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New state estimator for decentralized event-triggered consensus for multi-agent systems

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Abstract: This paper extends recent work of Garcia *et al.* on event-triggered communication to reach consensus in multi-agent systems. It proposes an improved agent state estimator as well as an estimator of the state estimation uncertainty to trigger communications. Convergence to consensus is studied. Simulations show the effectiveness of the proposed estimators in presence of state perturbations.

Keywords: Multi-agent systems, event-triggered, consensus, undirected communication graph.

1. INTRODUCTION

Consensus is an important problem in cooperative control, see Olfati-Saber et al. [2007], Wei [2008], Garcia et al. [2014c,b]. In this problem, several agents have to be synchronized to the same state. When the control is distributed, consensus usually requires significant exchange of information between neighbouring agents so that each agent can properly evaluate its control law. This communication may be either permanent, as in Olfati-Saber et al. [2007], Wei [2008], or may take place at discrete time instants, which is much more practical. In the latter case, communications may occur periodically, as in Garcia et al. [2014b], may be intermittent, as in Wen et al. [2012a,b, 2013], or may be event-triggered as in Dimarogonas and Johansson [2009], Jiangping et al. [2011], Dimarogonas et al. [2012], Fan et al. [2013], Garcia et al. [2014c], Zhang et al. [2015].

Event-triggered communication is the most promising approach to save communication energy, while allowing a consensus to be reached. To reduce the number of communications in a decentralized case, each agent estimates the state of its neighours to evaluate its control law. Additionally, each agent also estimates its own state with the information available to its neighbours. The error between this estimate and its actual state is then used to trigger a communication when it reaches some threshold. In Dimarogonas et al. [2012], the agent dynamic is a single integrator and the considered threshold decreases with time while reaching the consensus. This implies an increase of the frequency of communications. In Seyboth et al. [2013], the dynamic is a double integrator and the triggering condition depends on a state-independent and exponentially decreasing threshold. The communication frequency reduces compared to Dimarogonas et al. [2012] but still increases close to consensus. General linear dynamics are considered in Zhu et al. [2014], Garcia et al. [2014c,a]. State-dependent thresholds are then considered to ensure some convergence property for the system. These previous approaches were developed for noise-free dynamics and prove sensitive to perturbations. This issue has been partly addressed by Hu et al. [2014] and Cheng et al. [2014] who proposed an event-triggered method to mitigate the impact of perturbations in the case of dynamics described by simple integrators.

This paper addresses the problem of decentralized event-triggered communications for consensus of a multi-agent system with both general linear dynamics and state perturbations. This work extends results presented in Garcia et al. [2014c,a] by introducing a new estimator to take into account the control input of the agents. With this approach, estimates of the states of all the agents (not only neighboring ones) are required to evaluate all control laws. More estimates are performed, but this reduces the communication frequency. A convergence analysis is achieved while considering state perturbations composed of two components: one common to all agents, and one agent-specific.

After introducing some notations in Section 2, the problem statement is presented in Section 3. The new estimator is described in Section 4, along with a communication protocol. A second estimator to obtain a decentralized event-triggered strategy is presented in Section 5. Section 6 compares the performance of the proposed approach to state-of-the-art results from Garcia et al. [2014c,a].

2. NOTATIONS AND HYPOTHESES

Classical notations from Cortes and Martinez [2009] are first briefly recalled. Consider a network of N agents which topology is described by a fixed and undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. $\mathcal{N}$ is the set of agents and $\mathcal{E}$ describes the communication links between pair of agents. The set of neighbours of an Agent i is $\mathcal{N}_i = \{j \in \mathcal{N} \,|\, (i,j) \in \mathcal{E}, i \neq j\}$. N_i is the cardinal number of $\mathcal{N}_i$. Let $1_N = [1, 1, ..., 1]^T \in$

$\mathbb{R}^{N\times 1}$ be the all-one vector and $\mathbf{I}_N \in \mathbb{R}^{N\times N}$ be the identity matrix of size N. Since $\mathcal{G}$ is undirected, its Laplacian matrix L is symmetric. L also satisfies $L1_N = 0$ and has only one null eigenvalue $\lambda_1(L)$ and all its non-zeros eigenvalues $\lambda_2(L) \leq \lambda_3(L) \leq \ldots \leq \lambda_N(L)$ are strictly positive.

The Kronecker product is denoted by $\otimes$. For a matrix M, $\lambda_{\min}(M)$, $\lambda_{\min>0}(M)$, and $\lambda_{\max}(M)$ are respectively the smallest, the smallest strictly positive, and the largest eigenvalue of M. For a given vector x and a symmetric matrix M, $\|x\|_M = x^T M x$.

3. PROBLEM STATEMENT

Assume that the dynamic equations of Agent i are

$$\dot{x}_i(t) = \mathbf{A}x_i(t) + \mathbf{B}u_i(t) + d_i(t) \tag{1}$$

$$u_i(t) = c_1\mathbf{F}\sum_{j\in\mathcal{N}_i}\left(y_i^i(t) - y_j^i(t)\right), \tag{2}$$

where $x_i \in \mathbb{R}^n$ is the state of Agent i, $u_i \in \mathbb{R}^m$ is its control input evaluated using $y_j^i \in \mathbb{R}^n$, the estimate of x_j performed by Agent i as described in Section 4, and $d_i(t)$ is some state perturbation. $c_1 = c + c_2$ with $c = 1/\lambda_2(L)$ and $c_2 \geq 0$ is a design parameter. $\mathbf{F} = -\mathbf{B}^T\mathbf{P}$ where $\mathbf{P}$ is a symmetric positive semi-definite matrix, solution of the Riccati equation

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} - 2\mathbf{PBB}^T\mathbf{P} + 2\alpha\mathbf{P} < 0, \tag{3}$$

with $\alpha > 0$.

The additive perturbation is assumed to be such that

$$d_i(t) = m(t) + s_i(t), \tag{4}$$

where $m(t) \in \mathbb{R}^n$ is a bounded time-varying perturbation with $\|m(t)\| \leq M_{\max}$, identical for all agents and $s_i(t) \in \mathbb{R}^n$ is a bounded agent-specific perturbation, with for all $i = 1, \ldots, N$ $\|s_i(t)\| \leq S_{\max}$ $\forall t$. The vector of all state perturbations is denoted by

$$d(t) = 1_N \otimes m(t) + \left[s_1(t)^T \ldots s_N(t)^T\right]^T. \tag{5}$$

The problem considered consists in designing a control scheme to reach a bounded consensus, while limiting the communications between agents. For that purpose, communication time instants are chosen locally by Agent i using an event-triggered approach involving the state estimation error $e_i^i = y_i^j - x_i$, as detailed in Section 5.

In this paper, we suppose as in Garcia et al. [2014c] that there is no communication delay, and agents know perfectly their own state.

4. AGENT STATE ESTIMATION AND COMMUNICATION PROTOCOL

4.1 Agent state estimation

Define $t_{j,k}^i$ as the time at which the k-th message sent by Agent j has been received by Agent i. The time instant at which the k-th message has been sent by Agent j is denoted $t_{j,k}$. The time of reception by Agent i of the ℓ-th message is t_ℓ^i, whatever the sending agent.

In Garcia et al. [2014c], the estimate $y_j^i(t)$ of x_j performed by Agent i is

$$y_j^i\left(t_{j,k}^i\right) = x_j\left(t_{j,k}^i\right), \tag{6}$$

$$\dot{y}_j^i(t) = \mathbf{A}y_j^i(t), \quad t_{j,k}^i \leq t < t_{j,k+1}^i. \tag{7}$$

Let $y^i = \left[y_1^{iT}\; y_2^{iT}\; \ldots\; y_N^{iT}\right]^T \in \mathbb{R}^{Nn}$ be the vector gathering the estimates of the state of all agents performed by Agent i. The vector gathering the estimates of their own state by each agent is $y = \left[y_1^{1T}\; y_2^{2T}\; \ldots\; y_N^{NT}\right]^T \in \mathbb{R}^{Nn}$.

The first state estimator proposed here takes into account the control input of the agents and the way it is evaluated

$$\dot{y}_j^i(t) = \mathbf{A}y_j^i(t) + \mathbf{B}\tilde{u}_j^i(t), \quad t_{j,k}^i \leq t < t_{j,k+1}^i \tag{8}$$

$$\tilde{u}_j^i(t) = c_1\mathbf{F}\sum_{p\in\mathcal{N}_j}\left(y_j^i(t) - y_p^i(t)\right) \tag{9}$$

$$y_j^i\left(t_{j,k}^i\right) = x_j\left(t_{j,k}^i\right). \tag{10}$$

Considering all agents, (8)-(10) can be rewritten as

$$\dot{y}^i(t) = \mathbf{A}_c y^i(t), \quad t_{j,k}^i \leq t < t_{j,k+1}^i \tag{11}$$

$$y_j^i\left(t_{j,k}^i\right) = x_j\left(t_{j,k}^i\right), \tag{12}$$

where $\mathbf{A}_c = \overline{\mathbf{A}} + \overline{\mathbf{B}}_1$, $\overline{\mathbf{A}} = \mathbf{I}_N \otimes \mathbf{A}$, and $\overline{\mathbf{B}}_1 = c_1 L \otimes (\mathbf{BF})$. Each agent has then to estimate the states of all agents of the network to determine the control inputs applied by all other agents.

4.2 Communication protocol

As in Garcia et al. [2014c], the message broadcast by Agent i at $t_{i,k}$ contains its state $x_i(t_{i,k})$ and $t_{i,k}$. Agent j, $j \in \mathcal{N}_i$, uses $x_i(t_{i,k})$ to update its estimate y_i^j (6). Nevertheless, this is not possible when $j \notin \mathcal{N}_i$. The following protocol is proposed to address this issue and implement (8)-(10).

Let $\mathcal{T}^i = [t_{1,k_1}^i \ldots t_{N,k_N}^i]^T$ be the vector of reception times of $y_1^1 \ldots y_N^N$ by Agent i. When an agent broadcasts a message, it updates its own estimate y_i^i with x_i, *i.e.*, $y_i^i(t_{i,k}) = x_i(t_{i,k})$ and transmits y^i and $\mathcal{T}^i$ to its neighbours. Then, each neigbhour compares the time instants in $\mathcal{T}^i$ with those of its own $\mathcal{T}^j$. Only the components of y^j such that $t_{i,k} > t_{j,k}$, *i.e.*, corresponding to a more recent time instant, are replaced by those of y^i.

With this communication protocol, Agent j is thus able to update its estimate y_i^j of x_i even if $j \notin \mathcal{N}_i$.

4.3 Estimation v^i of y^i by Agent j

With the previously introduced communication protocol, y_j^i is only known by Agent i and cannot be used by Agent j in its communication triggering condition. To address this issue, each Agent i considers estimates $v^j = \left[v_1^{jT} \ldots v_N^{jT}\right]^T \in \mathbb{R}^{Nn}$ of y^j for all $j \in \mathcal{N}_i \cup \{i\}$, with the constraint that estimate v^i performed by Agent i and Agent j when $j \in \mathcal{N}_i$ have to be identical. As a consequence, it is less frequently updated than y^i and thus it is less accurate.

The dynamics of v^i is expressed as

$$\dot{v}^i(t) = \mathbf{A}_c v^i(t) \tag{13}$$

$$v^i(t_{i,k}) = y^i(t_{i,k}) \tag{14}$$

$$v_j^i\left(t_{j,k}\right)=y_j^j\left(t_{j,k}\right), \quad j \in \mathcal{N}_i \tag{15}$$

Using (14) one updates v^i when Agent i broadcasts some message.

5. DECENTRALIZED EVENT-TRIGGERED

Let $\widehat{L}=L\otimes\mathbf{P}$ and $\overline{L}=\widehat{L}\mathbf{A}_c+\mathbf{A}_c^T\widehat{L}$.

Theorem 1. Assume that $(\mathbf{A},\mathbf{B})$ is controllable and that the communication graph is connected and undirected with a fixed topology described by the Laplacian matrix L. Agents which dynamics is (1) achieve a bounded consensus with

$$\lim_{t\to\infty}\left\|x_i-x_j\right\|^2\leq\frac{N\eta}{\beta\lambda_{\min}\left(\mathbf{P}\right)}, \tag{16}$$

where $\eta>0$ is a design parameter and $\beta=\frac{\lambda_{\min>0}(-\bar{L})}{\lambda_{\max}(\widehat{L})}$, if the following condition on the perturbation bound is satisfied

$$S_{\max}\leq\frac{N\eta}{\lambda_{\max}\left(\mathbf{P}\right)\lambda_2\left(L\right)}\left\|-2\alpha\mathbf{P}+2\left(1-c_1\lambda_2(L)\right)\mathbf{M}\right\| \tag{17}$$

where $\mathbf{M}=\mathbf{PBB}^T\mathbf{P}$, and if the communication events are triggered when

$$\tilde{\delta}_i>\sigma z_i^T\Theta z_i+\eta \tag{18}$$

with

$$\Theta_i=\left(2c_2-b_iN_i\left(c_2-c\right)\right)\mathbf{M} \tag{19}$$

$$\begin{aligned}\tilde{\delta}_i=c_1&\left[\frac{1}{2b_{i2}}\left\|z_i-N_ie_i^i\right\|_{\mathbf{M}}+\frac{b_{i2}}{2}\sum_{j\in\mathcal{N}_i}N_j\left\|y_j^i-v_j^i\right\|_{\mathbf{M}}\right.\\&+\left(z_i-N_ie_i^i\right)^T\mathbf{M}\sum_{j\in\mathcal{N}_i}\left(v_i^j-y_i^i\right)\\&\left.+2\left(1+\frac{b_i}{2}\right)N_i\sum_{j\in\mathcal{N}_i}\left[\left\|v_i^j-y_i^i\right\|_{\mathbf{M}}+\left\|y_j^i-v_j^i\right\|_{\mathbf{M}}\right]\right]\\&+2\left(c_2-c\right)N_iz_i^T\mathbf{M}e_i^i+\left[2c\left(N_i\right)^2\left(1+b_i\right)+\frac{c_2-c}{b_i}N_i\right.\\&\left.+cN_i\left(N-1\right)\left(b_i+\frac{3}{b_i}\right)+c_1\frac{N_i}{2b_i}\right]\left\|e_i^i\right\|_{\mathbf{M}}\end{aligned} \tag{20}$$

and $z_i=\sum_{j\in\mathcal{N}_i}\left(y_i^i-y_j^i\right)$, $0<b_i<\frac{2c_2}{(c-c_2)N_i}$ if $c_2>c$, $b_i>0$ otherwise.

The proof is in Appendix A.

Remark 2. When an Agent i broadcasts a message, the event (18) in Theorem 1 stops to trigger because the estimate error e_i, the discrepancy $y_j^i\left(t\right)-v_j^i\left(t\right)$ and $\left(v_j^i-v_j^j\right)$ are reset to zero by (10), (14) and (15).

6. SIMULATION RESULTS

Consider a network of $N=5$ agents with state and control matrices given by

$$\mathbf{A}=\begin{bmatrix}0.48&0.29&-0.3\\0.13&0.23&0\\0&-1.2&-1\end{bmatrix}\quad\mathbf{B}=\begin{bmatrix}2&0\\-1.5&1\\0&1\end{bmatrix},$$

$\mathbf{P}$ is obtained by solving (3) with $\alpha=1$ and the Laplacian matrix associated to the graph is

$$\mathbf{P}=\begin{bmatrix}4.84&5.48&-1.11\\5.48&7.05&-1.43\\-1.11&-1.43&0.38\end{bmatrix}\quad L=\begin{bmatrix}1&-1&0&0&0\\-1&2&-1&0&0\\0&-1&2&-1&0\\0&0&-1&2&-1\\0&0&0&-1&1\end{bmatrix}.$$

The initial states

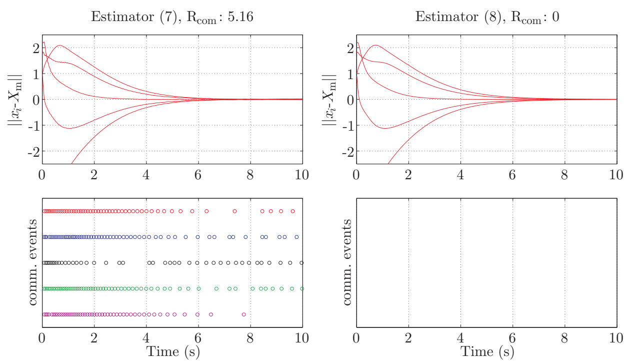
$$\begin{aligned}x\left(0\right)=\Bigg[&\begin{bmatrix}8.51\\-0.66\\0\end{bmatrix}^T\begin{bmatrix}1.74\\-0.19\\0\end{bmatrix}^T\begin{bmatrix}-0.03\\-0.47\\0\end{bmatrix}^T\dots\\&\dots\begin{bmatrix}-0.78\\-0.38\\0\end{bmatrix}^T\begin{bmatrix}-0.66\\1.51\\0\end{bmatrix}^T\Bigg]^T\end{aligned}$$

are considered to be known by all agents, *e.g.*, transmitted in some initialization phase.

The agent-specific component of the perturbation is $s_i(t)=[0,s_{i,2}(t),0]^T$ where $s_{i,2}(t)$ is a truncated zero-mean Gaussian noise of variance $\sigma_s^2=S_{\max}$ such that $|s_{i,2}|=\|s_i\|<S_{\max}=1.6$. The common component of the perturbation is $m\left(t\right)=\left[0,m_2\left(t\right),0\right]^T$. Two cases are considered: a truncated zero-mean Gaussian noise with variance σ_m^2, such that $|m_2|=\|m\|<M_{\max}$ (Figure 2 (a)), or a constant value $m_2(t)=M_{max}$ (Figure 2 (b)). We set $\eta=0.1$, $c=\frac{1}{\lambda_2(L)}$, $c_2=0.1$, $b_i=1.36$, and $b_{2i}=1$. The simulation duration is $T=5$ s. Euler integration with a step $dt=0.01$ s is used. As the system has been discretised, the minimum delay between the transmission of two messages by the same agent is set to dt.

The estimator (8) is compared to the reference estimator (7) considering the total number of messages $N_{\mathrm{m}}\leqslant\overline{N}_{\mathrm{m}}=NT/dt$. The reduction ratio of the number of broadcasted messages, expressed in %, is computed as

6.1 In absence of perturbations



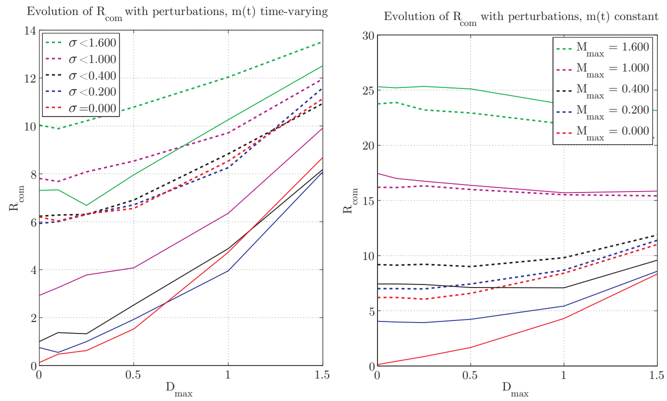
(a) reference estimator (7) (b) proposed estimator (8)

Fig. 1. Comparison between (7) and (8) without perturbation and $X_m=\frac{1}{N}\sum_{i=1}^{N}x_i$.

Figure 1 compares the performance in terms of consensus errors and number of event-triggered communications for the estimators (7) and (8). In absence of perturbations, the proposed estimator (8) limits the number of communications to one, corresponding to that occurring at initialization.

6.2 *In presence of perturbations*

Figure 2 shows that if the bounds on the perturbations are small, (8) allows a consensus with fewer communications than with (7). When the bounds on the perturbations increase, the performance gap of the consensus algorithms involving (7) and (8) decreases.



(a) with Gaussian $m(t)$ (b) with constant $m(t)$

Fig. 2. Comparison between estimators (7) (dashed) and (8) (plain).

7. CONCLUSION

This paper presents an event-triggered communication technique to reach consensus in multi-agent systems with a reduced need for communication compared to state-of-the-art techniques. This is obtained considereing two estimators. The first provides an improved agent state estimate, but does not coincide among all agents. The second is less accurate but their value is equal when two agents are neighbours. Both estimators are used to trigger communications. Convergence to consensus has been studied. Simulations have shown the effectiveness of the proposed estimators in presence of state perturbations.

Extensions of this work will consider the case of a time-varying topology, time delays in communications, and influence of packet drops during transmission of messages.

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Appendix A. PROOF OF THEOREM 1

The system gathering the dynamics of all the agents is

$$\dot{x}(t) = \bar{\mathbf{A}}x(t) + \tilde{\mathbf{B}}\tilde{y}(t) + d(t)$$

where $x = \left[x_1^T \dots x_N^T\right]^T$, $\bar{\mathbf{A}} = 1_N \otimes \mathbf{A}$, $\tilde{\mathbf{B}} = \mathbf{T}\left(\mathbf{I}_N \otimes \overline{\mathbf{B}}_1\right)$, $\overline{\mathbf{B}}_1 = c_1 L \otimes \mathbf{BF}$, and $\tilde{y} = \left[y^{1T} \dots y^{NT}\right]^T \in \mathbb{R}^{N^2 n}$ is the vector gathering the estimates of the states of Agents $1, \dots, N$ performed by all agents. Define $\tilde{e} = \tilde{y} - 1_N \otimes x = \left[e^{1T} \dots e^{NT}\right]^T \in \mathbb{R}^{N^2 n}$. $\mathbf{T} = \left(\left(\mathbf{I}_N \otimes 1_N\right) \circ \left(1_N \otimes \mathbf{I}_N\right)\right) \otimes 1_N$, with $\circ$ the entrywise matrix product, is a matrix such that $\mathbf{T}\tilde{y} = y$. $\mathbf{T}$ satisfies $\mathbf{T}\left(1_N \otimes y\right) = y$. Define the Lyapunov function $V = x^T \widehat{L} x$, with $\widehat{L} = L \otimes \mathbf{P}$. Since L is symmetric,

$$\dot{V} = 2\left(x^T \widehat{L}\left(\bar{\mathbf{A}}x + \tilde{\mathbf{B}}\tilde{y}\right) + d(t)^T \widehat{L} x\right). \tag{A.1}$$

Define $\dot{V}_1 = 2x^T \widehat{L}\left(\bar{\mathbf{A}}x + \tilde{\mathbf{B}}\tilde{y}\right)$ and $\dot{V}_2 = 2d(t)^T \widehat{L}x + x^T \bar{L} x$. Upper bounds on $\dot{V}_1$ and on $\dot{V}_2$ are derived in the two following sections.

Upper bound on $\dot{V}_1$ Let $\Delta_{ij} = y_i^j - y_i^i$ and $\Delta(t) = \left[\Delta_{11}^T(t)\ \Delta_{12}^T(t)\ \dots\ \Delta_{N,N-1}^T(t)\ \Delta_{NN}^T(t)\right]^T \in \mathbb{R}^{N^2 n}$. One has $\tilde{y} = 1_N \otimes y + \Delta$ and $\tilde{e} = 1_N \otimes e + \Delta$.

$$\begin{aligned}\dot{V}_1 &= 2x^T \widehat{L}\left(\bar{\mathbf{A}}x + \tilde{\mathbf{B}}\tilde{y}\right)\\ &= 2x^T \widehat{L}\left(\bar{\mathbf{A}}x + \tilde{\mathbf{B}}\left(1_N \otimes y + \Delta\right)\right)\end{aligned} \tag{A.2}$$

Since $\tilde{\mathbf{B}} = \mathbf{T}\left(\mathbf{I}_N \otimes \overline{\mathbf{B}}_1\right)$ and $\overline{\mathbf{B}}_1 = c_1 L \otimes (\mathbf{BF})$ as $\mathbf{T}\left(1_N \otimes y\right) = y$, one obtains

$$\begin{aligned}\tilde{\mathbf{B}}\left(1_N \otimes y\right) &= \mathbf{T}\left(\mathbf{I}_N \otimes \overline{\mathbf{B}}_1\right)\left(1_N \otimes y\right)\\ &= \mathbf{T}\left(\mathbf{I}_N \otimes \left(\overline{\mathbf{B}}_1 y\right)\right) = \overline{\mathbf{B}}_1 y\end{aligned}$$

and

$$\dot{V}_1 = 2x^T \widehat{L}\tilde{\mathbf{B}}\Delta + 2x^T \widehat{L}\left(\bar{\mathbf{A}}x + \overline{\mathbf{B}}_1 y\right) \tag{A.3}$$

Consider the two following terms

$$\dot{V}_{11} = 2x^T \widehat{L}\left(\bar{\mathbf{A}}x + \overline{\mathbf{B}}_1 y\right) \tag{A.4}$$

and

$$\dot{V}_{12} = 2x^T \widehat{L}\tilde{\mathbf{B}}\Delta. \tag{A.5}$$

The expression of $\dot{V}_{11}$ can be found in Garcia et al. [2014c] where it is shown that $\dot{V}_{11} = x^T \bar{L} x + \sum_{i=1}^N \left(\delta_i - z_i^T \Theta_i z_i\right)$. Consider now $\dot{V}_{12} = 2x^T \widehat{L}\tilde{\mathbf{B}}\Delta$,

$$\dot{V}_{12} = 2\left(\widehat{L}(y-e)\right)^T \mathbf{T}\left(\mathbf{I}_N \otimes \left(c_1 L \otimes (\mathbf{BF})\right)\right)\Delta. \tag{A.6}$$

Since $\mathbf{T}\tilde{y} = y$,

$$\mathbf{T}\left(\mathbf{I}_N \otimes \left(c_1 L \otimes (\mathbf{BF})\right)\right)\Delta = \begin{bmatrix} c_1 \mathbf{BF} \sum_{k \in \mathcal{N}_1} \left(\Delta_{11} - \Delta_{1k}\right) \\ \vdots \\ c_1 \mathbf{BF} \sum_{k \in \mathcal{N}_N} \left(\Delta_{NN} - \Delta_{Nk}\right) \end{bmatrix}$$

Since $\Delta_{ii} = 0$, $\widehat{L} = L \otimes \mathbf{P}$, and $\mathbf{F} = -\mathbf{B}^T \mathbf{P}$ one may rewrite $\dot{V}_{12}$ as

$$\begin{aligned}\dot{V}_{12} = c_1 \sum_{i=1}^N \Bigg[&\sum_{j \in \mathcal{N}_i} \left(y_i^i - y_j^j\right)^T \left(-\mathbf{PBB}^T\mathbf{P}\right) \sum_{k \in \mathcal{N}_i} \left(-\Delta_{ik}\right)\\ &- \sum_{j \in \mathcal{N}_i} \left(e_i^i - e_j^j\right)^T \left(-\mathbf{PBB}^T\mathbf{P}\right) \sum_{k \in \mathcal{N}_i} \left(-\Delta_{ik}\right)\Bigg]\end{aligned} \tag{A.7}$$

One may rewrite $\sum_{j \in \mathcal{N}_i} \left(y_i^i - y_j^j\right)^T$ as

$$\sum_{j \in \mathcal{N}_i} \left(y_i^i - y_j^j\right)^T = \sum_{j \in \mathcal{N}_i} \left(y_i^i - y_j^i + y_j^i - y_j^j\right)^T \tag{A.8}$$

$$= z_i^T + \sum_{j \in \mathcal{N}_i} \Delta_{ji}^T \tag{A.9}$$

Inserting this expression in (A.7) and defining $\mathbf{M} = \mathbf{PBB}^T\mathbf{P}$, one gets

$$\begin{aligned}\dot{V}_{12} \leq c_1 \sum_{i=1}^N \Bigg[&z_i^T \mathbf{M} \sum_{k \in \mathcal{N}_i} \Delta_{ik} + \sum_{j \in \mathcal{N}_i} \left(\Delta_{ji}\right)^T \mathbf{M} \sum_{k \in \mathcal{N}_i} \Delta_{ik}\\ &- N_i e_i^T \mathbf{M} \sum_{k \in \mathcal{N}_i} \Delta_{ik} + \sum_{j \in \mathcal{N}_i} e_j^T \mathbf{M} \sum_{k \in \mathcal{N}_i} \Delta_{ik}\Bigg]\end{aligned} \tag{A.10}$$

Using $|xy| \leq \frac{1}{2b_i} x^T x + \frac{b_i}{2} y^T y$, with $b_i > 0$, one obtains

$$\begin{aligned}\dot{V}_{12} \leq c_1 \sum_{i=1}^N \Bigg[&\left(z_i - N_i e_i^i\right)^T \mathbf{M} \sum_{k \in \mathcal{N}_i} \Delta_{ik} + \frac{N_i}{2b_i} e_i^{iT} \mathbf{M} e_i^i\\ &+ \sum_{j \in \mathcal{N}_i} N_i \left(1 + \frac{b_i}{2}\right) \Delta_{ij}^T \mathbf{M} \Delta_{ij}\Bigg]\end{aligned} \tag{A.11}$$

Expressing Δ_{ij} as $\Delta_{ij} = y_i^j - v_i^j + v_i^j - y_i^i$ in (A.11) and using $xy \leq \frac{1}{2b_{2i}} x^T x + \frac{b_{2i}}{2} y^T y$, with $b_{2i} > 0$, one gets

$$\begin{aligned}\dot{V}_{12} \leq c_1 \Bigg[&\frac{1}{2b_{i2}} \left\|z_i - N_i e_i^i\right\|_{\mathbf{M}} + \left(1 + \frac{b_i}{2}\right) N_i \sum_{j \in \mathcal{N}_i} \left\|\Delta_{ij}\right\|_{\mathbf{M}}\\ &+ \frac{b_{i2}}{2} \sum_{j \in \mathcal{N}_i} \left(y_i^j - v_i^j\right)^T \mathbf{M} \sum_{k \in \mathcal{N}_i} \left(y_i^k - v_i^k\right)\\ &+ \left(z_i - N_i e_i^i\right)^T \mathbf{M} \sum_{j \in \mathcal{N}_i} \left(v_i^j - y_i^i\right) + \frac{N_i}{2b_i} e_i^{iT} \mathbf{M} e_i^i\Bigg].\end{aligned} \tag{A.12}$$

Let $\dot{V}_{12a} = \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \left(y_i^j - v_i^j\right)^T \mathbf{M} \sum_{k \in \mathcal{N}_i} \left(y_i^k - v_i^k\right)$ and $\dot{V}_{12b} = \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} \left(\Delta_{ij}^T \mathbf{M} \Delta_{ij}\right)$. Using $xy \leq \frac{1}{2} x^T x + \frac{1}{2} y^T y$ and the fact that the communication graph is undirected, one gets

$$\dot{V}_{12a} \leq \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} N_j \left(y_j^i - v_j^i\right)^T \mathbf{M} \left(y_j^i - v_j^i\right) \tag{A.13}$$

Similarly,

$$\dot{V}_{12b} \leq \sum_{i=1}^N \left(2 \sum_{j \in \mathcal{N}_i} \left\|y_j^i - v_j^i\right\|_{\mathbf{M}} + 2 \sum_{j \in \mathcal{N}_i} \left\|v_i^j - y_i^i\right\|_{\mathbf{M}}\right) \tag{A.14}$$

Finally, injecting (A.13) and (A.14) in (A.12), one obtains

$$\begin{aligned}\dot{V}_{12} \leq c_1 \sum_{i=1}^N \Bigg[&\frac{1}{2b_{i2}} \left\|z_i - N_i e_i^i\right\|_{\mathbf{M}} + \frac{b_{i2}}{2} \sum_{j \in \mathcal{N}_i} N_j \left\|y_j^i - v_j^i\right\|_{\mathbf{M}}\\ &+ \left(z_i - N_i e_i^i\right)^T \mathbf{M} \sum_{j \in \mathcal{N}_i} \left(v_i^j - y_i^i\right) + \frac{N_i}{2b_i} e_i^{iT} \mathbf{M} e_i^i\\ &+ 2\left(1 + \frac{b_i}{2}\right) N_i \sum_{j \in \mathcal{N}_i} \left[\left\|v_i^j - y_i^i\right\|_{\mathbf{M}} + \left\|y_j^i - v_j^i\right\|_{\mathbf{M}}\right]\Bigg].\end{aligned} \tag{A.15}$$

The upper bound for $\dot{V}_1$ becomes

$$\dot{V}_1 \leq x^T \bar{L} x + \sum_{i=1}^{N} \left(\tilde{\delta}_i - \sigma z_i^T \Theta_i z_i \right).$$

Then $\dot{V}_1 \leq 0$ if, for $i, j = 1 \dots N$, the events are triggered when $\delta_i > \sigma z_i^T \Theta z_i$.

Remark 3. With $\delta_i > \sigma z_i^T \Theta z_i$ and no perturbation, $V_1(t)$ converges asymptotically to zero. In order to reduce the number of communications, a threshold η can be introduced so that $\delta_i > \sigma z_i^T \Theta z_i + \eta$.

Upper bounding of $\dot{V}_2$

$$\begin{aligned}\dot{V}_2 &= 2x^T \widehat{L} d + x^T \bar{L} x \\ &= 2x^T (L \otimes \mathbf{P})(1_N \otimes m + s) + x^T \bar{L} x \\ &= 2x^T \widehat{L} s + x^T \bar{L} x\end{aligned}$$

because $L 1_N = 0$ so $(L \otimes \mathbf{P})(1_N \otimes m) = ((L 1_N) \otimes (\mathbf{P} m)) = 0$. Let $\dot{V}_3 = 2x^T \widehat{L} s$ and $\dot{V}_4 = x^T \bar{L} x$.

$$\dot{V}_3 = \sum_{i=1}^{N} \left(2 \sum_{j \in \mathcal{N}_i} (x_i - x_j)^T \mathbf{P} s_i \right)$$

$$\dot{V}_3 \leq 2 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} \|x_i - x_j\| \lambda_{\max}(\mathbf{P}) S_{\max} \tag{A.16}$$

Bounding $\dot{V}_4$ requires first to note that

$$\begin{aligned}& L \otimes \left(\mathbf{PA} + \mathbf{A}^T \mathbf{P}\right) + (LL) \otimes \left(-2c\mathbf{PBB}^T\mathbf{P}\right) \\ &= \frac{1}{\lambda_2(L)} (\lambda_2(L) L) \otimes \left(\mathbf{PA} + \mathbf{A}^T \mathbf{P}\right) + (LL) \otimes (-2c\mathbf{M}) \\ &\leq \frac{1}{\lambda_2(L)} (LL) \otimes \left(\mathbf{PA} + \mathbf{A}^T \mathbf{P}\right) \\ &\quad + (LL) \otimes \left(-2 \frac{1}{\lambda_2(L)} \mathbf{M} + 2 \left(\frac{1}{\lambda_2(L)} - c \right) \mathbf{M} \right) \\ &\leq \frac{1}{\lambda_2(L)} (LL) \otimes \left[\mathbf{PA} + \mathbf{A}^T \mathbf{P} - 2\mathbf{M} + 2(1 - c\lambda_2(L)) \mathbf{M}\right] \\ &\leq \frac{1}{\lambda_2(L)} (LL) \otimes (-2\alpha \mathbf{P} + 2(1 - c\lambda_2(L)) \mathbf{M}) \end{aligned} \tag{A.17}$$

Since $c_1 \geq c \geq \frac{1}{\lambda_2(L)}$, $(1 - c_1 \lambda_2(L)) \leq 0$ and $\frac{1}{\lambda_2(L)} (LL) \otimes (-2\alpha \mathbf{P} + 2(1 - c_1 \lambda_2(L)) \mathbf{M})$ is semi-definite negative. It can be shown that $\dot{V}_4 = x^T \bar{L} x$ is equal to

$$\dot{V}_4 = x^T \left[L \otimes \left(\mathbf{PA} + \mathbf{A}^T \mathbf{P} \right) - (LL) \otimes \left(2c_1 \mathbf{PBB}^T \mathbf{P} \right) \right] x$$

Using (A.17)

$$\begin{aligned}\dot{V}_4 &\leq x^T \left[\frac{1}{\lambda_2(L)} (LL) \otimes (-2\alpha \mathbf{P} + 2(1 - c_1 \lambda_2(L)) \mathbf{M}) \right] x \\ &\leq \frac{1}{\lambda_2(L)} \sum_{i=1}^{N} \left[\sum_{j \in \mathcal{N}_i} (x_i - x_j)^T (-2\alpha \mathbf{P} \right. \\ &\quad \left. + 2(1 - c_1 \lambda_2) \mathbf{M}) \sum_{j \in \mathcal{N}_i} (x_i - x_j) \right] \\ &\leq \frac{1}{\lambda_2(L)} \sum_{i=1}^{N} \left[N\eta \|-2\alpha \mathbf{P} + 2(1 - c_1 \lambda_2(L)) \mathbf{M}\| N\eta \right]\end{aligned}$$

with η a positive constant threshold. As $\dot{V}_2 \leq \dot{V}_3 + \dot{V}_4$,

$$\begin{aligned}\dot{V}_2 \leq \sum_{i=1}^{N} &\left[\sum_{j \in \mathcal{N}_i} (x_i - x_j)^T (-2\alpha \mathbf{P} + 2(1 - c_1 \lambda_2(L)) \mathbf{M}) \right. \\ &\left. \times \sum_{j \in \mathcal{N}_i} (x_i - x_j) + \sum_{j \in \mathcal{N}_i} \|x_i - x_j\| \lambda_{\max}(\mathbf{P}) S_{\max} \right]\end{aligned} \tag{A.18}$$

The condition of Theorem 1 is then

$$S_{\max} \leq \frac{N\eta}{\lambda_{\max}(\mathbf{P}) \lambda_2(L)} \|-2\alpha \mathbf{P} + 2(1 - c_1 \lambda_2(L)) \mathbf{M}\|.$$

It may be rewritten as

$$\begin{aligned}2 \sum_{j \in \mathcal{N}_i} \|x_i - x_j\| \lambda_{\max}(\mathbf{P}) S_{\max} &\leq \sum_{j \in \mathcal{N}_i} \|x_i - x_j\| \\ \times \|-2\alpha \mathbf{P} + 2(1 - c_1 \lambda_2(L)) \mathbf{M}\| &\sum_{j \in \mathcal{N}_i} \|x_i - x_j\| \frac{1}{\lambda_2(L)}\end{aligned}$$

to become

$$\begin{aligned}2 \sum_{j \in \mathcal{N}_i} \|x_i - x_j\| \lambda_{\max}(\mathbf{P}) S_{\max} &\leq - \sum_{j \in \mathcal{N}_i} (x_i - x_j) \\ \times (-2\alpha \mathbf{P} + 2(1 - c_1 \lambda_2(L)) \mathbf{M}) &\sum_{j \in \mathcal{N}_i} (x_i - x_j) \frac{1}{\lambda_2(L)}\end{aligned}$$

Using it in (A.18), one obtains $\dot{V}_3 \leq 0$. The system converges thus to a bounded consensus.

Upper bound on $\dot{V}$ Assume there is no perturbation. The value of $\|x_i - x_j\|$ is now studied when the conditions of Theorem 1 are satisfied.

Since $x^T \widehat{L} x \geq 0$, one has $x^T \widehat{L} x \leq \lambda_{\max}\left(\widehat{L}\right) x^T x$ and $x^T \bar{L} x \leq 0$. As a consequence, $-x^T \bar{L} x \geq \lambda_{\min}\left(-\bar{L}\right) x^T x$. One deduces $x^T \widehat{L} x \frac{1}{\lambda_{\max}\left(\widehat{L}\right)} \leq x^T x \leq -x^T \bar{L} x \frac{1}{\lambda_{\min}\left(-\bar{L}\right)}$ and thus

$$x^T \bar{L} x \leq \frac{-\lambda_{\min}(-\bar{L})}{\lambda_{\max}\left(\widehat{L}\right)} x^T \widehat{L} x. \tag{A.19}$$

Define $\beta_2 = \frac{\lambda_{\min}(-\bar{L})}{\lambda_{\max}(\widehat{L})}$. With the triggering condition introduced in Theorem 1, one obtains

$$\begin{aligned}\dot{V}(t) &\leq x^T \bar{L} x + \sum_{i=1}^{N} \left(\delta_i - \sigma z_i^T \Theta_i z_i \right) \\ &\leq -\beta_2 V(t) + N\eta\end{aligned} \tag{A.20}$$

from which one deduces that $V(t) \leq V(0) e^{-\beta_2 t} + \frac{N\eta}{\beta_2}$. Consequently, $\lim_{t \to \infty} V(t) \leq \frac{N\eta}{\beta_2}$. According to Garcia et al. [2014c], $V(t)$ may also be expressed as $V(t) = \frac{1}{2} \sum_{i=1}^{N} \left[\sum_{k \in \mathcal{N}_i} (x_i - x_k)^T \mathbf{P} (x_i - x_k) \right]$ and a bound on the difference between any two states i,j can be obtained as follows

$$\lambda_{\min}(\mathbf{P}) \sum_{i=1}^{N} \|x_i - x_j\|^2 \leq \frac{N\eta}{\beta_2}$$

$$\|x_i - x_j\|^2 \leq \frac{N\eta}{\lambda_{\min}(\mathbf{P}) \beta_2}$$

The perturbation terms do not appear in $\bar{\delta}_i$ and Θ_i, but they will have an impact on the estimation error and on the communication triggering frequency.