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A constructive study of the module structure of rings of partial differential operators*

A. Quadrat[†] D. Robertz[‡]

Abstract

The purpose of this paper is to develop constructive versions of Stafford's theorems on the module structure of Weyl algebras $A_n(k)$ (i.e., the rings of partial differential operators with polynomial coefficients) over a base field k of characteristic zero. More generally, based on results of Stafford and Coutinho-Holland, we develop constructive versions of Stafford's theorems for very simple domains D . The algorithmization is based on the fact that certain inhomogeneous quadratic equations admit solutions in a very simple domain. We show how to explicitly compute a unimodular element of a finitely generated left D -module of rank at least two. This result is used to constructively decompose any finitely generated left D -module into a direct sum of a free left D -module and a left D -module of rank at most one. If the latter is torsion-free, then we explicitly show that it is isomorphic to a left ideal of D which can be generated by two elements. Then, we give an algorithm which reduces the number of generators of a finitely presented left D -module with module of relations of rank at least two. In particular, any finitely generated torsion left D -module can be generated by two elements and is the homomorphic image of a projective ideal whose construction is explicitly given. Moreover, a non-torsion but non-free left D -module of rank r can be generated by $r + 1$ elements but no fewer. These results are implemented in the STAFFORD package for $D = A_n(k)$ and their system-theoretical interpretations are given within a D -module approach. Finally, we prove that the above results also hold for the ring of ordinary differential operators with either formal power series or locally convergent power series coefficients and, using a result of Caro-Levcovitz, also for the ring of partial differential operators with coefficients in the field of fractions of the ring of formal power series or of the ring of locally convergent power series.

Keywords: Weyl algebras, Stafford's theorems, linear systems of partial differential equations, D -modules, mathematical systems theory, constructive algebra, symbolic computation.

1 Introduction

In his seminal paper [44], Stafford precisely described the module structure of the *Weyl algebra* $A_n(k)$ over a field k of characteristic zero, namely, of the ring of partial differential

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(PD) operators with coefficients that are polynomials over k . In particular, he proved his famous result asserting that every left/right ideal of $A_n(k)$ can be generated by two elements. Constructive proofs of this result were studied in [22, 26] and implemented in Macaulay2 ([27]) and in Maple ([37]). A consequence of Stafford's theorem is that every finitely generated *projective* (or, equivalently, *stably free*) left/right module over $A_n(k)$ of rank at least two is *free* ([44]). This result was made constructive in [20] by following Stafford's original proof and in [37] by developing a more general one based on the concept of *stable rank* (Stafford's theorem asserting that the stable rank of $A_n(k)$ is two). This last algorithm was implemented in the STAFFORD package ([37]) and used in mathematical systems theory ([37]).

Based on extensions of Stafford's results obtained in [18], we proved in [38] that the same results as the ones developed in [44] are valid for the ring of ordinary differential (OD) operators with coefficients in the ring of formal power series or in the ring of real or complex convergent power series. This result was applied to mathematical systems theory (e.g., reduction and decomposition problems, Serre's reduction, controllability, observability, Monge problem, computation of flat outputs and injective parametrizations of differentially flat systems, blowing-up of singularities). For more details, see [13, 15, 38].

Stafford's theorem has also been extended in [9] to the ring $\widehat{\mathcal{D}}_n(k)$ (resp., $\mathcal{D}_n(k)$) of PD operators with coefficients in the field of fractions of the ring of formal power series $\widehat{\mathcal{A}} = k[[x_1, \dots, x_n]]$ (resp., of the ring of locally convergent power series $\mathcal{A} = k\{x_1, \dots, x_n\}$). As explained in [37, 38], finitely generated projective modules of rank at least two over these algebras are free.

More results on the module structure of the Weyl algebra $D = A_n(k)$, obtained by Stafford in [44], have not been made constructive yet, and in particular the following ones (where every module is assumed or claimed to be finitely generated):

1. Every left D -module M can be decomposed as the direct sum of a free left D -module $D^{1 \times r}$ for some $r \in \mathbb{Z}_{\geq 0}$ and a left D -module M' of rank at most one, i.e., $M \cong D^{1 \times r} \oplus M'$, where $\text{rank}_D(M') \leq 1$.
2. Every torsion-free left D -module M can be decomposed as the direct sum of a free left D -module $D^{1 \times r}$ for some $r \in \mathbb{Z}_{\geq 0}$ and a left ideal I of D generated by two elements, i.e., $M \cong D^{1 \times r} \oplus I$, where $I = Dd_1 + Dd_2$ for certain $d_1, d_2 \in D$.
3. Every torsion left D -module T is the homomorphic image of a projective left ideal of D , and thus can be generated by two elements, i.e., $T \cong I/J$, where I is a projective left ideal of D and J a left D -submodule of I (which can also be generated by two elements).
4. Every non-torsion left D -module of rank r is either free or can be generated by $r + 1$ elements but no fewer.
5. If $\text{rank}_D(M) \geq 2$ and $M \oplus D \cong N \oplus D$, then $M \cong N$.

Similar results hold for finitely generated right D -modules ([44]).

The first (resp., last) point above can be traced back to *Serre's splitting-off theorem* ([41]) (resp., *Bass' cancellation theorem* ([5])), not requiring, however, any projectivity condition on M . These results are well-known in commutative algebra and they have been studied in [17, 28] (see also the references therein) within a constructive commutative algebra approach.

Versions of these results were studied in [42, 43] for noncommutative (simple) noetherian rings and in [18] for *very simple domains*. A *very simple ring* D is a *prime ring* D (namely, $d_1 D d_2 \neq 0$ for all $d_1, d_2 \in D \setminus \{0\}$) which is noetherian and satisfies

$$\forall a, b, c \in D, \forall d \in D \setminus \{0\}, \exists u, v \in D : D a + D b + D c = D (a + d u c) + D (b + d v c), \quad (1)$$

where $D s = \{r s \mid r \in D\}$ denotes the principal left ideal of D generated by $s \in D$. For more details, see [18]. If D is a very simple domain (and thus satisfies the right Ore condition), then one can easily show that for fixed $d_1, d_2 \in D \setminus \{0\}$, the following inhomogeneous quadratic equation

$$y_1 d_1 z_1 + y_2 d_2 z_2 = 1$$

admits solutions $(y_1 \ y_2 \ z_1 \ z_2)^T \in D^4$ [18, 44]. Solving inhomogeneous quadratic equations of the above form is the cornerstone of our approach. The Weyl algebra $A_n(k)$, where k is a field of characteristic zero, is a prototypical example of a very simple domain. The solvability of the above equation plays a fundamental role in certain problems studied in module theory (e.g., existence of *unimodular elements*, direct sum decomposition of modules). In particular, Stafford's theorems, stated above for $A_n(k)$, hold for a very simple domain D . More generally, it can easily be seen that the (category of) finitely generated modules over a very simple domain possess(es) some features that bear some resemblance with (the category of) finitely generated vector spaces over a division ring (e.g., a field).

The purpose of this paper is to study the above results for *very simple domains* from a constructive viewpoint, therefore completing our understanding of Stafford's results for $A_n(k)$ ([44]) in terms of symbolic computation. Using the recent extension of Stafford's main theorem to the rings $\widehat{D}_n(k)$ and $\mathcal{D}_n(k)$ obtained in [9], we also generalize Stafford's theorems to the algebras $\widehat{D}_n(k)$ and $\mathcal{D}_n(k)$. Finally, the algorithms we developed in the present paper for $A_n(\mathbb{Q})$ are implemented in the STAFFORD package ([37]). For more details, see also Appendix of [39].

From a more general perspective, the constructive approach to algebraic analysis initiated in [1, 3, 4, 7, 11, 12, 13, 14, 15, 16, 32, 33, 34, 35] relies on the solvability of inhomogeneous linear systems over the ring under consideration. Within category theory, this idea was axiomatized in [2] and applied, for instance, to sheaf theory. If D is a very simple domain, in this paper, we show that problems classically studied in module theory (e.g., Serre's splitting-off theorem, Swan's lemma, Bass' theorem), which are based on the solvability of inhomogeneous quadratic equations (see above), can be studied within the same constructive framework. As pointed out by a referee, if we are only interested in linear systems theory, i.e., in the category of finitely presented left D -modules, all the results of this paper also hold for a finitely presented left D -module M over a coherent domain D which satisfies both the left and right Ore conditions and (1).

One of the main motivations for making Stafford's results constructive is the fact that they have system-theoretic interpretations within the *algebraic analysis* or *D -module* approach ([6, 24, 29, 30]) (e.g., efficient generation of the set of autonomous elements ([11, 37]), computation of injective parametrizations ([37]), Serre's reduction ([7, 15]), decomposability of the solution space ([13])). These results apply to general linear PD systems which can be determined, overdetermined, or underdetermined depending on which result we consider. For instance, apart from many other results, we prove that a linear system of PDEs defined by a square full row rank matrix of PD operators (e.g., most of the classical linear systems of PDEs studied in

mathematical physics and engineering science) is equivalent to a linear system defined by two PDEs in two unknown functions (see Example 8). This result can be seen as a generalization for linear systems of PDEs of the well-known cyclic vector theorem in the theory of ODEs (see, e.g., [10] and the references therein). Moreover, these results can be used to study nonlinear PD systems by means of their linearizations around a polynomial/rational/formal power series/locally convergent power series solution (e.g., shallow water waves, Poiseuille flow, flexible thread attached to one point in a vertical equilibrium position).

We point out that the above results are not valid for a ring of PD operators with constant coefficients. But if one allows the use of non-constant coefficients, then we can always embed this ring into a ring of PD operators with polynomial/rational/formal power series/locally convergent power series coefficients, so that the above results hold (see, e.g., Example 5).

The paper is organized as follows. In Section 2, we shortly introduce basic ideas of algebraic analysis, review well-known results of noncommutative algebra, and explain how linear PD systems can be studied by means of module theory. The main results of the paper are presented in Section 3. More precisely, the fundamental role of the unimodular elements of a module is explained in Section 3.1. Section 3.3 shows how to combine Stafford's main theorem with constructive module-theoretic techniques to explicitly compute unimodular elements of (sub)modules. These results are then used in Section 4 to study Stafford's reduction and in Section 5 to examine questions about the efficient generation of modules. In Section 6, we study Stafford's cancellation theorem and its consequences. Finally, we conclude and open some perspectives in Section 7.

Notation. In what follows, $\mathbb{Z}_{\geq 0}$ will denote the set of non-negative integers, k a field of characteristic zero (e.g., $k \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$), D a *noetherian domain*, namely, a left and right noetherian ring with 1 and without zero-divisors, and $D^{q \times p}$ the set of $q \times p$ matrices with entries in D . We shall identify D^q and $D^{q \times 1}$ and we write I_r for the $r \times r$ identity matrix. Moreover,

$$\mathrm{GL}_r(D) = \{U \in D^{r \times r} \mid \exists V \in D^{r \times r} : UV = VU = I_r\}$$

is the *general linear group of degree r over D* and $U(D) = \mathrm{GL}_1(D)$ the group of units of D .

2 Algebraic analysis

Using ideas of *algebraic analysis* ([6, 24, 29, 30]), we can study linear systems over a noetherian domain D by means of module theory and homological algebra. We shortly recall a few results which will be useful in what follows. For more details and results, we refer to [11, 13, 37].

Let $R \in D^{q \times p}$ and $M := D^{1 \times p} / (D^{1 \times q} R)$ be the left D -module *finitely presented by R* . Let us describe M by means of *generators* and *relations*. Let $\{f_j\}_{j=1, \dots, p}$ be the *standard basis* of $D^{1 \times p}$, namely, f_j is the row vector of length p with 1 at the j^{th} position and 0 elsewhere. Moreover, let $\pi : D^{1 \times p} \rightarrow M$ be the left D -homomorphism defined by sending $\lambda \in D^{1 \times p}$ to its residue class $\pi(\lambda)$ in M . By definition of M , π is surjective since every $m \in M$ is the residue class of certain λ 's in $D^{1 \times p}$, i.e., $m = \pi(\lambda) = \pi(\lambda + \nu R)$ for all $\nu \in D^{1 \times q}$. If $y_j = \pi(f_j)$ for $j = 1, \dots, p$, then, for every $m \in M$, there exists $\lambda = (\lambda_1 \dots \lambda_p) \in D^{1 \times p}$ such that

$$m = \pi(\lambda) = \pi \left(\sum_{j=1}^p \lambda_j f_j \right) = \sum_{j=1}^p \lambda_j \pi(f_j) = \sum_{j=1}^p \lambda_j y_j,$$

which shows that $\{y_j\}_{j=1,\dots,p}$ is a family of generators of M . Let $R_{i\bullet}$ (resp., $R_{\bullet j}$) denote the i^{th} row (resp., j^{th} column) of R . The family of generators $\{y_j\}_{j=1,\dots,p}$ of M satisfies relations

$$\forall i = 1, \dots, q, \quad \sum_{j=1}^p R_{ij} y_j = \sum_{j=1}^p R_{ij} \pi(f_j) = \pi \left(\sum_{j=1}^p R_{ij} f_j \right) = \pi(R_{i\bullet}) = 0, \quad (2)$$

since $R_{i\bullet} \in D^{1 \times q} R$ for $i = 1, \dots, q$. Now, let \mathcal{F} be a left D -module and let

$$\ker_{\mathcal{F}}(R.) := \{\eta \in \mathcal{F}^p \mid R\eta = 0\}$$

be the *linear system* defined by R and \mathcal{F} . A simple but fundamental remark due to Malgrange ([30]) is that the linear system $\ker_{\mathcal{F}}(R.)$ is isomorphic to the abelian group $\text{hom}_D(M, \mathcal{F})$ of left D -homomorphisms from M to \mathcal{F} , i.e.,

$$\ker_{\mathcal{F}}(R.) \cong \text{hom}_D(M, \mathcal{F}) \quad (3)$$

as abelian groups, where \cong denotes an isomorphism (e.g., of abelian groups, left/right modules). This isomorphism can easily be described: if $\varphi \in \text{hom}_D(M, \mathcal{F})$, $\eta_j = \varphi(y_j)$ for $j = 1, \dots, p$, and $\eta = (\eta_1 \dots \eta_p)^T \in \mathcal{F}^p$, then using (2), $R\eta = 0$ since for $i = 1, \dots, q$:

$$\sum_{j=1}^p R_{ij} \eta_j = \sum_{j=1}^p R_{ij} \varphi(y_j) = \varphi \left(\sum_{j=1}^p R_{ij} y_j \right) = \varphi \left(\sum_{j=1}^p R_{ij} \pi(f_j) \right) = \varphi(\pi(R_{i\bullet})) = \varphi(0) = 0.$$

Moreover, we can show that given $\eta = (\eta_1 \dots \eta_p)^T \in \ker_{\mathcal{F}}(R.)$, the map $\varphi_{\eta} : M \rightarrow \mathcal{F}$ defined by $\varphi_{\eta}(\pi(\lambda)) = \lambda \eta$ for $\lambda \in D^{1 \times p}$ is a well-defined left D -homomorphism from M to \mathcal{F} , i.e., $\varphi_{\eta} \in \text{hom}_D(M, \mathcal{F})$. Finally, the abelian group homomorphism $\chi : \ker_{\mathcal{F}}(R.) \rightarrow \text{hom}_D(M, \mathcal{F})$ defined by $\chi(\eta) = \varphi_{\eta}$ is bijective. For more details, see [11, 13, 37]. Hence, (3) shows that the properties of the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of $\text{hom}_D(M, \mathcal{F})$, and thus by means of the left D -modules M and \mathcal{F} .

If M', M, M'' are three left/right D -modules, $f \in \text{hom}_D(M', M)$, and $g \in \text{hom}_D(M, M'')$, then $M' \xrightarrow{f} M \xrightarrow{g} M''$ is called a *complex* if $\text{im } f \subseteq \ker g$. This complex is said to be *exact at M* if $\ker g = \text{im } f$. A complex which is exact at any place is called an *exact sequence*. In particular, the complex $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is called a *short exact sequence* if f is injective, $\ker g = \text{im } f$, and g is surjective. For more details, see [40].

The definition of the left D -module $M = D^{1 \times p} / (D^{1 \times q} R)$ yields the following exact sequence

$$D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \rightarrow 0,$$

called a *presentation of M* , where $\cdot R : D^{1 \times q} \rightarrow D^{1 \times p}$ is defined by $(\cdot R)(\mu) = \mu R$ for all $\mu \in D^{1 \times q}$. Since equivalence of linear systems of equations is tantamount to isomorphism of the corresponding left D -modules, $\text{hom}_D(M, \mathcal{F})$ is a more intrinsic characterization of the linear system than $\ker_{\mathcal{F}}(R.)$ (e.g., it does not depend on the particular embedding of $\ker_{\mathcal{F}}(R.)$ into \mathcal{F}^p).

Example 1. Let A be a *differential ring*, namely, A is a commutative ring equipped with *commuting derivations* δ_i for $i = 1, \dots, n$, namely, maps $\delta_i : A \rightarrow A$ satisfying

$$\forall a_1, a_2 \in A, \quad \delta_i(a_1 + a_2) = \delta_i(a_1) + \delta_i(a_2), \quad \delta_i(a_1 a_2) = \delta_i(a_1) a_2 + a_1 \delta_i(a_2),$$

and $\delta_i \circ \delta_j = \delta_j \circ \delta_i$ for all $1 \leq i < j \leq n$. Moreover, let $D = A\langle \partial_1, \dots, \partial_n \rangle$ be the (not necessarily commutative) polynomial ring of PD operators in $\partial_1, \dots, \partial_n$ with coefficients in A , namely, every element $d \in D$ is of the form

$$d = \sum_{0 \leq |\mu| \leq r} a_\mu \partial^\mu, \quad r \in \mathbb{Z}_{\geq 0}, \quad a_\mu \in A, \quad \mu = (\mu_1 \dots \mu_n) \in \mathbb{Z}_{\geq 0}^{1 \times n},$$

where $\partial^\mu = \partial_1^{\mu_1} \dots \partial_n^{\mu_n}$ is a monomial in the pairwise commuting indeterminates $\partial_1, \dots, \partial_n$, and $\partial_i a = a \partial_i + \delta_i(a)$ for all $a \in A$. If k is a field and $A = k[x_1, \dots, x_n]$ (resp., $k(x_1, \dots, x_n)$), then $A\langle \partial_1, \dots, \partial_n \rangle$ is simply denoted by $A_n(k)$ (resp., $B_n(k)$) and is called the *polynomial* (resp., *rational*) *Weyl algebra*. If $R \in D^{q \times p}$ and \mathcal{F} is a left D -module (e.g., $\mathcal{F} = A$), then the linear PD system $\ker_{\mathcal{F}}(R) = \{\eta \in \mathcal{F}^p \mid R\eta = 0\}$ is intrinsically defined by $\text{hom}_D(M, \mathcal{F})$, where $M = D^{1 \times p} / (D^{1 \times q} R)$ is the left D -module finitely presented by R .

Within algebraic analysis, the study of the module structure of rings of PD operators plays a fundamental role for the study of linear systems of PD equations. In [11, 13, 34, 35, 37], we have initiated the constructive study of module theory and homological algebra over rings of *functional operators* such as rings of PD operators or *Ore algebras*.

Let us now recall a few classical definitions of module theory (see, e.g., [31, 40]).

Definition 1. Let D be a noetherian domain and M a finitely generated left D -module.

- M is *free* if there exists $r \in \mathbb{Z}_{\geq 0}$ such that $M \cong D^{1 \times r}$. In this case r is called the *rank* of M .
- M is *stably free* if there exist $r, s \in \mathbb{Z}_{\geq 0}$ such that $M \oplus D^{1 \times s} \cong D^{1 \times r}$, where \oplus denotes the direct sum of modules.
- M is *projective* if there exist $r \in \mathbb{Z}_{\geq 0}$ and a left D -module P such that $M \oplus P \cong D^{1 \times r}$.
- M is *reflexive* if the canonical left D -homomorphism $\varepsilon : M \rightarrow \text{hom}_D(\text{hom}_D(M, D), D)$ defined by $\varepsilon(m)(f) = f(m)$, for all $m \in M$ and for all $f \in \text{hom}_D(M, D)$, is an isomorphism of left D -modules.
- M is *torsion-free* if the *torsion* left D -submodule of M defined by

$$t(M) = \{m \in M \mid \exists d \in D \setminus \{0\} : dm = 0\}$$

is reduced to 0, i.e., $t(M) = 0$.

- M is *torsion* if $t(M) = M$.

Similar definitions hold for right D -modules.

See [11, 34, 37] for algorithms which test whether or not a finitely presented left D -module M is stably free, projective, reflexive, torsion-free, has torsion elements or is torsion.

Since D is a noetherian domain, D satisfies *the left* and *the right Ore properties* (see, e.g., [31]), namely, for all $d_1, d_2 \in D \setminus \{0\}$, there exist $e_1, e_2 \in D \setminus \{0\}$ such that $e_1 d_1 = e_2 d_2$ (resp., $d_1 e_1 = d_2 e_2$). This result yields the existence of the *division ring of fractions* $Q(D) = S^{-1} D = D S^{-1}$ of D , where $S = D \setminus \{0\}$ (see, e.g., [31]). If M is a finitely generated left (resp., right) D -module, then $Q(D) \otimes_D M$ (resp., $M \otimes_D Q(D)$) is a finitely generated left (resp., right) $Q(D)$ -vector space and $\text{rank}_D(M) := \dim_{Q(D)}(Q(D) \otimes_D M)$ (resp., $\text{rank}_D(M) := \dim_{Q(D)}(M \otimes_D Q(D))$).

Theorem 1 ([31, 40, 44]). *With the hypotheses of Definition 1, the following results hold:*

1. *The following implications hold for finitely generated left/right D -modules:*

$$\text{free} \Rightarrow \text{stably free} \Rightarrow \text{projective} \Rightarrow \text{reflexive} \Rightarrow \text{torsion-free}.$$

2. *If $D = A_1(k)$, then every finitely generated torsion-free left D -module is stably free.*

3. *If D is a principal ideal domain (e.g., $D = B_1(k)$), namely, every left ideal and every right ideal of D can be generated by one element, then every finitely generated torsion-free left D -module is free.*

4. *If $D = A_n(k)$, then every finitely generated projective left/right D -module is stably free and every stably free left D -module M with $\text{rank}_D(M) \geq 2$ is free (Stafford's theorem).*

Constructive proofs of Stafford's theorem (see 4 of Theorem 1) were given in [20, 37]. Computation of bases of free $A_n(\mathbb{Q})$ -modules is implemented in the STAFFORD package ([37]).

Finally, let us state a proposition that will be constantly used in what follows.

Proposition 1 ([13]). *Let $R \in D^{q \times p}$ and $R' \in D^{q' \times p}$ satisfy $D^{1 \times q} R \subseteq D^{1 \times q'} R'$, i.e., are such that $R = R'' R'$ for a certain $R'' \in D^{q \times q'}$. Moreover, let $R'_2 \in D^{r' \times q'}$ be a matrix such that $\ker_D(\cdot R') = D^{1 \times r'} R'_2$, and let π and π' be respectively the following canonical projections:*

$$\begin{aligned} \pi: D^{1 \times q'} R' &\longrightarrow P := (D^{1 \times q'} R') / (D^{1 \times q} R), \\ \pi': D^{1 \times q'} &\longrightarrow Q := D^{1 \times q'} / (D^{1 \times q} R'' + D^{1 \times r'} R'_2). \end{aligned}$$

Then, the left D -homomorphism ι defined by

$$\begin{aligned} Q &\xrightarrow{\iota} P \\ \pi'(\lambda) &\longmapsto \pi(\lambda R'), \end{aligned}$$

is an isomorphism and its inverse ι^{-1} is defined by:

$$\begin{aligned} P &\xrightarrow{\iota^{-1}} Q \\ \pi(\lambda R') &\longmapsto \pi'(\lambda). \end{aligned}$$

3 Unimodular elements and very simple domains

3.1 Unimodular elements

Definition 2. An element m^* of a left D -module M is called *unimodular* if there exists $\varphi \in \text{hom}_D(M, D)$ such that $\varphi(m^*) = 1$.

The set of unimodular elements of M is denoted by $U(M)$. Let us explain the significance of unimodular elements in module theory. If $m^* \in U(M)$, then there exists $\varphi \in \text{hom}_D(M, D)$ such that $\varphi(m^*) = 1$. If $\sigma: D \rightarrow M$ is the left D -homomorphism defined by $\sigma(1) = m^*$, then $(\varphi \circ \sigma)(d) = \varphi(d m^*) = d$ for all $d \in D$, i.e., $\varphi \circ \sigma = \text{id}_D$. In particular, φ is surjective and we have the following short exact sequence of left D -modules:

$$0 \longrightarrow \ker \varphi \longrightarrow M \xrightarrow{\varphi} D \longrightarrow 0. \quad (4)$$

Since $\varphi \circ \sigma = \text{id}_D$, the exact sequence (4) *splits* (see, e.g., [40]), i.e., $M = \ker \varphi \oplus D m^*$, where $D m^* = D \sigma(1) \cong D$ since σ is injective. Hence, the left D -module $D m^*$ generated by m^* is a direct summand of M . Thus, every unimodular element of M can be used to decompose M into a direct sum. Finally, the splitting of (4) yields the following unique decomposition of elements of M

$$\forall m \in M : \quad m = (m - \varphi(m) m^*) + \varphi(m) m^*, \quad (5)$$

where $m - \varphi(m) m^* \in \ker \varphi$ and $\varphi(m) m^* \in D m^*$.

If m is a torsion element of M , i.e., $d m = 0$ for some $d \in D \setminus \{0\}$, and if $f \in \text{hom}_D(M, D)$, then $d f(m) = f(d m) = f(0) = 0$, which shows that $f(m) = 0$ since D is a domain. Hence, a unimodular element of M cannot be a torsion element of M .

In the sequel we will frequently use the following characterization of torsion modules.

Lemma 1 ([11, 25]). *Let D be a noetherian domain and M a finitely generated left D -module. Then, the following statements are equivalent:*

1. M is torsion.
2. $\text{hom}_D(M, D) = 0$.
3. $\text{rank}_D(M) = 0$.

Let us now study the problem of computing unimodular elements of M . Let us consider a finitely presented left D -module $M = D^{1 \times p} / (D^{1 \times q} R)$. Malgrange's remark (see (3)) with $\mathcal{F} = D$ yields $\text{hom}_D(M, D) \cong \ker_D(R)$. More precisely, we have the following simple lemma.

Lemma 2. *Let $M = D^{1 \times p} / (D^{1 \times q} R)$ and $\pi : D^{1 \times p} \rightarrow M$ be the canonical projection onto M . Then, for every $\varphi \in \text{hom}_D(M, D)$, there exists $\mu \in \ker_D(R)$, i.e., $R \mu = 0$, such that:*

$$\forall \lambda \in D^{1 \times p}, \quad \varphi(\pi(\lambda)) = \lambda \mu. \quad (6)$$

Notation. We shall simply denote the left D -homomorphism defined by (6) by φ_μ .

As previously shown, if M is a torsion left D -module, then $U(M) = \emptyset$, i.e., a torsion left D -module does not contain unimodular elements. Hence, let us suppose that M is not a torsion left D -module. Then, 2 \Rightarrow 1 of Lemma 1 shows that $\ker_D(R) \cong \text{hom}_D(M, D) \neq 0$. Let $Q \in D^{p \times m}$ be such that $\ker_D(R) = Q D^m$. Since $R Q = 0$, we get the following complex:

$$D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\cdot Q} D^{1 \times m}.$$

Lemma 3 ([11]). *With the above notations, we have:*

$$\begin{cases} t(M) = \ker_D(\cdot Q) / \text{im}_D(\cdot R), \\ M/t(M) \cong D^{1 \times p} / \ker_D(\cdot Q). \end{cases}$$

In particular, $\pi(\lambda)$ is a torsion element of M if and only if $\lambda Q = 0$.

Remark 1. Using Lemma 2, $\varphi \in \text{hom}_D(M, D)$ has the form φ_μ for a certain $\mu \in \ker_D(R) = \text{im}_D(Q)$, i.e., $\mu = Q \xi$ for some $\xi \in D^m$. Hence, if $\pi(\lambda) \in M \setminus t(M)$, i.e., $\lambda Q \neq 0$ by Lemma 3, there exists $\xi \in D^m$ such that $\lambda Q \xi \neq 0$. Thus, $\mu := Q \xi \in \ker_D(R)$ yields $\varphi_\mu \in \text{hom}_D(M, D)$ such that $\varphi_\mu(\pi(\lambda)) \neq 0$.

The problem of finding a unimodular element $m^* = \pi(\lambda^*)$ of M and $\mu^* = Q \xi^* \in \ker_D(R.)$ such that $\varphi_{\mu^*} \in \text{hom}_D(M, D)$ satisfies $\varphi_{\mu^*}(m^*) = 1$ amounts to solving the following problem:

Problem 1. Find $\lambda^* \in D^{1 \times p}$ and $\xi^* \in D^m$ such that $\lambda^* Q \xi^* = 1$.

Remark 2. We note that Problem 1 corresponds to solving a certain inhomogeneous quadratic equation in the λ_i^* 's and the ξ_j^* 's. As for the problem of checking whether or not $\pi(\lambda^*)$ is a unimodular element of M , it is a linear problem: Check whether or not $\lambda^* Q \in D^{1 \times m}$ admits a right inverse in D . For instance, this can be answered constructively for (not necessarily commutative) polynomial rings which admit Gröbner basis techniques (see, e.g., [11, 12]).

If one entry of Q is invertible in D , then Problem 1 can be solved easily: if $Q_{ij} \in U(D)$ and $\{f_r\}_{r=1, \dots, p}$ (resp., $\{h_s\}_{s=1, \dots, m}$) is the standard basis of $D^{1 \times p}$ (resp., D^m), then $\lambda^* = f_i$ and $\xi^* = Q_{ij}^{-1} h_j$ are such that $\lambda^* Q \xi^* = 1$. Then, $m^* = \pi(f_i)$ is a unimodular element of M , $\mu^* = Q \xi^* \in \ker_D(R.)$, and $\varphi_{\mu^*}(m^*) = 1$.

More generally, if one row (resp., one column) of Q admits a right inverse (resp., a left inverse) over D , then Problem 1 can be solved easily. For instance, if the j^{th} column $Q_{\bullet j}$ of Q admits a left inverse $T \in D^{1 \times p}$, then considering $\lambda^* = T$ and $\xi^* = h_j$, where $\{h_s\}_{s=1, \dots, m}$ is the standard basis of D^m , and $\mu^* = Q \xi^*$, we get $\lambda^* \mu^* = 1$, which proves that $m^* = \pi(T) \in U(M)$ and $\varphi_{\mu^*}(m^*) = 1$. Now, if the i^{th} row $Q_{i \bullet}$ of Q admits a right inverse $S \in D^m$, then considering $\lambda^* = f_i$, where $\{f_r\}_{r=1, \dots, p}$ is the standard basis of $D^{1 \times p}$, $\xi^* = S$, and $\mu^* = Q S \in \ker_D(R.)$, we get $\lambda^* \mu^* = 1$, which shows that $m^* = \pi(\lambda^*) \in U(M)$ and $\varphi_{\mu^*}(m^*) = 1$.

Let us illustrate these results with two explicit examples.

Example 2. Let us consider the Weyl algebra $D = A_3(\mathbb{Q})$ and the finitely presented left D -module $M = D^{1 \times 3} / (D^{1 \times 3} R)$, where the matrix R is defined by:

$$R = \begin{pmatrix} \frac{1}{2} x_2 \partial_1 & x_2 \partial_2 + 1 & x_2 \partial_3 + \frac{1}{2} \partial_1 \\ -\frac{1}{2} x_2 \partial_2 - \frac{3}{2} & 0 & \frac{1}{2} \partial_2 \\ -\partial_1 - \frac{1}{2} x_2 \partial_3 & -\partial_2 & -\frac{1}{2} \partial_3 \end{pmatrix} \in D^{3 \times 3}.$$

The corresponding system defines the *infinitesimal transformations of the Lie pseudogroup formed by the contact transformations* (see Example V.1.84 in [32]). We can check that $\ker_D(R.) = Q D$, where the matrix $Q \in D^3$ is defined by:

$$Q = \begin{pmatrix} -\partial_2 \\ \partial_1 + x_2 \partial_3 \\ -x_2 \partial_2 - 2 \end{pmatrix}.$$

Moreover, Q admits the left inverse $T = \frac{1}{2}(x_2 \ 0 \ -1)$. Thus, if $\lambda^* = T$ and $\xi^* = 1$, i.e., $\mu^* = Q$, then $\lambda^* \mu^* = 1$, which shows that $m^* = \pi(\lambda^*)$ is a unimodular element of M and $\varphi_{\mu^*}(m^*) = 1$. Following [11], we can prove that M is a torsion-free left D -module, i.e., $t(M) = 0$. Then, using Lemma 3, we get $\ker \varphi_{\mu^*} = \ker_D(Q) / \text{im}_D(R) = t(M) = 0$, i.e., φ_{μ^*} is injective. Thus, $M = D m^* \oplus \ker \varphi_{\mu^*} = D m^* \cong D$, which proves that M is a free left D -module of rank one (compare with Example 1.5.1 of [34]).

Example 3. Let $D = A_3(\mathbb{Q})$ be the Weyl algebra and $M = D^{1 \times 3}/(DR)$ the left D -module finitely presented by $R = (\partial_1 + x_2 \quad \partial_2 \quad \partial_3 + x_1)$. Using the OREMODULES package ([12]), we get $\ker_D(R) = QD^4$, where the matrix $Q \in D^{3 \times 4}$ is defined by:

$$Q = \begin{pmatrix} \partial_2 + \partial_3 + x_1 & -(\partial_3 + x_1)^2 & -(\partial_1 + x_2)(\partial_3 + x_1) + 2 & 0 \\ -(\partial_1 + x_2) & 0 & 0 & \partial_3 + x_1 \\ -(\partial_1 + x_2) & (\partial_1 + x_2)(\partial_3 + x_1) + 1 & (\partial_1 + x_2)^2 & -\partial_2 \end{pmatrix}.$$

We can check that the last row of Q admits the right inverse $S = (\partial_3 + x_1 \quad 1 \quad 0 \quad 0)^T$. Thus, if $\lambda^* = (0 \quad 0 \quad 1)$, $\xi^* = S$, and $\mu^* = QS = (\partial_2(\partial_3 + x_1) \quad -(\partial_1 + x_2)(\partial_3 + x_1) \quad 1)^T$, then $\lambda^* \mu^* = 1$ and $R\mu^* = 0$. Hence, $m^* = \pi(\lambda^*) \in U(M)$ and $\varphi_{\mu^*}(m^*) = 1$. Therefore, $M = Dm^* \oplus \ker \varphi_{\mu^*}$ and $\ker \varphi_{\mu^*} = \ker_D(\cdot \mu^*)/\text{im}_D(\cdot R) = (D^{1 \times 2}T)/(DR)$, where:

$$T = \begin{pmatrix} 1 & 0 & -\partial_2(\partial_3 + x_1) \\ 0 & 1 & (\partial_1 + x_2)(\partial_3 + x_1) \end{pmatrix}.$$

Since $\ker_D(T) = 0$ and $R = CT$, where $C = (\partial_1 + x_2 \quad \partial_2)$, Proposition 1 then shows that $\ker \varphi_{\mu^*} \cong O := D^{1 \times 2}/(DC)$, which proves that $M \cong D \oplus O$. Now, since $\text{rank}_D(M) = 2$ and $(\partial_3 + x_1 \quad 0 \quad -(\partial_1 + x_2))^T$ is a right inverse of R , $M = D^{1 \times 3}/(DR)$ is a stably free left D -module of rank two (see [37] for more details), and thus a free left D -module of rank two by Stafford's theorem (see 4 of Theorem 1). Therefore, we get $D \oplus O \cong D^{1 \times 2}$, which is consistent with the fact that O is a stably free left D -module of rank one since $(-\partial_2 \quad \partial_1 + x_2)^T$ is a right inverse of C . Finally, following [37], we can prove that O is not a free left D -module of rank one, which shows that $D \oplus O \cong D^{1 \times 2}$ does not imply $O \cong D$.

In Section 3.3, we shall explain how unimodular elements of a finitely generated left D -module M of rank at least 2 can be computed when D is a *very simple* noetherian domain, a concept studied in the next section. In particular, in Section 3.2, it is shown that the Weyl algebras $A_n(k)$ and $B_n(k)$ satisfy this interesting property.

3.2 Very simple domains

Let us now introduce an important definition for what follows.

Definition 3 ([18]). A ring D is called *very simple* if D is prime, namely, $d_1 D d_2 \neq 0$ for all $d_1, d_2 \in D \setminus \{0\}$, noetherian and satisfies the following property:

$$\forall a, b, c \in D, \forall d \in D \setminus \{0\}, \exists u, v \in D : Da + Db + Dc = D(a + duc) + D(b + dvc). \quad (7)$$

If D is a very simple ring, choosing $a = b = 0$, $c = 1$, and $d \in D \setminus \{0\}$, there exist $u, v \in D$ such that $D = Ddu + Ddv$, which implies that there exist $s, t \in D$ such that $1 = sdu + tdv$, and thus $DD = D$. Hence, every two-sided ideal of D is trivial, i.e., D is a *simple* ring (see, e.g., [31, 40]).

Moreover, considering $d = 1$ in (7), every left ideal of D generated by three elements, and thus every finitely generated left ideal of D , can be generated by two elements.

Clearly, a domain is prime. A well-known result states that a left (resp., right) noetherian domain satisfies the left (resp., right) Ore condition (see, e.g., [25, 31]). Hence, if D is a very

simple domain and $d_1, d_2 \in D \setminus \{0\}$, there exist $e_1, e_2 \in D \setminus \{0\}$ satisfying $d := d_1 e_1 = d_2 e_2$ (see Section 2). Thus, (7) yields:

$$D a + D b + D c = D(a + d u c) + D(b + d v c) = D(a + d_1(e_1 u) c) + D(b + d_2(e_2 v) c).$$

Lemma 4 ([18]). If D is a very simple domain, then we have:

1. Every finitely generated left ideal of D can be generated by two elements.
2. The following property holds:

$$\forall a, b, c \in D, \forall d_1, d_2 \in D \setminus \{0\}, \exists u, v \in D : D a + D b + D c = D(a + d_1 u c) + D(b + d_2 v c). \quad (8)$$

Remark 3. If D is a very simple domain admitting an *involution* θ , namely, an *anti-automorphism of order two*, i.e., an additive map $\theta : D \rightarrow D$ satisfying $\theta(d_1 d_2) = \theta(d_2) \theta(d_1)$ for all $d_1, d_2 \in D$, and $\theta^2 = \text{id}_D$, then a right analogue of (8) holds, namely:

$$\forall a, b, c \in D, \forall d_1, d_2 \in D \setminus \{0\}, \exists u, v \in D : a D + b D + c D = (a + c u d_1) D + (b + c v d_2) D. \quad (9)$$

Indeed, applying (8) to $\theta(a), \theta(b), \theta(c) \in D$ and $\theta(d_1), \theta(d_2) \in D \setminus \{0\}$, there exist $u', v' \in D$ such that $D \theta(a) + D \theta(b) + D \theta(c) = D(\theta(a) + \theta(d_1) u' \theta(c)) + D(\theta(b) + \theta(d_2) v' \theta(c))$, i.e.,

$$\begin{cases} \theta(a) = \alpha_1 (\theta(a) + \theta(d_1) u' \theta(c)) + \alpha_2 (\theta(b) + \theta(d_2) v' \theta(c)), \\ \theta(b) = \beta_1 (\theta(a) + \theta(d_1) u' \theta(c)) + \beta_2 (\theta(b) + \theta(d_2) v' \theta(c)), \\ \theta(c) = \gamma_1 (\theta(a) + \theta(d_1) u' \theta(c)) + \gamma_2 (\theta(b) + \theta(d_2) v' \theta(c)), \end{cases}$$

for certain $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in D$. Hence, if $u = \theta(u')$ and $v = \theta(v')$, then we obtain

$$\begin{cases} a = (a + c u d_1) \theta(\alpha_1) + (b + c v d_2) \theta(\alpha_2), \\ b = (a + c u d_1) \theta(\beta_1) + (b + c v d_2) \theta(\beta_2), \\ c = (a + c u d_1) \theta(\gamma_1) + (b + c v d_2) \theta(\gamma_2), \end{cases}$$

which proves (9). Finally, we note that the ring $D = A\langle \partial_1, \dots, \partial_n \rangle$ of PD operators admits the standard involution θ defined by:

$$\forall a \in A, \quad \theta(a) = a, \quad \theta(\partial_i) = -\partial_i, \quad i = 1, \dots, n. \quad (10)$$

Theorem 2 ([44]). *The Weyl algebra $A_n(k)$ is a very simple domain.*

The computation of elements u and v defined in (8) is implemented in the STAFFORD package ([37]) based on algorithms developed in [22, 26] for the computation of two generators of left/right ideals generated by three elements.

Example 4. Let us consider $D = A_2(\mathbb{Q})$, $a = \partial_1$, $b = \partial_2$, $c = x_1$, $d_1 = x_2$, and $d_2 = x_1$. If $u = 0$ and $v = 1$, and $a_2 = a + d_1 u c = \partial_1$ and $b_2 = b + d_2 v c = \partial_2 + x_1^2$, then (8) holds, i.e.,

$$\begin{cases} a = a_2, \\ b = \frac{1}{2} x_1 (\partial_2 + x_1^2) a_2 + \frac{1}{2} (-x_1 \partial_1 + 2) b_2, \\ c = -\frac{1}{2} (\partial_2 + x_1^2) a_2 + \frac{1}{2} \partial_1 b_2, \end{cases}$$

which shows that $D a + D b + D c = D a_2 + D b_2$. Finally, using the involution θ of D defined by (10), the above identities yield $\theta(a) D + \theta(b) D + \theta(c) D = \theta(a_2) D + \theta(b_2) D$, where $\theta(a) = \theta(a_2) = -\partial_1$, $\theta(b) = -\partial_2$, $\theta(c) = x_1$, and $\theta(b_2) = -\partial_2 + x_1^2$.

Stafford's theorem, namely, Theorem 2, has recently been extended. Let us briefly review these extensions.

We denote by $\text{Kdim}(D)$ the *Krull dimension* of a noetherian ring D ([31]).

Theorem 3 (Proposition 1.3 of [18]). *If D is a simple noetherian ring of Krull dimension 1, then D is very simple.*

A ring D is *artinian* if every descending chain of left or right ideals of D is stationary (see, e.g., [40]).

Proposition 2 ((ii) of Corollary 6.6.7 of [31]). *Let A be a noetherian differential ring of finite Krull dimension and $D = A\langle\partial\rangle$ the ring of OD operators with coefficients in A . If A is not artinian and D is simple, then $\text{Kdim}(D) = \text{Kdim}(A)$.*

Theorem 4 ([37]). *The ring $A\langle\partial\rangle$ of OD operators with coefficients in the differential ring $A = k[[t]]$ (resp., $k\{t\}$, where $k \in \{\mathbb{R}, \mathbb{C}\}$) is a very simple domain.*

Proof. Let A be either $k[[t]]$ or $k\{t\}$ with $k \in \{\mathbb{R}, \mathbb{C}\}$ in the latter case. The ring A is not artinian because $(t) \supset (t^2) \supset (t^3) \supset \dots$ is an infinite strictly descending chain of ideals, i.e., there exists no $I \in \mathbb{Z}_{>0}$ such that $(t^i) = (t^{i+1})$ for all $i \geq I$. It is also well-known that A is a *local ring*, namely, $\mathfrak{m} := (t)$ is the only *maximal ideal* of A (namely, the only ideal \mathfrak{m} of A such that A/\mathfrak{m} is a field), and $D = A\langle\partial\rangle$ is a simple noetherian domain (see, e.g., [6, 29]).

Now, if A is a local ring, \mathfrak{m} the maximal ideal of A , and $k = A/\mathfrak{m}$ the residue field, then a standard result of commutative algebra asserts that $\text{Kdim}(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. See, e.g., [40], p. 487. Since $\dim_k((t)/(t^2)) = 1$, we get $\text{Kdim}(A) \leq 1$. Moreover, since an integral domain is a field if and only if its Krull dimension is 0 and the integral domain A is not a field, $\text{Kdim}(A) = 1$, which yields $\text{Kdim}(D) = 1$ by Proposition 2 and finally proves the result by Theorem 3. \square

Theorem 5 ([9]). *Let $A = k((x_1, \dots, x_n))$ (resp., $k\{\{x_1, \dots, x_n\}\}$) be the field of fractions of the ring of formal power series (resp., the ring of locally convergent power series) with coefficients in k (resp., $k \in \{\mathbb{R}, \mathbb{C}\}$). Then, the ring $A\langle\partial_1, \dots, \partial_n\rangle$ of PD operators with coefficients in A is a very simple domain.*

A consequence of D being very simple is that the *stable rank* of D is at most two, i.e., $\text{sr}(D) \leq 2$. See, e.g., [18, 31, 37]. This implies the following theorem.

Theorem 6 ([18, 37, 31]). *Let D be a very simple domain and M a finitely generated stably free left D -module. If $\text{rank}_D(M) \geq 2$, then M is free. If $\text{rank}_D(M) = 1$, then M is isomorphic to a left ideal of D which can be generated by two elements.*

In [37], we explained how to compute bases of a finitely generated left/right projective D -module M with $\text{rank}_D(M) \geq \text{sr}(D)$. The corresponding algorithm is implemented in the STAFFORD package ([37]) for $D = A_n(\mathbb{Q})$ (see 4 of Theorem 1). Using Theorems 4 and 5, the same algorithm is valid for $D = A\langle\partial\rangle$, where $A = k[[t]]$ or $A = k\{t\}$ and $k \in \{\mathbb{R}, \mathbb{C}\}$, and for $D = A\langle\partial_1, \dots, \partial_n\rangle$, where $A = k((x_1, \dots, x_n))$ or $A = k\{\{x_1, \dots, x_n\}\}$ with $k \in \{\mathbb{R}, \mathbb{C}\}$. For more details, see [38].

If D is a very simple domain and if we consider the particular case $a = b = 0$, $c = 1$, and $d_1, d_2 \in D \setminus \{0\}$ in (8), then there exist $u, v \in D$ such that $Dd_1u + Dd_2v = D$, i.e., such that $yd_1u + zd_2v = 1$ holds for some $y, z \in D$. We obtain the following corollary, on which the algorithmization of the above mentioned results of Stafford are based.

Corollary 1 ([44]). *Let D be a very simple domain and $d_1, d_2 \in D \setminus \{0\}$. Then, the following inhomogeneous quadratic equation*

$$y_1 d_1 z_1 + y_2 d_2 z_2 = 1 \tag{11}$$

admits a solution $(y_1 y_2 z_1 z_2)^T \in D^4$.

We emphasize that the algorithmization of Stafford's theorems stated in Section 1, which hold for a very simple domain D as we shall see in the sequel, is based on the solvability of (11). Hence, Corollary 1 plays a key role in the constructive versions of Stafford's theorems for very simple domains.

Elements y_1, y_2, z_1 , and z_2 as in Corollary 1 can be computed by the STAFFORD package for $D = A_n(\mathbb{Q})$ ([37]). However, due to involved computations required in the corresponding algorithm, solutions of the quadratic equation (11) can in general only be computed in reasonable time for low order and low degree PD operators d_1 and d_2 .

3.3 Computation of unimodular elements

The next lemma called *Serre's splitting-off theorem* due to its reminiscence of [41], states the existence of a unimodular element in a finitely generated module of sufficiently big rank.

Lemma 5. *Let D be a very simple domain and M a finitely generated left D -module. If $\text{rank}_D(M) \geq 2$, then M contains a unimodular element.*

Proof. Since M is a finitely generated left D -module over a left noetherian domain D , M is finitely presented, i.e., there exists $R \in D^{q \times p}$ such that $M = D^{1 \times p} / (D^{1 \times q} R)$. For more details, see, e.g., [40]. Let $\pi : D^{1 \times p} \rightarrow M$ be the canonical projection onto M .

By $1 \Rightarrow 3$ of Lemma 1, M is not a torsion left D -module, and thus there exists $m_1 \notin t(M)$. Moreover, since M is not a torsion left D -module, $\ker_D(R) \cong \text{hom}_D(M, D) \neq 0$ by $2 \Rightarrow 1$ of Lemma 1. Let $Q \in D^{p \times m}$ be such that $\ker_D(R) = Q D^m$. According to Lemma 3, $m_1 = \pi(\lambda_1)$ has to be chosen so that $\lambda_1 Q \neq 0$.

Using Remark 1, there exists $\mu_1 \in \ker_D(R) = Q D^m$ such that $\varphi_1 := \varphi_{\mu_1}$ satisfies $\varphi_1(m_1) \neq 0$. In particular, $\mu_1 = Q \xi_1$ has to be chosen so that $(\lambda_1 Q) \xi_1 \neq 0$. Then, the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow \cdot \mu_1 & & \downarrow \varphi_1 & & \\ 0 & \longrightarrow & D & \xrightarrow{\text{id}_D} & D & \longrightarrow & 0. \end{array}$$

We can easily check that $\ker \varphi_1 = \ker_D(\cdot \mu_1) / (D^{1 \times q} R)$. For more details, see [13]. Since $0 \neq D \varphi_1(m_1) \subseteq \varphi_1(M) = D^{1 \times p} \mu_1 \subseteq D$ and D is a noetherian domain, $\text{rank}_D(\varphi_1(M)) = 1$. Then, the short exact sequence $0 \rightarrow \ker \varphi_1 \rightarrow M \rightarrow \text{im } \varphi_1 \rightarrow 0$ yields:

$$\text{rank}_D(\ker \varphi_1) = \text{rank}_D(M) - \text{rank}_D(\text{im } \varphi_1) \geq 1.$$

For more details, see [31, 40]. Thus, by $1 \Rightarrow 3$ of Lemma 1, $\ker \varphi_1$ is not a torsion left D -module, and there exists $m_2 \in \ker \varphi_1$ such that $m_2 \notin t(M)$. We choose m_2 as follows. Let $S \in D^{r \times p}$ be such that $\ker_D(\cdot \mu_1) = D^{1 \times r} S$. If $\nu \in D^{1 \times r}$ satisfies $\nu(SQ) \neq 0$ and

$\lambda_2 = \nu S$, then Lemma 3 shows that we can take $m_2 = \pi(\lambda_2)$. Let $\xi_2 \in D^m$ be such that $(\lambda_2 Q) \xi_2 \neq 0$ and define $\mu_2 = Q \xi_2$. Then $\varphi_2 := \varphi_{\mu_2} \in \text{hom}_D(M, D)$ satisfies $\varphi_2(m_2) \neq 0$. Since $m_2 \in \ker \varphi_1$, we have $\varphi_1(m_2) = 0$.

If $\varphi_2(m_1) \neq 0$, then by the right Ore property of D , there exist $r_1, r_2 \in D \setminus \{0\}$ such that:

$$\varphi_1(m_1) r_1 + \varphi_2(m_1) r_2 = 0.$$

Recall that $\text{hom}_D(M, D)$ has a right D -module structure defined by $(\varphi r)(m) = \varphi(m) r$ for all $r \in D$ and for all $m \in M$ (see, e.g., [40]). Then $\varphi'_2 = \varphi_1 r_1 + \varphi_2 r_2 \in \text{hom}_D(M, D)$ satisfies:

$$\begin{cases} \varphi'_2(m_1) = \varphi_1(m_1) r_1 + \varphi_2(m_1) r_2 = 0, \\ \varphi'_2(m_2) = \varphi_1(m_2) r_1 + \varphi_2(m_2) r_2 = \varphi_2(m_2) r_2 \neq 0. \end{cases}$$

Therefore, without loss of generality, we may assume that $\varphi_2(m_1) = 0$. Then, we have:

$$\varphi_1(m_1) \neq 0, \quad \varphi_1(m_2) = 0, \quad \varphi_2(m_1) = 0, \quad \varphi_2(m_2) \neq 0.$$

Applying Corollary 1 to $d_1 := \varphi_1(m_1) \neq 0$ and $d_2 := \varphi_2(m_2) \neq 0$, there exist $y_1, y_2, z_1, z_2 \in D$ such that (11) holds, i.e., $y_1 \varphi_1(m_1) z_1 + y_2 \varphi_2(m_2) z_2 = 1$, or equivalently:

$$y_1 (\lambda_1 \mu_1) z_1 + y_2 (\lambda_2 \mu_2) z_2 = 1.$$

Let $\lambda^* = y_1 \lambda_1 + y_2 \lambda_2 \in D^{1 \times p}$, $m^* = \pi(\lambda^*) = y_1 m_1 + y_2 m_2 \in M$, $\mu^* = \mu_1 z_1 + \mu_2 z_2 \in \ker_D(R.)$, and $\varphi := \varphi_1 z_1 + \varphi_2 z_2 = \varphi_{\mu^*} \in \text{hom}_D(M, D)$. Then,

$$\begin{aligned} \varphi(m^*) &= (\varphi_1 z_1 + \varphi_2 z_2)(m^*) = \varphi_1(m^*) z_1 + \varphi_2(m^*) z_2 \\ &= (y_1 \varphi_1(m_1) + y_2 \varphi_1(m_2)) z_1 + (y_1 \varphi_2(m_1) + y_2 \varphi_2(m_2)) z_2 \\ &= y_1 \varphi_1(m_1) z_1 + y_2 \varphi_2(m_2) z_2 = 1, \end{aligned}$$

which shows that $m^* \in U(M)$. As explained at the beginning of Section 3.1, we then get $M = D m^* \oplus \ker \varphi \cong D \oplus \ker \varphi$, where $\ker \varphi = \ker_D(\cdot, \mu^*) / (D^{1 \times q} R)$ is a left D -submodule of M with $\text{rank}_D(\ker \varphi) = \text{rank}_D(M) - 1$. \square

We summarize the procedure indicated in the above proof in the following algorithm.

Algorithm 1 (UnimodularElement).

- **Input:** A very simple domain D and $R \in D^{q \times p}$ such that the finitely presented left D -module $M = D^{1 \times p} / (D^{1 \times q} R)$ has rank at least two, i.e., $\text{rank}_D(M) \geq 2$.
- **Output:** $\lambda^* \in D^{1 \times p}$ and $\mu^* \in \ker_D(R.)$ such that we have $m^* = \pi(\lambda^*) \in U(M)$ and:

$$\varphi_{\mu^*}(m^*) = 1.$$

1. Compute $Q \in D^{p \times m}$ such that $\ker_D(R.) = Q D^m$.
2. Pick $\lambda_1 \in D^{1 \times p}$ such that $\lambda_1 Q \neq 0$.
3. Find $\xi_1 \in D^m$ such that $(\lambda_1 Q) \xi_1 \neq 0$ and define $\mu_1 = Q \xi_1 \in \ker_D(R.)$.

4. Compute $S \in D^{r \times p}$ such that $\ker_D(\cdot \mu_1) = D^{1 \times r} S$.
5. Pick $\nu \in D^{1 \times r}$ such that $\nu(SQ) \neq 0$ and define $\lambda_2 = \nu S$.
6. Find $\xi_2 \in D^m$ such that $(\lambda_2 Q)\xi_2 \neq 0$ and define $\mu_2 = Q\xi_2 \in \ker_D(R)$.
7. If $\lambda_1 \mu_2 \neq 0$, then compute $r_1, r_2 \in D \setminus \{0\}$ such that $(\lambda_1 \mu_1)r_1 + (\lambda_1 \mu_2)r_2 = 0$ and replace μ_2 by $\mu_1 r_1 + \mu_2 r_2$.
8. Compute $y_1, y_2, z_1, z_2 \in D$ such that $y_1(\lambda_1 \mu_1)z_1 + y_2(\lambda_2 \mu_2)z_2 = 1$.
9. Return $\lambda^* = y_1 \lambda_1 + y_2 \lambda_2 \in D^{1 \times p}$ and $\mu^* = \mu_1 z_1 + \mu_2 z_2 \in \ker_D(R)$.

Steps 2 and 3 of Algorithm 1 can be replaced by selection of a non-zero entry of Q and defining λ_1 and ξ_1 as the row and column standard basis vector, respectively, which corresponds to the chosen row and column. But, in special cases, it is worth considering a vector λ_1 (resp., ξ_1) not occurring in the standard basis of $D^{1 \times p}$ (resp., D^m), as if instance $\lambda_1 Q$ (resp., $Q\xi_1$) is a unit of D (see Examples 2 and 3).

Lemma 5 yields the following important theorem.

Theorem 7 ([44]). *Let D be a very simple domain and M a finitely generated left D -module. Then there exists $r \in \mathbb{Z}_{\geq 0}$ and a finitely generated left D -module M' with $\text{rank}_D(M') \leq 1$ such that:*

$$M \cong D^{1 \times r} \oplus M'.$$

If M is torsion-free, then M' can be chosen as a left ideal of D (which can be generated by two elements).

Proof. If $\text{rank}_D(M) \leq 1$, then the result trivially holds with $r = 0$ and $M' = M$.

Let us now suppose that $\text{rank}_D(M) \geq 2$. Using Lemma 5, we obtain $M \cong D^{1 \times r} \oplus M'$, where $\text{rank}_D(M') \leq 1$.

Assuming $t(M) = 0$, we have $t(M') = 0$, and thus M' is a torsion-free left D -module of rank at most one. Using the concept of *minimal parametrization* ([11]), M' is then isomorphic to a left ideal of D . Indeed, if $M' = D^{1 \times l} / (D^{1 \times m} U)$ and $V \in D^l$ is a non-zero vector such that $V \in \ker_D(U)$, then Theorem 8 of [11] shows that M' is isomorphic to the left ideal of D generated by the entries of V . Since D is a very simple domain, M' can be generated by two elements by 1 of Lemma 4. \square

In the context of Theorem 7, let us compute a finite presentation R' of the kernel of $\varphi = \varphi_{\mu_1} z_1 + \varphi_{\mu_2} z_2$ and an isomorphism between M and $D \oplus L$, where L is the cokernel of the presentation matrix R' . This isomorphism is used in order to iterate Algorithm 1 and to explicitly describe the left D -isomorphism $M \cong D^{1 \times r} \oplus M'$.

Let $T \in D^{s \times p}$ be such that $\ker_D(\cdot \mu^*) = D^{1 \times s} T$. Thus, $\ker \varphi = (D^{1 \times s} T) / (D^{1 \times q} R)$. Moreover, let $T_2 \in D^{t \times s}$ be such that $\ker_D(T) = D^{1 \times t} T_2$, $C \in D^{q \times s}$ such that $R = CT$ and $R' := (C^T \ T_2^T)^T$. We define $O := D^{1 \times s} / (D^{1 \times (q+t)} R')$ and the canonical projection $\kappa : D^{1 \times s} \rightarrow O$ onto O . Then, Proposition 1 yields the following left D -isomorphism:

$$\begin{aligned} \phi : \ker \varphi &\longrightarrow O \\ \pi(\gamma T) &\longmapsto \kappa(\gamma). \end{aligned} \tag{12}$$

If $Z := (I_q \ 0)^T \in D^{(q+t) \times q}$, then we have

$$R'T = ZR, \quad (13)$$

which yields the following commutative diagram with exact rows and columns, where $\varepsilon : \ker \varphi \rightarrow M$ is the canonical injection:

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \downarrow & & \\
D^{1 \times (q+t)} & \xrightarrow{\cdot R'} & D^{1 \times s} & \xrightarrow{\cdot \kappa} & O & \longrightarrow & 0 \\
\downarrow \cdot Z & & \downarrow \cdot T & & \downarrow \varepsilon \circ \phi^{-1} & & \\
D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\cdot \pi} & M & \longrightarrow & 0 \\
\downarrow & & \downarrow \cdot \mu^* & & \downarrow \varphi & & \\
0 & \longrightarrow & D & \xrightarrow{\text{id}} & D & \longrightarrow & 0. \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Remark 4. We note that the short exact sequence $0 \rightarrow \ker_D(\cdot \mu^*) \rightarrow D^{1 \times p} \xrightarrow{\cdot \mu^*} D \rightarrow 0$ splits (see the paragraph after (4)), i.e., $D^{1 \times p} \cong D \oplus \ker_D(\cdot \mu^*)$, which shows that $\ker_D(\cdot \mu^*)$ is a stably free left D -module of rank $p-1$ (see Definition 1). If $\ker_D(\cdot \mu^*)$ is a free left D -module (e.g., if $D = A_n(k)$ and $p \geq 3$ by 4 of Theorem 1), then there exists a full row rank matrix $T \in D^{(p-1) \times p}$ (i.e., $\ker_D(\cdot T) = 0$) such that $\ker_D(\cdot \mu^*) = D^{1 \times (p-1)} T$, which yields $s = p-1$, $t = 0$, and $R' = C$ in the above diagram.

Now, using the split short exact sequence (4) and the decomposition (5), we get the following left D -isomorphism:

$$\begin{aligned}
\iota_1 : M &\longrightarrow D \oplus \ker \varphi \\
m = \pi(\lambda) &\longmapsto (\varphi(m), m - \varphi(m) m^*) = (\lambda \mu^*, \pi(\lambda (I_p - \mu^* \lambda^*))).
\end{aligned} \quad (14)$$

Since we have $\varphi(m - \varphi(m) m^*) = \lambda (I_p - \mu^* \lambda^*) \mu^* = 0$ for all $\lambda \in D^{1 \times p}$, $m = \pi(\lambda)$, and $\ker_D(\cdot \mu^*) = D^{1 \times s} T$, there exists $G \in D^{p \times s}$ such that:

$$I_p - \mu^* \lambda^* = GT. \quad (15)$$

In other words, we have the following split exact sequence:

$$D^{1 \times s} \begin{array}{c} \xrightarrow{\cdot T} \\ \xleftarrow{\cdot G} \end{array} D^{1 \times p} \begin{array}{c} \xrightarrow{\cdot \mu^*} \\ \xleftarrow{\cdot \lambda^*} \end{array} D \longrightarrow 0.$$

Now, using (12), we obtain the following left D -isomorphism

$$\begin{aligned}
\iota_2 : D \oplus \ker \varphi &\longrightarrow D \oplus O \\
(d, \pi(\gamma T)) &\longmapsto (d, \kappa(\gamma)),
\end{aligned}$$

which combined with (14) (using (15)) yields the following left D -isomorphism:

$$\begin{aligned}
\iota := \iota_2 \circ \iota_1 : M &\longrightarrow D \oplus O \\
m = \pi(\lambda) &\longmapsto (\lambda \mu^*, \kappa(\lambda G)).
\end{aligned}$$

To characterize ι^{-1} , we first need to derive a few identities.

Multiplying T (resp., λ^*) from the left to (15) and using $T\mu^* = 0$ and $\lambda^*\mu^* = 1$, we get:

$$TGT = T - T\mu^*\lambda^* = T, \quad (16)$$

$$\lambda^*GT = \lambda^* - (\lambda^*\mu^*)\lambda^* = 0. \quad (17)$$

(16) yields $(I_s - TG)T = 0$ and using $\ker_D(.T) = D^{1 \times t}T_2$, there exists $E \in D^{s \times t}$ such that:

$$I_s - TG = ET_2. \quad (18)$$

Combining (17) and $\ker_D(.T) = D^{1 \times t}T_2$, there exists $F \in D^{1 \times t}$ such that:

$$\lambda^*G = FT_2. \quad (19)$$

Now, multiplying R from the left to (15), we obtain $R = RGT$ because $\mu^* \in \ker_D(R)$. Using $R = CT$, we get $CT = RGT$, i.e., $(C - RG)T = 0$ and using $\ker_D(.T) = D^{1 \times t}T_2$, there exists $H \in D^{q \times t}$ such that:

$$C - RG = HT_2. \quad (20)$$

Let us now introduce the following map:

$$\begin{aligned} \varpi : D \oplus O &\longrightarrow M \\ (d, \kappa(\gamma)) &\longmapsto \pi(d\lambda^* + \gamma T). \end{aligned}$$

We can easily check that ϖ is a well-defined left D -homomorphism since, using (13), we have:

$$\begin{aligned} \forall \theta \in D^{1 \times (q+t)}, \quad \varpi((d, \kappa(\gamma + \theta R'))) &= \pi(d\lambda^* + (\gamma + \theta R')T) = \pi(d\lambda^* + \gamma T + (\theta Z)R) \\ &= \pi(d\lambda^* + \gamma T). \end{aligned}$$

Then, using (15), $\lambda^*\mu^* = 1$, $T\mu^* = 0$, (18), and (19), we have

$$\begin{aligned} (\varpi \circ \iota)(\pi(\lambda)) &= \varpi((\lambda\mu^*, \kappa(\lambda G))) = \pi(\lambda(\mu^*\lambda^* + GT)) = \pi(\lambda), \\ (\iota \circ \varpi)((d, \kappa(\gamma))) &= \iota(\pi(d\lambda^* + \gamma T)) = ((d\lambda^* + \gamma T)\mu^*, \kappa((d\lambda^* + \gamma T)G)) \\ &= (d, \kappa(dFT_2 + \gamma(I_s - ET_2))) = (d, \kappa(\gamma + (dF - \gamma E)T_2)) = (d, \kappa(\gamma)), \end{aligned}$$

which finally shows that $\iota^{-1} = \varpi$.

Let $\bar{R} = (0 \quad R') \in D^{(q+t) \times (1+s)}$, $\bar{M} = D^{1 \times (1+s)} / (D^{1 \times (q+t)}\bar{R})$ be the left D -module finitely presented by \bar{R} , and $\bar{\pi} = \text{id}_D \oplus \kappa : D^{1 \times (1+s)} \longrightarrow \bar{M}$ the canonical projection onto \bar{M} . We get $\bar{M} \cong D \oplus O \cong M$. If $P_1 := (\mu^* \quad G) \in D^{p \times (1+s)}$ and $P_2 := (I_q \quad -H) \in D^{q \times (q+t)}$, then using (20), we get $RP_1 = P_2\bar{R}$, which yields the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow \cdot P_2 & & \downarrow \cdot P_1 & & \downarrow f & & \\ D^{1 \times (q+t)} & \xrightarrow{\cdot \bar{R}} & D^{1 \times (1+s)} & \xrightarrow{\bar{\pi}} & \bar{M} & \longrightarrow & 0. \end{array}$$

Hence, the left D -isomorphism f is defined by:

$$\begin{aligned} f : M &\longrightarrow \bar{M} \\ \pi(\lambda) &\longmapsto \bar{\pi}(\lambda P_1) = (\lambda\mu^*, \kappa(\lambda G)). \end{aligned}$$

To finish, let us compute f^{-1} . Using the identity $R = CT$, we have:

$$\bar{R} \begin{pmatrix} \lambda^* \\ T \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} \lambda^* \\ T \end{pmatrix} = \begin{pmatrix} CT \\ 0 \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{pmatrix} I_q \\ 0 \end{pmatrix} R = ZR.$$

Hence, if $P'_1 := (\lambda^{*T} \ T^T)^T \in D^{(1+s) \times p}$ and $P'_2 := Z \in D^{(q+t) \times q}$, then we have $\bar{R}P'_1 = P'_2R$, which yields the following commutative diagram with exact rows

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & \uparrow \cdot P'_2 & \uparrow \cdot P'_1 & & & & \\ D^{1 \times (q+t)} & \xrightarrow{\cdot \bar{R}} & D^{1 \times (1+s)} & \xrightarrow{\bar{\pi}} & \bar{M} & \longrightarrow & 0, \end{array}$$

and the following left D -homomorphism:

$$\begin{aligned} g : \bar{M} &\longrightarrow M \\ \bar{\pi}((\theta_1 \ \theta_2)) &\longmapsto \pi((\theta_1 \ \theta_2) P'_1) = \pi(\theta_1 \lambda^* + \theta_2 T). \end{aligned}$$

Using (15), $\lambda^* \mu^* = 1$, (19), $T \mu^* = 0$, (18), and (20), we can easily check that

$$\begin{cases} P_1 P'_1 &= \mu^* \lambda^* + GT = I_p, \\ P'_1 P_1 &= \begin{pmatrix} \lambda^* \mu^* & \lambda^* G \\ T \mu^* & TG \end{pmatrix} = \begin{pmatrix} 1 & FT_2 \\ 0 & I_s - ET_2 \end{pmatrix} = I_{s+1} + \begin{pmatrix} 0 & F \\ 0 & -E \end{pmatrix} \bar{R}, \end{cases}$$

which yields $g \circ f = \text{id}_M$ and $f \circ g = \text{id}_{\bar{M}}$, and finally shows that $g = f^{-1}$.

Example 5. Let $D = A_3(\mathbb{Q})$ and $M = D^{1 \times 3}/(DR)$ be the left D -module finitely presented by the *divergence operator* in \mathbb{R}^3 , namely, $R = (\partial_1 \ \partial_2 \ \partial_3)$. Then, $\text{rank}_D(M) = 2$, which shows that there exists a non-trivial decomposition of M as in Theorem 7. We can check that $\ker_D(R.) = QD^3$, where Q is the matrix of PD operators defining the *curl operator*, namely:

$$Q = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}. \quad (21)$$

Since the entries of Q are not units of D and none of the rows (resp., columns) of Q admits a right inverse (resp., left inverse) over D , there is no easy way to detect unimodular elements of M . Then, let us apply Algorithm 1 to M . If we consider $\lambda_1 = (0 \ -1 \ 0)$, then $\lambda_1 Q = (-\partial_3 \ 0 \ \partial_1) \neq 0$. Now, taking $\xi_1 = (0 \ 0 \ 1)^T$, then $\mu_1 = Q \xi_1 = (\partial_2 \ -\partial_1 \ 0)^T$ and $d_1 := \lambda_1 \mu_1 = \partial_1$. Now, $\ker_D(\cdot \mu_1) = D^{1 \times 2} S$, where S and thus SQ are defined by:

$$S = \begin{pmatrix} \partial_1 & \partial_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad SQ = \begin{pmatrix} \partial_2 \partial_3 & -\partial_1 \partial_3 & 0 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}.$$

Then, considering $\nu = (0 \ 1)$, we get $\nu(SQ) \neq 0$ and $\lambda_2 = \nu S = (0 \ 0 \ 1)$. The choice $\xi_2 = (0 \ 1 \ 0)^T$ yields $\mu_2 = Q \xi_2 = (-\partial_3 \ 0 \ \partial_1)^T$ and $d_2 := \lambda_2 \mu_2 = \partial_1$. Moreover, $\lambda_1 \mu_2 = 0$. Computing a solution $(y_1 \ y_2 \ z_1 \ z_2)^T \in D^4$ of $y_1 d_1 z_1 + y_2 d_2 z_2 = 1$, we get:

$$y_1 = -x_1 - 1, \quad y_2 = 1, \quad z_1 = 1, \quad z_2 = x_1 + 1.$$

Thus, $\lambda^* = y_1 \lambda_1 + y_2 \lambda_2 = (0 \quad x_1 + 1 \quad 1)$ defines the unimodular element $m^* = \pi(\lambda^*)$ of M . Moreover, if $\mu^* = \mu_1 z_1 + \mu_2 z_2 = (\partial_2 - (x_1 + 1) \partial_3 \quad -\partial_1 \quad (x_1 + 1) \partial_1 + 1)^T$, then $\varphi_{\mu^*}(m^*) = 1$. Hence, $M = D m^* \oplus \ker \varphi_{\mu^*}$, and $\ker \varphi_{\mu^*} = \ker_D(\cdot \mu^*) / (D R)$. Remark 4 shows that $\ker_D(\cdot \mu^*)$ is a free left D -module of rank 2 and, computing a basis, we get $\ker_D(\cdot \mu^*) = D^{1 \times 2} T$, where:

$$T = \begin{pmatrix} 1 & -(x_1 + 1) (\partial_2 - (x_1 + 1) \partial_3) & (x_1 + 1) \partial_3 - \partial_2 \\ 0 & (x_1 + 1) \partial_1 + 2 & \partial_1 \end{pmatrix}.$$

The matrix T has full row rank, i.e., $\ker_D(\cdot T) = 0$. Let C be such that $R = C T$, namely:

$$C = (\partial_1 \quad \partial_2 - (x_1 + 1) \partial_3).$$

Then, $\ker \varphi_{\mu^*} \cong O := D^{1 \times 2} / (D C)$, which shows that $M \cong D \oplus O$. Let us give an explicit isomorphism. If $\bar{R} = (0 \quad C) \in D^{1 \times 3}$, $\bar{M} = D^{1 \times 3} / (D \bar{R})$, and $\bar{\pi} : D^{1 \times 3} \rightarrow \bar{M}$ the canonical projection onto \bar{M} , $P'_1 = (\lambda^{*T} \quad T^T)^T \in D^{3 \times 3}$, then $\bar{R} P'_1 = C T = R$, which shows that the following diagram is commutative with exact rows

$$\begin{array}{ccccccc} D & \xrightarrow{\cdot R} & D^{1 \times 3} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \uparrow \text{id}_D & & \uparrow P'_1 & & \uparrow g & & \\ D & \xrightarrow{\cdot \bar{R}} & D^{1 \times 3} & \xrightarrow{\bar{\pi}} & \bar{M} & \longrightarrow & 0, \end{array}$$

where $g(\bar{\pi}(\theta)) = \pi(\theta P'_1)$ for all $\theta \in D^{1 \times 3}$. Now, since $P'_1 \in \text{GL}_3(D)$, g is a D -isomorphism and $f = g^{-1}$ is defined by $f(\pi(\lambda)) = \bar{\pi}(\lambda P_1)$ for all $\lambda \in D^{1 \times 3}$, where:

$$P_1 := P'_1{}^{-1} = \begin{pmatrix} \partial_2 - (x_1 + 1) \partial_3 & 1 & 0 \\ -\partial_1 & 0 & 1 \\ (x_1 + 1) \partial_1 + 1 & 0 & -(x_1 + 1) \end{pmatrix}.$$

In other words, $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$ is equivalent to $\partial_1 w_1 + (\partial_2 - (x_1 + 1) \partial_3) w_2 = 0$ (and no condition on v) under the following invertible transformations:

$$\Leftrightarrow \begin{cases} v = (x_1 + 1) u_2 + u_3, \\ w_1 = u_1 - (x_1 + 1) (\partial_2 - (x_1 + 1) \partial_3) u_2 + ((x_1 + 1) \partial_3 - \partial_2) u_3, \\ w_2 = ((x_1 + 1) \partial_1 + 2) u_2 + \partial_1 u_3, \\ u_1 = (\partial_2 - (x_1 + 1) \partial_3) v + w_1, \\ u_2 = -\partial_1 v + w_2, \\ u_3 = ((x_1 + 1) \partial_1 + 1) v - (x_1 + 1) w_2. \end{cases}$$

Now, since M is torsion-free (see, e.g., [11]), so is O . Then, computing a minimal parametrization of O (see [11]), O is isomorphic to the left ideal $I = D \partial_1^2 + D((x_1 + 1) \partial_1 \partial_3 - \partial_1 \partial_2 - \partial_3)$ generated by two elements of D , which finally shows that $M \cong D \oplus I$.

Remark 5. Let us give system-theoretical applications of Theorem 7. Let M be a finitely generated left D -module with $\text{rank}_D(M) \geq 2$. Then, according to Theorem 7, there exist $r \in \mathbb{Z}_{\geq 0}$ and a left D -module M' with $\text{rank}_D(M') \leq 1$ such that $M \cong D^{1 \times r} \oplus M'$. If \mathcal{F} is a left D -module, then using the *additivity* of the functor $\text{hom}_D(\cdot, \mathcal{F})$ (see, e.g., [40]), we get:

$$\text{hom}_D(M, \mathcal{F}) \cong \text{hom}_D(D^{1 \times r}, \mathcal{F}) \oplus \text{hom}_D(M', \mathcal{F}) \cong \mathcal{F}^r \oplus \text{hom}_D(M', \mathcal{F}). \quad (22)$$

Significant information on the solution space $\text{hom}_D(M, \mathcal{F})$ is then contained in $\text{hom}_D(M', \mathcal{F})$. We refer to [11] for the relevance of the *extension functor* $\text{ext}_D^i(\cdot, \mathcal{F})$ in mathematical systems theory. More generally, since $\text{ext}_D^i(D^{1 \times r}, \mathcal{F}) = 0$ for $i \geq 1$ (see, e.g., [40]), we then have:

$$\forall i \geq 1, \quad \text{ext}_D^i(M, \mathcal{F}) \cong \text{ext}_D^i(M', \mathcal{F}). \quad (23)$$

A more general version of Theorem 7, namely, a *relative version* where M is replaced by $M \subseteq N$ or, more generally, by an injective homomorphism $\iota : M \rightarrow N$, is now given.

Theorem 8 ([44]). *Let M and N be two finitely generated left D -modules satisfying $M \subseteq N$ and $\text{rank}_D(M) \geq 2$. Then, there exists $m^* \in M$ which is a unimodular element of N . Hence, there exist a left D -submodule M' (resp., N') of M (resp., N) such that $M' = M \cap N'$ and:*

$$M = D m^* \oplus M' \subseteq N = D m^* \oplus N'.$$

Proof. Let us consider a presentation of the finitely generated left D -module M (resp., N) $M = D^{1 \times p} / (D^{1 \times q} R)$ (resp., $N = D^{1 \times p'} / (D^{1 \times q'} R')$), where $R \in D^{q \times p}$ (resp., $R' \in D^{q' \times p'}$). Let $\iota \in \text{hom}_D(M, N)$ be the injection of M into N . Firstly, $\iota \in \text{hom}_D(M, N)$ is defined by the following commutative diagram with exact rows

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow \cdot P' & & \downarrow \iota & & \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & N & \longrightarrow & 0, \end{array} \quad (24)$$

i.e., $\iota(\pi(\eta)) = \pi'(\eta P)$ for all $\eta \in D^{1 \times p}$, where $P \in D^{p \times p'}$ is such that $R P = P' R'$ for a certain $P' \in D^{q \times q'}$ and $\pi' : D^{1 \times p'} \rightarrow N$ is the canonical projection onto N (see [13]). Secondly, the injectivity of ι is equivalent to the fact that for all $S \in D^{s \times p}$ and for all $T \in D^{s \times q'}$ satisfying $S P = T R'$, there exists $L \in D^{s \times q}$ such that $S = L R$. For more details, see [13]. Finally, we have $\iota(M) = (D^{1 \times (p+q')} (P^T \quad R'^T)^T) / (D^{1 \times q'} R') \subseteq N = D^{1 \times p'} / (D^{1 \times q'} R')$ (see [13]).

Theorem 8 follows similar arguments as the ones used in the proof of Lemma 5. Let us shortly adapt the arguments of the proof of Lemma 5 to this more general situation. We note that Theorem 7 is a particular case of Theorem 8 where $N = M$ and $\iota = \text{id}_M$.

The hypothesis $\text{rank}_D(M) \geq 2$ implies that M is not torsion by 1 \Rightarrow 3 of Lemma 1. Then, there exists $m_1 = \pi(\eta_1) \notin t(M)$. Thus, if $\lambda_1 = \eta_1 P$, then $\iota(m_1) = \pi'(\lambda_1) \notin t(N)$. If $Q \in D^{p \times m}$ (resp., $Q' \in D^{p' \times m'}$) is such that $\ker_D(R.) = Q D^m$ (resp., $\ker_D(R'.) = Q' D^{m'}$), then $\lambda_1 \in D^{1 \times p}$ has to be chosen so that $\lambda_1 Q \neq 0$ (or such that $\lambda_1 P Q' \neq 0$). Now, Remark 1 shows that there exists $\mu_1 \in \ker_D(R'.) = Q' D^{m'}$ such that $\varphi_1 := \varphi_{\mu_1}$ satisfies $\varphi_1(\iota(m_1)) \neq 0$. More precisely, we have to choose $\xi_1 \in D^{m'}$ such that $\mu_1 = Q' \xi_1$ satisfies:

$$\varphi_1(\iota(m_1)) = \lambda_1 \mu_1 = (\lambda_1 Q') \xi_1 \neq 0.$$

Then, the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow \cdot (P \mu_1) & & \downarrow \varphi_1 \circ \iota & & \\ 0 & \longrightarrow & D & \xrightarrow{\text{id}} & D & \longrightarrow & 0. \end{array}$$

Since $\text{im}(\varphi_1 \circ \iota)$ is a left ideal of D containing $(\varphi_1 \circ \iota)(m_1) \neq 0$, $\text{rank}_D(\text{im}(\varphi_1 \circ \iota)) = 1$, which combined with the short exact sequence $0 \rightarrow \ker(\varphi_1 \circ \iota) \rightarrow M \rightarrow \text{im}(\varphi_1 \circ \iota) \rightarrow 0$ yields

$\text{rank}_D(\ker(\varphi_1 \circ \iota)) = \text{rank}_D(M) - 1 \geq 1$. Thus, $\ker(\varphi_1 \circ \iota)$ is not torsion by **1** \Rightarrow **3** of Lemma **1**. Since $t(\ker(\varphi_1 \circ \iota)) = t(M) \cap \ker(\varphi_1 \circ \iota)$, there exists $m_2 \in \ker(\varphi_1 \circ \iota)$ such that $m_2 \notin t(M)$, and thus $\iota(m_2) \notin t(N)$. Let $S \in D^{r \times p}$ be such that $\ker_D(\cdot(P \mu_1)) = D^{1 \times r} S$. Then, we have

$$\ker(\varphi_1 \circ \iota) = \ker_D(\cdot(P \mu_1)) / (D^{1 \times q} R) = (D^{1 \times r} S) / (D^{1 \times q} R),$$

and thus $m_2 = \pi(\eta_2)$ for some $\eta_2 = \nu S \in D^{1 \times p}$, where $\nu \in D^{1 \times r}$ has to be chosen so that $\nu(S P Q') \neq 0$. Define $\lambda_2 = \eta_2 P$. Considering $\xi_2 \in D^{m'}$ such that $(\lambda_2 Q') \xi_2 \neq 0$ and $\mu_2 := Q' \xi_2 \in D^{p'}$, then $\varphi_2 := \varphi_{\mu_2}$ satisfies $\varphi_2(\iota(m_2)) = \lambda_2 \mu_2 \neq 0$.

By construction, $m_2 \in \ker(\varphi_1 \circ \iota)$, which yields $\varphi_1(\iota(m_2)) = \lambda_2 \mu_1 = 0$. Now, if $\varphi_2(\iota(m_1)) = \lambda_1 \mu_2 \neq 0$, then by the right Ore property of D , there exist $r_1, r_2 \in D \setminus \{0\}$ such that $(\lambda_1 \mu_1) r_1 + (\lambda_1 \mu_2) r_2 = 0$. Let $\mu'_2 := \mu_1 r_1 + \mu_2 r_2 \in \ker_D(R')$ and define $\varphi'_2 := \varphi_{\mu_1} r_1 + \varphi_{\mu_2} r_2 = \varphi_{\mu'_2} \in \text{hom}_D(N, D)$. Then, we have:

$$\begin{cases} \varphi'_2(\iota(m_1)) = \lambda_1 \mu'_2 = \lambda_1 (\mu_1 r_1 + \mu_2 r_2) = 0, \\ \varphi'_2(\iota(m_2)) = \lambda_2 \mu'_2 = \lambda_2 (\mu_1 r_1 + \mu_2 r_2) = (\lambda_2 \mu_2) r_2 \neq 0. \end{cases}$$

Therefore, without loss of generality, we may assume that $\varphi_2(\iota(m_1)) = 0$.

Let $d_1 := \lambda_1 \mu_1 \neq 0$ and $d_2 := \lambda_2 \mu_2 \neq 0$. Then, Corollary **1** shows that there exists $(y_1 \ y_2 \ z_1 \ z_2)^T \in D^4$ satisfying $y_1 (\lambda_1 \mu_1) z_1 + y_2 (\lambda_2 \mu_2) z_2 = 1$. If we define

$$\begin{aligned} \eta^* &:= y_1 \eta_1 + y_2 \eta_2, & m^* &:= \pi(\eta^*) = y_1 \pi(\eta_1) + y_2 \pi(\eta_2) \in M, \\ \mu^* &:= \mu_1 z_1 + \mu_2 z_2 \in \ker_D(R'), & \varphi &:= \varphi_{\mu^*} \in \text{hom}_D(N, D), \end{aligned}$$

then we have:

$$\begin{aligned} \varphi(\iota(m^*)) &= \eta^* P \mu^* = (y_1 \eta_1 + y_2 \eta_2) P (\mu_1 z_1 + \mu_2 z_2) = (y_1 \lambda_1 + y_2 \lambda_2) (\mu_1 z_1 + \mu_2 z_2) \\ &= y_1 (\lambda_1 \mu_1) z_1 + y_1 (\lambda_1 \mu_2) z_2 + y_2 (\lambda_2 \mu_1) z_1 + y_2 (\lambda_2 \mu_2) z_2 \\ &= y_1 (\lambda_1 \mu_1) z_1 + y_2 (\lambda_2 \mu_2) z_2 = 1. \end{aligned}$$

Thus, $\iota(m^*) \in \iota(M)$ is a unimodular element of N , which yields $N = D \iota(m^*) \oplus \ker \varphi$. Moreover, since $\psi := \varphi|_{\iota(M)} \in \text{hom}_D(\iota(M), D)$ satisfies $\psi(\iota(m^*)) = 1$, $\iota(m^*)$ is also a unimodular element of $\iota(M)$, which shows that $\iota(M) = D \iota(m^*) \oplus \ker \psi$, and finally proves

$$\iota(M) = D \iota(m^*) \oplus M' \subseteq N = D \iota(m^*) \oplus N',$$

where $N' := \ker \varphi$ and $M' := \ker \varphi|_{\iota(M)} = \ker \varphi \cap \iota(M)$. □

Remark 6. We note that a *division ring* D is characterized by the result of Theorem **8** for finitely generated left D -modules $M \subseteq N$ if the condition $\text{rank}_D(M) \geq 2$ is dropped. Indeed, Theorem **8** holds for non-zero finitely generated D -vector spaces $M \subseteq N$.

A unimodular element, whose existence is proved in Theorem **8**, can be computed by the following algorithm.

Algorithm 2 (UnimodularElementInSubmodule).

- **Input:** A very simple domain D and $R \in D^{q \times p}$ such that the finitely presented left D -module $M = D^{1 \times p} / (D^{1 \times q} R)$ has rank at least two, i.e., $\text{rank}_D(M) \geq 2$, a matrix $R' \in D^{q' \times p'}$ presenting a left D -module $N = D^{1 \times p'} / (D^{1 \times q'} R')$, and a matrix $P \in D^{p \times p'}$ defining an injective homomorphism $\iota \in \text{hom}_D(M, N)$, i.e., $\iota(\pi(\eta)) = \pi'(\eta P)$ for all $\eta \in D^{1 \times p}$, where π (resp., π') is the canonical projection onto M (resp., N).
 - **Output:** $\eta^* \in D^{1 \times p}$ and $\mu^* \in \ker_D(R')$ such that $\iota(\pi(\eta^*)) = \pi'(\eta^* P) \in U(N)$ and $\varphi_{\mu^*} \in \text{hom}_D(N, D)$ such that $\varphi_{\mu^*}(\iota(\pi(\eta^*))) = 1$.
1. Compute $Q' \in D^{p' \times m'}$ such that $\ker_D(R') = Q' D^{m'}$.
 2. Pick $\eta_1 \in D^{1 \times p}$ such that $\eta_1 (P Q') \neq 0$ and define $\lambda_1 = \eta_1 P$.
 3. Find $\xi_1 \in D^{m'}$ such that $(\lambda_1 Q') \xi_1 \neq 0$ and define $\mu_1 = Q' \xi_1 \in \ker_D(R')$.
 4. Compute $S \in D^{r \times p}$ such that $\ker_D((P \mu_1)) = D^{1 \times r} S$.
 5. Pick $\nu \in D^{1 \times r}$ such that $\nu (S P Q') \neq 0$ and define $\eta_2 = \nu S$ and $\lambda_2 = \eta_2 P$.
 6. Find $\xi_2 \in D^{m'}$ such that $(\lambda_2 Q') \xi_2 \neq 0$ and define $\mu_2 = Q' \xi_2 \in \ker_D(R')$.
 7. If $\lambda_1 \mu_2 \neq 0$, then compute $r_1, r_2 \in D \setminus \{0\}$ such that $(\lambda_1 \mu_1) r_1 + (\lambda_1 \mu_2) r_2 = 0$ and replace μ_2 by $\mu_1 r_1 + \mu_2 r_2$.
 8. Compute $y_1, y_2, z_1, z_2 \in D$ such that $y_1 (\lambda_1 \mu_1) z_1 + y_2 (\lambda_2 \mu_2) z_2 = 1$.
 9. Return $\eta^* = y_1 \eta_1 + y_2 \eta_2 \in D^{1 \times p}$ and $\mu^* = \mu_1 z_1 + \mu_2 z_2 \in \ker_D(R')$.

Example 6. Let $D = A_1(\mathbb{Q})$ and $M = D^{1 \times 3} / (D R)$ and $N = D^{1 \times 6} / (D^{1 \times 3} R')$ be left D -modules finitely presented respectively by:

$$R = (\partial \quad 0 \quad -t), \quad R' = \begin{pmatrix} \partial & -t & 0 & 0 & 0 & -1 \\ 0 & \partial & 0 & -t & 0 & 0 \\ 0 & 0 & \partial & 0 & -t & 0 \end{pmatrix}.$$

Since R and R' have full row rank, we have $\text{rank}_D(M) = 2$ and $\text{rank}_D(N) = 3$. Now, if

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad P' = (0 \quad 0 \quad 1),$$

then we have $RP = P'R'$, i.e., the diagram (24) is commutative with exact rows, where $\iota \in \text{hom}_D(M, N)$ is defined by $\iota(\pi(\eta)) = \pi'(\eta P)$ for all $\eta \in D^{1 \times 3}$, and $\pi : D^{1 \times 3} \rightarrow M$ (resp., $\pi' : D^{1 \times 6} \rightarrow N$) is the canonical projection onto M (resp., N). Then, ι is injective because:

$$\ker_D((P^T \quad R'^T)^T) = D (S \quad -T), \quad S = (\partial \quad 0 \quad -t) = R, \quad T = (0 \quad 0 \quad 1).$$

We can check that $\ker_D(R') = Q' D^5$, where:

$$Q' = \begin{pmatrix} -1 & t^2 & 0 & 0 & 0 & 0 \\ 0 & t\partial - 1 & -t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t^2 & t\partial - 1 & 0 \\ 0 & \partial^2 & -t\partial - 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t\partial - 2 & \partial^2 & 0 \\ -\partial & 3t & t^3 & 0 & 0 & 0 \end{pmatrix}.$$

If $\eta_1 = (0 \ 1 \ 0)$ and $\xi_1 = (-1 \ 0 \ 0 \ 0 \ 0)^T$, then $\mu_1 = Q' \xi_1 = (1 \ 0 \ 0 \ 0 \ 0 \ \partial)^T$ and $d_1 = \eta_1 P \mu_1 = \partial + 1$. Now, $\ker_D(\cdot(P \mu_1)) = D^{1 \times 2} S$, where:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $\nu = (0 \ 1)$ and $\xi_2 = (0 \ 0 \ 0 \ 0 \ 1)^T$, then $\eta_2 = \nu S = (0 \ 0 \ 1)$, $\mu_2 = Q' \xi_2 = (0 \ 0 \ t \partial - 1 \ 0 \ \partial^2 \ 0)^T$, $d_2 = \eta_2 P \mu_2 = \partial^2$, and $\eta_1 P \mu_2 = 0$. Then, $z_1 = 1$, $z_2 = t + 1$, $y_1 = -(t + 1) \partial^2 - 3 \partial + 1$, and $y_2 = \partial + 1$ are such that $y_1 d_1 z_1 + y_2 d_2 z_2 = 1$, which yields

$$\begin{cases} \eta^* = y_1 \eta_1 + y_2 \eta_2 = (0 & -(t + 1) \partial^2 - 3 \partial + 1 & \partial + 1), \\ \mu^* = \mu_1 z_1 + \mu_2 z_2 = (1 & 0 & t(t + 1) \partial - 1 & 0 & ((t + 1) \partial + 2) \partial & \partial)^T, \end{cases}$$

which satisfy $\eta^* P \mu^* = 1$. Thus, $m' = \pi'(\eta^* P) \in U(N)$ because $\varphi(m') = 1$, where $\varphi := \varphi_{\mu^*}$. Moreover, if $m^* = \pi(\eta^*) \in M$, then $\varphi_{|\iota(M)}(m^*) = 1$, i.e., $\iota(m^*) = \pi'(\eta^* P) = m' \in U(\iota(M))$, which yields $\iota(M) = D m' \oplus M' \subseteq N = D m' \oplus N'$, where $N' = \ker \varphi$ and $M' = \ker \varphi_{|\iota(M)}$.

4 Stafford's reduction

Let us give now an important application of Theorem 8 and Algorithm 2. Let us consider a finitely presented left D -module $L = D^{1 \times p'} / (D^{1 \times p} P)$ and the left D -modules $K = D^{1 \times p} P$ and $N = D^{1 \times p'}$. Let $R \in D^{q \times p}$ (possibly $R = 0$ and $q = 0$) be such that $\ker_D(\cdot P) = D^{1 \times q} R$. Then, $M := D^{1 \times p} / (D^{1 \times q} R) \cong K$ and the injection $\iota : M \rightarrow D^{1 \times p'}$ is defined by

$$\forall \eta \in D^{1 \times p}, \quad \iota(\pi(\eta)) = \eta P,$$

where $\pi : D^{1 \times p} \rightarrow M$ is the canonical projection onto M . Note that $K = \iota(M)$. The following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow \cdot P & & \downarrow \iota & & \\ 0 & \longrightarrow & D^{1 \times p'} & \xrightarrow{\text{id}} & D^{1 \times p'} & \longrightarrow & 0. \end{array}$$

Now, if $\text{rank}_D(K) = \text{rank}_D(M) \geq 2$, then Theorem 8 is applicable. Since $\text{hom}_D(N, D) \cong D^{p'}$, we can take $Q' = I_{p'}$ in Algorithm 2 and we obtain $\eta^* \in D^{1 \times p}$ and $\xi^* \in D^{p'}$ such that $m^* = \pi(\eta^*) \in M$ satisfies that $\iota(m^*) = \eta^* P \in U(D^{1 \times p'})$ and $\varphi := \varphi_{\xi^*}$ satisfies $\varphi(\iota(m^*)) = \eta^* P \xi^* = 1$. If $\lambda^* := \eta^* P \in D^{1 \times p'}$, then we get:

$$D^{1 \times p'} = D \iota(m^*) \oplus \ker \varphi = D \lambda^* \oplus \ker \varphi.$$

Since $\iota(m^*) = \lambda^*$ is also a unimodular element of $K = \iota(M)$, we get

$$K = D^{1 \times p} P = D \iota(m^*) \oplus \ker \varphi_{|K} = D \lambda^* \oplus \ker \varphi_{|K},$$

where $\ker \varphi_{|K} = \ker \varphi \cap K$. Hence, we obtain

$$L = D^{1 \times p'} / (D^{1 \times p} P) = (D \lambda^* \oplus \ker \varphi) / (D \lambda^* \oplus \ker \varphi_{|K}) \cong \ker \varphi / \ker \varphi_{|K}, \quad (25)$$

where $\text{rank}_D(\ker \varphi) = p' - 1$ and $\text{rank}_D(\ker \varphi|_K) = \text{rank}_D(K) - 1$.

Let us now characterize the left D -module $L' := \ker \varphi / \ker \varphi|_K$ and the isomorphism $L \cong L'$ appearing in (25). We have $\ker \varphi = \ker_D(\cdot \xi^*) = D^{1 \times r} X$ for a certain $X \in D^{r \times p'}$, and $\ker \varphi|_K = \{\eta P \mid \eta \in D^{1 \times p} : (\eta P) \xi^* = 0\} = \ker_D(\cdot (P \xi^*)) P$. If $Y \in D^{s \times p}$ is such that $\ker_D(\cdot (P \xi^*)) = D^{1 \times s} Y$ and $Z = Y P \in D^{s \times p'}$, then $\ker \varphi|_K = D^{1 \times s} Z$, which yields:

$$L' = (D^{1 \times r} X) / (D^{1 \times s} Z).$$

We note that $\text{rank}_D(D^{1 \times r} X) = p' - 1$ and $\text{rank}_D(D^{1 \times s} Z) = \text{rank}_D(K) - 1$.

If $F \in D^{s \times r}$ (resp., $X_2 \in D^{t \times r}$) is such that $Z = F X$ (resp., $\ker_D(\cdot X) = D^{1 \times t} X_2$), then Proposition 1 shows that:

$$L' = (D^{1 \times r} X) / (D^{1 \times s} Z) \cong L'' := D^{1 \times r} / (D^{1 \times (s+t)} (F^T \ X_2^T)^T).$$

Now, let $\tau : D^{1 \times p'} \rightarrow L$ (resp., $\kappa : \ker \varphi \rightarrow L'$) be the canonical projection onto L (resp., L'). Then, the following diagram is commutative with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker \varphi|_K & \longrightarrow & D^{1 \times p} P & \xrightarrow{\varphi|_K} & D \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker \varphi & \longrightarrow & D^{1 \times p'} & \xrightarrow{\varphi} & D \longrightarrow 0, \\ & & \downarrow \kappa & & \downarrow \tau & & \downarrow \\ 0 & \longrightarrow & L' & \xrightarrow{\alpha} & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where $\alpha : L' \rightarrow L$ is the left D -isomorphism defined by:

$$\forall \nu \in \ker \varphi, \quad \alpha(\kappa(\nu)) = \tau(\nu). \quad (26)$$

If $\sigma : D^{1 \times r} \rightarrow L'' = D^{1 \times r} / (D^{1 \times (s+t)} (F^T \ X_2^T)^T)$ is the canonical projection onto L'' , then composing the left D -isomorphism $\beta : L'' \rightarrow L'$ defined by $\beta(\sigma(\theta)) = \kappa(\theta X)$ for all $\theta \in D^{1 \times r}$ (see Proposition 1) with α , we get the following left D -isomorphism:

$$\begin{array}{ccc} \gamma := \alpha \circ \beta : L'' & \longrightarrow & L \\ \sigma(\theta) & \longmapsto & \tau(\theta X). \end{array} \quad (27)$$

Hence, using $Y P = Z = F X$, the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} D^{1 \times (s+t)} & \xrightarrow{\cdot \begin{pmatrix} F \\ X_2 \end{pmatrix}} & D^{1 \times r} & \xrightarrow{\sigma} & L'' & \longrightarrow & 0 \\ \downarrow \cdot (Y^T \ 0^T)^T & & \downarrow \cdot X & & \downarrow \gamma & & \\ D^{1 \times p} & \xrightarrow{\cdot P} & D^{1 \times p'} & \xrightarrow{\tau} & L & \longrightarrow & 0. \end{array} \quad (28)$$

Now, using the identity $\lambda^* \xi^* = 1$, we obtain $(I_{p'} - \xi^* \lambda^*) \xi^* = 0$, which shows that $D^{1 \times p'} (I_{p'} - \xi^* \lambda^*) \subseteq \ker_D(\cdot \xi^*) = D^{1 \times r} X$, i.e., there exists $U \in D^{p' \times r}$ such that:

$$\xi^* \lambda^* + U X = I_{p'}. \quad (29)$$

Using $\lambda^* = \eta^* P$, (29) implies that $UX + (\xi^* \eta^*) P = I_{p'}$ and 4 of Lemma 2 of [16] then yields:

$$\begin{aligned} \gamma^{-1} : L &\longrightarrow L'' \\ \tau(\lambda) &\longmapsto \sigma(\lambda U). \end{aligned}$$

Finally, let us point out an interesting application of the above results. We first note that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \ker_D(.P\xi^*) & \longrightarrow & D^{1 \times p} & \xrightarrow{.(P\xi^*)} & D & \longrightarrow & 0 \\ & & \downarrow .P & & \downarrow .P & & \downarrow \text{id}_D & & \\ 0 & \longrightarrow & \ker_D(. \xi^*) & \longrightarrow & D^{1 \times p'} & \xrightarrow{. \xi^*} & D & \longrightarrow & 0. \\ & & \downarrow \kappa & & \downarrow \tau & & \downarrow & & \\ 0 & \longrightarrow & L' & \xrightarrow{\alpha} & L & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since $\eta^* P \xi^* = 1$, the left D -homomorphisms $.(\eta^* P) : D \longrightarrow D^{1 \times p'}$ and $. \eta^* : D \longrightarrow D^{1 \times p}$ satisfy $. \xi^* \circ .(\eta^* P) = \text{id}_D$ and $.(P \xi^*) \circ . \eta^* = \text{id}_D$, and thus the following short exact sequences

$$0 \longrightarrow \ker_D(. \xi^*) \longrightarrow D^{1 \times p'} \xrightarrow{. \xi^*} D \longrightarrow 0, \quad 0 \longrightarrow \ker_D(. (P \xi^*)) \longrightarrow D^{1 \times p} \xrightarrow{.(P \xi^*)} D \longrightarrow 0$$

split (see [40]). Thus, $D^{1 \times p'} \cong D \oplus \ker_D(. \xi^*)$ and $D^{1 \times p} \cong D \oplus \ker_D(. (P \xi^*))$, which shows that $\ker_D(. \xi^*)$ (resp., $\ker_D(. (P \xi^*))$) is a stably free left D -module of rank $p' - 1$ (resp., $p - 1$).

If $p' \geq 3$, then using Theorem 6, $\ker \varphi = \ker_D(. \xi^*)$ is a free left D -module of rank $p' - 1$. Considering a basis of $\ker \varphi$, there exists a full row rank matrix $X \in D^{(p'-1) \times p'}$ such that $\ker \varphi = D^{1 \times (p'-1)} X$. Thus, $r = p' - 1$, $X_2 = 0$, $t = 0$, and (28) becomes the following commutative diagram with exact rows

$$\begin{array}{ccccccc} D^{1 \times s} & \xrightarrow{.F} & D^{1 \times (p'-1)} & \xrightarrow{\sigma} & L'' & \longrightarrow & 0 \\ \downarrow .Y & & \downarrow .X & & \downarrow \gamma & & \\ D^{1 \times p} & \xrightarrow{.P} & D^{1 \times p'} & \xrightarrow{\tau} & L & \longrightarrow & 0, \end{array}$$

where γ is the left D -isomorphism defined by (27). Hence, we have

$$L = D^{1 \times p'} / (D^{1 \times p} P) \cong L'' = D^{1 \times (p'-1)} / (D^{1 \times s} F),$$

which shows that one generator of L can be removed from the presentation matrix P .

Moreover, if $p \geq 3$, then using Theorem 6, $\ker_D(. (P \xi^*))$ is a free left D -module of rank $p - 1$. Considering a basis of $\ker_D(. (P \xi^*))$, there exists a full row rank matrix $Y \in D^{(p-1) \times p}$ such that $\ker_D(. (P \xi^*)) = D^{1 \times (p-1)} Y$. Thus, $s = p - 1$, and (28) becomes the following commutative diagram with exact rows

$$\begin{array}{ccccccc} D^{1 \times (p-1)} & \xrightarrow{.F} & D^{1 \times (p'-1)} & \xrightarrow{\sigma} & L'' & \longrightarrow & 0 \\ \downarrow .Y & & \downarrow .X & & \downarrow \gamma & & \\ D^{1 \times p} & \xrightarrow{.P} & D^{1 \times p'} & \xrightarrow{\tau} & L & \longrightarrow & 0, \end{array}$$

where γ is the left D -isomorphism defined by (27). Hence, we have

$$L = D^{1 \times p'} / (D^{1 \times p} P) \cong L'' = D^{1 \times (p'-1)} / (D^{1 \times (p-1)} F),$$

which shows that one generator and one relation of L can be removed from P .

Remark 7. Since $\text{rank}_D(K) = \text{rank}_D(D^{1 \times p} P) \leq p$ and $\text{rank}_D(K) \geq 2$ so that Theorem 8 is applicable, then we necessarily have $p \geq 2$.

We obtain the following theorem which surprisingly does not appear in [44].

Theorem 9. *Let D be a very simple domain, $P \in D^{p \times p'}$, and $L = D^{1 \times p'} / (D^{1 \times p} P)$ a finitely presented left D -module. If $\text{rank}_D(D^{1 \times p} P) \geq 2$ and $p' \geq 3$, then there exists $\bar{P} \in D^{s \times (p'-1)}$ such that $L \cong \bar{L} := D^{1 \times (p'-1)} / (D^{1 \times s} \bar{P})$. Moreover, if $p \geq 3$, then \bar{P} can be chosen so that $s = p - 1$, i.e., $L \cong \bar{L} := D^{1 \times (p'-1)} / (D^{1 \times (p-1)} \bar{P})$.*

Algorithm 3 (StaffordReduction).

- **Input:** A very simple domain D and $P \in D^{p \times p'}$ such that:

$$\text{rank}_D(D^{1 \times p} P) \geq 2.$$

- **Output:** Two matrices $\bar{P} \in D^{s \times r}$ and $X \in D^{r \times p'}$ such that the left D -homomorphism

$$\begin{aligned} \gamma : \bar{L} := D^{1 \times r} / (D^{1 \times s} \bar{P}) &\longrightarrow L := D^{1 \times p'} / (D^{1 \times p} P) \\ \sigma(\theta) &\longmapsto \tau(\theta X), \end{aligned} \quad (30)$$

is a left D -isomorphism, where σ (resp., τ) is the canonical projection onto \bar{L} (resp., L). If $p' \geq 3$, then $r = p' - 1$ and, moreover, if $p \geq 3$, then $s = p - 1$.

1. Pick $\eta_1 \in D^{1 \times p}$ such that $\lambda_1 := \eta_1 P \neq 0$.
2. Find $\xi_1 \in D^{p'}$ such that $\lambda_1 \xi_1 \neq 0$.
3. Compute $S \in D^{l \times p}$ such that $\ker_D(. (P \xi_1)) = D^{1 \times l} S$.
4. Pick $\nu \in D^{1 \times l}$ such that $\nu(S P) \neq 0$ and define $\eta_2 = \nu S$.
5. Find $\xi_2 \in D^{p'}$ such that $(\eta_2 P) \xi_2 \neq 0$ and define $\lambda_2 = \eta_2 P$.
6. If $\lambda_1 \xi_2 \neq 0$, then compute $r_1, r_2 \in D \setminus \{0\}$ such that $(\lambda_1 \xi_1) r_1 + (\lambda_1 \xi_2) r_2 = 0$ and replace ξ_2 by $\xi_1 r_1 + \xi_2 r_2$.
7. Compute $y_1, y_2, z_1, z_2 \in D$ such that $y_1 (\lambda_1 \xi_1) z_1 + y_2 (\lambda_2 \xi_2) z_2 = 1$.
8. Compute $\eta^* = y_1 \eta_1 + y_2 \eta_2 \in D^{1 \times p}$ and $\xi^* = \xi_1 z_1 + \xi_2 z_2 \in D^{p'}$.
9. If $p' \leq 2$, then compute $X \in D^{r \times p'}$ such that $\ker_D(. \xi^*) = D^{1 \times r} X$. Else, compute a basis of the free left D -module $\ker_D(. \xi^*)$ of rank $p' - 1$ to get a full row rank matrix $X \in D^{(p'-1) \times p'}$ such that $\ker_D(. \xi^*) = D^{1 \times (p'-1)} X$, and set $r = p' - 1$.
10. If $p \leq 2$, then compute $Y \in D^{s' \times p}$ such that $\ker_D(. (P \xi^*)) = D^{1 \times s'} Y$. Else, compute a basis of the free left D -module $\ker_D(. (P \xi^*))$ of rank $p - 1$ to get a full row rank matrix $Y \in D^{(p-1) \times p}$ such that $\ker_D(. (P \xi^*)) = D^{1 \times (p-1)} Y$, and set $s' = p - 1$.

11. Compute $Z = Y P$.
12. Left factorize Z by X to get $F \in D^{s' \times r}$ such that $Z = F X$.
13. If $p \leq 2$, then find $X_2 \in D^{t \times r}$ (possibly reduced to 0) such that $\ker_D(.X) = D^{1 \times t} X_2$ and define $\bar{P} = (F^T \quad X_2^T)^T$, else $\bar{P} = F$ and set $t = 0$.
14. Return (\bar{P}, X) defining respectively the left D -module $\bar{L} = D^{1 \times (p'-1)} / (D^{1 \times (s'+t)} \bar{P})$ and the left D -isomorphism $\gamma : \bar{L} \rightarrow L$ given by (27).

Example 7. Let $D = A_3(\mathbb{Q})$ and $L = D^{1 \times 3} / (D^{1 \times 3} P)$ be the left D -module finitely presented by the curl operator (21). We can easily check that $\ker_D(.P) = D R$, where $R = (\partial_1 \quad \partial_2 \quad \partial_3)$ is the divergence operator (see Example 5). Let us apply Algorithm 2 to the left D -modules $M = D^{1 \times 3} / (D R) \cong K = D^{1 \times 3} P$ and $N = D^{1 \times 3}$. If we consider $\eta_1 = (0 \quad 0 \quad 1)$ and $\xi_1 = (0 \quad 1 \quad 0)^T$, then $d_1 := \eta_1 P \xi_1 = \partial_1$. Moreover, $\ker_D(. (P \xi_1)) = D^{1 \times 2} S$, where:

$$S = \begin{pmatrix} \partial_1 & 0 & \partial_3 \\ 0 & 1 & 0 \end{pmatrix}.$$

If we consider $\nu = (0 \quad 1)$, $\eta_2 = \nu S = (0 \quad 1 \quad 0)$, and $\xi_2 = (0 \quad 0 \quad -1)^T$, then $d_2 := \eta_2 P \xi_2 = \partial_1$. Since $\eta_1 P \xi_2 = 0$, then we need to compute a solution $(y_1 \ y_2 \ z_1 \ z_2)^T \in D^4$ of (11). We get $z_1 = 1$, $z_2 = x_1 + 1$, $y_1 = -(x_1 + 1)$, and $y_2 = 1$, which yields:

$$\begin{cases} \eta^* = y_1 \eta_1 + y_2 \eta_2 = (0 \quad 1 \quad -(x_1 + 1)), \\ \xi^* = \xi_1 z_1 + \xi_2 z_2 = (0 \quad 1 \quad -(x_1 + 1))^T. \end{cases}$$

Hence, $\lambda^* := \eta^* P = ((x_1 + 1) \partial_2 + \partial_3 \quad -(x_1 + 1) \partial_1 \quad -\partial_1) \in U(D^{1 \times 3})$ and $\varphi := \varphi_{\xi^*}$ satisfies $\varphi(\lambda^*) = \lambda^* \xi^* = 1$. In other words, λ^* is a certain left D -combination of the rows of P which admits a right inverse over D . Thus, $D^{1 \times 3} = D \lambda^* \oplus \ker \varphi$, where $\ker \varphi = \ker_D(. \xi^*) = D^{1 \times 2} X$ and:

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1 + 1 & 1 \end{pmatrix}.$$

Since X has full row rank, the rows of X define a basis of the free left D -module $\ker_D(. \xi^*)$ of rank two. Now, λ^* is a unimodular element of K , and thus $D^{1 \times 3} P = D \lambda^* \oplus \ker \varphi|_K$, where

$$\ker \varphi|_K = \{\eta P \mid \eta \in D^{1 \times 3} : (\eta P) \xi^* = 0\} = \ker_D(. (P \xi^*)) P,$$

and $\ker_D(. (P \xi^*)) = D^{1 \times 2} Y$, where the matrix $Y \in D^{2 \times 3}$ is defined by:

$$Y = \begin{pmatrix} -1 & -(x_1 + 1) \partial_2 - \partial_3 & (x_1 + 1) ((x_1 + 1) \partial_2 + \partial_3) \\ 0 & -\partial_1 & (x_1 + 1) \partial_1 + 2 \end{pmatrix}.$$

Since Y has full row rank, the rows of Y define a basis of the free left D -module $\ker_D(. (P \xi^*))$ of rank two. Then, $\ker \varphi|_K = D^{1 \times 2} Z$, where $Z = Y P$. Now, let $F \in D^{2 \times 2}$ be the matrix such that $Z = F X$, i.e.:

$$F = \begin{pmatrix} -(\partial_3 + (x_1 + 1) \partial_2)^2 & ((x_1 + 1) \partial_2 + \partial_3) \partial_1 - \partial_2 \\ -((x_1 + 1) \partial_2 + \partial_3) \partial_1 - 2 \partial_2 & \partial_1^2 \end{pmatrix}. \quad (31)$$

Since $\ker_D(.X) = 0$, Proposition 1 shows that:

$$L = D^{1 \times 3} / (D^{1 \times 3} P) \cong L' := (D^{1 \times 2} X) / (D^{1 \times 2} Z) \cong \bar{L} := D^{1 \times 2} / (D^{1 \times 2} F).$$

The left D -isomorphism $\gamma : \bar{L} \rightarrow L$ is then defined by $\gamma(\sigma(\theta)) = \tau(\theta X)$ for all $\theta \in D^{1 \times 2}$, where $\sigma : D^{1 \times 2} \rightarrow \bar{L}$ (resp., $\tau : D^{1 \times 3} \rightarrow L$) is the canonical projection onto \bar{L} (resp., L). Moreover, $\gamma^{-1} : L \rightarrow \bar{L}$ is defined by $\gamma^{-1}(\tau(\lambda)) = \sigma(\lambda U)$ for all $\lambda \in D^{1 \times 3}$, where:

$$U = \begin{pmatrix} 1 & 0 \\ -(x_1 + 1) \partial_2 - \partial_3 & \partial_1 \\ (x_1 + 1) ((x_1 + 1) \partial_2 + \partial_3) & -(x_1 + 1) \partial_1 + 1 \end{pmatrix}. \quad (32)$$

Finally, since $\ker_D(.F) = D(-\partial_1 \quad \partial_3 + (x_1 + 1) \partial_2)$, we have $\text{rank}_D(D^{1 \times 2} F) = 1$, and Algorithm 2 cannot be applied again to \bar{L} .

Finally, let us compute the inverse of the left D -isomorphism $\alpha : L' \rightarrow L$ defined by $\alpha(\kappa(\nu)) = \tau(\nu)$ for all $\nu \in D^{1 \times r} X$ (see (26)). This result will be used in the next section.

Now, using the identity $\eta^* P \xi^* = 1$, we obtain $(I_p - P \xi^* \eta^*) P \xi^* = 0$, which shows that $D^{1 \times p} (I_p - P \xi^* \eta^*) \subseteq \ker_D(. (P \xi^*)) = D^{1 \times s} Y$, i.e., there exists $V \in D^{p \times s}$ such that:

$$P \xi^* \eta^* + V Y = I_p. \quad (33)$$

Multiplying P on the right of (33) and on the left of (29) and subtracting the result, we get:

$$P U X = V Y P. \quad (34)$$

Now, we claim that α^{-1} is the left D -homomorphism defined by:

$$\begin{aligned} \omega : L &\rightarrow L' \\ \tau(\lambda) &\mapsto \kappa(\lambda U X). \end{aligned} \quad (35)$$

Using (34) and $Z = Y P$, we first check that ω is well-defined:

$$\forall \eta \in D^{1 \times p}, \quad \omega(\tau(\lambda + \eta P)) = \kappa(\lambda U X) + \kappa((\eta V) Z) = \kappa(\lambda U X) = \omega(\tau(\lambda)).$$

Finally, using (29), $\lambda^* = \eta^* P$, and $X \xi^* = 0$, we can check that

$$\begin{aligned} (\alpha \circ \omega)(\tau(\lambda)) &= \alpha(\kappa(\lambda U X)) = \tau(\lambda U X) = \tau(\lambda) - \tau((\lambda \xi^* \eta^*) P) = \tau(\lambda), \\ (\omega \circ \alpha)(\kappa(\theta X)) &= \omega(\tau(\theta X)) = \kappa(\theta X U X) = \kappa(\theta X (I_{p'} - \xi^* \eta^* P)) = \kappa(\theta X), \end{aligned}$$

which shows that $\omega = \alpha^{-1}$.

5 Efficient generation of finitely generated modules

We have the following interesting corollary of Theorem 9.

Corollary 2. *Let $L = D^{1 \times p'} / (D^{1 \times p'} P)$ be a torsion left D -module.*

1. *If $p' \geq 3$, then there exists $\bar{P} \in D^{l \times 2}$ such that $L \cong \bar{L} := D^{1 \times 2} / (D^{1 \times l} \bar{P})$. In particular, L can be generated by two elements. Moreover, l can be chosen such that:*

$$l = p - p' + 2.$$

2. There exist a projective left ideal I of D and a left submodule J of I such that $L \cong I/J$. In other words, L is a homomorphic image of a projective left ideal of D (which can be generated by two elements).

Proof. 1. Since L is a torsion left D -module, by 1 \Rightarrow 3 of Lemma 1, $\text{rank}_D(L) = 0$, which yields $\text{rank}_D(D^{1 \times p} P) = p' \geq 3$. Applying $p' - 2$ times Theorem 9, we obtain $\bar{P} \in D^{l \times 2}$ such that $L \cong \bar{L} := D^{1 \times 2} / (D^{1 \times l} \bar{P})$, which proves the first part of 1. Finally, using that

$$p \geq \text{rank}_D(D^{1 \times p} P) = p' \geq 3,$$

then Theorem 9 shows that l can be chosen such that $l = p - p' + 2$.

2. If $p' = 1$, then $L = D / (D^{1 \times p} P)$, which proves 2 with $I = D$ and $J = D^{1 \times p} P$. Now, if $p' > 1$, using 1, we may assume that $p' = 2$, i.e., $L = D^{1 \times 2} / (D^{1 \times p} P)$. Let $R \in D^{q \times p}$ (possibly $R = 0$ and $q = 0$) be such that $\ker_D(.P) = D^{1 \times q} R$, $M = D^{1 \times p} / (D^{1 \times q} R)$, π the canonical projection onto M ($\pi = \text{id}_M$ if $R = 0$), and $\iota : M \rightarrow D^{1 \times 2}$ the injection defined by $\iota(\pi(\lambda)) = \lambda P$ for all $\lambda \in D^{1 \times p}$. The following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{.R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow .P & & \downarrow \iota & & \\ 0 & \longrightarrow & D^{1 \times 2} & \xrightarrow{\text{id}} & D^{1 \times 2} & \longrightarrow & 0. \end{array}$$

Since $\text{rank}_D(D^{1 \times p} P) = 2$, the proof of Theorem 8 for $K = \iota(M)$ shows that there exist $\eta^* \in D^{1 \times p}$ and $\xi^* \in D^2$ such that $\varphi := \varphi_{\xi^*}$ satisfies:

$$\varphi(\iota(\pi(\eta^*))) = \eta^* P \xi^* = 1.$$

Thus, if $\lambda^* := \eta^* P$, then $D^{1 \times p} P = D \lambda^* \oplus N' \subseteq D^{1 \times 2} = D \lambda^* \oplus N$, where $N := \ker \varphi = \ker_D(. \xi^*)$ and $N' := \ker \varphi|_K = \ker_D(. (P \xi^*)) P$. This implies:

$$L = (D \lambda^* \oplus N) / (D \lambda^* \oplus N') \cong L' := N / N'.$$

Let $\kappa : N \rightarrow L' = N / N'$ (resp., $\tau : D^{1 \times 2} \rightarrow L$) be the canonical projection onto L' (resp., L). Then, the above left D -isomorphism is defined by (26), namely:

$$\begin{array}{ccc} \alpha : L' & \longrightarrow & L \\ \kappa(\nu) & \longmapsto & \tau(\nu). \end{array}$$

Since $D^{1 \times 2} = D \lambda^* \oplus N \cong D \oplus N$, N is a projective left D -module of rank one, and thus N is isomorphic to a projective left ideal of D , which can be generated by two elements (see Section 3.2). Note that if N is a free left D -module of rank one, then N is isomorphic to a principal left ideal of D (i.e., generated by one element). Let $X \in D^{r \times 2}$ (resp., $Y \in D^{s \times p}$) be such that $N = \ker_D(. \xi^*) = D^{1 \times r} X$ (resp., $\ker_D(. (P \xi^*)) = D^{1 \times s} Y$). If $Z = Y P \in D^{s \times 2}$, then $N' = \ker_D(. (P \xi^*)) P = D^{1 \times s} Z$, and thus $N / N' = (D^{1 \times r} X) / (D^{1 \times s} Z)$. Since $N' \subseteq N$, there exists $F \in D^{s \times r}$ such that $Z = F X$ and the injection $i : N' \rightarrow N$ is defined by:

$$\forall \theta \in D^{1 \times s}, \quad i(\theta Z) = (\theta F) X. \quad (36)$$

Let $X_2 \in D^{t \times r}$ (resp., $Z_2 \in D^{u \times s}$) be such that $\ker_D(. X) = D^{1 \times t} X_2$ (resp., $\ker_D(. Z) = D^{1 \times u} Z_2$). Then, the following diagram is commutative with exact rows

$$\begin{array}{ccccccc} D^{1 \times u} & \xrightarrow{.Z_2} & D^{1 \times s} & \xrightarrow{.Z} & N' & \longrightarrow & 0 \\ \downarrow .G & & \downarrow .F & & \downarrow i & & \\ D^{1 \times t} & \xrightarrow{.X_2} & D^{1 \times r} & \xrightarrow{.X} & N & \longrightarrow & 0, \end{array}$$

where $G \in D^{u \times t}$ satisfies $Z_2 F = G X_2$. Since N and N' are two left D -submodules of $D^{1 \times 2}$, they are torsion-free. Hence, $N_2 := D^{1 \times r} / (D^{1 \times t} X_2) \cong D^{1 \times r} X = N$ and $N'_2 := D^{1 \times s} / (D^{1 \times u} Z_2) \cong D^{1 \times s} Z = N'$ are also torsion-free left D -modules of rank one. Considering a minimal parametrization of N_2 and of N'_2 (see [11]), there exist two matrices $B \in D^r$ and $C \in D^s$ such that $N_2 \cong I := D^{1 \times r} B$ and $N'_2 \cong H := D^{1 \times s} C$, where I and H are two finitely generated left ideals of D . To get such minimal parametrizations, we need to pick a non-zero $B \in D^r$ (resp., a non-zero $C \in D^s$) such that $B D \subseteq \ker_D(X_2)$ (resp., $C D \subseteq \ker_D(Z_2)$). Then, by definition of the minimal parametrizations, the next diagram is commutative with exact rows

$$\begin{array}{ccccccc} D^{1 \times u} & \xrightarrow{\cdot Z_2} & D^{1 \times s} & \xrightarrow{\cdot C} & H & \longrightarrow & 0 \\ \downarrow \cdot G & & \downarrow \cdot F & & \downarrow j & & \\ D^{1 \times t} & \xrightarrow{\cdot X_2} & D^{1 \times r} & \xrightarrow{\cdot B} & I & \longrightarrow & 0, \end{array}$$

where $j \in \text{hom}_D(H, I)$ is defined by:

$$\forall \theta \in D^{1 \times s}, \quad j(\theta C) = (\theta F) B. \quad (37)$$

Let us note that we then have the following left D -isomorphisms

$$\begin{array}{ccc} \varepsilon : H & \longrightarrow & N' \\ \theta C & \longmapsto & \theta Z, \end{array} \quad \begin{array}{ccc} \varepsilon' : I & \longrightarrow & N \\ v B & \longmapsto & v X, \end{array}$$

which satisfy $\varepsilon' \circ j = i \circ \varepsilon$, where i and j are respectively defined by (36) and (37).

If we consider the left D -submodule $J = \text{im } j = D^{1 \times s} (F B)$ of I , then we have:

$$\text{coker } j = I/J = (D^{1 \times r} B) / (D^{1 \times s} (F B)).$$

Let $\delta : I \longrightarrow I/J$ be the canonical projection onto I/J . Then, the following diagram is commutative with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H & \xrightarrow{\varepsilon} & N' & \longrightarrow & 0 \\ & & \downarrow j & & \downarrow i & & \\ 0 & \longrightarrow & I & \xrightarrow{\varepsilon'} & N & \longrightarrow & 0, \\ & & \downarrow \delta & & \downarrow \kappa & & \\ & & I/J & \xrightarrow{\varpi} & L' & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where $\varpi \in \text{hom}_D(I/J, L')$ is the left D -isomorphism defined by:

$$\begin{array}{ccc} \varpi : I/J & \longrightarrow & L' \\ \delta(v B) & \longmapsto & \kappa(\varepsilon'(v B)) = \kappa(v X). \end{array}$$

Composing ϖ with the left D -isomorphism $\alpha \in \text{hom}_D(L', L)$ defined by (26), we then obtain the following left D -isomorphism:

$$\begin{array}{ccc} \chi = \alpha \circ \varpi : I/J & \longrightarrow & L \\ \delta(v B) & \longmapsto & \tau(v X). \end{array}$$

Let us now explicitly characterize the inverse $\chi^{-1} = \varpi^{-1} \circ \alpha^{-1}$ of χ . We first note that $\varpi^{-1} : L' \rightarrow I/J$ is defined by $\varpi^{-1}(\kappa(vX)) = \delta(vB)$ for all $v \in D^{1 \times r}$. Secondly, α^{-1} is defined by (35), i.e., $\alpha^{-1}(\tau(\lambda)) = \kappa(\lambda UX)$, where $U \in D^{2 \times r}$ is defined by (29), i.e., by $I_2 - \xi^* \lambda^* = UX$. Thus, the left D -isomorphism χ^{-1} is finally defined by:

$$\forall \lambda \in D^{1 \times 2}, \quad \chi^{-1}(\tau(\lambda)) = \varpi^{-1}(\alpha^{-1}(\tau(\lambda))) = \varpi^{-1}(\kappa(\lambda UX)) = \delta((\lambda UB)).$$

□

Remark 8. We note that 1 of Corollary 2 does not give in general the most accurate bound for the number of generators of a torsion left D -module because *cyclic* torsion left D -modules exist (namely, torsion left D -modules of the form D/J , where J is a left ideal of D).

Example 8. Many linear PD systems coming from mathematical physics or engineering science are defined by means of a square full row rank matrix $P \in D^{p' \times p'}$ of PD operators. Applying 1 of Corollary 2 to $P \in D^{p' \times p'}$, we obtain that these linear PD systems are equivalent to linear PD systems defined by $\bar{P} \in D^{2 \times 2}$, i.e., to linear PD systems defined by (at most) two unknowns and two PD equations. For an algorithm computing $\bar{P} \in D^{2 \times 2}$, see [7, 15].

Example 9. Let $D = A\langle\partial\rangle$ be a ring of OD operators, where $A = k[t]$ or $A = k[[t]]$ and k is a field of characteristic zero, or $A = k\{t\}$, where $k \in \{\mathbb{R}, \mathbb{C}\}$, $E \in A^{n \times n}$, $R = \partial I_n - E \in D^{n \times n}$, and $M = D^{1 \times n}/(D^{1 \times n}R)$. If $\{f_j\}_{j=1, \dots, n}$ is the standard basis of $D^{1 \times n}$, $\pi : D^{1 \times n} \rightarrow M$ the canonical projection onto M , $x_j = \pi(f_j)$ for $j = 1, \dots, n$, and $x = (x_1 \dots x_n)^T$, then $Rx = 0$ (see Section 2). By 1 of Corollary 2, the torsion left D -module M can be generated by two elements $y_1 = \pi(G_1) = G_1 x$ and $y_2 = \pi(G_2) = G_2 x$, where $G_i \in D^{1 \times n}$ for $i = 1, 2$. Equivalently, the matrix $(G_1^T \ G_2^T \ R^T)^T$ admits a left inverse (see, e.g., [7, 15]), i.e., there exist $S_1 \in D^n$, $S_2 \in D^n$, and $S_3 \in D^{n \times n}$ such that:

$$S_1 G_1 + S_2 G_2 + S_3 R = I_n. \quad (38)$$

Now, if we consider the linear OD control system $\dot{x} = Ex + Fu$ (see, e.g., [23]), where $F \in A^{n \times m}$, the above remark shows that there exists $G = (G_1^T \ G_2^T)^T \in A^{2 \times n}$ such that (38) holds. Thus, the following linear OD control system

$$\begin{cases} \dot{x} = Ex + Fu, \\ y = Gx, \end{cases} \quad (39)$$

is *observable* (see, e.g., [23]), namely, the state x of (39) can be expressed in terms of y_1, y_2, u , and their derivatives. Indeed, combining $Rx = Fu$ and (38), we get $x = S_1 y_1 + S_2 y_2 + S_3 Fu$. Hence, 1 of Corollary 2 shows that there exist two outputs $y_1 = G_1 x$ and $y_2 = G_2 x$ of the linear OD control system $\dot{x} = Ex + Fu$ such that its state x is observable by means of y_1, y_2 , and u .

Algorithm 4 (ProjectiveIdealPresentation).

- **Input:** A very simple domain D and $P \in D^{p \times p'}$ such that the finitely presented left D -module $L = D^{1 \times p'}/(D^{1 \times p}P)$ is torsion.

- **Output:** Three matrices $B \in D^x$, $B' \in D^y$, and $W \in D^{x \times p'}$ such that

$$\begin{aligned} \chi : K := (D^{1 \times x} B) / (D^{1 \times y} B') &\longrightarrow L \\ \delta(v B) &\longmapsto \tau(v W) \end{aligned}$$

is an isomorphism of left D -modules, where $\delta : D^{1 \times x} B \longrightarrow K$ (resp., $\tau : D^{1 \times p'} \longrightarrow L$) is the canonical projection onto K (resp., L).

1. If $p' = 1$, then return $B' = P$, $B = 1$, $W = 1$ (i.e., $x = 1$, $y = p$).
2. Apply Algorithm 3 to P to obtain two matrices $\bar{P} \in D^{s \times 2}$ and $X \in D^{2 \times p'}$ such that the left D -homomorphism $\gamma : \bar{L} = D^{1 \times 2} / (D^{1 \times s} \bar{P}) \longrightarrow L$ defined by (30) is an isomorphism.
3. Apply Algorithm 2 to the matrices $R := \bar{R} \in D^{t \times s}$ satisfying $\ker_D(\bar{P}) = D^{1 \times t} \bar{R}$, $R' = 0$, and $P := \bar{P}$ to obtain $\eta^* \in D^{1 \times s}$ and $\xi^* \in D^2$ such that $\eta^* \bar{P} \xi^* = 1$.
4. Compute $\bar{X} \in D^{x \times 2}$ such that $\ker_D(\xi^*) = D^{1 \times x} \bar{X}$.
5. Compute $\bar{Y} \in D^{y \times s}$ such that $\ker_D(\bar{P} \xi^*) = D^{1 \times y} \bar{Y}$.
6. Compute $\bar{Z} = \bar{Y} \bar{P} \in D^{y \times 2}$.
7. Left factorize \bar{Z} by \bar{X} to get $\bar{F} \in D^{y \times x}$ such that $\bar{Z} = \bar{F} \bar{X}$.
8. Compute $\bar{X}_2 \in D^{z \times x}$ such that $\ker_D(\bar{X}) = D^{1 \times z} \bar{X}_2$.
9. Compute a non-zero $B \in D^x$ such that $B D \subseteq \ker_D(\bar{X}_2)$.
10. Compute $B' = \bar{F} B \in D^y$.
11. Compute $W = \bar{X} X \in D^{x \times p'}$.
12. Return the matrices B , B' , and W .

Example 10. Let $D = A_1(\mathbb{Q})$ and $L = D^{1 \times 2} / (D^{1 \times 2} P)$ be the left D -module finitely presented by the matrix P defined by:

$$P = \begin{pmatrix} t^2 & t \\ t\partial + 2 & \partial \end{pmatrix}.$$

Since P has full row rank, L is a torsion left D -module. Corollary 2 then shows that $L \cong I/J$, where I and J are two left ideals of D such that $J \subseteq I$, both of which can be generated by two elements. Let us compute I and J . If we consider $\eta^* = (\partial \quad -t)$ and $\xi^* = (0 \quad 1)^T$, then $\eta^* P \xi^* = 1$. Now, $\ker_D(\xi^*) = D X$, $\ker_D(\bar{P} \xi^*) = D^{1 \times 2} Y$, $Z = Y P$, $Z = F X$, where:

$$X = (1 \quad 0), \quad Y = \begin{pmatrix} t\partial - 1 & -t^2 \\ \partial^2 & -t\partial - 2 \end{pmatrix}, \quad Z = \begin{pmatrix} -t^2 & 0 \\ -t\partial - 2 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} -t^2 & \\ -t\partial - 2 & \end{pmatrix}.$$

Since $\ker_D(\bar{P} \xi^*) = 0$, i.e., $X_2 = 0$, then $\ker_D(X_2) = D$, which shows that $B = 1$ is a minimal parametrization of $N = D X \cong D$ and $I = D B = D$. Thus, $B' = F B = -(t^2 \quad t\partial + 2)^T$, which shows that $L \cong K := D/J$, where $J = D t^2 + D(t\partial + 2)$. We note that J is the cyclic left D -module which is the annihilator of the derivative of the *Dirac distribution* (see

Example 1.2.9 of [34]). The left D -isomorphism $\chi : K \rightarrow L$ is defined by $\chi(\delta(v)) = \tau(vX)$ for all $v \in D$, where $\delta : D \rightarrow K$ (resp., $\tau : D^{1 \times 2} \rightarrow L$) is the canonical projection onto K (resp., L). Thus, L is generated by $\tau(X) = \chi(\delta(1))$ and we can easily check that $\tau((0 \ 1)) = 0$ (for more details, see [37]). Finally, the matrix $U = (1 \ 0)^T$ satisfies $I_2 - (\xi^* \eta^*)P = UX$, which shows that $\chi^{-1} : L \rightarrow K$ is defined by $\chi^{-1}(\tau(\lambda)) = \delta(\lambda U)$ for all $\lambda \in D^{1 \times 2}$.

Example 11. Corollary 2 can be understood as a generalization of the standard *cyclic vector theorem* for linear OD systems (see [10] and the references therein). 1 of Corollary 2 shows that every finitely generated torsion left D -module L can be generated by two elements. Moreover, 2 of Corollary 2 states that $L \cong I/J$, where I is a projective left ideal of D and J is a left D -submodule of I . Now, if $D = A\langle\partial\rangle$ is a ring of OD operators with coefficients in a differential field A , then D is a principal ideal domain (see, e.g., [31]), and thus I and J are two principal ideals of D , i.e., there exist $d_1, d_2 \in D$ such that $L \cong (Dd_1)/(Dd_2) \cong D/(Dd_3)$, where $d_3 \in D$ is such that $d_1 = d_3d_2$. The element d_3 , which is not uniquely determined by L , can be computed by means of a *Jacobson normal form* of the presentation matrix P of L .

Let us illustrate with a simple example that Algorithm 4 can be used to compute d_3 . Let $D = B_1(\mathbb{Q})$, $a_{ij} \in A = \mathbb{Q}(t)$ for $i, j = 1, 2$, $E = (a_{ij})_{i,j=1,2}$, and $L = D^{1 \times 2}/(D^{1 \times 2}P)$, where:

$$P = \partial I_2 - E = \begin{pmatrix} \partial - a_{11} & -a_{12} \\ -a_{21} & \partial - a_{22} \end{pmatrix}.$$

The left D -module L corresponds to the linear OD system $\dot{x} = Ex$. Let us suppose that $a_{12} \neq 0$ (a similar result holds if $a_{21} \neq 0$), which implies $a_{12}^{-1} \in A$. Choosing $\eta^* = (1 \ 0)$, $\xi^* = (0 \ -a_{12}^{-1})^T$, we get $\eta^*P\xi^* = 1$. Then, $\ker_D(\cdot\xi^*) = DX$, where $X = (1 \ 0)$, and $\ker_D(\cdot(P\xi^*)) = DY$, where $Y = ((\partial - a_{22})a_{12}^{-1} \ 1)$, since $P\xi^* = (1 \ -(\partial - a_{22})a_{12}^{-1})^T$. Then, $Z = YP = ((\partial - a_{22})a_{12}^{-1}(\partial - a_{11}) - a_{21} \ 0) = FX$, where $F = (\partial - a_{22})a_{12}^{-1}(\partial - a_{11}) - a_{21}$, which shows that:

$$L \cong (D^{1 \times 2}X)/(D^{1 \times 2}Z) \cong D/(DF).$$

In case $a_{11} \neq a_{22}$ and $a_{12} = a_{21} = 0$, then considering $\eta_1 = (1 \ 0)$, $\xi_1 = (1 \ 0)^T$, $d_1 := \eta_1 P \xi_1 = \partial - a_{11}$, $\ker_D(\cdot(P\xi_1)) = D(0 \ 1)$, $\eta_2 = (0 \ 1)$, $\xi_2 = (0 \ 1)^T$, $d_2 := \eta_2 P \xi_2 = \partial - a_{22}$, $\eta_2 P \xi_1 = 0$, $y_1 d_1 z_1 + y_2 d_2 z_2 = 1$, where $-y_1 = y_2 = (a_{11} - a_{22})^{-1}$, $z_1 = z_2 = 1$, and thus $\eta^* = y_1 \eta_1 + y_2 \eta_2 = (a_{11} - a_{22})^{-1}(-1 \ 1)$ and $\xi = \xi_1 z_1 + \xi_2 z_2 = (1 \ 1)^T$ are such that $\eta^* P \xi = 1$. Then, Algorithm 4 gives $\ker_D(\cdot\xi) = DX$, where $X = (1 \ -1)$, $\ker_D(\cdot(P\xi)) = DY$, where $Y = (\partial - a_{22} \ -\partial + a_{11})(a_{11} - a_{22})^{-1}$, $Z = YP = FX$, where $F = (\partial - a_{22})(a_{11} - a_{22})^{-1}(\partial - a_{11}) = (\partial - a_{11})(a_{11} - a_{22})^{-1}(\partial - a_{22})$, and thus we obtain:

$$L \cong (DX)/(DZ) \cong D/(DF).$$

Now, if $a_{11} = a_{22}$ and $a_{12} = a_{21} = 0$, then considering $\eta_1 = (1 \ 0)$, $\xi_1 = (1 \ 0)^T$, $d_1 := \eta_1 P \xi_1 = \partial - a_{11}$, $\ker_D(\cdot(P\xi_1)) = D(0 \ 1)$, $\eta_2 = (0 \ 1)$, $\xi_2 = (0 \ 1)^T$, $d_2 := \eta_2 P \xi_2 = \partial - a_{11}$, $\eta_2 P \xi_1 = 0$, $y_1 d_1 z_1 + y_2 d_2 z_2 = 1$, where $y_1 = z_2 = 1$, $z_1 = t$, $y_2 = -t$, and thus $\eta^* = y_1 \eta_1 + y_2 \eta_2 = (1 \ -t)$ and $\xi = \xi_1 z_1 + \xi_2 z_2 = (t \ 1)^T$ are such that $\eta^* P \xi = 1$. Then, Algorithm 4 gives $\ker_D(\cdot\xi) = DX$, where $X = (-1 \ t)$, $\ker_D(\cdot(P\xi)) = DY$, where $Y = (-\partial - a_{11} \ t\partial - a_{11}t + 2)$, $Z = YP = FX$, where $F = (\partial - a_{11})^2$, and thus we obtain:

$$L \cong (DX)/(DZ) \cong D/(DF).$$

Corollary 3. Let $P \in D^{p \times p'}$ be a full row rank matrix and $L = D^{1 \times p'} / (D^{1 \times p} P)$. Then, there exists $\bar{P} \in D^{2 \times (p' - p + 2)}$ such that:

$$L \cong \bar{L} := D^{1 \times (p' - p + 2)} / (D^{1 \times 2} \bar{P}).$$

If $p' - p \geq 1$ and the right D -module $\text{ext}_D^1(L, D) := D^p / (P D^p)$ is cyclic, then there exists $R' \in D^{1 \times (p' - p + 1)}$ such that $L \cong M' := D^{1 \times (p' - p + 1)} / (D R')$.

Proof. Applying the contravariant left exact functor $\text{hom}_D(\cdot, D)$ to the short exact sequence

$$0 \longrightarrow D^{1 \times p} \xrightarrow{P} D^{1 \times p'} \xrightarrow{\sigma} L \longrightarrow 0,$$

we get the following long exact sequence of right D -modules (see, e.g., [11, 40]):

$$0 \longleftarrow \text{ext}_D^1(L, D) \xleftarrow{\varpi} D^p \xleftarrow{P} D^{p'} \longleftarrow \text{hom}_D(L, D) \longleftarrow 0.$$

Now, by a version of (3) for right D -modules, we have $\text{hom}_D(\text{ext}_D^1(L, D), D) \cong \ker_D(\cdot P) = 0$, which shows that $\text{ext}_D^1(L, D)$ is a torsion right D -module by 2 \Rightarrow 1 of Lemma 1. Thus, it can be generated by two elements by 1 of Corollary 2. Let $\Lambda \in D^{p \times 2}$ be such that $\{\varpi(\Lambda_{\bullet, i})\}_{i=1,2}$ generates $\text{ext}_D^1(L, D)$. Theorem 3.6 of [7] then proves that the left D -module $E = D^{1 \times (p' + 2)} / (D^{1 \times p} Q)$, where $Q := (P \quad -\Lambda) \in D^{p \times (p' + 2)}$, is stably free of rank $p' + 2 - p$. Since $\text{rank}_D(L) = p' - p \geq 0$, then $p' + 2 - p \geq 2$, which shows that E is a finitely generated free left D -module by Theorem 6. Finally, the result follows from Theorem 4.1 of [7].

Finally, if the right D -module $\text{ext}_D^1(L, D)$ is cyclic, then there exists $\Lambda \in D^q$ such that $\text{ext}_D^1(L, D) = \varpi(\Lambda) D$. Using Theorem 3.6 of [7], the left D -module $E' = D^{1 \times (p' + 1)} / (D^{1 \times p} Q')$, where $Q' := (P \quad -\Lambda) \in D^{p \times (p' + 1)}$, is stably free of rank $p' + 1 - p$. If $p' + 1 - p \geq 2$, i.e., $p' - p \geq 1$, then E' is free by Theorem 6 and the result follows from Theorem 4.1 of [7]. \square

Since *holonomic modules* over the standard rings of PD operators are cyclic (see, e.g., [6, 29]), the second part of Corollary 3 yields Theorem 30 of [15].

Remark 9. Corollary 3 is most interesting in the case of a finitely generated left D -module M satisfying $\text{rank}_D(M) \geq 1$, i.e., for underdetermined linear systems. Indeed, if $P \in D^{p' \times p'}$ has full row rank, i.e., $\text{rank}_D(M) = 0$, then Corollary 3 yields again 1 of Corollary 2.

Example 12. Since the *global dimension* of the rings $A_1(k)$, $B_1(k)$, $k[[t]]\langle \partial \rangle$, and $k\{t\}\langle \partial \rangle$ where $k \in \{\mathbb{R}, \mathbb{C}\}$, is one (see, e.g., [6, 29]), Section 3 of [37] shows that the first part of Corollary 3 holds for these rings. Hence, any underdetermined linear system of OD equations (namely, $\text{rank}_D(M) = p' - p \geq 1$) with either polynomial, rational, formal power series or locally convergent power series coefficients can be reduced to an equivalent linear OD system defined by at most two OD equations.

Following Stafford ([44]), let us now show how to use Corollary 2 to obtain the minimal number of generators of a finitely generated left D -module L which is not torsion.

First of all, if L can be generated by r elements, then there exists a surjective left D -homomorphism $\omega : D^{1 \times r} \longrightarrow L$ defined by mapping the i^{th} element of the standard basis of $D^{1 \times r}$ to the i^{th} generator of L , i.e., we have the following short exact sequence:

$$0 \longrightarrow \ker \omega \longrightarrow D^{1 \times r} \xrightarrow{\omega} L \longrightarrow 0. \quad (40)$$

If $\text{rank}_D(L) = r$, then the equality $\text{rank}_D(L) + \text{rank}_D(\ker \omega) = r$ yields $\text{rank}_D(\ker \omega) = 0$, i.e., $\ker \omega$ is either 0 or a torsion left D -module. But, $\ker \omega$ is a left D -submodule of the free, and thus of the torsion-free left D -module $D^{1 \times r}$, which yields $\ker \omega = 0$, and proves that $L \cong D^{1 \times r}$, i.e., L is a free left D -module of rank r . Conversely, if L is a finitely generated free left D -module of rank r , then L can be generated by r elements. Hence, a finitely generated left D -module L can be generated by $r := \text{rank}_D(L)$ elements if and only if $L \cong D^{1 \times r}$.

We note that a finitely generated left D -module L cannot be generated by fewer elements than $\text{rank}_D(L)$ since the short exact sequence (40) yields $r - \text{rank}_D(L) = \text{rank}_D(\ker \omega) \geq 0$.

Now, let us suppose that L is not free, i.e., cannot be generated by $r := \text{rank}_D(L)$ elements. By Theorem 7, there exists a left D -module L' with $\text{rank}_D(L') \leq 1$ such that $L \cong D^{1 \times s} \oplus L'$. If $\text{rank}_D(L') = 0$, i.e., L' is a torsion left D -module, then $r = s$, which yields $L \cong D^{1 \times (r-1)} \oplus (D \oplus L')$, where $\text{rank}_D(D \oplus L') = 1$. Now, if $\text{rank}_D(L') = 1$, then $s = r - 1$. Hence, with a change of notations, we can always suppose that $L \cong D^{1 \times (r-1)} \oplus L'$, where $\text{rank}_D(L') = 1$. Since L' is not a torsion left D -module, there exists $l' \in L' \setminus t(L')$. Thus, $\text{ann}_D(l') = \{d \in D \mid dl' = 0\} = 0$, i.e., $Dl' \cong D/\text{ann}_D(l') \cong D$. Now, the following short exact sequence

$$0 \longrightarrow Dl' \longrightarrow L' \longrightarrow L'/(Dl') \longrightarrow 0$$

yields $\text{rank}_D(L'/(Dl')) = 0$, i.e., $L'/(Dl')$ is a torsion left D -module. Now, 2 of Corollary 2 shows that there exist a projective left ideal I of D and a left D -submodule J of I such that $L'/(Dl') \cong I/J$. Thus, we get the short exact sequence $0 \longrightarrow Dl' \xrightarrow{\alpha} L' \xrightarrow{\beta} I/J \longrightarrow 0$. Hence, the following diagram has an exact row and exact columns

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & J & & \\ & & 0 & & \downarrow & & \\ & & \downarrow & & I & & \\ & & D & & \downarrow & & \\ & & \downarrow & & I/J & & \\ 0 & \longrightarrow & Dl' & \xrightarrow{\alpha} & L' & \xrightarrow{\beta} & I/J \longrightarrow 0, \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

which can be completed to the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} 0 & \longrightarrow & D & \xrightarrow{u} & D \oplus I & \xrightarrow{v} & I \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Dl' & \xrightarrow{\alpha} & L' & \xrightarrow{\beta} & I/J \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

For more details, see, e.g., [40]. We now note that $D \oplus I$ is a projective left D -module with $\text{rank}_D(D \oplus I) = 2$, and thus $D \oplus I$ is a free left D -module of rank two by Theorem 6, i.e., $D \oplus I \cong D^{1 \times 2}$. Therefore, L' can be defined by two generators (but no fewer), which finally shows that L can be generated by $r - 1 + 2 = r + 1$ elements but no fewer.

Lemma 6. *Let M be a non-free left D -module with $\text{rank}_D(M) = 1$. Then, M can be generated by two elements and no fewer.*

Let us explain how to get a presentation of $L' = D^{1 \times r}/(D^{1 \times s} Q)$ defined by only two generators. Since $\text{rank}_D(L') = 1$, $\text{rank}_D(D^{1 \times s} Q) = r - 1$. If $r - 1 = 1$, i.e., $r = 2$, then nothing has to be done. Let us now suppose that $r - 1 \geq 2$, i.e., $r \geq 3$. Then, applying Algorithm 3, we get $L' \cong L'' = D^{1 \times (r-1)}/(D^{1 \times t} Q')$ where $\text{rank}_D(D^{1 \times t} Q') = r - 2$. Repeating the same argument, we finally get $L' \cong \bar{L} = D^{1 \times 2}/(D^{1 \times l} \bar{Q})$.

Example 13. Using standard results on the minimal number of generators of finitely presented modules over a commutative ring based on *Fitting ideals* (see, e.g., [19]), we can easily check that the $E = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$ -module $L = E^{1 \times 3}/(E^{1 \times 3} R)$, where R is the matrix (21) defining the curl operator in \mathbb{R}^3 (see Example 7), can be generated by three elements but no fewer. Now, if we consider the left $D = A_3(\mathbb{Q})$ -module $M = D^{1 \times 3}/(D^{1 \times 3} R) \cong D \otimes_E L$, then $\text{rank}_D(M) = 1$ and Lemma 6 shows that M can be generated by two elements but no fewer. If $\{f_j\}_{j=1,2,3}$ is the standard basis of $D^{1 \times 3}$, $\pi : D^{1 \times 3} \rightarrow M$ the canonical projection onto M , $\{y_j = \pi(f_j)\}_{j=1,2,3}$ a family of generators of M , and $y = (y_1 \ y_2 \ y_3)^T$, then it was shown in Example 7 that $\{z_1 = y_1, z_2 = (x_1 + 1)y_2 + y_3\}$ is also a family of generators of M and $y = Uz$, where the matrix U is defined by (32) and $z = (z_1 \ z_2)^T$.

Let us sum up the above remarks in the next theorem due to Stafford.

Theorem 10 ([44]). *Let L be a finitely generated left D -module that is not a torsion module, i.e., $\text{rank}_D(L) \geq 1$. Then, the following statements are equivalent:*

1. $\text{rank}_D(L) = r$.
2. *Either L is a free left D -module of rank r , i.e., $L \cong D^{1 \times r}$, and thus L can be generated by r elements, or L is not a free left D -module and L can be generated by $r + 1$ elements but no fewer.*

Example 14. If $M = D^{1 \times 3}/(DR)$ is the left $D = A_3(\mathbb{Q})$ -module finitely presented by the divergence operator R , then it is well-known that M is torsion-free but not free (see [11, 37]). Since $\text{rank}_D(M) = 2$, by Theorem 10, M can be generated by three elements. Hence, R is a *minimal presentation* of M , namely, a presentation of M having the minimal number of generators $\mu(M)$ of M .

Remark 10. Corollary 2 shows that a finitely generated torsion left D -module M is either cyclic, i.e., generated by one element, or can be generated by two elements. Theorem 10 (see also Lemma 6) shows that a finitely generated left D -module of rank one is either free, i.e., cyclic, or can be generated by two elements and no fewer. Hence, a finitely generated left D -module M satisfying $\text{rank}_D(M) \leq 1$ can always be generated by one or two elements. From a systems theory viewpoint, Remark 5 and the above results show that the study of $\text{ext}_D^i(M, \mathcal{F})$ for $i \geq 0$ ($\text{ext}_D^0(M, \mathcal{F}) = \text{hom}_D(M, \mathcal{F})$) can be reduced to $\text{ext}_D^i(M', \mathcal{F})$ for $i \geq 0$, where $\text{rank}_D(M') \leq 1$, and thus to the case of a left D -module M' which is either zero, can be generated by one or by two elements. Finally, Theorem 10 also shows that a finitely generated left D -module M which cannot be generated by two elements satisfies $\text{rank}_D(M) \geq 2$.

6 Cancellation theorem

6.1 Main result

Let us state a result due to Stafford (which can be traced back to Swan [45]).

Lemma 7 ([44]). *Let D be a very simple domain satisfying (9), M a finitely generated left D -module with $\text{rank}_D(M) \geq 2$, $d^* \in D$, $m^* \in M$ such that (d^*, m^*) is a unimodular element of $D \oplus M$. Then, there exists $\vartheta \in \text{hom}_D(D, M)$ such that:*

$$\vartheta(d^*) + m^* \in U(M).$$

Proof. Since $(d^*, m^*) \in U(D \oplus M)$, there exist $\omega_1 \in \text{hom}_D(D, D)$ and $\omega_2 \in \text{hom}_D(M, D)$ such that $\omega = \omega_1 \oplus \omega_2 \in \text{hom}_D(D \oplus M, D)$ satisfies:

$$\omega((d^*, m^*)) = \omega_1(d^*) + \omega_2(m^*) = 1. \quad (41)$$

Let $R \in D^{q \times p}$ be such that $M = D^{1 \times p} / (D^{1 \times q} R)$ and $\pi : D^{1 \times p} \rightarrow M$ the canonical projection onto M . Since $\text{rank}_D(M) \geq 2$, applying the first seven steps of Algorithm 1, there exist two elements $m_1 = \pi(\lambda_1)$ and $m_2 = \pi(\lambda_2)$ of M , where $\lambda_1, \lambda_2 \in D^{1 \times p}$, and two elements μ_1 and μ_2 of D^p such that $\varphi_i := \varphi_{\mu_i} \in \text{hom}_D(M, D)$, $i = 1, 2$, satisfy:

$$\varphi_1(m_1) \neq 0, \quad \varphi_1(m_2) = 0, \quad \varphi_2(m_1) = 0, \quad \varphi_2(m_2) \neq 0. \quad (42)$$

Using (9), there exist $r, s \in D$ such that:

$$\varphi_1(m^*) D + \varphi_2(m^*) D + d^* D = (\varphi_1(m^*) + d^* r \varphi_1(m_1)) D + (\varphi_2(m^*) + d^* s \varphi_2(m_2)) D. \quad (43)$$

In particular, there exist $\alpha, \beta \in D$ such that:

$$d^* = (\varphi_1(m^*) + d^* r \varphi_1(m_1)) \alpha + (\varphi_2(m^*) + d^* s \varphi_2(m_2)) \beta. \quad (44)$$

Using the right D -module structure of $\text{hom}_D(M, D)$, let $\chi := \varphi_1 \alpha + \varphi_2 \beta \in \text{hom}_D(M, D)$ be defined by $\chi(m) = \varphi_1(m) \alpha + \varphi_2(m) \beta$ for all $m \in M$. Let us also consider $\vartheta \in \text{hom}_D(D, M)$ defined by $\vartheta(d) = d(r m_1 + s m_2)$ for all $d \in D$. Then, using (42), we get

$$\begin{aligned} \chi(\vartheta(d^*)) &= \chi(d^* \vartheta(1)) = d^* \chi(r m_1 + s m_2) \\ &= d^* (\varphi_1(r m_1 + s m_2) \alpha + \varphi_2(r m_1 + s m_2) \beta) \\ &= d^* (r \varphi_1(m_1) \alpha + s \varphi_2(m_2) \beta), \end{aligned}$$

which combined with (44) yields:

$$\begin{aligned} \chi(m^* + \vartheta(d^*)) &= \varphi_1(m^*) \alpha + \varphi_2(m^*) \beta + d^* (r \varphi_1(m_1) \alpha + s \varphi_2(m_2) \beta) \\ &= (\varphi_1(m^*) + d^* r \varphi_1(m_1)) \alpha + (\varphi_2(m^*) + d^* s \varphi_2(m_2)) \beta = d^*. \end{aligned} \quad (45)$$

Let us define $\psi := (\omega_1 - \omega_2 \circ \vartheta) \oplus \omega_2 \in \text{hom}_D(D \oplus M, D)$. Then, using (41), we get

$$\psi((d^*, m^* + \vartheta(d^*))) = \omega_1(d^*) - \omega_2(\vartheta(d^*)) + \omega_2(m^* + \vartheta(d^*)) = \omega_1(d^*) + \omega_2(m^*) = 1,$$

which shows that $(d^*, m^* + \vartheta(d^*)) \in U(D \oplus M)$. Let us define $t := \omega_1(1) - \omega_2(\vartheta(1)) \in D$ and $\varphi := \omega_2 + \chi t \in \text{hom}_D(M, D)$. Thus, using (41) and (45), we obtain

$$\begin{aligned} \varphi(m^* + \vartheta(d^*)) &= (\omega_2 + \chi t)(m^* + \vartheta(d^*)) \\ &= \omega_2(m^*) + \omega_2(\vartheta(d^*)) + \chi(m^* + \vartheta(d^*)) t \\ &= 1 - \omega_1(d^*) + \omega_2(d^* \vartheta(1)) + d^* t \\ &= 1 - d^* \omega_1(1) + d^* \omega_2(\vartheta(1)) + d^* t \\ &= 1 - d^* t + d^* t = 1, \end{aligned}$$

which finally shows that $\tilde{m} := m^* + \vartheta(d^*) \in U(M)$ and $\varphi(\tilde{m}) = 1$. \square

If D admits an involution, then (9) is a direct consequence of (7) as noticed in Remark 3.

Note that if $d^* = 0$, then $m^* \in \text{U}(M)$ and we can choose $\vartheta = \text{id}_D$.

Remark 11. If $m^* \notin t(M)$, then we can choose $m_1 = m^*$ so that $\varphi_2(m_1) = \varphi_2(m^*) = 0$ and (43) simplifies to $\varphi_1(m^*) D + d^* D = (1 + d^* r) \varphi_1(m^*) D + d^* s \varphi_2(m_2) D$, i.e., we need to find $r, s, \alpha, \beta \in D$ such that $d^* = (1 + d^* r) \varphi_1(m^*) \alpha + d^* s \varphi_2(m_2) \beta$.

Remark 12. If $\text{rank}_D(M) \geq 2$ and $m \in M$, then considering $\omega_1 = \text{id}_D$ and $\omega_2 = 0$, then $(1, m) \in \text{U}(D \oplus M)$. Lemma 7 then shows that there exists $\vartheta \in \text{hom}_D(D, M)$ such that $m' := \vartheta(1) + m \in \text{U}(M)$, and thus $M = D m' \oplus M'$ for a certain left D -module M' , which proves again Theorem 7. If $m \notin t(M)$, then using Remark 11 with $d^* = 1$, the unimodular element m' can be obtained by computing a solution of the following inhomogeneous quadratic equation:

$$1 = (1 + r) \varphi_1(m_1) \alpha + s \varphi_2(m_2) \beta.$$

We find again the main argument of the proof of Theorem 7.

Algorithm 5 (ReductionOfUnimodularElement).

- **Input:** A very simple domain D satisfying (9), $R \in D^{q \times p}$ a matrix such that the left D -module $M = D^{1 \times p} / (D^{1 \times q} R)$ has rank at least two, i.e., $\text{rank}_D(M) \geq 2$, $d^* \in D$, $\lambda^* \in D^{1 \times p}$ such that $(d^*, m^*) \in \text{U}(D \oplus M)$, where $m^* = \pi(\lambda^*)$ and $\pi : D^{1 \times p} \rightarrow M$ is the canonical projection, and $e \in D$ and $\mu \in \ker_D(R.)$ such that $\omega \in \text{hom}_D(D \oplus M, D)$ defined by

$$\forall d \in D, \quad \forall \lambda \in D^{1 \times p}, \quad \omega((d, \pi(\lambda))) = d e + \lambda \mu,$$

satisfies $\omega((d^*, m^*)) = 1$, i.e., $d^* e + \lambda^* \mu = 1$.

- **Output:**

1. $\bar{\lambda} \in D^{1 \times p}$ such that $\vartheta \in \text{hom}_D(D, M)$ defined by $\vartheta(d) = d \bar{m}$ for all $d \in D$, where $\bar{m} := \pi(\bar{\lambda})$, satisfies $m^* + \vartheta(d^*) = m^* + d^* \bar{m} \in \text{U}(M)$.
2. $\bar{\mu} \in \ker_D(R.)$ such that $\varphi := \varphi_{\bar{\mu}} \in \text{hom}_D(M, D)$ satisfies $\varphi(m^* + \vartheta(d^*)) = 1$, i.e.:

$$(\lambda^* + d^* \bar{\lambda}) \bar{\mu} = 1. \tag{46}$$

1. Applying steps 1-7 of **UnimodularElement** (see Algorithm 1) to R , we get $\lambda_1, \lambda_2 \in D^{1 \times p}$ and $\mu_1, \mu_2 \in \ker_D(R.)$ such that $d_1 := \varphi_1(m_1) \neq 0$, $\varphi_1(m_2) = 0$, $d_2 := \varphi_2(m_2) \neq 0$, and $\varphi_2(m_1) = 0$, where $\varphi_i := \varphi_{\mu_i} \in \text{hom}_D(M, D)$, and $m_i = \pi(\lambda_i)$ for $i = 1, 2$. Note that if $\pi(\lambda) \notin t(M)$, then we can take $\lambda_1 := \lambda$.
2. Using (9), compute $r, s \in D$ such that:

$$(\lambda^* \mu_1) D + (\lambda^* \mu_2) D + d^* D = (\lambda^* \mu_1 + d^* r \lambda_1 \mu_1) D + (\lambda^* \mu_2 + d^* s \lambda_2 \mu_2) D.$$

3. Define $\bar{\lambda} := r \lambda_1 + s \lambda_2$.
4. Right factorize d^* by $S = ((\lambda^* + d^* r \lambda_1) \mu_1 \quad (\lambda^* + d^* s \lambda_2) \mu_2)$ to get $d^* = S (\alpha \quad \beta)^T$, where $\alpha, \beta \in D$.
5. Define $\mu^* := \mu_1 \alpha + \mu_2 \beta \in \ker_D(R.)$, $t := e - \bar{\lambda} \mu$, and $\bar{\mu} := \mu + \mu^* t \in \ker_D(R.)$.

6. Return $\bar{\lambda}$ and $\bar{\mu}$.

Let us check (46). Using $\lambda_1 \mu_2 = 0$, $\lambda_2 \mu_1 = 0$, and 4 of Algorithm 5, we first get:

$$\begin{aligned} (\lambda^* + d^*(r \lambda_1 + s \lambda_2))(\mu_1 \alpha + \mu_2 \beta) &= \lambda^*(\mu_1 \alpha + \mu_2 \beta) + d^*(r \lambda_1 + s \lambda_2)(\mu_1 \alpha + \mu_2 \beta) \\ &= \lambda^*(\mu_1 \alpha + \mu_2 \beta) + d^*(r \lambda_1 \mu_1 \alpha + s \lambda_2 \mu_2 \beta) \\ &= (\lambda^* + d^* r \lambda_1) \mu_1 \alpha + (\lambda^* + d^* s \lambda_2) \mu_2 \beta = d^*. \end{aligned}$$

With the notations of 5 of Algorithm 5, using $\bar{\mu} = \mu + (\mu_1 \alpha + \mu_2 \beta) t$, $t = e - (r \lambda_1 + s \lambda_2) \mu$, $d^* e + \lambda^* \mu = 1$, and the above identity, it follows:

$$\begin{aligned} (\lambda^* + d^* \bar{\lambda}) \bar{\mu} &= (\lambda^* + d^*(r \lambda_1 + s \lambda_2))(\mu + (\mu_1 \alpha + \mu_2 \beta) t) \\ &= \lambda^* \mu + d^*(r \lambda_1 + s \lambda_2) \mu + (\lambda^* + d^*(r \lambda_1 + s \lambda_2))(\mu_1 \alpha + \mu_2 \beta) t \\ &= 1 - d^* e + d^*(r \lambda_1 + s \lambda_2) \mu + d^* t \\ &= 1 - d^*(e - (r \lambda_1 + s \lambda_2) \mu) + d^* t = 1. \end{aligned}$$

Remark 13. If $M = D^{1 \times r}$ with $r \geq 2$, then Theorem 7 and Algorithm 5 show that for every $(d^* \ \lambda^*) \in U(D^{1 \times (1+r)})$, where $d^* \in D$ and $\lambda^* \in D^{1 \times r}$, i.e., such that $(d^* \ \lambda^*)$ admits a right inverse over D , there exists $\bar{\lambda} \in D^{1 \times r}$ such that $\tilde{\lambda} := \lambda + d^* \bar{\lambda} \in U(D^{1 \times r})$, i.e., such that $\tilde{\lambda}$ admits a right inverse over D . This result comes from the fact that the *stable rank* of D is at most 2, i.e., $\text{sr}(D) \leq 2$, which is a direct consequence of (7) as explained in [37].

Example 15. Let us consider again Example 5. Let $d^* = \partial_1$, $\lambda^* = (0 \ 0 \ -x_1)$, and $m^* = \pi(\lambda^*)$. If $e = x_1$ and $\mu = (-\partial_3 \ 0 \ \partial_1)^T \in \ker_D(R.)$, then $d^* e + \lambda^* \mu = 1$, which implies that $d^* \oplus m^* \in U(D \oplus M)$ since $\varphi \in \text{hom}_D(D \oplus M, D)$ defined by $\varphi(d \oplus \pi(\lambda)) = d e + \lambda \mu$ for all $d \in D$ and for all $\lambda \in D^{1 \times 3}$ satisfies $\varphi(d^* \oplus m^*) = d^* e + \lambda^* \mu = 1$. By Lemma 7, there exists $\bar{\lambda} \in D^{1 \times 3}$ such that $\vartheta \in \text{hom}_D(D, M)$ defined by $\vartheta(d) = d \pi(\bar{\lambda})$ for all $d \in D$ satisfies that $m^* + \vartheta(d^*) = \pi(\lambda^* + d^* \bar{\lambda}) \in U(M)$. Using Algorithm 5, let us compute $\bar{\lambda}$.

Let $\lambda_1 = (0 \ -1 \ 0)$, $\lambda_2 = (0 \ 0 \ 1)$, $\mu_1 = (\partial_2 \ -\partial_1 \ 0)^T$, and $\mu_2 = (-\partial_3 \ 0 \ \partial_1)^T$. Then, using Example 5, we have $d_1 := \varphi_1(m_1) = \partial_1$, $\varphi_1(m_2) = 0$, $d_2 := \varphi_2(m_2) = \partial_1$, and $\varphi_2(m_1) = 0$, where $\varphi_i := \varphi_{\mu_i} \in \text{hom}_D(M, D)$, and $m_i = \pi(\lambda_i)$ for $i = 1, 2$. Then, $d_3 := \varphi_1(m^*) = \lambda^* \mu_1 = 0$ and $d_4 := \varphi_2(m^*) = \lambda^* \mu_2 = -x_1 \partial_1$. Using Remark 3, we can compute $r, s \in D$ satisfying (43) by computing $u, v \in D$ such that

$$D \theta(d_3) + D \theta(d_4) + D \theta(d^*) = D(\theta(d_3) + \theta(d_1) u \theta(d^*)) + D(\theta(d_4) + \theta(d_2) v \theta(d^*)),$$

where $\theta(d_3) = 0$, $\theta(d_4) = x_1 \partial_1 + 1$, and $\theta(d^*) = -\partial_1$, and then defining $r = \theta(u)$ and $s = \theta(v)$. We can take $u = 1$ and $v = 0$, and thus $r = 1$ and $s = 0$. Then, we obtain $\bar{\lambda} = \lambda_1$ and $\Lambda := \lambda^* + d^* \bar{\lambda} = (0 \ -\partial_1 \ -x_1)$ is such that $\pi(\Lambda) \in U(M)$.

Let us compute $\bar{\mu} \in \ker_D(R.)$ such that $\Lambda \bar{\mu} = 1$, i.e., such that $\varphi \in \text{hom}_D(M, D)$ defined by $\varphi(\pi(\lambda)) = \lambda \bar{\mu}$ for all $\lambda \in D^{1 \times 3}$ satisfies $\varphi(\pi(\Lambda)) = 1$. Right factoring $d^* = \partial_1$ by the matrix $S = (\partial_1^2 \ -x_1 \partial_1)$, we get $d^* = S(\alpha \ \beta)^T$, where $\alpha = \frac{1}{2} x_1$ and $\beta = \frac{1}{2} \partial_1$, which yields $\mu^* = \frac{1}{2} (-\partial_1 \partial_3 + x_1 \partial_2 \ -x_1 \partial_1 - 1 \ \partial_1^2)^T$, $t = x_1$, and:

$$\bar{\mu} = \frac{1}{2} (-x_1 \partial_1 \partial_3 + x_1^2 \partial_2 - 3 \partial_3 \ -x_1(x_1 \partial_1 + 2) \ (x_1 \partial_1 + 4) \partial_1)^T.$$

Finally, we note that we can also use Remark 2 to compute $\bar{\mu}$.

Lemma 7 is the key result for the so-called (*Bass'*) *cancellation theorem* ([44]) which states that $M \oplus D \cong N \oplus D$ implies $M \cong N$ when $\text{rank}_D(M) \geq 2$. Let us explicitly describe this last isomorphism following the proof of Corollary 12.6 of [45] as pointed out in [44].

Let $f : D \oplus M \rightarrow D \oplus N$ be a left D -isomorphism and $(d^*, n^*) := f((1, 0))$. Now, if $g : D \oplus M \rightarrow D$ is defined by $g((d, m)) = d$ for all $d \in D$ and for all $m \in M$, then $(g \circ f^{-1})(d^*, n^*) = g((1, 0)) = 1$, which shows that $(d^*, n^*) \in U(D \oplus N)$. By Lemma 7, there exists $\vartheta \in \text{hom}_D(D, N)$ such that $\tilde{n} := n^* + \vartheta(d^*) \in U(N)$. Thus, there exists $\varphi \in \text{hom}_D(N, D)$ such that $\varphi(\tilde{n}) = 1$. Now, let us consider the following left D -homomorphisms:

$$\begin{aligned} k : D &\rightarrow D & h : D \oplus N &\rightarrow D \oplus N & l : D \oplus N &\rightarrow D \oplus N \\ d &\mapsto dd^*, & (d, n) &\mapsto (d, n + \vartheta(d)), & (d, n) &\mapsto (d - (k \circ \varphi)(n), n). \end{aligned} \tag{47}$$

Since h and l are automorphisms of $D \oplus N$, $i := l \circ h \circ f : D \oplus M \rightarrow D \oplus N$ is a left D -isomorphism. Moreover, $i((1, 0)) = (l \circ h)((d^*, n^*)) = l((d^*, \tilde{n})) = (d^* - (k \circ \varphi)(\tilde{n}), \tilde{n}) = (0, \tilde{n})$.

Now, $N = D\tilde{n} \oplus \ker \varphi \cong D \oplus \ker \varphi$ since $\tilde{n} \in U(N)$. If $X := D \oplus N = D \oplus (D\tilde{n} \oplus \ker \varphi)$, then $Y := X / (0 \oplus D\tilde{n}) \cong D \oplus \ker \varphi \cong D\tilde{n} \oplus \ker \varphi = N$. Using (5), we have the direct sum decomposition

$$\forall n \in N, \quad n = \varphi(n)\tilde{n} + (n - \varphi(n)\tilde{n}), \quad \varphi(n)\tilde{n} \in D\tilde{n}, \quad n - \varphi(n)\tilde{n} \in \ker \varphi,$$

and the left D -isomorphism $N \cong Y$ is then defined by

$$\begin{aligned} \varepsilon : N &\rightarrow Y & \varepsilon^{-1} : Y &\rightarrow N \\ n &\mapsto \sigma_1((\varphi(n), n)), & \sigma_1((d, n)) &\mapsto n + (d - \varphi(n))\tilde{n}, \end{aligned}$$

where $\sigma_1 : X = D \oplus N \rightarrow Y$ is the canonical projection onto Y .

If $i_1 \in \text{hom}_D(D, D \oplus N)$ (resp., $i_2 \in \text{hom}_D(D, D \oplus M)$) is defined by $i_1(1) = (0, \tilde{n})$ (resp., $i_2(1) = (1, 0)$), then $\text{im } i_1 = D(0, \tilde{n})$ and $\text{im } i_2 = D(1, 0)$, which yields

$$\text{coker } i_1 = Y, \quad Z := \text{coker } i_2 = (D \oplus M) / (D(1, 0)) \cong M,$$

where the last left D -isomorphism is defined by

$$\begin{aligned} \kappa : M &\rightarrow Z \\ m &\mapsto \sigma_2((0, m)), \end{aligned}$$

where $\sigma_2 : D \oplus M \rightarrow Z$ is the canonical projection onto Z .

Using $i \circ i_2 = i_1$, the following diagram is commutative with exact rows and columns

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D & \xrightarrow{\text{id}_D} & D & \longrightarrow & 0 \\ & & \downarrow i_2 & & \downarrow i_1 & & \\ 0 & \longrightarrow & D \oplus M & \xrightarrow{i} & D \oplus N & \longrightarrow & 0, \\ & & \downarrow \sigma_2 & & \downarrow \sigma_1 & & \\ & & Z & \xrightarrow{\iota} & Y & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where $\iota(\sigma_2((d, m))) = \sigma_1(i((d, m)))$ for all $d \in D$ and for all $m \in M$. With the notation $(e, n) := i((0, m))$, the left D -isomorphism $\varpi := \varepsilon^{-1} \circ \iota \circ \kappa \in \text{hom}_D(M, N)$ is defined by:

$$\begin{aligned} \varpi : M &\longrightarrow N \\ m &\longmapsto n + (e - \varphi(n)) \tilde{n}. \end{aligned} \quad (48)$$

Theorem 11 ([44]). *Let D be a very simple domain satisfying (9), M and N two finitely generated left D -modules, and $\text{rank}_D(M) \geq 2$. Then, $M \oplus D \cong N \oplus D$ implies $M \cong N$.*

A similar result holds for right D -modules.

Let us give an explicit description of ϖ . Let $M = D^{1 \times p} / (D^{1 \times q} R)$, $N = D^{1 \times p'} / (D^{1 \times q'} R')$, $\pi : D^{1 \times p} \rightarrow M$ (resp., $\pi' : D^{1 \times p'} \rightarrow N$) be the canonical projection onto M (resp., N), $P = \begin{pmatrix} 0 & R \end{pmatrix} \in D^{q \times (1+p)}$, $P' = \begin{pmatrix} 0 & R' \end{pmatrix} \in D^{q' \times (1+p')}$, $L = D^{1 \times (1+p)} / (D^{1 \times q} P) \cong D \oplus M$, $L' = D^{1 \times (1+p')} / (D^{1 \times q'} P') \cong D \oplus N$, and $\tau : D^{1 \times (1+p)} \rightarrow L$ (resp., $\tau' : D^{1 \times (1+p')} \rightarrow L'$) the canonical projection onto L (resp., L'). Let f be a left D -isomorphism defined by

$$\begin{aligned} f : L &\longrightarrow L' \\ \tau(\zeta) &\longmapsto \tau'(\zeta X), \end{aligned} \quad (49)$$

where:

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in D^{(1+p) \times (1+p')}, \quad X_{11} \in D, \quad X_{12} \in D^{1 \times p'}, \quad X_{21} \in D^p, \quad X_{22} \in D^{p \times p'}.$$

Then, $f(\tau \begin{pmatrix} 1 & 0 \end{pmatrix}) = \tau'(\begin{pmatrix} X_{11} & X_{12} \end{pmatrix})$. Now, applying Lemma 7, there exist $\bar{\lambda} \in D^{1 \times p'}$ and $\bar{\mu} \in \ker_D(R')$ such that $(X_{12} + X_{11} \bar{\lambda}) \bar{\mu} = 1$. The left D -isomorphism i is then defined by

$$\begin{aligned} i : L &\longrightarrow L' \\ \tau(\zeta) &\longmapsto \tau'(\zeta X'), \end{aligned}$$

where:

$$X' = \begin{pmatrix} X'_{11} & X'_{12} \\ X'_{21} & X'_{22} \end{pmatrix} := \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} 1 & \bar{\lambda} \\ 0 & I_{p'} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{\mu} X_{11} & I_p \end{pmatrix} \quad (50)$$

$$= \begin{pmatrix} 0 & X_{11} \bar{\lambda} + X_{12} \\ X_{21} - (X_{21} \bar{\lambda} + X_{22}) \bar{\mu} X_{11} & X_{21} \bar{\lambda} + X_{22} \end{pmatrix}. \quad (51)$$

We note that the last matrix in (50) defines an automorphism of L' since $R' \bar{\mu} = 0$. Using (48), the left D -isomorphism ϖ is then defined by:

$$\begin{aligned} \varpi : M &\longrightarrow N \\ \pi(\lambda) &\longmapsto \pi'(\lambda (X'_{22} + (X'_{21} - X'_{22} \bar{\mu}) X'_{12})). \end{aligned} \quad (52)$$

Remark 14. We note that the matrix defining (52) can be obtained from X' as the entry at position (2, 2) of the following matrix:

$$\begin{pmatrix} 1 & 0 \\ X'_{21} - X'_{22} \bar{\mu} & I_p \end{pmatrix} \begin{pmatrix} 0 & X'_{12} \\ X'_{21} & X'_{22} \end{pmatrix} = \begin{pmatrix} 0 & X'_{12} \\ X'_{21} & X'_{22} + (X'_{21} - X'_{22} \bar{\mu}) X'_{12} \end{pmatrix}.$$

Let us explicitly describe the algorithm realizing Stafford's cancellation theorem (see Theorem 11).

Algorithm 6 (CancellationTheorem).

- **Input:** A very simple domain D satisfying (9), $R \in D^{q \times p}$ such that the rank of the left D -module $M = D^{1 \times p} / (D^{1 \times q} R)$ is at least two, $R' \in D^{q' \times p'}$, $P = \begin{pmatrix} 0 & R \end{pmatrix} \in D^{q \times (1+p)}$, $P' = \begin{pmatrix} 0 & R' \end{pmatrix} \in D^{q' \times (1+p')}$, and $X \in D^{(1+p) \times (1+p')}$ defining the left D -isomorphism (49).
- **Output:** $Y \in D^{p \times p'}$ defining the left D -isomorphism (52).

1. Select the first row $(X_{11} \ X_{12})$ of X , where $X_{11} \in D$ and $X_{12} \in D^{1 \times p'}$.
2. Compute $Q' \in D^{p' \times m'}$ such that $\ker_D(R'.) = Q' D^{m'}$.
3. Compute a right inverse $(\tilde{e} \ \tilde{\mu}^T)^T \in D^{1+m'}$ of $(X_{11} \ X_{12} Q')$.
4. Apply Algorithm 5 to $(d^*, m^*) = (X_{11}, \pi'(X_{12})) \in U(D \oplus N)$, $e = \tilde{e}$, $\mu = Q' \tilde{\mu}$, where $N = D^{1 \times p'} / (D^{1 \times q'} R')$ and $\pi' : D^{1 \times p'} \rightarrow N$ is the canonical projection onto N , to get $\bar{\lambda} \in D^{1 \times p'}$ and $\bar{\mu} \in \ker_D(R'.)$ such that:

$$(X_{12} + X_{11} \bar{\lambda}) \bar{\mu} = 1.$$

5. Compute the following matrices:

$$X'_{12} := X_{11} \bar{\lambda} + X_{12}, \quad X'_{21} := X_{21} - (X_{21} \bar{\lambda} + X_{22}) \bar{\mu} X_{11}, \quad X'_{22} := X_{21} \bar{\lambda} + X_{22}.$$

6. Return $Y := X'_{22} + (X'_{21} - X'_{22} \bar{\mu}) X'_{12}$.

Algorithm 6 will be illustrated in the next section.

6.2 Applications

6.2.1 Computation of bases

Let D be a very simple domain satisfying (9) and M a finitely generated stably free left D -module. By definition (see Definition 1), there exist $r, s \in \mathbb{Z}_{\geq 0}$ such that $M \oplus D^{1 \times s} \cong D^{1 \times r}$. Since D is a noetherian domain, we have $0 \leq s \leq r$ (see, e.g., [25]). If $\text{rank}_D(M) = r - s \geq 2$, then applying Stafford's cancellation theorem (see Theorem 11), we obtain $M \cong D^{1 \times (r-s)}$, i.e., M is a free left D -module of rank $r - s \geq 2$, which gives a proof of Theorem 6.

This approach to compute bases of free left $A_n(k)$ -modules of rank at least two was pursued in [20]. Below, we give a proof which is different from the one in [20]. Indeed, the approach of [20] is based on versions of Lemma 7 and Theorem 11 for a left/right D -submodule M of a finitely generated free left/right D -module, and not for a general finitely generated (i.e., presented) left/right D -module M as it is done in this paper. In other words, [20] considers $\ker_D(R.) \subseteq D^p$ whereas we consider the finitely presented left D -module $\text{coker}_D(.R) = D^{1 \times p} / (D^{1 \times q} R)$.

Given a presentation matrix of a finitely generated stably free left D -module L , in [37], it is shown how to compute a matrix $R \in D^{q \times p}$ which admits a right inverse $S \in D^{p \times q}$, i.e.,

$RS = I_q$, and is such that the left D -module $M := D^{1 \times p}/(D^{1 \times q}R)$ is isomorphic to L . Then, the following short exact sequence

$$0 \longrightarrow D^{1 \times q} \xrightarrow{\cdot R} D^{1 \times p} \xrightarrow{\pi} M \longrightarrow 0$$

splits, namely, $D^{1 \times q} \oplus M \cong D^{1 \times p}$ (see, e.g., [40]), and this left D -isomorphism is defined by:

$$\begin{aligned} g : D^{1 \times q} \oplus M &\longrightarrow D^{1 \times p} \\ (\theta, \pi(\lambda)) &\longmapsto (\theta \ \lambda) \begin{pmatrix} R \\ I_p - SR \end{pmatrix}, \quad g^{-1} : D^{1 \times p} \longrightarrow D^{1 \times q} \oplus M \\ &\quad \lambda \longmapsto (\lambda S, \pi(\lambda)). \end{aligned}$$

For more details, see [36]. Let $P = (0 \ R) \in D^{q \times (q+p)}$, $P' = 0$, $X = (R^T \ (I_p - SR)^T)^T$ the above matrix defining the left D -isomorphism g , $L = D^{1 \times (q+p)}/(D^{1 \times q}P)$, $L' = D^{1 \times p}$, $\tau : D^{1 \times (q+p)} \longrightarrow L$ and $\tau' = \text{id}_{L'}$. Using the left D -isomorphism $L \cong D^{1 \times q} \oplus M$ defined by sending $\tau((\theta \ \lambda))$ to $(\theta, \pi(\lambda))$ for all $\theta \in D^{1 \times q}$ and for all $\lambda \in D^{1 \times p}$, then we get the following left D -isomorphism:

$$\begin{aligned} f : L &\longrightarrow L' = D^{1 \times p} \\ \tau((\theta \ \lambda)) &\longmapsto (\theta \ \lambda) \begin{pmatrix} R \\ I_p - SR \end{pmatrix}. \end{aligned} \tag{53}$$

If $\text{rank}_D(M) = p - q \geq 2$, then applying q times Algorithm 6, we obtain a left D -isomorphism $h : M \longrightarrow D^{1 \times (p-q)}$ defined by $h(\pi(\lambda)) = \lambda Q$ for a certain matrix $Q \in D^{p \times (p-q)}$ (called an *injective parametrization* in [11, 37]). Moreover, the matrix Q necessarily admits a left inverse $T \in D^{(p-q) \times p}$, i.e., $TQ = I_{p-q}$, and $\{\pi(T_{i\bullet})\}_{i=1, \dots, p-q}$ forms a basis of the free left D -module M of rank $p - q$, where $T_{i\bullet}$ denotes the i^{th} row of T . For more details, see [37].

Remark 15. Theorem 11 is based on solving quadratic equations of the form (11). As already noticed in [37], for the computation of bases of finitely generated free modules, we only need to find $\alpha, \beta \in D$ such that

$$d_1 D + d_2 D + d_3 D = (d_1 + d_3 \alpha) D + (d_2 + d_3 \beta) D, \tag{54}$$

which is generally faster and gives smaller results than solving equations of the form (11). Indeed, $f(\tau(1 \ 0)) = R_{1\bullet} \in U(D^{1 \times p})$ (i.e., $R_{1\bullet}$ admits a right inverse, e.g., the first column of S) and Algorithm 5 aims at finding $\bar{\lambda} \in D^{1 \times (p-1)}$ such that $X_{12} + X_{11} \bar{\lambda} \in U(D^{1 \times (p-1)})$, where $R_{1\bullet} = (X_{11} \ X_{12})$, $X_{11} \in D$, and $X_{12} \in D^{1 \times (p-1)}$. If d_1 and d_2 are two non-zero entries of X_{12} and $d_3 = X_{11}$, then there exist $\alpha, \beta \in D$ such that (54) holds. Since the right ideal generated by the entries of $R_{1\bullet}$ is D , so is the right ideal generated by the entries of X_{12} but where d_1 (resp., d_2) is replaced by $d_1 + d_3 \alpha$ (resp., $d_2 + d_3 \beta$). Then, $\bar{\lambda}$ can be chosen as the vector defined by α (resp., β) at the position corresponding to the one of d_1 (resp., d_2), and 0 elsewhere. Compare with [37].

Example 16. Let $D = A_1(\mathbb{Q})$, $R = (\partial \ t \ 0)$, and $M = D^{1 \times 3}/(DR)$. Since R admits the right inverse $S = (t \ -\partial \ 0)^T$, M is a stably free left D -module of rank 2, i.e., a free left D -module of rank 2 (see 4 of Theorem 1). Using Algorithm 6, let us compute a basis. Let $P = (0 \ R) \in D^{1 \times 4}$, $L = D^{1 \times 4}/(DP)$, $\tau : D^{1 \times 4} \longrightarrow L$ be the canonical projection onto L , $L' = D^{1 \times 3}$, and the left D -isomorphism $f : L \longrightarrow L'$ defined by (53), i.e., which sends

$\tau((\theta \ \lambda))$ to $(\theta \ \lambda) X$, where:

$$X = \begin{pmatrix} & R \\ I_p - S R & \end{pmatrix} = \begin{pmatrix} \partial & t & 0 \\ -t\partial + 1 & -t^2 & 0 \\ \partial^2 & t\partial + 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us define:

$$X_{11} = \partial, \quad X_{12} = (t \ 0), \quad X_{21} = \begin{pmatrix} -t\partial + 1 \\ \partial^2 \\ 0 \end{pmatrix}, \quad X_{22} = \begin{pmatrix} -t^2 & 0 \\ t\partial + 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using Remark 15 and $tD + 0D + \partial D = tD + (0 + \partial)D$, we can take $\alpha = 0$ and $\beta = 1$ and get $\bar{\lambda} = (0 \ 1)$ which is such that $X_{12} + X_{11}\bar{\lambda} = (t \ \partial) \in U(D^{1 \times 2})$ since $X_{12} + X_{11}\bar{\lambda}$ admits the right inverse $\bar{\mu} = (-\partial \ t)^T$. Computing the matrix X' defined by (51), we get

$$X'_{11} = 0, \quad X'_{12} = (t \ \partial), \quad X'_{21} = \begin{pmatrix} -t\partial + 1 \\ \partial^2 \\ -t\partial \end{pmatrix}, \quad X'_{22} = \begin{pmatrix} -t^2 & -t\partial + 1 \\ t\partial + 2 & \partial^2 \\ 0 & 1 \end{pmatrix},$$

which yields:

$$Q = X'_{22} + (X'_{21} - X'_{22}\bar{\mu})X'_{12} = \begin{pmatrix} -t^2(\partial + 1) & -t\partial^2 + (1 - t)\partial + 1 \\ t\partial^2 + (t + 2)\partial + 2 & \partial^2(\partial + 1) \\ -t(t(\partial + 1) + 1) & -t\partial(\partial + 1) + 1 \end{pmatrix}.$$

We can check that $\ker_D(\cdot Q) = DR$, and Q admits the following left inverse:

$$T = \begin{pmatrix} 0 & t\partial + t + 2 & \partial(\partial + 1) \\ t & -t^2 & -t(\partial + 1) + 1 \end{pmatrix}.$$

Therefore, we have $M \cong D^{1 \times 3}Q = D^{1 \times 2}$ and $\{\pi(T_{i\bullet})\}_{i=1,2}$ is a basis of M .

6.2.2 Rank reduction compatible with isomorphism

Let us consider a finitely generated left D -module M with $\text{rank}_D(M) \geq 2$. Using Theorem 7, there exist $r \in \mathbb{Z}_{\geq 0}$ and a finitely generated left D -module M' with $\text{rank}_D(M') \leq 1$ such that:

$$M \cong D^{1 \times r} \oplus M'.$$

Let us also suppose that we have another direct sum decomposition $M \cong D^{1 \times s} \oplus M''$, where $\text{rank}_D(M'') \leq 1$. Let us now study the relations between M' and M'' . The left D -isomorphism $D^{1 \times r} \oplus M' \cong D^{1 \times s} \oplus M''$ and the fact that D is a noetherian domain imply:

$$\text{rank}_D(M) = r + \text{rank}_D(M') = s + \text{rank}_D(M'').$$

Then, applying Stafford's cancellation theorem (see Theorem 11), one of the following implications holds:

1. If M' and M'' are two torsion left D -modules, then $r = s \geq 2$ and:

$$D^{1 \times (r-2)} \oplus (D^{1 \times 2} \oplus M') \cong D^{1 \times (r-2)} \oplus (D^{1 \times 2} \oplus M'') \Rightarrow D^{1 \times 2} \oplus M' \cong D^{1 \times 2} \oplus M''.$$

2. If $\text{rank}_D(M') = \text{rank}_D(M'') = 1$, then $r = s \geq 1$ and:

$$D^{1 \times (r-1)} \oplus (D \oplus M') \cong D^{1 \times (r-1)} \oplus (D \oplus M'') \Rightarrow D \oplus M' \cong D \oplus M''.$$

3. If $\text{rank}_D(M') = 1$ and M'' is a torsion left D -module, then $s = r + 1 \geq 2$ and:

$$D^{1 \times (r-1)} \oplus (D \oplus M') \cong D^{1 \times (s-2)} \oplus (D^{1 \times 2} \oplus M'') \Rightarrow D \oplus M' \cong D^{1 \times 2} \oplus M''.$$

4. If M' is a torsion left D -module and $\text{rank}_D(M'') = 1$, then $r = s + 1 \geq 2$ and:

$$D^{1 \times (r-2)} \oplus (D^{1 \times 2} \oplus M') \cong D^{1 \times (s-1)} \oplus (D \oplus M'') \Rightarrow D^{1 \times 2} \oplus M' \cong D \oplus M''.$$

6.2.3 Auslander transposes

Auslander transposes play a fundamental role in the study of the module structure of *Auslander regular rings*. For more details, see [11, 16, 35] and the references therein. Let us introduce this concept and its main properties.

Theorem 12 ([16]). *Let us consider the following finite presentations of a left D -module M :*

$$\begin{array}{ccccccc} D^{1 \times q} & \xrightarrow{\cdot R} & D^{1 \times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ D^{1 \times q'} & \xrightarrow{\cdot R'} & D^{1 \times p'} & \xrightarrow{\pi'} & M & \longrightarrow & 0. \end{array}$$

Let us consider the following two finitely presented right D -modules:

$$N := D^q / (R D^p), \quad N' := D^{q'} / (R' D^{p'}).$$

Then, we have

$$D^{q+p} \oplus N \cong D^{q+p'} \oplus N' \tag{55}$$

*as right D -modules. Hence, N and N' are projectively equivalent, which shows that M defines a unique projective equivalence class of right D -modules. Any representative of this projective equivalence class is called an *Auslander transpose* of M . In particular, we have:*

$$\text{rank}_D(N) + q' + p = \text{rank}_D(N') + q + p'.$$

We refer to [16] for the explicit construction of a right D -isomorphism as in (55).

The Auslander transpose of a finitely presented left D -module is a finitely presented right D -module. If $D = A\langle \partial_1, \dots, \partial_n \rangle$, then using the involution θ of D defined by (10), the right D -module structure on an Auslander transpose can be turned into a left D -module structure. More precisely, if we introduce the so-called *adjoint matrix* $\tilde{R} := (\theta(R_{ij}))^T \in D^{p \times q}$ of $R \in D^{q \times p}$, then the left D -module $\tilde{N} := D^{1 \times q} / (D^{1 \times p} \tilde{R})$ is called the *adjoint* of the left D -module $M = D^{1 \times p} / (D^{1 \times q} R)$. Then, (55) yields:

$$D^{1 \times (q'+p)} \oplus \tilde{N} \cong D^{1 \times (q+p')} \oplus \tilde{N}'. \tag{56}$$

Turning right D -modules into left D -modules by means of an involution of the ring is usually useful in practice since many computer algebra systems (e.g., Maple) are only equipped with left Gröbner basis techniques. In mathematical physics, adjoint modules play an important role as shown in [32] (see also the references therein and [16]).

Corollary 4. *Let D be a very simple domain. With the notations of Theorem 12, if $r = q' + p$, $r' = q + p'$, and $s \in \mathbb{Z}_{\geq 0}$ is such that*

$$\text{rank}_D(N) + r - s = \text{rank}_D(N') + r' - s \geq 2,$$

(55) yields:

$$D^{r-s} \oplus N \cong D^{r'-s} \oplus N'.$$

Moreover, if D admits an involution, then:

$$D^{1 \times (r-s)} \oplus \tilde{N} \cong D^{1 \times (r'-s)} \oplus \tilde{N}'.$$

For more examples illustrating Theorem 11, see [39].

7 Conclusion

The constructive study of Stafford's theorems ([44]) on the module structure of the Weyl algebras $A_n(k)$ over a field k of characteristic zero was initiated in [22, 26] and continued in [20, 37]. In the present paper, we have studied the rest of Stafford's theorems from a constructive viewpoint. The corresponding algorithms have been implemented in the STAFFORD package ([37, 39]) and the main results of the paper were illustrated with explicit and classical linear systems of PD equations. Finally, we have also explained that Stafford's theorems could be extended to the case of rings of PD operators with coefficients in various differential rings or fields, or more generally, to any very simple domain. Since many linear functional systems of practical interest can be defined by means of a finitely presented left module over an Ore extension [31], a quantum algebra [31] or a skew PBW extension [21], an interesting problem is to investigate when these rings are very simple domains.

We hope that the results of this paper and those of [20, 37] will motivate the symbolic computation community to pay more attention to constructive versions of Stafford's main theorem (see (7)). To be able to efficiently compute two "simple generators" of ideals of $A_n(k)$ is the main issue for studying the module structure of $A_n(k)$ (e.g., by means of the algorithms in the present paper) and using it in the study of linear PD systems within algebraic analysis or D -module theory.

The feasibility of an implementation of the extensions of Stafford's theorems (see Theorems 4 and 5) in a computer algebra system also needs to be studied in the future. For analytic linear OD systems, an extension of the STAFFORD package called STAFFORDANALYTIC is under development.

Stafford's theorems ([44]) have been generalized in another important direction in [18] where it was shown that the module structure of the ring $\mathcal{D}(X)$ of PD operators on a smooth irreducible affine variety X over a field k of characteristic zero is similar to the one of the Weyl algebra $A_n(k) = \mathcal{D}(\mathbb{A}^n(k))$, where $\mathbb{A}^n(k)$ denotes affine space of dimension n over k . The only differences are that one-sided ideals can now be generated by three elements instead of two and the condition $\text{rank}_{A_n(k)}(M) \geq 2$ in the theorems of [44] and of this paper has to be replaced by $\text{rank}_{\mathcal{D}(X)}(M) \geq 3$. These results can also be extended to certain singular affine algebraic varieties (see [18]). It was also shown in [8] that statements similar to the ones for $\mathcal{D}(X)$ hold for $A_n(K)$, where K is a division ring of characteristic zero (e.g., the quaternions \mathbb{H}

or the division ring of fractions of $A_m(k)$ (see Section 2)). Based on the constructive versions of Stafford’s theorems developed in this paper, we are currently studying these extensions.

Finally, the properties of (the category of) finitely generated modules over a very simple domain should be studied in detail (as shown in [18], it has good functorial properties such as respecting localizations with respect to Ore sets).

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