, $v(x)$ at $x = L/2$ tends ∞. This indicates that the maximum axial compressive load of the column is given by

$$P_{\max}=\frac{\pi^2(S_oS_2-S_1^2)}{S_oL^2} \tag{17}$$

Clearly, P_{max} is also the critical load of the column for the buckling about z-axis. It is obvious that if the temperature is uniform, then $S_1 = 0$ and P_{max} reduces to the Euler critical buckling load.

The curvature and the membrane strain can be calculated using Eqs. (14) and (11) as follows:

$$\kappa_{xy}=\frac{d^2v}{dx^2}=\frac{S_1(S_{To}-P)-S_oS_{T1}}{S_oS_2-S_1^2}\left(\frac{1-\cos kL}{\sin kL}\sin kx+\cos kx\right) \tag{18}$$

$$\varepsilon_o=\frac{du}{dx}=\frac{S_{To}-P}{S_o}+\frac{S_1^2(S_{To}-P)-S_oS_1S_{T1}}{S_o^2S_2-S_oS_1^2}\left(\frac{1-\cos kL}{\sin kL}\sin kx+\cos kx\right) \tag{19}$$

The pre-buckling stress distribution in flanges and web can be determined using Eq. (3) as follows:

At the lower flange:

$$\begin{aligned}\sigma_1=&E_1\left[\frac{S_{To}-P}{S_o}-\alpha(T_1-T_o)\right]\\&+E_1\left(\frac{S_1}{S_o}+\frac{h}{2}\right)\frac{S_1(S_{To}-P)-S_oS_{T1}}{S_oS_2-S_1^2}\left[\frac{1-\cos(kL)}{\sin(kL)}\sin(kx)+\cos(kx)\right]\end{aligned} \tag{20}$$

At the upper flange:

$$\begin{aligned}\sigma_2=&E_2\left[\frac{S_{To}-P}{S_o}-\alpha(T_2-T_o)\right]\\&+E_2\left(\frac{S_1}{S_o}-\frac{h}{2}\right)\frac{S_1(S_{To}-P)-S_oS_{T1}}{S_oS_2-S_1^2}\left[\frac{1-\cos(kL)}{\sin(kL)}\sin(kx)+\cos(kx)\right]\end{aligned} \tag{21}$$

In the web:

$$\sigma_3 = \left(\frac{E_1+E_2}{2} + \frac{y(E_2-E_1)}{h_w}\right)\left[\frac{S_{To}-P}{S_o} - \alpha\left(\frac{T_1+T_2}{2} + \frac{y(T_2-T_1)}{h_w} - T_o\right)\right] + \left(\frac{E_1+E_2}{2} + \frac{y(E_2-E_1)}{h_w}\right)\left(\frac{S_1}{S_o} - y\right)\frac{S_1(S_{To}-P) - S_o S_{T1}}{S_o S_2 - S_1^2}\left[\frac{1-\cos(kL)}{\sin(kL)}\sin(kx) + \cos(kx)\right] \tag{22}$$

Note that the membrane strain and curvature are expressed as functions of x-coordinate because of the beam-column effect. Therefore the pre-buckling stress varies not only with y- but also with x-coordinates. However, it can be seen from Eqs. (20)-(22) that, for a given cross section, the pre-buckling stress is constant in each of the two flanges, whereas it varies parabolically in the web. The variation of pre-buckling stresses along x-axis is largely dependent on the value of k, which can be expressed as $k = \frac{\pi}{L}\sqrt{\frac{P}{P_{\max}}}$. If $k \to 0$, then $\left[\frac{1-\cos(kL)}{\sin(kL)}\sin(kx) + \cos(kx)\right] \to 1$. This indicates that, if P is much smaller than P_{max}, then the variation of pre-buckling stresses with x-axis can be ignored. However, if P is close to P_{max}, the variation of pre-buckling stresses with x-axis becomes infinite.

3. Torsional and flexural-torsional buckling analysis

The aforementioned buckling is presented under the assumption that the column will buckle in the plane of principal axis without accompanying rotation of the cross section. This assumption appears reasonable for the doubly symmetric cross section but becomes doubtful if cross-sections have only one axis of symmetry or none at all. Geometrically, I-section columns are doubly symmetric about the two principal axes. However, when the temperature is not uniformly distributed in their cross sections their mechanical properties are not symmetric. Experience has revealed that columns having open section with only one or no axis of symmetry show a tendency to bend and twist simultaneously under axial compression.[1-3] The ominous nature of this type of failure lies in the fact that the actual critical load of such columns may be less than that predicted by the buckling load shown in the above section due to their small torsional rigidities.

Since the I-section discussed here is symmetric about y-axis, but not about z-axis because of the temperature variation along the y-axis. For the convenience of analysis, two parallel reference axes are used. One is the z-axis of passing through geometric centroid o and the other is the z_s-axis of passing through shear centre s (see Fig. 3). Let y_s be the distance between the centroid and shear centre, which can be expressed as follows,

$$y_s = \frac{h(E_1 - E_2)}{2(E_2 + E_1)} \tag{23}$$

When the column has a torsional or a flexural-torsional buckling, the strain energy stored in the column in the adjacent equilibrium configuration can be calculated based on the sum of strain energies of the upper flange, lower flange and web. According to the displacement components defined for a buckled column as shown in Fig. 3, the following strain energy expressions can be obtained:

For the lower flange

$$U_1 = \frac{E_1 I_{y1}}{2}\int_o^L \left[\frac{d^2 w_s}{dx^2} - \left(\frac{h}{2} - y_s\right)\frac{d^2\phi}{dx^2}\right]^2 dx + \frac{G_1 J_1}{2}\int_o^L \left(\frac{d\phi}{dx}\right)^2 dx \tag{24}$$

For the upper flange

$$U_2 = \frac{E_2 I_{y2}}{2}\int_o^L \left[\frac{d^2 w_s}{dx^2} + \left(\frac{h}{2} + y_s\right)\frac{d^2\phi}{dx^2}\right]^2 dx + \frac{G_2 J_2}{2}\int_o^L \left(\frac{d\phi}{dx}\right)^2 dx \tag{25}$$

For the web

$$U_3 = \frac{(E_1 + E_2) I_{y3}}{4}\int_o^L \left(\frac{d^2 w_s}{dx^2} + (y_s - y_{ws})\frac{d^2\phi}{dx^2}\right)^2 dx + \frac{(G_1 + G_2) J_3}{4}\int_o^L \left(\frac{d\phi}{dx}\right)^2 dx \tag{26}$$

where w_s is the lateral translation displacement of the section, ϕ the angle of twist of the section about the shear centre, $I_{y1} = I_{y2} = \frac{b_f^3 t_f}{12}$ the second moment of area of the flange about y-axis, $I_{y3} = \frac{t_w^3 h_w}{12}$ the second moment of area of the web about y-axis, $J_1 = J_2 = \frac{t_f^3 b_f}{3}$ the torsional constant of the flange, $J_3 = \frac{t_w^3 h_w}{3}$ the torsional constant of the web, $G_1 = \frac{E_1}{2(1+\nu)}$ and $G_2 = \frac{E_2}{2(1+\nu)}$ are the shear moduli of the lower and upper flanges; respectively, y_{ws} the distance between the centroid and shear centre of the web element, which can be expressed as

$$y_{ws} = \frac{h_w (E_1 - E_2)}{6(E_2 + E_1)} \tag{27}$$

There is no warping strain energy in the strain energy expression of each component. This is because the flanges and web are treated independently here and each has a rectangular cross-section for which the warping constant is very small and can be ignored. However, this does not mean that there is no warping for the whole section. The warping strain energy of the section is represented by the bending strain energy related to the angle-of-twist terms in the strain energy expression of each component. This is a novelty and it avoids the difficulty in dealing with the warping of the I-section when the mechanical properties are not uniform in the section.

The loss of potential energy of external loads during buckling is equal to the product of the loads and the distances they travel as the column takes an adjacent equilibrium position, which are expressed as follows:[9,14,15]

For the lower flange

$$W_1 = \frac{1}{2}\int_o^L \sigma_1 b_f t_f \left\{\left[\frac{dw_s}{dx} - \left(\frac{h}{2} - y_s\right)\frac{d\phi}{dx}\right]^2 + r_f^2\left(\frac{d\phi}{dx}\right)^2\right\} dx \tag{28}$$

For the upper flange

$$W_2 = \frac{1}{2}\int_o^L \sigma_2 b_f t_f \left\{\left[\frac{dw_s}{dx} + \left(\frac{h}{2} + y_s\right)\frac{d\phi}{dx}\right]^2 + r_f^2\left(\frac{d\phi}{dx}\right)^2\right\} dx \tag{29}$$

For the web

$$W_3 = \frac{1}{2}\int_o^L \int_{-h_w/2}^{h_w/2} \sigma_3 t_w \left[\frac{dw_s}{dx} + (y_s + y)\frac{d\phi}{dx}\right]^2 dxdy \tag{30}$$

where $r_f = \frac{1}{2}\sqrt{\frac{b_f^2 + t_f^2}{3}}$ is the polar radius of gyration of the flange section with respect to its own centroid. The variation of the pre-buckling stresses can have significant effect on the buckling behaviour[16-20] and thus it is important to split the section into components for which the potential energy of pre-buckling stresses can be calculated directly.

For the column with simply supported ends, the torsional and/or flexural-torsional buckling displacements $w_s(x)$ and $\phi(x)$, that satisfy the simply supported boundary conditions at $x = 0$ and $x = L$, can be assumed to be

$$w_s(x) = \frac{(C_1 - y_s C_2)L}{\pi}\sin\frac{\pi x}{L} \tag{31}$$

$$\phi(x) = \frac{C_2 L}{\pi}\sin\frac{\pi x}{L} \tag{32}$$

where C_1 and C_2 are constants to be determined. Eqs. (31) and (32) are substituted into Eqs. (24)-(26) to furnish

$$U_1 = \frac{E_1 I_{y1} L}{4}\left(C_1 - \frac{hC_2}{2}\right)^2\left(\frac{\pi}{L}\right)^2 + \frac{G_1 J_1 L}{4}C_2^2 \tag{33}$$

$$U_2 = \frac{E_2 I_{y2} L}{4}\left(C_1 + \frac{hC_2}{2}\right)^2\left(\frac{\pi}{L}\right)^2 + \frac{G_2 J_2 L}{4}C_2^2 \tag{34}$$

$$U_3 = \frac{(E_1 + E_2) I_{y3} L}{8}\left(C_1 - y_{ws} C_2\right)^2\left(\frac{\pi}{L}\right)^2 + \frac{(G_1 + G_2) J_3 L}{8}C_2^2 \tag{35}$$

Eqs. (20)-(22), (30) and (31) are substituted into (27)-(29) to furnish

$$\begin{aligned} W_1 = {} & \frac{b_f t_f L E_1}{4}\left[\frac{S_{To} - P}{S_o} - \alpha(T_1 - T_o)\right]\left[\left(C_1 - \frac{hC_2}{2}\right)^2 + r_f^2 C_2^2\right] + \frac{b_f t_f L E_1}{2\lambda\pi} \times \\ & \left(\frac{S_1}{S_o} + \frac{h}{2}\right)\left(\frac{S_1(S_{To} - P) - S_o S_{T1}}{S_o S_2 - S_1^2}\right)\left[\left(C_1 - \frac{hC_2}{2}\right)^2 + r_f^2 C_2^2\right]\left(\frac{4 - 2\lambda^2}{4 - \lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right) \end{aligned} \tag{36}$$

$$\begin{aligned} W_2 = {} & \frac{b_f t_f L E_2}{4}\left[\frac{S_{To} - P}{S_o} - \alpha(T_2 - T_o)\right]\left[\left(C_1 + \frac{hC_2}{2}\right)^2 + r_f^2 C_2^2\right] + \frac{b_f t_f L E_2}{2\lambda\pi} \times \\ & \left(\frac{S_1}{S_o} - \frac{h}{2}\right)\left(\frac{S_1(S_{To} - P) - S_o S_{T1}}{S_o S_2 - S_1^2}\right)\left[\left(C_1 + \frac{hC_2}{2}\right)^2 + r_f^2 C_2^2\right]\left(\frac{4 - 2\lambda^2}{4 - \lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right) \end{aligned} \tag{37}$$

$$W_3 = \frac{h_w t_w L}{4}\left[\frac{S_{To}-P}{S_o} - \alpha\left(\frac{T_1+T_2}{2} - T_o\right)\right]\left[\frac{E_1+E_2}{2}\left(C_1^2 + \frac{h_w^2}{12}C_2^2\right) + \frac{h_w(E_2-E_1)}{6}C_1C_2\right]$$
$$- \frac{h_w t_w L}{4}\alpha(T_2-T_1)\left[\frac{(E_1+E_2)C_1C_2h_w}{12} + \frac{(E_2-E_1)}{4}\left(\frac{C_1^2}{3} + \frac{(C_2h_w)^2}{20}\right)\right]$$
$$+ \frac{h_w t_w L}{2\lambda\pi}\left(\frac{S_1(S_{To}-P) - S_oS_{T1}}{S_oS_2 - S_1^2}\right)\left\{\frac{S_1}{S_o}\left[\frac{E_1+E_2}{2}\left(C_1^2 + \frac{h_w^2}{12}C_2^2\right) + \frac{(E_2-E_1)}{6}C_1C_2h_w\right]\right.$$
$$\left. - h_w\left[\frac{(E_1+E_2)C_1C_2h_w}{12} + \frac{(E_2-E_1)}{4}\left(\frac{C_1^2}{3} + \frac{(C_2h_w)^2}{20}\right)\right]\right\}\left(\frac{4-2\lambda^2}{4-\lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right) \quad (38)$$

where $\lambda = \sqrt{(P/P_{max})}$ is the dimensionless load. From the principle of minimum potential energy, when the torsional or flexural-torsional buckling occurs, the following equation holds,

$$F(P) = \frac{\partial^2\Pi}{\partial C_1^2}\frac{\partial^2\Pi}{\partial C_2^2} - \left(\frac{\partial^2\Pi}{\partial C_1\partial C_2}\right)^2 = 0 \quad (39)$$

where $\Pi = (U_1 + U_2 + U_3) + (W_1 + W_2 + W_3)$. The substitution of Eqs. (33)-(38) into Eq. (39) yields

$$F(P) = (A_{11} + B_{11})(A_{22} + B_{22}) - (A_{12} + B_{12})^2 = 0 \quad (40)$$

in which

$$A_{11} = \left(\frac{\partial^2 U_1}{\partial C_1^2} + \frac{\partial^2 U_2}{\partial C_1^2} + \frac{\partial^2 U_3}{\partial C_1^2}\right) = \frac{L}{2}\left(\frac{\pi}{L}\right)^2\left(E_1I_{y1} + E_2I_{y2} + \frac{(E_1+E_2)I_{y3}}{2}\right)$$

$$B_{11} = \frac{\partial^2 W_1}{\partial C_1^2} + \frac{\partial^2 W_2}{\partial C_1^2} + \frac{\partial^2 W_3}{\partial C_1^2}$$
$$= \frac{b_f t_f L}{2}\left\{\left(\frac{S_{To}-P}{S_o}\right)(E_1+E_2) - \alpha\left[(T_1-T_o)E_1 + (T_2-T_o)E_2\right]\right\} +$$
$$\frac{h_w t_w L}{4}\left\{\left(\frac{S_{To}-P}{S_o}\right)(E_1+E_2) - \alpha\left[\frac{(2T_1+T_2-3T_0)E_1 + (T_1+2T_2-3T_0)E_2}{3}\right]\right\} +$$
$$\frac{b_f t_f L}{\lambda\pi}\left(\frac{S_1(S_{To}-P) - S_oS_{T1}}{S_oS_2 - S_1^2}\right)\left(\frac{4-2\lambda^2}{4-\lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right)\left[E_1\left(\frac{S_1}{S_o} + \frac{h}{2}\right) + E_2\left(\frac{S_1}{S_o} - \frac{h}{2}\right)\right] +$$
$$\frac{h_w t_w L}{2\lambda\pi}\left(\frac{S_1(S_{To}-P) - S_oS_{T1}}{S_oS_2 - S_1^2}\right)\left(\frac{4-2\lambda^2}{4-\lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right)\left[\frac{S_1}{S_o}(E_1+E_2) - h_w\left(\frac{E_2-E_1}{6}\right)\right]$$

$$A_{12} = \frac{\partial^2 U_1}{\partial C_1\partial C_2} + \frac{\partial^2 U_2}{\partial C_1\partial C_2} + \frac{\partial^2 U_3}{\partial C_1\partial C_2} = \frac{L}{4}\left(\frac{\pi}{L}\right)^2\left[(E_2I_{y2} - E_1I_{y1})h - (E_1+E_2)I_{y3}y_{ws}\right]$$

$$B_{12}=\frac{\partial^2 W_1}{\partial C_1\partial C_2}+\frac{\partial^2 W_2}{\partial C_1\partial C_2}+\frac{\partial^2 W_3}{\partial C_1\partial C_2}$$

$$=\frac{b_f t_f Lh}{4}\left\{\left(\frac{S_{To}-P}{S_o}\right)(E_2-E_1)-\alpha\left[(T_2-T_0)E_2-(T_1-T_0)E_1\right]\right\}+$$

$$\frac{h_w^2 t_w L}{4}\left\{\left(\frac{S_{To}-P}{S_o}\right)\left(\frac{E_2-E_1}{6}\right)-\alpha\left[\frac{(T_2-T_0)E_2-(T_1-T_0)E_1}{6}\right]\right\}+$$

$$\frac{b_f t_f Lh}{2\lambda\pi}\times\left(\frac{S_1(S_{To}-P)-S_oS_{T1}}{S_oS_2-S_1^2}\right)\left(\frac{4-2\lambda^2}{4-\lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right)\left[E_2\left(\frac{S_1}{S_o}-\frac{h}{2}\right)-E_1\left(\frac{S_1}{S_o}+\frac{h}{2}\right)\right]+$$

$$\frac{h_w^2 t_w L}{2\lambda\pi}\left(\frac{S_1(S_{To}-P)-S_oS_{T1}}{S_oS_2-S_1^2}\right)\left(\frac{4-2\lambda^2}{4-\lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right)\left[\frac{S_1}{S_o}\left(\frac{E_2-E_1}{6}\right)-h_w\left(\frac{E_1+E_2}{12}\right)\right]$$

$$A_{22}=\frac{\partial^2 U_1}{\partial C_2^2}+\frac{\partial^2 U_2}{\partial C_2^2}+\frac{\partial^2 U_3}{\partial C_2^2}$$

$$=\frac{L}{8}\left(\frac{\pi}{L}\right)^2\left[(E_1I_{y1}+E_2I_{y1})h^2+2(E_1+E_2)I_{y3}y_{ws}^2\right]+\frac{L}{4}\left[2(G_1J_1+G_2J_2)+(G_1+G_2)J_3\right]$$

$$B_{22}=\frac{\partial^2 W_1}{\partial C_2^2}+\frac{\partial^2 W_2}{\partial C_2^2}+\frac{\partial^2 W_3}{\partial C_2^2}$$

$$=\frac{b_f t_f L}{2}\left(\frac{h^2}{4}+r_f^2\right)\left\{\left(\frac{S_{To}-P}{S_o}\right)(E_1+E_2)-\alpha\left[(T_1-T_o)E_1+(T_2-T_o)E_2\right]\right\}+$$

$$\frac{h_w^3 t_w L}{4}\left[\left(\frac{S_{To}-P}{S_o}\right)\left(\frac{E_1+E_2}{12}\right)-\alpha\left(\frac{(4T_1+T_2-5T_0)E_1+(T_1+4T_2-5T_0)E_2}{60}\right)\right]+$$

$$\frac{b_f t_f L}{\lambda\pi}\left(\frac{S_1(S_{To}-P)-S_oS_{T1}}{S_oS_2-S_1^2}\right)\left(\frac{4-2\lambda^2}{4-\lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right)\left(\frac{h^2}{4}+r_f^2\right)\left[E_1\left(\frac{S_1}{S_o}+\frac{h}{2}\right)+E_2\left(\frac{S_1}{S_o}-\frac{h}{2}\right)\right]+$$

$$\frac{h_w^3 t_w L}{2\lambda\pi}\left(\frac{S_1(S_{To}-P)-S_oS_{T1}}{S_oS_2-S_1^2}\right)\left(\frac{4-2\lambda^2}{4-\lambda^2}\right)\tan\left(\frac{\lambda\pi}{2}\right)\left[\frac{S_1}{S_o}\left(\frac{E_1+E_2}{12}\right)-h_w\left(\frac{E_2-E_1}{40}\right)\right]$$

Eq. (40) is a nonlinear algebraic equation about the axial compressive load P. For a given column with given distributions of temperature and mechanical properties, one can find the roots, P, of Eq. (40). The lowest root represents the critical buckling load.

It should be pointed out here that, due to the bending effect involved in the pre-buckling analysis, the term related to the loss of potential energy of external loads is not linearly proportional to the axial compressive load P. Thus, the classical method of buckling analysis which is to find the smallest eigenvalue in the eigen-equation cannot be used directly here.

4. Numerical examples

A commercially available I-section column with section dimensions, h_w = 138.8 mm, b_f = 152.2 mm, t_f = 6.8 mm, t_w = 5.8 mm, is considered herein for numerical illustration. The reduction of Young's modulus due to elevated temperatures is given in Table 1, which is obtained from steel design manual. Four different temperature distributions defined in Table

2 are discussed in this numerical example. The thermal expansion coefficient and ambient temperature are taken as $\alpha = 1.4\text{x}10^{-5}$ and T_o = 20 °C in all cases.

Figure 4 shows the pre-buckling stress distributions on the end cross-section of the column in four different temperature distribution cases, in which the axial compressive load is taken as $P = \sigma_y(2A_f + A_w)$ where σ_y = 275 MPa is the yield strength of steel. It can be seen from the figure that the stresses in the two flanges are very close although they have different temperatures. The variation of stresses in the web depends on the temperature difference between T_2 and T_1. The larger difference between T_2 and T_1 leads to a larger variation in web stress. The highest stress is found at near the geometric centre of the web element. Figure 5 shows the pre-buckling stress distributions on the middle cross-section of the column in four different temperature distribution cases. Owing to the bending effect, the stress distributions in the middle section of the column are quite different from those in the end section of the column. In the former the stress is approximately symmetric about z-axis, indicating that the column is nearly in a pure compression; while in the latter the stress in the flange of low temperature is much greater than that in the flange of high temperature, indicating that the column is subjected to not only compression but also bending.

Under uniform temperature an I-section column will always buckle in a flexural mode. Whether the flexural mode is bending about y-axis or z-axis depends on which flexural rigidity is weaker. When the temperature is non-uniform, however, the I-section column will buckle in the flexural-torsional mode owing to the non-uniform mechanical properties induced by the non-uniform temperature. To demonstrate this, Fig. 6 shows the critical loads of the column in four different temperature distribution cases, in which P_{max} and P_{min} represent the critical loads of the flexural buckling about z-axis defined by Eq. (17) and y-axis defined by Eq. (41), P_{cr1} and P_{cr2} represent the critical loads of the flexural-torsional buckling calculated from Eq. (40) with and without taking into account the bending effect in pre-buckling stress (i.e. for P_{cr2} the pre-buckling stress is taken at the end section of the column and its variation with x-axis is ignored),

$$P_{\min} = \frac{\pi^2}{L^2}\left(E_1 I_{y1} + E_2 I_{y1} + \frac{(E_1 + E_2) I_{y3}}{2} \right) \tag{41}$$

For convenience, all these four critical loads are normalised using P_{maxo}, which is the value of P_{max} at ambient temperature, i.e. when $T_2 = T_1$ = 20 °C. It can be seen from Fig. 6 that, among the three critical loads, P_{cr1} is always smallest. This demonstrates that when there is a temperature difference between the two flanges, the column will buckle in the flexural-torsional mode. It is worth noting that if the bending effect is ignored in the calculation of pre-buckling stresses, the critical load of flexural-torsional buckling coincides with that of flexural buckling about the y-axis unless there is a huge temperature difference between the two flanges. This highlights the importance of considering bending effect in pre-buckling analysis.

5. Conclusions

This paper has presented an analytical study on the buckling analysis of axially loaded columns of thin-walled open section with non-uniform sectional properties. Critical loads related to flexural, torsional and flexural-torsional buckling of I-section columns subjected to axial compressive loads applied at geometric centroid, under linearly varied non-uniform temperature distribution scenarios have been derived. The influences of thermal strain, non-uniform distribution of pre-buckling stresses, and variation of pre-buckling stresses along the longitudinal axis of the column on critical buckling loads have been discussed using numerical examples. From the obtained results the following conclusions can be drawn:

- Non-uniform distribution of temperature can lead to non-uniform distribution of mechanical properties. The non-uniform of both the temperature and mechanical properties can significantly affect the pre-buckling stress in axially loaded columns.
- When the bending effect is taken into account in the pre-buckling stress analysis, the buckling analysis becomes a nonlinear problem, which cannot be treated using the classical eigenvalue analysis method.
- The doubly symmetric I-section column subjected to an axial compressive load applied at its geometric centroid will buckle in the flexural-torsional mode when the temperature distribution is not uniform. The critical load of the flexural-torsional buckling is smaller than the critical load of the flexural buckling about either principal axis.
- The pre-buckling bending has significant influence not only on the value of the critical load but also on the mode of flexural-torsional buckling.
- Although the present study focuses on the flexural, torsional, and flexural-torsional buckling of columns caused due to linearly varied non-uniform temperature along the axis parallel to web line, the concept and the method itself can be applied to the general columns with non-uniform mechanical properties when subjected to axial compression.

Finally, it should be pointed out that the study presented in the paper deals with only the elastic buckling and there is no material yield. For some cases, however, material yield may occur prior to the buckling because of the effect of high temperature. In this case, a full nonlinear analysis is needed in order to calculate the failure load.

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Table 1. Reduction of Young's modulus at different temperatures

T (°C)	20	100	200	300	400	500	600	700	800
E (GPa)	210	210	210	168	147	126	65.1	27.3	18.9

Table 2. Parametric values employed in different cases

Case	T_1 (°C)	T_2 (°C)	E_1 (GPa)	E_2 (GPa)
1	200	300	210	168
2	200	400	210	147
3	200	500	210	126
4	200	600	210	65.1

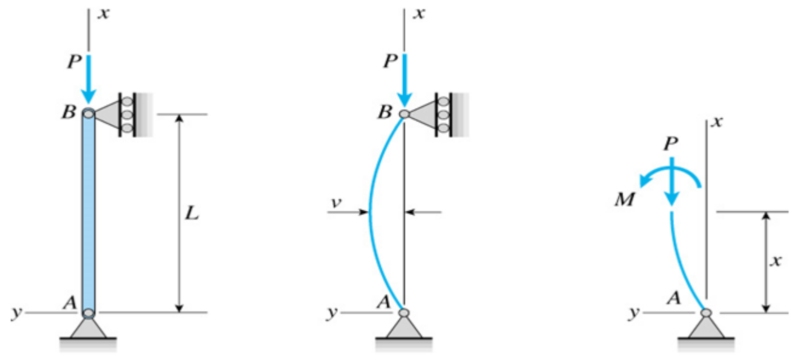

Figure 1. (a) Column with axial compression. (b) Deformed shape.
(c) Definition of internal forces.

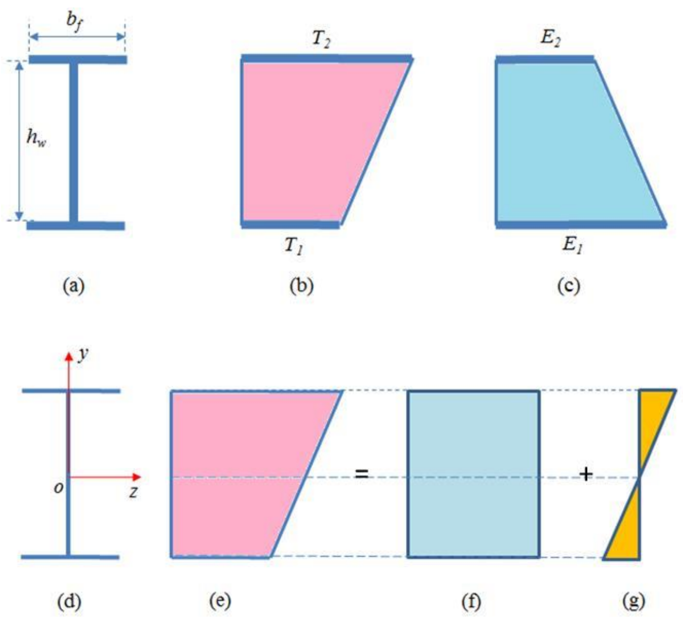

Figure 2. (a) Cross-section. (b) Temperature distribution. (c) Young's modulus distribution. (d) Coordinate system. (e) Strain distribution. (f) Membrane strain. (g) Bending strain.

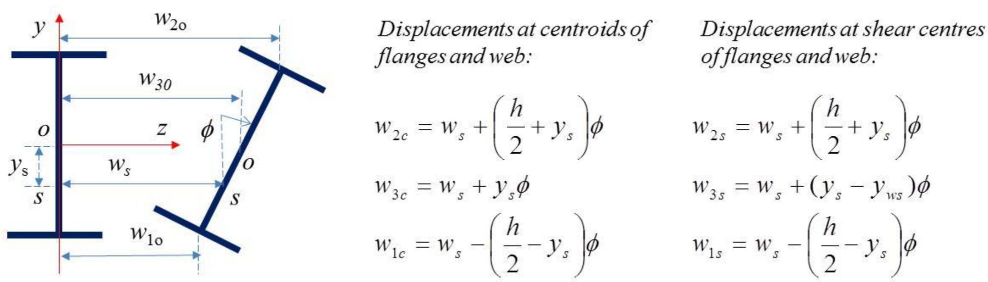

Figure 3. Definition of displacements in flexural-torsional buckling (points o and s represent the geometric centroid and shear centre of the cross-section).

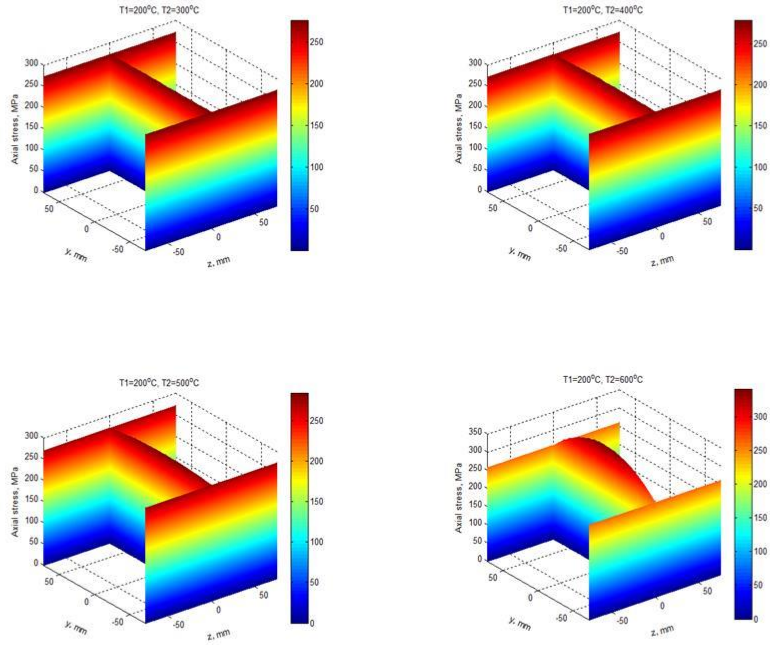

Figure 4. Axial stress distribution on the end section of column (section dimensions: h_w = 138.8 mm, b_f = 152.2 mm, t_f = 6.8 mm, t_w = 5.8 mm, L = 3000 mm). (a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4.

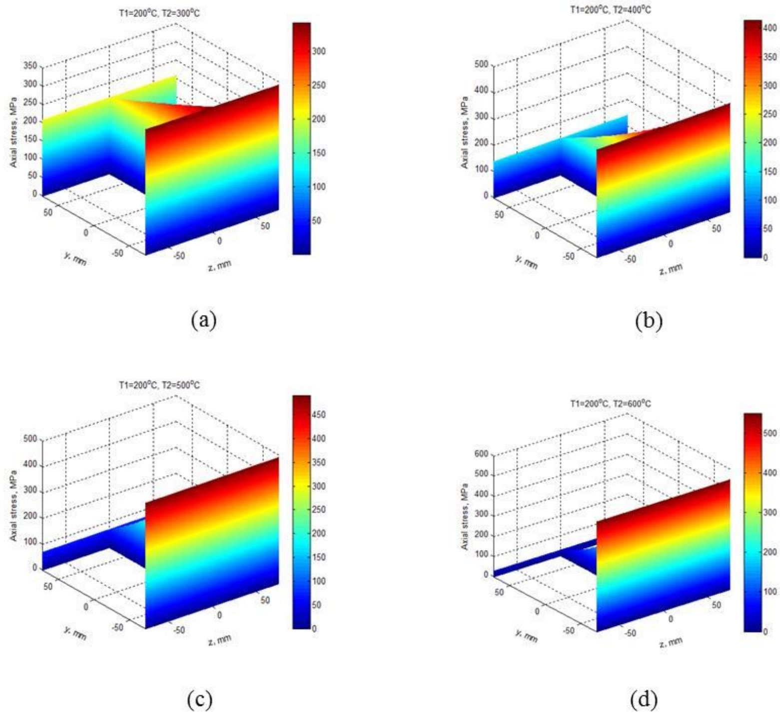

Figure 5. Axial stress distribution on the middle section of column (section dimensions: h_w = 138.8 mm, b_f = 152.2 mm, t_f = 6.8 mm, t_w = 5.8 mm, L = 3000 mm). (a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4.

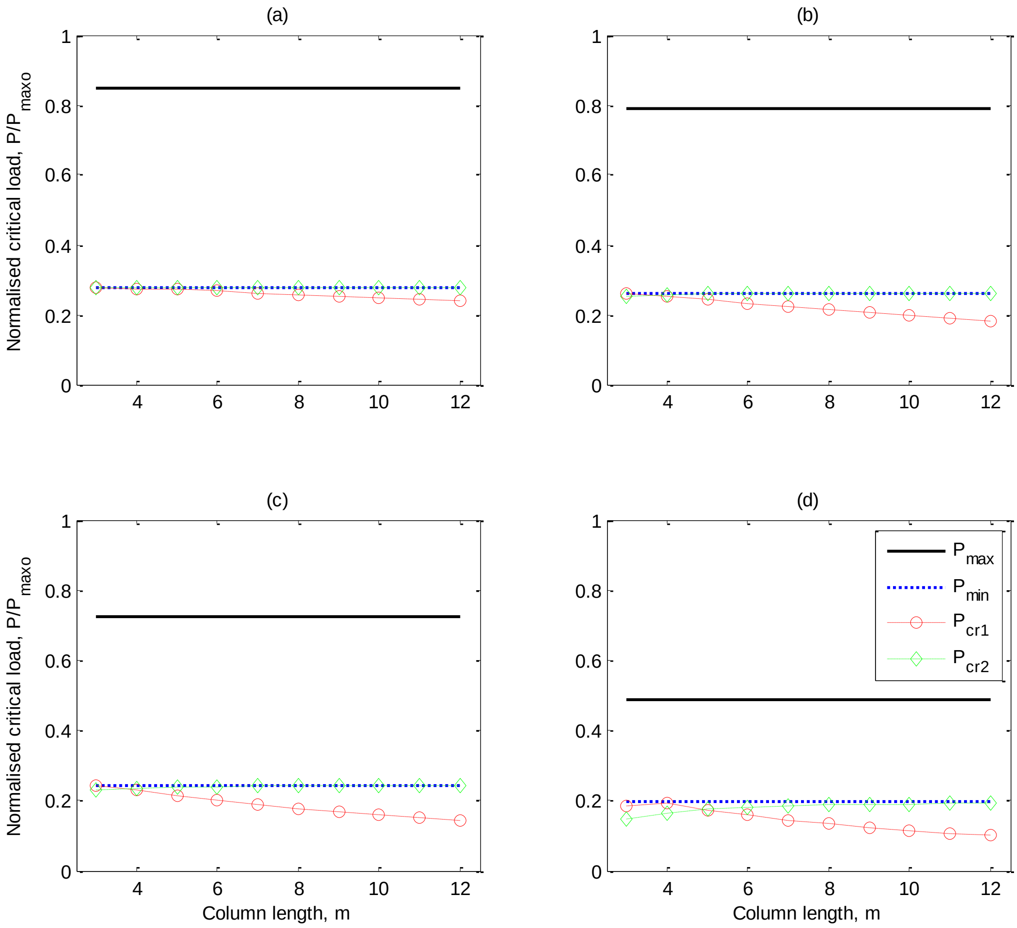

Figure 6. Variation of critical load with column length (section dimensions: h_w = 138.8 mm, b_f = 152.2 mm, t_f = 6.8 mm, t_w = 5.8 mm). (a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4.